Chapter 10 Solutions

Problem 10.1. Let H_0 be the event that a 0 is transmitted and let R_0 be the event that a 0 is received. Define H_1 and R_1, similarly for a 1. Express the BER in terms of the probability of these events when:
(a) The probability of a 1 error is the same as the probability of a 0 error.
(b) The probability of a 1 being transmitted is not the same as the probability of a 0 being transmitted.

Solution
In both cases, the probability of error may be expressed as

$$\mathbf{P}[error] = \mathbf{P}(R_0|H_1)\mathbf{P}(H_1) + \mathbf{P}(R_1|H_0)\mathbf{P}(H_0) \qquad (1)$$

(a) The BER is the same as the $\mathbf{P}[error]$ and with $\mathbf{P}(R_0|H_1) = \mathbf{P}(R_1/H_0) = p$ then

$$\mathbf{P}[error] = p[\mathbf{P}(H_1) + \mathbf{P}(H_0)] = p$$

since $\mathbf{P}(H_1)+\mathbf{P}(H_0) = 1$.

(b) With $\mathbf{P}(H_0) \neq \mathbf{P}(H_1)$, the answer is given by the general result of Eq. (1).

Problem 10.2. Suppose that in Eq. (10.4), $r(t)$ represents a complex baseband signal instead of a real signal. What would be the ideal choice for g(t) in this case? Justify your answer.

Solution
Inspecting the Schwarz inequality of Eq. (10.12), we see that equality is achieved with

$$g(T-t) = cs^*(t)$$

if $s(t)$ is complex.

Problem 10.3 If $g(t) = c\ \mathrm{rect}\left[\frac{\alpha(t-T/2)}{T}\right]$, determine c such g(t) satisfies Eq. (10.10) where $\alpha > 1$.

Solution
From the definition of the rect(.) function,

$$g(t) = c\ \text{rect}\left(\frac{\alpha(t - T/2)}{T}\right)$$
$$= \begin{cases} c & |t - T/2| < T/(2\alpha) \\ 0 & \text{otherwise} \end{cases}$$

Substituting this into Eq. (10.10)

$$T = \int_0^T |g\ \ t\ |^2 dt$$
$$= c^2 \int_{T/2-T/(2\alpha)}^{T/2+T/(2\alpha)} 1^2 dt$$
$$= c^2 T/\alpha$$

And so $c = \sqrt{\alpha}$.

Problem 10.4. Show that with on-off signaling, the probability of a Type II error in Eq.(10.23) is given by

$$P[Y > \gamma \mid H_0] = Q\left(\frac{\gamma}{\sigma}\right)$$

<u>Solution</u>

A Type II error probability is

$$P[Y > \gamma \mid H_0] = \frac{1}{\sqrt{2\pi}\sigma} \int_\gamma^{+\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

Let $s = \dfrac{y}{\sigma}$, and then

$$P[Y > \gamma \mid H_0] = \frac{1}{\sqrt{2\pi}} \int_{\gamma/\sigma}^{+\infty} \exp\left(-\frac{s^2}{2}\right) ds = Q\left(\frac{\gamma}{\sigma}\right)$$

using the definition of the Q-function given in Section 8.4.

Problem 10.5 Prove the property of root-raised cosine pulse shape $p(t)$ given by Eq. (10.32), using the following steps:

(a) If $R(f)$ is the Fourier transform representation of $p(t)$, what is the Fourier transform representation of $p(t\text{-}lT)$?

(b) What is the Fourier transform of $q(\tau) = \int_{-\infty}^{\infty} p(\tau - t)p(t - lT)dt$? What spectral shape does it have?

(c) What $q(\tau)$? What is $q(kT)$?

Use these results to show that Eq. (10.32) holds.

Solution

(a) From the time-shifting property of Fourier transforms (see Section 2.2), we have that

$$\mathbf{F}\left[p(t-lT)\right] = R(f)\exp\left(-j2\pi flT\right)$$

(b) From the convolution property of Fourier transforms (See Section 2.2) we have that

$$\begin{aligned} Q(f) &= \mathbf{F}\left[q(\tau)\right] \\ &= \mathbf{F}\left[p(t)\right]\mathbf{F}\left[p(t-lT)\right] \\ &= R^2(f)\exp\left(-j2\pi flT\right) \end{aligned}$$

(c) Since $R(f)$ is the root-raised cosine spectrum, $R^2(f)$ is the raised cosine spectrum and so $q(\tau)$ corresponds to a raised cosine pulse. In particular, using the time-shifting property of inverse Fourier transforms

$$q(\tau) = m(\tau - lT)$$

where $m(\tau)$ is the raised cosine pulse shape. Using the properties of the raised cosine pulse shape (see Section 6.4)

$$\begin{aligned} q(kT) &= m(kT - lT) \\ &= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \\ &= \delta(k-l) \end{aligned}$$

and Eq. (10.32) holds.

Problem 10.6 Compare the transmission bandwidth required for binary PAM and BPSK modulation, if both signals have a data rate of 9600 bps and use root-raised cosine pulse spectrum with a roll-off factor of 0.5.

Solution

For BPSK modulation (bandpass signal), the transmission bandwidth is $B_T = 2\times\frac{1+\beta}{2T}$, where β is the roll-off factor (0.5) and T is the symbol duration (1/9600 sec). Therefore, B_T = (1+0.5)×9600 = 14.4 kHz.

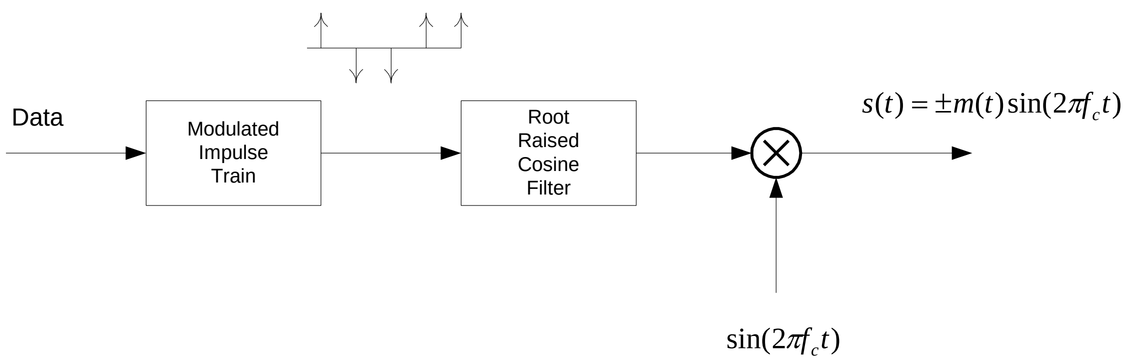
For binary PAM modulation (baseband signal), $B_T = \frac{1+\beta}{2T}$ = 7.2 kHz.

Problem 10.7 Sketch a block diagram of a transmission system including both transmitter and receiver for BPSK modulation with root-raised cosine pulse shaping.

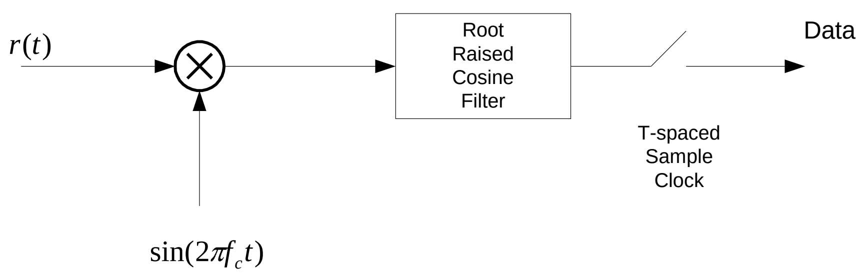
Solution

The BPSK transmitter with root-raised cosine pulse shaping is shown in (a), and the corresponding BPSK receiver is shown in (b).

(a)



(b)



Problem 10.8 Show that the integral of the high frequency term in Eq. (10.53) is approximately zero.

Solution

Consider the integral over the period from 0 to T of the high frequency term in Eq. (10.53):

$$\int_0^T \frac{A_C^2}{2}\cos\left(4\pi f_c t + 2\phi(t)\right) dt = \frac{A_c^2}{8\pi f_c}\sin\left(4\pi f_c t + 2\phi(t)\right)\Big|_0^T$$
$$= \frac{A_c^2}{8\pi f_c}\left[\sin\left(4\pi f_c T + 2\phi(T)\right) - \sin\left(2\phi(0)\right)\right]$$
$$< \frac{A_c^2}{4\pi f_c}$$

where the first line follows since $\phi(t)$ is constant over a symbol interval. By the bandpass assumption $f_c >> 1$, so this last line is small.

Problem 10.9. Use Eqs. (10.61), (10.64), and (10.66) to show that N_1 and N_2 are uncorrelated and therefore independent Gaussian random variables. Compute the variance of N_1-N_2.

Solution

The correlation of N_1 and N_2 is

$$\begin{aligned}
\mathbf{E}\left[N_1 N_2\right] &= \mathbf{E}\left[2\int_0^T\int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right) ds dt\right] \\
&= 2\int_0^T\int_0^T \mathbf{E}\left[w(s)w(t)\right]\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right) ds dt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 s\right) ds\, dt \\
&= N_0\int_0^T \cos\left(2\pi f_1 t\right)\cos\left(2\pi f_2 t\right) dt \\
&= 0
\end{aligned}$$

where the last line follows from Eq.(10.61). Since N_1 and N_2 are uncorrelated

$$\begin{aligned}
\mathbf{E}\left[\left(N_1 - N_2\right)^2\right] &= \mathbf{E}\left[\left(N_1\right)^2\right] + 2\mathbf{E}\left[N_1 N_2\right] + \mathbf{E}\left[\left(N_2\right)^2\right] \\
&= \mathbf{E}\left[\left(N_1\right)^2\right] + \mathbf{E}\left[\left(N_2\right)^2\right]
\end{aligned}$$

The variance of the N_1 term is

$$
\begin{aligned}
\mathbf{E}\left(N_1 N_1\right) &= \mathbf{E}\left[2\int_0^T \int_0^T w(s)w(t)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right) ds dt\right] \\
&= 2\int_0^T \int_0^T \mathbf{E}\left(w(s)w(t)\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right) ds dt \\
&= 2\frac{N_0}{2}\int\int \delta\left(t-s\right)\cos\left(2\pi f_1 t\right)\cos\left(2\pi f_1 s\right) ds\, dt \\
&= N_0 \int_0^T \cos^2\left(2\pi f_0 t\right) dt
\end{aligned}
$$

Using the double angle formula $2\cos^2\theta = 1+2\cos\theta$, we have

$$
\begin{aligned}
\mathbf{E}\left[\left(N_1\right)^2\right] &= \frac{N_0}{2}\int_0^T \left(1+\cos 4\pi f t\right) dt \\
&= \frac{N_0 T}{2}
\end{aligned}
$$

The derivation of the variance of N_2 is similar and the combined variance is $N_0 T$.

