Chapter 3
Amplitude Modulation

Problem 3.1.
For 100 percent modulation, is it possible for the envelope of AM to become zero for some time t? Justify your answer.

Solution
By definition, the envelope of AM signals is $A_c|t + k_a m(t)|$, where A_c is the carrier amplitude, k_a is the amplitude sensitivity of the modulator, and $m(t)$ is the message signal. The envelope will assure the zero value if and only if

$$k_a m(t) = 1$$

So long as this condition is satisfied, then the envelope of the AM signal will assume the value zero.

Problem 3.2.
For a particular case of AM using sinusoidal modulating wave, the percentage modulation is 20 percent. Calculate the average power in (a) the carrier and (b) each side frequency, expressing your results as percentages of total transmitted power.

Solution
For sinusoidal modulation, the AM wave is defined by

$$s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$$

For $m(t) = A_m\cos(2\pi f_m t)$, we have (see Example 3.1)

$$s(t) = A_c\cos(2\pi f_c t) + \frac{1}{2}\mu A_c\cos[2\pi(f_c + f_m)t] + \frac{1}{2}\mu A_c\cos[2\pi(f_c - f_m)t]$$

(a) The average power in the carrier, expressed as a percentage of the total transmitted power, is (with $\mu = 20\%$)

$$\frac{\frac{1}{2}A_c^2}{\frac{1}{2}A_c^2 + \frac{1}{4}\mu^2 A_c^2} = \frac{1}{1 + 0.5\mu^2} = \frac{1}{1 + 0.5 \times 0.2^2} = \frac{1}{1 + 0.02} \approx 0.98$$

Expressing this result as a percentage, the result reads as 98%.

(b) The average power in each side frequency is therefore approximately 1%.

Problem 3.3
In AM, *spectral overlap* is said to occur if the lower sideband for positive frequencies overlaps with its *image* for negative frequencies. What condition must the modulated wave satisfy if we are to avoid spectral overlap? Assume that the message signal $m(t)$ is of a low-pass kind with bandwidth W.

Solution

The lowest frequency of the lower sideband is f_c - W, where f_c is the carrier frequency and W is the message bandwidth. To avoid spectral overlap, we must therefore satisfy the condition:

$$f_c - W > 0$$

Hence, f_c must always be greater than the message bandwidth W.

Problem 3.4

A *square-law modulator* for generating an AM wave relies on the use of a nonlinear device (e.g., diode); Fig. 3.8 depicts the simplest form of such a modulator. Ignoring higher order terms, the input-output characteristic of the diode-load resistor combination in this figure is represented by the *square law*:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where

$$v_1(t) = A_c \cos(2\pi f_c t) + m(t)$$

is the input signal, $v_2(t)$ is the output signal developed across the load resistor, and a_1 and a_2 are constants.

(a) Determine the spectral content of the output signal $v_2(t)$.

(b) To extract the desired AM wave from $v_2(t)$, we need a band-pass filter (not shown in Fig. 3.8). Determine the cutoff frequencies of the required filter, assuming that the message signal is limited to the band $-W \le f \le W$.

(c) To avoid *spectral distortion* by the presence of undesired modulation products in $v_2(t)$, the condition $f_c > 2W$ must be satisfied; validate this condition.

Solution

The output signal is

$$\begin{aligned} v_2(t) &= a_1 v_1(t) + a_2 v_1^2(t) \\ &= a_1(A_c \cos(2\pi f_c t) + m(t)) + a_2(A_c \cos(2\pi f_c t) + m(t))^2 \\ &= [a_1 + 2a_2 m(t)] A_c \cos(2\pi f_c t) \\ &\quad + [a_1 m(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + a_2 m^2(t)] \end{aligned} \tag{1}$$

(a) The expression inside the first set of square brackets defines the desired AM wave:

$$\begin{aligned} s(t) &= A_c[a_1 + 2a_2 m(t)] \cos(2\pi f_c t) \\ &= a_1 A_c \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos(2\pi f_c t) \end{aligned}$$

which represents an AM wave with

$$k_a = \frac{2a_2}{a_1}$$

defining the amplitude sensitivity of the modulator.

(b) The required band-pass filter must have a passband centered on f_c and a bandwidth equal to $2W$.

(c) The expression inside the second set of square brackets of Eq. (1) defines the undesired modulated products. The terms that matter are:

- The term $a_2 m^2(t)$, whose highest frequency component is 2W.
- The term $a_2 A_c^2 \cos^2(2\pi f_c t)$, whose frequency is $2f_c$.

To extract the desired AM wave we therefore require:
Condition 1:
$(f_c + W) < 2f_c$
or $f_c > W$
Condition 2:
$(f_c - W) > 2W$
or $f_c > 3W$
If therefore we satisfy condition 2, then condition 1 is automatically satisfied.

Problem 3.5
For the sinusoidally DSB-SC modulation considered in Problem 3.5, what is the average power in the lower or upper side-frequency, expressed as a percentage of the average power in the DSB-SC modulated wave?

Solution
With the carrier suppressed at the modulator output, the average power in either side frequency is 50% of the average power of the modulated wave.

Problem 3.6.
The sinusoidally modulated DSB-SC wave of Example 3.2 is applied to a product modulator using a locally generated sinusoid of unit amplitude, and which is synchronous with the carrier used in the modulation.
(a) Determine the output of the product modulator, denoted by $v(t)$.
(b) Identify the two sinusoidal terms in $v(t)$ that are produced by the upper side frequency of the DSB-SC modulated wave, and the remaining two sinusoidal terms produced by the lower side frequency.

Solution
(a) From Example 3.2, the DSB-SC modulated is defined by

$$s(t) = \frac{1}{2}A_c A_m \cos(2\pi(f_c + f_m)t) + \frac{1}{2}A_c A_m \cos(2\pi(f_c - f_m)t)$$

Applying $s(t)$ and $\cos(2\pi f_c t)$ to a product modulator yields

$$\begin{aligned} v(t) &= s(t)\cos(2\pi f_c t) \\ &= \frac{1}{2}A_c A_m \cos(2\pi(f_c + f_m)t)\cos(2\pi f_c t) + \frac{1}{2}A_c A_m \cos(2\pi(f_c - f_m)t)\cos(2\pi f_c t) \end{aligned}$$

$$= \frac{1}{4}A_c A_m[\cos(2\pi(2f_c + f_m)t) + \cos(2f_m t)]$$

$$+ \frac{1}{4}A_c A_m[\cos(2\pi(2f_c - f_m)t) + \cos(2f_m t)] \qquad (1)$$

(b) The two sinusoidal terms inside the first set of square brackets in Eq. (1) are produced by the upper side frequency at $f_c + f_m$. The other two sinusoidal terms inside the second set of brackets are produced by the lower side frequency $f_c - f_m$.

Note that with $f_c > f_m$, the first and third terms in $v(t)$, both of which relate to carrier frequency $2f_c$ are removed by a low-pass filter. This would then leave the second and fourth sinusoidal terms, both of frequency f_m, as the only output of the filter. The coherent detector thus reproduces the original modulating wave of frequency f_m, with the output consisting of two contributions, one due to the upper side frequency and the other due to the lower side frequency.

Problem 3.7

The coherent detector for the demodulation of DSB-SC fails to operate satisfactorily if the modulator experiences spectral overlap. Explain the reason for this failure.

Solution

The DSB-SC modulated wave is defined by

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

Spectral overlap occurs if the condition $f_c > W$ is violated, in which case the lower sideband overlaps with its image.

However, when $s(t)$ is applied to a coherent detector, the resulting output is

$$v(t) = s(t)\cos(2\pi f_c t)$$

$$= A_c m(t)\cos^2(2\pi f_c t)$$

$$= \frac{1}{2}A_c m(t)[1 + \cos(4\pi f_c t)]$$

The spectral description of $v(t)$ is shown in Fig. 1, assuming that $f_c < W$:

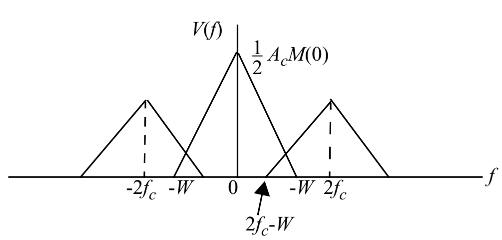


Figure 1

Recovery of the original signal is possible only if

$$2f_c - W > W$$

or

$f_c > W$

But this condition is being violated because of the spectral overlap.

Hence, once spectral overlap is permitted, no coherent detector can recover the original modulating signal.

Problem 3.8

As just mentioned, the phase discriminator in the Costas receiver of Fig. 3.16 consists of a multiplier followed by a time-averaging unit. Referring to this figure, do the following:

(a) Assuming that the phase error ϕ is small compared to one radian, show that the output $g(t)$ of the multiplier component is approximately $\frac{1}{4}\phi m^2(t)$.

(b) Furthermore, passing $g(t)$ through the time-averaging unit defined by

$$\frac{1}{2T}\int_{-T}^{T} g(t)dt$$

where the averaging interval $2T$ is long enough compared to the reciprocal of the bandwidth of $g(t)$, show that the output of the phase discriminator is proportional to the phase-error ϕ multiplied by the dc (direct current) component of $m^2(t)$. The amplitude of this signal (acting as the control signal applied to the voltage-controlled oscillator in Fig. 3.16) will therefore always have the same algebraic sign as that of the phase error ϕ, which is how it should be.

