Problem 10.27. A BPSK signal is applied to a matched-filter receiver that lacks perfect phase synchronization with the transmitter. Specifically, it is supplied with a local carrier whose phase differs from that of the carrier used in the transmitter by ϕ radians.

(a) Determine the effect of the phase error ϕ on the average probability of error of this receiver.

(b) As a check on the formula derived in part (a), show that when the phase error is zero the formula reduces to the same form as in Eq. (10.44).

Solution

(a) With BPSK, assume the transmitted signal is (10.36):

$$s(t) = A_c \sum_{k=0}^{N} b_k h(t-kT)\cos(2\pi f_c t),$$

where b_k = +1 for a 1 and b_k = -1 for a 0, $h(t)$ is the rectangular pulse $\text{rect}\left(\frac{t-T/2}{T}\right)$.

The received signal is

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c \sum_{k=0}^{N} b_k h(t-kT)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

The receiver matched filter is the integrate-and-dump filter. The output for the k^{th} symbol after down-conversion with phase error ϕ and match filtering is:

$$\begin{aligned} Y_k &= \int_{(k-1)T}^{kT} x(t)\cos(2\pi f_c t + \phi)dt \\ &= \int_{(k-1)T}^{kT} \left[A_c b_k + n_I(t)\right]\cos(2\pi f_c t)\cos(2\pi f_c t + \phi)dt - \int_{(k-1)T}^{kT} n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t+\phi)dt \\ &= \int_{(k-1)T}^{kT} \frac{1}{2}[A_c b_k + n_I(t)][\cos\phi + \cos(4\pi f_c t + \phi)]dt - \int_{(k-1)T}^{kT} \frac{1}{2} n_Q(t)[\sin(4\pi f_c t + \phi) + \sin(-\phi)]dt \\ &\cong \frac{T}{2} A_c b_k \cos\phi + \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt \\ &= \frac{T}{2} A_c b_k \cos\phi + N_k \end{aligned}$$

where we define

$$N_k = \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt$$

Continued on next slide

Problem 10.27 continued

The random variable N_k has zero mean and variance

$$\begin{aligned}\text{var}[N_k] &= \cos^2\phi \frac{N_0 T}{4} + \sin^2\phi \frac{N_0 T}{4} \\ &= \frac{N_0 T}{4} = \sigma^2\end{aligned}$$

Let $\mu = \frac{T}{2} A_c \cos\phi$. Then the probability of bit error P_e is

$$\begin{aligned}P_e &= \mathbf{P}[b_k = 1]\mathbf{P}[Y_k < 0 \mid b_k = 1] + \mathbf{P}[b_k = -1]\mathbf{P}[Y_k > 0 \mid b_k = -1] \\ &= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{\sigma^2}\right\} dy + \frac{1}{2}\int_{0}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y+\mu)^2}{\sigma^2}\right\} dy \\ &= Q\left(\frac{\mu}{\sigma}\right)\end{aligned}$$

with $E_b = \frac{A_c^2 T}{2}, \mu = \frac{T}{2} A_c \cos\phi, \sigma = \frac{1}{2}\sqrt{N_0 T}$, we have $P_e = Q\left(\sqrt{\frac{2E_b \cos\phi}{N_0}}\right)$

(b) When the phase error ϕ=0, $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, as the same as Eq. (10.44).