Problem 10.10. Plot the BER performance of differential BPSK and compare the results to Fig. 10.16.

Solution

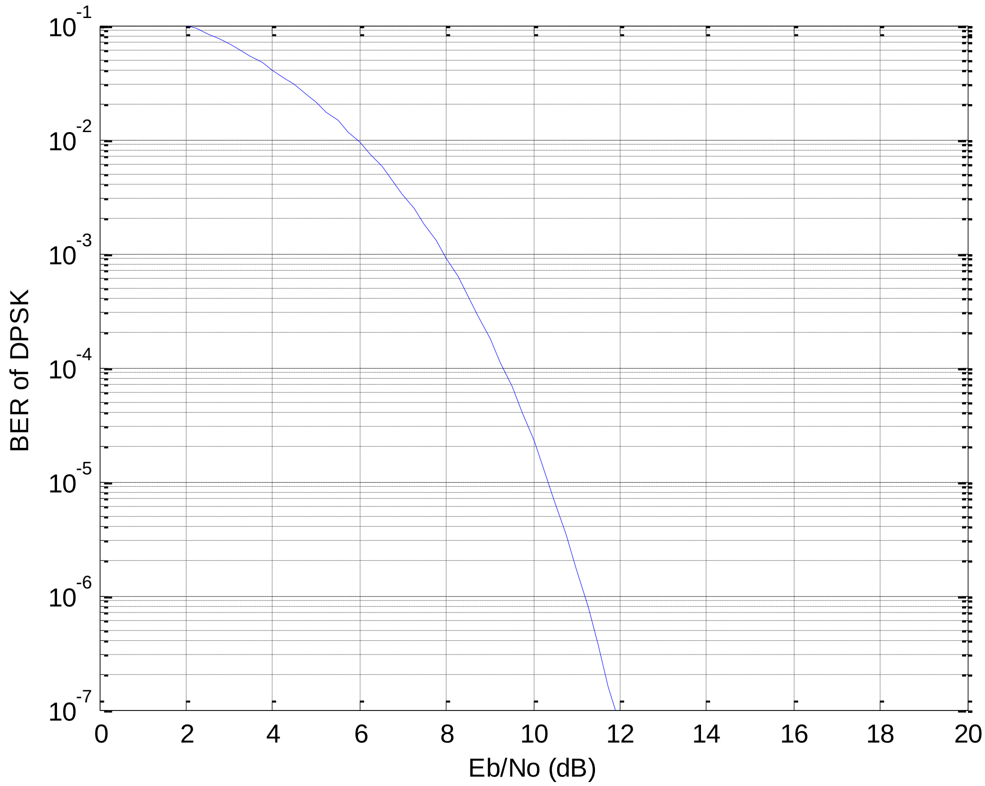
The bit error probability of differential BPSK is (Eq. (10.75))

$$
P_e^{DPSK} = 0.5\exp\left(-\frac{E_b}{N_0}\right).
$$

The following Matlab script plots this performance

```
EbNodB=[0:0.25:12];
EbNo = 10.^(EbNodB/10);
BER = 0.5*exp(-EbNo);
semilogy(EbNodB,BER)
grid
xlabel('Eb/No (dB)')
ylabel('BER of DPSK')
axis([0 20 1E-7 0.1])
```

This script produces the following plot.



The performance of DPSK is slightly worse than BPSK and QPSK. The relative loss with DPSK is less than 1 dB at E_b/N_0 of 8 dB and higher. The loss at lower E_b/N_0 ratios is greater.

Problem 10.11. A communication system that transmits single isolated pulses is subject to multipath such that, if the transmitted pulse is $p(t)$ of length T, the received signal is

$$s(t) = p(t) + \alpha p(t - \tau)$$

Assuming that α and τ are known, determine the optimum receiver filter for signal in the presence of white Gaussian noise of power spectral density $N_0/2$. What is the post-detection SNR at the output of this filter?

Solution

We first note that the pulse is non-zero over the interval $0 \le t \le T + \tau$. From Section 10.2 the appropriate linear receiver is

$$Y = \int_0^{T+\tau} g(T + \tau - u) r(u) du$$

and the optimum choice for $g(t)$ is

$$g(T+\tau-t) = c \;\; p(t) + \alpha p(t-\tau)$$

where c is chosen such that

$$\int_0^{T+\tau} |g(t)|^2 \, dt = T+\tau$$

With this filtering arrangement, if follows from the modified Eq. (10.9) that

$$\mathbf{E}\left[N^2\right] = \frac{N_0 \;\; T+t}{2}$$

The corresponding signal level S is

$$\begin{aligned} S &= c \int_0^{T+\tau} g(T-t) \;\; p(t) + \alpha p(t+\tau) \;\; dt \\ &= c \int_0^{T+\tau} \;\; p(t) + \alpha p(t+\tau) \;\; ^2 \, dt \\ &= T+\tau \end{aligned}$$

which follows from the normalization properties of c. The received signal to noise is then

$$\mathrm{SNR} = \frac{S^2}{\mathbf{E}\left[N^2\right]} = \frac{T+\tau}{N_0/2}$$

Although the units on this expression may appear unusual, note that the units of N_0 are (volt)^2/Hz = (volt)^2-sec. The units of the numerator are also (volt)^2-sec, although the (volt)^2 has been suppressed. Consequently, the SNR is dimensionless, as it should be.

Problem 10.12. The impulse response corresponding to a root-raised cosine spectrum, normalized to satisfy Eq.(10.10), is given by

$$g(t) = \frac{4\alpha}{\pi} \frac{\cos\left[\frac{(1+\alpha)\pi t}{T}\right] + \frac{T}{4\alpha t}\sin\left[\frac{(1-\alpha)\pi t}{T}\right]}{1-\left(\frac{4\alpha t}{T}\right)^2}$$

where $T = 1/2B_0$ is the symbol period and α is the roll-off factor. Obtain a discrete-time representation of this impulse response by sampling it at $t = 0.1nT$ for integer n such that

$-3T < t < 3T$. Numerically approximate match filtering (e.g. with Matlab) by performing the discrete-time convolution

$$q_k = 0.1 \sum_{n=-60}^{60} g_n g_{k-n}$$

where $g_n = g(0.1nT)$. What is the value of $q_k = q(0.1kT)$ for $k = \pm 20$, ± 10, and 0?

Solution

A Matlab script for this problem is shown below. Note the starting time of -3.01 is used to avoid divide-by-zero problems. Using the *filter* function is just one way the discrete convolution can be performed.

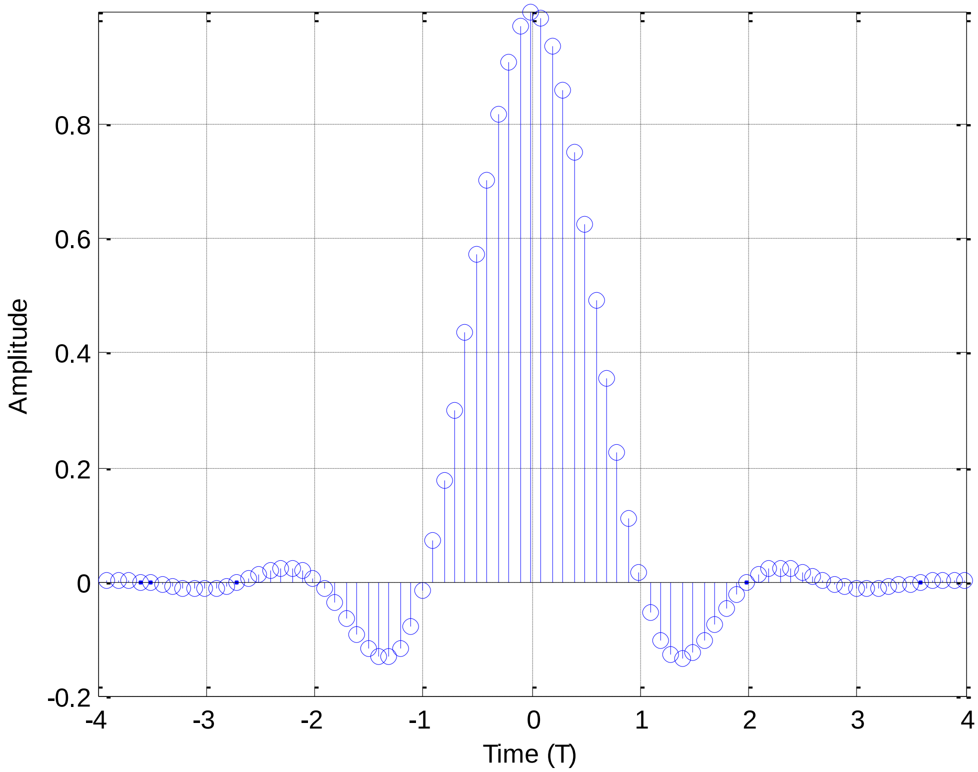
```
alpha = 0.5;
B0    = 0.5;
T     = 1/(2*B0);

t = [-3.01: 0.1 :3] * T;

%-- root raised cosine impulse response
g = cos( (1+alpha)*pi*t/T) + (T/4/alpha) ./ t .* sin( (1-
alpha)*pi*t/T);
g = g ./ (1 - (4*alpha*t/T).^2 );
g = 4*alpha/pi * g;

%--- discrete convolution -----
q = 0.1*filter(g,1, [g zeros(1,60)]);
tp = [-6.01:0.1:6] * T;
stem(tp,q)
xlabel('Time (T)')
ylabel('Amplitude')
axis([-4 4 -.2 1]), grid on
```

The plot of q_k is shown below for $\alpha = 0.5$. At $k = \pm 20$ and ± 10, the amplitude is approximately zero. At $k = 0$ the amplitude is 1.



Problem 10.13. Determine the discrete-time autocorrelation function of the noise sequence $\{N_k\}$ defined by Eq. (10.34)

$$N_k = \int_{-\infty}^{\infty} p(kT - t)w(t)dt$$

where $w(t)$ is a white Gaussian noise process and the pulse $p(t)$ corresponds to a root-raised cosine spectrum. How are the noise samples corresponding to adjacent bit intervals related?

