جامعة بيرزيت

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Electromagnetic Theory I

Basics of Vector Calculus

- **Line, Surface and Volume integrals**

EXAMPLE 3.1 Consider the object shown in Figure 3.7. Calculate

(a) The distance BC
(b) The distance CD
(c) The surface area $ABCD$
(d) The surface area ABO
(e) The surface area $AOFD$
(f) The volume $ABDCFO$

Solution:

Although points A, B, C, and D are given in Cartesian coordinates, it is obvious that the object has cylindrical symmetry. Hence, we solve the problem in cylindrical coordinates The points are transformed from Cartesian to cylindrical coordinates as follows:

$$A(5, 0, 0) \rightarrow A(5, 0°, 0)$$

$$B(0, 5, 0) \rightarrow B\left(5, \frac{\pi}{2}, 0\right)$$

$$C(0, 5, 10) \rightarrow C\left(5, \frac{\pi}{2}, 10\right)$$

$$D(5, 0, 10) \rightarrow D(5, 0°, 10)$$

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Basics of Vector Calculus

- **Line, Surface and Volume integrals**

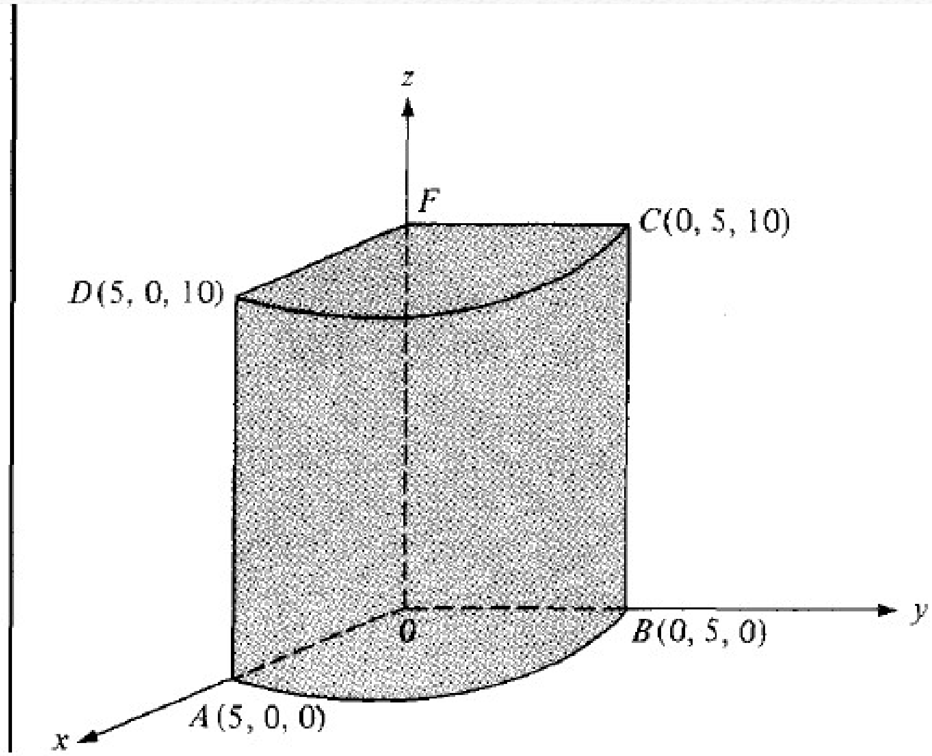

Figure 3.7 For Example 3.1.

Basics of Vector Calculus

- **Line, Surface and Volume integrals**

(a) Along BC, $dl = dz$; hence,

$$BC = \int dl = \int_0^{10} dz = 10$$

(b) Along CD, $dl = \rho\, d\phi$ and $\rho = 5$, so

$$CD = \int_0^{\pi/2} \rho\, d\phi = 5\,\phi \Big|_0^{\pi/2} = 2.5\pi$$

(c) For $ABCD$, $dS = \rho\, d\phi\, dz$, $\rho = 5$. Hence,

$$\text{area } ABCD = \int dS = \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz = 5 \int_0^{\pi/2} d\phi \int_0^{10} dz \Big|_{\rho=5} = 25\pi$$

(d) For ABO, $dS = \rho\, d\phi\, d\rho$ and $z = 0$, so

$$\text{area } ABO = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{5} \rho\, d\phi\, d\rho = \int_0^{\pi/2} d\phi \int_0^{5} \rho\, d\rho = 6.25\pi$$

