Chapter 4
Angle Modulation

Problem 4.1.
Using Eq. (4.7), show that FM waves also violate the principle of superposition.

Solution
From Eq. (4.7), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

Suppose $m(t) = m_1(t) + m_2(t)$. Then,

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_1(\tau)d\tau + 2\pi k_f \int_0^t m_2(\tau)d\tau\right] \quad (1)$$

Suppose next the two message signals $m_1(t)$ and $m_2(t)$ are applied individually to the frequency modulator. Then in response to $m_1(t)$, we have

$$s_1(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_1(\tau)d\tau\right] \quad (2)$$

Likewise, for $m_2(t)$ we have

$$s_2(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m_2(\tau)d\tau\right] \quad (3)$$

From Eqs. (1) through (3), we readily see that

$$s(t) \neq s_1(t) + s_2(t)$$

In other words, the principle of superposition (basic to linear systems) is violated. Hence, frequency modulation is a nonlinear process.

Problem 4.2
Suppose that the linear modulating wave

$$m(t) = \begin{cases} at & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

is applied to the scheme shown in Fig. 4.3(a). The phase modulator is defined by Eq. (4.4). Show that if the resulting FM wave is to have exactly the form as that defined in Eq. (4.7), then the phase-sensitivity factor k_p of the phase modulator is related to the frequency sensitivity factor k_f in Eq. (4.7) by the formula

$$k_p = 2\pi k_f T$$

where T is the interval over which the integration in Fig. 4.3(a) is performed. Justify the dimensionality of this expression.

Solution
According to Fig. 4.3(a), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + \frac{1}{T} k_p \int_0^t m(\tau) d\tau\right] \quad (1)$$

where T is an integration constant.

According to Eq. (4.7), the FM wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right] \quad (2)$$

If Eqs. (1) and (2) are to be identical, then we require that

$$k_p = 2\pi k_f T$$

Dimensionality of this expression is justified as follows:

1. k_f is measured in hertz per volt. Therefore, $2\pi k_f T$ has the dimensions of cycles, per volt and therefore radians per volt.
2. k_p is itself measured in radians per volt.

Problem 4.3

The *Cartesian baseband representation* of band-pass signals discussed in Section 3.8.1 is well-suited for linear modulation schemes exemplified by the amplitude modulation family. On the other hand, the *polar baseband representation*

$$s(t) = a(t) \cos[2\pi f_c t + \phi(t)]$$

is well-suited for nonlinear modulation schemes exemplified by the angle modulation family. The $a(t)$ in this new representation is the envelope of $s(t)$ and $\phi(t)$ is its phase.

Starting with the baseband representation [see Eq. (3.39)]

$$s(t) = s_I(t) \cos 2\pi f_c t - s_Q(t) \sin(2\pi f_c t)$$

where $s_I(t)$ is the in-phase component and $s_Q(t)$ is the quadrature component, we may write

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Show that the polar representation of $s(t)$ in terms of $a(t)$ and $\phi(t)$ is exactly equivalent to its Cartesian representation in terms of $s_I(t)$ and $s_Q(t)$.

Solution

We are given

$$a(t) = [s_I^2(t) + s_Q^2(t)]^{1/2}$$

and

$$\phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

Hence, expanding the polar representation of $s(t)$, we write

$$s(t) = a(t) \cos[\theta t]$$
$$= a(t) \cos[2\pi f_c t + \phi(t)]$$

$$= a(t)\cos(\phi(t))\cos(2\pi f_c t) - a(t)\sin(\phi(t))\sin(2\pi f_c t) \quad (1)$$

Since $\tan[\phi(t)] = \left[\frac{s_Q(t)}{s_I(t)}\right]$, it follows that

$$\cos\phi(t) = \frac{s_I(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_I(t)}{a(t)}$$

and

$$\sin\phi(t) = \frac{s_Q(t)}{[s_I^2(t) + s_Q^2(t)]^{1/2}} = \frac{s_Q(t)}{a(t)}$$

Hence,

$$a(t)\cos\phi(t) = s_I(t) \quad (2)$$

and

$$a(t)\sin\phi(t) = s_Q(t) \quad (3)$$

Substituting Eqs. (2) and (3) into (1), we get

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

which is the Cartesian representation of $s(t)$.

Problem 4.4

Consider the narrow-band FM wave approximately defined by Eq. (4.17). Building on Problem 4.3, do the following:

(a) Determine the envelope of this modulated wave. What is the ratio of the maximum to the minimum value of this envelope?

(b) Determine the average power of the narrow-band FM wave, expressed as a percentage of the average power of the unmodulated carrier wave.

(c) By expanding the angular argument $\theta(t) = 2\pi f_c t + \phi(t)$ of the narrow-band FM wave $s(t)$ in the form of a power series and restricting the modulation index β to a maximum value of 0.3 radian, show that

$$\theta(t) \approx 2\pi f_c t + \beta\sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

What is the value of the harmonic distortion for $\beta = 0.3$ radian?

Hint: For small x, the following power series approximation

$$\tan^{-1}(x) \approx x - \frac{1}{3}x^3$$

holds. In this approximation, terms involving x^5 and higher order ones are ignored, which is justified when x is small compared to unity.

