جامعة بيرزيت
BIRZEIT UNIVERSITY

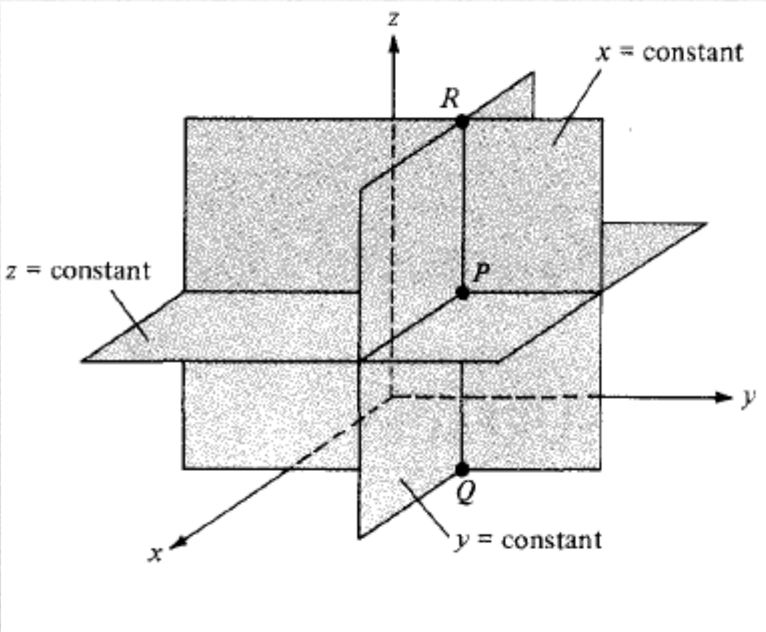
Dr.Ashraf Al-Rimawi

Electromagnetic Theory I

Second Semester, 2017/2018

Coordinate Systems and Transformation

- **Cartesian Coordinates**



$$a_x \bullet a_y = 0,\ a_y \bullet a_z = 0,\ a_z \bullet a_x = 0$$

$$a_x \bullet a_x = 1,\ a_y \bullet a_y = 1,\ a_z \bullet a_z = 1$$

We study right handed coordinates system xyz:

$$a_x \times a_y = a_z,\ a_y \times a_z = a_x,\ a_z \times a_x = a_y$$

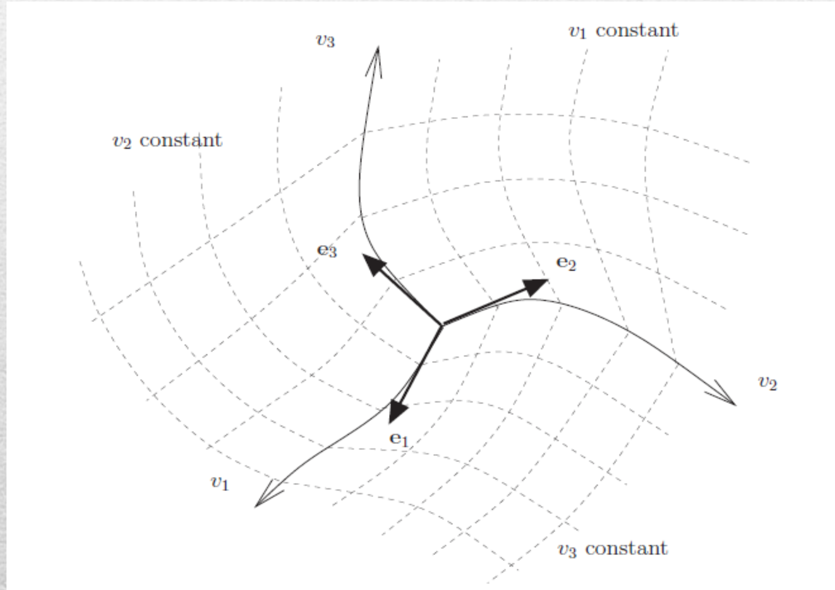
$$x = \text{constant},\ y = \text{constant},\ z = \text{constant}$$

Coordinate Systems and Transformation

- **Orthogonal Curvilinear Coordinates: it is a coordinate system in which the points in space are specified as the intersection of three curved planes (curved lines), which are mutually perpendicular to each other. (3 orthogonal planes, basis vectors needed)**

Goals:

- **Define new coordinate systems for cylindrical and spherical symmetrical problems(plane equations and basis vectors for which only one coordinate changes when moving along)**
- **To get the differential length differential surface and differential volume in different coordinate systems, because we need them to perform line surface and volume integrals for problems with different symmetries**
- **To convert from one coordinate system to another**



Coordinate Systems and Transformation

- **Sometimes presenting space coordinates in a different coordinate system than Cartesian coordinates can be beneficial. Specifically, in problems where spherical or cylindrical symmetry can lead to simplified calculations and treatment of the problem in hand. That is why, the more general curvilinear coordinate system is presented here and the two most important curvilinear coordinates (spherical and cylindrical) are treated in details. To define a coordinate system, the planes need to be defined and basis vectors.**

