Chapter 8 Solutions

Problem 8.1 An information packet contains 200 bits. This packet is transmitted over a communications channel where the probability of error for each bit is 10^{-3}. What is the probability that the packet is received error-free?

Solution

Recognizing that the number of errors has a binomial distribution over the sequence of 200 bits, let x represent the number of errors with $p = 0.001$ and $n = 200$. Then the probability of no errors is

$$\begin{aligned}\mathbf{P}[x=0] &= (1-p)^{n}\\ &= (1-.001)^{200}\\ &= .999^{200}\\ &= 0.82\end{aligned}$$

Problem 8.2 Suppose the packet of the Problem 8.1 includes an error-correcting code that can correct up to three errors located anywhere in the packet. What is the probability that a particular packet is received in error in this case?

Solution

The probability of a packet error is equal to the probability of more than three bit errors. This is equivalent to 1 minus the probability of 0, 1, 2, or 3 errors:

$$\begin{aligned}1-\mathbf{P}[x\le 3] &= 1-\left(\mathbf{P}[x=0]+\mathbf{P}[x=1]+\mathbf{P}[x=2]+\mathbf{P}[x=3]\right)\\ &= 1-(1-p)^n-\binom{n}{1}p(1-p)^{n-1}-\binom{n}{2}p^2(1-p)^{n-2}-\binom{n}{3}p^3(1-p)^{n-3}\\ &= 1-(1-p)^{n-3}\left[(1-p)^3+np(1-p)^2+\frac{n(n-1)}{2}p^2(1-p)+\frac{n(n-1)(n-2)}{6}p^3\right]\\ &= 5.5\times10^{-5}\end{aligned}$$

Problem 8.3 Continuing with Example 8.6, find the following conditional probabilities: $\mathbf{P}[X=0|Y=1]$ and $\mathbf{P}[X=1|Y=0]$.

Solution

From Bayes' Rule

$$\begin{aligned}\mathbf{P}[X=0|Y=1] &= \frac{\mathbf{P}[Y=1|X=0]\,\mathbf{P}[X=0]}{\mathbf{P}[Y=1]}\\ &= \frac{pp_0}{pp_0+(1-p)p_1}\end{aligned}$$

$$\mathbf{P}[X=1|Y=0] = \frac{\mathbf{P}[Y=0|X=1]\mathbf{P}[X=1]}{\mathbf{P}[Y=0]}$$

$$= \frac{pp_1}{pp_1 + (1-p)p_0}$$

Problem 8.4 Consider a binary symmetric channel for which the conditional probability of error $p = 10^{-4}$, and symbols 0 and 1 occur with equal probability. Calculate the following probabilities:

a) The probability of receiving symbol 0.
b) The probability of receiving symbol 1.
c) The probability that symbol 0 was sent, given that symbol 0 is received
d) The probability that symbol 1 was sent, given that symbol 0 is received.

Solution

(a)

$$\mathbf{P}[Y=0] = \mathbf{P}[Y=0 \mid X=0]\mathbf{P}[X=0] + \mathbf{P}[Y=0 \mid X=1]\mathbf{P}[X=1]$$
$$= (1-p)p_0 + pp_1$$
$$= .9999\,\tfrac{1}{2} + .0001\,\tfrac{1}{2}$$
$$= \tfrac{1}{2}$$

(b)

$$\mathbf{P}[Y=1] = 1 - \mathbf{P}[Y=0]$$
$$= \tfrac{1}{2}$$

(c) From Eq.(8.30)

$$\mathbf{P}[X=0|Y=0] = \frac{(1-p)p_0}{(1-p)p_0 + pp_1}$$
$$= \frac{(1-10^{-4})\tfrac{1}{2}}{(1-10^{-4})\tfrac{1}{2} + 10^{-4}\,\tfrac{1}{2}}$$
$$= 1 - 10^{-4}$$

(d) From Prob. 8.3

$$\mathbf{P}[X=1|Y=0] = \frac{pp_1}{pp_1 + (1-p)p_0}$$
$$= \frac{10^{-4}\,\tfrac{1}{2}}{10^{-4}\,\tfrac{1}{2} + (1-10^{-4})\,\tfrac{1}{2}}$$
$$= 10^{-4}$$

Problem 8.5 Determine the mean and variance of a random variable that is uniformly distributed between *a* and *b*.

Solution

The mean of the uniform distribution is given by

$$\begin{aligned}
\mu = \mathbf{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_a^b x \frac{1}{b-a} dx \\
&= \left.\frac{x^2}{2(b-a)}\right|_a^b \\
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}$$

The variance is given by

$$\begin{aligned}
\mathbf{E}\left[(X-\mu)^2\right] &= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx \\
&= \int_a^b \frac{(x-\mu)^2}{b-a} dx \\
&= \frac{1}{b-a}\left(\frac{(b-\mu)^3}{3} - \frac{(a-\mu)^3}{3}\right)
\end{aligned}$$

If we substitute $\mu = \frac{b+a}{2}$ then

$$\begin{aligned}
\mathbf{E}\left[(X-\mu)^2\right] &= \frac{1}{b-a}\left[\frac{(b-a)^3}{24} - \frac{(a-b)^3}{24}\right] \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

Problem 8.6 Let X be a random variable and let $Y = (X-\mu_X)/\sigma_X$. What is the mean and variance of the random variable Y?

Solution

$$\mathbf{E}[Y] = \mathbf{E}\left[\frac{X-\mu_X}{\sigma_X}\right] = \frac{\mathbf{E}[X]-\mu_X}{\sigma_X} = \frac{0}{\sigma_X} = 0$$

$$\mathbf{E}\left[(Y-\mu_Y)^2\right] = \mathbf{E}\left[Y^2\right] = \mathbf{E}\left(\frac{X-\mu_X}{\sigma_X}\right)^2$$

$$= \frac{\mathbf{E}\left[(X-\mu_X)^2\right]}{\sigma_X{}^2} = \frac{\sigma_X{}^2}{\sigma_X{}^2} = 1$$

Problem 8.7 What is the probability density function of the random variable *Y* of Example 8.8? Sketch this density function.

Solution

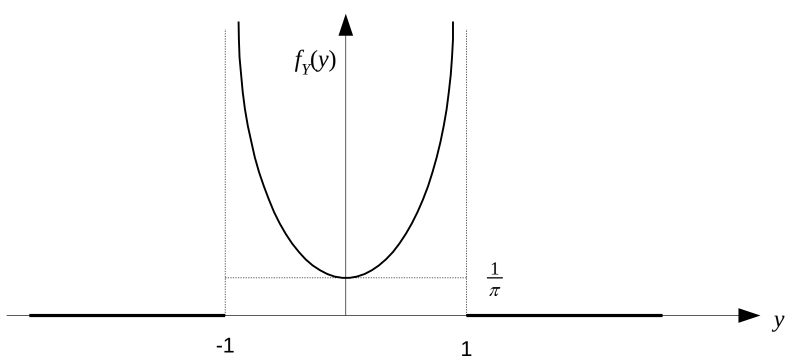
From Example 8.8, the distribution of *Y* is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \dfrac{2\pi - 2\cos^{-1}(y)}{2\pi} & |y| < 1 \\ 1 & y > 1 \end{cases}$$

Thus, the density of *Y* is given by

$$\frac{dF_Y(y)}{dy} = \begin{cases} 0 & y < -1 \\ \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & y > 1 \end{cases}$$

This density is sketched in the following figure.



Problem 8.8 Show that the mean and variance of a Gaussian random variable *X* with the density function given by Eq. (8.48) are μ_X and σ_X^2 .

Solution

Consider the difference $\mathbf{E}[X]-\mu_X$:

$$\mathbf{E}[X]-\mu_X = \int_{-\infty}^{\infty} \frac{(x-\mu_X)}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\} dx$$

Let $y = x-\mu_X$ and substitute

$$\begin{aligned}\mathbf{E}[X]-\mu_X &= \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}\sigma_X} \exp\left(-y^2 \big/ 2\sigma_X{}^2\right) dy \\ &= 0\end{aligned}$$

since integrand has odd symmetry. This implies $E[X]=\mu_X$. With this result

$$\begin{aligned}\operatorname{Var}(X) &= \mathbf{E}(x-\mu_X)^2 \\ &= \int_{-\infty}^{\infty} \frac{(x-\mu_X)^2}{\sqrt{2\pi}\sigma_X} \exp\left\{\frac{-(x-\mu_X)^2}{2\sigma_X{}^2}\right\} dx\end{aligned}$$

In this case let

$$y = \frac{x-\mu_X}{\sigma_X}$$

and making the substitution, we obtain

$$\operatorname{Var}(X) = \sigma_X{}^2 \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} \exp\left\{\frac{-y^2}{2}\right\} dy$$

Recalling the integration-by-parts, i.e., $\int u dv = uv - \int v du$, let $u = y$ and $dv = y \exp\left(\frac{-y^2}{2}\right) dy$. Then

$$\begin{aligned}\operatorname{Var}(X) &= \sigma_X{}^2 \frac{(-y)}{\sqrt{2\pi}} \exp\left(-y^2 \big/ 2\right)\Bigg|_{-\infty}^{\infty} + \sigma_X{}^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= 0 + \sigma_X{}^2 \bullet 1 \\ &= \sigma_X{}^2\end{aligned}$$

where the second integral is one since it is integral of the normalized Gaussian probability density.

Problem 8.9 Show that for a Gaussian random variable X with mean μ_X and variance σ_X^2 the transformation $Y = (X - \mu_X)/\sigma_X$, converts X to a normalized Gaussian random variable.

