Chapter 5
Pulse Modulation: Transition from Analog to Digital Communications

Problem 5.1

(a) Using the material presented in Section 2.5, justify the mathematical relationships listed at the bottom of the left-hand side of Table 5.1, which pertain to ideal sampling in the frequency domain.

(b) Applying the duality property of the Fourier transform to part (a), justify the mathematical relationships listed at the bottom of the right-hand side of this table, which pertain to ideal sampling in the time-domain.

Solution

1. Entry 1 on the left-hand side of Table 5.1:
   - The relationship

     $$\sum_{m=\infty}^{\infty} g(t-mT_s) = f_s \sum_{n=-\infty}^{\infty} G(nf_s)e^{j2\pi nf_st}$$

     where $g(t) \rightleftharpoons G(f)$ and $f_s = 1/T_s$, is a rewrite of Eq. (2.87) with one trivial change, namely, the replacements of T_o and f_o by T_s and f_s, respectively.
   - The Fourier transform pair

     $$\sum_{m=\infty}^{\infty} g(t)-m(T_s) \rightleftharpoons f_s \sum_{n=-\infty}^{\infty} G(nf_s)\delta(f-f_s)$$

     is also a rewrite of Eq. (2.88) except for the replacement of T_o and f_o with T_s and f_s, respectively.
2. Entry 2 on the right-hand side of Table 5.2:
   - The relationship

     $$\sum_{n=\infty}^{\infty} g(nT_s)e^{j2\pi nf_st} = f_s \sum_{m=-\infty}^{\infty} G(nf_sf-mf_s)$$

     is an exact reproduction of the equality in Eq. (5.2).
   - The Fourier-transform pair

     $$\sum_{n=\infty}^{\infty} g(nT_s)\delta(t-nT_s) \rightleftharpoons f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)$$

     is an exact reproduction of the Fourier-transform pair listed in Eq. (5.2).

Problem 5.2

Show that as the sampling period T_s approaches zero, the formula for the discrete-time Fourier transform $G_\delta(f)$ approaches the Fourier transform $G(f)$.

Solution

From Eq. (5.3), we have

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right)$$

The sampling period $T_s = -1/(2W)$. We may therefore rewrite this equation as

$$G_\delta(f) = \sum_{n=\infty}^{\infty} g(nT_s)\exp(-j2\pi nT_s f)$$

In the limit, as T_s approaches zero, the discrete time nT_s, approaches continuous time t. Moreover, the summation over n approaches the integral

$$\int_{-\infty}^{\infty} g(t)\exp(-j2\pi tf)dt$$

Correspondingly, $G_\delta(f)$ approaches the continuous Fourier transform $G(f)$. We may therefore state that the formula for the discrete Fourier transform $G_\delta(f)$ given in Eq. (5.3) approaches the formula for the Fourier transform:

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi tf)dt$$

as the sampling period T_s approaches zero.

Problem 5.3

Show that

$$\begin{aligned}\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]df &= \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \\ &= \mathrm{sinc}(2Wt-n)\end{aligned}$$

Solution

$$\begin{aligned}\frac{1}{2W}\int_{-W}^{W} \exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]df &= \frac{1}{2W}\cdot\frac{1}{j2\pi(t-n/2W)}\cdot\exp\left[j2\pi f\left(t-\frac{n}{2W}\right)\right]_{f=-W}^{W} \\ &= \frac{1}{j4\pi W(t-n/2W)}\cdot[\exp j\pi(2Wt-n) - \exp(-j\pi(2Wt-n))] \\ &= \frac{\sin(\pi(2Wt-n))}{\pi(2Wt-n)} \\ &= \mathrm{sin}c(2Wt-n)\end{aligned}$$

Problem 5.4

This problem is intended to identify a linear filter for satisfying the interpolation formula of Eq. (5.7), albeit in a non-physically realizable manner. Equation (5.7) is based on the premise that the signal $g(t)$ is strictly limited to the band $-W \le f \le W$. With this specification in mind, consider an ideal low-pass filter whose frequency response $H(f)$ is as depicted in Fig. 5.2(c). The impulse response of this filter is defined by (see Eq. (2.25))

$$h(t) = \mathrm{sinc}(2Wt), \qquad -\infty < t < \infty$$

Suppose that the correspondingly instantaneously sampled signal $g_\delta(t)$ defined in Eq. (5.1) is applied to this ideal low-pass filter. With this background, use the convolution integral to show that the resulting output of the filter is defined exactly by the interpolation formula of Eq. (5.7).

Solution

From Eq. (5.5), we have

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

According to this equation, $G(f)$ is low-pass with its frequency content confined to the range $-W < f < W$. Since $G(f)$ is the Fourier transform of $g(t)$, we can also write

$$\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wt - n) \rightleftharpoons \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W$$

Hence, the reconstruction filter defined by the left-hand side of this Fourier-transform pair is low-pass with its passband confined to the range $W < f < W$.

Problem 5.5

Specify the Nyquist rate and the Nyquist interval for each of the following signals:

(a) $g(t) = \mathrm{sinc}(200t)$

(b) $g(t) = \mathrm{sinc}^2(200t)$

(c) $g(t) = \mathrm{sinc}(200t) + \mathrm{sinc}^2(200t)$

Solution

(a) The highest frequency component of

$$g(t) = \mathrm{sinc}(200t) = \frac{\sin(200\pi t)}{200\pi t}$$

is 100 Hz. Hence, the Nyquist rate is 200 Hz and the Nyquist interval is 5 ms.

(b) The highest frequency component of

$$g(t) = \mathrm{sinc}^2(200t)$$

is twice that of $g(t)$ in part (a); it is so because squaring a band-limited single has the effect of doubling its highest frequency component. Hence, the Nyquist rate of

$$g(t) = \mathrm{sinc}^2(2Wt)$$

is 400 Hz and the Nyquist interval is 2.5 ms.

(c) The highest frequency component of the composite signal

$$g(t) = \mathrm{sinc}(200t) + \mathrm{sinc}^2(200t)$$

is determined by the component $\mathrm{sinc}^2(200t)$. Hence, the Nyquist rate of this third signal is 400 Hz and the Nyquist interval is 2.5 ms.

Problem 5.6

Consider uniform sampling of the sinusoidal wave

$g(t) = \cos(\pi t)$

Determine the Fourier transform of the sampled waveform for the following sampling period:

(a) $T_s = 0.25$s

(b) $T_s = 1$s

(c) $T_s = 1.5$s

Solution

We are given

$g(t) = \cos(\pi t)$

the frequency of which is 0.5 Hz.

(a) For the sampling period $T_s = 0.25$, we have

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} \cos\left(\frac{n\pi}{4}\right)\delta(t - nT_s)$$

(b) For $T_s = 1$s,

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} \cos(n\pi)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n\delta(t - nT_s)$$

(c) For $T_s = 1.5$,

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} \cos(1.5n\pi)\delta(t - nT_s)$$

Problem 5.7

Consider a continuous-time signal defined by

$$g(t) = \frac{\sin(\pi t)}{\pi t}$$

The signal $g(t)$ is uniformly sampled to produce the infinite sequence $\{g(nT_s)\}_{n=-\infty}^{\infty}$. Determine the condition which the sampling period T_s must satisfy so that the signal $g(t)$ is uniquely recovered from the sequence $\{g(nT_s)\}$.