Solution

(a) From Eq. (4.17), the narrow-band FM wave is approximately defined by

$$s(t) \approx A_c\cos((2\pi f_c t) - \beta A_c\sin(2\pi f_c t)\sin(2\pi f_m t)) \quad (1)$$

The envelope of $s(t)$ is therefore

$$a(t) = A_c(1 + \beta^2 \sin^2(2\pi f_m t))^{1/2}$$

$$\approx A_c\left(1 + \frac{1}{2}\beta^2 \sin^2(2\pi f_m t)\right)^{1/2} \qquad \text{for small } \beta$$

The maximum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 1$, yielding

$$A_{\max} \approx A_c\left(1 + \frac{1}{2}\beta^2\right)$$

The minimum value of $a(t)$ occurs when $\sin^2(2\pi f_m t) = 0$, yielding

$$A_{\min} = A_c$$

The ratio of the maximum to the minimum value is therefore

$$\frac{A_{\max}}{A_{\min}} \approx \left(1 + \frac{1}{2}\beta^2\right)$$

(b) Expanding Eq. (1) into its individual frequency components, we may write

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c \cos(2\pi(f_c - f_m)t)$$

The average power of $s(t)$ is therefore

$$P_{\text{av}} = \frac{1}{2}A_c^2 + \left(\frac{1}{2}\beta A_c\right)^2 + \left(\frac{1}{2}\beta A_c\right)^2$$

$$= \frac{1}{2}A_c^2(1 + \beta^2)$$

The average power of the unmodulated carrier is

$$P_c = \frac{1}{2}A_c^2$$

Hence,

$$\frac{P_{\text{av}}}{P_c} = 1 + \beta^2$$

(c) The angle $\theta(t)$ is defined by

$$\theta(t) = 2\pi f_c t + \phi(t)$$

$$= 2\pi f_c t + \tan^{-1}(\beta \sin(2\pi f_m t))$$

Setting $\beta = \sin(2\pi f_m t)$

and using the approximation (based on the Hint), we may approximate $\theta(t)$ as

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{1}{3}\beta^3 \sin(2\pi f_m t)$$

Ideally, we should have (see Eq. (4.15))

$$\theta(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

The harmonic distortion produced by using the narrow-band approximation is therefore

$$D(t) = \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

The maximum absolute value of $D(t)$ for $\beta = 0.3$ is therefore

$$D_{\max} = \frac{\beta^3}{3}$$

$$= \frac{0.3^3}{3} = 0.009 \approx 1\%$$

which is small enough for it to be ignored in practice.

Problem 4.5

Strictly speaking, the FM wave of Eq. (4.15) produced by a sinusoidal modulating wave is a nonperiodic function of time t. Demonstrate this property of frequency modulation.

Solution

Starting with Eq. (4.15) we write

$$s(t) = A_c[\cos(2\pi f_c t) + \beta\sin(2\pi f_m t)] \tag{1}$$

For the FM wave $s(t)$ to be a periodic function of time, we require that the condition

$$s\left(t + \frac{1}{f_m}\right) = s(t) \tag{2}$$

be satisfied for a period equal to $1/f_m$. Replacing t with $t + (1/f_m)$ in Eq. (1), we write

$$s\left(t + \frac{1}{f_m}\right) = A_c\cos\left[2\pi f_c\left(t + \frac{1}{f_m}\right) + \beta\sin\left(2\pi f_m t + \frac{1}{f_m}\right)\right]$$

$$= A_c\cos[2\pi f_c + (2\pi f_c/f_m) + \beta\sin(2\pi f_m t + 2\pi)]$$

$$= A_c\cos[2\pi f_c + (2\pi f_c/f_m) + \beta\sin(2\pi f_m t)] \tag{3}$$

In general, the carrier frequency f_c is a noninteger multiple of the modulation frequency f_m. Accordingly, $s(t + (1/f_m)) \neq s(t)$ and therefore the condition of Eq. (2) for periodicity is violated.

Problem 4.6

Using a well-known trigonometric identity involving the product of the sine of an angle and the cosine of another angle, demonstrate the two results just described under points 1 and 2.

Solution

The incoming FM wave is defined by (see Eq. (4.57))

$$s(t) = A_v\cos[2\pi f_c t + \phi_1(t)] \tag{1}$$

The internally generated output of the VCO is defined by (see Eq. (4.59))

$$r(t) = A_v\cos[2\pi f_c t + \phi_2(t)] \tag{2}$$

Multiplying $s(t)$ by $r(t)$ yields

$$s(t)r(t) = A_c A_v\sin[2\pi f_c t + \phi_1(t)]\cos[2\pi f_c t + \phi_2(t)] \tag{3}$$

Using the trigonometric identity

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

we may rewrite Eq. (3) as

$$s(t)r(t) = \frac{1}{2}A_c A_v \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)]$$
$$+ \frac{1}{2}A_c A_v \sin[\phi_1(t) - \phi_2(t)] \quad (4)$$

Except for a scaling factor, the first term of Eq. (4) defines the double-frequency term (identified under point 1 on page 179) and the second term of the equation defines the difference-frequency term (identified under point 2 of the same page).

Problem 4.7

Using the linearized model of Fig. 4.15(a), show that the model is approximately governed by the integro-differential equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau \approx \frac{d\phi_1(t)}{dt}$$

Hence, derive the following two approximate results in the frequency domain:

(a) $\Phi_e(f) = \frac{1}{1+L(f)}\Phi_1(f)$

(b) $V(f) = \frac{jf}{k_v}\frac{L(f)}{1+L(f)}\Phi_1(f)$

where

$$L(f) = K_0 \frac{H(f)}{jf}$$

is the open-loop transfer function. Finally, show that when $L(f)$ is large compared with unity for all frequencies inside the message band, the time-domain version of the formula in part (b) reduces to the approximate form in Eq. (4.68).