Solution

Let $y = \frac{x-\mu_X}{\sigma_X}$. Then

$$\begin{aligned}\mathbf{E}[Y] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\exp\left(-y^2/2\right)dy \\ &= 0\end{aligned}$$

by the odd symmetry of the integrand. If $\mathbf{E}[Y] = 0$, then from the definition of Y, $\mathbf{E}[X] = \mu_X$. In a similar fashion

$$\begin{aligned}\mathbf{E}[Y^2] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2\exp\left(-\frac{y^2}{2}\right)dy \\ &= \frac{(-)y}{\sqrt{2\pi}}\cdot\exp\left\{-\frac{y^2}{2}\right\}\Bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-y^2/2\right)dy \\ &= 1\end{aligned}$$

where we use integration by parts as in Problem 8.8. This result implies

$$E\left(\frac{x-\mu_X}{\sigma_X}\right)^2 = 1$$

and hence $\mathbf{E}[(X-\mu_X)^2] = \sigma_X^2$

Problem 8.10 Determine the mean and variance of the sum of five independent uniformly-distributed random variables on the interval from -1 to +1.

Solution

Let X_i be the individual uniformly distributed random variables for i = 1,..,5, and let Y be the random variable representing the sum:

$$Y = \sum_{i=1}^{5} X_i$$

Since X_i has zero mean and $\text{Var}(X_i)$ = 1/3 (see Problem 8.5), we have

$$\mathbf{E}[Y] = \sum_{i=1}^{5} \mathbf{E}[X_i] = 0$$

and

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathbf{E}\left[(Y-\mu_Y)^2\right] = \mathbf{E}\left[Y^2\right] \\
&= \mathbf{E}\left[\left(\sum X_i\right)^2\right] \\
&= \sum_{i=1}^{5} \mathbf{E}\left[X_i^2\right] + \sum_{i \neq j} \mathbf{E}\left[X_i X_j\right]
\end{aligned}$$

Since the X_i are independent, we may write this as

$$\begin{aligned}
\mathrm{Var}(Y) &= 5\left(\tfrac{1}{3}\right) + \sum \mathbf{E}[X_i]\,\mathbf{E}[X_j] \\
&= \tfrac{5}{3} + 0 \\
&= \tfrac{5}{3}
\end{aligned}$$

Problem 8.11 A random process is defined by the function

$$X(t,\theta) = A\cos(2\pi f t + \theta)$$

where A and f are constants, and θ is uniformly distributed over the interval 0 to 2π. Is X stationary to the first order?

Solution

Denote

$$Y = X(t_1,\theta) = A\cos(2\pi f t_1 + \theta)$$

for any t_1. From Problem 8.7, the distribution of Y and therefore of X for any t_1 is

$$F_{X(t_1)}(y) = \begin{cases} 0 & y < -A \\ \dfrac{2\pi - 2\cos^{-1}(y/A)}{2\pi} & |y| < A \\ 1 & y > A \end{cases}$$

Since the distribution is independent of t it is stationary to first order.

Problem 8.12 Show that a random process that is stationary to the second order is also stationary to the first order.

Solution

Let the distribution F be stationary to second order

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2)$$

Then,

$$\begin{aligned} F_{X(t_1)X(t_2)}(x_1, \infty) &= F_{X(t_1)}(x_1) \\ &= F_{X(t_1+\tau)X(t_2+\tau)}(x_1, \infty) \\ &= F_{X(t_1+\tau)}(x_1) \end{aligned}$$

Thus the first order distributions are stationary as well.

Problem 8.13 Let $X(t)$ be a random process defined by

$$X(t) = A\cos(2\pi f t)$$

where A is uniformly distributed between 0 and 1, and f is constant. Determine the autocorrelation function of X. Is X wide-sense stationary?

Solution

$$\begin{aligned} \mathbf{E}[X(t_1)X(t_2)] &= \mathbf{E}[A^2]\cos(2\pi f t_1)\cos(2\pi f t_2) \\ &= \mathbf{E}[A^2][\cos(2\pi f(t_1 - t_2)) + \cos 2\pi f(t_1 + t_2)] \end{aligned}$$

$$\mathbf{E}[A^2] = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

Since the autocorrelation function depends on $t_1 + t_2$ as well as $t_1 - t_2$, the process is not wide-sense stationary.

Problem 8.14 A discrete-time random process $\{Y_n: n = \ldots,-1,0,1,2,\ldots\}$ is defined by

$$Y_n = \alpha_0 Z_n + \alpha_1 Z_{n-1}$$

where $\{Z_n\}$ is a random process with autocorrelation function $R_Z(n) = \sigma^2\delta(n)$. What is the autocorrelation function $R_Y(n,m) = \mathbf{E}[Y_n Y_m]$? Is the process $\{Y_n\}$ wide-sense stationary?

Solution

We implicitly assume that Z_n is stationary and has a constant mean μ_Z. Then the mean of Y_n is given by

$$\begin{aligned}\mathbf{E}[Y_n] &= \alpha_0\mathbf{E}[Z_n] + \alpha_1\mathbf{E}[Z_{n-1}] \\ &= (\alpha_0 + \alpha_1)\mu_Z\end{aligned}$$

The autocorrelation of Y is given by

$$\begin{aligned}\mathbf{E}[Y_nY_m] &= \mathbf{E}[(\alpha_0 Z_n + \alpha_0 Z_{n-1})(\alpha_0 Z_m + \alpha_1 Z_{m-1})] \\ &= \alpha_0^2\mathbf{E}[Z_nZ_m] + \alpha_1\alpha_0\mathbf{E}[Z_nZ_{m-1}] + \alpha_0\alpha_1\mathbf{E}[Z_{n-1}Z_m] + \alpha_1^2\mathbf{E}[Z_{m-1}Z_{n-1}] \\ &= \alpha_0^2\sigma^2\delta(n-m) + \alpha_1\alpha_0\sigma^2\delta(m-1-n) + \alpha_0\alpha_1\sigma^2\delta(n-1-m) + \alpha_1^2\delta(m-1-(n-1)) \\ &= (\alpha_0^2+\alpha_1^2)\sigma^2\delta(n-m) + \alpha_0\alpha_1\sigma^2[\delta(n-m-1) + \delta(m-n-1)]\end{aligned}$$

Since the autocorrelation only depends on the time difference *n-m*, the process is wide-sense stationary with

$$R_Y(n) = (\alpha_0^2 + \alpha_1^2)\sigma^2\delta(n) + \alpha_0\alpha_1\sigma^2(\delta(n-1) + \delta(n+1))$$

Problem 8.15 For the discrete-time process of Problem 8.14, use the discrete Fourier transform to approximate the corresponding spectrum. That is,

$$S_Y(k) = \sum_{n=0}^{N-1} R_Y(n)W^{kn}$$

If the sampling in the time domain is at n/T_s where $n = 0, 1, 2, \ldots, N$-1. What frequency does k correspond to?

Solution

Let $\beta_0 = (\alpha_0^2 + \alpha_1^2)\sigma^2$ and $\beta_1 = \alpha_0\alpha_1\sigma^2$. Then

$$\begin{aligned}S_Y(k) &= \sum_{n=0}^{N-1}[\beta_0\delta(n) + \beta_1(\delta(n-1) + \delta(n+1))]W^{kn} \\ &= \beta_0 W^0 + \beta_1(W^{-k} + W^{+k}) \\ &= \beta_0 + \beta_1\left(e^{\frac{-j2\pi k}{N}} + e^{\frac{+j2\pi k}{N}}\right) \\ &= \beta_0 + 2\beta_1\cos\left(\frac{2\pi k}{N}\right)\end{aligned}$$

The term $S_Y(k)$ corresponds to frequency $\frac{kf_s}{N}$ where $f_S = \frac{1}{T_S}$.

Problem 8.16 Is the discrete-time process $\{Y_n: n = 1,2,\ldots\}$ defined by: $Y_0 = 0$ and

$$Y_{n+1} = \alpha Y_n + W_n,$$

a Gaussian process, if W_n is Gaussian?

<u>Solution</u>

(Proof by mathematical induction.) The first term $Y_1 = \alpha Y_0 + W_0$ is Gaussian since $Y_0 = 0$ and W_0 are Gaussian. The second term $Y_2 = \alpha Y_1 + W_1$ is Gaussian since Y_1 and W_1 are Gaussian. Assume Y_n is Gaussian. Then $Y_{n+1} = \alpha Y_n + W_n$ is Gaussian since Y_n and W_n are both Gaussian.

Problem 8.17 A discrete-time white noise process $\{W_n\}$ has an autocorrelation function given by $R_W(n) = N_0\delta(n)$.

(a) Using the discrete Fourier transform, determine the power spectral density of $\{W_n\}$.

(b) The white noise process is passed through a discrete-time filter having a discrete-frequency response

$$H(k) = \frac{1-(\alpha W^k)^N}{1-\alpha W^k}$$

where, for a N-point discrete Fourier transform, $W = \exp\{j2\pi/N\}$. What is the spectrum of the filter output?

<u>Solution</u>

The spectrum of the discrete white noise process is

$$\begin{aligned} S(k) &= \sum_{n=0}^{N-1} R(n) W^{nk} \\ &= \sum_{n=0}^{N-1} N_0 \delta(n) W^{nk} \\ &= N_0 \end{aligned}$$

The spectrum of the process after filtering is

$$\begin{aligned} S_Y(k) &= |H(k)|^2 S(k) \\ &= N_0 \left| \frac{1-(\alpha W^k)^N}{1-\alpha W^k} \right|^2 \end{aligned}$$

Problem 8.18 Consider a deck of 52 cards, divided into four different suits, with 13 cards in each suit ranging from the two up through the ace. Assume that all the cards are equally likely to be drawn.