Solution

The signal

$$g(t) = \frac{\sin(\pi t)}{\pi t} = \text{sinc}(t)$$

is limited to the band $-0.5 < f < 0.5$ Hz. The Nyquist rate for this signal must therefore exceed 2 x 0.5 = 1 Hz. Correspondingly, the permissible sampling interval must satisfy the condition $T_s < 1$s.

Problem 5.8

Starting with Eq. (5.9), show that the Fourier transform of the rectangular pulse $h(t)$ is given by

$$H(f) = T\,\text{sinc}(fT)\exp(-j\pi fT)$$

What happens to $H(f)/T$ as the pulse duration T approaches zero?

Solution

Given

$$h(t) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases}$$

the required Fourier transform is

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \\ &= \int_0^T 1 \cdot \exp(-j2\pi ft)dt \\ &= \left[\frac{\exp(-j2\pi ft)}{-j2\pi f}\right]_{t=0}^T \\ &= \frac{1}{j2\pi f} - \frac{\exp(-j2\pi fT)}{j2\pi f} \\ &= \frac{\exp(-j2\pi fT)}{j2\pi f}[\exp(j\pi fT) - \exp(-j\pi fT)] \end{aligned}$$

Since

$$\sin(\pi fT) = \frac{1}{2j}[\exp(j\pi fT) - \exp(-j\pi fT)]$$

it follows that

$$\begin{aligned} H(f) &= \frac{\sin(\pi fT)}{\pi f}\exp(-j\pi fT) \\ &= T \cdot \frac{\sin(\pi fT)}{\pi fT}\exp(-j\pi fT) \\ &= T\,\text{sinc}(fT)\exp(-j\pi fT) \end{aligned}$$

Problem 5.9

Using Eqs. (5.23) and (5.25), respectively, derive the slope characteristics of Eqs. (5.24) and (5.26).

Solution

(a) The logarithmic law is defined by (see Eq. (5.23)

$$|v| = \frac{\log(1+\mu|m|)}{\log(1+\mu)}$$

Therefore, differentiation with respect to $|m|$ yields

$$\frac{d|v|}{d|m|} = \frac{1}{\log(1+\mu)} \cdot \frac{\mu}{1+\mu|m|}$$

Equivalently, we may write

$$\frac{dm}{d|v|} = \log(1+\mu)\frac{1+\mu|m|}{\mu}$$

(b) The A-law is defined by (see Eq. (5.25):

$$|v| = \begin{cases} \dfrac{A|m|}{1+\log A}, & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{1+\log(A|m|)}{1+\log A}, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Hence, differentiation of $|v|$ with respect to $|m|$ yields

$$\frac{d|v|}{d|m|} = \begin{cases} \dfrac{A}{1+\log A}, & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{A}{|m|(1+\log A)}, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Equivalently, we may write

$$\frac{d|m|}{d|v|} = \begin{cases} \dfrac{1+\log A}{A}, & 0 \le |m| \le \dfrac{1}{A} \\ \left(\dfrac{1+\log A}{A}\right)|m|, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Problem 5.10

The best that a linear delta modulator can do is to provide a compromise between slope-overload distortion and granular noise. Justify this statement.

Solution

(a) In linear delta modulation, if we make the step-size Δ too small, then the system suffers from slope overload distortion.

(b) On the other hand, if we make the step-size Δ too large relative to the local slope characteristic of the message signal $m(t)$, then the system suffers from granular distortion.

For a *fixed* sampling rate $1/T_s$ and with Δ *as the only variable*, the best that the linear delta modulator can do is to choose a step-size Δ that will provide a compromise between these two forms of quantization noise.

Problem 5.11

Justify the two statements just made on sources of noise in a DPCM system.

Solution

1. When the step-size is too small and the input signal is changing too rapidly, the DPCM is unable to track the input signal, resulting in slope-overload distortion similar to linear delta modulation.
2. DPCM uses a quantizer in the transmitter. Hence, like pulse-code modulation, DPCM suffers from quantization noise.

Problem 5.12

(a) The PAM wave is defined by

$$s(t) = \sum_{n=-\infty}^{\infty} [1 + \mu m'(nT_s)]g(t - nT_s), \tag{1}$$

where $g(t)$ is the pulse shape, $m'(t) = m(t)/A_m = \cos(2\pi f_m t)$ and μ is the modulation factor. The PAM wave is equivalent to the convolution of the instantaneously sampled signal $[1 + \mu m'(t)]$ and the pulse shape $g(t)$, as shown by

$$s(t) = \left\{ \sum_{n-\infty}^{\infty} [1 + \mu m'(nT_s)]\delta(t - nT_s) \right\} \star g(t)$$

$$= \left\{ 1 + \mu m'(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} \star g(t) \tag{2}$$

Let $m'(t) \rightleftharpoons M'(f)$, $g(t) \rightleftharpoons G(f)$, and $s(t) \rightleftharpoons S(f)$.

The spectrum of the PAM wave is therefore,

$$S(f) = \left\{ [\delta(f) + \mu M'(f)] \star \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_s}\right) \right\} G(f)$$

$$= \frac{1}{T_s} G(f) \sum_{m=-\infty}^{\infty} \left[\delta\left(f - \frac{m}{T_s}\right) + \mu M'\left(f - \frac{m}{T_s}\right) \right] \tag{3}$$

For a rectangular pulse $g(t)$ of duration $T = 0.45$s and with $AT = 1$, we have

$$G(f) = AT\,\mathrm{sinc}\,(fT)$$

$$= \mathrm{sinc}\,(0.45f)$$

For $m'(t) = \cos(2\pi fmt)$ and $f_m = 0.25$ Hz, we have

$$M'(f) = \frac{1}{2}[\delta(f-0.25)+\delta(f+0.25)]$$

For $T_s = 1s$, the ideally sampled spectrum is

$$S_\delta(f) = \sum_{m=-\infty}^{\infty}[\delta(f-m)+\mu M'(f-m)] \tag{4}$$

which is plotted in Fig. 2(c).

The actual sampled spectrum is defined by

$$S(f) = \sum_{m=-\infty}^{\infty}\operatorname{sinc}(0.45f)[\delta(f-m)+\mu M'(f-m)] \tag{5}$$

which is plotted in Fig. 1(b).

(b) The ideal reconstruction filter would retain the centre 3 delta functions of $S(f)$. With no aperture effect, the two outer delta functions would have amplitude $\mu/2$. The aperture effect distorts the reconstructed signal by attenuating the high-frequency portion of the message signal.