Solution

(i) Referring to the Costas receiver in Fig. 3.16 in the text, we see that the output of the in-phase channel is $\frac{1}{2}A_c\cos\phi m(t)$ and the output of the quadrature channel is $\frac{1}{2}A_c\sin\phi m(t)$. The output of the multiplier in the phase discriminator is therefore

$$g(t) = \left(\frac{1}{2}A_c\cos\phi m(t)\right)\left(\frac{1}{2}A_c\sin\phi m(t)\right)$$

$$= \frac{1}{4}\sin\phi\cos\phi m^2(t) \qquad (1)$$

If the phase error ϕ is small compared to one radian, we may use the approximations:

$\sin\phi \approx \phi$

$\cos\phi \approx 1$

in which case the multiplier output $g(t)$ simplifies approximately to $\frac{1}{4}\phi m^2(t)$.

(ii) Passing $g(t)$ through the time-averaging unit yields the phase discriminator output

$$v(t) = \frac{1}{2T}\int_{-T}^{T} g(t)dt$$

$$\approx \frac{1}{2T}\int_{-T}^{T} \frac{1}{4}\phi m^2(t)dt$$

$$= \frac{\phi}{8T}\int_{-T}^{T} m^2(t)dt$$

$$= \frac{1}{4}\phi P_0$$

where

$$P_0 = \frac{1}{2T}\int_{-T}^{T} m^2(t)dt$$

is the dc component of $m^2(t)$ or, equivalently, the average power of $m(t)$.

Problem 3.9

Verify that the outputs of the receiver in Fig. 3.17(b) are as indicated in the figure, assuming perfect synchronism between the receiver and transmitter.

Solution

The transmitted signal is

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Hence, the product modulator output of upper channel in Fig. 3.17(b) is

$$\begin{aligned} v_1(t) &= A'_c \cos(2\pi f_c t)s(t) \\ &= A_c A'_c m_1(t)\cos(2\pi f_c t) + A_c A'_c m_2(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}A_c A'_c m(t)[1 + \cos(4\pi f_c t)] + \frac{1}{2}A_c A'_c m(t)\sin(4\pi f_c t) \end{aligned}$$

Passing $v_1(t)$ through the low-pass filter yields $\frac{1}{2}A_c A'_c m(t)$, so long as there is no spectral overlap, that is, $f_c > W$.

Consider next the lower channel of the figure. The product-modulator output is

$$\begin{aligned} v_2(t) &= A'_c \cos(2\pi f_c t)s(t) \\ &= A_c A'_c m_1(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + A_c A'_c m_2(t)\sin^2(2\pi f_c t) \\ &= \frac{1}{2}A_c A'_c m(t)\sin(4\pi f_c t) + \frac{1}{2}A_c A'_c m_2(t)[1 - \cos(4\pi f_c t)] \end{aligned}$$

Passing $v_2(t)$ through the low-pass filter yields $\frac{1}{2}A_c A'_c m(t)$, as indicated in the figure.

Problem 3.10

Using Eqs. (3.22) and (3.23), show that for positive frequencies the spectra of the two kinds of SSB modulated waves are defined as follows:

(a) For the upper SSB,

$$S(f) = \begin{cases} \frac{A_c}{2}M(f - f_c) & \text{for } f \geq f_c \\ 0 & \text{for } 0 < f \leq f_c \end{cases}$$

(b) For the lower SSB,

$$S(f) = \begin{cases} 0 & \text{for} \quad f > f_c \\ \frac{A_c}{2} M(f - f_c) & \text{for } 0 < f \le f_c \end{cases}$$

(c) Write down the formulas for these two kinds of SSB modulation that pertain to negative frequencies.

Solution

According to Eq. (3.24):

$$s(t) = \frac{A_c}{2} m(t) \cos(2\pi f_c t) \mp \frac{A_c}{2} \hat{m}(t) \sin(2\pi f_c t)$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. Taking the Fourier transform of $s(t)$:

$$S(f) = \frac{A_c}{4}(M(f - f_c) + M(f + f_c)) \pm \frac{A_c}{4j}(\hat{M}(f - f_c) - \hat{M}(f + f_c))$$

From Eq. (3.22):

$$\hat{M}(f) = -jM(f)\operatorname{sgn}(f)$$

Hence,

$$S(f) = \frac{A_c}{4}(M(f\text{-}f_c) + M(f\text{+}f_c)) \pm (\frac{A_c}{4j}(-jM(f\text{-}f_c)\operatorname{sgn}(f\text{-}f_c) + jM(f\text{+}f_c)\operatorname{sgn}(f\text{+}f_c))$$

$$= \frac{A_c}{4}(M(f - f_c) \mp M(f - f_c)\operatorname{sgn}(f - f_c)) + \frac{A_c}{4}(M(f + f_c) \ \pm \ M(f + f_c)\operatorname{sgn}(f + f_c))$$

$$= \frac{A_c}{4}(1 \mp \operatorname{sgn}(f - f_c))M(f - c_c) + \frac{A_c}{4}(1 \pm \operatorname{sgn}(f + f_c))M(f + f_c) \qquad (1)$$

By definition:

$$\operatorname{sgn}(f - f_c) = \begin{cases} 1 & \text{for } f > f_c \\ -1 & \text{for } f < f_c \end{cases}$$

and

$$\operatorname{sgn}(f + f_c) = \begin{cases} 1 & \text{for } f > -f_c \\ -1 & \text{for } f < -f_c \end{cases}$$

(a) From Eq. (3.24), recall that the minus sign in this formula corresponds to upper SSB. Hence, for the upper SSB, we have

$$S(f) = \frac{A_c}{4}(1 + \operatorname{sgn}(f - f_c))M(f - f_c) + \frac{A_c}{4}(1 - \operatorname{sgn}(f + f_c))M(f + f_c) \qquad (2)$$

where the term containing $M(f - f_c)$ pertains to positive frequencies and the term containing $M(f + f_c)$ pertains to negative frequencies. Therefore for positive frequencies and $f \geq f_c$, Eq. (2) simplifies to

$$S(f) = \frac{A_c}{4}(1+1)M(f - f_c)$$

$$= \frac{A_c}{2}M(f - f_c) \qquad (3)$$

For $0 \leq f \leq f_c$, $S(f) = 0$.

(b) From Eq. (3.24), also recall that the plus sign in this formula corresponds to the lower SSB, for which we find that for $f \leq f_c$:

$$S(f) = \frac{A_c}{4}(1 - \text{sgn}(f - f_c))M(f - f_c) + \frac{A_c}{4}(1 + \text{sgn}(f + f_c))M(f + f_c)$$

Therefore for positive frequencies and $f \leq f_c$, we have

$$S(f) = \frac{A_c}{4}(1+1)M(f - f_c)$$

$$= \frac{A_c}{2}M(f - f_c) \qquad (4)$$

On the other hand, for $f > f_c$ we have $S(f) = 0$.

(c) For negative frequencies, we focus on terms containing $M(f + f_c)$, in light of which we get the following results:

(i) For upper SSB:

$$S(f) = \begin{cases} \frac{A_c}{4}M(f + f_c) & \text{for } f \leq -f_c \\ 0 & \text{for } -f_c < f < 0 \end{cases} \qquad (5)$$

(ii) For lower SSB:

$$S(f) = \begin{cases} 0 & \text{for } f < -f_c \\ \frac{A_c}{4}M(f + f_c) & \text{for } -f_c < f < 0 \end{cases} \qquad (6)$$

Problem 3.11

Show that if the message signal $m(t)$ is low-pass, then the Hilbert transform $\hat{m}(t)$ is also low-pass with the same bandwidth as $m(t)$.

Solution

The Fourier transform of the Hilbert transform $\hat{m}(t)$ is defined by

$\hat{M}(f) = -j\text{sgn}(f)M(f)$

where $M(f)$ = **F**[$m(t)$]. To illustrate, let the spectrum $M(f)$ be as shown in Fig. 1(a). Then, the corresponding spectrum $\hat{M}(f)$ is as shown in part (b) of the figure. The spectrum $\hat{M}(f)$ is therefore also low-pass, occupying the frequency band $-W \le f \le W$ just like $M(f)$.

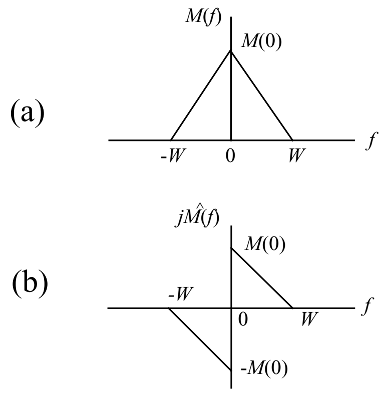


Figure 1

Problem 3.12

Starting with Eq. (3.23) for a SSB modulated wave, show that the product-modulator output in the coherent detector of Fig. 3.12 (assuming perfect synchronism with the transmitter) in response to this modulated wave contains a new SSB modulated wave with carrier frequency $2f_c$.

Solution

Suppose we focus on the upper SSB in Eq. (3.23) (i.e., use the minus sign in this formula). Then

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) - \frac{A_c}{2}\hat{m}(t)\sin(2\pi f_c t)$$

Applying this modulated wave to the coherent detector of Fig. 3.12 with the phase error $\phi = 0$, we first get the product-modulated output

$$\begin{aligned}
v(t) &= A'_c s(t)\cos(2\pi f_c t) \\
&= \frac{A_c A'_c}{2}m(t)\cos^2(2\pi f_c t) - \frac{A_c A'_c}{2}\hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\
&= \frac{A_c A'_c}{4}m(t)[1 + \cos(4\pi f_c t)] - \frac{A_c A'_c}{4}\hat{m}(t)\sin(4\pi f_c t) \\
&= \frac{A_c A'_c}{4}m(t) + \frac{A_c A'_c}{4}[m(t)\cos(4\pi f_c t) - \hat{m}(t)\sin(4\pi f_c t)]
\end{aligned}$$

Comparing this formula with that for $s(t)$, we see that $v(t)$ contains a new upper SSB modulated wave with carrier frequency $2f_c$. This same statement also applies to the lower SSB modulated wave $s(t)$.