(a) Suppose that a single card is drawn from a full deck. What is the probability that this card is the ace of diamonds? What is the probability that the single card drawn is an ace of any one of the four suits?

(b) Suppose that two cards are drawn from the full deck. What is the probability that the cards drawn are an ace and a king, not necessarily the same suit? What if they are of the same suit?

<u>Solution</u>

(a)

$$\mathbf{P}[\text{Ace of diamonds}] = \frac{1}{52}$$

$$\mathbf{P}[\text{Any ace}] = \frac{1}{13}$$

(b)

$$\begin{aligned}\mathbf{P}[\text{Ace and king}] &= \mathbf{P}[\text{Ace on first draw}]\mathbf{P}[\text{King on second}] + \mathbf{P}[\text{King on first draw}]\mathbf{P}[\text{Ace on seco}\\ &= \frac{1}{13}\times\frac{4}{51} + \frac{1}{13}\times\frac{4}{51}\\ &= \frac{8}{663}\end{aligned}$$

$$\begin{aligned}\mathbf{P}[\text{Ace and king of same suit}] &= \frac{1}{13}\times\frac{1}{51} + \frac{1}{13}\times\frac{1}{51}\\ &= \frac{1}{663}\end{aligned}$$

Problem 8.19 Suppose a player has one red die and one white die. How many outcomes are possible in the random experiment of tossing the two dice? Suppose the dice are indistinguishable, how many outcomes are possible?

<u>Solution</u>

The number of possible outcomes is $6\times 6 = 36$, if distinguishable.
If the die are indistinguishable then the outcomes are

(11) (12)…(16)
(22)(23)…(26)
(33)(34)…(36)
(44)(45)(46)
(55)(56)
(66)

And the number of possible outcomes are 21.

Problem 8.20 Refer to Problem 8.19.
(a) What is the probability of throwing a red 5 and a white 2?
(b) If the dice are indistinguishable, what is the probability of throwing a sum of 7? If they are distinguishable, what is this probability?

Solution

(a) $\mathbf{P}[\text{Red 5 and white 2}] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(b) The probability of the sum does not depend upon whether the die are distinguishable or not. If we consider the distinguishable case the possible outcomes are (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1) so

$$\mathbf{P}[\text{sum of } 7] = \frac{6}{36} = \frac{1}{6}$$

Problem 8.21 Consider a random variable X that is uniformly distributed between the values of 0 and 1 with probability ¼ takes on the value 1 with probability ¼ and is uniformly distributed between values 1 and 2 with probability ½ . Determine the distribution function of the random variable X.

Solution

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x/4 & 0 < x < 1 \\ 1/2 & x = 1 \\ \frac{1}{2} + \frac{1}{2}(x-1) & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$$

Problem 8.22 Consider a random variable X defined by the double-exponential density where a and b are constants.

$$f_X(x) = a\exp(-b|x|) \qquad -\infty < x < \infty$$

(a) Determine the relationship between a and b so that $f_X(x)$ is a probability density function.
(b) Determine the corresponding distribution function $F_X(x)$.
(c) Find the probability that the random variable X lies between 1 and 2.

Solution
(a)

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad \Rightarrow 2\int_{0}^{\infty} a\exp(-bx)dx = 1$$

$$-\frac{2a}{b}\exp(-bx)\Big|_{0}^{\infty} = 1$$

$$\Rightarrow \frac{2a}{b} = 1 \;\; or \quad b = 2a$$

(b)

$$F_X(x) = \int_{-\infty}^{x} a\exp(-b|s|)ds$$

$$= \begin{cases} -\frac{a}{b}\exp(-b(-s))\Big|_{-\infty}^{x} & -\infty < x < 0 \\ \frac{1}{2} + -\frac{a}{b}\exp(-bs)\Big|_{0}^{x} & 0 < x < \infty \end{cases}$$

$$= \begin{cases} \frac{a}{b}\exp(bs) & -\infty < x < 0 \\ \frac{1}{2} + \frac{a}{b} - \frac{a}{b}\exp(-bs) & 0 \le x < \infty \end{cases}$$

$$= \begin{cases} \frac{1}{2}\exp(bx) & -\infty < x < 0 \\ 1 - \frac{1}{2}\exp(-bx) & 0 \le x < \infty \end{cases}$$

(c) The probability that $1 \le X \le 2$ is

$$F_X(2) - F_X(1) = \frac{1}{2}\left[\exp(-b) - \exp(-2b)\right]$$

Problem 8.23 Show that the expression for the variance of a random variable can be expressed in terms of the first and second moments as

$$\text{Var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2$$

<u>Solution</u>

$$\begin{aligned} \text{Var}(X) &= \mathbf{E}\left[(X - \mathbf{E}(X))^2\right] \\ &= \mathbf{E}\left(X^2 - 2X\mathbf{E}(X) + (\mathbf{E}[X])^2\right) \\ &= \mathbf{E}\left[X^2\right] - 2\mathbf{E}[X]\mathbf{E}[X] + (\mathbf{E}[X])^2 \\ &= \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 \end{aligned}$$

Problem 8.24 A random variable R is Rayleigh distributed with its probability density function given by

$$f_R(r) = \begin{cases} \frac{r}{b}\exp\left(-r^2/2b\right) & 0 \le r < \infty \\ 0 & otherwise \end{cases}$$

(a) Determine the corresponding distribution function

(b) Show that the mean of R is equal to $\sqrt{b\pi/2}$

(c) What is the mean-square value of R?

(d) What is the variance of R?

Solution

(a) The distribution of R is

$$\begin{aligned} F_R(r) &= \int_0^r f_R(s)\,ds \\ &= \int_0^r \frac{s}{b}\exp\left(-\frac{s^2}{2b}\right)ds \\ &= -\exp\left(-\frac{s^2}{2b}\right)\Bigg|_0^r \\ &= 1-\exp\left(-r^2/2b\right) \end{aligned}$$

(b) The mean value of R is

$$\begin{aligned} \mathbf{E}[R] &= \int_0^\infty s f_R(s)\,ds \\ &= \int_0^\infty \frac{s^2}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\ &= \frac{1}{b}\sqrt{2\pi b}\left[\frac{1}{\sqrt{2\pi b}}\int_0^\infty s^2\exp\left(\frac{-s^2}{2b}\right)ds\right] \end{aligned}$$

The bracketed expression is equivalent to the evaluation of the half of the variance of a zero-mean Gaussian random variable which we know is b in this case, so

$$\mathbf{E}[R] = \frac{\sqrt{2\pi b}}{b}\frac{1}{2}(b) = \sqrt{\frac{\pi b}{2}}$$

(c) The second moment of R is

$$
\begin{aligned}
\mathbf{E}[R^2] &= \int_0^\infty s^2 f_R(s)\,ds \\
&= \int_0^\infty \frac{s^3}{b}\exp\left(\frac{-s^2}{2b}\right)ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s\; F_R(s)\,ds \\
&= s^2 F_R(s)\Big|_0^\infty - \int_0^\infty 2s\left(1-\exp\left(\frac{s^2}{2b}\right)\right)ds \\
&= s^2\left(F_R(s)-1\right)\Big|_0^\infty + 2b\int_0^\infty f_R(s)\,ds \\
&= 2b
\end{aligned}
$$

(d) The variance of R is

$$
\begin{aligned}
\mathrm{Var}(R) &= \mathbf{E}[R^2] - (\mathbf{E}[R])^2 \\
&= 2b - \left(\sqrt{b\pi}\big/2\right)^2 \\
&= b\left(2 - \pi\big/2\right)
\end{aligned}
$$

Problem 8.25 Consider a uniformly distributed random variable Z, defined by

$$
f_Z(z) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le z < 2\pi \\ 0, & \text{otherwise} \end{cases}
$$

The two random variables X and Y are related to Z by $X = \sin(Z)$ and $Y = \cos(Z)$.

(a) Determine the probability density functions of X and Y.
(b) Show that X and Y are uncorrelated random variables.
(c) Are X and Y statistically independent? Why?

<u>Solution</u>

(a) The distribution function of X is formally given by

$$
F_X(x) = \begin{cases} 0 & x \le -1 \\ \mathbf{P}[-1 \le X \le x] & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}
$$

Analogous to Example 8.8, we have

$$\mathbf{P}(-1 \le X \le x) = \begin{cases} \mathbf{P}\left[\pi - \sin^{-1}(x) \le Z \le 2\pi + \sin^{-1}(x)\right] & -1 \le x \le 0 \\ \frac{1}{2} + \mathbf{P}\left[0 \le Z \le \sin^{-1}(x)\right] + \mathbf{P}\left[\pi - \sin^{-1}(x) \le Z \le \pi\right] & 0 \le x \le 1 \end{cases}$$

$$= \begin{cases} \frac{\pi + 2\sin^{-1}(x)}{2\pi} & -1 \le x \le 0 \\ \frac{1}{2} + \frac{2\sin^{-1}(x)}{2\pi} & 0 \le x \le 1 \end{cases}$$

$$= \frac{1}{2} + \frac{\sin^{-1}(x)}{\pi} \qquad -1 \le x \le 1$$

where the second line follows from the fact that the probability for a uniform random variable is proportional to the length of the interval. The distribution of Y follows from a similar argument (see Example 8.8).