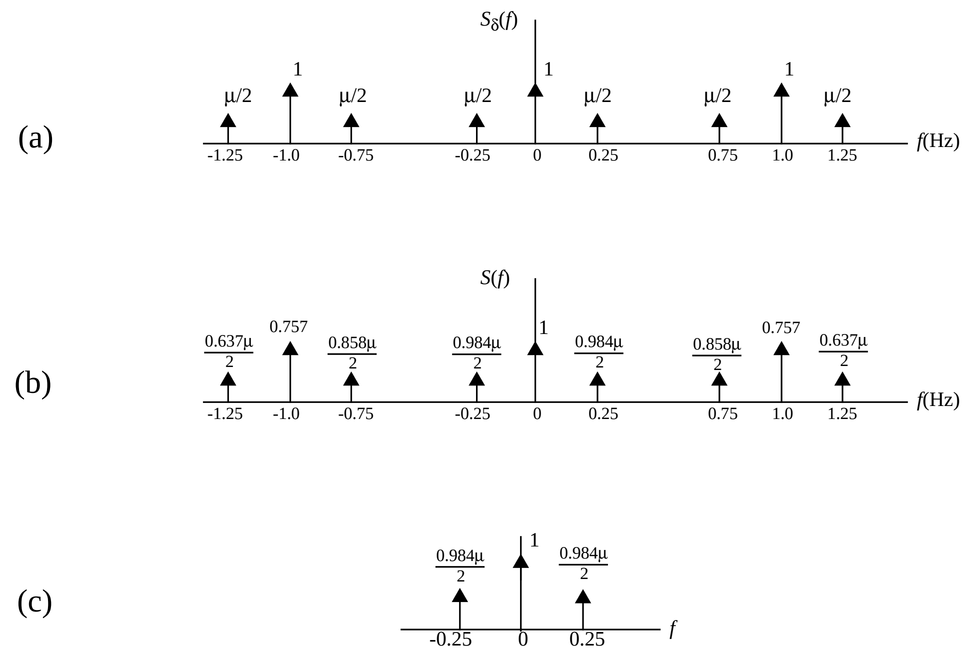


Figure 1

Problem 5.13

At $f = 1/2T_s$, which corresponds to the highest frequency component of the message signal for a sampling rate equal to the Nyquist rate, we find from Eq. (5.17) that the amplitude response of the equalizer normalized to that at zero frequency is defined by

$$\frac{1}{\operatorname{sinc}(0.5T/T_s)} = \frac{(\pi/2)(T/T_s)}{\sin[(\pi/2)(T/T_s)]} \tag{1}$$

where the ratio T/T_s is equal to the duty cycle. In Fig. 1, Eq. (1) is plotted as a function of T/T_s. Ideally, the graph should be equal to one for all values of T/T_s, as indicated by the dashed horizontal line in Fig. 1. For a duty cycle of 25 percent, it is approximately equal to 1.04, which exceeds the ideal case by about 4%.

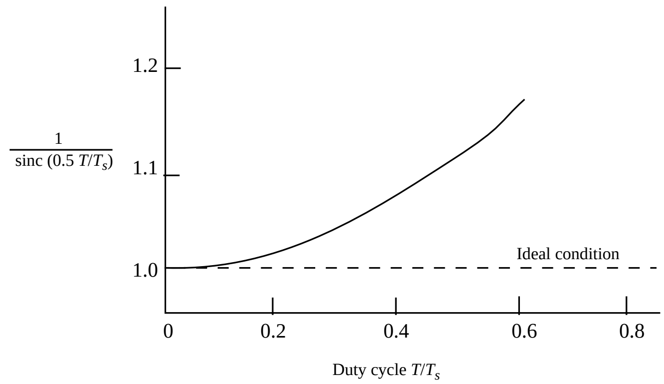


Figure 1

Problem 5.14

(a) The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, the factor $\frac{2400}{2^R}$ must be greater than 160, and the maximum R is 3.

(b) With R = 3, we may use the following signal format displayed in Fig. 1 to multiplex the signals $s_1(t)$ and $s_2(t)$ into a new signal, and then multiplex $s_3(t)$ and $s_4(t)$ and $s_5(t)$ including markers for synchronization.

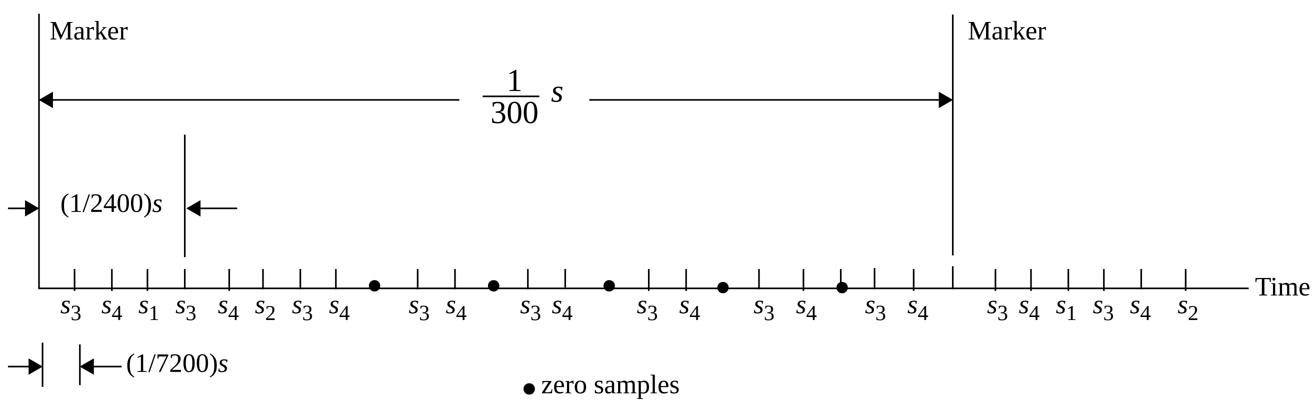


Figure 1

Based on the signal format shown in Fig. 1, we may develop the multiplexing system shown in Fig. 2.

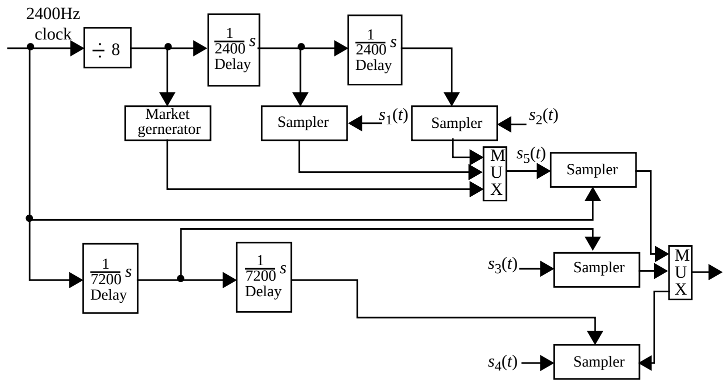
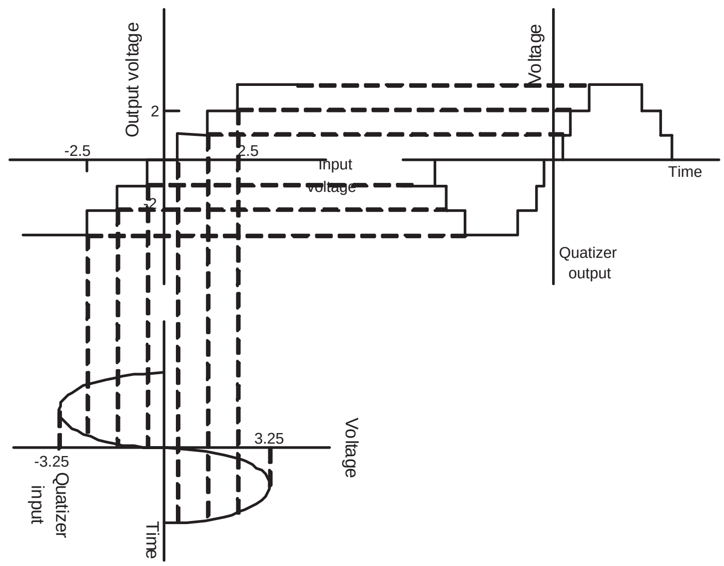
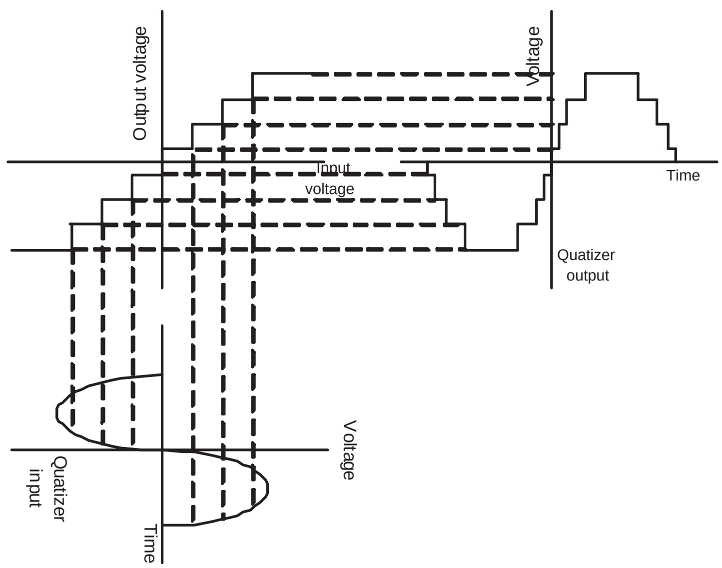


