Problem 10.22. Show that the choice $\gamma = \mu/2$ minimizes the probability of error given by Eq. (10.26). Hint: The *Q*-function is continuously differentiable.

Solution

From (10.26), we have the average probability of error as:

$$P_{e}(\gamma) = \frac{1}{2}Q\left(\frac{\mu - \gamma}{\sigma}\right) + \frac{1}{2}Q\left(\frac{\gamma}{\sigma}\right)$$

Recall the definition of *Q*-function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} \exp(-s^{2}/2) ds$$

(let $u = -s$)
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp(-u^{2}/2) du$$

So the derivative is given by

$$\frac{dQ(x)}{dx} = \frac{-1}{\sqrt{2\pi}} \exp(-x^2/2) \le 0$$

Substituting this result into the definition of $P_e(\gamma)$ we obtain

$$\frac{dP_e(\gamma)}{d\gamma} = \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(\mu - \gamma)^2}{2\sigma^2}\right) \cdot \frac{-1}{\sigma} + \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma}$$
$$= \frac{1}{2\sqrt{2\pi\sigma}} \left\{ \exp\left(-\frac{(\mu - \gamma)^2}{2\sigma^2}\right) - \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \right\}$$

Setting $\frac{dP_e(\gamma)}{d\gamma} = 0$ implies $\exp\left(-\frac{(\mu - \gamma)^2}{2\sigma^2}\right) = \exp\left(-\frac{\gamma^2}{2\sigma^2}\right)$ $(\mu - \gamma)^2 = \gamma^2$ $\gamma = \mu/2$

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Problem 10.22 continued

Checking the second derivative, we have

$$\frac{d^2 P_e(\gamma)}{d^2 \gamma} = \frac{1}{2\sqrt{2\pi\sigma}} \left[\frac{2(\mu - \gamma)}{2\sigma^2} \cdot \exp\left(-\frac{(\mu - \gamma)^2}{2\sigma^2}\right) + \frac{2\gamma}{2\sigma^2} \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \right]$$

> 0

when $\gamma = \mu/2$. Therefore at $\gamma = \mu/2$, $P_e(\gamma)$ has a minimum value.

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