**Problem 10.9.** Use Eqs. (10.61), (10.64), and (10.66) to show that  $N_1$  and  $N_2$  are uncorrelated and therefore independent Gaussian random variables. Compute the variance of  $N_1$ - $N_2$ .

## **Solution**

The correlation of  $N_1$  and  $N_2$  is

$$\mathbf{E}(N_1N_2) = \mathbf{E}\left[2\int_0^T \int_0^T w(s)w(t)\cos\left(2\pi f_1t\right)\cos\left(2\pi f_2s\right)dsdt\right]$$
$$= 2\int_0^T \int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1t\right)\cos\left(2\pi f_2s\right)dsdt$$
$$= 2\frac{N_0}{2}\int\int\delta(t-s)\cos\left(2\pi f_1t\right)\cos\left(2\pi f_2s\right)dsdt$$
$$= N_0\int_0^T\cos\left(2\pi f_1t\right)\cos\left(2\pi f_2t\right)dt$$
$$= 0$$

where the last line follows from Eq.(10.61). Since  $N_1$  and  $N_2$  are uncorrelated

$$\mathbf{E}\left[\left(N_{1}-N_{2}\right)^{2}\right] = \mathbf{E}\left[\left(N_{1}\right)^{2}\right] + 2\mathbf{E}\left[N_{1}N_{2}\right] + \mathbf{E}\left[\left(N_{2}\right)^{2}\right]$$
$$= \mathbf{E}\left[\left(N_{1}\right)^{2}\right] + \mathbf{E}\left[\left(N_{2}\right)^{2}\right]$$

The variance of the  $N_1$  term is

$$\mathbf{E}(N_{1}N_{1}) = \mathbf{E}\left[2\int_{0}^{T}\int_{0}^{T}w(s)w(t)\cos(2\pi f_{1}t)\cos(2\pi f_{1}s)dsdt\right]$$
  
=  $2\int_{0}^{T}\int_{0}^{T}\mathbf{E}[w(s)w(t)]\cos(2\pi f_{1}t)\cos(2\pi f_{1}s)dsdt$   
=  $2\frac{N_{0}}{2}\int\int\delta(t-s)\cos(2\pi f_{1}t)\cos(2\pi f_{1}s)dsdt$   
=  $N_{0}\int_{0}^{T}\cos^{2}(2\pi f_{0}t)dt$ 

Using the double angle formula  $2\cos^2\theta = 1 + 2\cos\theta$ , we have

$$\mathbf{E}\left[\left(N_{1}\right)^{2}\right] = \frac{N_{0}}{2} \int_{0}^{T} \left(1 + \cos 4\pi ft\right) dt$$
$$= \frac{N_{0}T}{2}$$

The derivation of the variance of  $N_2$  is similar and the combined variance is  $N_0T$ .

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