Basics of Vector Calculus

- **Line, Surface and Volume integrals**

(e) For $AOFD$, $dS = d\rho\, dz$ and $\phi = 0°$, so

$$\text{area } AOFD = \int_{\rho=0}^{5} \int_{z=0}^{10} dp\, dz = 50$$

(f) For volume $ABDCFO$, $dv = \rho\, d\phi\, dz\, d\rho$. Hence,

$$v = \int dv = \int_{\rho=0}^{5} \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz\, d\rho = \int_{0}^{10} dz \int_{0}^{\pi/2} d\phi \int_{0}^{5} \rho\, d\rho = 62.5\pi$$

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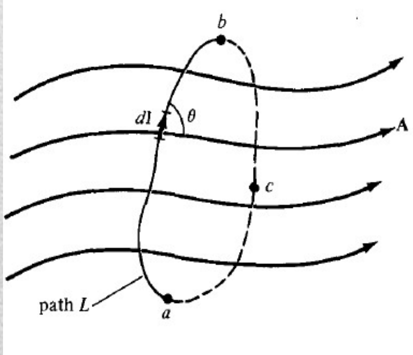
Basics of Vector Calculus

- **Line, Surface and Volume integrals**
  - **The third category is to calculate the integral of a vector field over a line, surface, or volume. In this case a vector field enters the integral. Ex. Calculating work done, the potential, the flux...**

The **line integral** $\int_L \mathbf{A} \cdot d\mathbf{l}$ is the integral of the tangential component of $\mathbf{A}$ along curve L.

Work done: $W = \int \mathbf{F} \cdot d\mathbf{l}.$

For closed path integral, it is called circulation $\oint_L \mathbf{A} \cdot d\mathbf{l}$



Normally the result depends on the path, if the result is independent of the path the vector field is called conservative. And its integral over closed path is zero.

Basics of Vector Calculus

- **Surface integral example**

EXAMPLE 3.2 Given that $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - y^2\mathbf{a}_z$, calculate the circulation of $\mathbf{F}$ around the (closed) path shown in Figure 3.10.

Solution:

The circulation of $\mathbf{F}$ around path L is given by

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \left(\int_1 + \int_2 + \int_3 + \int_4 \right) \mathbf{F} \cdot d\mathbf{l}$$

where the path is broken into segments numbered 1 to 4 as shown in Figure 3.10.

For segment 1, $y = 0 = z$

$$\mathbf{F} = x^2\mathbf{a}_x, \qquad d\mathbf{l} = dx\,\mathbf{a}_x$$

Notice that $d\mathbf{l}$ is always taken as along $+\mathbf{a}_x$ so that the direction on segment 1 is taken care of by the limits of integration. Thus,

$$\int_1 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 x^2 dx = \frac{x^3}{3}\bigg|_1^0 = -\frac{1}{3}$$

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Basics of Vector Calculus

- **Surface integral example**

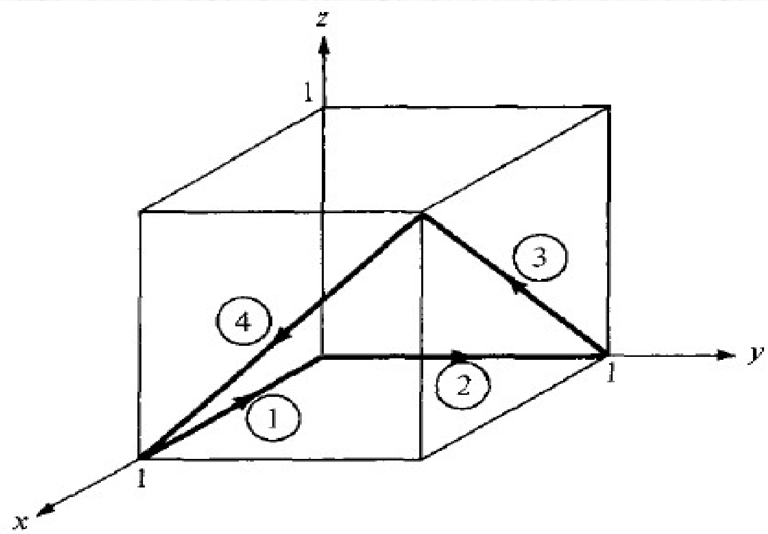


Figure 3.10 For Example 3.2.