Problem 3.13

For the low-pass filter in Fig. 3.12 (assuming perfect synchronism) to suppress the undesired SSB wave, the following condition must hold

$f_c > W, f_c$ = carrier frequency, and W = message bandwidth
Justify this condition

Solution
Continuing with the solution to Problem 3.12, we see that the product-modulator output $v(t)$ also contains a scaled version of the original message signal $m(t)$. For positive frequencies,the highest frequency component of $m(t)$ is W, and the lowest frequency of the new upper SSB modulated wave is $2f_c$ - W. For the low-pass filter to reject this SSB modulated wave, we require that
$2f_c - W > W$, or simply $f_c > W$. Under this condition, the detector output is

$$v_o(t) = \frac{A_c A'_c}{4} m(t)$$

Problem 3.14
Validate the statement that the high-frequency components in Eq. (3.36) represent a VSB wave modulated onto a carrier of frequency $2f_c$.

Solution
The high-frequency components in Eq. (3.36) are defined by the formula

$$G(f) = \frac{1}{4} A_c A'_c [M(f-2f_c)H(f-f_c) + M(f+2f_c)H(f+f_c)] \qquad (1)$$

Referring to Eq. (3.33), the spectrum of the incoming VSB modulated wave is

$$S(f) = \frac{1}{2} A_c [M(f-f_c) + M(f+f_c)] H(f)$$
$$= \frac{1}{2} A_c M(f-f_c)H(f) + \frac{1}{2} A_c M(f+f_c)H(f) \qquad (2)$$

Examining Eqs. (1) and (2), as labelled here, we see that (ignoring the scaling factors)
1. The first term in Eq. (1), namely, $M(f - 2f_c)H(f - f_c)$ is equal to the first term in Eq. (2), namely $M(f - f_c)H(f)$ shifted to the right by f_c.
2. By the same token, the second term in Eq. (1), namely, $M(f + 2f_c)H(f + f_c)$ is equal to the second term in Eq. (2), namely, $M(f + f_c)H(f)$ shifted to the left by f_c.

Since Eq. (2) represents a VSB wave modulated onto carrier frequency f_c, it follows that Eq. (1) represents a VSB wave modulated onto the new carrier frequency $2f_c$.

Problem 3.15
Derivation of the synthesizer depicted in Fig. 3.25(b) follows directly from Eq. (3.39). However, derivation of the analyzer depicted in Fig. 3.25(a) requires more detailed consideration. Given that $f_c > W$ and

$$\cos^2(2\pi f_c t) = \frac{1}{2}[1 + \cos(4\pi f_c t)]$$

and

$$\sin(2\pi f_c t)\cos(2\pi f_c t) = \frac{1}{2}\sin(4\pi f_c t),$$

show that the analyzer of Fig. 3.25(a) yields $s_I(t)$ and $s_Q(t)$ as its two outputs.

Solution

Consider first the upper channel in Fig. 3.25(a). Multiplying (see Eq. (3.39))

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q\sin(2\pi f_c t)$$

by the carrier $2\cos(2\pi f_c t)$, we get

$$\begin{aligned} v_1(t) &= 2s(t)\cos(2\pi f_c t) \\ &= 2s_I(t)\cos^2(2\pi f_c t) - 2s_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= s_I(t)[1 + \cos(4\pi f_c t)] - s_Q(t)\sin(4\pi f_c t) \\ &= s_I(t) + s'(t) \end{aligned}$$

where

$$s'(t) = s_I(t)\cos(4\pi f_c t) - s_Q(t)\sin(4\pi f_c t)$$

represents a new linearly modulated signal with carrier frequency $2f_c$. Provided that both $s_I(t)$ and $s_Q(t)$ are limited to the band $-W \le f \le W$ and we pass $v_1(t)$ through a low-pass filter of cutoff frequency W as in Fig. 3.25(a), then $s'(t)$ is rejected provided that $f_c > W$.

Consider next the lower channel in Fig. 3.25(a). Multiplying $s(t)$ by $-2\sin(2\pi f_c t)$, we get

$$\begin{aligned} v_2(t) &= -2s(t)\sin(2\pi f_c t) \\ &= -2s_I(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + 2s_Q(t)\sin^2(2\pi f_c t) \\ &= -s_I(t)\sin(4\pi f_c t) + [1 - \cos(4\pi f_c t)]s_Q(t) \\ &= s_Q(t) - s''(t) \end{aligned}$$

where

$$s''(t) = s_I(t)\sin(4\pi f_c t) + s_Q(t)\cos(4\pi f_c t)$$

is a new linearly modulated signal with carrier frequency $2f_c$. Hence, passing $v_2(t)$ through a low-pass filter as in Fig. 3.25(a), $s''(t)$ is rejected again provided that the cutoff frequency W of the low-pass filter satisfies the condition $f_c > W$.

Problem 3.16

Starting with the complex low-pass system depicted in Fig. 3.26(c), show that the $y(t)$ derived in Eq. (3.45) is identical to the actual output $y(t)$ in Fig. 3.26(a).

Solution

According to Fig. 3.25(a), we have

$$Y(f) = H(f)S(f) \tag{1}$$

and according to Fig. 3.25(b),

$$2\tilde{Y}(f) = \tilde{H}(f)\tilde{S}(f) \tag{2}$$

From Eq. (3.44) we note that

$$\tilde{H}(f - f_c) = 2H(f) \qquad \text{for } f > 0 \tag{3}$$

Therefore, substituting Eq. (3) into (2) and cancelling the common factor 2, we get

$$\tilde{Y}(f - f_c) = H(f)\tilde{S}(f - f_c), \qquad f > 0 \tag{4}$$

Finally, noting that for $f > 0$

$$Y(f) = \tilde{Y}(f - f_c)$$

and

$$S(f) = \tilde{S}(f - f_c),$$

we readily see that Eq. (3) is a rewrite of Eq. (1), which validates the outputs displayed in Fig. 3.26.

Problem 3.17

(a) We are given

$$c(t) = A_c \sin(2\pi f_c t)$$

and

$$m(t) = A_m \sin(2\pi f_m t)$$

Invoking the definition of AM wave

$$s(t) = [1 + k_a m(t)]c(t)$$

we now write

$$\begin{aligned} s(t) &= A_c[1 + k_a A_m \sin(2\pi f_m t)]\sin(2\pi f_c t) \\ &= A_c \sin(2\pi f_c t) + \mu A_c \sin(2\pi f_m t)\sin(2\pi f_c t) \end{aligned} \tag{1}$$

where

$$\mu = k_a A_m$$

is the modulation factor. Next, we use the trigonometric identity

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

Hence, we may rewrite Eq. (1) as

$$s(t) = A_c \sin(2\pi f_c t) + \frac{1}{2}\mu A_c[\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)] \tag{2}$$

The spectrum of the AM wave s(t) consists of three components:

(i) Carrier: $A_c \sin(2\pi f_c t)$

(ii) Lower side-frequency: $\frac{1}{2}\mu A_c \cos(2\pi(f_c - f_m)t)$

(iii) Upper side-frequency: $-\frac{1}{2}\mu A_c \cos(2\pi(f_c + f_m)t)$

This spectrum is depicted in Fig. 1.

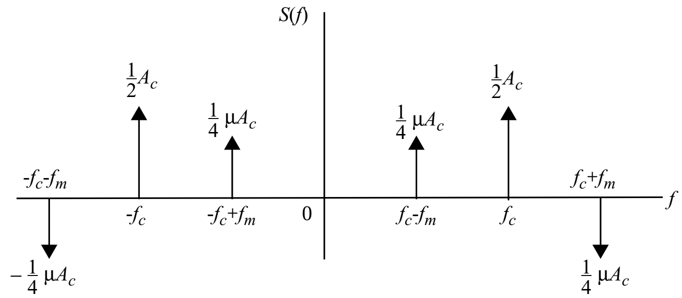


Figure 1

(b) Comparing the AM spectrum of Fig. 1 with the corresponding AM spectrum of Fig. 3.3(c) on page 105 of the text, we may make two observations:
- The frequency locations of the spectral components of these two AM waves are identical.
- The only difference between them is that the upper side-frequency $f_c + f_m$ in Fig. 1 is the negative of the upper side-frequency $f_c + f_m$ in Fig. 3.3(c).

Note: The following correction in the first printing of the book should be made. The modulating wave should read as follows:

$$m(t) = A_m \sin(2\pi f_m t)$$

Problem 3.18

(a) We are given

$c(t) = 50\cos(100\pi t)$ volts, $\quad f_c = 50$Hz

and

$m(t) = 20\cos(2\pi t)$ volts, $\quad f_m = 1$Hz

The resulting AM wave is

$$s(t) = [1 + k_a m(t)]c(t)$$
$$= 50[1 + 20k\cos(2\pi t)]\cos(100\pi t) \qquad (1)$$

A percentage modulation of 75% corresponds

$20k = 0.75$

or

$k = 0.0375$

Accordingly, we may rewrite Eq. (1) as

$$s(t) = 50[1 + 0.75\cos(2\pi t)]\cos(100\pi t) \qquad (2)$$

Equation (2) is plotted in Fig. 1.