Solution

The autocorrelation function of the noise at samples spaced by T is

$$\begin{aligned} R_N(n) &= \mathbf{E}\;\; N_k N_{k+n} \\ &= \mathbf{E}\left[\int_{-\infty}^{\infty} p(kT-t)w(t)dt \cdot \int_{-\infty}^{\infty} p((k+n)T-s)w(s)ds \right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-s)\mathbf{E}\;\; w(t)w(s)\;\; dtds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-s)\frac{N_0}{2}\delta(t-s)dtds \end{aligned}$$

where we have interchanged integration and expectation on the third line, and the fourth line follows from the uncorrelated properties of the white noise. We next apply the sifting property of the delta function to obtain

$$\begin{aligned} R_N(n) &= \int_{-\infty}^{\infty} p(kT-t)\,p((k+n)T-t)\frac{N_0}{2}\,dt \\ &= \frac{N_0}{2}\int_{-\infty}^{\infty} p(kT-t)\,p(t-(k+n)T)\,dt \\ &= \frac{N_0}{2}\delta(n) \end{aligned}$$

where the second line follows from the even symmetry property of the raised cosine pulse, and third line follows from Eq. (10.32). Therefore, noise samples corresponding to adjacent bit intervals are not correlated.

Problem 10.14. Draw the Gray-encoded constellation (signal-space diagram) for 16-QAM and for 64-QAM. Can you suggest a constellation for 32-QAM?

<u>Solution</u>

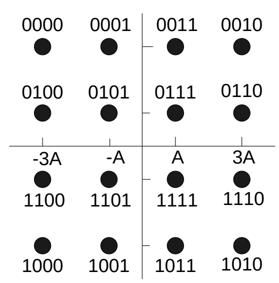
A general hint for Gray encoding is to

(a) first Gray encode two bits and assign one pair of the resulting encoding to each quadrant.

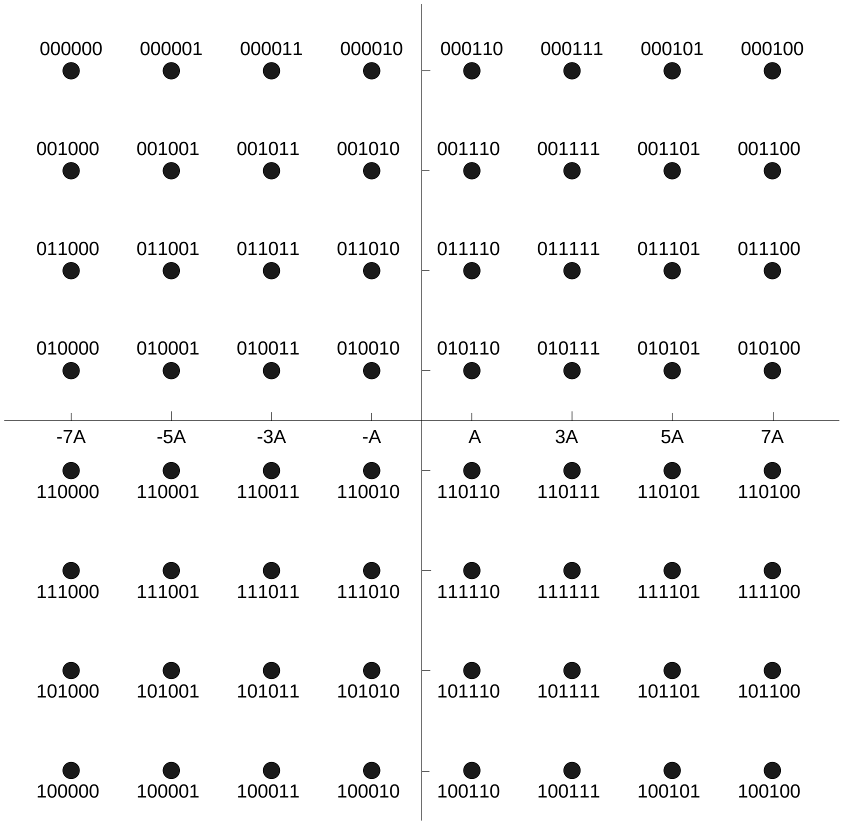
(b) Gray encode the remaining bits within one of the quadrants.

(c) obtain the Gray encodings for the remaining quadrants by reflecting the result across the in-phase and quadrature axes.

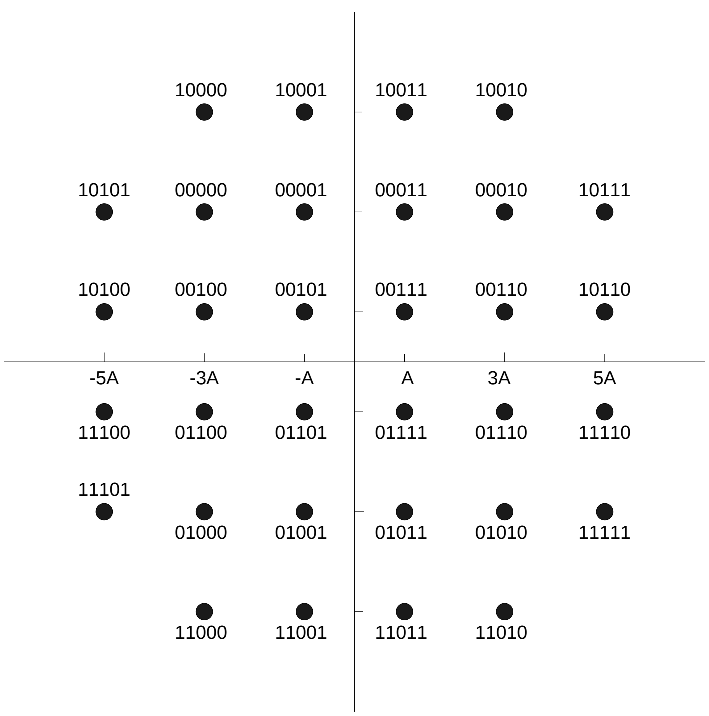
16-QAM constellation:



64-QAM constellation:



32-QAM constellation: (There does not appear to be a Gray encoding for 32-QAM)



Problem 10.15. Write the defining equation for a QAM-modulated signal. Based on the discussion of QPSK and multi-level PAM, draw the block diagram for a coherent QAM receiver.

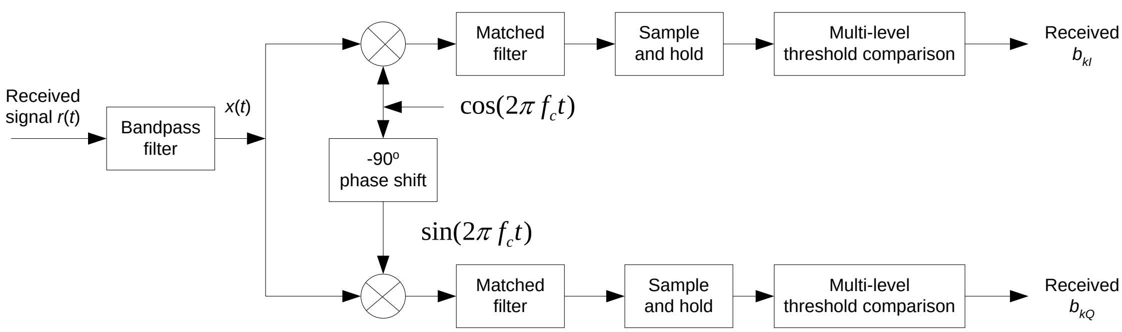
Solution

The QAM modulated signal can be defined as:

$$s(t) = \sum_k \left[b_{kI} h(t-kT)\cos(2\pi f_c t) + b_{kQ} h(t-kT)\sin(2\pi f_c t) \right],$$

where b_{kI} , b_{kQ} are different modulation levels on the I and Q channels, respectively. T is the QAM symbol duration, $h(t)$ is the pulse shape and is nonzero during $0 \le t < T$, and f_c is the carrier frequency.

The block diagram for a coherent QAM receiver is



Problem 10.16. Show that if T is a multiple of the period of f_c, then the terms $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $t_0, T+t_0$.

Solution

$$\int_{t_0}^{T+t_0} \sin(2\pi f_c t)\cos(2\pi f_c t)dt = \int_{t_0}^{T+t_0} \frac{1}{2}\sin(4\pi f_c t)dt$$

$$= \frac{1}{8\pi f_c} \; -\cos(4\pi f_c t) \Big|_{t_0}^{T+t_0}$$

$$= -\frac{1}{8\pi f_c} \; \cos(4\pi f_c (t_0+T)) - \cos(4\pi f_c t_0)$$

$$= \frac{-1}{4\pi f_c}\sin(4\pi f_c t_0 + 2\pi f_c T)\cdot\sin(2\pi f_c T)$$

where we have used the equivalence cos$A$ - cos$B$ = 2sin[(A+B)/2]sin[(B-A)/2)]. If T is a multiple of the period of f_c, then $f_c T$ = integer, and $\sin(2\pi f_c T) = 0$.

Therefore, $\int_{t_0}^{t_0+T} \sin(2\pi f_c t)\cos(2\pi f_c t)dt = 0$. That is, $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$ are orthogonal over the interval $[t_0, t_0+T]$.

Problem 10.17. For a rectangular pulse shape, by how much does null-to-null transmission bandwidth increase, if the transmission rate is increased by a factor of three?

Solution

Without loss of generality, consider the baseband BPSK signal:

$$s(t) = \sum_k b_k h(t - kT),$$

where T is the symbol duration, b_k = +1 or -1 for transmitted 1 or 0, respectively. The pulse $h(t)$ is rectangular,

$$h(t) = \text{rect}\left(\frac{t - T/2}{T}\right).$$

The Fourier transform $H(f)$ of $h(t)$ is

$$\begin{aligned} H(f) &= T\text{sinc}(fT) \cdot e^{-j2\pi fT/2} \\ &= T\frac{\sin(\pi fT)}{\pi fT} e^{-j\pi fT} \end{aligned} .$$

Inspecting a plot of the sinc function, we see the null-to-null transmission bandwidth of $H(f)$ is $B = 2/T$. When the transmission rate is increased by a factor three, we have the new symbol duration $T' = T/3$. The null-to-null bandwidth $B' = 2/T' = 3B$, increased by a factor of 3.

Problem 10.18. Under the bandpass assumptions, determine the conditions under which the two signals $\cos(2\pi f_0 t)$ and $\cos(2\pi f_1 t)$ are orthogonal over the interval from 0 to T.