- **Any curved plane can be presented as a function of the x, y, z coordinates.**
- **Suppose a new coordinate curvilinear coordinate system is to be defined by three curved planes (u_1 = constant , u_2 = constant, u_3 =constant)**

- **$u_1 = f_1 (x, y, z)$, $u_2 = f_2 (x, y, z)$, $u_3 = f_3 (x, y, z)$**
- **Inversion of the functions: $x = g_1 (u_1, u_2, u_3)$, $y = g_1 (u_1, u_2, u_3)$, $z = g_1 (u_1, u_2, u_3)$**

To represent any vector in the new coordinate system two steps are required:

$\mathbf{A} = A_x(x, y, z)\mathbf{a}_{x+} A_z(x, y, z)\mathbf{a}_{y+} A_z(x, y, z)\mathbf{a}_z$

1. Substitute **$x = g_1 (u_1, u_2, u_3)$, $y = g_1 (u_1, u_2, u_3)$, $z = g_1 (u_1, u_2, u_3)$**
2. Find the relation between the directions **$\mathbf{a}_{u1}$, $\mathbf{a}_{u2}$, $\mathbf{a}_{u3}$ and $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$**

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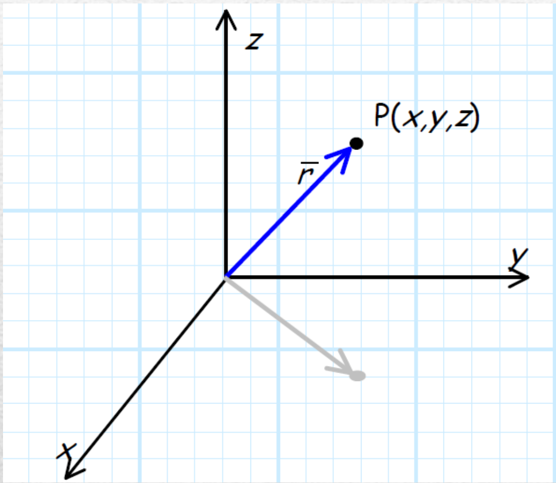
Coordinate Systems and Transformation

- **Important:** the **position vector** is very helpful in determining the relation between the directions of the two coordinate systems.



R= x**a**$_x$+ y**a**$_y$+ z**a**$_z$

R= g_1 (u_1, u_2, u_3) **a**$_x$+ g_2 (u_1, u_2, u_3) **a**$_y$ +g_3 (u_1, u_2, u_3) **a**$_z$

Note: The unit vectors can be only in the Cartesian coordinates for the position vector, because the directions in other coordinate system are dependent of space, whereas, they are independent from the the position of the point in space.

The derivative of the position vector with respect to a coordinate (keeping the other two coordinates constant) is in the direction of that coordinate:

$$\frac{\partial R}{\partial u_1} = \breve{a}_{u_1}, \quad \frac{\partial R}{\partial u_2} = \breve{a}_{u_2}, \quad \frac{\partial R}{\partial u_3} = \breve{a}_{u_3}$$

$\breve{a}_{u_1}$ might not be a unit vector, we define a unit vector a_{u_1} by dividing by the magnitude of $\breve{a}_{u_1}$

Coordinate Systems and Transformation

- **The unit vectors are:**

$$\mathbf{a}_{u_1} = \frac{\vec{d}_{u_1}}{\left|\frac{\partial R}{\partial u_1}\right|}, \mathbf{a}_{u_2} = \frac{\vec{d}_{u_2}}{\left|\frac{\partial R}{\partial u_2}\right|}, \mathbf{a}_{u_3} = \frac{\vec{d}_{u_3}}{\left|\frac{\partial R}{\partial u_3}\right|}$$

$$h_1 = \left|\frac{\partial R}{\partial u_1}\right|, \; h_2 = \left|\frac{\partial R}{\partial u_2}\right|, \; h_3 = \left|\frac{\partial R}{\partial u_3}\right|$$

$$\mathbf{a}_{u_1} = \frac{1}{h_1}\frac{\partial R}{\partial u_1}, \; \mathbf{a}_{u_2} = \frac{1}{h_2}\frac{\partial R}{\partial u_2}, \; \mathbf{a}_{u_3} = \frac{1}{h_3}\frac{\partial R}{\partial u_3}$$

h_1, h_2, h_3 are called the conversion metric coefficients

The differential length vector is important for finding integrals (such as work, and potential difference), It is also helpful in finding the differential area and volume which is also essential to perform integrals in electromagnetic theory. The conversion factors are necessary to convert a non-length coordinate like angle to a differential length in the direction of that coordinate.