For segment 2, $x = 0 = z$, $\mathbf{F} = -y^2\mathbf{a}_z$, $d\mathbf{l} = dy\,\mathbf{a}_y$, $\mathbf{F}\cdot d\mathbf{l} = 0$. Hence,

$$\int_2 \mathbf{F}\cdot d\mathbf{l} = 0$$

Basics of Vector Calculus

- **Surface integral example**

For segment 3, $y = 1$, $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - \mathbf{a}_z$, and $d\mathbf{l} = dx\,\mathbf{a}_x + dz\,\mathbf{a}_z$, so

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int (x^2 dx - dz)$$

But on 3, $z = x$; that is, $dx = dz$. Hence,

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int_0^1 (x^2 - 1)\,dx = \frac{x^3}{3} - x \Big|_0^1 = -\frac{2}{3}$$

For segment 4, $x = 1$, so $\mathbf{F} = \mathbf{a}_x - z\mathbf{a}_y - y^2\mathbf{a}_z$, and $d\mathbf{l} = dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$. Hence,

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int (-z\,dy - y^2 dz)$$

Basics of Vector Calculus

- **Surface integral example**

But on 4, $z = y$; that is, $dz = dy$, so

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 (-y - y^2)\, dy = -\frac{y^2}{2} - \frac{y^3}{3}\Big|_1^0 = \frac{5}{6}$$

By putting all these together, we obtain

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6} = -\frac{1}{6}$$

Basics of Vector Calculus

- **Line integral example**

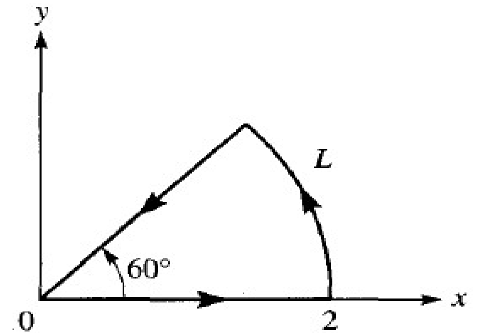
PRACTICE EXERCISE 3.2

Calculate the circulation of

$$\mathbf{A} = \rho \cos \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in Figure 3.11.

Answer: 1.

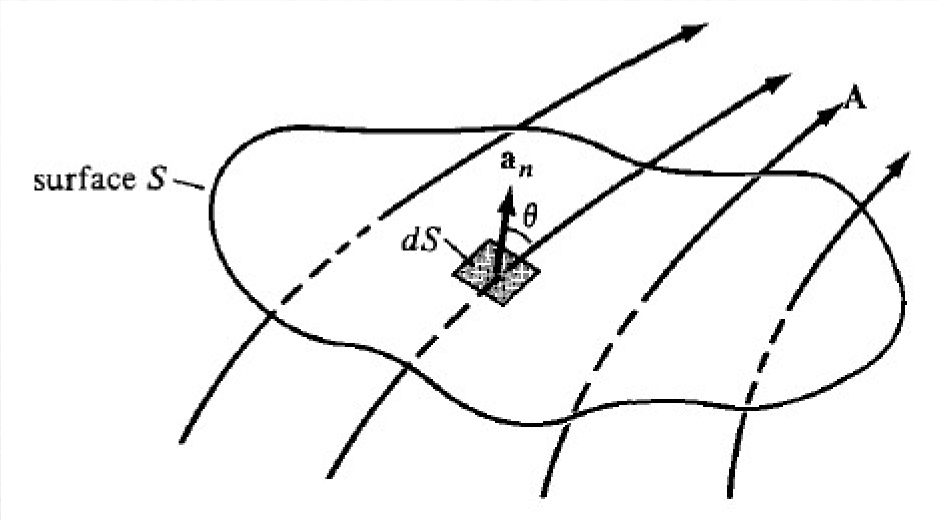


Basics of Vector Calculus

- **Line, Surface and Volume integrals**

Surface integral of vector field: Flux of A through the given area

$$\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}$$



$$\Psi = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

Surface integral of vector field over closed area
It is also called the net outward flux.

