Problem 10.24. Consider binary FSK transmission where $(f_1 - f_2)T$ is not an integer.

(a) What is the mean output of the upper correlator of Fig. 10.12, if a 1 is transmitted? What is the mean output of the lower correlator?

(b) Are the random variables N_1 and N_2 independent under these conditions? What is the variance of $N_1 - N_2$?

(c) Describe the properties of the random variable D of Fig. 10.12 in this case.

Solution:

(a) If a 1 is transmitted,

$$r(t) = A_c \cos(2\pi f_1 t) + n(t)$$

where $n(t)$ is a narrow band Gaussian noise. The output of the upper correlator is Y_1:

$$\begin{aligned}
Y_1 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_1 t)dt \\
&= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_1 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt \\
&\cong \frac{1}{\sqrt{2}} A_c T + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt
\end{aligned}$$

The expected value of Y_1 is $\mathbf{E}[Y_1] = \frac{1}{2} A_c T$, since $n(t)$ has zero mean.

The output of the lower correlator is Y_2:

$$\begin{aligned}
Y_2 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_2 t)dt \\
&= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \\
&= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 + f_2)t)dt + \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \sqrt{2}\int_0^T n(t)\cos(2\pi f_2 t)dt \\
&\cong \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt
\end{aligned}$$

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Problem 10.24 continued

where the first term of the third line is negligible due to the bandpass assumption. The expected value of Y_2 is

$$\mathbf{E}[Y_2] = \frac{A_c}{\sqrt{2}} \int_0^T \cos(2\pi(f_1 - f_2)t)dt$$

$$= \frac{A_c}{\sqrt{2}} \cdot \frac{1}{2\pi\left(f_1 - f_2\right)} \sin\left[2\pi\left(f_1 - f_2\right)t\right]\Big|_0^T$$

$$= \frac{A_c}{2\sqrt{2}\pi\left(f_1 - f_2\right)} \sin(2\pi\left(f_1 - f_2\right)T)$$

which clearly differs from the orthogonal case.

(b) The random variables N_1 and N_2 are given by

$$N_1 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt$$

$$N_2 = \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt$$

Since $n(t)$ is a Gaussian process, both N_1 and N_2 are Gaussian. To show N_1 and N_2 are correlated consider

$$\mathbf{E}[N_1 N_2] = \mathbf{E}\left[\int_0^T n(t)\cos(2\pi f_1 t)dt \cdot \int_0^T n(\tau)\cos(2\pi f_2 \tau)d\tau\right]$$

$$= \int_0^T \int_0^T \mathbf{E}[n(t)n(\tau)]\cos(2\pi f_1 t)\cos(2\pi f_2 \tau)dtd\tau$$

$$= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\tau)\cos(2\pi f_1 t)\cos(2\pi f_2 \tau)dtd\tau$$

$$= \frac{N_0}{2}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt$$

$$= \frac{N_0}{4}\int_0^T \left[\cos(2\pi(f_1 + f_2)t) + \cos(2\pi(f_1 - f_2)t)\right]dt$$

$$= \frac{N_0}{4} \frac{\sin\left(2\pi(f_1 + f_2)t\right)}{2\pi(f_1 + f_2)}\Big|_0^T + \frac{\sin\left(2\pi(f_1 - f_2)t\right)}{2\pi(f_1 - f_2)}\Big|_0^T$$

$$\cong \frac{N_0}{4}\operatorname{sinc}(2(f_1 - f_2)T)$$

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Problem 10.24 continued

where the first term of the second last line is assumed negligible due to the bandpass assumption. Since N_1 and N_2 are correlated, they are not independent. The variance of (N_1-N_2) is

$$\begin{aligned}\text{var}[N_1 - N_2] &= \text{var}[N_1] + \text{var}[N_2] - 2\mathbf{E}[N_1 N_2] \\ &= N_0 - \frac{N_0}{2}\text{sinc}\left(2(f_1 - f_2)T\right)\end{aligned}$$

(c) The random variable D is Gaussian with zero mean and variance var[N_1-N_2].