Problem 10.9. Use Eqs. (10.61), (10.64), and (10.66) to show that N_1 and N_2 are uncorrelated and therefore independent Gaussian random variables. Compute the variance of N_1-N_2.

Solution

The correlation of N_1 and N_2 is

$$\begin{aligned}
\mathbf{E}\left(N_1N_2\right) &= \mathbf{E}\left[2\int_0^T\int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)dsdt\right] \\
&= 2\int_0^T\int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)dsdt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right)ds\,dt \\
&= N_0\int_0^T \cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 t\right)dt \\
&= 0
\end{aligned}$$

where the last line follows from Eq.(10.61). Since N_1 and N_2 are uncorrelated

$$\begin{aligned}
\mathbf{E}\left[\left(N_1-N_2\right)^2\right] &= \mathbf{E}\left[\left(N_1\right)^2\right]+2\mathbf{E}\left[N_1N_2\right]+\mathbf{E}\left[\left(N_2\right)^2\right] \\
&= \mathbf{E}\left[\left(N_1\right)^2\right]+\mathbf{E}\left[\left(N_2\right)^2\right]
\end{aligned}$$

The variance of the N_1 term is

$$\begin{aligned}
\mathbf{E}\left(N_1N_1\right) &= \mathbf{E}\left[2\int_0^T\int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)dsdt\right] \\
&= 2\int_0^T\int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)dsdt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right)ds\,dt \\
&= N_0\int_0^T \cos^2\left(2\pi f_0 t\right)dt
\end{aligned}$$

Using the double angle formula $2\cos^2\theta = 1+2\cos\theta$, we have

$$\begin{aligned}
\mathbf{E}\left[\left(N_1\right)^2\right] &= \frac{N_0}{2}\int_0^T\left(1+\cos 4\pi ft\right)dt \\
&= \frac{N_0T}{2}
\end{aligned}$$

The derivation of the variance of N_2 is similar and the combined variance is N_0T.