Solution

For two signals to be orthogonal over the interval from 0 to T, they must satisfy

$$\int_0^T \cos(2\pi f_0 t)\cos(2\pi f_1 t)dt = 0 .$$

To verify this we perform the integration as follows:

$$\int_0^T \cos(2\pi f_0 t)\cos(2\pi f_1 t)dt = \frac{1}{2}\int_0^T \cos(2\pi(f_0+f_1)t) + \cos(2\pi(f_0-f_1)t)\ dt$$

$$= \frac{1}{4\pi(f_0+f_1)}\sin(2\pi(f_0+f_1))\Big|_0^T + \frac{1}{4\pi(f_0-f_1)}\sin(2\pi(f_0-f_1))\Big|_0^T$$

$$= \frac{1}{4\pi(f_0+f_1)}\sin(2\pi(f_0+f_1)T) + \frac{1}{4\pi(f_0-f_1)}\sin(2\pi(f_0-f_1)T)$$

By the bandpass assumption (f_0+f_1) >> 1 so the first term in the last line is negligible. For the second term to be zero it must satisfy

$$2\pi(f_0 - f_1)T = n\pi$$

where n is an integer. This implies that (f_0 - f_1)= $n/2T$.

Problem 10.19. Encode the sequence 1101 with a Hamming (7,4) block code.

Solution

Coded bit sequence **c** = $\mathbf{x}\cdot\mathbf{G}$, where **G** is defined by (10.89).

$$\mathbf{c} = [1101]\cdot\begin{bmatrix} 1000101 \\ 0100111 \\ 0010110 \\ 0001011 \end{bmatrix}$$
$$= [1101001]$$

Problem 10.20. The Hamming (7,4) encoded sequence 1001000 was received. If the number of transmission errors is less than two, what was the transmitted sequence?

Solution

The syndrome of the received sequence is $\mathbf{S} = \mathbf{R}\Box\mathbf{H}$ where **H** is defined by (10.92).

$$\mathbf{S} = \mathbf{R} \cdot \mathbf{H}$$

$$= [1001000] \cdot \begin{bmatrix} 101 \\ 111 \\ 110 \\ 011 \\ 100 \\ 010 \\ 001 \end{bmatrix}$$

$$= [110]$$

Based on Table 10.4, the error vector **E** = [0010000]. The transmitted sequence is **E** ⊕ **R** = [1011000].

Problem 10.21. A Hamming (15,11) block code is applied to a BPSK transmission scheme. Compare the block error rate performance of the uncoded and coded systems. Explain how this would differ if the modulation strategy was QPSK.

<u>Solution</u>

1) For the uncoded system, the probability of a bit error with BPSK is

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The probability of a block error with block length of 15 bits, assuming independent errors is:

$$P_b^{uncoded} = 1 - (1 - P_e)^{15}$$

2) For the coded system, with a (15,11) Hamming code, the probability of block error is

$$P_b^{coded} = 1 - (1 - P_e')^{15} - \binom{15}{1}(1 - P_e')^{14} P_e',$$

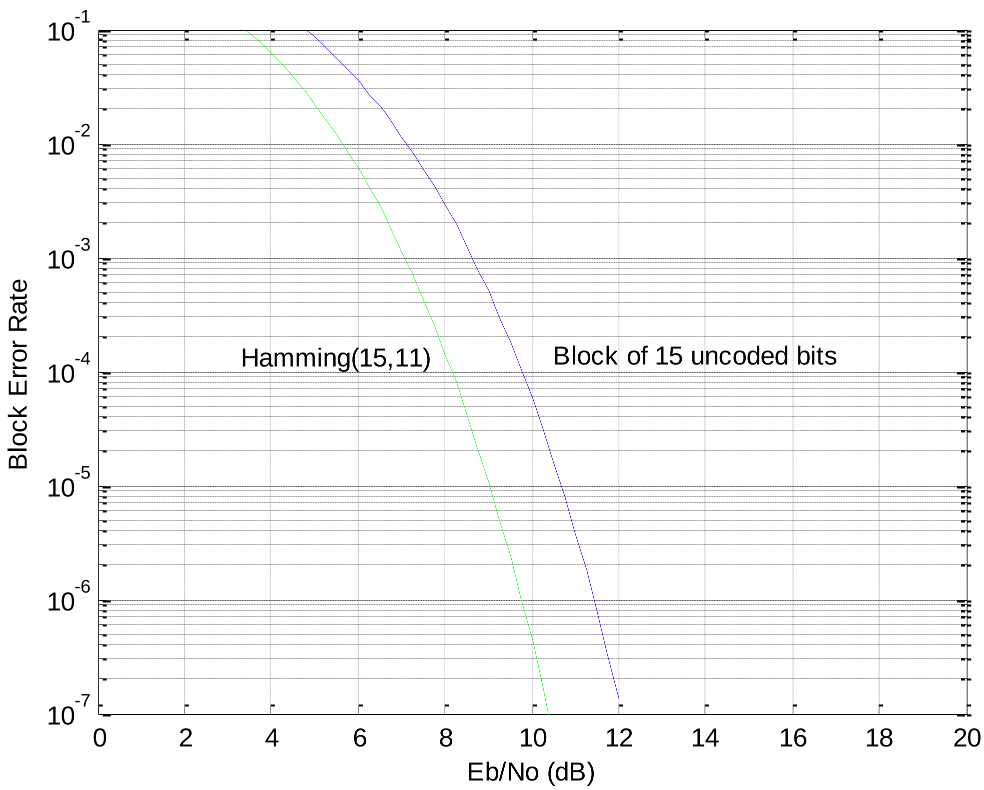
where P_e' is the bit error probability of coded bit, since the code can correct a single bit error. The probability of bit error in this case is:

$$P_e' = Q\left(\sqrt{\frac{2E_c}{N_0}}\right),$$

where E_c is the coded bit energy, and E_c = 11/15E_b. Therefore

$$P_e' = Q\left(\sqrt{\frac{22E_b}{15N_0}}\right)$$

To compare the block error probabilities of uncoded and coded systems, we use Matlab to plot the block error rate curves for $P_b^{uncoded}$ and P_b^{coded} versus E_b/N_0 (dB), as shown below



The Matlab script that generates the above plot is

```
EbNodB=[0:0.25:12];
EbNo     = 10.^(EbNodB/10);
Pe       = 0.5*erfc(sqrt(EbNo));
Puncoded = 1 - (1-Pe).^15;
EcNo     = 11/15 * EbNo;
Peprime  = 0.5*erfc(sqrt(EcNo));
Pcoded   = 1 - (1-Peprime).^15 - 15*(1-Peprime).^14.*Peprime;
semilogy(EbNodB,Puncoded)
grid
xlabel('Eb/No (dB)')
ylabel('Block Error Rate')
axis([0 20 1E-7 0.1])
hold on, semilogy(EbNodB,Pcoded,'g'), hold off
```

3) Since for QPSK modulation, bit error probabilities of uncoded bits P_e and coded bits $P_e^{'}$ are unchanged compared with BPSK modulation, the block error probabilities of two systems are also the same as those of BPSK modulation.

Problem 10.22. Show that the choice $\gamma = \mu / 2$ minimizes the probability of error given by Eq. (10.26). Hint: The Q-function is continuously differentiable.

Solution

From (10.26), we have the average probability of error as:

$$P_e(\gamma) = \frac{1}{2} Q\left(\frac{\mu - \gamma}{\sigma}\right) + \frac{1}{2} Q\left(\frac{\gamma}{\sigma}\right)$$

Recall the definition of Q-function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp(-s^2/2) ds$$

$$\text{(let } u = -s\text{)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp(-u^2/2) du$$

So the derivative is given by

$$\frac{dQ(x)}{dx} = \frac{-1}{\sqrt{2\pi}} \exp(-x^2/2) \le 0$$

Substituting this result into the definition of $P_e(\gamma)$ we obtain

$$\frac{dP_e(\gamma)}{d\gamma} = \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) \cdot \frac{-1}{\sigma} + \frac{1}{2} \cdot \frac{-1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \cdot \frac{1}{\sigma}$$

$$= \frac{1}{2\sqrt{2\pi}\sigma} \left\{ \exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) - \exp\left(-\frac{\gamma^2}{2\sigma^2}\right) \right\}$$

Setting $\frac{dP_e(\gamma)}{d\gamma} = 0$ implies

$$\exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right) = \exp\left(-\frac{\gamma^2}{2\sigma^2}\right)$$

$$(\mu - \gamma)^2 = \gamma^2$$

$$\gamma = \mu / 2$$

Checking the second derivative, we have

$$\frac{d^2 P_e(\gamma)}{d^2\gamma} = \frac{1}{2\sqrt{2\pi}\sigma}\left[\frac{2(\mu-\gamma)}{2\sigma^2}\cdot\exp\left(-\frac{(\mu-\gamma)^2}{2\sigma^2}\right)+\frac{2\gamma}{2\sigma^2}\exp\left(-\frac{\gamma^2}{2\sigma^2}\right)\right]$$
$$> 0$$

when $\gamma = \mu/2$. Therefore at $\gamma = \mu/2$, $P_e(\gamma)$ has a minimum value.

Problem 10.23. For *M*-ary PAM,

(a) Show that the formula for probability of error, namely,

$$P_e = 2\left(\frac{M-1}{M}\right)Q\left(\frac{A}{\sigma}\right)$$

holds for M = 2, 3, and 4. By mathematical induction, show that it holds for all M.

(b) Show the formula for average power, namely,

$$P = \frac{(M^2-1)A^2}{3}$$

holds for M = 2, and 3. Show it holds for all M.

<u>Solution</u>

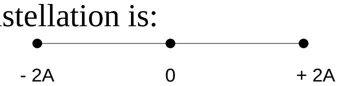
(a) *M*-ary PAM with the separation between nearest neighbours as 2A. Assume that all M symbols are equally transmitted.

(i) For M=2, we have the result given in the text for binary PAM

$$P_e^{2PAM} = Q\left(\frac{A}{\sigma}\right)$$
$$= 2\frac{M-1}{M}Q\left(\frac{A}{\sigma}\right)$$

for M = 2.

(ii) For M = 3, the constellation is:



$$P_e = \frac{1}{3} P\ \ y > -A \,|\, (-2A) \text{ is transmitted } + \frac{1}{3} P\ \ y > A \text{ or } y < -A \,|\, 0 \text{ is transmitted}$$

$$+\frac{1}{3} P\ \ y < A \,|\, (2A) \text{ is transmitted}$$

$$= \frac{1}{3}\int_{-A}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+2A)^2}{2\sigma^2}\right) dy + \frac{1}{3}\int_{A}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$+\frac{1}{3}\int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy + \frac{1}{3}\int_{-\infty}^{A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-2A)^2}{2\sigma^2}\right) dy$$

$$= \frac{4}{3} Q\left(\frac{A}{\sigma}\right)$$

From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=3, $P_e = \frac{4}{3} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 3.