Solution

According to condition 1 stated on p.178 of the text, the frequency of the VCO is set equal to the carrier frequency f_c. According to condition 2 on the same page, the VCO output has a 90° phase shift with respect to the unmodulated carrier. In light of these two conditions, we note starting with the equation

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau \approx \frac{d\phi_1(t)}{dt}$$

the integral in the left-hand side of the equation is the convolution of $\phi_e(t)$ and $h(t)$. Therefore, applying the Fourier transform to this equation and using two properties of the fourier transform pertaining to differentiation and convolution, we get

$$j2\pi f \Phi_e(f) + 2\pi K_0 \Phi_e(f) H(f) \approx j2\pi f \Phi_1(f) \quad (1)$$

where

$$\Phi_e(f) = \mathbf{F}[\phi_e(t)] \text{ and } \Phi_1(f) = \mathbf{F}[\phi_1(t)]$$

(a) Solving Eq. (1) for $\Phi_e(f)$, we get

$$\Phi_e(f) \approx \frac{j2\pi f}{j2\pi f + 2\pi K_0 H(f)}\Phi_1(f)$$

$$= \frac{1}{1 + K_0 \frac{H(f)}{jf}}\Phi_1(f)$$

$$= \frac{1}{1 + L(f)}\Phi_1(f) \qquad (2)$$

where $L(f) = \frac{H(f)}{jf}$

(b) Next, from Eq. (4.63) we have

$$e(t) = \frac{K_0}{k_v}\phi_e(t)$$

Therefore

$$E(f) = \frac{K_0}{k_v}\Phi_e(f)$$

And, from Eq. (4.65) we have

$$v(t) = \int_{-\infty}^{\infty} e(\tau)h(t-\tau)d\tau$$

Therefore

$$V(f) = E(f)H(f)$$

Eliminating $E(f)$ between these two transform-related equations, we get

$$V(f) = \frac{K_0}{k_v}H(f)\Phi_e(f) \qquad (3)$$

Eliminating $\Phi_e(f)$ between Eqs. (1) and (3), we get

$$V(f) = \frac{K_0}{k_v}H(f) \cdot \frac{1}{1 + L(f)}\Phi_1(f)$$

Since

$$L(f) = K_0 \frac{H(f)}{jf}$$

then

$$\frac{K_0}{k_v}H(f) = \frac{jf}{k_v}L(f)$$

and so we get the desired result

$$V(f) \approx \frac{jf}{k_v} \frac{L(f)}{1 + L(f)}\Phi_1(f) \qquad (4)$$

Finally, when $L(f) >> 1$ for all f, Eq. (4) simplifies further as

$$(f) \approx \frac{jf}{k_v}\Phi_1(f) \approx \frac{j2\pi f}{2\pi k_v}\Phi_1(f)$$

The time-domain version of this formula reads as follows

$$v(t) \approx \frac{1}{2\pi k_v} \frac{d\phi_1(t)}{dt}$$

which is a repeat of Eq. (4.67).

Problem 4.8

For the PM case, we have by definition

$s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$.

whose angle is

$\theta_i(t) = 2\pi f_c t + k_p m(t)$.

The instantaneous frequency is therefore

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}, \\ &= f_c + \frac{k_p}{2\pi} \frac{dm(t)}{dt} \\ &= f_c + \frac{Ak_p}{2\pi T_0} - \frac{Ak_p}{2\pi}\sum_n \delta(t - nT_0) \end{aligned} \qquad (1)$$

which is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_p A/T_0$ radians.Accordingly, the PM wave has the waveform depicted in Fig. 1

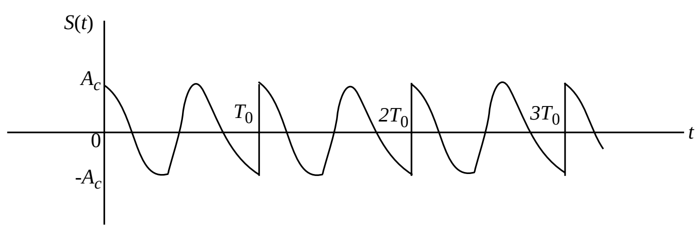

Figure 1

For the FM case, we have

$f_i(t) = f_c + k_f m(t)$

and the modulated wave is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) dt\right]$$

The modulated wave is therefore depicted in Fig. 2.

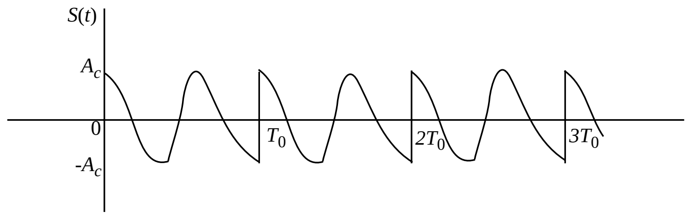

Figure 2

Problem 4.9

The instantaneous frequency of the mixer output is as shown in Fig. 1:

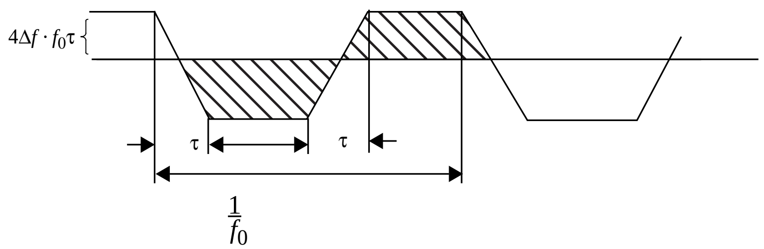

Figure 1

The presence of negative frequency merely indicates that the phasor representing the difference frequency at the mixer output has reversed its direction of rotation.

