جامعة بيرزيت

BIRZEIT UNIVERSITY

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Electromagnetic Theory I

Lecture Notes

Motivation:

What is Engineering?

Creativity in applying scientific theories and principles in creating and designing practical systems with certain specifications for different puposes. Additionally, to be able to predict the system behaviour. Systems like, machines, buildings, devices, materials, and manufacturing processes, etc ...

What is Electrical Engineering?

Applying electric theories and principles to design electrical systems, such as, communication systems, computer systems, power systems, etc… and the analysis of systems to predicts their behavior.

Knowledge of electrical elements and electrical theories and field theory is indispensable for electrical engineers

Course Goals:

Intended Learning Outcomes (ILO's)

1. To understand the math modeling of fields and waves in different coordinate systems.
2. To understand the laws of Electrostatic and Magneto-static fields, both in different materials.
3. To understand the electric and magnetic energy, and to be able to solve boundary value problems.
4. To learn Maxwell's equation.
5. To understand the dynamic fields through Maxwell's equations.
6. To understand plane waves and the concept of wave propagation.
7. To understand the equations of transmission line and their relation to Maxwell's equations.

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SYLLABUS:

Required Textbook:

"Elements of Electromagnetics" Matthew O. Sadiku, 6th Edition, Oxford University Press, USA, ISBN 978-0199321384

REFERENCE LIST:

"Introduction to Electrodynamics" by David J. Griffiths" 3rd Edition, Benjamin Cummings,
"Div, Grad, Curl, and All That: An Informal Text on Vector Calculus" 4th Edition, H. M. Schey

COURSE OUTLINE

1. Coordinate System and Vector calculus
2. Electrostatic Fields
3. Electric Fields in Material Space
4. Electrostatic Boundary-Value Problem
5. Magnetostatic Fields
6. Magnetic Forces, Materials and Devices
7. Maxwell's Equations
8. Electromagnetic Wave

GRADING

Homework and Quizes	15%
First Exam	15%
Second Exam	15%
Second Exam	15%
Final Exam	40%

The dates of the first and second exams will be announced in class at least two weeks before the exam, and will depend on the course progress.

Deifionitions:

- **What is a field?**

  Field is a spatial distribution of a quantity, which may or may not be a function, of time

  Scalar field: a field of quantity that can be specified at a point by only its magnitude
  Vector field: a field of a quantity that can be specified at a point in space by its magnitude and direction.

- **What is electromagnetics?**

  It is the study of charges at rest and at motion, and their effect on the surroundings. Charges are positive or negative. At rest charges affect their surrounding with an electric force, this electric force can be presented as and electric field. Where as, moving charges affect their surrounding by electric and magnetic forces which can be represented by *electromagnetic field*.

- **Why do we study electromagnetics?**

  The study of static electric and magnetic fields is necessary for understanding electric machines, cathode ray tubes, avoiding the electrostatic damage of electronic circuits,… The study of electromagnetic fields is necessary for understanding AC electric machines, wireless communication systems, remote sensing, and optical fiber …

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Electromagnetic model

- **Electromagnetic theory model:**

1. **Source, field quantities and universal constants**

- **Sources:** charges at rest q and in motion I: $q = -1.6 \times 10^{-19}\,C$

- **Field quantities** electric field intensity **E** and magnetic field intensity **H** as model variables.

2. **Partial differential equations** and **vector calculus** in time and space as tools

3. **Maxwell's equations**

Electromagnetic model

- **Electromagnetic theory model:**

1. Source, field quantities and universal constants

- **Sources: a)** charges at rest $: q = -1.6\times 10^{-19} C$

Conservation of electric charge is a fundamental postulate in physics that must be satisfied at all times. Mathematically, it is presented by the equation of continuity. Ex. KCL satisfies it.

$$\rho_V = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} \ \text{C/m}^3$$

Volume charge density

$$\rho_S = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} \ \text{C/m}^2$$

Surface charge density

$$\rho_L = \lim_{\Delta L \to 0} \frac{\Delta q}{\Delta L} \ \text{C/m}$$

line charge density

b) Charges in motion I

$$I = \frac{dq}{dt} \ \text{Ampere (C/s)}$$

Electromagnetic model

- **Electromagnetic theory model:**

1. **Source and field quantities**

- **Sources: b)** Current is a scalar quantity and needs an area or length to be defined
We define current density which is a vector showing the direction and magnitude of the current and is a point function(is defined at each point in space)