(b) To show X and Y are uncorrelated, consider

$$\begin{aligned} \mathbf{E}[XY] &= \mathbf{E}[\sin(Z)\cos(Z)] \\ &= \mathbf{E}\left[\frac{\sin(2Z)}{2}\right] \\ &= \frac{1}{4\pi}\int_0^{2\pi} \sin(2z)\,dz \\ &= -\frac{1}{8\pi}\cos(2z)\Big|_0^{2\pi} = 0 \end{aligned}$$

Thus X and Y are uncorrelated.

(c) The random variables X and Y are not statistically independent since

$$\Pr[X|Y] \ne \Pr[X]$$

Problem 8.26 A Gaussian random variable has zero mean and a standard deviation of 10 V. A constant voltage of 5 V is added to this random variable.

(a) Determine the probability that a measurement of this composite signal yields a positive value.

(b) Determine the probability that the arithmetic mean of two independent measurements of this signal is positive.

Solution

(a) Let Z represent the initial Gaussian random variable and Y the composite random variable. Then

$$Y = 5 + Z$$

and the density function of Y is given by

$$f_y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-(y-\mu)^2/2\sigma^2\right\}$$

where μ corresponds to a mean of 5V and σ corresponds to a standard deviation of 10V. The probability that Y is positive is

$$\begin{aligned} \mathbf{P}[Y > 0] &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty \exp\left\{-(y-\mu)^2/2\sigma^2\right\} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\mu/\sigma}^\infty \exp\left(-s^2/2\right) ds \\ &= Q\left(\frac{-\mu}{\sigma}\right) \end{aligned}$$

where, in the second line, we have made the substitution

$$s = \frac{y-\mu}{\sigma}$$

Making the substitutions for μ and σ, we have $\mathbf{P}[Y>0] = Q(-\frac{1}{2})$. We note that in Fig. 8.11, the values of $Q(x)$ are not shown for negative x; to obtain a numerical result, we use the fact that $Q(-x) = 1 - Q(x)$. Consequently, $Q(-\frac{1}{2}) = 1 - 0.3 = 0.7$.

(b) Let W represent the arithmetic mean of two measurements Y_1 and Y_2, that is

$$W = \frac{Y_1 + Y_2}{2}$$

It follows that W is a Gaussian random variable with $\mathbf{E}[W] = \mathbf{E}[Y] = 5$. The variance of W is given by

$$\mathrm{Var}(W) = \mathbf{E}\left[\left(W - \mathbf{E}(W)\right)^2\right]$$

$$= \mathbf{E}\left[\left(\frac{Y_1 + Y_2}{2} - \frac{\mathbf{E}(Y_1) + \mathbf{E}(Y_2)}{2}\right)^2\right]$$

$$= \frac{1}{4}\left(\mathbf{E}\left[\left(Y_1 - \mathbf{E}(Y_1)\right)^2 + \left(Y_2 - \mathbf{E}(Y_2)\right)^2 + 2\left(Y_1 - \mathbf{E}(Y_1)\right)\left(Y_2 - \mathbf{E}(Y_2)\right)\right]\right)$$

The first two terms correspond to the variance of Y. The third term is zero because the measurements are independent. Making these substitutions, the variance of W reduces to

$$\mathrm{Var}\left[W\right] = \sigma^2/2$$

Using the result of part (a), we then have

$$\mathbf{P}\left[W > 0\right] = Q\left(\frac{-\mu}{\left(\sigma/\sqrt{2}\right)}\right) = Q\left(-\frac{1}{\sqrt{2}}\right)$$

Problem 8.27 Consider a random process defined by

$$X(t) = \sin\left(2\pi W t\right)$$

in which the frequency W is a random variable with the probability density function

$$f_W(w) = \begin{cases} \dfrac{1}{B} & 0 < w < B \\ 0 & \text{otherwise} \end{cases}$$

Show that $X(t)$ is nonstationary.

<u>Solution</u>

At time $t = 0$, $X(0)=0$ and the distribution of $X(0)$ is

$$F_{X(0)}\left(x\right) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

At time $t = 1$, $X(1) = \sin\left(2\pi w\right)$, and the distribution of X(1) is clearly not a step function so

$$F_{X(1)}\left(x\right) \neq F_{X(0)}\left(x\right)$$

And the process $X(t)$ is not first-order stationary, and hence nonstationary.

Problem 8.28 Consider the sinusoidal process

$$X(t) = A\cos(2\pi f_c t)$$

where the frequency is constant and the amplitude A is uniformly distributed:

$$f_A(a) = \begin{cases} 1 & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine whether or not this process is stationary in the strict sense.

<u>Solution</u>

At time $t = 0$, $X(0) = A$, and $F_{X(0)}(0)$ is uniformly distributed over 0 to 1.
At time $t = (4f_c)^{-1}$, $X((4f_c)^{-1}) = 0$ and

$$F_{X\left(\frac{1}{4f_c}\right)}(x) = \delta(x)$$

Thus, $F_{X(0)}(x) \neq F_{X(1/4f_c)}(x)$ and the process $X(t)$ is not stationary to first order. Hence not strictly stationary.

Problem 8.29 A random process is defined by

$$X(t) = A\cos(2\pi f_c t)$$

where A is a Gaussian random variable of zero mean and variance σ^2. This random process is applied to an ideal integrator, producing an output $Y(t)$ defined by

$$Y(t) = \int_0^t X(\tau)d\tau$$

(a) Determine the probability density function of the output at a particular time.
(b) Determine whether or not is stationary.

<u>Solution</u>

(a) The output process is given by

$$\begin{aligned} Y(t) &= \int_0^t X(\tau)d\tau \\ &= \int_0^t A\cos(2\pi f_c \tau)d\tau \\ &= \frac{A}{2\pi f_c}\sin(2\pi f_c t) \end{aligned}$$

At time t_0, it follows that $Y(t_0)$ is Gaussian with zero mean, and variance

$$\frac{\sigma^2}{(2\pi f_c)^2}\sin^2(2\pi f_c t_0)$$

(b) No, the process $Y(t)$ is not stationary as $F_{Y(t_0)} \neq F_{Y(t_1)}$ for all t_1 and t_0.

Problem 8.30 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(a) If $X(t)$ contains a dc component equal to A, then $R_X(\tau)$ contains a constant component equal to A^2.

(b) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ also contains a sinusoidal component of the same frequency.

Solution

(a) Let $Y(t) = X(t) - A$ and $Y(t)$ is a random process with zero *dc* component. Then

$$\mathbf{E}[X(t)] = A$$

and

$$\begin{aligned}
R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\
&= \mathbf{E}[(X(t) - A + A)(X(t+\tau) - A + A)] \\
&= \mathbf{E}[(X(t) - A)(X(t+\tau) - A)] + \mathbf{E}(X(t+\tau) - A)A + \mathbf{E}(X(t) - A) A + A^2 \\
&= R_Y(\tau) + 0 + 0 + A^2
\end{aligned}$$

And thus $R_X(\tau)$ has a constant component A^2.

(b) Let $X(t) = Y(t) + A\sin(2\pi f_c t)$ where $Y(t)$ does not contain a sinusoidal component of frequency f_c.

$$\begin{aligned}
R_X(\tau) &= \mathbf{E}[X(t)X(t+\tau)] \\
&= \mathbf{E}[Y(t) + A\sin(2\pi f_c t))(Y(t+\tau) + A\sin(2\pi f_c t))] + \mathbf{E}[A^2 \sin(2\pi f_c t)\sin(2\pi f_c(t+\tau))] \\
&= R_Y(\tau) + \ldots + \frac{A^2}{2}[\cos 2\pi f_c t + \cos 2\pi f_c(t+\tau) + \theta] \\
&= R_Y(\tau) + \frac{A^2}{2}\cos(2\pi f_c \tau)
\end{aligned}$$

And thus $R_X(\tau)$ has a sinusoidal component at f_c.

Problem 8.31 A discrete-time random process is defined by

$$Y_n = \alpha Y_{n-1} + W_n \qquad n = \ldots, -1, 0, +1, \ldots$$

where the zero-mean random process W_n is stationary with autocorrelation function $R_W(k) = \sigma^2\delta(k)$. What is the autocorrelation function $R_y(k)$ of Y_n? Is Y_n a wide-sense stationary process? Justify your answer.

Solution

We partially address the question of whether Y_n is wide-sense stationary (WSS) first by noting that

$$\begin{aligned}\mathbf{E}[Y_n] &= \mathbf{E}[\alpha Y_{n-1} + W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}] + \mathbf{E}[W_n] \\ &= \alpha\mathbf{E}[Y_{n-1}]\end{aligned}$$

since $\mathbf{E}[W_n] = 0$. To be WSS, the mean of the process must be constant and consequently, we must have that $\mathbf{E}[Y_n] = 0$ for all n, to satisfy the above relationship.