$$dR = \frac{\partial R}{\partial u_1}du_1 + \frac{\partial R}{\partial u_2}du_2 + \frac{\partial R}{\partial u_3}du_3$$

$$dR = h_1 du_1 \mathbf{a}_{u_1} + h_2 du_2 \mathbf{a}_{u_2} + h_3 du_3 \mathbf{a}_{u_3}$$

Can be found graphical or using this approach, for more complicated curvilinear coordinates this is easier

Coordinate Systems and Transformation

- **Differential length area and volume**

Differential length

$$dL = h_1 du_1 \mathrm{a}_{u_1} + h_2 du_2 \mathrm{a}_{u_2} + h_3 du_3 \mathrm{a}_{u_3}$$

$$dL = \sqrt{(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2}$$

Differential area

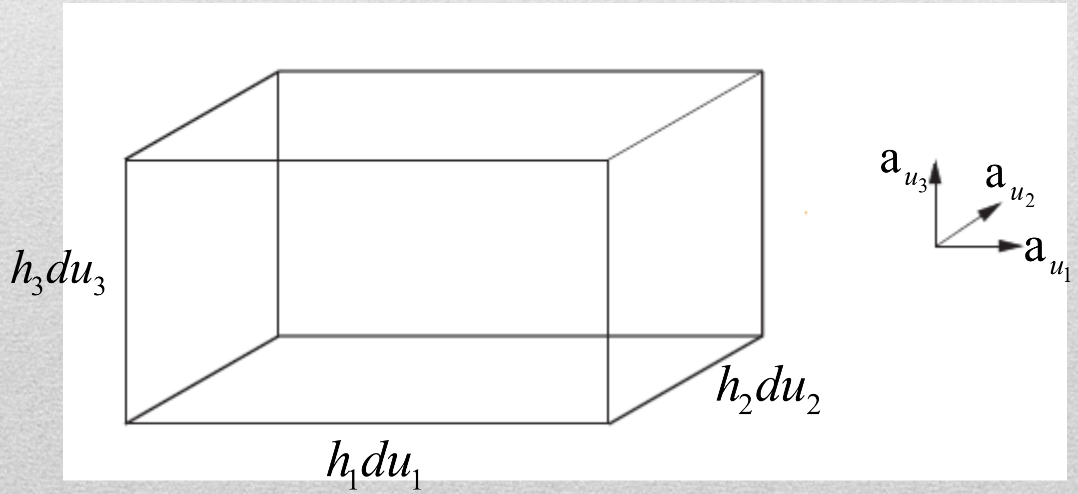
$$dS_{u_1} = h_2 du_2 h_3 du_3 \mathrm{a}_{u_1}$$

