Problem 10.23. For *M*-ary PAM,

(a) Show that the formula for probability of error, namely,

$$P_e = 2\left(\frac{M-1}{M}\right)Q\left(\frac{A}{\sigma}\right)$$

holds for *M* = 2, 3, and 4. By mathematical induction, show that it holds for all *M*.

(b) Show the formula for average power, namely,

$$P = \frac{(M^2-1)A^2}{3}$$

holds for *M* = 2, and 3. Show it holds for all *M*.

<u>Solution</u>

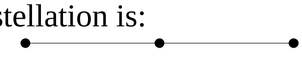
(a) *M*-ary PAM with the separation between nearest neighbours as 2A. Assume that all *M* symbols are equally transmitted.

(i) For *M*=2, we have the result given in the text for binary PAM

$$\begin{aligned} P_e^{2PAM} &= Q\left(\frac{A}{\sigma}\right) \\ &= 2\frac{M-1}{M}Q\left(\frac{A}{\sigma}\right) \end{aligned}$$

for *M* = 2.

(ii) For *M* = 3, the constellation is:



$$\begin{aligned} P_e &= \frac{1}{3}P\left[y > -A \mid (-2A) \text{ is transmitted}\right] + \frac{1}{3}P\left[y > A \text{ or } y < -A \mid 0 \text{ is transmitted}\right] \\ &\quad + \frac{1}{3}P\left[y < A \mid (2A) \text{ is transmitted}\right] \\ &= \frac{1}{3}\int_{-A}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y+2A)^2}{2\sigma^2}\right)dy + \frac{1}{3}\int_{A}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy \\ &\quad + \frac{1}{3}\int_{-\infty}^{-A}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy + \frac{1}{3}\int_{-\infty}^{A}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-2A)^2}{2\sigma^2}\right)dy \\ &= \frac{4}{3}Q\left(\frac{A}{\sigma}\right) \end{aligned}$$

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Problem 10.23 continued

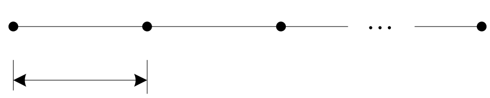
From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=3, $P_e = \frac{4}{3} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 3.

(iii) For M = 4, the constellation is:

$$P_e = \frac{1}{4} P\left[y > -2A \,|\, (-3A) \text{ is transmitted}\right] + \frac{1}{4} P\left[y < -2A \text{ or } y > 0 \,|\, -A \text{ is transmitted}\right]$$
$$+ \frac{1}{4} P\left[y < 0 \text{ or } y > 2A \,|\, +A \text{ is transmitted}\right] + \frac{1}{4} P\left[y < 2A \,|\, (3A) \text{ is transmitted}\right]$$
$$= \frac{1}{4} \int_{-2A}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+3A)^2}{2\sigma^2}\right) dy$$
$$+ 2\left[\frac{1}{4} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+A)^2}{2\sigma^2}\right) dy + \frac{1}{4} \int_{-\infty}^{-2A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+A)^2}{2\sigma^2}\right) dy\right]$$
$$+ \frac{1}{4} \int_{-\infty}^{A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-3A)^2}{2\sigma^2}\right) dy$$
$$= \frac{6}{4} Q\left(\frac{A}{\sigma}\right)$$

where the factor 2 in the third last line, comes from the symmetry of the second and third terms of the first equation. From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=4, $P_e = \frac{6}{4} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 4.

(iv) Assume that the formula of P_e holds for (M-1)-ary PAM. By mathematical induction, we need to show it also holds for M-ary PAM. The (M-1)-ary PAM constellation may be illustrated as shown:



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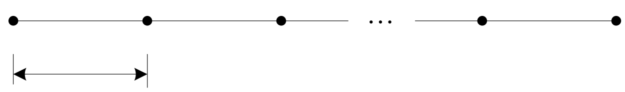
Problem 10.23 continued

By adding one point P2 on the (M-1)-ary PAM constellation, which has the distance 2A from point P1, we obtain M-ary PAM constellation as follows (in practice, the average or dc level may be adjusted as well but this has no effect on the symbol error rate):