Let N denote the number of beat cycles in one period. Then, noting that N is equal to the number of shaded areas shown in Fig. 1, we deduce that

$$N = 2\left[4\Delta f \cdot f_0\tau\left(\frac{1}{2f_0} - \tau\right) + 2\Delta f \cdot f_0\tau^2\right]$$

$$= 4\Delta f \cdot \tau(1 - f_0\tau)$$

Since $f_0\tau << 1$, we have the approximate result

$$N \approx 4\Delta f \cdot \tau$$

Therefore, the number of beat cycles counted over one second is equal to

$$\frac{N}{1/f_0} = 4\Delta f \cdot f_0\tau.$$

Problem 4.10

By definition, the instantaneous frequency f_i is related to the phase $\theta(t)$ as

$$f_i = \frac{1}{2\pi}\frac{d\theta}{dt}$$

which may be rewritten as

$$f_i \approx \frac{1}{2\pi}\frac{\Delta\theta}{\Delta t} \qquad (1)$$

where $\Delta\theta$ and Δt are small changes in the phase $\theta(t)$ and time t. We are given

$$\theta(t + \Delta t) - \theta(t) = \pi$$

from which we infer that

$$\Delta\theta = \pi \tag{2}$$

Substituting Eq. (2) into (1) yields

$$f_i \approx \frac{1}{2\pi} \cdot \frac{\Delta\theta}{\Delta t} = \frac{1}{2\Delta t}$$

which is the desired result.

Problem 4.11

The phase-modulated wave is defined by

$$\begin{aligned} s(t) &= A_c \cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)] \\ &= A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)], \qquad \beta_p = k_p A_m \\ &= A_c \cos(2\pi f_c t) \cos[\beta_p \cos(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta_p \cos(2\pi f_m t)] \qquad (1) \end{aligned}$$

If $\beta_p \le 0.3$, then for all time t we approximately have

$$\cos[\beta_p \cos(2\pi f_m t)] \approx 1$$

$$\sin[\beta_p \cos(2\pi f_m t)] \approx \beta_p \cos(2\pi f_m t)$$

Correspondingly, we may approximate Eq. (1) as follows:

$$\begin{aligned} s(t) &\approx A_c \cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t) \cos(2\pi f_m t) \\ &= A_c \cos(2\pi f_c t) - \frac{1}{2}\beta_p A_c \sin[2\pi(f_c + f_m)t] - \frac{1}{2}\beta_p A_c \sin[2\pi(f_c - f_m)t] \qquad (2) \end{aligned}$$

The spectrum of $s(t)$ is therefore

$$\begin{aligned} S(f) \approx\ & \frac{1}{2}A_c[\delta(f - f_c) + \delta(f + f_c)] \\ & -\frac{1}{4j}\beta_p A_c[\delta(f - f_c - f_m) - \delta(f + f_c + f_m)] \\ & -\frac{1}{4j}\beta_p A_c[\delta(f - f_c + f_m) - \delta(f + f_c - f_m)] \end{aligned}$$

Problem 4.12

(a) From Table A3.1 in Appendix 3, we find (by interpolation) that $J_0(\beta)$ is zero for the following values of modulation index:

$\beta = 2.44$,
$\beta = 5.52$,
$\beta = 8.65$,
$\beta = 11.8$,
and so on.

(b) The modulation index is defined by

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Therefore, the frequency sensitivity factor is

$$k_f = \frac{\beta f_m}{A_m} \tag{1}$$

We are given f_m = 1 kHz and A_m = 2 volts. Hence, with $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, the use of Eq. (1) yields

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

Problem 4.13

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

Using Carson's rule, the transmission bandwidth of the FM wave is therefore

$$B_T = 2f_m(1 + \beta) = 2 \times 100(1 + 5) = 1200\text{kHz} = 1.2\text{MHz}$$

(b) Using the universal curve of Fig. 4.9, we find that for $\beta = 5$:

$$\frac{B_T}{\Delta f} = 3$$

Therefore, the transmission bandwidth is

$$B_T = 3 \times 500 = 1500\text{kHz} = 1.5\text{MHz}$$

which is greater than the value calculated by Carson's rule.

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f = 1\text{MHz} \text{ and } \beta = 10$$

Thus, using Carson's rule we now obtain the transmission bandwidth

$$B_T = 2 \times 100(1 + 10) = 2200\text{kHz} = 2.2\text{MHz}$$

On the other hand, using the universal curve of Fig. 4.9, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and $B_T = 2.75$ MHz.

(d) If f_m is doubled, $\beta = 2.5$. Then, using Carson's rule, $B_T = 1.4$ MHz. Using the universal curve, $(B_T/\Delta f) = 4$, and

$$B_T = 4\Delta f = 2\text{MHz}$$

Problem 4.14

(a) The angle of the PM wave is defined by

$$\begin{aligned}\theta_i(t) &= 2\pi f_c t + k_p m(t)\\ &= 2\pi f_c t + k_p A_m \cos(2\pi f_m t)\\ &= 2\pi f_c t + \beta_p \cos(2\pi f_m t)\end{aligned}$$

where $\beta_p = k_p A_m$. The instantaneous frequency of the PM wave is therefore

$$\begin{aligned}f_i(t) &= \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}\\ &= f_c - \beta_p f_m \sin(2\pi f_m t) \qquad (1)\end{aligned}$$

Based on Eq. (1), we see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when β_p is small compared to unity)

$$B_T \approx 2(f_m + \beta_p f_m) = 2f_m(1 + \beta_p) \approx 2f_m\beta_p. \qquad (2)$$

Equation (2) shows that B_T varies linearly with the modulation frequency f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, assuming that the modulation index β is small compared to unity. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 4.15

Consider first the action of the mixer with the two inputs: voltage-controlled oscillator (VCO) output and crystal oscillator output. The mixer produces an output of its own whose frequency is the difference between the instantaneous frequency of the VCO and the crystal oscillator frequency.

The mixer output is applied to the frequency discriminator followed by a low-pass filter. By design, the output produced by the frequency discriminator has an instantaneous amplitude that is proportional to the instantaneous frequency of the FM signal applied to its input. Accordingly, the amplitude of the signal produced by the frequency discriminator is proportional to the difference between the VCO frequency and the crystal oscillator frequency.