We consider the autocorrelation of Y_n in steps. First note that $R_Y(0)$ is given by

$$R_Y(0) = \mathbf{E}[Y_n Y_n] = \mathbf{E}[Y_n^2]$$

and that $R_Y(1)$ is

$$\begin{aligned}R_Y(1) &= \mathbf{E}[Y_n Y_{n+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_n + W_n)] \\ &= \alpha\mathbf{E}[Y_n^2] + \mathbf{E}[Y_n W_n]\end{aligned}$$

Although not explicitly stated in the problem, we assume that W_n is independent of Y_n, thus $\mathbf{E}[Y_n W_n] = \mathbf{E}[Y_n]\mathbf{E}[W_n] = 0$, and so

$$R_Y(1) = \alpha R_Y(0)$$

We prove the result for general positive k by assuming $R_Y(k) = \alpha^k R_Y(0)$ and then noting that

$$\begin{aligned}R_Y(k+1) &= \mathbf{E}[Y_n Y_{n+k+1}] \\ &= \mathbf{E}[Y_n(\alpha Y_{n+k} + W_{n+k})] \\ &= \alpha\mathbf{E}[Y_n Y_{n+k}] + \mathbf{E}[Y_n W_{n+k}]\end{aligned}$$

To evaluate this last expression, we note that, since

$$
\begin{aligned}
Y_n &= \alpha Y_{n-1} + W_n \\
&= \alpha^2 Y_{n-2} + \alpha W_{n-1} + W_n \\
&= \alpha^3 Y_{n-3} + \alpha^2 W_{n-2} + \alpha W_{n-1} + W_n \\
&= \cdots
\end{aligned}
$$

we see that Y_n only depends on W_k for $k \le n$. Thus $\mathbf{E}[Y_n W_{n+k}] = 0$. Thus, for positive k, we have

$$
\begin{aligned}
R_Y(k+1) &= \alpha R_Y(k) \\
&= \alpha^{k+1} R_Y(0)
\end{aligned}
$$

Using a similar argument, a corresponding result can be shown for negative k. Combining the results, we have

$$R_Y(k) = \alpha^{|k|} R_Y(0)$$

Since the autocorrelation only depends on the time difference k, and the process is wide-sense stationary.

Problem 8.32 Find the power spectral density of the process that has the autocorrelation function

$$
R_X(\tau) = \begin{cases} \sigma^2 \left(1 - |\tau|^2\right) & |\tau| < 1 \\ 0 & \text{otherwise} \end{cases}
$$

Solution

The Wiener-Khintchine relations imply the power spectral density is given by the Fourier transform of $R_X(\tau)$, which is (see Appendix 6)

$$S_X(f) = \sigma^2 \mathrm{sinc}^2(f)$$

Problem 8.33. A random pulse has amplitude A and duration T but starts at an arbitrary time t_0. That is, the random process is defined as

$$X(t) = A\,\mathrm{rect}(t + t_0)$$

where rect(t) is defined in Section 2.9. The random variable t_0 is assumed to be uniformly distributed over $[0,T]$ with density

$$f_{t_0}(s) = \begin{cases} \dfrac{1}{T} & 0 \le s \le T \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the autocorrelation function of the random process $X(t)$?
(b) What is the spectrum of the random process $X(t)$?

Solution

First note that the process $X(t)$ is not stationary. This may be demonstrated by computing the mean of $X(t)$ for which we use the fact that

$$f_X(x) = \int_{-\infty}^{\infty} f_X(x \mid s) f_{t_0}(s) ds$$

combined with the fact that

$$\begin{aligned} \mathbf{E}[X(t) \mid t_0] &= \int_{-\infty}^{\infty} x f_X(x \mid t_0) dx \\ &= \begin{cases} A & t_0 \le t \le t_0 + T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathbf{E}[X(t)] &= \int_{-\infty}^{\infty} \mathbf{E}[X(t) \mid s] f_{t_0}(s) ds \\ &= \begin{cases} 0 & t < 0 \\ At/T & 0 \le t \le T \\ A(2 - t/T) & T < t \le 2T \\ 0 & t > 2T \end{cases} \end{aligned}$$

Thus the mean of the process is dependent on t, and the process is nonstationary.

We take a similar approach to compute the autocorrelation function. First we break the situation into a number of cases:

i) For any $t < 0$, $s < 0$, $t > 2T$, or $s > 2T$, we have that

$$\mathbf{E}[X(t)X(s)] = 0$$

ii) For $0 \le t < s \le 2T$, we first assume t_0 is known

$$\mathbf{E}\left[X(t)X(s)\,|\,t_0\right] = \begin{cases} A^2 & t > t_0,\ s < t_0 + T,\ 0 < t_0 < T \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} A^2 & \max(s - T, 0) < t_0 < \min(t, T) \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the unconditional expectation, we have

$$\mathbf{E}\left[X(t)X(s)\right] = \int_{-\infty}^{\infty} \mathbf{E}\left[X(t)X(s)\,|\,w\right] f_{t_0}(w)\,dw$$
$$= \int_{\max(0, s-T)}^{\min(t,T)} A^2 \left(\frac{1}{T}\right) dw$$
$$= \frac{A^2}{T} \max\left\{\min(t,T) - \max(0, s-T), 0\right\}$$

where the second maximum takes care of the case where the lower limit on the integral is greater than the upper limit.

iii) For $0 \le s < t \le 2T$, we use a similar argument to obtain

$$\mathbf{E}\left[X(t)X(s)\,|\,t_0\right] = \begin{cases} A^2 & \max(t - T, 0) < t_0 < \min(s, T) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{E}\left[X(t)X(s)\right] = \frac{A^2}{T} \max\left\{\min(s,T) - \max(0, t-T), 0\right\}$$

Combining all of these results we have the autocorrelation is given by

$$\mathbf{E}\left[X(t)X(s)\right] = \begin{cases} \frac{A^2}{T} \max\left\{\min(t,T) - \max(0, s-T), 0\right\} & 0 \le t < s \le 2T \\ \frac{A^2}{T} \max\left\{\min(s,T) - \max(0, t-T), 0\right\} & 0 \le s < t \le 2T \\ 0 & \text{otherwise} \end{cases}$$

This result depends upon both t and s, not just t-s, as one would expect for a non-stationary process.

Problem 8.34 Given that a stationary random process $X(t)$ has an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$, show that:

(a) The autocorrelation function of $dX(t)/dt$, the first derivative of $X(t)$ is equal to the negative of the second derivative of $R_X(\tau)$.
(b) The power spectral density of $dX(t)/dt$ is equal to $4\pi^2 f^2 S_X(f)$.

Hint: Use the results of Problem 2.24.

Solution

(a) Let $Y(t) = \frac{dX}{dt}(t)$, and from the Wiener-Khintchine relations, we know the autocorrelation of $Y(t)$ is the inverse Fourier transform of the power spectral density of Y. Using the results of part (b),

$$\begin{aligned} R_Y(f) &= \mathbf{F}^{-1}\left\{S_Y(f)\right\} \\ &= \mathbf{F}^{-1}\left\{4\pi^2 f^2 S_X(f)\right\} \\ &= -\mathbf{F}^{-1}\left\{(j2\pi f)^2 S_X(f)\right\} \end{aligned}$$

from the differential properties of the Fourier transform, we know that differentiation in the time domain corresponds to multiplication by $j2\pi f$ in the frequency domain. Consequently, we conclude that

$$\begin{aligned} R_Y(f) &= -\mathbf{F}^{-1}\left\{(j2\pi f)^2 S_X(f)\right\} \\ &= -\frac{d^2}{d\tau^2} R_X(\tau) \end{aligned}$$

(b) Let $Y(t) = \frac{dX}{dt}(t)$, then the spectrum of $Y(t)$ is given by (see Section 8.8)

$$S_Y(f) = \lim_{T\to\infty} \frac{1}{2T}\mathbf{E}\left[\left|H_T^Y(f)\right|^2\right]$$

where $H_T^Y(f)$ is the Fourier transform of $Y(t)$ from $-T$ to $+T$. By the properties of Fourier transforms $H_T^Y(f) = (j2\pi f)H_T^X(f)$ so we have

$$\begin{aligned} S_Y(f) &= \lim_{T\to\infty} \frac{1}{2T}\mathbf{E}\left[\left|H_T^Y(f)\right|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T}\mathbf{E}\left[\left|(j2\pi f)H_T^X(f)\right|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T}\left(4\pi^2 f^2\right)\mathbf{E}\left[\left|H_T^X(f)\right|^2\right] \\ &= 4\pi^2 f^2 S_X(f) \end{aligned}$$

Note that the expectation occurs at a particular value of *f*; frequency plays the role of an index into a family of random variables.

Problem 8.35 Consider a wide-sense stationary process $X(t)$ having the power spectral density $S_X(f)$ shown in Fig. 8.26. Find the autocorrelation function $R_X(\tau)$ of the process $X(t)$.

<u>Solution</u>

The Wiener-Khintchine relations imply the autocorrelation is given by the inverse Fourier transform of the power spectral density, thus

$$
\begin{aligned}
R(\tau) &= \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f t) df \\
&= \int_0^1 (1-f) \cos(2\pi f t) df
\end{aligned}
$$

where we have used the symmetry properties of the spectrum to obtain the second line. Integrating by parts, we obtain

$$
\begin{aligned}
R_X(\tau) &= (1-f)\frac{\sin(2\pi f\tau)}{2\pi\tau}\bigg|_0^1 + \int_0^1 \frac{\sin(2\pi f\tau)}{2\pi\tau} df \\
&= 0 + \frac{-\cos(2\pi f\tau)}{(2\pi\tau)^2}\bigg|_0^1 \\
&= \frac{1-\cos(2\pi\tau)}{(2\pi\tau)^2}
\end{aligned}
$$

Using the half-angle formula $\sin^2(\theta) = ½(1-\cos(2\theta))$, this result simplifies to

$$
\begin{aligned}
R_X(\tau) &= \frac{2\sin^2(\pi\tau)}{(2\pi\tau)^2} \\
&= \tfrac{1}{2}\,\mathrm{sinc}^2(\tau)
\end{aligned}
$$

where $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$.

Problem 8.36 The power spectral density of a random process $X(t)$ is shown in Fig. 8.27.

(a) Determine and sketch the autocorrelation function $R_X(\tau)$ of the $X(t)$.
(b) What is the dc power contained in $X(t)$?
(c) What is the ac power contained in $X(t)$?
(d) What sampling rates will give uncorrelated samples of X(t)? Are the samples statistically independent?