(b) Expanding the AM wave s(t) of Eq. (2) into its spectral components, we write

$$s(t) = 50\cos(100\pi t) + 37.5\cos(2\pi t)\cos(100\pi t)$$
$$= 50\cos(100\pi t) + 18.75[\cos(102\pi t) + \cos(98\pi t)]\text{volts}$$

The power developed across a 100-ohm load by this AM wave is therefore

$$P = \frac{1}{2}\ \frac{(50)^2}{100} + \frac{1}{2}\ \frac{(18.75)^2}{100} + \frac{1}{2}\ \frac{(18.75)^2}{100}$$

$$= 12.5 + 3.426$$

$$= 15.926 \text{ watts}$$

This result shows that the carrier contributes about 80% of the power delivered to the load.

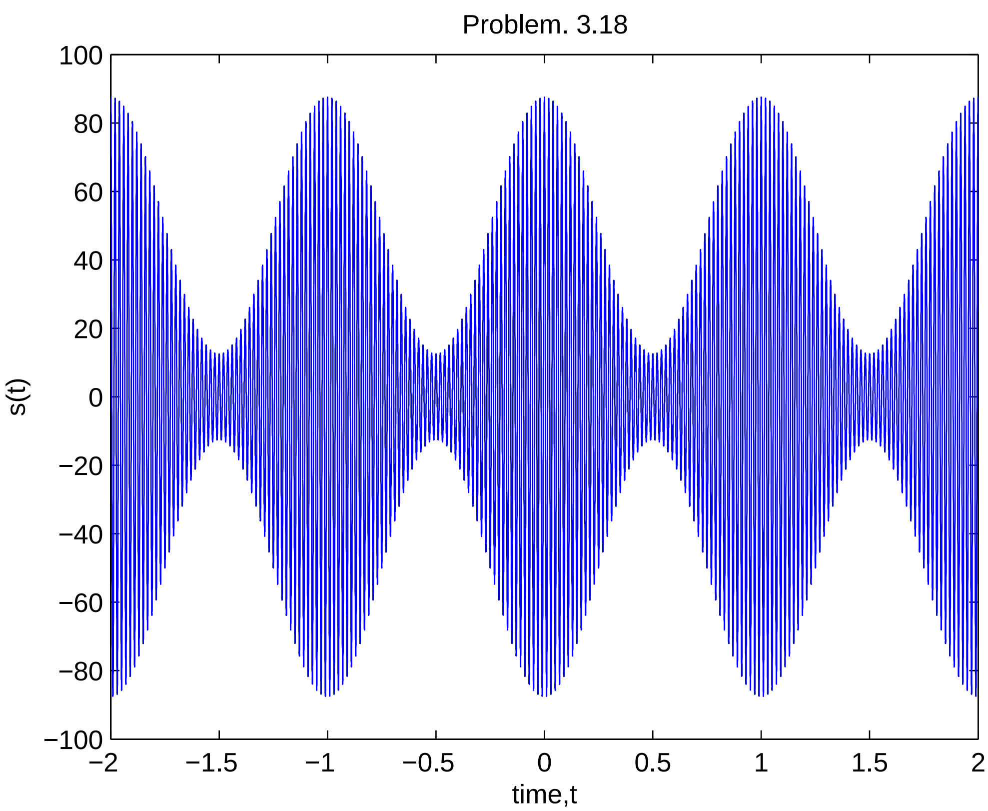


Figure 1: Problem 3.18

Problem 3.19

We are given

$$m(t) = \frac{t}{1+t^2} \qquad (1)$$

The AM wave is therefore defined by

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$
$$= A_c\left(1 + \frac{k_a t}{1+t^2}\right)\cos(2\pi f_c t) \qquad (2)$$

The message signal $m(t)$ is plotted in Fig. 1(a) with its maximum value of 1/2 and minimum value of -1/2 at $t = 1$ and $t = -1$, respectively.

(a) Percentage modulation = 50%

$$k_a|m(t)|_{\max} = 0.5$$

with $|m(t)|_{\max} = \frac{1}{2}$, it follows that $k_a = 1$ for 50% modulation. For this example, Eq. (1) takes the form

$$s(t) = A_c\left(1 + \frac{t}{1+t^2}\right)\cos(2\pi f_c t) \qquad (3)$$

Let t be measured in seconds. Then, for the envelope of the AM wave to be clearly visible, the period of the carrier, $1/f_c$, must be small compared to the time taken for the message signal $m(t)$ to reach its peak value. To satisfy this requirement, we let

$$\frac{1}{f_c} = 10\text{Hz}$$

which corresponds to

$$f_c = 10\text{Hz}$$

Setting the carrier amplitude $A_c = 1$ volt, and $f_c = 10$Hz, Eq. (3) is plotted in part (b) of Fig. 1.

(b) Percentage modulation = 100%

In this case, we have $k_a = 2$. Correspondingly, Eq. (1) assumes the form

$$s(t) = A_c\left(1 + \frac{2t}{1+t^2}\right)\cos(2\pi f_c t) \qquad (4)$$

Keeping $A_c = 1$ volt, and $f_c = 10$Hz as in case (a), Eq. (4) is plotted in Fig. 1(c).

(c) Percentage modulation = 125%

In this third and final example, we have $k_a = 2.5$. Hence, Eq. (1) now assumes the form

$$s(t) = A_c\left(1 + \frac{2.5t}{1+t^2}\right)\cos(2\pi f_c t) \qquad (5)$$

Keeping $_c = 1$ volt, and $f_c = 10$Hz as before, Eq. (5) is plotted in Fig. 1(d).

Comparing the AM waveforms plotted in parts (b), (c) and (d) of Fig. 1, we may make the following observations:

- The AM wave of Fig. 1(b) is undermodulated
- The AM wave of Fig. 1(c) is on the verge of overmodulation
- The AM wave of Fig. 1(d) is overmodulated

(a)

(b)

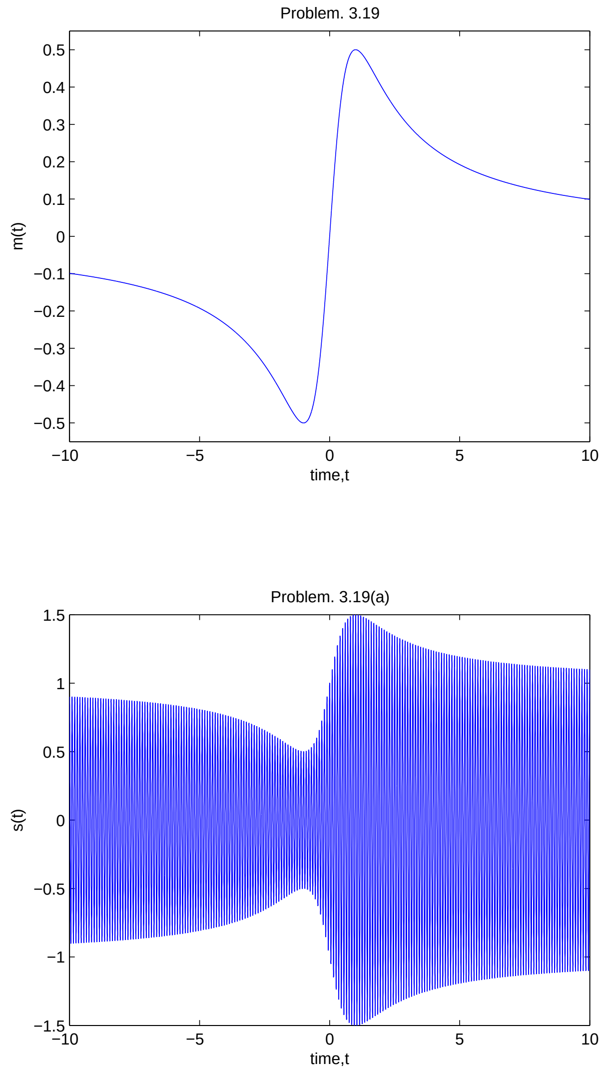
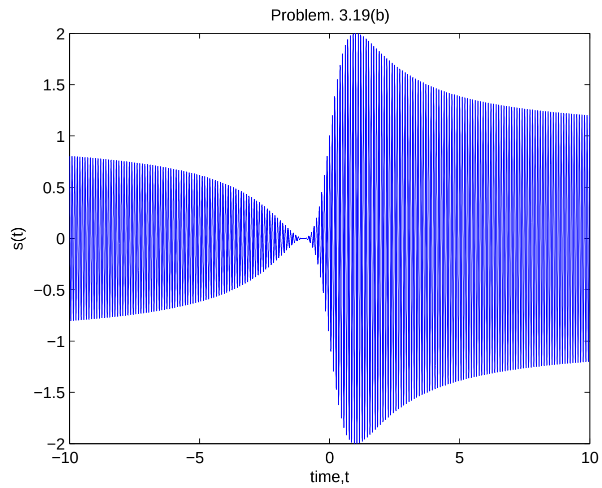


Figure 1: Problem 3.19

(c)



(d)