(iii) For M = 4, the constellation is:

$$P_e = \frac{1}{4} P\ \ y > -2A \,|\, (-3A) \text{ is transmitted } + \frac{1}{4} P\ \ y < -2A \text{ or } y > 0 \,|\, -A \text{ is transmitted}$$

$$+\frac{1}{4} P\ \ y < 0 \text{ or } y > 2A \,|\, +A \text{ is transmitted } + \frac{1}{4} P\ \ y < 2A \,|\, (3A) \text{ is transmitted}$$

$$= \frac{1}{4}\int_{-2A}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y+3A)^2}{2\sigma^2}\right) dy$$

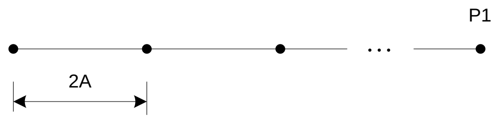
$$+ 2\left[\frac{1}{4}\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y+A\ {}^2}{2\sigma^2}\right) dy + \frac{1}{4}\int_{-\infty}^{-2A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y+A\ {}^2}{2\sigma^2}\right) dy\right]$$

$$+\frac{1}{4}\int_{-\infty}^{A} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-3A)^2}{2\sigma^2}\right) dy$$

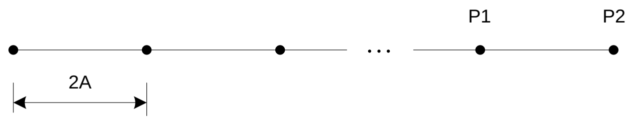
$$= \frac{6}{4} Q\left(\frac{A}{\sigma}\right)$$

where the factor 2 in the third last line, comes from the symmetry of the second and third terms of the first equation. From the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$, when M=4, $P_e = \frac{6}{4} Q\left(\frac{A}{\sigma}\right)$. Thus the formula $P_e = \frac{2(M-1)}{M} Q\left(\frac{A}{\sigma}\right)$ holds for M = 4.

(iv) Assume that the formula of P_e holds for (M-1)-ary PAM. By mathematical induction, we need to show it also holds for M-ary PAM. The (M-1)-ary PAM constellation may be illustrated as shown:



By adding one point P2 on the (M-1)-ary PAM constellation, which has the distance 2A from point P1, we obtain M-ary PAM constellation as follows (in practice, the average or dc level may be adjusted as well but this has no effect on the symbol error rate):



Since error probabilities of P1 symbol on the (M-1)-ary PAM is the same as that of P2 point on the M-ary PAM, the error probability of M-ary PAM is

$$P_e^{M-ary} = \frac{M-1}{M} P_e^{(M-1)-ary} + \frac{1}{M} \cdot \text{symbol error prob. of P1 symbol on } M\text{-ary} \quad (1)$$

where $1/M$ is the probability that P1 is transmitted and $(M-1)/M$ is the probability that one of the other constellation points is transmitted. The probability of error formula for (M-1)-ary PAM is given by

$$P_e^{(M-1)-ary} = 2\frac{(M-2)}{(M-1)} Q\left(\frac{A}{\sigma}\right). \quad (2)$$

The symbol error rate of P1 symbol on M-ary PAM is

$$P_{P1} = \frac{1}{M} P\{ y < (\mu - A), \text{or } y > (\mu + A) \,|\, \text{P1 is transmitted}\}$$

where μ is the signal level of P1 symbol.

$$\begin{aligned} P_{P1} &= \frac{1}{M}\int_{-\infty}^{\mu-A} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy + \frac{1}{M}\int_{\mu+A}^{+\infty} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy \\ &= \frac{2}{M} Q\left(\frac{A}{\sigma}\right) \end{aligned} \quad (3)$$

Substituting Eqs. (2) and (3) into (1), we obtain the symbol error probability of M-ary PAM

$$
\begin{aligned}
P_e^{M-PAM} &= \frac{M-1}{M}\cdot 2\cdot\frac{M-2}{M-1}Q\left(\frac{A}{\sigma}\right)+\frac{2}{M}Q\left(\frac{A}{\sigma}\right) \\
&= 2\frac{M-1}{M}Q\left(\frac{A}{\sigma}\right)
\end{aligned}
.
$$

The formula holds for M-ary PAM. Therefore, by mathematical induction, the formula holds for all M.

(b) To compute the average symbol power we note:

i) For $M = 2$, the average symbol power is A^2 and the formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=2.

ii) For $M = 3$, the average symbol energy is

$$
P = \frac{1}{3}\ (2A)^2 + 0^2 + (2A)^2 \ \ = \frac{8}{3}A^2 .
$$

The formula $P = \frac{(M^2-1)A^2}{3}$ holds for M=3.

iii) For general even M, the M-ary PAM constellation points are

$$
-(M-1)A, \cdots, -3A, -A, A, 3A, \cdots, (M-1)A \ .
$$

The average symbol energy is

$$
\begin{aligned}
P &= \frac{2\left[(M-1)^2+(M-3)^2+\cdots+3^2+1\right]}{M}A^2 \\
&= \frac{2A^2}{M}\sum_{k=1}^{M/2}(2k-1)^2 \\
&= \frac{2A^2}{M}\left[2^2\sum_{k=1}^{M/2}k^2 - 4\sum_{k=1}^{M/2}k + \sum_{k=1}^{M/2}1\right] \\
&= \frac{2A^2}{M}\left[4\frac{M(M/2+1)(M+1)}{2\cdot 6} - 4\frac{M(M/2+1)}{2\cdot 2} + \frac{M}{2}\right] \\
&= \frac{(M^2-1)A^2}{3}
\end{aligned}
.
$$

where we have used the summation formulas of Appendix 6.

iv) For general odd M, the M-ary PAM constellation points are

$$-(M-1)A, \cdots, -2A, 0, 2A, \cdots (M-1)A \quad .$$

The average symbol energy is

$$\begin{aligned}
P &= \frac{2\left[(M-1)^2 + (M-3)^2 + \cdots + 2^2\right]}{M} A^2 \\
&= \frac{2A^2}{M} 2^2 \left[\left(\frac{M-1}{2}\right)^2 + \left(\frac{M-3}{2}\right)^2 + \ldots + 1^2 \right] \\
&= \frac{8A^2}{M} \sum_{k=1}^{(M-1)/2} k^2 \\
&= \frac{8A^2}{M} \frac{(M-1)(M+1)(M)}{2 \cdot 2 \cdot 6} \\
&= \frac{(M^2-1)A^2}{3}
\end{aligned}$$

where the fourth line uses the summation formula found in Appendix 6.

Problem 10.24. Consider binary FSK transmission where $(f_1 - f_2)T$ is not an integer.

(a) What is the mean output of the upper correlator of Fig. 10.12, if a 1 is transmitted? What is the mean output of the lower correlator?
(b) Are the random variables N_1 and N_2 independent under these conditions? What is the variance of $N_1 - N_2$?
(c) Describe the properties of the random variable D of Fig. 10.12 in this case.

Solution:

(a) If a 1 is transmitted,

$$r(t) = A_c \cos(2\pi f_1 t) + n(t)$$

where $n(t)$ is a narrow band Gaussian noise. The output of the upper correlator is Y_1:

$$\begin{aligned}
Y_1 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_1 t)dt \\
&= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_1 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt \\
&\cong \frac{1}{\sqrt{2}} A_c T + \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt
\end{aligned}$$

The expected value of Y_1 is $\mathbf{E}[Y_1] = \frac{1}{2} A_c T$, since $n(t)$ has zero mean.

The output of the lower correlator is Y_2:

$$
\begin{aligned}
Y_2 &= \int_0^T r(t)\sqrt{2}\cos(2\pi f_2 t)dt \\
&= \int_0^T \sqrt{2}A_c \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt \\
&= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 + f_2)t)dt + \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \sqrt{2}\int_0^T n(t)\cos(2\pi f_2 t)dt \\
&\cong \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt + \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt
\end{aligned}
$$

where the first term of the third line is negligible due to the bandpass assumption. The expected value of Y_2 is

$$
\begin{aligned}
\mathbf{E}[Y_2] &= \frac{A_c}{\sqrt{2}}\int_0^T \cos(2\pi(f_1 - f_2)t)dt \\
&= \frac{A_c}{\sqrt{2}} \cdot \frac{1}{2\pi \quad f_1 - f_2}\sin\left[2\pi \quad f_1 - f_2 \quad t\right]\Big|_0^T \\
&= \frac{A_c}{2\sqrt{2}\pi \quad f_1 - f_2}\sin(2\pi \quad f_1 - f_2 \quad T)
\end{aligned}
$$

which clearly differs from the orthogonal case.

(b) The random variables N_1 and N_2 are given by

$$
\begin{aligned}
N_1 &= \int_0^T \sqrt{2}n(t)\cos(2\pi f_1 t)dt \\
N_2 &= \int_0^T \sqrt{2}n(t)\cos(2\pi f_2 t)dt
\end{aligned}
$$

Since $n(t)$ is a Gaussian process, both N_1 and N_2 are Gaussian. To show N_1 and N_2 are correlated consider

$$
\begin{aligned}
\mathbf{E}[N_1N_2] &= \mathbf{E}\left[\int_0^T n(t)\cos(2\pi f_1 t)dt \cdot \int_0^T n(\tau)\cos(2\pi f_2\tau)d\tau\right] \\
&= \int_0^T \int_0^T \mathbf{E}[n(t)n(\tau)]\cos(2\pi f_1 t)\cos(2\pi f_2\tau)dtd\tau \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\tau)\cos(2\pi f_1 t)\cos(2\pi f_2\tau)dtd\tau \\
&= \frac{N_0}{2}\int_0^T \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt \\
&= \frac{N_0}{4}\int_0^T \left[\cos(2\pi(f_1+f_2)t)+\cos(2\pi(f_1-f_2)t)\right] dt \\
&= \frac{N_0}{4}\left[\left.\frac{\sin\left(2\pi(f_1+f_2)t\right)}{2\pi(f_1+f_2)}\right|_0^T + \left.\frac{\sin\left(2\pi(f_1-f_2)t\right)}{2\pi(f_1-f_2)}\right|_0^T\right] \\
&\cong \frac{N_0}{4}\operatorname{sinc}(2(f_1-f_2)T)
\end{aligned}
$$

where the first term of the second last line is assumed negligible due to the bandpass assumption. Since N_1 and N_2 are correlated, they are not independent. The variance of (N_1-N_2) is

$$
\begin{aligned}
\operatorname{var}[N_1-N_2] &= \operatorname{var}[N_1]+\operatorname{var}[N_2]-2\mathbf{E}[N_1N_2] \\
&= N_0 - \frac{N_0}{2}\operatorname{sinc}\left[2(f_1-f_2)T\right]
\end{aligned}
$$

(c) The random variable D is Gaussian with zero mean and variance var[N_1-N_2].