<u>Solution</u>

(a) Using the results of Problem 8.35, and the linear properties of the Fourier transform

$$R(\tau) = 1 + \tfrac{1}{2}\,\mathrm{sinc}^2(f_0\tau)$$

(b) The *dc* power is given by power centered on the origin

$$\begin{aligned} dc\text{ power} &= \lim_{\delta\to 0}\int_{-\varepsilon}^{\varepsilon} S_X(f)\,df \\ &= \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \delta(f)\,df \\ &= 1 \end{aligned}$$

(c) The *ac* power is the total power minus the dc power

$$\begin{aligned} ac\ power &= R_X(0) - dc\text{ power} \\ &= R_X(0) - 1 \\ &= \tfrac{1}{2} \end{aligned}$$

(d) The correlation function $R_X(\tau)$ is zero if samples are spaced at multiples of $1/f_0$.

Problem 8.37 Consider the two linear filters shown in cascade as in Fig. 8.28. Let $X(t)$ be a stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.

(a) Find the autocorrelation function of $V(t)$.
(b) Find the autocorrelation function of $Y(t)$.

Solution

Expressing the first filtering operation in the frequency domain, we have

$$V(f) = H_1(f)X(f)$$

where $H_1(f)$ is the Fourier transform of $h_1(t)$. From Eq. (8.87) it follows that the spectrum of $V(t)$ is

$$S_V(f) = |H_1(f)|^2 S_X(f)$$

By analogy, we have

$$S_Y(f) = |H_2(f)|^2 S_V(f)$$

$$= |H_2(f)|^2 |H_1(f)|^2 S_X(f)$$

Consequently, apply the convolution properties of the Fourier transform, we have

$$R_Y(\tau) = g_2(t) * g_1(t) * R_X(f)$$

where * denotes convolution; $g_2(t)$ and $g_1(t)$ are the inverse Fourier transforms of $|H_2(f)|^2$ and $|H_1(f)|^2$, respectively.

Problem 8.38 The power spectral density of a narrowband random process $X(t)$ is as shown in Fig. 8.29. Find the power spectral densities of the in-phase and quadrature components of $X(t)$, assuming $f_c = 5$ Hz.

Solution

From Section 8.11, the power spectral densities of the in-phase and quadrature components are given by

$$S_{N_1}(f) = S_{N_0}(f) = \begin{cases} S(f + f_c) + S(f - f_c) & |f| < B \\ 0 & 0 \geq B \end{cases}$$

Evaluating this expression for Fig. 8.29, we obtain

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} 1 - \dfrac{|f|}{2} & 1 < |f| < 2 \\ \left(2 - 3\dfrac{|f|}{2}\right) & 0 < |f| < 1 \\ 0 & otherwise \end{cases}$$

Problem 8.39 Assume the narrow-band process $X(t)$ described in Problem 8.38 is Gaussian with zero mean and variance σ_X^2.

(a) Calculate σ_X^2.
(b) Determine the joint probability density function of the random variables Y and Z obtained by observing the in-phase and quadrature components of $X(t)$ at some fixed time.

Solution

(a) The variance is given by

$$\begin{aligned} \sigma_X^2 &= R(0) = \int_{-\infty}^{\infty} S(f)\,df \\ &= 2\left(\frac{1}{2}b_1h_1 + \frac{1}{2}b_2h_2\right) \\ &= 2\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 1\right) \\ &= 3\ watts \end{aligned}$$

(b) The random variables Y and Z have zero mean, are Gaussian and have variance σ_X^2. If Y and Z are independent, the joint density is given by

$$f_{Y,Z}(Y,Z) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-y^2 / 2\sigma_X^2\right) \cdot \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-z^2 / 2\sigma_X^2\right)$$
$$= \frac{1}{2\pi\sigma_X^2}\exp\left(-\frac{y^2+z^2}{2\sigma_X^2}\right)$$

Problem 8.40 Find the probability that the last two digits of the cube of a natural number (1, 2, 3, …) will be 01.

Solution

Let a natural number be represented by concatenation xy where y represents last two digits and x represents the other digits. For example, the number 1345 has $y = 45$ and $x = 13$. Then

$$(xy)^3 = (x00 + y)^3 = (x^3 000000) + 3(x^2 0000)y + 3(x00)y^2 + y^3$$

where we have used the binomial expansion of $(a+b)^3$. The last digits of the first three terms on the right are clearly 00. Consequently, it is the last two digits of y^3 which determines the last two digits of $(xy)^3$. Checking the cube of all two digit numbers for 00 to 99, we find that: (a) y^3 ends in 1, only if y ends in 1; and (b) only the number $(01)^3$ gives 01 as the last two digits. From this counting argument, the probability is 0.01.

Problem 8.41 Consider the random experiment of selecting a number uniformly distributed over the range {1, 2, 3, …, 120}. Let A, B, and C be the events that the selected number is a multiple of 3, 4, and 6, respectively.

a) What is the probability of event A, i.e. **P**[A]?
b) What is **P**[B]?
c) What is $\mathbf{P}[A \cap B]$?
d) What is $\mathbf{P}[A \cup B]$?
e) What is $\mathbf{P}[A \cap C]$?

Solution

(a) From a counting argument, $\mathbf{P}(A) = \frac{40}{120} = \frac{1}{3}$

(b) $\mathbf{P}(B) = \frac{30}{120} = \frac{1}{4}$

(c) $\mathbf{P}(A \cap B) = \frac{12}{120} = \frac{1}{10}$

(d)

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$
$$= 1/3 + 1/4 - 1/10 = \frac{20 + 15 - 6}{60} = \frac{29}{60}$$

(e) $\mathbf{P}(A \cap C) = \mathbf{P}(C) = \frac{20}{120} = \frac{1}{6}$

Problem 8.42 A message consists of ten "0"s and "1"s.

a) How many such messages are there?
b) How many such messages are there that contain exactly four "1"s?
c) Suppose the 10th bit is not independent of the others but is chosen such that the modulo-2 sum of all the bits is zero. This is referred to as an even parity sequence. How many such even parity sequences are there?
d) If this ten-bit even-parity sequence is transmitted over a channel that has a probability of error p for each bit. What is the probability that the received sequence contains an undetected error?

<u>Solution</u>

(a) A message corresponds to a binary number of length 10, there are thus 2^{10} possibilities.

(b) The number of messages with four "1"s is

$$\binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} = 10 \times 3 \times 7 = 210$$

(c) Since there are only 9 independent bits in this case, the number of such message is 2^9.

(d) The probability of an undetected error corresponds to the probability of 2, 4, 6, 8, or 10 errors. The received message corresponds to a Bernoulli sequence, so the corresponding error probabilities are given by the binomial distribution and is

$$\binom{10}{2} p^2 (1-p)^8 + \binom{10}{4} p^4 (1-p)^6 + \binom{10}{6} p^6 (1-p)^4 + \binom{10}{8} p^8 (1-p)^2 + \binom{10}{10} p^{10}$$

Problem 8.43 The probability that an event occurs at least once in four independent trials is equal to 0.59. What is the probability of occurrence of the event in one trial, if the probabilities are equal in all trials?

<u>Solution</u>

The probability that the event occurs on a least one trial is 1 minus the probability that the event does not occur at all. Let p be the probability of occurrence on a single trial, so 1-p is the probability of not occurring on a single trial. Then

$$\mathbf{P}[\text{at least one occurence}] = 1 - (1 - p)^4$$
$$0.59 = 1 - (1 - p)^4$$

Solving for p gives a probability on a single trial of 0.20.

Problem 8.44 The arrival times of two signals at a receiver are uniformly distributed over the interval [0,T]. The receiver will be jammed if the time difference in the arrivals is less than τ. Find the probability that the receiver will be jammed.

<u>Solution</u>

Let X and Y be random variables representing the arrival times of the two signals. The probability density functions of the random variables are

$$f_X(x) = \begin{cases} \dfrac{1}{T} & 0 < x < T \\ 0, & \text{otherwise} \end{cases}$$

and $f_Y(y)$ is similarly defined. Then the probability that the time difference between arrivals is less than τ is given by

$$\begin{aligned} \mathbf{P}[|X - Y| < \tau] &= \mathbf{P}[|X - Y| < \tau \mid X > Y]\,\mathbf{P}[X > Y] + \mathbf{P}[|X - Y| < \tau \mid Y > X]\,\mathbf{P}[Y > X] \\ &= \mathbf{P}[|X - Y| < \tau \mid X > Y] \end{aligned}$$

where the second line follows from the symmetry between the random variables X and Y, namely, $\mathbf{P}[X > Y] = \mathbf{P}[Y > X]$. If we only consider the case $X > Y$, then we have the conditions: $0 < X < T$ and $0 < Y < X < \tau + Y$. Combining these conditions we have $Y < X < \min(T, \tau + Y)$. Consequently,

$$\mathbf{P}[|X - Y| < \tau] = \int_0^T \int_y^{\min(T,\tau+y)} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_0^T \int_y^{\min(T,\tau+y)} \left(\frac{1}{T}\right)^2 dx\,dy$$

$$= \frac{1}{T^2} \int_0^T \{\min(T, \tau + y) - y\}\,dy$$

Combining the two terms of the integrand,

$$P\left[|X-Y|<\tau\right] = \frac{1}{T^2}\int_0^T \min(T-y,\tau)\,dy$$
$$= \frac{1}{T^2}\min\left(Ty-\frac{y^2}{2}, \tau y\right)\Big|_0^T$$
$$= \min\left(\frac{1}{2}, \frac{\tau}{T}\right)$$

Problem 8.45 A telegraph system (an early version of digital communications) transmits either a dot or dash signal. Assume the transmission properties are such that 2/5 of the dots and 1/3 of the dashes are received incorrectly. Suppose the ratio of transmitted dots to transmitted dashes is 5 to 3. What is the probability that a received signal as the transmitted if:

a) The received signal is a dot?
b) The received signal is a dash?