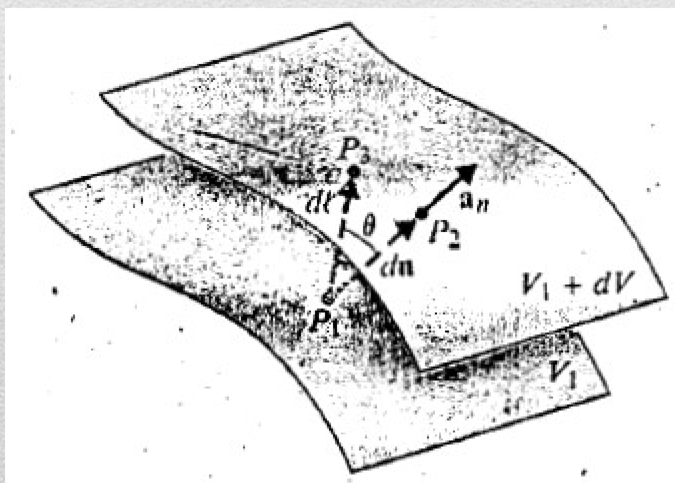
Basics of Vector Calculus

- **Spatial derivatives (differential calculus)**

Question: Suppose we have a function of one variable: $f(x)$. What does the derivative, df/dx, do for us? *Answer:* It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

1. **Differentiation of a scalar field with multivariable, leads to gradient of a scalar field**

   Gradient is a vector which tells us the direction and the amount of maximum rate of change of the scalar field at a certain point in space.



$$\nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}.$$

$$dV = (\nabla V) \cdot d\boldsymbol{\ell},$$

$$\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial \ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial \ell_3}$$

$$\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3\, \partial u_3}.$$

The derivation depends on the parallel and normal component to the constant surface

Basics of Vector Calculus

Gradiant in coordinates systems

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Cartesian

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Cylindrical

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

Spherical

Gradient Properties

(a) $\nabla(V + U) = \nabla V + \nabla U$

(b) $\nabla(VU) = V\nabla U + U\nabla V$

(c) $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$

(d) $\nabla V^n = nV^{n-1}\nabla V$

The del Operator (Nabla)

$$\nabla \equiv \left(\mathbf{a}_{u_1}\frac{\partial}{h_1\,\partial u_1} + \mathbf{a}_{u_2}\frac{\partial}{h_2\,\partial u_2} + \mathbf{a}_{u_3}\frac{\partial}{h_3\,\partial u_3}\right)$$

The projection (or component) of ∇V in the direction of a unit vector $\mathbf{a}$ is $\nabla V \cdot \mathbf{a}$ and is called the *directional derivative* of V along $\mathbf{a}$. This is the rate of change of V in the direction of $\mathbf{a}$.

Basics of Vector Calculus

Gradiant Example 1

EXAMPLE 3.3 Find the gradient of the following scalar fields:

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2\theta \cos\phi$

Solution:

(a)
$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= 2e^{-z}\cos 2x \cosh y\,\mathbf{a}_x + e^{-z}\sin 2x \sinh y\,\mathbf{a}_y - e^{-z}\sin 2x \cosh y\,\mathbf{a}_z$$

(b)
$$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$
$$= 2\rho z \cos 2\phi\,\mathbf{a}_\rho - 2\rho z \sin 2\phi\,\mathbf{a}_\phi + \rho^2 \cos 2\phi\,\mathbf{a}_z$$

(c)
$$\nabla W = \frac{\partial W}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$
$$= 10 \sin^2\theta \cos\phi\,\mathbf{a}_r + 10 \sin 2\theta \cos\phi\,\mathbf{a}_\theta - 10 \sin\theta \sin\phi\,\mathbf{a}_\phi$$

Basics of Vector Calculus

Gradiant Example 2

EXAMPLE 3.4 Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:

$$\nabla W = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$

$$= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$

Hence,

$$\frac{dW}{dl} = \nabla W \cdot \mathbf{a}_l = (4, -8, -2) \cdot \frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

Basics of Vector Calculus

Fundamental theory of gradient:

$$\int_{\mathbf{a}\atop\mathcal{P}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).$$

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

Basics of Vector Calculus

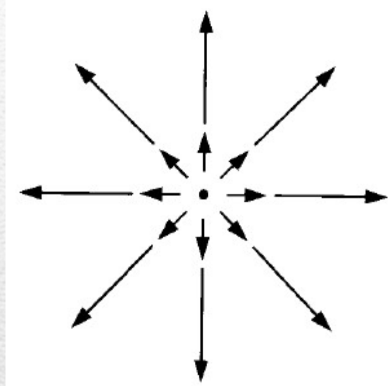
2. **Differentiation of a vector field field with multivariable, leads to divergence and curl of a vector field**