Volume current density : the current flowing through an area **J** A/m^2
Surface current density: in some cases the current flows on the surface of the conductor and as a result this quantity is defined $\mathbf{J_s}$ **A/m**

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Electromagnetic model

- **Electromagnetic theory model:**

1. Source and field quantities

- **Field Quantities:** **Electric and magnetic field quantities**

Quantities	Field Quantities	Symbols	Units
Electric Quantities	Electric Field Intensity	**E**	V/m
	Electric Field Density (Electric Displacement) (Electric Flux Density)	**D**	C/m^2
Magnetic Quantities	Magnetic Field Intensity	**H**	A/m
	Magnetic Field Density	**B**	T

In static fields, electric and magnetic fields are independent from each other
In time variant fields, electric and magnetic fields are coupled (they depend on each other)

Electromagnetic model

- **Electromagnetic theory model:**

1. **Source and field quantities**

- **Universal Constants:**

Universal Constants	Symbol	Value	Unit
Speed of light	c	3 x 10^8	m/s
Permeability	μ_o	$4\pi \times 10^{-7}$	H/m
Permitivity	ε_o	$(1/36\pi)\times 10^{-9}$	F/m

2. Vector algebra and calculus will be the tools (to be explained next)
3. Maxwell's Equations are the postulates…

Basics of Vector Calculus

- Electromagnetic theory vector quantities: E, D, B, H, J
- Electromagnetic theory scalar quantities: I, Q, Energy (E) e.g. Q= 1 uC

- Because we are dealing with fields, space location must be specified, to do that, we define an origin and a coordinates system to define the location of each point in the space of the specified problem, which is to be solved.

- In EM theory, problems involving objects are normally needed to be solved, depending on the shape of such objects, a coordinate system is used. E.g. circular discs are better specified in polar or cylindrical coordinates, whereas spherical ball, is better analyzed in spherical coordinates system.

1. Vector Algebra: addition, subtraction, multiplication, division
2. Curvilinear Coordinate Systems: Spherical, Cylindrical, Cartesian
3. Vector Calculus: Divergence, Curl, Gradient, Surface Integrals, Line integrals, Volume Integrals

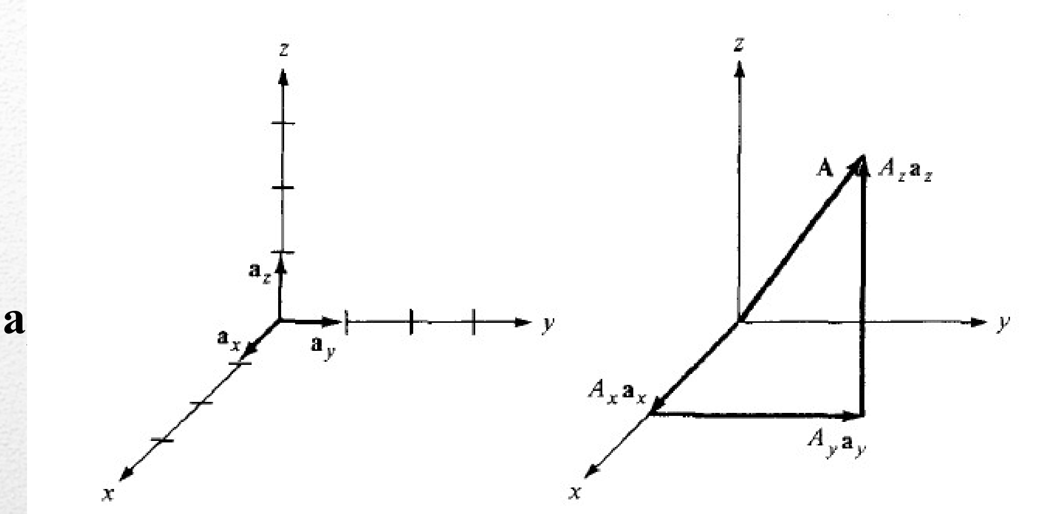
Basics of Vector Calculus

- **Vector Algebra**

  A vector in Cartesian(Rectangular) coordinates can be expresses as:

  $\mathbf{A}=(A_x, A_y, A_z)$ or $\mathbf{A}=A_x\,\mathbf{a_x}+A_y\,\mathbf{a_y}+A_z\,\mathbf{a}$