Figure 2

Problem 5.15



Problem:(5.15a)



Problem:(5.15b)

Problem 5.16

(a) <u>An alternating sequence of 1's and 0's</u>

On-off signaling: The signal $g(t)$ consists of a periodic train of rectangular pulses with pulse duration $T = T_0/2$, where T_0 is the period.

Bipolar return-to-zero signaling: The signal $g(t)$ consists of a periodic train of pulses of duration T and of alternating polarity.

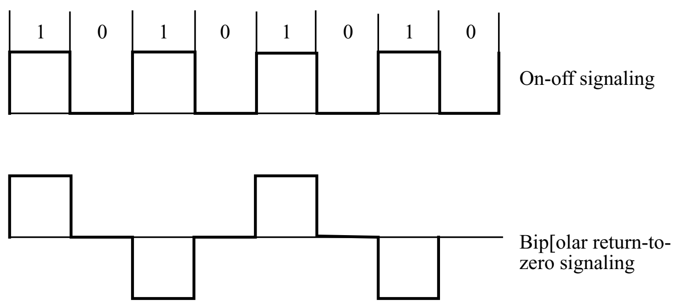
(b) <u>A long sequence of 1’s followed by a long sequence of 0’s</u>
On-off signaling: The signal $g(t)$ consists of a unit step function defined for negative time, that is, $u(-t)$.
Bipolar return-to-zero signaling: The signal $g(t)$ consists of pulses of alternating polarity, followed by a long period of zero volts.

(c) <u>An alternating sequence of 1’s followed by a single 0 and then a long sequence of 1’s</u>
On-off signaling: The signal $g(t)$ consists of a dc component minus a rectangular pulse (of the same amplitude as the dc component).
Bipolar return-to-signal signaling: The signal $g(t)$ consists of two identically periodic sequences of pulses separated by a period of zero volts.

The line codes just described are plotted in Fig. 1.

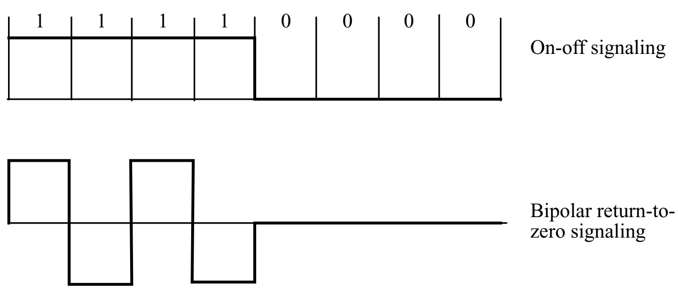
(a) An alternating sequence of 1’s and 0’s

(a)



(b) A long sequence of l’s followed by a long sequence of 0’s

(b)



(c) A long sequence of 1's followed by a single 0 and then a long sequence of 1's

(c)

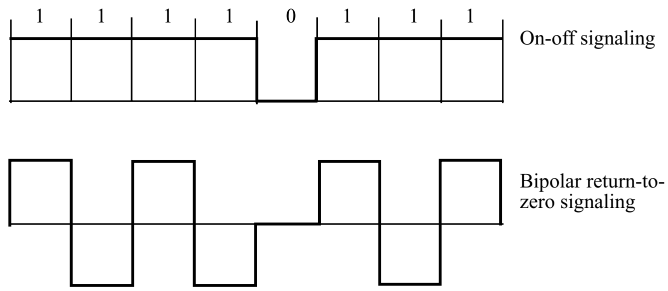


Figure 1

Problem 5.17

The quantizer has the following input-output curve plotted in Fig. 1

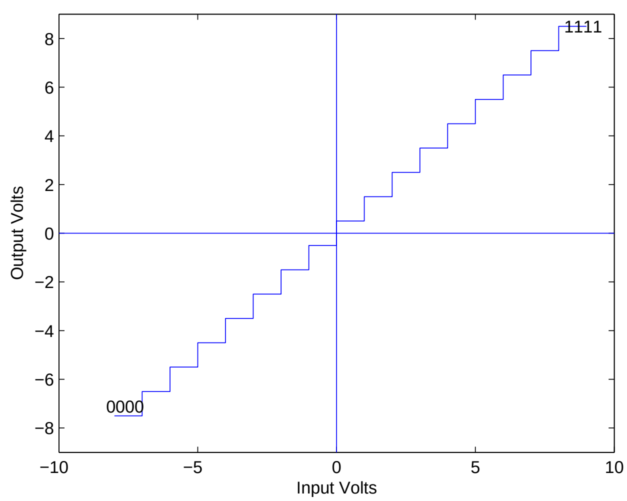


Figure 1

At the sampling instants we have:

t	$m(t)$	code
-3/8	$-3\sqrt{2}$	0011
-1/8	$-3\sqrt{2}$	0011
+1/8	$3\sqrt{2}$	1100
+3/8	$3\sqrt{2}$	1100

And the coded waveform is (assuming on-off signaling):

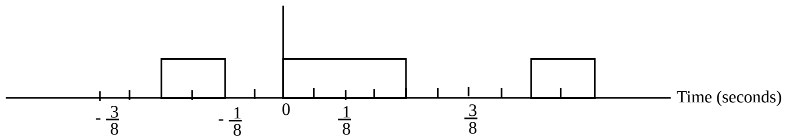


Figure 2

Problem 5.18

We are given
- Audio signal bandwidth, W = 15 kHz
- Number of uniform quantization levels = 512 levels
- Encoding : binary

(a) The Nyquist rate is $2W$ = 30 kHz.

(b) To accommodate 512 quantization levels, we require a binary code with B bits, which would have to satisfy the following requirement:
$2^B = 512$
Hence, B = 9. The sampling period T_s = 1/30 milliseconds must be divided into 9 bits. The minimum sampling rate is therefore
30 x 9 = 270 kilobits/second
= 0.27 megabits/second

Problem 5.19

(a) We are given
- Video bandwidth = 4.5 MHz
- Sampling rate = 15% in excess of the Nyquist rate
- Uniform quantization using 1024 levels
- Binary encoding

(b) The Nyquist rate is 2 x 4.5 = 9 MHz.
Actual sampling rate = 9 x 1.15 = 10.35 MHz
The sampling period is therefore

$$T_s = \frac{1}{10.35} \mu s$$

This sampling rate must be divided into B bits, where
$2^B = 1024$
Hence, B = 10. The bit duration is therefore

$$\frac{T_s}{10} = \frac{1}{103.5} \mu s$$

The permissible bit rate is therefore 103.5 megabits/s.