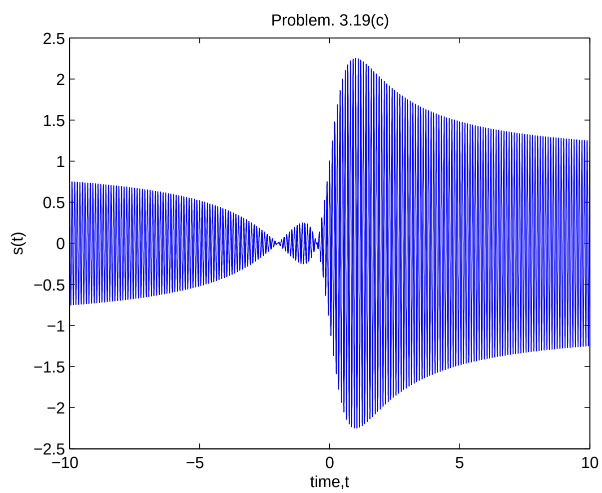


Figure 1 (continued): Problem 3.19

```
close all;
 clear all;
 clc;


 Sampling_freq = 10;      %in kHz
 t = -2:1/(Sampling_freq*1e3):2;
 figure;
 s1  = 50*(1 + 0.75*cos(2*pi*t)).*cos(100*pi*t);
 plot(t,s1);
 xlabel('time,t');
 ylabel('s(t)');
 title('Problem. 3.18');


 figure;
 Sampling_freq = 1;
 t = [];
 t = -10:1/(Sampling_freq*1e3):10;
 m = t./(1 +t.^2);
 plot(t,m);
 axis([ -10 10  -0.55 0.55]);
 xlabel('time,t');
 ylabel('m(t)');
 title('Problem. 3.19');


 figure;
 Sampling_freq = 1;
 t = [];
 t = -10:1/(Sampling_freq*1e3):10;
 s2 = (1 + t./(1 +t.^2)).*cos(20*pi*t);
 plot(t,s2);
 xlabel('time,t');
 ylabel('s(t)');
 title('Problem. 3.19(a)');


 figure;
 Sampling_freq = 1;
 t = [];
 t = -10:1/(Sampling_freq*1e3):10;
 s3 = ( 1 + 2*t./(1 +t.^2)).*cos(20*pi*t);
 plot(t,s3);
 xlabel('time,t');
 ylabel('s(t)');
 title('Problem. 3.19(b)');


 figure;
 Sampling_freq = 1;
 t = [];
 t = -10:1/(Sampling_freq*1e3):10;
 s3 = ( 1 + 2.5*t./(1 +t.^2)).*cos(20*pi*t);
```

```
plot(t,s3);
xlabel('time,t');
ylabel('s(t)');
title('Problem. 3.19(c)');



Sampling_freq = 10;
t = [];
t = -1:1/(Sampling_freq*1e3):1;
for i=1:4
    s5 = (1+ 0.5*cos(2*pi*t)).*cos(10*pi*t + (i-1)*pi/4);
    figure;
    plot(t,s5);
    xlabel('time,t');
    ylabel('s(t)');
    axis([ -1.10 1.10  -1.6 1.6]);
    title(['Theta=',num2str((i-1)*45)]);
    grid on;
end
```

Problem 3.20

(a) Let the input voltage v_i consist of a sinusoidal wave of frequency $\frac{1}{2}f_c$ (i.e., half the desired carrier frequency) and the message signal $m(t)$, as shown by

$$v_i = A_c\cos(\pi f_c t) + m(t) \qquad (1)$$

Then, the output current i_o is

$$i_o = a_1 v_i + a_3 v_i^3$$
$$= a_1[A_c\cos(\pi f_c t) + m(t)] + a_3[A_c\cos(\pi f_c t) + m(t)]^3$$
$$= a_1[A_c\cos(\pi f_c t) + m(t)] + \frac{1}{4}a_3A_c^3[\cos 3(\pi f_c t) + 3\cos(\pi f_c t)]$$
$$+\frac{3}{2}a_3A_c^2 m(t)[1+\cos(2\pi f_c t)] + 3a_3A_c\cos(\pi f_c t)m^2(t) + a_3m^3(t)$$

Assume that $m(t)$ occupies the frequency interval $-W \le f \le W$. Then, the amplitude spectrum of the output current i_o is as shown Fig. 1:

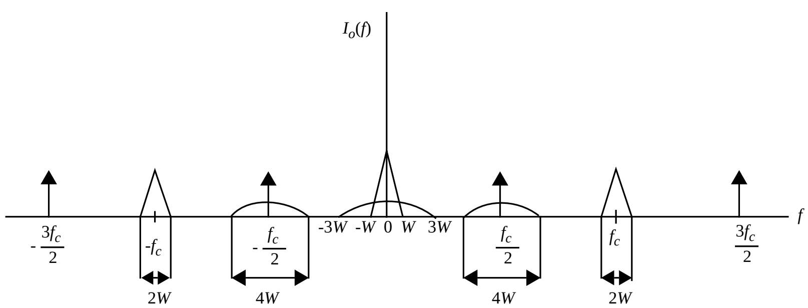


Figure 1

From this spectrum we see that in order to extract a DSB-SC wave with carrier frequency f_c from i_o, we need a bandpass filter with mid-band frequency f_c and bandwidth $2W$, the two of which satisfy the requirement:

$$f_c - W > \frac{f_c}{2} + 2W$$

that is, $f_c > 6W$

Therefore, to use the given nonlinear device as a product modulator, we may use the configuration: shown in Fig. 2.

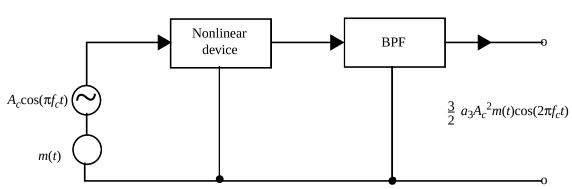


Figure 2

(b) To generate an AM wave with carrier frequency f_c, we require a sinusoidal component of frequency f_c to be added to the DSB-SC generated in the manner described under (a). To achieve this requirement, we may use a configuration involving a pair of the nonlinear devices and a pair of identical bandpass filters, as depicted in Fig. 3.

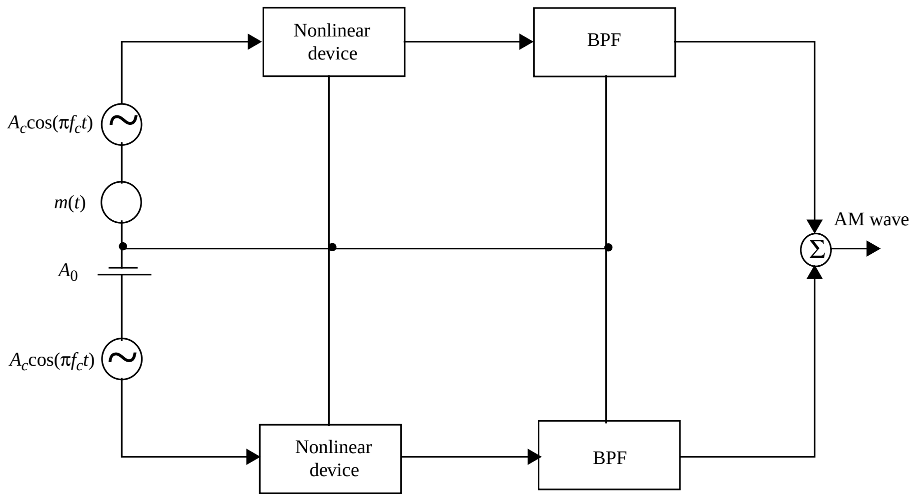


Figure 3

The resulting AM wave is therefore $\frac{3}{2}a_3A_c^2[A_0 + m(t)]\cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

The nonlinear device defined in Eq. (1) cannot be used for demodulation. The reason for saying so is that Eq. (1) lacks a square-law term, which is essential for demonstration (i.e., recovery of the message signal from an incoming AM wave).

Problem 3.21

We are given

$m(t) = A_c\cos(2\pi f_m t)$

and

$c(t) = A_c\cos(2\pi f_c t + \phi)$

The AM wave is therefore

$$\begin{aligned} s(t) &= [1 + k_a m(t)]c(t) \\ &= A_c[1 + k_a A_m\cos(2\pi f_m t)]\cos(2\pi f_c t + \phi) \\ &= A_c[1 + \mu\cos(2\pi f_m t)]\cos(2\pi f_c t + \phi) \qquad (1) \end{aligned}$$

The focus in this problem is to see how varying the phase ϕ affects the waveform of the AM signal $s(t)$. We may thus set the following parameters:

$\mu = k_a A_m = 0.5$

$A_c = 1$ volt

f_m = 1Hz
and
f_c = 5Hz
Then, Eq. (1) assumes the form

$$s(t) = [1 + 0.5\cos(2\pi t)]\cos(10\pi t + \phi) \tag{2}$$

Equation (2) is plotted in Fig. 1 for the prescribed values $\phi = 0^o, 45^o, 90^o$, and 135^o. Examining these four waveforms, we may make the following observation:

- Insofar as the envelope of the AM wave is concerned, varying the carrier phase ϕ has no effect whatsoever on the waveform of the envelope, which is intuitively satisfying.
- The only visible effect of varying the carrier phase ϕ is a shift in the uniformly spaced zero-crossings of the AM wave.