  $\mathbf{a_x}$, $\mathbf{a_y}$, $\mathbf{a_z}$ are called Cartesian coordinates unit vectors



Cartesian(rectangular) coordinates

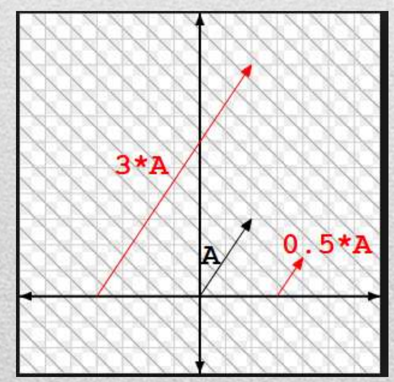
1. **Vector magnitude and direction:**

   $\mathbf{A}= A\mathbf{a_A}$, $A=|\mathbf{A}|$, $\mathbf{a_A}=\mathbf{A}/|\mathbf{A}|$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ **magnitude**

$$\mathbf{a}_A = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$ **Unit vector (direction)**

2. **Vector scaling: $\mathbf{A}= kA\mathbf{a_A}$**



Vector Scaling

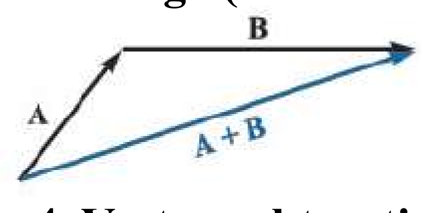
Basics of Vector Calculus

- **Vector Algebra**

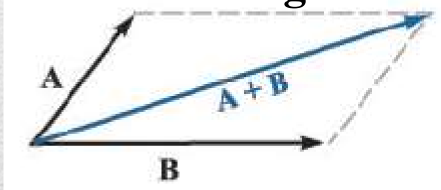
3. Vector addition : C= A+B
adding to vector is carried out component by component , If A=(A_x, A_y ,A_z) , **B**=(B_x, B_y ,B_z) then:

$$\mathbf{C} = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z$$

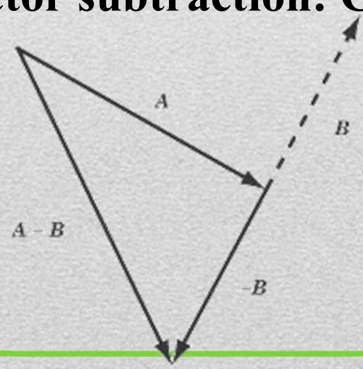
Triangle(head-tale)



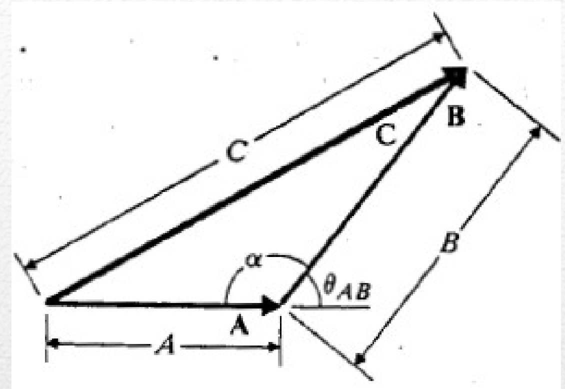
Parallelogram



4. Vector subtraction: C= A-B



Law of cosines: For any triangle

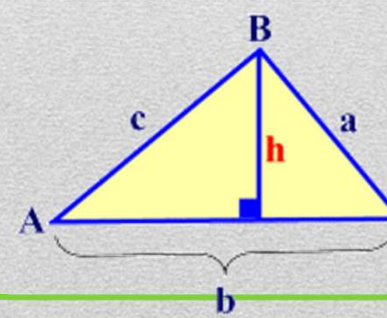


$$C = \sqrt{A^2 + B^2 - 2AB\cos\alpha}$$

Law of sines: For any traingle

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

True for ALL triangles!



