Problem 10.24. Consider binary FSK transmission where $(f_1 - f_2)T$ is not an integer.

- (a) What is the mean output of the upper correlator of Fig. 10.12, if a 1 is transmitted? What is the mean output of the lower correlator?
- (b) Are the random variables N_1 and N_2 independent under these conditions? What is the variance of $N_1 - N_2$?
- (c) Describe the properties of the random variable *D* of Fig. 10.12 in this case.

Solution:

(a) If a 1 is transmitted,

$$
r(t) = A_c \cos(2\pi f_1 t) + n(t)
$$

where $n(t)$ is a narrow band Gaussian noise. The output of the upper correlator is Y_1 :

$$
Y_1 = \int_0^T r(t)\sqrt{2}\cos(2\pi f_1 t)dt
$$

= $\int_0^T \sqrt{2}A_c \cos(2\pi f_1 t) \cos(2\pi f_1 t)dt + \int_0^T \sqrt{2}n(t) \cos(2\pi f_1 t)dt$
 $\approx \frac{1}{\sqrt{2}}A_cT + \int_0^T \sqrt{2}n(t) \cos(2\pi f_1 t)dt$

The expected value of Y_1 is $\mathbf{E}[Y_1] = \frac{1}{2} A_c T$, since $n(t)$ has zero mean.

The output of the lower correlator is Y_2 :

$$
Y_2 = \int_0^T r(t)\sqrt{2}\cos(2\pi f_2 t)dt
$$

\n
$$
= \int_0^T \sqrt{2}A_c\cos(2\pi f_1 t)\cos(2\pi f_2 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt
$$

\n
$$
= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 + f_2)t)dt + \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 - f_2)t)dt + \sqrt{2}\int_0^T n(t)\cos(2\pi f_2 t)dt
$$

\n
$$
\approx \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi (f_1 - f_2)t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt
$$

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Problem 10.24 continued

where the first term of the third line is negligible due to the bandpass assumption. The expected value of Y_2 is

$$
\begin{aligned} \mathbf{E}[Y_2] &= \frac{A_c}{\sqrt{2}} \int_0^T \cos(2\pi (f_1 - f_2)t) dt \\ &= \frac{A_c}{\sqrt{2}} \cdot \frac{1}{2\pi (f_1 - f_2)} \sin\left[2\pi (f_1 - f_2)t\right]_0^T \\ &= \frac{A_c}{2\sqrt{2}\pi (f_1 - f_2)} \sin(2\pi (f_1 - f_2)T) \end{aligned}
$$

which clearly differs from the orthogonal case.

(b) The random variables N_1 and N_2 are given by

$$
N_1 = \int_0^T \sqrt{2n(t)\cos(2\pi f_1 t)}dt
$$

$$
N_2 = \int_0^T \sqrt{2n(t)\cos(2\pi f_2 t)}dt
$$

Since $n(t)$ is a Gaussian process, both N_1 and N_2 are Gaussian. To show N_1 and N_2 are correlated consider

$$
\mathbf{E}[N_1N_2] = \mathbf{E} \bigg[\int_0^T n(t) \cos(2\pi f_1 t) dt \cdot \int_0^T n(\tau) \cos(2\pi f_2 \tau) d\tau \bigg]
$$

\n
$$
= \int_0^T \int_0^T \mathbf{E}[n(t)n(\tau)] \cos(2\pi f_1 t) \cos(2\pi f_2 \tau) dt d\tau
$$

\n
$$
= \int_0^T \int_0^T \frac{N_0}{2} \delta(t-\tau) \cos(2\pi f_1 t) \cos(2\pi f_2 \tau) dt d\tau
$$

\n
$$
= \frac{N_0}{2} \int_0^T \cos(2\pi f_1 t) \cos(2\pi f_2 t) dt
$$

\n
$$
= \frac{N_0}{4} \int_0^T \big[\cos(2\pi (f_1 + f_2)t) + \cos(2\pi (f_1 - f_2)t) \big] dt
$$

\n
$$
= \frac{N_0}{4} \frac{\sin(2\pi (f_1 + f_2)t)}{2\pi (f_1 + f_2)} \bigg|_0^T + \frac{\sin(2\pi (f_1 - f_2)t)}{2\pi (f_1 - f_2)} \bigg|_0^T
$$

\n
$$
\approx \frac{N_0}{4} \operatorname{sinc}(2(f_1 - f_2)T)
$$

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Problem 10.24 continued

where the first term of the second last line is assumed negligible due to the bandpass assumption. Since N_1 and N_2 are correlated, they are not independent. The variance of $(N_1 - N_2)$ is

$$
var[N_1 - N_2] = var[N_1] + var[N_2] - 2E[N_1N_2]
$$

= $N_0 - \frac{N_0}{2} sinc(2(f_1 - f_2)T)$

(c) The random variable *D* is Gaussian with zero mean and variance var[N_1 - N_2].

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