In light of these considerations, we may now make the following statements:

- When the FM signal $s(t)$ produced at the VCO output has exactly the correct frequency, the low-pass filter output is zero.
- Deviations in the carrier frequency of the FM signal $s(t)$ from its assigned value will cause the frequency discriminator-filter output to produce a dc output with a polarity determined by the sense of the carrier-frequency drift in the FM signal $s(t)$. This dc signal, after suitable amplification is, in turn, applied to the VCO in such a way as to modify the instantaneous frequency of the VCO in a direction that tends to restore the carrier frequency of the FM signal $s(t)$ to its correct value.

In summary, the application of feedback applied to the VCO in the manner described in Fig. 4.19 has the beneficial effect of stabilizing the carrier frequency of the FM signal produced at the VCO output.

Problem 4.16

From Fig. 4.20, we see that the envelope detector input is

$$\begin{aligned} v(t) &= s(t) - s(t-T) \\ &= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)] \end{aligned}$$

Using a well-known trigonometric identity, we write

$$v(t) = -2A_c \sin\left[\frac{2\pi f_c (2t-T) + \phi(t) + \phi(t-T)}{2}\right] \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1)$$

For $\phi(t)$, we have

$$\phi(t) = \beta \sin(2\pi f_m t)$$

Correspondingly, the phase difference $\phi(t)$ - $\phi(t - T)$ is given by

$$\begin{aligned} \phi(t) - \phi(t-T) &= \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)] \\ &= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t)\cos(2\pi f_m T) + \cos(2\pi f_c t)\sin(2\pi f_m T)] \qquad (2) \end{aligned}$$

Using the approximations:

$$\cos(2\pi f_m T) \approx 1$$

$$\sin(2\pi f_m T) \approx 2\pi f_m T$$

we may approximate Eq. (2) as

$$\begin{aligned} \phi(t) - \phi(t-T) &\approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)] \\ &= 2\pi \Delta f T \cos(2\pi f_m t) \qquad (3) \end{aligned}$$

where

$$\Delta f = \beta f_m.$$

Therefore, recognizing that $2\pi f_c T = \pi/2$, we may write

$$\begin{aligned} \sin\left(\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right) &\approx \sin(\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)) \\ &= \sin\left(\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)\right) \end{aligned}$$

$$= \sqrt{2}\cos(\pi\Delta fT\cos(2\pi f_m t)) + \sqrt{2}\sin(\pi\Delta fT\cos(2\pi f_m t))$$

$$= \sqrt{2} + \sqrt{2}\pi\Delta fT\cos(2\pi f_m t)$$

where we have made use of the fact that $\pi\Delta fT << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}A_c(1 + \pi\Delta fT\cos(2\pi f_m t))\sin\left(\pi f_c(2t - T) + \frac{\phi(t) + \phi(t-T)}{2}\right) \tag{4}$$

Accordingly, the envelope detector output is the envelope of $v(t)$, namely,

$$a(t) \approx 2\sqrt{2}A_c(1 + \pi\Delta fT\cos(2\pi f_m t))$$

which, except for a bias term, is proportional to the modulating wave.

Problem 4.17

Consider first the message signal

$$m_1(t) = \begin{cases} a_1 t + a_0, & t \geq 0 \\ 0, & t = 0 \end{cases}$$

applied to a frequency modulator. The signal produced by this modulator is defined by

$$s_1(t) = A_c\cos\left[2\pi f_c t + 2\pi k_f\int_0^t m_1(\tau)d\tau\right]$$

$$= A_c\cos\left[2\pi f_c t + 2\pi k_f\int_0^t (a_1 t + a_0)d\tau\right]$$

$$= A_c\cos\left[2\pi f_c t + 2\pi k_f\left(\frac{1}{2}a_1 t^2 + a_0 t + C\right)\right], \qquad t \geq 0 \tag{1}$$

where C is the constant of integration.

Consider next the message signal

$$m_2(t) = \begin{cases} b_2 t^2 + b_1 t + b_0 & t \geq 0 \\ 0, & t = 0 \end{cases}$$

applied to a phase modulator. The signal produced by this second modulator is defined by

$$s_2(t) = A_c\cos[2\pi f_c t + k_p m_2(t)]$$

$$= A_c\cos[2\pi f_c t + k_p(b_2 t^2 + b_1 t + b_0)], \qquad t \geq 0 \tag{2}$$

For the FM signal $s_1(t)$ of Eq. (1) and the PM signal of Eq. (2) to be exactly equal for $t \geq 0$, we require that the following conditions be satisfied:

(i) $\pi k_f a_1 = k_p b_2$

(ii) $2\pi k_f a_0 = k_p b_1$

(iii) $2\pi k_f C = k_p b_0$

Problem 4.18

We are given that the IF filter has a bandwidth of 200 kHz centered on the frequency f_{IF} = 10.7 MHz. This filter will therefore pass frequencies inside the range defined by the two extremes:

low-end: 10.7 - 0.2 = 10.5 MHz

high-end: 10.7 + 0.2 = 10.9 MHz

The image lies inside the band 109.4 to 129.4 MHz, which is positioned well outside the passband of the IF filter. Therefore, the IF filter will suppress the translated band centered on the image frequency f_{image}.