$$dS_{u_2} = h_1 du_1 h_3 du_3 \mathrm{a}_{u_2}$$

$$dS_{u_3} = h_1 du_1 h_2 du_2 \mathrm{a}_{u_3}$$

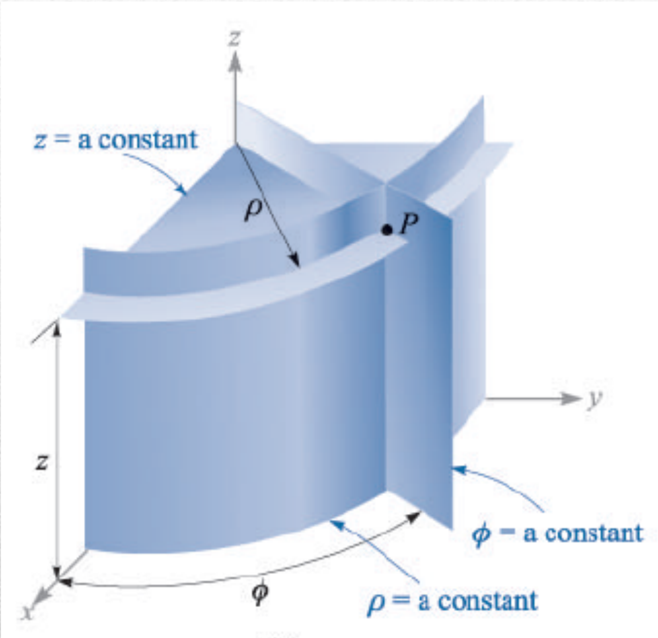
Differential Volume

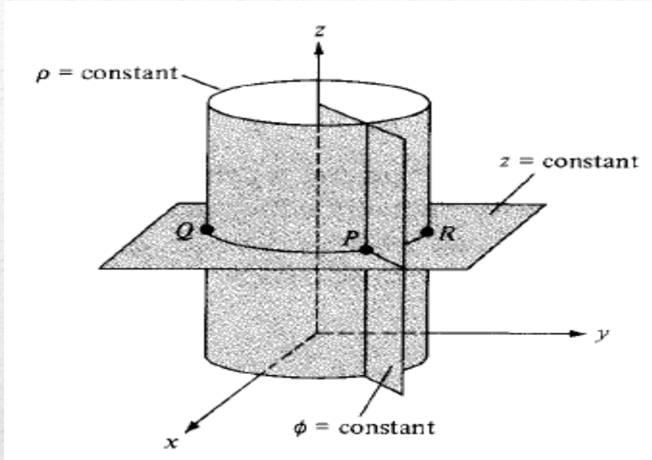
$$dV = h_1 du_1 h_2 du_2 h_3 du_3$$



Coordinate Systems and Transformation

- **Circular cylindrical coordinate system**

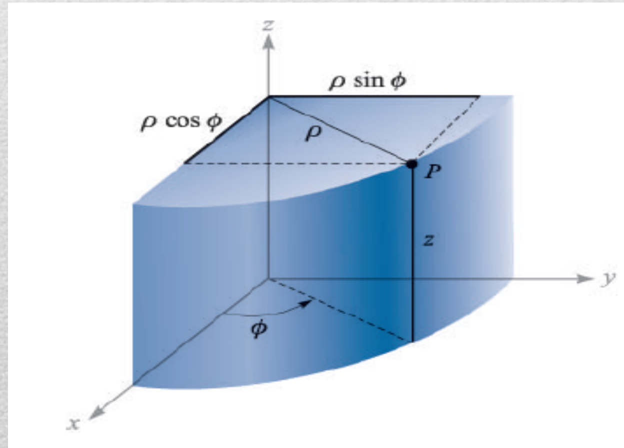




$$z = \text{constant},\ \rho = \text{constant},\ \varphi = \text{constant}$$

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



Coordinate Systems and Transformation

- **Circular cylindrical coordinate system**

$a_\rho \bullet a_\varphi = 0,\ a_\varphi \bullet a_z = 0,\ a_z \bullet a_\rho = 0$

$a_\rho \bullet a_\rho = 1,\ a_\varphi \bullet a_\varphi = 1,\ a_z \bullet a_z = 1$

$R = x a_x + y a_y + z a_z$

$R = \rho\cos\varphi a_x + \rho\sin\varphi a_y + z a_z$

$(\rho,\varphi,z) \Rightarrow a_\rho \times a_\varphi = a_z,\ a_\varphi \times a_z = a_\rho,\ a_z \times a_\rho = a_\varphi$

Finding the directions and metric coefficients:

$$\frac{\partial R}{\partial \rho} = \cos\varphi a_x + \sin\varphi a_y = \vec{a}_\rho \qquad \frac{\partial R}{\partial \varphi} = -\rho\sin\varphi a_x + \rho\cos\varphi a_y = \vec{a}_\varphi \qquad \frac{\partial R}{\partial z} = a_z = \vec{a}_z$$

$$\left|\frac{\partial R}{\partial \rho}\right| = h_\rho = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 \qquad \left|\frac{\partial R}{\partial \varphi}\right| = h_\varphi = \sqrt{\rho^2(\cos^2\varphi + \sin^2\varphi)} = \rho \qquad \left|\frac{\partial R}{\partial z}\right| = h_z = \sqrt{1} = 1$$

$$a_\rho = \cos\varphi a_x + \sin\varphi a_y \qquad a_\varphi = -\sin\varphi a_x + \cos\varphi a_y \qquad a_z = a_z$$

Solving the three equations:

$$a_x = \cos\varphi a_\rho - \sin\varphi a_\varphi \qquad a_y = \sin\varphi a_\rho + \cos\varphi a_\varphi \qquad a_z = a_z$$

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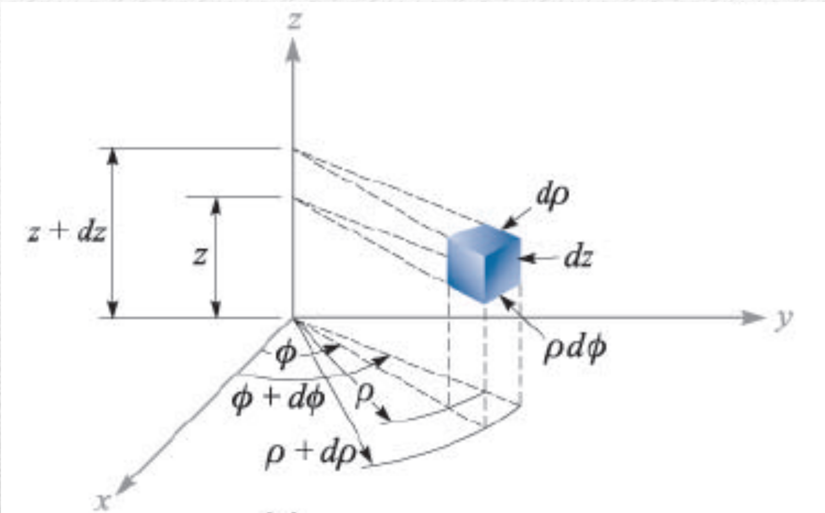
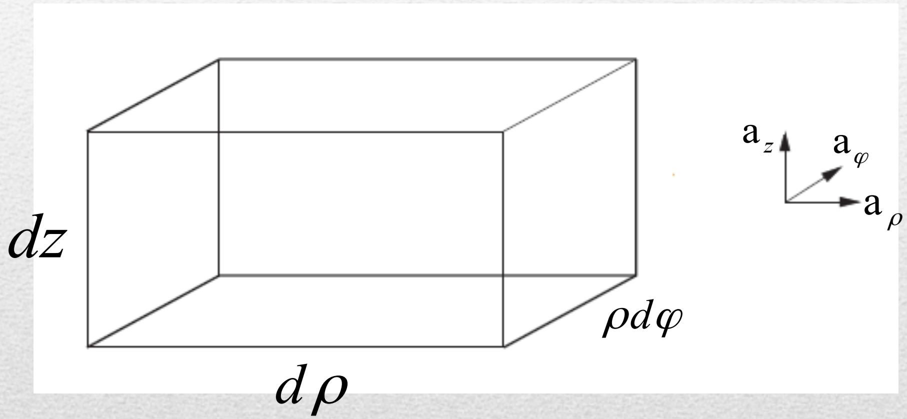
Coordinate Systems and Transformation