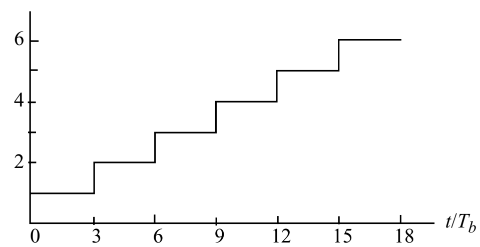
Problem 5.20

The transmitted code words, representing the PCM waveform

t/T_b	code
1	001
2	010
3	011
4	100
5	101
6	110

Accordingly, the sampled analog signal from which these code words are derived is shown in Fig. 1.

Figure 1



Problem 5.21

The modulating wave is

$m(t) = A_m \cos(2\pi f_m t)$

The slope of $m(t)$ is given by

$$\frac{dm(t)}{dt} = -2\pi f_m A_m \sin(2\pi f_m t)$$

The maximum slope of $m(t)$ is therefore equal to $2\pi f_m A_m$.

The maximum average slope of the approximating signal $m_q(t)$ produced by the delta modulator is δ/T_s, where δ is the step size and T_s is the sampling period. The limiting value of A_m is therefore given by

$$2\pi f_m A_m > \frac{\delta}{T_s}$$

or

$$A_m > \frac{\delta}{2\pi f_m T_s}$$

Assuming a load of 1 ohm, the transmitted power is $A_m^2/2$. Therefore, the maximum power that may be transmitted without slope-overload distortion is equal to $8^2/(\delta\pi^2 f_m^2 T_s^2)$.

Problem 5.22

Sampling rate = 64 kHz
Voice signal bandwidth = W = 3.1 kHz
Maximum signal amplitude $A_{\max}$ = 10 volts

(a) To avoid slope overload, we must satisfy the following requirement (see Problem 5.21)

$$A_{\max} < \frac{\Delta}{2\pi W T_s}$$

Solving for the step size Δ, we write

$$\Delta > \frac{1}{2\pi W T_s A_{\max}} = \frac{f_s}{2\pi W A_{\max}} \tag{1}$$

Substituting the given values into Eq. (1) yields

$$\Delta > \frac{64}{2\pi \times 3.1 \times 10}$$

or

$$\Delta > 0.33 \quad \text{volts}$$

Effectively, provided that the step size Δ is 0.33 volt, then slope-overload distortion is avoided.

(b) Let $\in(t)$ denote the granular noise, viewed as a function of time t. The average power of granular noise (analogous to quantization noise in PCM), is defined by

$$\begin{aligned} P_g &= \frac{2}{\Delta}\int_{-\Delta/2}^{\Delta/2} \in^2 d\in \\ &= \frac{2}{\Delta}\left[\frac{\in^3}{3}\right]_{\in=-\Delta/2}^{\Delta/2} \\ &= \frac{\Delta^2}{3} \end{aligned}$$

With Δ set at 0.33 volt, the average power of granular noise is therefore 0.03 watts (assuming that the power is calculated for a load of 1 ohm).

(c) The minimum channel bandwidth needed to transmit the DM encoded signal is the inverse of the sampling rate, that is, 64 kHz.

Problem 5.23

The values calculated in parts (a), (b) and (c) of Problem 5.22 also hold for a sinusoidal signal of peak amplitude 10 volts and frequency 3.1 kHz.

Problem 5.24

The transmitting prediction filter operates on exact samples of the signal while the receiving prediction filter operates on quantized samples. Hence, unlike the DPCM system described in Section 5.8, the prediction filters in the transmitter and receiver of Fig. 5.26 operate on different signals.

Problem 5.25

(a) In theory, any physical signal (exemplified by audio and video signals) has a spectrum that gradually decreases towards zero. From Fourier transform theory, we know that any signal cannot simultaneously have finite duration and finite bandwidth. Therefore, theoretically speaking, given a physical signal of finite duration, the band of frequencies occupied by that signal is infinitely large. Accordingly, when the signal is sampled in accordance with the Nyquist sampling theorem, there will always be some distortion produced by sampling the signal due to the aliasing phenomenon.

(b) In practice, however, we usually limit the sampling rate to some finite value, depending on the application of interest. For example, for telephonic communication, it has been found experimentally that 3.1 kHz is considered to be adequate for describing the "effective" bandwidth of a voice signal, be that for a male or female. Thus, choosing a rate of 8 kHz is considered to be adequate for the uniform sampling of a voice signal in telephonic communication. In reality, there is some distortion produced by the sampling process, but for all practical purposes, the distortion is not significant enough to be perceived by a human listener. Indeed, it is for this reason that a sampling rate of 8 kHz is the universally accepted standard for the sampling of voice signals transmitted over a telephone line.

Similar remarks apply to the sampling of video signals; naturally, the sampling rate used for video signals is much higher than 8 kHz,

Problem 5.26

Let $2W$ denote the bandwidth of a narrowband signal with carrier frequency f_c. The in-phase and quadrature components of this signal are both low-pass signals with a common bandwidth of W. According to the sampling theorem, there is no information loss if the in-phase and quadrature components are sampled at a rate higher than $2W$. For the problem at hand, we have

f_c = 100 kHz

$2W$ = 10 kHz

Hence, W = 5 kHz, and the minimum rate at which it is permissible to sample the in-phase and quadrature components is 10 kHz.

From the sampling theorem, we also know that a physical waveform can be represented over the interval $-\infty < t < \infty$ by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(t) \qquad (1)$$

where $\{\phi_n(t)\}$ is a set of orthogonal functions defined as

$$\phi_n(t) = \frac{\sin\{\pi f_s(t - n/f_s)\}}{\pi f_s(t - n/f_s)}$$

where n is an integer and f_s is the sampling frequency. If $g(t)$ is a low-pass signal limited to W Hz, and $f_s \geq 2W$, then the coefficient a_n can be shown to equal $g(n/f_s)$. That is, for $f_s \geq 2W$, the orthogonal coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ second.

As already mentioned, the narrowband signal is two-dimensional, consisting of in-phase and quadrature components. In light of Eq. (1), we may represent them as follows, respectively:

$$g_I(t) = \sum_{n=-\infty}^{\infty} g_I(n/f_s)\phi_n(t)$$

$$g_Q(t) = \sum_{n=-\infty}^{\infty} g_Q(n/f_s)\phi_n(t)$$

Hence, given the in-phase samples $g_I\left(\frac{n}{f_s}\right)$ and quadrature samples $g_Q\left(\frac{n}{f_s}\right)$, we may reconstruct the narrowband signal $g(t)$ as follows:

$$\begin{aligned} g(t) &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\ &= \sum_{n=-\infty}^{\infty}\left[g_I\left(\frac{n}{f_s}\right)\cos(2\pi f_c t) - g_Q\left(\frac{n}{f_s}\right)\sin(2\pi f_c t)\right]\phi_n(t) \end{aligned}$$

where f_c = 100 kHz and $f_s \geq$ 10 kHz, and where the same set of orthonormal basis functions is used for reconstructing both the in-phase and quadrature components.