Solution

(a) Let X represent the transmitted signal and Y represent the received signal. Then by application of Bayes' rule

$$\mathbf{P}(Y=\text{dot}) = \mathbf{P}(X=\text{dot}\,|\,\text{No error})\mathbf{P}(\text{No dot error}) + \mathbf{P}(X=\text{dash}\,|\,\text{error})\mathbf{P}(\text{dash error})$$
$$= \tfrac{5}{8}\left(\tfrac{3}{5}\right) + \left(\tfrac{3}{8}\right)\left(\tfrac{1}{3}\right)$$
$$= \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{1}{2}$$

(b) Similarly,

$$\mathbf{P}[Y=\text{dash}] = \mathbf{P}[X=\text{dash}\,|\,\text{no error}]\mathbf{P}(\text{no dash error}) + \mathbf{P}(X=\text{dot})\mathbf{P}[\text{dot error}]$$
$$= \tfrac{3}{8}\cdot\tfrac{2}{3} + \left[\tfrac{5}{8}\right]\tfrac{2}{5}$$
$$= \tfrac{2}{8} + \tfrac{2}{8} = \tfrac{1}{2}$$

Problem 8.46 Four radio signals are emitted successively. The probability of reception for each of them is independent of the reception of the others and equal, respectively, 0.1, 0.2, 0.3 and 0.4. Find the probability that k signals will be received where k = 1, 2, 3, 4.

Solution

For one successful reception, the probability is given by the sum of the probabilities of the four mutually exclusive cases

$$
\begin{aligned}
P = {} & p_1(1-p_2)(1-p_3)(1-p_4)+ \\
& (1-p_1)p_2(1-p_3)(1-p_4)+ \\
& (1-p_1)(1-p_2)p_3(1-p_4)+ \\
& (1-p_1)(1-p_2)(1-p_3)p_4 \\
= {} & .1\cdot.8\cdot.7\cdot.6+.9\cdot.2\cdot.7\cdot.6+.9\cdot.8\cdot.3\cdot.6+.9\cdot.8\cdot.7\cdot.4 \\
= {} & 0.4404
\end{aligned}
$$

For $k = 2$, there six mutually exclusive cases

$$
\begin{aligned}
P = {} & p_1p_2(1-p_3)(1-p_4)+ \\
& p_1(1-p_2)p_3(1-p_4)+ \\
& p_1(1-p_2)(1-p_3)p_4+ \\
& (1-p_1)p_2p_3(1-p_4)+ \\
& (1-p_1)p_2(1-p_3)p_4+ \\
& (1-p_1)(1-p_2)p_3p_4 \\
= {} & .1\cdot.2\cdot.7\cdot.6+.1\cdot.8\cdot.3\cdot.6+.1\cdot.8\cdot.7\cdot.4+.9\cdot.2\cdot.3\cdot.6+.9\cdot.2\cdot.7\cdot.4+.9\cdot.8\cdot.3\cdot.4 \\
= {} & 0.2144
\end{aligned}
$$

For k =3 there are four mutually exclusive cases

$$
\begin{aligned}
P = {} & p_1p_2p_3(1-p_4)+ \\
& p_1(1-p_2)p_3p_4+ \\
& p_1p_2(1-p_3)p_4+ \\
& (1-p_1)p_2p_3p_4 \\
= {} & .1\cdot.2\cdot.3\cdot.6+.1\cdot.8\cdot.3\cdot.4+.1\cdot.2\cdot.7\cdot.4+.9\cdot.2\cdot.3\cdot.4 \\
= {} & 0.0404
\end{aligned}
$$

For k = 4 there is only one term

$$
\begin{aligned}
P &= p_1p_2p_3p_4 \\
&= .1\cdot.2\cdot.3\cdot.4 \\
&= 0.0024
\end{aligned}
$$

Problem 8.47 In a computer-communication network, the arrival time τ between messages is modeled with an exponential distribution function, having the density

$$
f_T(\tau)=\begin{cases}\dfrac{1}{\lambda}e^{-\lambda\tau} & \tau\ge 0\\ 0 & otherwise\end{cases}
$$

a) What is the mean time between messages with this distribution?
b) What is the variance in this time between messages?

Solution (Typo in problem statement, should read $f_T(\tau)=(1/\lambda)\exp(-\tau/\lambda)$ for $\tau>0$)
(a) The mean time between messages is

$$\begin{aligned}
\mathbf{E}[T] &= \int_0^\infty \tau f_T(\tau)d\tau \\
&= \int_0^\infty \frac{\tau}{\lambda}\exp(-\tau/\lambda)d\tau \\
&= -\tau\exp(-\tau/\lambda)\Big|_0^\infty + \int_0^\infty \exp(-\tau/\lambda)d\tau \\
&= 0 - \lambda\exp(-\tau/\lambda)\Big|_0^\infty \\
&= \lambda
\end{aligned}$$

where the third line follows by integration by parts.

(b) To compute the variance, we first determine the second moment of T

$$\begin{aligned}
\mathbf{E}[T^2] &= \int_0^\infty \tau^2 f_T(\tau)d\tau \\
&= \int_0^\infty \frac{\tau^2}{\lambda}\exp(-\tau/\lambda)d\tau \\
&= -\tau^2\exp(-\tau/\lambda)\Big|_0^\infty + 2\int_0^\infty \tau\exp(-\tau/\lambda)d\tau \\
&= 0 + 2\lambda\mathbf{E}[T] \\
&= 2\lambda^2
\end{aligned}$$

The variance is then given by the difference of the second moment and the first moment squared (see Problem 8.23)

$$\begin{aligned}
\mathrm{Var}(T) &= \mathbf{E}[T^2] - (\mathbf{E}[T])^2 \\
&= 2\lambda^2 - \lambda^2 \\
&= \lambda^2
\end{aligned}$$

Problem 8.48 If X has a density $f_X(x)$, find the density of Y where

a) $Y = aX + b$ for constants a and b.

b) $Y = X^2$.

c) $Y = \sqrt{X}$, assuming X is a non-negative random variable.

Solution

(a) If $Y = aX + b$, using the results of Section 8.3 for $Y = g(X)$

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right|$$
$$= f_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

(b) If $Y = X^2$, then

$$f_Y(y) = \left(f_X(\sqrt{y}) + f_X(\sqrt{y})\right)\left(\frac{1}{2\sqrt{y}}\right)$$

(c) If $Y = \sqrt{X}$, then we must assume X is positive valued so, this is a one-to-one mapping and

$$f_Y(y) = f_X\left(y^2\right)2y$$

Problem 8.49 Let X and Y be independent random variables with densities $f_X(x)$ and $f_Y(y)$, respectively. Show that the random variable $Z = X+Y$ has a density given by

$$f_Z(z) = \int_{-\infty}^{z} f_Y(z-s) f_X(s)ds$$

Hint: $\mathbf{P}[Z \le z] = \mathbf{P}[X \le z, Y \le z - X]$

Solution (*Typo in problem statement - should be "positive" independent random variables*)

Using the hint, we have that $F_Z(z) = \mathbf{P}[Z \le z]$ and

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{z-x} f_X(x) f_Y(y)dydx$$

To differentiate this result with respect to z, we use the fact that if

$$g(z) = \int_a^b h(x,z)dx$$

then

$$\frac{\partial g(z)}{dz} = \int_a^b \frac{\partial}{\partial z} h(x,z)dx + h(b,z)\frac{db}{dz} - h(a,z)\frac{da}{dz} \qquad (1)$$

Inspecting $F_Z(z)$, we identify $h(x,z)$

$$h(x,z) = \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy$$

and $a = -\infty$ and $b = z$. We then obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \int_{-\infty}^{z}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f_X(x) f_Y(y) dy\right]dx + \int_{-\infty}^{z-z} f_X(z) f_Y(y) dy \frac{dz}{dz} - \int_{-\infty}^{z-(-\infty)} f_X(-\infty) f_Y(y) dy \cdot 0 \\ &= \int_{-\infty}^{z}\left[\frac{d}{dz}\int_{-\infty}^{z-x} f_X(x) f_Y(y) dy\right]dx \end{aligned}$$

where the second term of the second line is zero since the random variables are positive, and the third term is zero due to the factor zero. Applying the differentiation rule a second time, we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{z}\left[0 + f_X(x) f_Y(z-x)\frac{d(z-x)}{dz} - f_X(x) f_Y(-\infty)\frac{d(-\infty)}{dz}\right]dx \\ &= \int_{-\infty}^{z} f_X(x) f_Y(z-x) dx \end{aligned}$$

which is the desired result.

An alternative solution is the following: we note that

$$\begin{aligned} \mathbf{P}[Z \le z \mid X = x] &= \mathbf{P}[X + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z \mid X = x] \\ &= \mathbf{P}[x + Y \le z] \\ &= \mathbf{P}[Y \le z - x] \end{aligned}$$

where the third equality follows from the independence of X and Y. By differentiating both sides with respect to z, we see that

$$f_{Z|X}(z\,|\,x) = f_Y(z-x)$$

By the properties of conditional densities

$$f_{Z,X}(z,x) = f_X(x)\,f_{Z|X}(z\,|\,x) = f_X(x)\,f_Y(z-x)$$

Integrating to form the marginal distribution, we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\,f_Y(z-x)\,dx$$

If Y is a positive random variable then $f_Y(z\text{-}x)$ is zero for $x > z$ and the desired result follows.