- **Circular cylindrical coordinate system**





$$dL = d\rho \mathrm{a}_\rho + \rho d\varphi \mathrm{a}_\varphi + dz \mathrm{a}_z$$

$$|dL| = \sqrt{(d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2}$$

$$dS_\rho = \rho d\varphi dz \mathrm{a}_\rho$$

$$dS_\varphi = d\rho dz \mathrm{a}_\varphi$$

$$dS_z = d\rho\rho d\varphi \mathrm{a}_z = \rho d\rho d\varphi \mathrm{a}_z$$

$$dV = \rho d\rho d\varphi dz$$

Coordinate Systems and Transformation

- **Conversion between Circular cylindrical coordinates and Cartesian coordinates**

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \quad \xleftrightarrow{\text{Conversion}} \quad \mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

To find any desired component of a vector, we recall from the discussion of the dot product that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction. Hence,

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi$$

Expanding these dot products, we have

$$A_\rho = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\rho = A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho$$

$$A_\phi = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\phi = A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi$$

$$A_z = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_z = A_z\mathbf{a}_z\cdot\mathbf{a}_z = A_z$$

$$a_x = \cos\varphi\, a_\rho - \sin\varphi\, a_\varphi$$

$$a_y = \sin\varphi\, a_\rho + \cos\varphi\, a_\varphi$$

$$a_z = a_z$$

$$\Rightarrow$$

$$a_x \bullet a_\rho = \cos\varphi$$

$$a_y \bullet a_\rho = \sin\varphi$$

$$a_x \bullet a_\varphi = -\sin\varphi$$

$$a_y \bullet a_\varphi = \cos\varphi$$

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

Coordinate Systems and Transformation

- **Conversion between Circular cylindrical coordinates and Cartesian coordinates in matrix form**

In matrix form, we have the transformation of vector **A** from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

From cylindrical to cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Coordinate Systems and Transformation

- **Conversion between Circular cylindrical coordinates and Cartesian coordinates**

Dot products of unit vectors in cylindrical and cartesian coordinate systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1

Example:

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$
$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$
$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Thus,

$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

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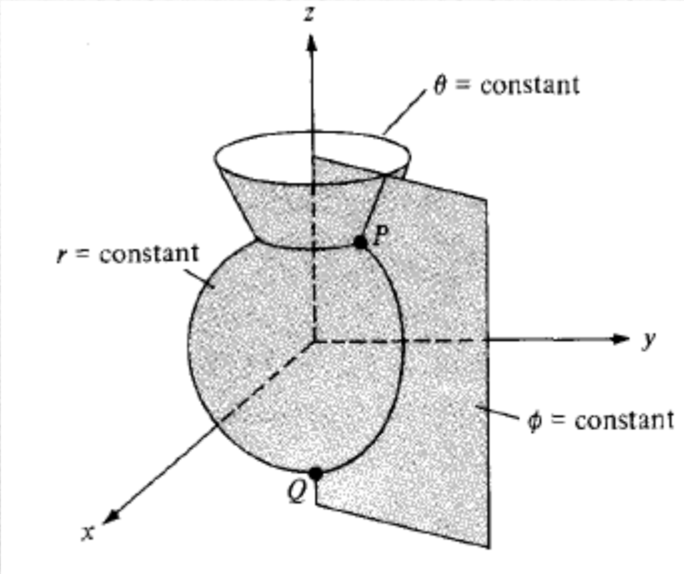
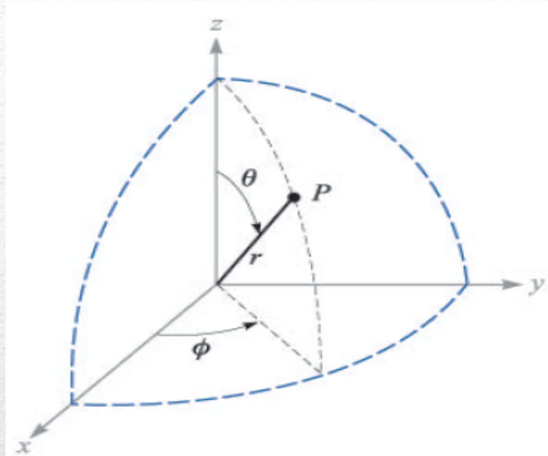
Coordinate Systems and Transformation