Divergence in coordinates systems

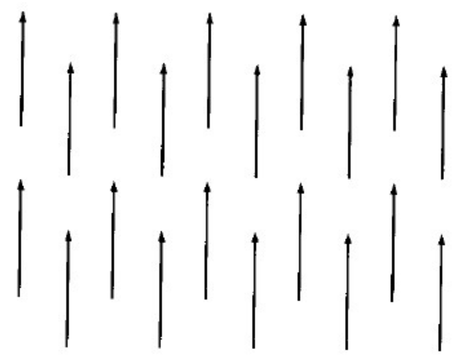
$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

It is very important to note that derivative is a point function in contrast to integrals. Divergence indicates physically a flow source or sink, in contrast to vortex source which is represented in terms of curl.

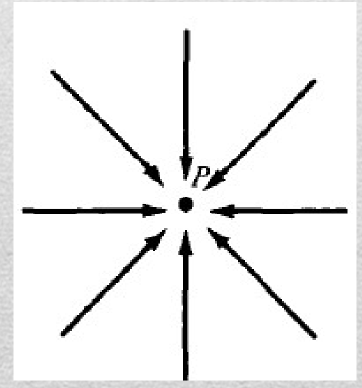
To test if there is divergence at a point we take a small volume and check the flux through its closed surface area, and get the flux as we shrink the volume to zero to accurately represent the point



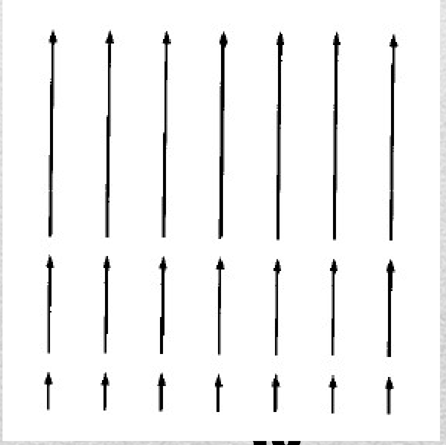
Positive divergence



zero divergence



negative divergence



Positive divergence

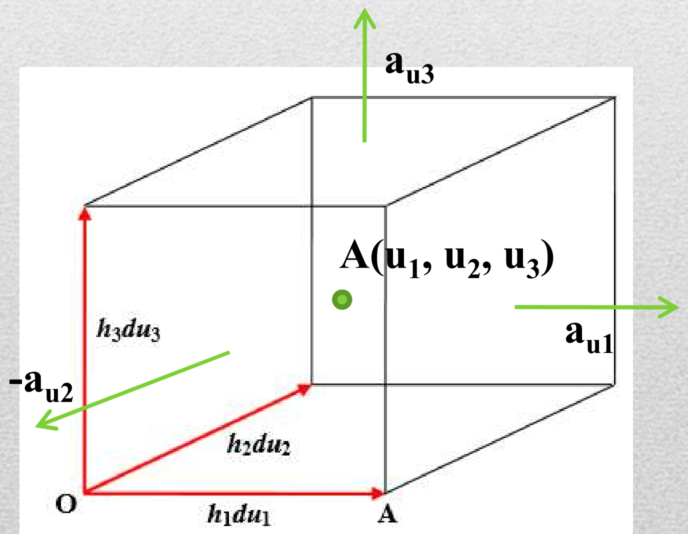
Basics of Vector Calculus

Divergence in coordinates systems

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

The **divergence** of **A** at a given point *P* is the *outward* flux per unit volume as the volume shrinks about *P*.

The name divergence comes from the fact fields which spreads out have a flow source, not necessarily at each point in space, but at least at a single point. A divergenceless vector field is called solenoidal. Divergence converts a vector field into a scalar field.