Problem 5.27

(a) The commutator at the output of the bipolar chopper switches between the direct path and inverted path at the frequency f_s. In effect, every $1/f_s$ seconds, the output of the chopper consists of the input $x(t)$ -- via the direct path -- for $1/2f_s$ seconds followed by the inverted version of $x(t)$ -- via the inverted path -- for the remaining $1/2f_s$ seconds of the commutation period. For one period of the commutation process, we may thus write

$$y(t) = \begin{cases} x(t) & \text{for} \quad 0 \leq t \leq 1/(2f_s) \\ -x(t) & \text{for} \quad 1/(2f_s) \leq t \leq 1/f_s \end{cases} \tag{1}$$

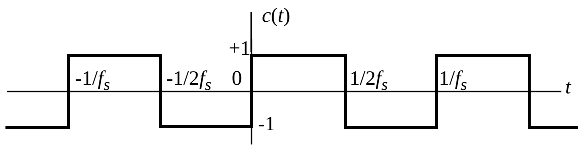
Equation (1) repeats itself every $1/f_s$ seconds.

(b) Equation (1) may be equivalently expressed as follows:

$$y(t) = c(t)x(t) \tag{2}$$

where $c(t)$ consists of the square wave (see Fig. 1)

$$c(t) = \begin{cases} 1 + & \text{for} \quad 0 \le t \le 1/(2f_s) \\ -1 & \text{for} \ 1/(2f_s) \le t \le 1/f_s \end{cases} \tag{3}$$



By inspection, we may make three observations from Fig. 1:

(i) The dc component of $c(t)$ is zero.

(ii) The Fourier series representation of $c(t)$ consists of sine components with a fundamental frequency f_s.

(iii) The even harmonic components of $c(t)$ are all zero.

Accordingly, we may represent $c(t)$ by the Fourier series:

$$c(t) = b_1 \sin(2\pi f_s t) + b_3 \sin(6\pi f_s t) + b_5 \sin(10\pi f_s t) + \dots \tag{4}$$

where b_n is defined by

$$\begin{aligned} b_n &= f_s \int_0^{1/f_s} c(t) \sin(2\pi n f_s t) dt \\ &= f_s \int_0^{1/2f_s} \sin(2\pi n f_s t) dt - f_s \int_{-1/2f_s}^{1/f_s} \sin(2\pi n f_s t) dt \\ &= \frac{-1}{2\pi n}[\cos(2\pi n f_s t)]_{t=0}^{1/2f_s} + \frac{1}{2\pi n}[\cos(2\pi n f_s t)]_{t=(1/2f_s)}^{1/f_s} \\ &= -\frac{1}{2\pi n}(\cos(n\pi) - 1) + \frac{1}{2\pi n}(\cos(n\pi) - \cos(n\pi)) \\ &= \begin{cases} \dfrac{2}{\pi n} & \text{for } n = 1, 3, 5, \dots \\ 0 & \text{for } n = 0, 2, 4, \dots \end{cases} \end{aligned} \tag{5}$$

We may thus express the Fourier series of the commutation function $c(t)$ as

$$c(t) = \frac{2}{\pi}\sin(2\pi f_s t) + \frac{2}{3\pi}\sin(6\pi f_s t) + \frac{2}{5\pi}\sin(10\pi f_s t) + \dots \tag{6}$$

Using Eq. (6) in (2) yields

$$y(t) = \frac{2}{\pi}\sin(2\pi f_s t)x(t) + \frac{2}{3\pi}\sin(6\pi f_s t)x(t) + \frac{2}{5\pi}\sin(10\pi f_s t) + \dots \tag{7}$$

The Fourier transform of $y(t)$ is therefore defined by

$$Y(f) = \frac{1}{j\pi}[X(f - f_s) - X(f + f_s)]$$

$$+ \frac{1}{j3\pi}[X(f - 3f_s) - X(f + 3f_s)]$$

$$+ \frac{1}{j5\pi}[X(f - 5f_s) - X(f + 5f_s)] + \dots \qquad (8)$$

where $X(f)$ is the Fourier transform of the input $x(t)$.

Figure 2 displays the relationship between the two Fourier transforms: $X(f)$ and $Y(f)$. Note that $X(f)$ can only be recovered from $Y(f)$ only through a band-pass filter with bandwidth $2W$ centered on f_s.

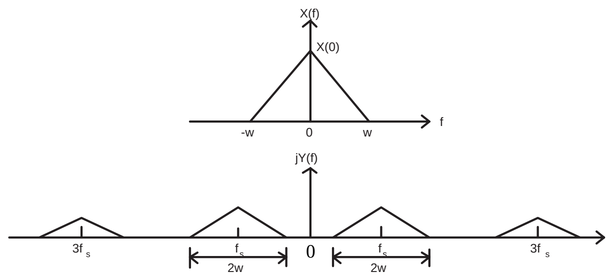


Figure 2

Problem 5.28

(a) Consider a periodic waveform $x(t)$ whose Fourier transform is defined by

$$X(f) = \sum_{k=-m}^{m} c_k \delta(f - kf_0) \qquad (1)$$

where f_0 is the fundamental frequency of $x(t)$. In effect, we are assuming that $x(t)$ is the result of prefiltering a periodic signal with period $1/f_0$ and all harmonic components in excess of the mth component have been suppressed. The highest frequency of $x(t)$ is therefore mf_0.

Suppose now $x(t)$ is purposely sampled at the rate

$$f_s = (1 - a)f_0 \qquad (2)$$

where $0 < a < 1$. The sampling rate f_s is clearly less than the Nyquist rate $2mf_0$, hence the possibility of aliasing. From Eq. (5.2) in the text, recall that the Fourier transform of the sampled version of $x(t)$ is defined by

$$\frac{1}{f_s}X_\delta(f) = \sum_{i=-\infty}^{\infty} X(f - if_s)$$

$$= \sum_{i=-\infty}^{\infty} X(f - if_0 + aif_0) \qquad (3)$$

Substituting Eq. (1) into (3) yields

$$\frac{1}{f_s}X_\delta(f) = \sum_{i=-\infty}^{\infty} \sum_{k=-m}^{\infty} c_k \delta(f - (i + k)f_0 + aif_0) \qquad (4)$$

To proceed further with this equation, we will use *induction* to solve Problem 5.28.

(i) Let m = 1, for which Eq. (1) reads as

$$X(f) = c_0\delta(f) + c_1[\delta(f - f_0) + \delta(f + f_0)] \tag{5}$$

This spectrum represents a sinusoidal wave of amplitude $2c_1$, superimposed on a dc bias of c_0; see Fig. 1(a). For this case, Eq. (4) simplifies to

$$\begin{aligned}\frac{1}{f_s}X_\delta(f) &= \sum_{i=-\infty}^{\infty}\sum_{k=-1}^{\infty} c_k\delta(f - (i+k)f_0 + af_0) \\ &= \sum_{i=-\infty}^{\infty}[c_0\delta(f - if_0 + aif_0) \\ &\qquad + c_i\delta(f - (i+1)f_0 + aif_0) \\ &\qquad + c_i\delta(f - (i-1)f_0 + aif_0)]\end{aligned} \tag{6}$$

Evaluating Eq. (5) yields the sampled spectrum depicted in Fig. 1(b).