Problem 10.25. Show that the noise variance of the in-phase component n_I(t) of the band-pass noise is the same as the band-pass noise $n(t)$ variance; that is, for a band-pass noise bandwidth B_N

$$
\mathbf{E}\left[n_I^2(t)\right] = N_0 B_N
$$

__**Solution**__

Recall the spectra of narrowband noise $n(t)$ and its in-phase component $n_I(t)$ shown in Figure 8.23. The variance of a random process $x(t) = R_x(0) = \int_{-\infty}^{\infty} X(f)df$, where $X(f)$ is the power spectral density of $x(t)$. Therefore,

$$
\begin{aligned}
\mathrm{Var}[n(t)] &= \mathbf{E}\left[n^2(t)\right] \\
&= \int_{-\infty}^{\infty} S_N(f)df \\
&= 2\cdot\frac{N_0}{2}\cdot 2B \\
&= N_0\cdot 2B
\end{aligned}
$$

Where we have used the fact that for a bandpass signal $B_T = 2B$, that is twice the lowpass bandwidth. Similarly, the variance of the in-phase noise is

$$
\begin{aligned}
\mathrm{Var}[n_I(t)] &= \mathbf{E}\left[n_I^2(t)\right] \\
&= \int_{-\infty}^{\infty} S_{n_I}(f)df \\
&= N_0\cdot 2B
\end{aligned}
$$

Problem 10.26 In this problem, we investigate the effects when transmit and receive filters do not combine to form an ISI-free pulse shape. To be specific, data is transmitted at baseband using binary PAM with an exponential pulse shape $g(t)=\exp(-t/T)u(t)$ where T is the symbol period (see Example 2.2). The receiver detects the data using an integrate-and-dump detector.

(a) With data represented as ±1, what is magnitude of the signal component at the output of the detector.

(b) What is the worst case magnitude of the intersymbol interference at the output of the detector. (Assume the data stream has infinite length.) Using the value obtained in part (a) as a reference, by what percentage is the eye opening reduced by this interference.

(c) What is the rms magnitude of the intersymbol interference at the output of the detector? If this interference is treated as equivalent to noise, what is the equivalent signal-to-noise ratio at the output of the detector? Comment on how this would affect bit error rate performance of this system when there is also receiver noise present.

(Typo in problem statement, there should be minus sign in exponential.)

Solution

(a) For a data pulse

$$
g(t) = A\exp(-t/T)u(t)
$$

where A is the binary PAM symbol (±1). The desired output of an integrate-and-dump filter in the n^{th} symbol period is

$$
\begin{aligned}
G_n &= \int_{nT}^{(n+1)T} g(t-nT)dt \\
&= \int_0^T A_n \exp(-t/T)dt \\
&= A_n T \;\; 1-\exp(-1)
\end{aligned}
$$

If the data is either ±1, then magnitude of the output is $T(1\text{-}e^{-1})$.

(b) In the n^{th} symbol period the received signal is

$$
y(t) = \sum_{k=-\infty}^{\infty} A_n g(t-kT)
$$

The output of the detection filter in the n^{th} symbol period is

$$
\begin{aligned}
Y_n &= \int_{nT}^{(n+1)T} y(t)dt \\
&= \int_{nT}^{(n+1)T} \sum_{k=-\infty}^{\infty} A_k \exp \;\; -(t-kT)/T \;\; dt \\
&= \int_{nT}^{(n+1)T} A_n \exp \;\; -(t-nT)/T \;\; dt + \sum_{k=1}^{\infty} \int_{nT}^{(n+1)T} A_{n-k} \exp \;\; -(t-(n-k)T)/T \;\; dt
\end{aligned}
$$

where, due to the causality of the pulse shape, the symbols A_{n+1} and later due not cause intersymbol interference into symbol A_n. The first term in the above is the desired signal and the second term is the intersymbol interference. By letting $s = t-(n\text{-}k)T$, we can express this interference as

$$
\begin{aligned}
J_n &= \sum_{k=1}^{\infty} \int_{kT}^{(k+1)T} A_{n-k} \exp \;\; -t/T \;\; dt \\
&= \sum_{k=1}^{\infty} A_{n-k} T \;\; \exp(-k) - \exp(-(k+1))
\end{aligned}
$$

where each term in the summation corresponds to the interference caused by a previous symbol. For worst case interference we assume that all of the A_{n-k} have the same sign. Then this worst case interference is given by

$$
\begin{aligned}
J_n &= \sum_{k=1}^{\infty} A_{n-k} T \;\; \exp(-k) - \exp(-(k+1)) \\
&\le T \;\; 1-\exp(-1) \;\; \sum_{k=1}^{\infty} \exp(-k)
\end{aligned}
$$

To simplify the notation, we let $\alpha = \exp(-1)$. Then

$$
\begin{aligned}
J_n^{\max} &= T(1-\alpha)\sum_{k=1}^{\infty}\alpha^k \\
&= T(1-\alpha)\frac{\alpha}{1-\alpha} \\
&= \alpha T
\end{aligned}
$$

Comparing this worst case interference to the desired signal level G_n, the eye-opening is reduced by

$$
\frac{J_n^{\max}}{G_n}\times 100 = \frac{T\alpha}{T(1-\alpha)}\times 100 = 58\%
$$

(c) From part (b), we found that k^{th} preceding symbol contributes an interference

$$
I_n^k = A_{n-k}(1-\alpha)\alpha^k
$$

The total interference is

$$
\begin{aligned}
J_n &= \sum_{k=1}^{\infty} I_n^k \\
&= \sum_{k=1}^{\infty} A_{n-k}T(1-\alpha)\alpha^k
\end{aligned}
$$

Since all symbol intervals are equivalent, we drop the subscript n on J_n. The mean value of this interference is $\mathbf{E}[J] = 0$ since $\mathbf{E}[A_{n-k}] = 0$. The variance of this interference is

$$
\begin{aligned}
Var(J) &= \mathbf{E}\left[J^2\right] \\
&= \sum_{k=1}^{\infty}\mathbf{E}\left[A_{n-k}^2\right]T^2(1-\alpha)^2\alpha^{2k} \\
&= T^2(1-\alpha)^2\frac{\alpha^2}{1-\alpha^2} \\
&= (\alpha T)^2\frac{1-\alpha}{1+\alpha}
\end{aligned}
$$

where we have assumed the symbols are independent so that $\mathbf{E}[A_iA_j] = 0$ if $i \neq j$. The *rms* interference is given by the square root of the variance so

$$
\begin{aligned}
J_{rms} &= \alpha T\sqrt{\frac{1-\alpha}{1+\alpha}} \\
&= 0.25T
\end{aligned}
$$

which is clearly less than the worst case interference $J^{\max}$.

If we represent the signal power by S, the noise power by N, then the equivalent signal-to-noise ratio taking account of the intersymbol interference is

$$SNR = \frac{S}{N + J_{rms}^2}$$

The intersymbol interference will further degrade performance. In fact, if the worst case interference is large enough such that the eye closes, it will result in a lower limit on the bit error rate regardless of how little noise there is.

Problem 10.27. A BPSK signal is applied to a matched-filter receiver that lacks perfect phase synchronization with the transmitter. Specifically, it is supplied with a local carrier whose phase differs from that of the carrier used in the transmitter by ϕ radians.

(a) Determine the effect of the phase error ϕ on the average probability of error of this receiver.

(b) As a check on the formula derived in part (a), show that when the phase error is zero the formula reduces to the same form as in Eq. (10.44).

<u>Solution</u>

(a) With BPSK, assume the transmitted signal is (10.36):

$$s(t) = A_c \sum_{k=0}^{N} b_k h(t - kT)\cos(2\pi f_c t),$$

where b_k = +1 for a 1 and b_k = -1 for a 0, $h(t)$ is the rectangular pulse $\text{rect}\left(\frac{t - T/2}{T}\right)$.

The received signal is

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c \sum_{k=0}^{N} b_k h(t - kT)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}$$

The receiver matched filter is the integrate-and-dump filter. The output for the k^{th} symbol after down-conversion with phase error ϕ and match filtering is:

$$
\begin{aligned}
Y_k &= \int_{(k-1)T}^{kT} x(t)\cos(2\pi f_c t+\phi)dt \\
&= \int_{(k-1)T}^{kT} A_c b_k + n_I(t)\ \cos(2\pi f_c t)\cos(2\pi f_c t+\phi)dt - \int_{(k-1)T}^{kT} n_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t+\phi)dt \\
&= \int_{(k-1)T}^{kT} \frac{1}{2}[A_c b_k + n_I(t)][\cos\phi + \cos(4\pi f_c t+\phi)]dt - \int_{(k-1)T}^{kT} \frac{1}{2} n_Q(t)[\sin(4\pi f_c t+\phi)+\sin(-\phi)]dt \\
&\cong \frac{T}{2}A_c b_k \cos\phi + \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt \\
&= \frac{T}{2}A_c b_k \cos\phi + N_k
\end{aligned}
$$

where we define

$$N_k = \frac{1}{2}\int_{(k-1)T}^{kT} n_I(t)\cos\phi dt + \frac{1}{2}\int_{(k-1)T}^{kT} n_Q(t)\sin\phi dt$$

The random variable N_k has zero mean and variance

$$
\begin{aligned}
\mathrm{var}[N_k] &= \cos^2\phi\,\frac{N_0 T}{4} + \sin^2\phi\,\frac{N_0 T}{4} \\
&= \frac{N_0 T}{4} = \sigma^2
\end{aligned}
$$

Let $\mu = \frac{T}{2}A_c\cos\phi$. Then the probability of bit error P_e is

$$
\begin{aligned}
P_e &= \mathbf{P}[b_k = 1]\mathbf{P}[Y_k < 0 \mid b_k = 1] + \mathbf{P}[b_k = -1]\mathbf{P}[Y_k > 0 \mid b_k = -1] \\
&= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(y-\mu)^2}{\sigma^2}\right\}dy + \frac{1}{2}\int_{0}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(y+\mu)^2}{\sigma^2}\right\}dy \\
&= Q\left(\frac{\mu}{\sigma}\right)
\end{aligned}
$$

with $E_b = \frac{A_c^2 T}{2}, \mu = \frac{T}{2}A_c\cos\phi, \sigma = \frac{1}{2}\sqrt{N_0 T}$, we have $P_e = Q\left(\sqrt{\frac{2E_b\cos\phi}{N_0}}\right)$

(b) When the phase error ϕ=0, $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, as the same as Eq. (10.44).

Problem 10.28. A binary FSK system transmits data at the rate of 2.5 megabits per second. During the course of transmission, white Gaussian noise of zero mean and power spectral density 10^{-20} watts per hertz is added to the signal. In the absence of noise, the amplitude of the received signal is 1 μV across 50 ohm impedance. Determine the average probability of error assuming coherent detection of the binary FSK signal.