(a)

(b)

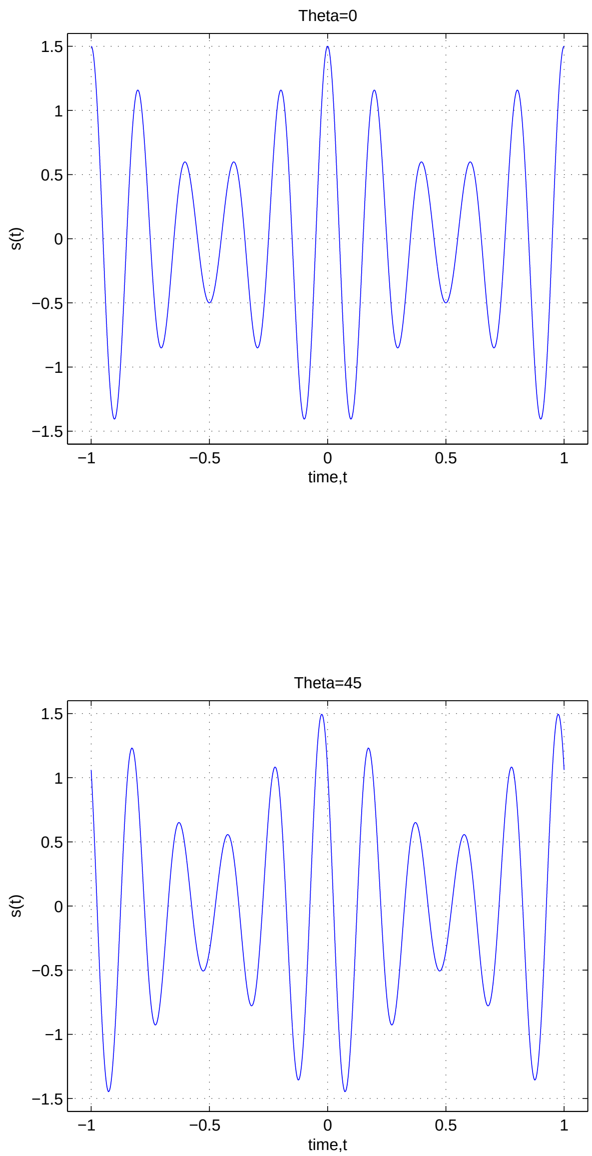
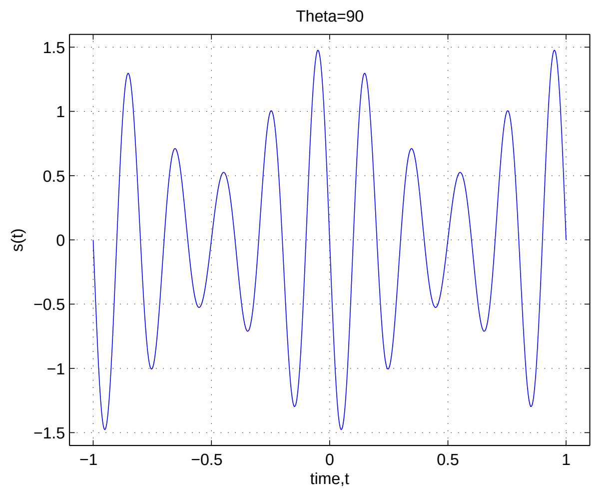


Figure 1: Problem 3.21

(c)



(d)

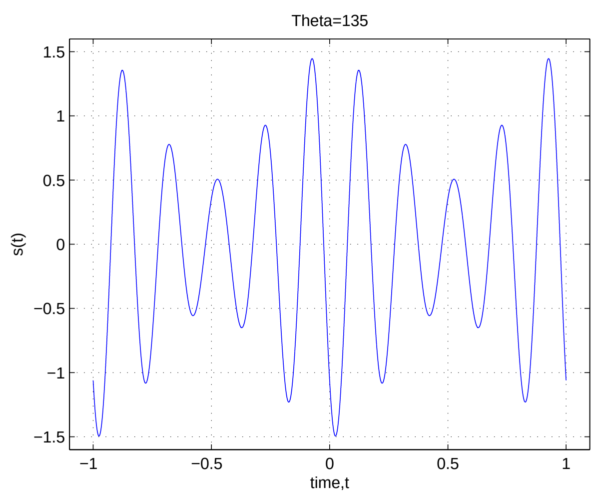


Figure 1 (continued): Problem 3.21

Problem 3.22

For the solution to this problem, see Fig. 2 in the solution to Problem 3.20.

Problem 3.23

(a) For f_c = 1.25 kHz, the spectra of the message signal $m(t)$, the product modulator output $s(t)$, and the coherent detector output $v(t)$ are as shown in Fig. 1, respectively:

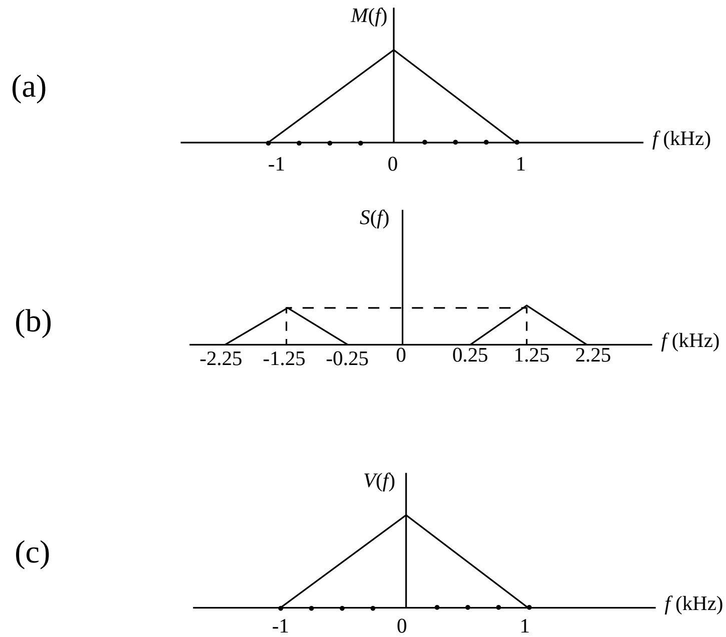


Figure 1

(b) For the case when f_c = 0.75, the respective spectra are as shown in Fig. 2:

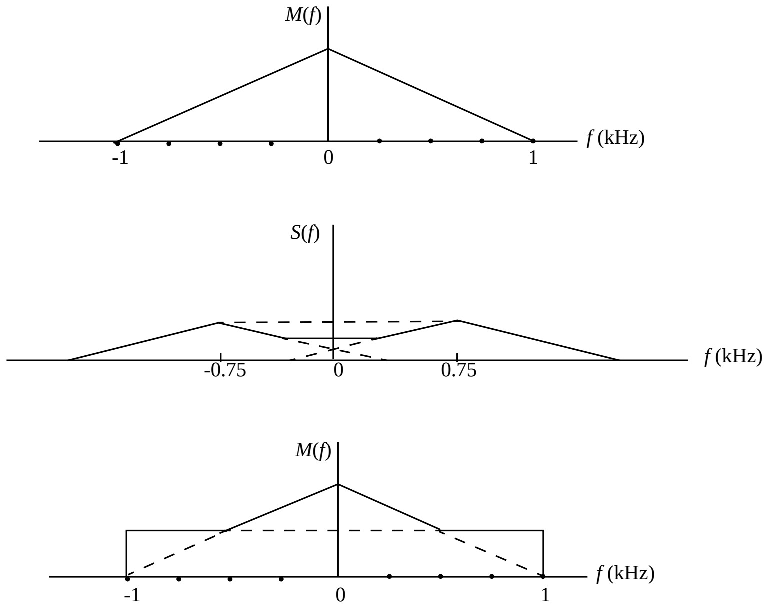


Figure 2

To avoid sideband-overlap, the carrier frequency f_c must be equal to or greater than 1 kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave $s(t)$ to be uniquely determined by $m(t)$.

Problem 3.24

The noncoherent carrier is

$$c(t) = A_c \cos(2\pi f_c t + \phi)$$

and the DSB-SC modulated wave is $m(t)\cos(2\pi f_c t)$. The composite signal is therefore

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \phi) + m(t)\cos(2\pi f_c t) \\ &= [A_c \cos\phi + m(t)]\cos(2\pi f_c t) - A_c \sin\phi \sin(2\pi f_c t) \\ &= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t) \end{aligned}$$

where

$$s_I(t) = A_c \cos\phi + m(t)$$

$$s_Q(t) = A_c \sin\phi$$

Applying the composite signal $s(t)$ to an ideal envelope detector produces the output

$$\begin{aligned} a(t) &= [s_I^2(t) + s_Q^2(t)]^{1/2} \\ &= [(A_c \cos\phi + m(t))^2] + (A_c \sin\phi)^{2\,1/2} \\ &= [A_c^2 \cos^2\phi + 2A_c \cos\phi\, m(t) + m^2(t) + A_c^2 \sin^2\phi]^{1/2} \end{aligned}$$

$$= [A_c^2 + 2A_c \cos\phi\, m(t) + m^2(t)]^{1/2} \qquad (1)$$

(a) For $\phi = 0$, Eq. (1) reduces to

$$a(t) = [A_c^2 + 2A_c m(t) + m^2(t)]^{1/2}$$
$$= A_c + m(t) \qquad (2)$$

which consists of the message signal $m(t)$ plus a dc bias equal to the carrier amplitude.

(b) For $\phi \neq 0$ and $|m(t)| << A_c/2$, we may approximate Eq. (1) as follows:

$$a(t) \approx [A_c^2 + 2A_c \cos\phi\, m(t)]^{1/2}$$
$$= A_c\left[1 + \frac{2}{A_c}\cos\phi\, m(t)\right]^{1/2} \qquad (3)$$

With $|\cos\phi| \leq 1$, and $|m(t)| << A_c/2$, we may approximate Eq. (3) further as

$$a(t) \approx A_c\left[1 + \frac{1}{A_c}\cos\phi\, m(t)\right]$$
$$= A_c + \cos\phi\, m(t) \qquad (4)$$

When ϕ is close to zero, the detector output in Eq. (4) is very close to the value defined in Eq. (2). However, when ϕ approaches 90^o, $\cos\phi$ approach zero, then the envelope detector output in Eq. (4) reduces to a dc component equal to A_c with no significant trace of the message signal $m(t)$ being visible. If therefore the phase error ϕ is variable, then the envelope detector output a(t) varies in a corresponding way, which could be undesirable.