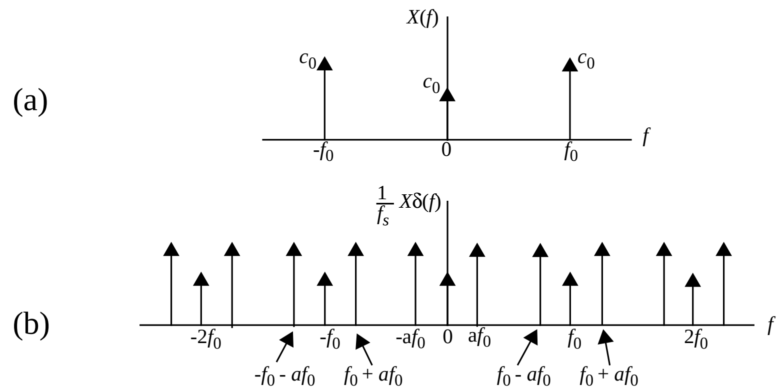


Figure 1

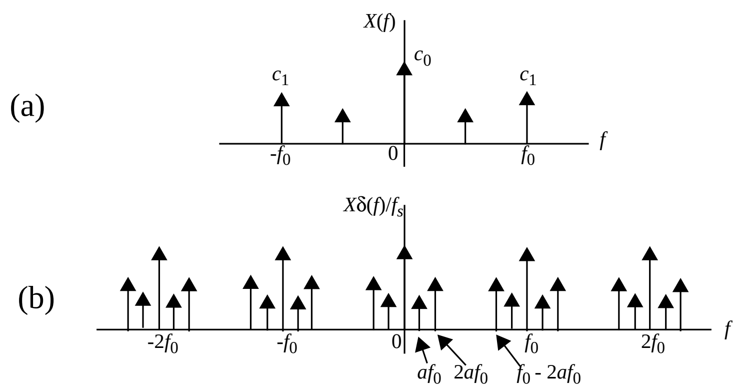


Figure 2

(ii) Next, let $m = 2$, for which we deduce that the relationship between the original spectrum $X(f)$ and the sampled spectrum $X_\delta(f)/f_s$ is pictured as shown in Fig. 2. The results displayed here follow from the evaluation of Eq. (4) for $m = 2$.

Based on the results depicted in Figs. 1 and 2, we may draw the following conclusions:

- The part of the spectrum $X_\delta(f)/f_s$ centered on the origin $f = 0$ is a compressed version of the original spectrum $X(f)$.
- The original spectrum $X(f)$ can be recovered from $X_\delta(f)/f_s$ by using a low-pass filter, provided there is no spectral overlap. In both figures, there is no spectral overlap. For this to be so, in Fig. 1(b) with $m = 1$ we must choose

  $$(f_0 - af_0) > af_0$$

  or

  $$a < \frac{1}{2} \tag{7}$$

  In the case of Fig. 2(b) with $m = 2$, we must choose

  $$(f_0 - 2af_0) > 2af_0$$

  or

  $$a < \frac{1}{4} \tag{8}$$

  Generalizing these two results, we may say that spectral overlap in the sampled spectrum $X_\delta(f)/f_s$ is avoided provided that we choose

  $$a < \frac{1}{2m}$$

  However, the choice of $1/2m$ does not leave any room for the design of a realizable low-pass reconstruction filter. This last provision is made by choosing

  $$a < \frac{1}{2M + 1} \tag{9}$$

- From Fourier transform theory, we recall that spectral compression in the frequency domain corresponds to signal expansion in the frequency domain. We therefore conclude that provided the choice of parameter a satisfies Eq. (9), then we may use the scheme described in Fig. 5.28 to expand the time display of a periodic waveform with highest frequency component mf_0 and do so with a realizable reconstruction filter, provided that parameter a satisfies the condition of Eq. (9).

Problem 5.29

Consider Fig. 1(a) that shows the mirror rotating counter clockwise about the horizontal axis at a rate of $2\pi f$ radians per second. At a given time t, the angular position of the position of the narrow horizontal strip on the television screen as seen in the mirror forms an angle of $2\pi ft$ with respect to the coordinate axes. The position of the narrow strip relative to the origin as seen in the mirror is described by

$$x(T_s) = \exp(j2\pi f T_s)$$

which is the sampled version of the complex exponential

$$x(t) = \exp(j2\pi ft)$$

(a) If there is exactly one revolution of the mirror between frames on the television screen, then the rotation speed of the mirror matches the sampling rate of the video signal. In this situation, the horizontal strip on the television screen does *not* appear to be rotating, as illustrated in Fig. 1(a).

(b) If however the mirror rotates at an angle less than π radians between television frames, then the rotation of the narrow strip as seen in the mirror appears like a left-to-right motion (i.e., backwards), as illustrated in Fig. 1(c). This situation implies that

$$2\pi f T_s < \pi$$

That is, with T_s = 1/60 seconds, the rotation rate of the mirror defined by $w = 2\pi f$ is

$$w < 60\pi \text{ radians/second}$$

which is one half of the television's sampling rate. If the rotation rate of the mirror satisfies this condition, then no aliasing occurs and the rotation of the mirror is visually consistent with the left-to-right motion.

On the other hand, if the mirror rotates between π and 2π radians between television frames, then the rotation of the mirror appears to be visually inconsistent with linear motion, as illustrated in Fig. 1(d). This inconsistent situation occurs when

$$\pi < 2\pi f T_s < 2\pi$$

or, with T_s = 1/60 seconds,

$$30 < f < 60 \text{ hertz}$$

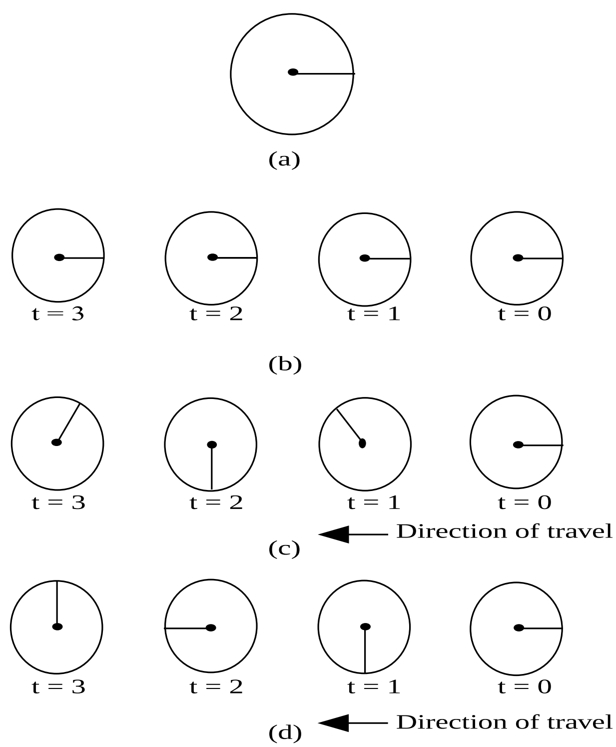


Figure 1

Problem 5.30

The first-order hold corresponds to extrapolating into the future with a straight line, as shown in Fig.1.

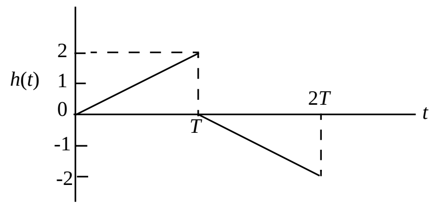


Figure 1

Specifically, the impulse response of the first-order hold may be expressed as

$$h(t) = \begin{cases} (t+T)/T & \text{for} \quad 0 \le t \le T \\ -(t-T)/T & \text{for} \quad T \le t \le 2T \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

Equivalently, we may express $h(t)$ as

$$\begin{aligned} h(t) &= u(t) + \frac{t}{T}u(t) - 2u(t-T) \\ &\quad - 2\frac{t-T}{T}u(t-T) + u(t-2T) + \frac{t-2T}{T}u(t-2T) \end{aligned} \tag{2}$$

where $u(t)$ is the unit step function.