Problem 8.50 Find the spectral density $S_Z(f)$ if

$$Z(t) = X(t)Y(t)$$

where $X(t)$ and $Y(t)$ are independent zero-mean random processes with

$$R_X(\tau) = a_1 e^{-\alpha_1|\tau|} \quad \text{and} \quad R_Y(\tau) = a_2 e^{-\alpha_2|\tau|}.$$

Solution

The autocorrelation of $Z(t)$ is given by

$$\begin{aligned} R_Z(\tau) &= \mathbf{E}\left[Z(t)Z(t+\tau)\right] \\ &= \mathbf{E}\left[X(t)X(t+\tau)Y(t)Y(t+\tau)\right] \\ &= \mathbf{E}\left[X(t)X(t+\tau)\right]\mathbf{E}\left[Y(t)Y(t+\tau)\right] \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

By the Wiener-Khintchine relations, the spectrum of $Z(t)$ is given by

$$\begin{aligned} S_Z(f) &= \mathbf{F}^{-1}\left[R_X(\tau)R_Y(\tau)\right] \\ &= \mathbf{F}^{-1}\left[a_1 a_2 \exp\left(-(\alpha_1+\alpha_2)|\tau|\right)\right] \\ &= \frac{2a_1a_2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)^2 + (2\pi f)^2} \end{aligned}$$

where the last line follows from the Fourier transform of the double-sided exponential (See Example 2.3).

Problem 8.51 Consider a random process $X(t)$ defined by

$$X(t) = \sin(2\pi f_c t)$$

where the frequency f_c is a random variable uniformly distributed over the interval $[0, W]$. Show that $X(t)$ is nonstationary. *Hint*: Examine specific sample functions of the random process $X(t)$ for, say, the frequencies $W/4$, $W/2$, and W.

Solution

To be stationary to first order implies that the mean value of the process $X(t)$ must be constant and independent of t. In this case,

$$\begin{aligned}
\mathbf{E}[X(t)] &= \mathbf{E}[\sin(2\pi f_c t)] \\
&= \frac{1}{W}\int_0^W \sin(2\pi w t)\,dw \\
&= \frac{-\cos(2\pi w t)}{2\pi W t}\bigg|_0^W \\
&= \frac{1-\cos(2\pi W t)}{2\pi W t}
\end{aligned}$$

This mean value clearly depends on t, and thus the process $X(t)$ is nonstationary.

Problem 8.52 The oscillators used in communication systems are not ideal but often suffer from a distortion known as phase noise. Such an oscillator may be modeled by the random process

$$Y(t) = A\cos(2\pi f_c t + \phi(t))$$

where $\phi(t)$ is a slowly varying random process. Describe and justify the conditions on the random process $\phi(t)$ such that $Y(t)$ is wide-sense stationary.

Solution

The first condition for wide-sense stationary process is a constant mean. Consider $t = t_0$, then

$$\mathbf{E}[Y(t_0)] = \mathbf{E}[A\cos(2\pi f_c t_0 + \phi(t_0))]$$

In general, the function $\cos\theta$ takes from values -1 to +1 when θ varies from 0 to 2π. In this case θ corresponds to $2\pi f_c t_0 + \phi(t_0)$. If $\phi(t_0)$ varies only by a small amount then θ will be biased toward the point $2\pi f_c t_0 + \mathbf{E}[\phi(t_0)]$, and the mean value of $\mathbf{E}[Y(t_0)]$ will depend upon the choice of t_0. However, if $\phi(t_0)$ is uniformly distributed over $[0, 2\pi]$ then $2\pi f_c t_0 + \phi(t_0)$ will be uniformly distributed over $[0, 2\pi]$ when considered modulo 2π, and the mean $\mathbf{E}[Y(t_0)]$ will be zero and will not depend upon t_0.

Thus the first requirement is that $\phi(t)$ must be uniformly distributed over $[0,2\pi]$ for all t.

The second condition for a wide-sense stationary $Y(t)$ is that the autocorrelation depends only upon the time difference

$$\begin{aligned}\mathbf{E}\left[Y(t_1)Y(t_2)\right] &= \mathbf{E}\left[A\cos\left(2\pi f_c t_1 + \phi(t_1)\right)A\cos\left(2\pi f_c t_2 + \phi(t_2)\right)\right] \\ &= \frac{A^2}{2}\mathbf{E}\left[\cos\left(2\pi f_c(t_1+t_2) + \phi(t_1) + \phi(t_2)\right) + \cos\left(2\pi f_c(t_1-t_2) + \phi(t_1) - \phi(t_2)\right)\right]\end{aligned}$$

where we have used the relation $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$. In general, this correlation does not depend solely on the time difference t_2-t_1. However, if we assume:

We first note that if $\phi(t_1)$ and $\phi(t_2)$ are both uniformly distributed over $[0,2\pi]$ then so is $\psi = \phi(t_1) + \phi(t_2)$ (modulo 2π), and

$$\begin{aligned}\mathbf{E}\left[\cos\left(2\pi f_c(t_1+t_2) + \psi\right)\right] &= \frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_c(t_1+t_2)+\psi)\,d\psi \\ &= 0\end{aligned} \tag{1}$$

We consider next the term $R_Y(t_1,t_2) = \mathbf{E}\left[\cos\left(2\pi f_c(t_1-t_2) + \phi(t_1) - \phi(t_2)\right)\right]$ and three special cases:

(a) if $\Delta t = t_1 - t_2$ is small then $\phi(t_1) \approx \phi(t_2)$ since $\phi(t)$ is a slowly varying process, and

$$R_Y(t_1,t_2) = \frac{A^2}{2}\cos\left(2\pi f_c(t_1-t_2)\right)$$

(b) if Δt is large then $\phi(t_1)$ and $\phi(t_2)$ should be approximately independent and $\phi(t_1) - \phi(t_2)$ would be approximately uniformly distributed over $[0,2\pi]$. In this case

$$R_Y(t_1,t_2) \approx 0$$

using the argument of Eq. (1).

(c) for intermediate values of Δt, we require that

$$\phi(t_1) - \phi(t_2) \approx g(t_1 - t_2)$$

for some arbitrary function $g(t)$.

Under these conditions the random process $Y(t)$ will be wide-sense stationary.

Problem 8.53 A baseband signal is disturbed by a noise process $N(t)$ as shown by

$$X(t) = A\sin(0.3\pi t) + N(t)$$

where $N(t)$ is a stationary Gaussian process of zero mean and variance σ^2.

(a) What are the density functions of the random variables X_1 and X_2 where

$$X_1 = X(t)\big|_{t=1}$$

$$X_2 = X(t)\big|_{t=2}$$

(b) The noise process $N(t)$ has an autocorrelation function given by

$$R_N(\tau) = \sigma^2 \exp(-|\tau|)$$

What is the joint density function of X_1 and X_2, that is, $f_{X_1,X_2}(x_1, x_2)$?

Solution

(a) The random variable X_1 has a mean

$$\begin{aligned} \mathbf{E}[X(t_1)] &= \mathbf{E}[A\sin(0.3\pi) + N(t_1)] \\ &= A\sin(0.3\pi) + \mathbf{E}[N(t_1)] \\ &= A\sin(0.3\pi) \end{aligned}$$

Since X_1 is equal to $N(t_1)$ plus a constant, the variance of X_1 is the same as that of $N(t_1)$. In addition, since $N(t_1)$ is a Gaussian random variable, X_1 is also Gaussian with a density given by

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-(x-\mu_1)/2\sigma^2\right\}$$

where $\mu_1 = \mathbf{E}[X(t_1)]$. By a similar argument, the density function of X_2 is

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-(x-\mu_2)/2\sigma^2\right\}$$

where $\mu_2 = A\sin(0.6\pi)$.

(b) First note that since the mean of $X(t)$ is not constant, $X(t)$ is not a stationary random process. However, $X(t)$ is still a Gaussian random process, so the joint distribution of N Gaussian random variables may be written as Eq. (8.90). For the case of $N = 2$, this equation reduces to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi|\Lambda|^{1/2}} \exp\left\{-(\mathbf{x}-\boldsymbol{\mu})\Lambda^{-1}(\mathbf{x}-\boldsymbol{\mu})^T/2\right\}$$

where Λ is the 2x2 covariance matrix. Recall that cov(X_1,X_2) =**E**[(X_1-μ_1)(X_2-μ_2)], so that

$$\Lambda = \begin{bmatrix} \text{cov}(X_1, X_1) & \text{cov}(X_1, X_2) \\ \text{cov}(X_2, X_1) & \text{cov}(X_2, X_2) \end{bmatrix}$$

$$= \begin{bmatrix} R_N(0) & R_N(1) \\ R_N(1) & R_N(0) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2 \exp(-1) \\ \sigma^2 \exp(-1) & \sigma^2 \end{bmatrix}$$

If we let ρ = exp(-1) then

$$|\Lambda| = \sigma^4 (1 - \rho^2)$$

and

$$\Lambda^{-1} = \frac{1}{\sigma^2 (1 - \rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

Making these substitutions into the above expression, we obtain upon simplification

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2 \sqrt{1 - \rho^2}} \exp\left\{ -\frac{(x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 - 2\rho(x_1 - \mu_1)(x_2 - \mu_2)}{2\sigma^2 (1 - \rho^2)} \right\}$$