- **Spherical coordinate system** θ = constant, r = constant, φ = constant





$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

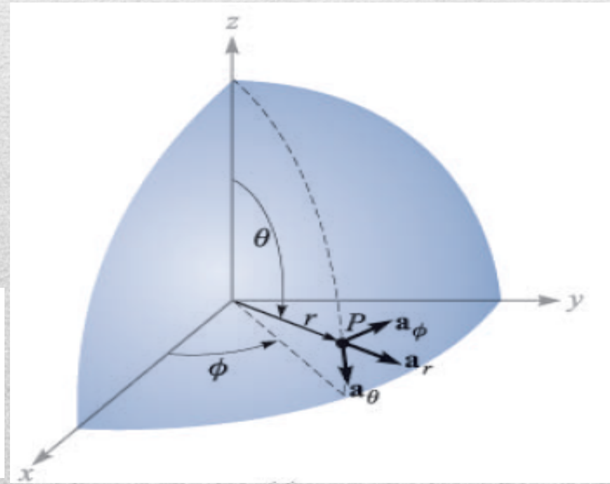
$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$



Coordinate Systems and Transformation

- **Spherical coordinate system**

$$R = x\mathrm{a}_x + y\mathrm{a}_y + z\mathrm{a}_z$$

$$R = r\sin\theta\cos\varphi\mathrm{a}_x + r\sin\theta\sin\varphi\mathrm{a}_y + r\cos\theta\mathrm{a}_z$$

$$\mathrm{a}_r \bullet \mathrm{a}_\theta = 0,\ \mathrm{a}_\theta \bullet \mathrm{a}_\varphi = 0,\ \mathrm{a}_\varphi \bullet \mathrm{a}_r = 0$$

$$\mathrm{a}_r \bullet \mathrm{a}_r = 1,\ \mathrm{a}_\theta \bullet \mathrm{a}_\theta = 1,\ \mathrm{a}_\varphi \bullet \mathrm{a}_\varphi = 1$$

$$(\mathrm{r},\theta,\varphi) \Rightarrow \mathrm{a}_r \times \mathrm{a}_\theta = \mathrm{a}_\varphi,\ \mathrm{a}_\theta \times \mathrm{a}_\varphi = \mathrm{a}_r,\ \mathrm{a}_\varphi \times \mathrm{a}_r = \mathrm{a}_\theta$$

Finding the directions and metric coefficients:

$$\frac{\partial R}{\partial r} = \sin\theta\cos\varphi\mathrm{a}_x + \sin\theta\sin\varphi\mathrm{a}_y + \cos\theta\mathrm{a}_z = \breve{a}_r$$

$$\left|\frac{\partial R}{\partial r}\right| = h_r = \sqrt{\sin^2\theta\left(\cos^2\varphi + \sin^2\varphi\right) + \cos^2\theta} = 1$$

$$\mathrm{a}_r = \sin\theta\cos\varphi\mathrm{a}_x + \sin\theta\sin\varphi\mathrm{a}_y + \cos\theta\mathrm{a}_z$$

$$\frac{\partial R}{\partial \theta} = r\cos\theta\cos\varphi\mathrm{a}_x + r\cos\theta\sin\varphi\mathrm{a}_y - r\sin\theta\mathrm{a}_z = \breve{a}_\theta$$

$$\left|\frac{\partial R}{\partial \theta}\right| = h_\theta = \sqrt{r^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + r^2\sin^2\theta} = r$$

$$\mathrm{a}_\theta = \cos\theta\cos\varphi\mathrm{a}_x + \cos\theta\sin\varphi\mathrm{a}_y - \sin\theta\mathrm{a}_z$$