Solution

The average probability of error for coherent FSK is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

from Eq. (10.68). For this example, we have noise power spectral density is

$$N_0 = 2\times 10^{-20}\,\text{watts}/\text{Hz}$$

and the energy per bit is

$$E_b = \frac{1}{2}\frac{A_c^2 T}{R},$$

In the text, we have nominally assumed the resistance is 1 ohm and omitted it. In this problem we use the resistance of R = 50 ohms. The symbol duration is $T = \dfrac{1}{2.5\times 10^6}$ seconds and the amplitude of received signal is A_c = 1μV. Therefore,

$$\begin{aligned} E_b &= \frac{1}{2}\times\frac{1\times 10^{-12}}{50}\times\frac{1}{2.5\times 10^6} \\ &= 4\times 10^{-21}\,\text{watts}/\text{Hz} \end{aligned}$$

Substituting the above values into the expression for P_e and we have the probability of error is

$$\begin{aligned} P_e &= Q\ \ 0.2 \\ &\cong 0.26 \end{aligned}$$

Problem 10.29. One of the simplest forms of forward error correction code is the repetition code. With an N-repetition code, the same bit is sent N times, and the decoder decides in favor of the bit that is detected on the majority of trials (assuming N is odd). For a BPSK transmission scheme, determine the BER performance of a 3-repetition code.

Solution

With 3-repetition code, the decoder will output the correct bit if there are one or fewer errors in the 3-bit code. Thus, assuming bit errors are independent, the bit error rate is

$$
\begin{aligned}
P_b^{coded} &= (1-P_e)^3 + \binom{3}{1} P_e (1-P_e)^2 \\
&= (1-P_e)^2 (1+2P_e)
\end{aligned},
$$

where P_e is the bit error rate of channel bit. With BPSK, the formula for bit error probability is

$$
\begin{aligned}
P_e &= Q\left(\sqrt{\frac{2E_c}{N_0}}\right) \\
&= Q\left(\sqrt{\frac{2E_b}{3N_0}}\right)
\end{aligned},
$$

since ratio of channel bit energy to information bit energy is given by $E_c = 1/3E_b$. Therefore, the bit error probability of the 3-repetition code is

$$
P_b^{coded} = \left(1 - Q\left(\sqrt{\frac{2E_b}{3N_0}}\right)\right)^2 \left(1 + 2Q\left(\sqrt{\frac{2E_b}{3N_0}}\right)\right)
$$

Problem 10.30 In this experiment, we simulate the performance of bipolar signalling in additive white Gaussian noise. The Matlab script included in Appendix 7 for this experiment:

- generates a random sequence with rectangular pulse shaping
- adds Gaussian noise
- detects the data with a simulated integrate-and-dump detector

With this Matlab script

(a) Compute the spectrum of the transmitted signal and compare to the theoretical.

(b) Explain the computation of the noise variance given an E_b/N_0 ratio.

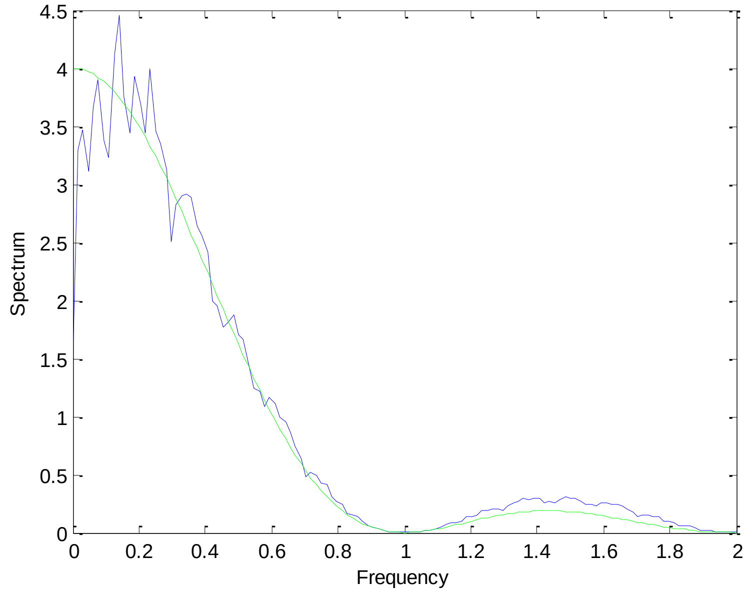
(c) Confirm the theoretically predicted bit error rate for E_b/N_0 from 0 to 10 dB.

Solution

(a) The provided script plots the simulated spectrum before noise is added. If we add the statement

*hold on, plot(F, abs(2*sinc(F)).^2,'g'), hold off*

at the same point, we obtain the following comparison graph. The two graphs agree reasonably well. There are two reasons for the differences observed with the simulated spectrum. The first is the relatively short random sequence used for generating the plot and the second is an aliasing effect.



(b) The calculation of the noise variance in a discrete time simulation proceeds as follows. We are given the sampling rate F_s and the required E_b/N_0 to simulation. We then note that

$$\begin{aligned} E_b &= \int_{-\infty}^{\infty} |p(t)|^2 \, dt \\ &\approx \sum |p_k|^2 T_s \end{aligned} \qquad (1)$$

where $p(t)$ is the pulse shape, $\{p_k\}$ is its sample version and $T_s = 1/F_s$ is the sample interval. On the other hand, if generate noise of variance σ^2, due to Nyquist considerations this can only be distributed over a bandwidth Fs, thus the noise spectral density is

$$\frac{N_0}{2} = \frac{\sigma^2}{F_{s\ s}} \qquad (2)$$

Re-arranging Eq. (2) and substituting Eq. (1) and the knowns, we have

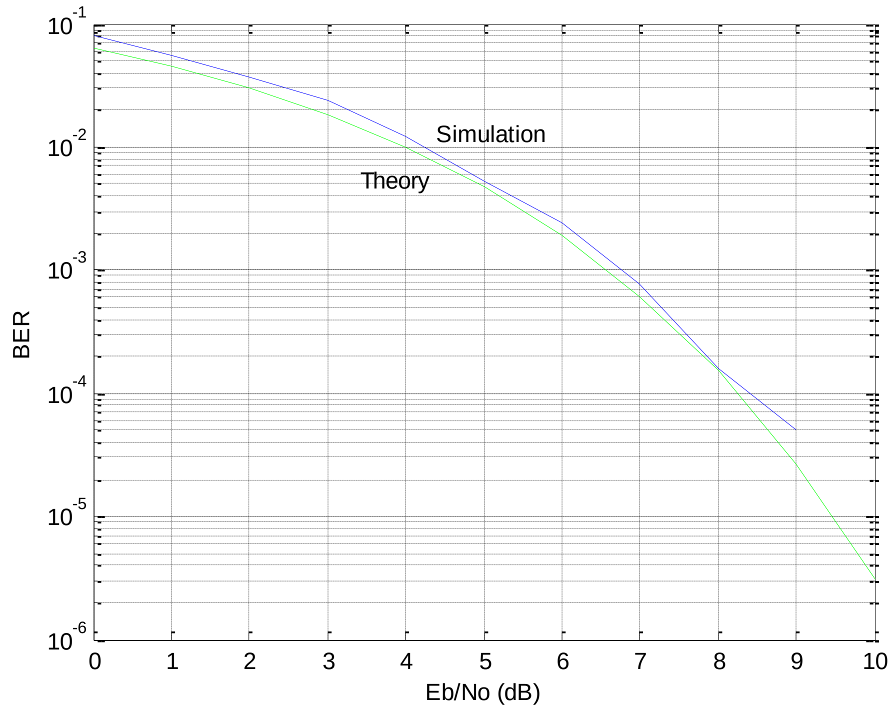
$$
\begin{aligned}
\sigma^2 &= \frac{N_0}{2} F_s \\
&= \left(\frac{F_s}{2}\right)\left(\frac{E_b}{N_0}\right)^{-1} E_b \\
&= \left(\frac{F_s}{2}\right)\left(\frac{E_b}{N_0}\right)^{-1} \sum |p_k|^2 T_s \\
&= \frac{1}{2}\left(\frac{E_b}{N_0}\right)^{-1} \sum |p_k|^2
\end{aligned}
$$

which agrees with what is used in the script (except that in the script we have suppressed F_s and T_s, knowing they would cancel).

(c) To compute the bit error rate for 0 to 10 dB, we add the following statements around the provided script

```
for kk = 0:10
        Eb_N0 = 10^(kk/10);
        Nbits = 100000; %  increase for higher Eb/N0
        …(provided script)
        BER(kk+1)   = Nerrs/Nbits
end
semilogy([0:10], BER)
xlabel('Eb/No (dB)')
ylabel('BER')
grid on
hold on, semilogy([0:10], 0.4*erfc(sqrt(10.^([0:10]/10))),'g')
```

The following plot is then produced by the Matlab script which shows good agreement between theory and simulation.



Problem 10.31 In this experiment, we simulate the performance of bipolar signalling in additive white Gaussian noise but with root-raised-cosine pulse shaping. A Matlab script is included in Appendix 7 for doing this. With this simulation:

(a) Compute the spectrum of the transmitted signal and compare to the theoretical. Also compare to the transmit spectrum with rectangular pulse shaping

(b) Plot the eye diagram of the received signal under no noise conditions. Explain the relationship of the eye opening to bit error rate performance.

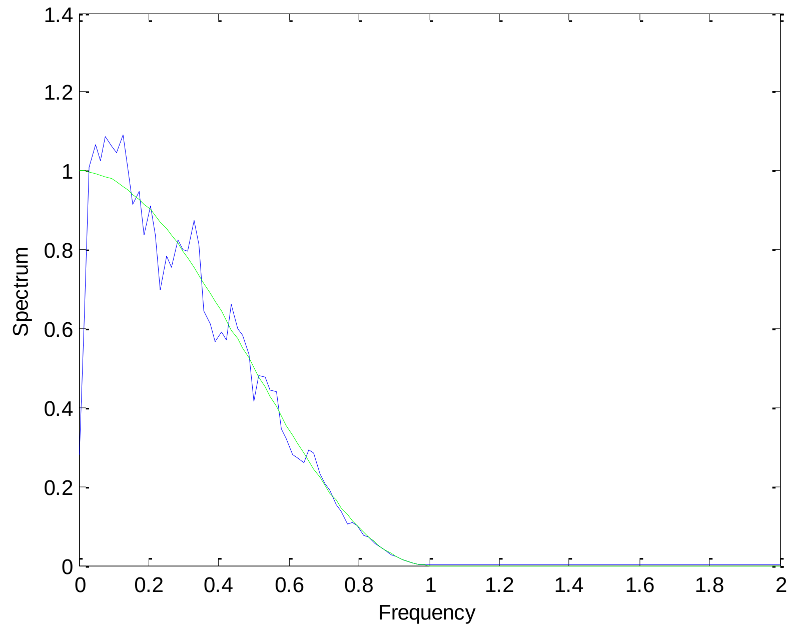
(c) Confirm the theoretically predicted bit error rate for E_b/N_0 from 0 to 10 dB.