Problem 4.19

The instantaneous frequency of the modulated wave $s(t)$ is shown in Fig. 1

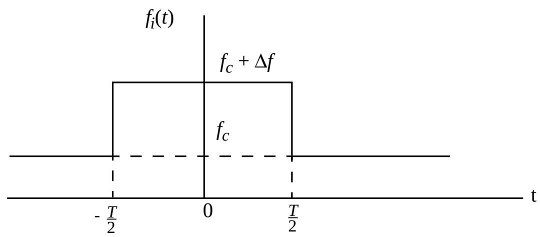

Figure 1

We may thus express $s(t)$ as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c + \Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos(2\pi f_c t), & \frac{T}{2} < t \end{cases} \quad (1)$$

The Fourier transform of $s(t)$ is therefore

$$\begin{aligned} S(f) &= \int_{-\infty}^{-T/2} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{-T/2} \cos[2\pi(f_c + \Delta f)t]\exp(-j2\pi ft)dt \\ &\quad + \int_{T/2}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &= \int_{-\infty}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{-T/2} \{\cos[2\pi(f_c + \Delta f)t - \cos(2\pi f_c t)]\}\exp(-j2\pi ft)dt \end{aligned} \quad (2)$$

The second term of Eq. (2) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c + \Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_c T >> 1$, we may express the Fourier transform $S(f)$ of Eq. (2) as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\delta(f - f_c) + \frac{T}{2}\text{sinc}\,[T(f - f_c - \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f - f_c)], & f > 0 \\ \frac{1}{2}\delta(f + f_c) + \frac{T}{2}\text{sinc}\,[T(f + f_c + \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f + f_c)], & f < 0 \end{cases} \quad (3)$$

Problem 4.20

The filter input is

$$v_1(t) = g(t)s(t)$$

$$= g(t)\cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t)\exp(-j\pi k t^2)$$

The impulse response $h(t)$ of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathbf{Re}[\tilde{h}(t)\exp(j2\pi f_c t)]$$

With $h(t)$ defined by

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore (except for a scaling factor)[1]

$$\tilde{v}_o(t) = \tilde{h}(t) \star \tilde{v}_i(t)$$

$$= \int_{-\infty}^{\infty} g(\tau)\exp(-j\pi k \tau^2)\exp[j\pi k(t - \tau)]^2 d\tau$$

$$= \exp(j\pi k t^2)\int_{-\infty}^{\infty} g(\tau)\exp(-2j\pi k t \tau)d\tau$$

$$= \exp(j\pi k t^2)G(kt) \quad (1)$$

where in the last line we have used the definition of the Fourier transform to write

$$G(kt) = \int_{-\infty}^{\infty} g(\tau)\exp(-j2\pi k t \tau)d\tau$$

Hence, from Eq. (1), we obtain the

---

1. It turns out that the scaling factor equals 1/2; to be exact, we should write

$$\tilde{v}_o(t) = \frac{1}{2}\tilde{h}(t) \star \tilde{v}_i(t)$$

For details, see the 4th edition of the book:
S. Haykin, Communication Systems, pp. 725-734, 4th edition, Wiley.

$$\tilde{v}_0(t) = |G(kt)| \tag{2}$$

Equation (2) shows that the envelope of the filter output is, except for a scaling factor, equal to the magnitude of the Fourier transform of the input signal $g(t)$ with kt playing the role of frequency f.

Problem 4.21

For convenience of the discussion, we assume time-domain symmetry around the origin $t = 0$. Accordingly, in theory, the signal produced by the amplitude limiter component of the band-pass limiter due to $s_1(t)$ consists of an infinite sequence of harmonically related angle-modulated components with two properties:

- The components are centered on odd multiples of the carrier frequency f_c.
- The components have progressively decreasing amplitudes.

Typically, the carrier frequency f_c of an FM signal is large compared to the transmission bandwidth B_T of the FM signal. It follows therefore that provided this condition is satisfied, that is, f_c is large enough compared to B_T, then the filter component of the band-pass limiter will effectively suppress all the spectral components coming out of the limiter except for the one component centered on f_c.

In light of these observations that are intuitively satisfying, we may now state that if f_c is large enough compared to B_T, then the output $s_2(t)$ produced by the band-pass limiter in response to the input $s_1(t)$ is defined by the FM signal

$$s_2(t) = A\cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

where the amplitude A is a constant.

Problem 4.22

(a) Starting with Eq. (4.15) for sinusoidal FM, we write

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \tag{1}$$

where f_m is the modulation frequency and β is the modulation index. Correspondingly, the Fourier transform of $s(t)$ is defined for an arbitrary value of β (see Eq. (4.31))

$$S(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \tag{2}$$

where $J_n(\beta)$ is the nth order Bessel function of the first kind. Passing $s(t)$ through a linear channel of transfer function $H(f)$ produces an output signal $y(t)$ whose Fourier transform is defined by

$$\begin{aligned} Y(f) &= H(f)S(f) \\ &= \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c+nf_m)\delta(f-f_c-nf_m) + H(-f_c-nf_m)\delta(f+f_c+nf_m)] \end{aligned} \tag{3}$$

Applying the inverse Fourier transform to $Y(f)$ yields the output signal

$$y(t) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)]$$

$$+ H(-f_c - nf_m)\exp(-j2\pi(f_c + nf_m)t) \tag{4}$$

For a channel with real-valued impulse response, we have $H(f) = H^*(-f)$ where the asterisk denotes complex conjugation. We may therefore rewrite Eq. (4) as

$$y(t) \approx \frac{1}{2}A_c \sum_{n=-\infty}^{\infty} J_n(\beta)[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)]$$

$$+ H^*(f_c + nf_m)\exp(-j2\pi(f_c + nf_m)t)$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\mathbf{Re}[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)] \tag{5}$$

where **Re** denotes the real-time operator.