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Basics of Vector Calculus

- **Vector Algebra**

  Associative law: A+(B+C)=(A+B)+C
  Commutative law: A+B=B+A

Example:

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

Basics of Vector Calculus

- **Vector Algebra**

D1.2. A vector field $\mathbf{S}$ is expressed in cartesian coordinates as $\mathbf{S} = \{125/[(x-1)^2 + (y-2)^2 + (z+1)^2]\}\{(x-1)\mathbf{a}_x + (y-2)\mathbf{a}_y + (z+1)\mathbf{a}_z\}$. (*a*) Evaluate $\mathbf{S}$ at $P(2, 4, 3)$. (*b*) Determine a unit vector that gives the direction of $\mathbf{S}$ at P. (*c*) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$.

Ans. $5.95\mathbf{a}_x + 11.90\mathbf{a}_y + 23.8\mathbf{a}_z$; $0.218\mathbf{a}_x + 0.436\mathbf{a}_y + 0.873\mathbf{a}_z$; $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$

Example:

If $\mathbf{A} = 10\mathbf{a}_x - 4\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y$, find: (a) the component of $\mathbf{A}$ along $\mathbf{a}_y$, (b) the magnitude of $3\mathbf{A} - \mathbf{B}$, (c) a unit vector along $\mathbf{A} + 2\mathbf{B}$.

Basics of Vector Calculus

- **Vector Algebra**

Solution:

(a) The component of $\mathbf{A}$ along $\mathbf{a}_y$ is $A_y = -4$.

(b)
$$\begin{aligned} 3\mathbf{A} - \mathbf{B} &= 3(10, -4, 6) - (2, 1, 0) \\ &= (30, -12, 18) - (2, 1, 0) \\ &= (28, -13, 18) \end{aligned}$$

Hence

$$\begin{aligned} |3\mathbf{A} - \mathbf{B}| &= \sqrt{28^2 + (-13)^2 + (18)^2} = \sqrt{1277} \\ &= 35.74 \end{aligned}$$

(c) Let $\mathbf{C} = \mathbf{A} + 2\mathbf{B} = (10, -4, 6) + (4, 2, 0) = (14, -2, 6)$.

A unit vector along $\mathbf{C}$ is

$$\mathbf{a}_c = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{(14, -2, 6)}{\sqrt{14^2 + (-2)^2 + 6^2}}$$

or

$$\mathbf{a}_c = 0.9113\mathbf{a}_x - 0.1302\mathbf{a}_y + 0.3906\mathbf{a}_z$$

Note that $|\mathbf{a}_c| = 1$ as expected.

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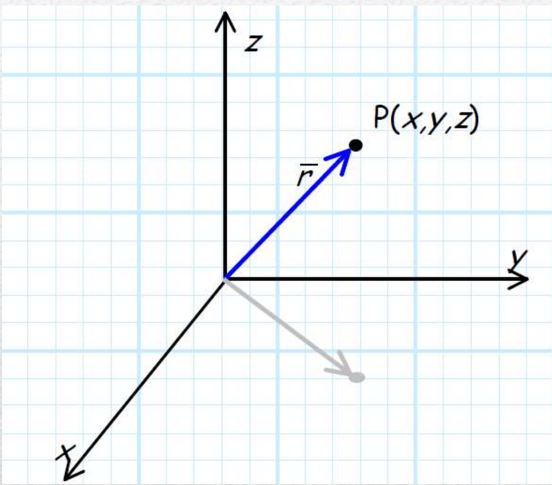
Basics of Vector Calculus

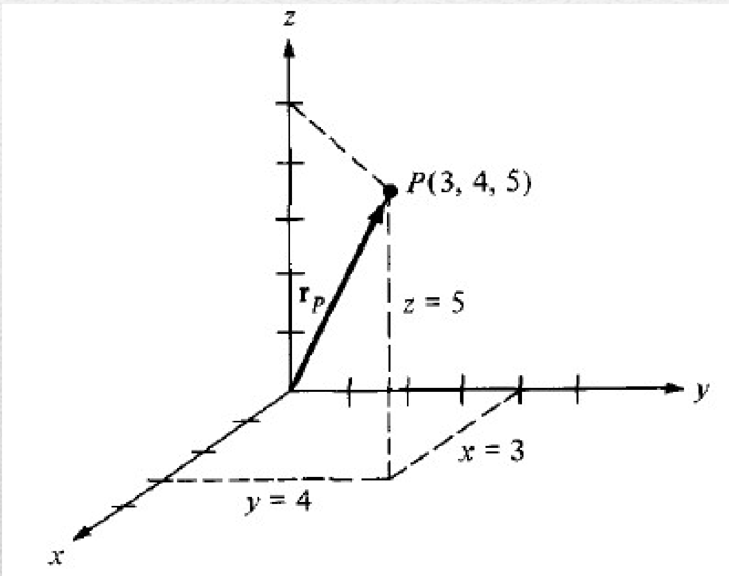
- **Vector Algebra**

Position Vector: The vector connecting the origin to a specific point X= $x\mathbf{a}_{x} + y\mathbf{a}_{y} + z\mathbf{a}_{z}$