(a) Taking the Fourier transform of Eq. (2) and using the Fourier-transform pairs of Table A6.2, we may therefore express the frequency response of the first-order hold as

$$\begin{aligned} H(f) &= \frac{1}{j2\pi f} + \frac{1}{T(j2\pi f)^2} - \frac{2}{j2\pi f}\exp(-j2\pi fT) \\ &\quad - \frac{2}{T(j2\pi f)^2} + \frac{1}{j2\pi f}\exp(-j4\pi fT) + \frac{1}{T(j2\pi f)^2}\exp(-j4\pi fT) \end{aligned}$$

which, after collecting and simplifying terms, yields

$$H(f) = T(1 + j2\pi fT)\left(\frac{1-\exp(-j2\pi fT)}{j2(\pi fT)}\right)^2 \tag{3}$$

(b) Figure 2 shows the magnitude and phase responses of the first-order hold.

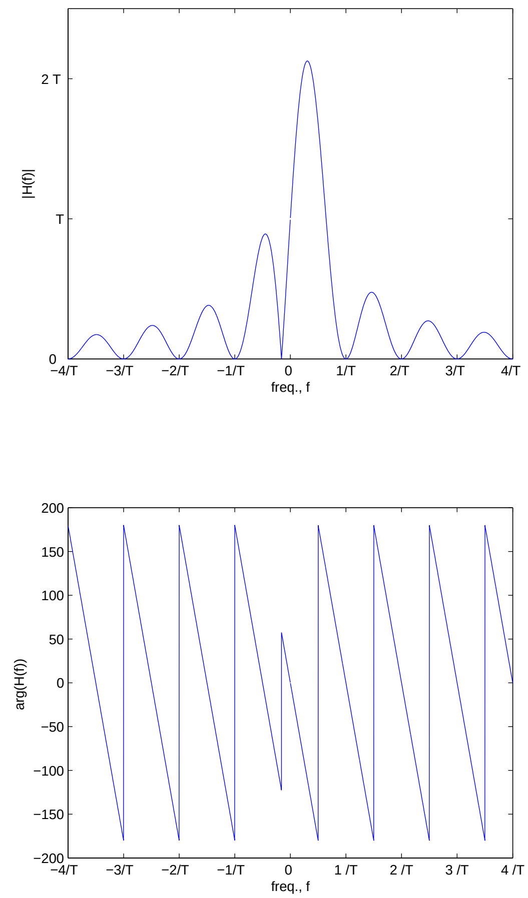


Figure 2

(c) For perfect reconstruction of the original analog signal, we need an equalizer whose transfer function is the inverse of $H(f)$ of Eq. (3), as shown by

$$\begin{aligned} H_{eq}(f) &= \frac{1}{H(f)} \\ &= \frac{1}{T(1 + j2\pi fT)}\left(\frac{j2\pi fT}{1 - \exp(-j2\pi fT)}\right)^2 \end{aligned} \tag{4}$$

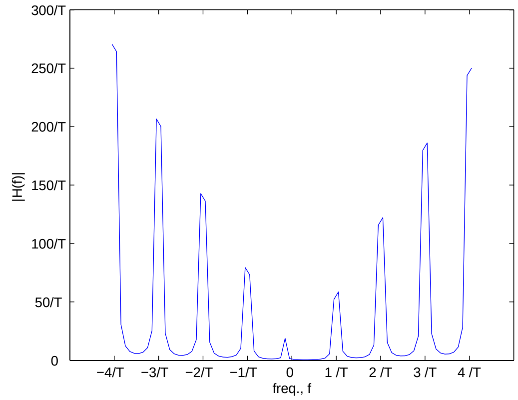
For a duty cycle $(T/T_s) = 0.1$, the use of Eq. (4) yields

$$H_{eq}(f_s) = \frac{1}{T}(0.8732 + 0.0589)$$

(d) For the sinusoidal input

$$x(t) = \cos(50t)$$

and f_s = 100Hz and T - 0.01, Fig. 3(c) shows the response produced by the first-order hold. Part (b) of the figure shows the corresponding response of the sample-and-hold filter. Comparing these two parts of Fig. 3, we may make the following observations:



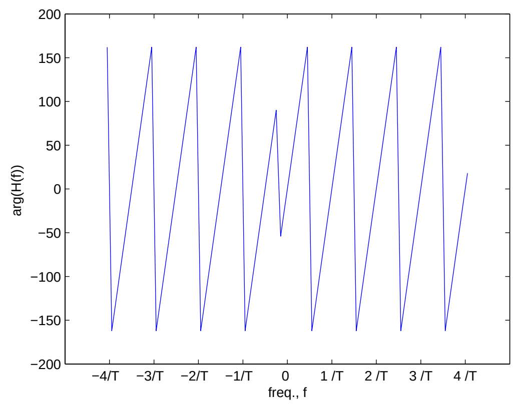


Figure 3

Problem 5.31

(a) Starting with the Fourier-transform pair

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2) \qquad (1)$$

and applying the differentiation property of the Fourier transform to Eq. (1), we write

$$\frac{d}{dt}\exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2)$$

or, equivalently

$$-2\pi t \exp(-\pi t^2) \rightleftharpoons j2\pi f \exp(-\pi f^2) \qquad (2)$$

Multiplying the left-hand side of Eq. (1) by A and invoking the linearity property of the Fourier transform, we go on to write

$$-2\pi t A \exp(-\pi t^2) \rightleftharpoons j2\pi f A \exp(-\pi f^2)$$

Simplifying terms:

$$tA \exp(-\pi t^2) \rightleftharpoons jfA \exp(-\pi f^2) \qquad (3)$$

Finally, applying the dilation property of the Fourier transform to Eq. (3). we get

$$A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \rightleftharpoons -j\tau f\, A\exp(-\pi f^2\tau^2) \tag{4}$$

The left-hand side of this transform pair is recognized as the time function (see Eq. (5.39))

$$v(t) = A\left(\frac{t}{\tau}\right)\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) \tag{5}$$

From Fig. 5.22, we see that the maximum value of $v(t)$ is +1. To find this maximum, we differentiate $v(t)$ with respect to time t and set the result equal to zero, obtaining

$$\frac{A}{\tau}\exp\left(-\pi\left(\frac{t}{\tau}\right)^2\right) - A\left(\frac{t}{\tau}\right)(2\pi t/\tau)\exp(-\pi t/\tau^2) = 0$$

Cancelling common terms and solving for t_{max}/τ, we get

$$\frac{t_{max}}{\tau} = \left(\frac{1}{2\pi}\right)^{1/2} \tag{6}$$

Using this value in Eq. (5):

$$v(t_{max}) = A\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}\right)$$

With $v(t_{max}) = 1$, it follows that

$$A = (2\pi)^{1/2}\exp\left(\frac{1}{2}\right) = 4.1327$$

(b) The formula used to plot the spectrum of Fig. 5.23 is defined by the Fourier transform on the right-hand side of Eq. (4), that is,

$$V(f) = -j2\pi f A\exp(-\pi f^2\tau^2) \tag{7}$$