Solution

(a) We compare the spectra by inserting the following statements prior to noise being added to the signal

```
[P,F] = spectrum(S,256,0,Hanning(256),Fs);
plot(F,P(:,1));
midpt = floor(length(F)/2);
hold on, plot(F, abs([(1+cos(pi*F(1:midpt)))/2; 0*F(midpt+1:end)]),'g'), hold off
xlabel('Frequency'), ylabel('Spectrum')
```

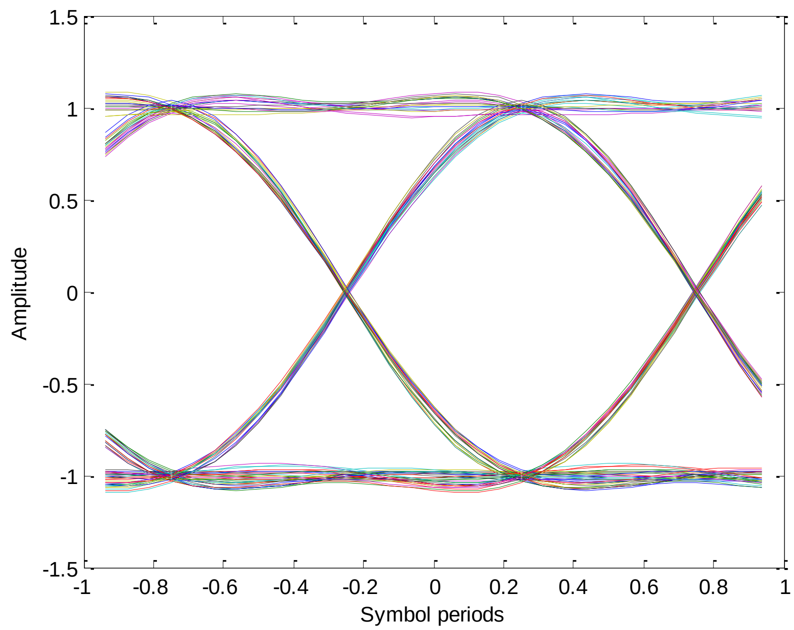
The comparison plot is shown below.



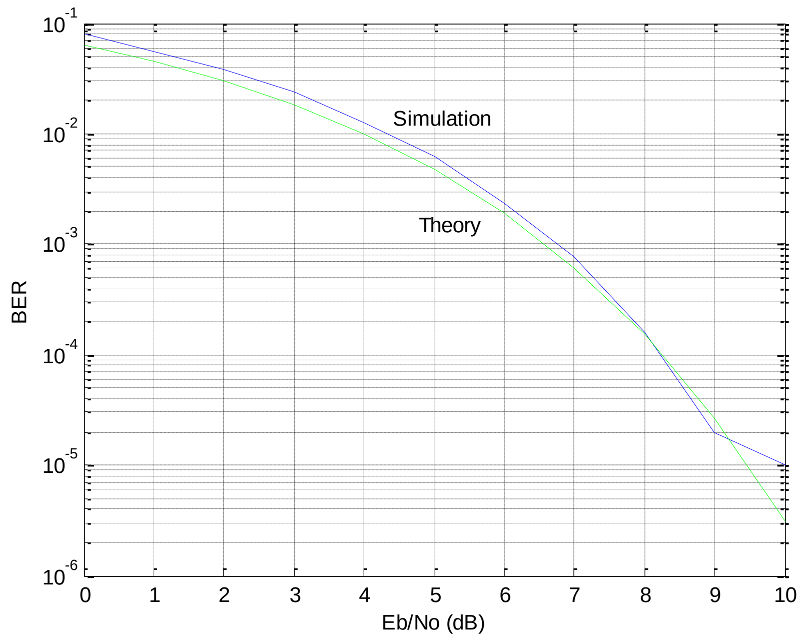
(b) To plot the eye diagram we eliminate the noise by setting E_b/N_0 to a high value

Eb_N0 = 2000;

Then running the Matlab script produces the following eye diagram.



(c) We simulate the bit error rate by commenting out the plotting statements and adding a set of statements similar to those used in Problem 10.30.



Problem 10.32 In this experiment, we simulate the effect of various mismatches in the communication system and their effect on performance. In particular, modify the MatLab scripts of the two preceding problems to:
(a) Simulate the performance of a system using rectangular pulse shaping at the transmitter and raised cosine pulse shaping at the receiver. Comment on the performance degradation.
(b) In the case of matched root-raised cosine filtering, include a complex phase rotation $\exp(j\theta)$ in the channel. Plot the resulting eye diagram for θ being the equivalent of 5, 10, 20, and 45°. Compare to the case of 0°. Do likewise for the BER performance. What modification to the theoretical BER formula would accurately model this behaviour?

Solution

(a) We can create this mismatch by inserting the statements:

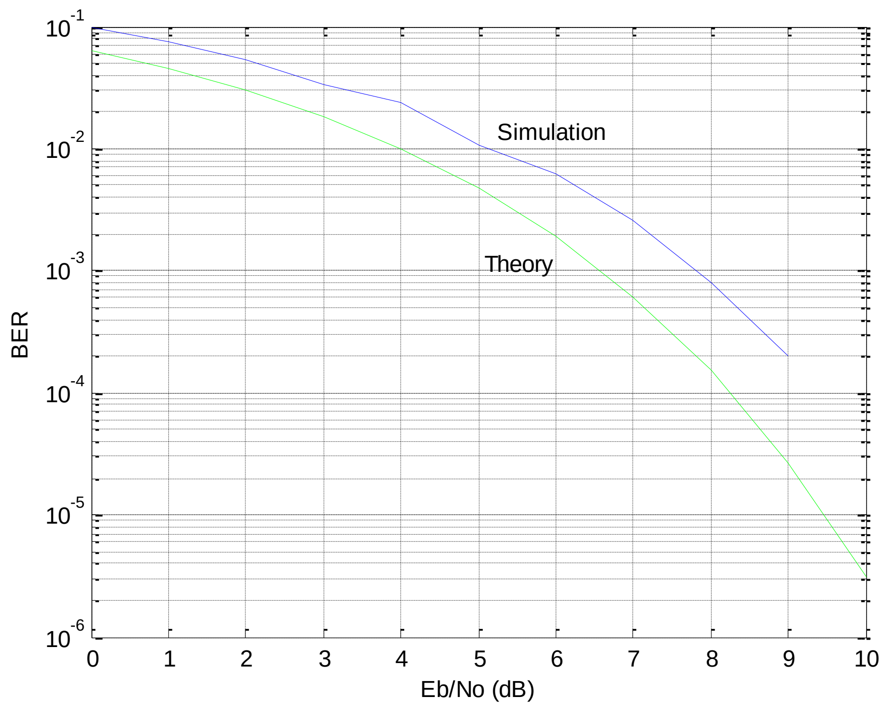
```
pulseTx = ones(1,Fs);

pulseRx = [  0.0064    0.0000   -0.0101    0.0000    0.0182   -0.0000   -0.0424 ...
             0.0000    0.2122    0.5000    0.6367    0.5000    0.2122   -0.0000 ...
            -0.0424    0.0000    0.0182   -0.0000   -0.0101    0.0000    0.0064 ];
Delay = floor((length(pulseTx)-1)/2 + (length(pulseRx)-1)/2 + 1);
Eb    = sum(pulseTx.^2);
```

And by modifying the statements

```
S      = filter(pulseTx,1,[b_delta zeros(1,Delay)]);
De = filter(pulseRx,1,R);
```

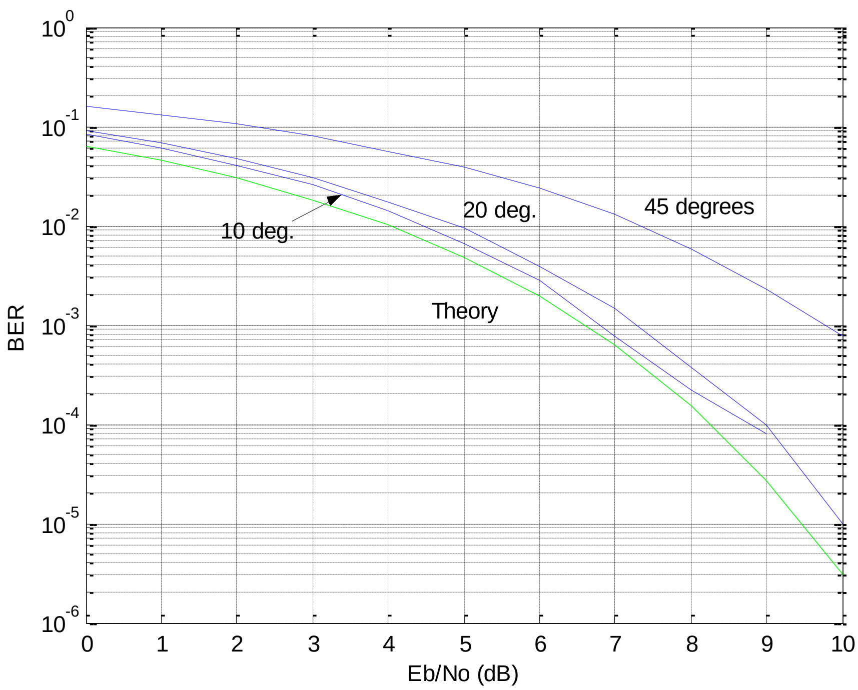
Then we obtain the performance shown below. Part of the loss seen is due to the filter mismatch but part of it is also due to a timing error; with the arrangement of the simulation the optimum sampling point for the data falls between the discrete samples. This sampling time loss could be recovered by interpolation.



(b) Implementation of the phase rotation requires simulation of the complete complex baseband. To do this we must modify the channel portion of the simulation to the following

```
%--- add Gaussian noise ----
Noise = sqrt(N0/2)*(randn(size(S))+j*randn(size(S)));
R  = S + Noise;
R  = R*exp(j*10/180*pi);
R  = real(R);
```

Where we have now included the quadrature component of the noise. Note the receiver only uses the in-phase portion (real part) of the signal to characterize this degradation. The resulting performance for rotations of 10, 20 and 45° are shown below. Note that the 45° rotation results in a 3 dB loss in performance.



BER
Eb/No (dB)
10 deg.
20 deg.
45 degrees
Theory