Example: the position vector of the point (3,4,5) is:

$$\mathbf{r}_P = 3\mathbf{a}_x + 4\,\mathbf{a}_y + 5\,\mathbf{a}_z$$



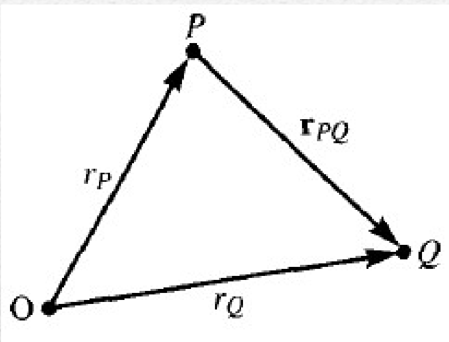
position vector of the point (3,4,5)

Basics of Vector Calculus

- **Vector Algebra**

Displacement (distance) vector: It is the vector directed from one point to another

$\mathbf{r}_{PQ} = (x_Q - x_P)\mathbf{a}_x + (y_Q - y_P)\mathbf{a}_y + (z_Q - z_P)\mathbf{a}_z$



The infinitesimal displacement (distance) vector

(x, y, z) to $(x + dx, y + dy, z + dz)$

$\mathbf{dl} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$

Example:

Points P and Q are located at $(0, 2, 4)$ and $(-3, 1, 5)$. Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magntude of 10

Basics of Vector Calculus

Solution:

(a) $\mathbf{r}_P = 0\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z = 2\mathbf{a}_y + 4\mathbf{a}_z$

(b) $\mathbf{r}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$
or $\mathbf{r}_{PQ} = -3\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z$

(c) Since $\mathbf{r}_{PQ}$ is the distance vector from P to Q, the distance between P and Q is the magnitude of this vector; that is,

$$d = |\mathbf{r}_{PQ}| = \sqrt{9 + 1 + 1} = 3.317$$

Alternatively:

$$d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$$
$$= \sqrt{9 + 1 + 1} = 3.317$$

(d) Let the required vector be $\mathbf{A}$, then

$$\mathbf{A} = A\mathbf{a}_A$$

where $A = 10$ is the magnitude of $\mathbf{A}$. Since $\mathbf{A}$ is parallel to PQ, it must have the same unit vector as $\mathbf{r}_{PQ}$ or $\mathbf{r}_{QP}$. Hence,

$$\mathbf{a}_A = \pm\frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = \pm\frac{(-3, -1, 1)}{3.317}$$

and

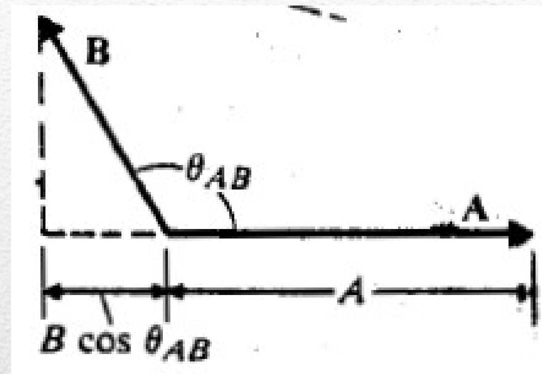
$$\mathbf{A} = \pm\frac{10(-3, -1, -1)}{3.317} = \pm(-9.045\mathbf{a}_x - 3.015\mathbf{a}_y + 3.015\mathbf{a}_z)$$

