Problem 10.22. Show that the choice $\gamma = \mu / 2$ minimizes the probability of error given by Eq. (10.26). Hint: The *Q*-function is continuously differentiable.

Solution

From (10.26), we have the average probability of error as:

$$P_e(\gamma) = \frac{1}{2} Q\left(\frac{\mu - \gamma}{\sigma}\right) + \frac{1}{2} Q\left(\frac{\gamma}{\sigma}\right)$$

Recall the definition of *Q*-function:

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp(-s^2/2)ds \\ &(\text{let } u = -s) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp(-u^2/2)du \end{aligned}$$

So the derivative is given by

$$\frac{dQ(x)}{dx} = \frac{-1}{\sqrt{2\pi}} \exp(-x^2/2) \le 0$$

Substituting this result into the definition of $P_e(\gamma)$ we obtain

$$\begin{aligned} \frac{dP_e(\gamma)}{d\gamma} &= \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) \cdot \frac{-1}{\sigma} + \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma} \\ &= \frac{1}{2\sqrt{2\pi}\sigma} \left\{ \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) - \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \right\} \end{aligned}$$

Setting $\frac{dP_e(\gamma)}{d\gamma} = 0$ implies

$$\begin{aligned} \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) &= \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \\ (\mu-\gamma)^2 &= \gamma^2 \\ \gamma &= \mu/2 \end{aligned}$$

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Problem 10.22 continued

Checking the second derivative, we have

$$\frac{d^2P_e(\gamma)}{d^2\gamma} = \frac{1}{2\sqrt{2\pi}\sigma}\left[\frac{2(\mu-\gamma)}{2\sigma^2}\cdot\exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right)+\frac{2\gamma}{2\sigma^2}\exp\left(-\frac{\gamma^2}{2\sigma^2}\right)\right]$$

$$> 0$$

when $\gamma = \mu/2$. Therefore at $\gamma = \mu/2$, $P_e(\gamma)$ has a minimum value.