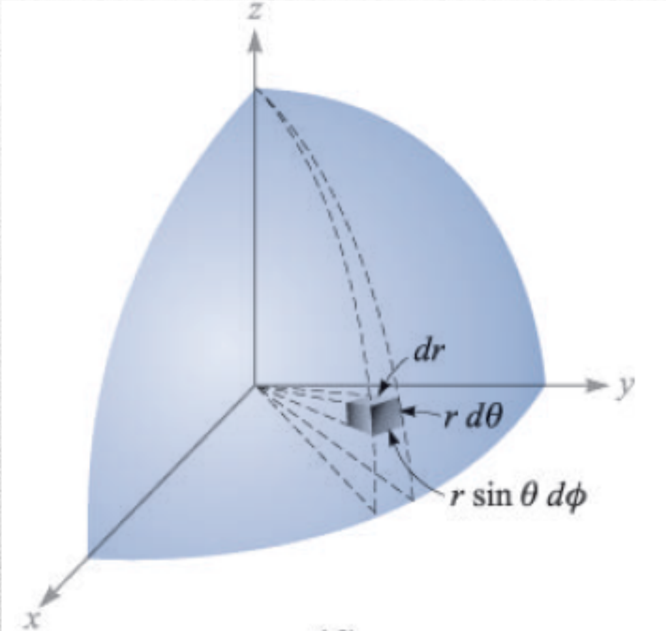
$$\frac{\partial R}{\partial \varphi} = -r\sin\theta\sin\varphi\mathrm{a}_x + r\sin\theta\cos\varphi\mathrm{a}_y = \breve{a}_\theta$$

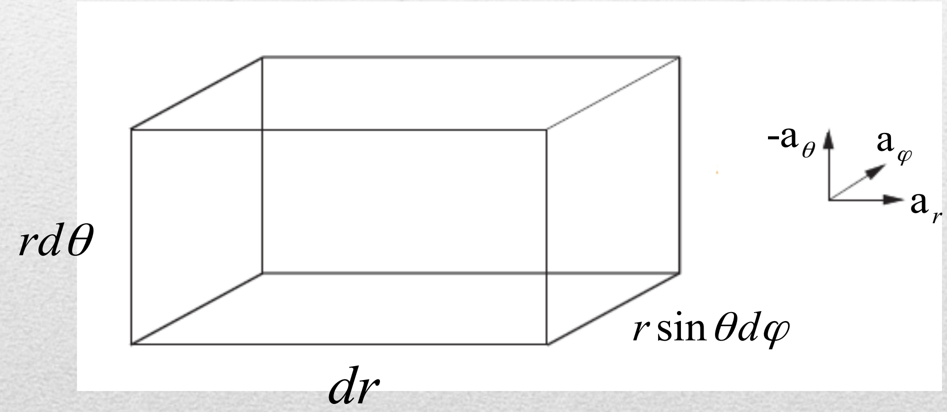
$$\left|\frac{\partial R}{\partial \varphi}\right| = h_\varphi = \sqrt{r^2\sin^2\theta(\cos^2\varphi + \sin^2\varphi)} = r\sin\theta$$

$$\mathrm{a}_\varphi = -\sin\varphi\mathrm{a}_x + \cos\varphi\mathrm{a}_y$$

Coordinate Systems and Transformation

- **Spherical coordinate system**





$$dL = dr\mathrm{a}_r + rd\theta\mathrm{a}_\theta + r\sin\theta d\varphi\mathrm{a}_\varphi$$

$$|dL| = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta d\varphi)^2}$$

$$dS_r = r^2\sin\theta d\theta d\varphi\mathrm{a}_r \quad dV = r^2\sin\theta dr d\theta d\varphi$$

$$dS_\theta = r\sin\theta dr d\varphi\mathrm{a}_\theta$$

$$dS_\varphi = rdrd\theta\mathrm{a}_\varphi$$

Coordinate Systems and Transformation

- **Spherical coordinate system**

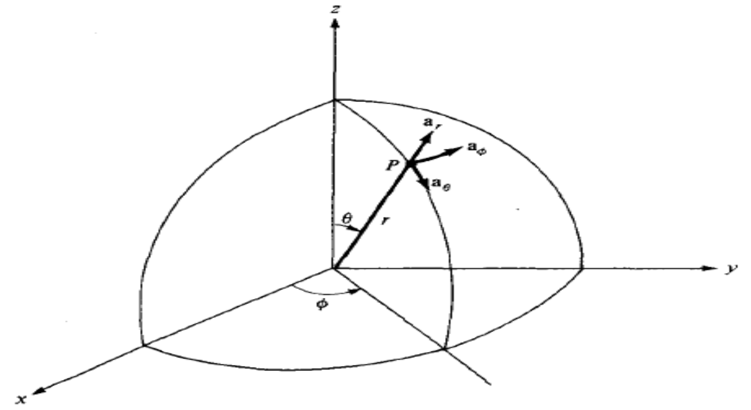


Figure 2.4 Point *P* and unit vectors in spherical coordinates.

TABLE 1.2
Dot products of unit vectors in spherical and cartesian coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x\cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y\cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z\cdot$	$\cos\theta$	$-\sin\theta$	0

Coordinate Systems and Transformation

- **Spherical coordinate system conversion to and from Cartesian**

In matrix form, the $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{2.27}$$

The inverse transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ is similarly obtained, or we obtain it from eq. (2.23). Thus,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \tag{2.28}$$

Coordinate Systems and Transformation

- **Example**

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cyl
and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and sp
systems.