Basics of Vector Calculus

- **Vector Algebra**

5. Vector Product

a. Dot Product: $\mathbf{C=A.B=|A||B|\cos\theta_{AB}}$



$B_A = \mathbf{a_A.B} = |\mathbf{B}||\mathbf{a_A}|\cos\theta_{AB} = |\mathbf{B}||\mathbf{a_A}|\cos\theta_{AB}$ **=>** $\mathbf{B_A} = B_A\,\mathbf{a_A}$

is the change of B in the direction of A

θ_{AB} is the smaller angle between the two vectors

In terms of the vectors' components

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{a}_x\cdot\mathbf{a}_y = \mathbf{a}_y\cdot\mathbf{a}_z = \mathbf{a}_z\cdot\mathbf{a}_x = 0$$

$$\mathbf{a}_x\cdot\mathbf{a}_x = \mathbf{a}_y\cdot\mathbf{a}_y = \mathbf{a}_z\cdot\mathbf{a}_z = 1$$

(i) *Commutative law:*

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

(ii) *Distributive law:*

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

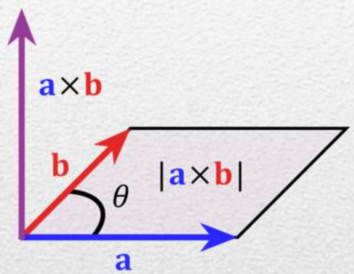
$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2$$

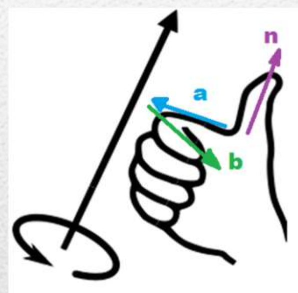
Basics of Vector Calculus

- **Vector Algebra**

5. Vector Product

b. Cross Product: $C = A \times B = |A||B| \sin\theta_{AB}\, a_N$





Distributive law: $A \times (B+C) = A \times B + A \times C$
Commutative law: $A \times B = -B \times A$
Not Asscoiative: $A \times (B \times C) \neq (A \times B) \times C$

The magnitude is the area of the parallelogram
The direction is determined by the right hand rule.

In terms of vectors components: it is the following determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

Basics of Vector Calculus

- **Vector Algebra**

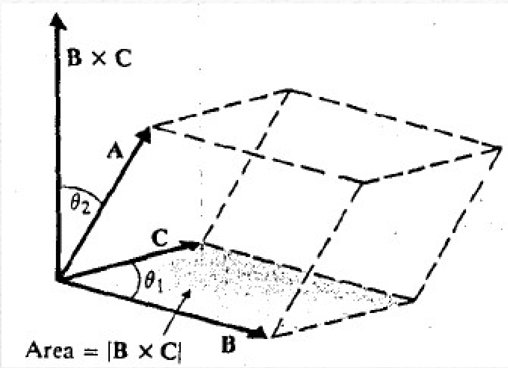
  5. Vector Product

  - **Product of three vectors**

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

In terms of vectors components

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



The volume of the parallelepiped

  - **Triple Cross product rule (BAC-CAB rule)**

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Basics of Vector Calculus

- **Vector Algebra**

Note that the cross product has the following basic properties:

(i) It is not commutative:

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

It is anticommutative:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

(ii) It is not associative:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

(iii) It is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

(iv)

$$\mathbf{A} \times \mathbf{A} = 0$$

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$$

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Basics of Vector Calculus

- **Vector Algebra**

Example:

Given vectors $\mathbf{A} = 3\mathbf{a}_x + 4\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_y - 5\mathbf{a}_z$, find the angle between **A** and **B**.

Solution:

The angle θ_{AB} can be found by using either dot product or cross product.

$$\mathbf{A} \cdot \mathbf{B} = (3, 4, 1) \cdot (0, 2, -5) = 0 + 8 - 5 = 3$$

$$|\mathbf{A}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$

$$|\mathbf{B}| = \sqrt{0^2 + 2^2 + (-5)^2} = \sqrt{29}$$

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{3}{\sqrt{(26)(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1} 0.1092 = 83.73°$$

Basics of Vector Calculus

- **Vector Algebra**

Alternatively:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix}$$

$$= (-20 - 2)\mathbf{a}_x + (0 + 15)\mathbf{a}_y + (6 - 0)\mathbf{a}_z$$

$$= (-22, 15, 6)$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-22)^2 + 15^2 + 6^2} = \sqrt{745}$$

$$\sin \theta_{AB} = \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$

$$\theta_{AB} = \cos^{-1} 0.994 = 83.73°$$

Basics of Vector Calculus

- **Vector Algebra**

Example

Three field quantities are given by

$$\mathbf{P} = 2\mathbf{a}_x - \mathbf{a}_z$$

$$\mathbf{Q} = 2\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{R} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$$

