**Problem 10.24**. Consider binary FSK transmission where  $(f_1 - f_2)T$  is not an integer.

- (a) What is the mean output of the upper correlator of Fig. 10.12, if a 1 is transmitted? What is the mean output of the lower correlator?
- (b) Are the random variables  $N_1$  and  $N_2$  independent under these conditions? What is the variance of  $N_1 N_2$ ?
- (c) Describe the properties of the random variable D of Fig. 10.12 in this case.

## Solution:

(a) If a 1 is transmitted,

$$r(t) = A_c \cos(2\pi f_1 t) + n(t)$$

where n(t) is a narrow band Gaussian noise. The output of the upper correlator is  $Y_1$ :

$$Y_{1} = \int_{0}^{T} r(t)\sqrt{2}\cos(2\pi f_{1}t)dt$$
  
=  $\int_{0}^{T}\sqrt{2}A_{c}\cos(2\pi f_{1}t)\cos(2\pi f_{1}t)dt + \int_{0}^{T}\sqrt{2}n(t)\cos(2\pi f_{1}t)dt$   
$$\cong \frac{1}{\sqrt{2}}A_{c}T + \int_{0}^{T}\sqrt{2}n(t)\cos(2\pi f_{1}t)dt$$

The expected value of  $Y_1$  is  $\mathbf{E}[Y_1] = \frac{1}{2}A_cT$ , since n(t) has zero mean.

The output of the lower correlator is  $Y_2$ :

$$\begin{split} Y_2 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_2 t)dt \\ &= \int_0^T \sqrt{2}A_c\cos(2\pi f_1 t)\cos(2\pi f_2 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \\ &= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 + f_2)t)dt + \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 - f_2)t)dt + \sqrt{2}\int_0^T n(t)\cos(2\pi f_2 t)dt \\ &\cong \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 - f_2)t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \end{split}$$

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## **Problem 10.24 continued**

where the first term of the third line is negligible due to the bandpass assumption. The expected value of  $Y_2$  is

$$\mathbf{E}[Y_2] = \frac{A_c}{\sqrt{2}} \int_0^T \cos(2\pi (f_1 - f_2)t) dt$$
  
=  $\frac{A_c}{\sqrt{2}} \cdot \frac{1}{2\pi (f_1 - f_2)} \sin\left[2\pi (f_1 - f_2)t\right] \Big|_0^T$   
=  $\frac{A_c}{2\sqrt{2}\pi (f_1 - f_2)} \sin(2\pi (f_1 - f_2)T)$ 

which clearly differs from the orthogonal case.

(b) The random variables  $N_1$  and  $N_2$  are given by

$$N_1 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt$$
$$N_2 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt$$

Since n(t) is a Gaussian process, both  $N_1$  and  $N_2$  are Gaussian. To show  $N_1$  and  $N_2$  are correlated consider

$$\begin{split} \mathbf{E}[N_1 N_2] &= \mathbf{E}\bigg[\int_0^T n(t)\cos(2\pi f_1 t)dt \cdot \int_0^T n(\tau)\cos(2\pi f_2 \tau)d\tau\bigg] \\ &= \int_0^T \int_0^T \mathbf{E}[n(t)n(\tau)]\cos(2\pi f_1 t)\cos(2\pi f_2 \tau)dtd\tau \\ &= \int_0^T \int_0^T \frac{N_0}{2} \,\delta(t-\tau)\cos(2\pi f_1 t)\cos(2\pi f_2 \tau)dtd\tau \\ &= \frac{N_0}{2} \int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt \\ &= \frac{N_0}{4} \int_0^T \bigg[\cos(2\pi (f_1 + f_2)t) + \cos(2\pi (f_1 - f_2)t)\bigg]dt \\ &= \frac{N_0}{4} \frac{\sin(2\pi (f_1 + f_2)t)}{2\pi (f_1 + f_2)}\bigg|_0^T + \frac{\sin(2\pi (f_1 - f_2)t)}{2\pi (f_1 - f_2)}\bigg|_0^T \\ &\cong \frac{N_0}{4} \operatorname{sinc}(2(f_1 - f_2)T) \end{split}$$

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## **Problem 10.24 continued**

where the first term of the second last line is assumed negligible due to the bandpass assumption. Since  $N_1$  and  $N_2$  are correlated, they are not independent. The variance of  $(N_1-N_2)$  is

$$\operatorname{var}[N_{1} - N_{2}] = \operatorname{var}[N_{1}] + \operatorname{var}[N_{2}] - 2\mathbf{E}[N_{1}N_{2}]$$
$$= N_{0} - \frac{N_{0}}{2}\operatorname{sinc}(2(f_{1} - f_{2})T)$$

(c) The random variable D is Gaussian with zero mean and variance  $var[N_1-N_2]$ .

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