(b) Following the development of the universal curve plotted in Fig. 4.9, let $n_{\max}$ denote the largest value of n in Eq. (5) for which the condition

$|J_n(\beta)| > 0.1$

is satisfied so as to preserve the effective frequency content of the FM signal $s(t)$. We may then approximate Eq. (5) as

$$y(t) \approx A_c \sum_{n=-n_{\max}}^{n_{\max}} J_n(\beta)\mathbf{Re}[H(f_c + nf_m)\exp(j2\pi(f_c + nf_m)t)] \tag{6}$$

Expressing the transfer function $H(f)$ in the polar form

$$H(f) = |H(f)|\exp(j\phi(f)) \tag{7}$$

we may rewrite Eq. (6) as

$$y(t) \approx A_c \sum_{n=-n_{\max}}^{n_{\max}} J_n(\beta)|H(f_c + nf_m)|\cos(2\pi(f_c + nf_m)t + \phi(f_c + nf_m)) \tag{8}$$

From the discussion presented in Section 2.7, recall that the transmission of a signal through a linear channel (filter) is distortionless provided that two conditions are satisfied:

(i) The amplitude response $|H(f)|$ is constant over the band $-B \le f \le B$, where B is the channel bandwidth.

(ii) The phase response $\phi(f)$ is a linear function of the frequency f inside the band $-B \le f \le B$.

Accordingly, in the context of our present discussion, the FM transmission through the channel of transfer function $H(f)$ introduces two forms of linear distortion:

(i) *Amplitude distortion*, which arises when the condition

$|H(f_c + nf_m)|$ is constant for $0 \le n \le n_{\max}$

is violated.

(ii) *Phase distortion*, when the condition

$\theta(f_c + nf_m)$ is a linear function of n for $0 \le n \le n_{\max}$

is violated.

Problem 4.23

(a) Consider the FM version of angle modulation. Let the instantaneous frequency of the modulator be a linear function of the first derivative of the message signal $m(t)$, as shown by

$$f_i(t) = f_c + k_1 \frac{d}{dt} m(t)$$

Then, correspondingly, the instantaneous phase is defined by

$$\begin{aligned}\theta_i(t) &= 2\pi \int_0^t f_i(t) dt \\ &= 2\pi f_c t + 2\pi k_1 m(t)\end{aligned}$$

where it is assumed that $m(0) = 0$. In this scenario, the modulated signal is defined by

$$\begin{aligned}s(t) &= A_c \cos(\theta_i(t)) \\ &= A_c \cos[2\pi f_c t + \theta_i(t) m(t)]\end{aligned}$$

which is recognized as phase modulation.

Suppose next that $f_i(t)$ is a linear function of the second derivative of $m(t)$, as shown by

$$f_i(t) = f_c + k_2 \frac{d^2 m(t)}{dt^2}$$

Correspondingly, we have

$$\theta_i(t) = 2\pi f_c t + 2\pi k_2 \frac{dm(t)}{dt}$$

where it is assumed that $dm(t)/dt$ is zero at $t = 0$. The modulated wave assumes the new form

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_2 \frac{dm(t)}{dt}\right)$$

We may generalize these results by stating that if the input to a frequency modulator is the nth derivative of the message signal $m(t)$, then the corresponding modulated wave is defined by

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_n \frac{d^{n-1} m(t)}{dt^{n-1}}\right)$$

where it is assumed that $d^{n-1}m(t)/dt^{n-1}$ is zero at time $t = 0$.

Consider next the scenario where the input to the frequency modulator involves integrals of the message signal $m(t)$. Starting with $\int_0^t m(\tau) d\tau$ as the input to the modulator, we write

$$f_i(t) = f_c + c_1 \int_0^t m(\tau) d\tau$$

and, correspondingly,

$$\theta_i(t) = 2\pi f_c t + 2\pi c_1 \left(\int_0^\lambda m(\tau) d\tau\right) d\lambda$$

The resulting modulated signal is defined by

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi c_1 \int_0^t \left(\int_0^\lambda m(\tau)d\tau\right)d\lambda\right]$$

Unlike the modulation scenario involving derivatives of $m(t)$, we can see that when considering the scenario involving integrals of $m(t)$, mathematical formulation of the modulated signal $s(t)$ becomes increasingly more complicated.

(b) There could be a practical benefit from using a frequency-modulation strategy involving integrals of the message signal $m(t)$ if $m(t)$ happens to be corrupted by additive noise. In such a scenario, the integration process tends to reduce the corruptive influence of the additive noise by smoothing it out. However, the drawback of such a modulation strategy is two-fold:

(i) Mathematical analysis of ordinary FM is complicated enough. Using integrals of the message signal as the input to the frequency modulator makes the problem even more complicated.

(ii) Likewise, designs of the transmitter and receiver become even more complicated.

Statements similar to (i) and (ii) apply to the use of second and higher derivatives of the message signal $m(t)$ as the input to the frequency modulator. The only exception here is the first derivative of $m(t)$, in which case the frequency modulator produces a phase modulated version of the signal. One other point to note is that if the message signal $m(t)$ is corrupted by additive noise, the operation of differentiation will enhance the presence of the noise component, which is undesirable.

To conclude, the "simple" forms of angle modulation exemplified by the ordinary FM and ordinary PM discussed in the chapter are good enough from a theoretical as well as practical perspective.

Problem 4.24

(a) We are given a nonlinear channel's input-output relation:

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \quad (1)$$

where $v_i(t)$ is the input and $v_o(t)$ is the output; a_1, a_2, and a_3 are fixed parameters. The input signal is defined by

$$v_i(t) = A_c \cos(2\pi f_c t + \phi(t)) \quad (2)$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau \quad (3)$$

where $m(t)$ is the message signal and k_f is the frequency sensitivity of the frequency modulator. Substituting Eq. (2) into (1) yields

$$v_o(t) = a_1 A_c \cos(2\pi f_c t + \phi(t)) + a_2 A_c^2 \cos^2(2\pi f_c t + \phi(t))$$
$$+ a_3 A_c^3 \cos^3(2\pi f_c t + \phi(t)) \quad (4)$$

Using the trigonometric identities:

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\cos^3\theta = \frac{1}{4}(1 + \cos(3\theta))$$

we may rewrite Eq. (4) as

$$v_o(t) = \frac{1}{2}a_2 A_c^2 + \left(a_1 A_c + \frac{3}{4}a_3 A_c^3\right)\cos(2\pi f_c t + \phi(t))$$

$$+ \frac{1}{2}a_2 A_c^2 \cos(4\pi f_c t + 2\phi(t)) + \frac{1}{4}a_3 A_c^3 \cos(6\pi f_c t + 3\phi(t)) \quad (5)$$

Equation (5) shows that the channel output consists of the following components:

- A dc component, $\frac{1}{2}a_2 A_c^2$
- Frequency modulated component of frequency f_c, phase $\phi(t)$ and amplitude $\left(a_1 A_c + \frac{3}{4}a_3 A_c^3\right)$
- Frequency modulated component of frequency $2f_c$, phase $2\phi(t)$ and amplitude $\frac{1}{2}a_2 A_c^2$
- Frequency modulated component of frequency $3f_c$, phase $3\phi(t)$ and amplitude $\frac{1}{4}a_3 A_c^3$

(b) To remove the nonlinear distortion and thereby extract a replica of the original FM signal $v_i(t)$, it is necessary to separate the FM component with carrier frequency f_c in $v_o(t)$ from the higher order FM components. Let Δf denote the frequency deviation of the original FM signal and W denote the highest frequency component of the message signal $m(t)$. Then, using Carson's rule and noting that the frequency deviation above $2f_c$ is doubled (which is the component nearest to the original FM signal), we find that the necessary condition for separating the desired FM signal with carrier frequency f_c from that with carrier frequency $2f_c$ is

$$2f_c - (2\Delta f + W) > f_c + \Delta f - W$$

or

$$f_c > 3\Delta f + 2W \quad (6)$$

(c) To extract a replica of the original FM signal $v_i(t)$, we need to pass the channel output $v_o(t)$ through a band-pass filter of midband frequency f_c and bandwidth $2(\Delta f + W)$. The resulting filter output is

$$v'_o(t) = \left(a_1 A_c + \frac{3}{4}a_3 A_c^3\right)\cos(2\pi f_c t + \phi(t)) \quad (7)$$

where $\phi(t)$ is defined by Eq. (3).

Problem 4.25

(a) The loop filter in the second-order phase-locked loop (PLL) is defined by

$$H(f) = 1 + \frac{a}{jf} \tag{1}$$

where a is a filter parameter. The Fourier transform of phase error $\phi_e(t)$ (i.e., the phase difference between the phase of the FM signal applied to the PLL and the phase of the FM signal produced by the VCO) in the PLL is defined by (see part (a) of Problem 4.7)

$$\Phi_e(f) = \frac{1}{1 + L(f)}\Phi_1(f) \tag{2}$$

where the loop transfer function is itself defined by

$$L(f) = K_0\frac{H(f)}{jf} \tag{3}$$

The $\Phi_1(f)$ in Eq. (2) is the Fourier transform of the angle $\phi_1(t)$ in the FM signal applied to the PLL. Substituting Eq. (1) into (3) and expanding terms, we get

$$\Phi_e(f) = \left(\frac{(jf)^2/aK_0}{1 + [(jf)/a] + [(jf)^2/aK_0]}\right)\Phi_1(f) \tag{4}$$

Define the *natural frequency* of the loop

$$f_n = \sqrt{aK_0} \tag{5}$$

and the *dampling factor*

$$\zeta = \sqrt{\frac{K_0}{4a}} \tag{6}$$

We may then recast Eq. (4) in terms of the loop parameters f_n and ζ as follows:

$$\Phi_e(f) = \left(\frac{(jf/f_n)}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2}\right) \tag{7}$$

(b) Suppose the FM signal applied to the PLL is a single-tone modulating signal, for which the phase input is defined by

$$\phi_1(t) = \beta\sin(2\pi f_m t) \tag{8}$$

Then, invoking the use of Eq. (7), we find that the corresponding phase error $\phi_e(t)$ is defined by

$$\phi_e(t) = \phi_{eo}\cos(2\pi f_m t + \psi) \tag{9}$$

where

$$\phi_{eo} = \frac{(\Delta f/f_m)(f_m/f_n)}{[(1-(f_m/f_n)^2)^2 + 4\zeta^2(f_m/f_n)^2]^{1/2}} \tag{10}$$

and

$$\psi = \frac{\pi}{2} - \tan^{-1}\left[\frac{2\zeta(f_m/f_n)}{1-(f_m/f_n)^2}\right] \tag{11}$$

One other thing we need to do is to evaluate the Fourier transform of the PLL output $v(t)$. For this purpose, we first note that the Fourier transform of $v(t)$ is related to $\Phi_1(f)$ as follows (see part (b) of Problem 4.7)

$$V(f) = \frac{jf}{k_v} \frac{L(f)}{1 + L(f)} \Phi_1(f) \tag{12}$$

where k_v is the frequency sensitivity of the VCO. Using Eqs. (1), (3), and (12), we may write

$$V(f) = \left(\frac{(jf/f_n)[1 + 2\zeta(jf/f_n)]}{1 + 2\zeta(jf/f_n) + (jf/f_n)^2} \right) \Phi_1(f) \tag{13}$$

In light of the PLL theory presented herein, we may make two important observations for an incoming FM signal of fixed frequency deviation produced by a sinusoidal modulating signal $m(t)$:

(i) The frequency response that defines the phase error $\phi_e(t)$ is representative of a band-pass filter, as shown by Eq. (10).

(ii) The frequency response of the PLL output $v(t)$ is representative of a low-pass filter, as shown by Eq. (13).

Therefore, by appropriately choosing the damping factor ζ and natural frequency f_n, which determine the frequency response of the PLL, it is possible to restrain the phase error $\phi_e(t)$ to always remain small and yet, at the same time, the modulating (message) signal is reproduced at the PLL output with minimum distortion.