Problem 3.25

(a) The effect of a frequency error Δf in the local oscillator used in the coherent detector shows itself as follows:

$$c'(t) = \cos(2\pi(f_c + \Delta f)t)$$

Applying the DSB-SC modulated wave $s(t)$

$$s(t) = A_c \cos(2\pi f_c t) m(t)$$

to a coherent detector employing $c'(t)$ yields the product modulator output (see Fig. 1)

$$v(t) = s(t)c'(t)$$
$$= A_c \cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi\Delta f t) m(t) \qquad (1)$$

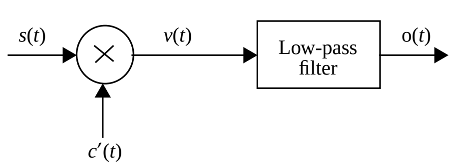


Figure 1

Using the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

we may rewrite Eq. (1) as

$$v(t) = \frac{1}{2}A_c[\cos(4\pi f_c t + 2\pi\Delta f t) + \cos(2\pi\Delta f t)]m(t) \qquad (2)$$

Next, passing $v(t)$ through the low-pass filter in Fig. 1 removes the high-frequency component, producing the output

$$o(t) = \frac{1}{2}A_c \cos(2\pi\Delta f t)m(t) \qquad (3)$$

which exhibits beats at the error frequency Δf.

Problem 3.26

The message signal is defined by the rectangular pulse

$$m(t) = \begin{cases} A, & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases} \qquad (1)$$

The SSB modulated wave is defined by

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) \mp \frac{A_c}{2}\hat{m}(t)\sin(2\pi f_c t)$$

where $\hat{m}(t)$ is the Hilbert transform of $m(t)$. The in-phase and quadrature components of $s(t)$ are respectively defined by

$$s_I(t) = \frac{A_c}{2}m(t)$$

$$s_Q(t) = \pm\frac{A_c}{2}\hat{m}(t)$$

The envelope of $s(t)$ is therefore

$$a(t) = [s_I^2(t) + S_Q^2(t)]^{1/2}$$

$$= \frac{A_c}{2}[m^2(t) + \hat{m}^2(t)]^{1/2} \qquad (2)$$

The Hilbert transform of the rectangular pulse of Eq. (1) was determined in Problem 2.52 of Chapter 2; it is reproduced here for a pulse of unit amplitude and duration T:

$$\hat{m}(t) = -\frac{1}{\pi}\ln\left|\frac{t-(T/2)}{t+(T/2)}\right| \qquad (3)$$

where ln denotes the natural logarithm. From Eq. (3) we see that $\hat{m}^2(t)$ assumes an infinitely large value at $t = T/2$ and $t = -T/2$. Correspondingly, the envelope of the SSB modulator exhibits peaks at the beginning and end of the input pulse.

Problem 3.27

(a) The frequency error $\Delta f = 20$ Hz. Since this frequency error is positive and the incoming SSB wave contains the upper sideband, the frequency components of the demodulated signal are shifted downward by Δf, compared with the message signal. The demodulated signal therefore consists of three frequency components: 80, 180, and 380 Hz.

(b) When the lower sideband is transmitted, the frequency components of the demodulated signal are shifted upward by Δf, compared with the message signal. The demodulated signal therefore consists of three frequency components: 120, 220, and 420 Hz.

Problem 3.28

The energy of the carrier over a bit duration is defined by

$$E = \int_0^{T_b} c^2(t)dt$$

$$= A_c^2 \int_0^{T_b} \cos^2(2\pi f t)dt \qquad (1)$$

Using the identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

we rewrite Eq. (1) as

$$E = \frac{1}{2}A_c^2 \int_0^{T_b} [1 + \cos(4\pi f_c t)]dt$$

$$= \frac{1}{2}A_c^2 \int_0^{T_b} dt + \frac{1}{2}A_c^2 \int_0^{T_b} \cos(4\pi f_c t)dt \qquad (2)$$

Typically, the carrier frequency f_c is high compared to the bit rate $1/T_b$; we may therefore set the integral term in Eq. (2) approximately equal to zero, in which case we write

$$E \approx \frac{1}{2}A_c^2 \int_0^{T_b} dt$$

$$= \frac{1}{2}A_c^2 T_b \qquad (3)$$

For the energy E to equal unity, we may solve Eq. (3) for the carrier amplitude A_c, obtaining

$$A_c = \sqrt{\frac{2}{T_b}}$$

which is the desired result. On this basis, we express the carrier as

$$c(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

Problem 3.29

(a) Using the terminated series expansion $\exp(-x) \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ we may express the diode current i, normalized with respect to I_0, as

$$\frac{i}{I_0} = \exp\left(-\frac{v}{V_T}\right) - 1$$

$$= -\frac{v}{V_T} + \frac{1}{2}\left(\frac{v}{V_T}\right)^2 - \frac{1}{6}\left(\frac{v}{V_T}\right)^3 \qquad (1)$$

(b) Given

$$\frac{v}{V_T} = \frac{0.01}{0.026}[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$\approx 0.385[\cos(2\pi f_m t) + \cos(2\pi f_c t)] \qquad (2)$$

we find that substitution of Eq. (2) into (1) yields

$$\frac{i}{I_0} \approx -0.385[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$+ 0.074[\cos(2\pi f_m t) + \cos(2\pi f_c t)]^2$$

$$- 0.0095[\cos(2\pi f_m t) + \cos(2\pi f_c t)]^3 \qquad (3)$$

Next, using the identities

$$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$$

$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta + \phi) + \cos(\theta - \phi)]$$

we may rewrite Eq. (3) in the form:

$$\frac{i}{I_0} = 0.074 - 0.406[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$$

$$+ 0.037\{\cos(4\pi f_m t) + \cos(4\pi f_c t) + \cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\}$$

$$- 0.0016[\cos(6\pi f_m t) + \cos(6\pi f_c t)]$$

$$- 0.0071\{\cos[2\pi(f_c + 2f_m)t] + \cos[2\pi(f_c - 2f_m)t]$$

$$+ \cos[2\pi(2f_c + f_m)t] + \cos[2\pi(2f_c - f_m)t]\}$$

For f_m - 1 kHz and f_c = 100 kHz, we thus find that the discrete amplitude spectrum of the diode current i (for $f \geq 0$) is as shown in Fig. 1.

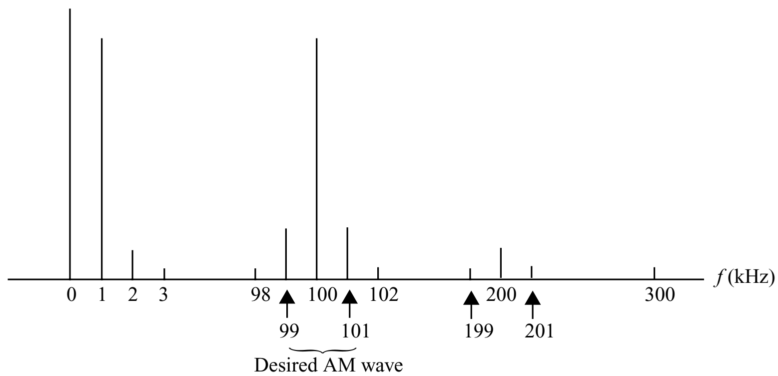


Figure 1

(c) From the amplitude spectrum of Fig. 1 we see that in order to extract an AM wave with carrier frequency f_c from the diode current i, we need a band-pass filter that passes only the frequency components: 99, 100 and 101 kHz, corresponding to f_c - f_m, f_c, and $f_c + f_m$, respectively. We therefore require a band-pass filter with center frequency 100 kHz and bandwidth 2 kHz.

(d) The resulting band-pass filter output is

$$\frac{i}{I_0} = -0.406\cos(2\pi f_c t) + 0.148\cos(2\pi f_c t)\cos(2\pi f_m t)$$

$$= -0.406[1 - 0.362\cos(2\pi f_m t)]\cos(2\pi f_c t)$$

The percentage modulation is therefore 36.2 percent.

Problem 3.30

The multiplexed signal is defined by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Therefore, the spectrum of $s(t)$ is

$$S(f) = \frac{A_c}{2}[M_1(f - f_c) + M_1(f + f_c)] + \frac{A_c}{2j}[M_2(f - f_c) - M_2(f + f_c)]$$

where $M_1(f) = \mathbf{F}(m_1(t)]$ and $M_2(f) = \mathbf{F}(m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2}H(f)\left[M_1(f - f_c) + M_1(f + f_c) + \frac{1}{j}M_2(f - f_c) - \frac{1}{j}M_2(f + f_c)\right]$$

To recover $m_1(t)$, we multiply $r(t)$ [i.e., the inverse Fourier transform of $R(f)$] by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, which is designed to have a cutoff frequency equal to the message bandwidth W. The signal produced at the filter output has the following spectrum

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{1}{2}[R(f - f_c) + R(f + f_c)]$$

$$= \frac{A_c}{4}H(f-f_c)[M_1(f-2f_c)+M_1(f)+\frac{1}{j}M_2(f-2f_c)-\frac{1}{j}M_2(f)]$$

$$+\frac{A_c}{4}H(f+f_c)\left[M_1(f)+M_1(f+2f_c)+\frac{1}{j}M_2(f)-\frac{1}{j}M_2(f+2f_c)\right] \quad (1)$$

The condition $H(f_c + f) = H^*(f_c - f)$ is equivalent to $H(f + f_c) = H(f - f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f) = H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$\mathbf{F}[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2}H(f-f_c)M_1(f)$$

$$+\frac{A_c}{4}H(f-f_c)\left[M_1(f-2f_c)+\frac{1}{j}M_2(f-2f_c)+M_1(f+2f_c)-\frac{1}{j}M_2(f+2f_c)\right]$$

The low-pass filter output therefore has a spectrum equal to $(A_c/2)H(f - f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)H(f - f_c)M_2(f)$.