Since error probabilities of P1 symbol on the (M-1)-ary PAM is the same as that of P2 point on the M-ary PAM, the error probability of M-ary PAM is

$$P_e^{M-ary} = \frac{M-1}{M} P_e^{(M-1)-ary} + \frac{1}{M} \cdot \text{symbol error prob. of P1 symbol on } M\text{-ary} \qquad (1)$$

where $1/M$ is the probability that P1 is transmitted and $(M-1)/M$ is the probability that one of the other constellation points is transmitted. The probability of error formula for (M-1)-ary PAM is given by

$$P_e^{(M-1)-ary} = 2\frac{(M-2)}{(M-1)} Q\left(\frac{A}{\sigma}\right). \qquad (2)$$

2A

The symbol error rate of P1 symbol on M-ary PAM is

$$P_{P1} = \frac{1}{M} P\left[y < (\mu - A), \text{or } y > (\mu + A) \mid \text{P1 is transmitted} \right]$$

where μ is the signal level of P1 symbol.

$$\begin{aligned} P_{P1} &= \frac{1}{M}\int_{-\infty}^{\mu-A} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy + \frac{1}{M}\int_{\mu+A}^{+\infty} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \\ &= \frac{2}{M} Q\left(\frac{A}{\sigma}\right) \end{aligned} \qquad (3)$$

Substituting Eqs. (2) and (3) into (1), we obtain the symbol error probability of M-ary PAM

$$\begin{aligned} P_e^{M-PAM} &= \frac{M-1}{M} \cdot 2 \cdot \frac{M-2}{M-1} Q\left(\frac{A}{\sigma}\right) + \frac{2}{M} Q\left(\frac{A}{\sigma}\right) \\ &= 2\frac{M-1}{M} Q\left(\frac{A}{\sigma}\right) \end{aligned}.$$

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Problem 10.23 continued

The formula holds for M-ary PAM. Therefore, by mathematical induction, the formula holds for all M.

(b) To compute the average symbol power we note:

i) For $M = 2$, the average symbol power is A^2 and the formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=2.

ii) For $M = 3$, the average symbol energy is

$$P = \frac{1}{3}\left((2A)^2 + 0^2 + (2A)^2\right) = \frac{8}{3}A^2 .$$

The formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=3.

iii) For general even M, the M-ary PAM constellation points are

$$\{-(M-1)A, \cdots, -3A, -A, A, 3A, \cdots, (M-1)A\} .$$

The average symbol energy is

$$\begin{aligned} P &= \frac{2\left[(M-1)^2 + (M-3)^2 + \cdots + 3^2 + 1\right]}{M} A^2 \\ &= \frac{2A^2}{M}\sum_{k=1}^{M/2}(2k-1)^2 \\ &= \frac{2A^2}{M}\left[2^2\sum_{k=1}^{M/2}k^2 - 4\sum_{k=1}^{M/2}k + \sum_{k=1}^{M/2}1\right] \\ &= \frac{2A^2}{M}\left[4\frac{M(M/2+1)(M+1)}{2\cdot 6} - 4\frac{M(M/2+1)}{2\cdot 2} + \frac{M}{2}\right] \\ &= \frac{(M^2-1)A^2}{3} \end{aligned} .$$

where we have used the summation formulas of Appendix 6.

iv) For general odd M, the M-ary PAM constellation points are

$$\{-(M-1)A, \cdots, -2A, 0, 2A, \cdots (M-1)A\} .$$

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Problem 10.23 continued

The average symbol energy is

$$
\begin{aligned}
P &= \frac{2\left[(M-1)^2 + (M-3)^2 + \cdots + 2^2\right]}{M} A^2 \\
&= \frac{2A^2}{M} 2^2 \left[\left(\frac{M-1}{2}\right)^2 + \left(\frac{M-3}{2}\right)^2 + \ldots + 1^2 \right] \\
&= \frac{8A^2}{M} \sum_{k=1}^{(M-1)/2} k^2 \\
&= \frac{8A^2}{M} \frac{(M-1)(M+1)(M)}{2 \cdot 2 \cdot 6} \\
&= \frac{(M^2-1)A^2}{3}
\end{aligned}
$$

where the fourth line uses the summation formula found in Appendix 6.