Determine

(a) $(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q})$

(b) $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$

(c) $\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$

(d) $\sin \theta_{QR}$

(e) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R})$

(f) A unit vector perpendicular to both **Q** and **R**

(g) The component of **P** along **Q**

Basics of Vector Calculus

- **Vector Algebra**

Solution:

(a)

$$(\mathbf{P} + \mathbf{Q}) \times (\mathbf{P} - \mathbf{Q}) = \mathbf{P} \times (\mathbf{P} - \mathbf{Q}) + \mathbf{Q} \times (\mathbf{P} - \mathbf{Q})$$
$$= \mathbf{P} \times \mathbf{P} - \mathbf{P} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P} - \mathbf{Q} \times \mathbf{Q}$$
$$= 0 + \mathbf{Q} \times \mathbf{P} + \mathbf{Q} \times \mathbf{P} - 0$$
$$= 2\mathbf{Q} \times \mathbf{P}$$
$$= 2\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$
$$= 2(1 - 0)\,\mathbf{a}_x + 2(4 + 2)\,\mathbf{a}_y + 2(0 + 2)\,\mathbf{a}_z$$
$$= 2\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z$$

(b) The only way $\mathbf{Q} \cdot \mathbf{R} \times \mathbf{P}$ makes sense is

$$\mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = (2, -1, 2) \cdot \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$
$$= (2, -1, 2) \cdot (3, 4, 6)$$
$$= 6 - 4 + 12 = 14.$$

Basics of Vector Calculus

- **Vector Algebra**

(c) From eq. (1.28)

$$\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{Q} \cdot (\mathbf{R} \times \mathbf{P}) = 14$$

or

$$\begin{aligned}\mathbf{P} \cdot (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \cdot (5, 2, -4) \\ &= 10 + 0 + 4 \\ &= 14\end{aligned}$$

(d)

$$\begin{aligned}\sin \theta_{QR} &= \frac{|\mathbf{Q} \times \mathbf{R}|}{|\mathbf{Q}||\mathbf{R}|} = \frac{|(5, 2, -4)|}{|(2, -1, 2)||(2, -3, 1)|} \\ &= \frac{\sqrt{45}}{3\sqrt{14}} = \frac{\sqrt{5}}{\sqrt{14}} = 0.5976\end{aligned}$$

(e)

$$\begin{aligned}\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= (2, 0, -1) \times (5, 2, -4) \\ &= (2, 3, 4)\end{aligned}$$

Basics of Vector Calculus

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Alternatively, using the bac-cab rule,

$$\begin{aligned}\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) &= \mathbf{Q}(\mathbf{P} \cdot \mathbf{R}) - \mathbf{R}(\mathbf{P} \cdot \mathbf{Q}) \\ &= (2, -1, 2)(4 + 0 - 1) - (2, -3, 1)(4 + 0 - 2) \\ &= (2, 3, 4)\end{aligned}$$

(f) A unit vector perpendicular to both **Q** and **R** is given by

$$\begin{aligned}\mathbf{a} &= \frac{\pm \mathbf{Q} \times \mathbf{R}}{|\mathbf{Q} \times \mathbf{R}|} = \frac{\pm(5, 2, -4)}{\sqrt{45}} \\ &= \pm(0.745, 0.298, -0.596)\end{aligned}$$

Note that $|\mathbf{a}| = 1$, $\mathbf{a} \cdot \mathbf{Q} = 0 = \mathbf{a} \cdot \mathbf{R}$. Any of these can be used to check **a**.

(g) The component of **P** along **Q** is

$$\begin{aligned}\mathbf{P}_Q &= |\mathbf{P}| \cos \theta_{PQ} \mathbf{a}_Q \\ &= (\mathbf{P} \cdot \mathbf{a}_Q)\mathbf{a}_Q = \frac{(\mathbf{P} \cdot \mathbf{Q})\mathbf{Q}}{|\mathbf{Q}|^2} \\ &= \frac{(4 + 0 - 2)(2, -1, 2)}{(4 + 1 + 4)} = \frac{2}{9}(2, -1, 2) \\ &= 0.4444\mathbf{a}_x - 0.2222\mathbf{a}_y + 0.4444\mathbf{a}_z.\end{aligned}$$