Problem 3.31

(a) The SSB wave $s_u(t)$ is defined by

$$s_u(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)-\hat{m}(t)\sin(2\pi f_c t)] \quad (1)$$

and its Hilbert transform is defined by

$$s_u(t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)-\hat{m}(t)\cos(2\pi f_c t)] \quad (2)$$

In Eq. (2), we have used the following properties of the Hilbert transform:

(a) The Hilbert transform of $m(t)\cos(2\pi f_c t)$ is $m(t)\sin(2\pi f_c t)$

(b) The Hilbert transform of $\hat{m}(t)\sin(2\pi f_c t)$ is $-\hat{m}(t)\cos(2\pi f_c t)$

We may therefore use Eqs. (1) and (2) to write

$$s_u(t) = \cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos^2(2\pi f_c t)-\hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \quad (3)$$

$$s_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin^2(2\pi f_c t)+\hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \quad (4)$$

Adding Eqs. (3) and (4) and solving for $m(t)$, we get

$$m(t) = \frac{A_c}{2}[s_u(t)\cos(2\pi f_c t)+\hat{s}_u(t)\sin(2\pi f_c t)] \quad (5)$$

Next, we use Eqs. (1) and (2) to write

$$s_u(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)\sin(2\pi f_c t)-\hat{m}(t)\sin^2(2\pi f_c t)] \quad (6)$$

$$s_u(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)\cos(2\pi f_c t) + \hat{m}(t)\cos^2(2\pi f_c t)] \quad (7)$$

Subtracting Eq. (6) from Eq. (7) and then solving for $\hat{m}(t)$, we get

$$\hat{m}(t) = \frac{2}{A_c}[\hat{s}_u(t)\cos(2\pi f_c t) - s_u(t)\sin(2\pi f_c t)] \quad (8)$$

Equations (5) and (8) are the desired results for part (a) of the problem.

(b) The SSB wave $s_l(t)$ is defined by

$$s_l(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)] \quad (9)$$

and its Hilbert transform is defined by

$$\hat{s}_l(t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)] \quad (10)$$

where again we have made use of the above-mentioned properties of the Hilbert transform. Therefore, using Eqs. (9) and (10) we write

$$s_l(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos^2(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t)] \quad (11)$$

$$\hat{s}_l(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin^2(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)\sin(2\pi f_c t)] \quad (12)$$

Adding Eqs. (11) and (12) and then solving for $m(t)$, we get

$$m(t) = \frac{2}{A_c}[s_l(t)\cos(2\pi f_c t) + \hat{s}_l(t)\sin(2\pi f_c t)] \quad (13)$$

Next, we use Eqs. (11) and (12) to write

$$s_l(t)\sin(2\pi f_c t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + \hat{m}(t)\sin^2(2\pi f_c t)] \quad (14)$$

$$\hat{s}_l(t)\cos(2\pi f_c t) = \frac{A_c}{2}[m(t)\sin(2\pi f_c t)\cos(2\pi f_c t) - \hat{m}(t)\cos^2(2\pi f_c t)] \quad (15)$$

Subtracting Eq. (15) from Eq. (14) and then solving for $\hat{m}(t)$, we get

$$\hat{m}(t) = \frac{2}{A_c}[s_l(t)\sin(2\pi f_c t) - \hat{s}_l(t)\cos(2\pi f_c t)] \quad (16)$$

Equations (13) and (16) are the desired results for part (b) of the problem.

(c) From Eqs. (15) and (16), we see that the message signal $m(t)$ may be recovered from $s_u(t)$ or $s_l(t)$ by using the scheme shown in Fig. 1.

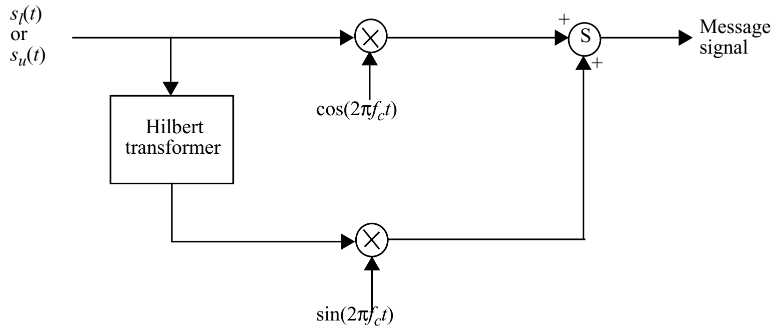


Figure 1

Problem 3.32

We will approach the solution to this problem by showing that, as postulated in the problem, if the in-phase component $H_I(f)$ of the complex low-pass filter's transfer function and its quadrature component $H_Q(f)$ satisfy the following relations

$$H_I(f) = 1 \qquad \text{for } -W \le f \le W \tag{1}$$

and

$$H_Q(-f) = -H_Q(f) \qquad \text{for } -W \le f \le W \tag{2}$$

then, starting with the frequency-discrimination basis for generating a VSB modulated wave $s(t)$, we may express $s(t)$ containing a vestige of the lower sideband as follows:

$$s(t) = \frac{A_c}{2}m(t)\cos(2\pi f_c t) - \frac{A_c}{2}m'(t)\sin(2\pi f_c t) \tag{3}$$

where $m'(t)$ is obtained by passing the message signal $m(t)$ through the quadrature filter defined by $H_Q(f)$.

To proceed, from Eq. (3.44) in the text, recall the relation

$$\frac{1}{2}\tilde{H}(f - f_c) = H(f), \qquad f > 0 \tag{4}$$

The corresponding relation for negative frequencies is described by

$$\frac{1}{2}\tilde{H}^*(f + f_c) = H(f), \qquad f < 0 \tag{5}$$

Using frequency discrimination as the basis for generating the VSB modulated wave $s(t)$, we express the spectrum of $s(t)$ as

$$S(f) = \frac{A_c}{2}[M(f - f_c) + M(f + f_c)]H(f) \tag{6}$$

where $M(f) = \mathbf{F}[s(t)]$. Next, using Eqs., (4) and (5) in (6), we write

$$S(f) = \frac{A_c}{4}[M(f - f_c) + M(f + f_c)][\tilde{H}(f - f_c)\tilde{H}^*(f + f_c)]$$

$$= \frac{A_c}{4}M(f - f_c)\tilde{H}(f - f_c) + \frac{A_c}{4}M(f + f_c)\tilde{H}^*(f + f_c) \tag{7}$$

where it is recognized that the cross-product terms

$M(f - f_c)\tilde{H}^*(f + f_c)$ and $M(f + f_c)\tilde{H}^*(f - f_c)$ are both zero, because the individual factors in each product term occupy completely disjoint frequency bands. Setting

$$\tilde{H}(f) = H_I(f) + jH_Q(f)$$

and

$$\tilde{H}^*(f) = H_I(f) - jH_Q(f)$$

we expand Eq. (7) as

$$S(f) = \frac{A_c}{4}[M(f - f_c)H_I(f - f_c) + M(f + f_c)H_I(f + f_c)]$$

$$+ j\frac{A_c}{4}[M(f - f_c)H_Q(f - f_c) - M(f + f_c)H_Q(f + f_c)] \quad (8)$$

Using the all-pass property of $H_I(f)$ defined in Eq. (1) and the odd-function property of $H_Q(f)$ defined in Eq. (2), we may simplify Eq. (8) as

$$S(f) = \frac{A_c}{4}[M(f - f_c) + M(f + f_c)]$$

$$+ j\frac{A_c}{4}[M(f - f_c) - M(f + f_c)]H_Q(f) \quad (9)$$

Transforming Eq. (9) into the time domain, we obtain the formula of Eq. (3) for the VSB modulated wave $s(t)$.

As noted earlier, $m'(t)$ is obtained by passing the message signal $m(t)$ through the quadrature filter. In accordance with the description of $H_Q(f)$ depicted in the problem, we may depict the frequency response of the quadrature filter as in Fig. 1, where f_v denotes the vestigial bandwidth.

The important point to note from the solution to this problem is that Eq. (3) includes SSB modulation as a special case. Specifically, if $f_v = 0$, then the frequency response depicted in Fig. 1 simplifies to a signum function. Correspondingly, Eq. (3) reduces to a SSB modulated wave containing the upper sideband.

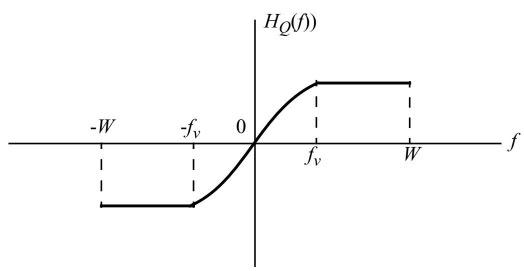


Figure 1