Solution:

At point P: $x = -2$, $y = 6$, $z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62°$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the Cartesian system, $\mathbf{A}$ at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

Coordinate Systems and Transformation

- **Cont…**

For vector $\mathbf{A}$, $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

or

$$A_\rho = y\cos\phi + (x + z)\sin\phi$$

$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$

$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\mathbf{A} = (A_\rho, A_\phi, A_z) = [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$\mathbf{A} = \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi$$

$$= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi$$

Coordinate Systems and Transformation

- **Cont…**

But $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi)\\ &= r[\sin^2\theta \cos\phi \sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r\\ &\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta\\ &\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi\end{aligned}$$

At P

$$r = 7, \qquad \tan\phi = \frac{6}{-2}, \qquad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence,

Coordinate Systems and Transformation

- **Cont…**

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}, \qquad \cos\theta = \frac{3}{7}, \qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\begin{aligned}
\mathbf{A} &= 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_r \\
&+ 7\cdot\left[\frac{\sqrt{40}}{7}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\theta \\
&+ 7\cdot\left[\frac{-\sqrt{40}}{7}\cdot\frac{36}{40} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi \\
&= \frac{-6}{7}\mathbf{a}_r - \frac{18}{7\sqrt{40}}\mathbf{a}_\theta - \frac{38}{\sqrt{40}}\mathbf{a}_\phi \\
&= -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi
\end{aligned}$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

Coordinate Systems and Transformation

- **Example: from spherical to cylindrical**

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

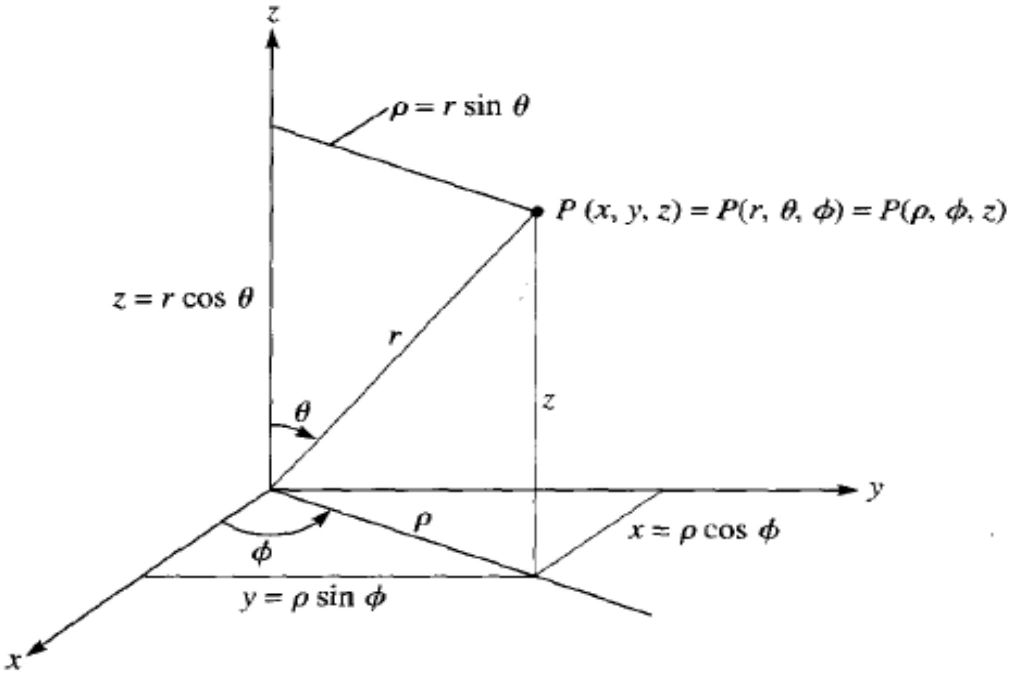


Figure 2.5 Relationships between space variables (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).

Coordinate Systems and Transformation

- **Distance and vector magnitude in coordinate systems**

Important: the magnitude of the vector is the same in all coordinate systems. This can be used as a way to confirm the correctness of conversion.

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\mathbf{A}| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$$

$$|\mathbf{A}| = (A_r^2 + A_\theta^2 + A_\theta^2)^{1/2}$$

The distance between two points is usually necessary in EM theory. The distance d between two points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is generally given by

$$d = |\mathbf{r}_2 - \mathbf{r}_1|$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \text{ (Cartesian)}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \text{ (cylindrical)}$$

$$d^2 = r_2^2 + r_1^2 - 2r_1r_2 \cos\theta_2 \cos\theta_1 - 2r_1r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1) \text{ (spherical)}$$