$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right].$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Basics of Vector Calculus

Divergence in coordinates systems

EXAMPLE 3.6

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho\sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$\begin{aligned}\nabla\cdot\mathbf{P} &= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z\\ &= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)\\ &= 2xyz + x\end{aligned}$$

(b)
$$\begin{aligned}\nabla\cdot\mathbf{Q} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}Q_\phi + \frac{\partial}{\partial z}Q_z\\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)\\ &= 2\sin\phi + \cos\phi\end{aligned}$$

Basics of Vector Calculus

Divergence in coordinates systems

(c) $$\nabla \cdot \mathbf{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(T_\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

$$= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0$$

$$= 2\cos\theta\cos\phi$$

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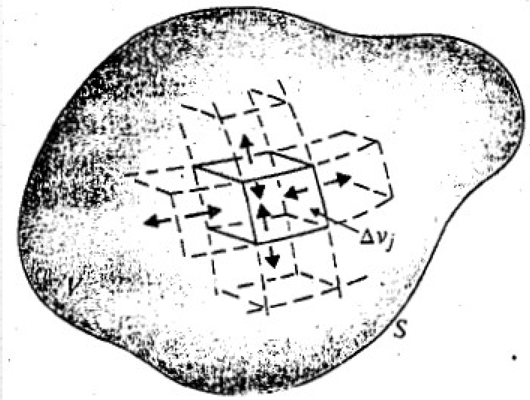
Basics of Vector Calculus

Divergence Theorem (Gauss's theorem):

$$\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}.$$

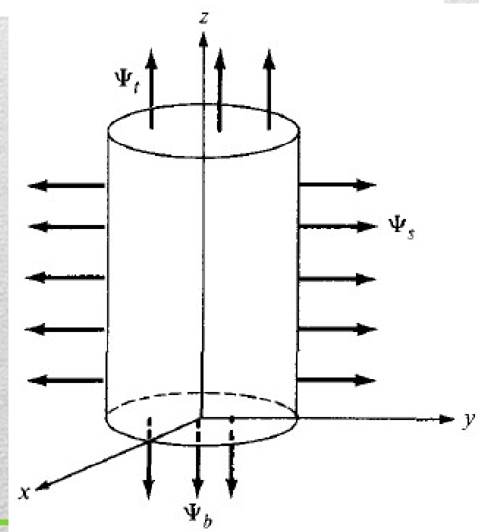


EXAMPLE 3.7 If $\mathbf{G}(r) = 10e^{-2z}(\rho \mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \leq z \leq 1$. Confirm the result using the divergence theorem.

Solution:

If Ψ is the flux of $\mathbf{G}$ through the given surface, shown in Figure 3.17, then

$$\Psi = \oint \mathbf{G} \cdot d\mathbf{S} = \Psi_t + \Psi_b + \Psi_s$$



Basics of Vector Calculus

Divergence Theorem (Gauss's theorem):

where Ψ_t, Ψ_b, and Ψ_s are the fluxes through the top, bottom, and sides (curved surface) of the cylinder as in Figure 3.17.

For Ψ_t, $z = 1$, $d\mathbf{S} = \rho\, dp\, d\phi\, \mathbf{a}_z$. Hence,

$$\Psi_t = \int \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2} \rho\, d\rho\, d\phi = 10e^{-2}(2\pi) \frac{\rho^2}{2} \bigg|_0^1$$
$$= 10\pi e^{-2}$$

For Ψ_b, $z = 0$ and $d\mathbf{S} = \rho\, d\rho\, d\phi(-\mathbf{a}_z)$. Hence,

$$\Psi_b = \int_b \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{0} \rho\, d\rho\, d\phi = -10(2\pi) \frac{\rho^2}{2} \bigg|_0^1$$
$$= -10\pi$$

For Ψ_s, $\rho = 1$, $d\mathbf{S} = \rho\, dz\, d\phi\, \mathbf{a}_\rho$. Hence,

$$\Psi_s = \int_s \mathbf{G} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2z} \rho^2\, dz\, d\phi = 10(1)^2(2\pi) \frac{e^{-2z}}{-2} \bigg|_0^1$$
$$= 10\pi(1 - e^{-2})$$

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Basics of Vector Calculus

Divergence Theorem (Gauss's theorem):

Thus,

$$\Psi = \Psi_t + \Psi_b + \Psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

Alternatively, since S is a closed surface, we can apply the divergence theorem:

$$\Psi = \oint_S G \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{G})\, dv$$

But

$$\nabla \cdot \mathbf{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}G_\phi + \frac{\partial}{\partial z}G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 10 e^{-2z}) - 20e^{-2z} = 0$$

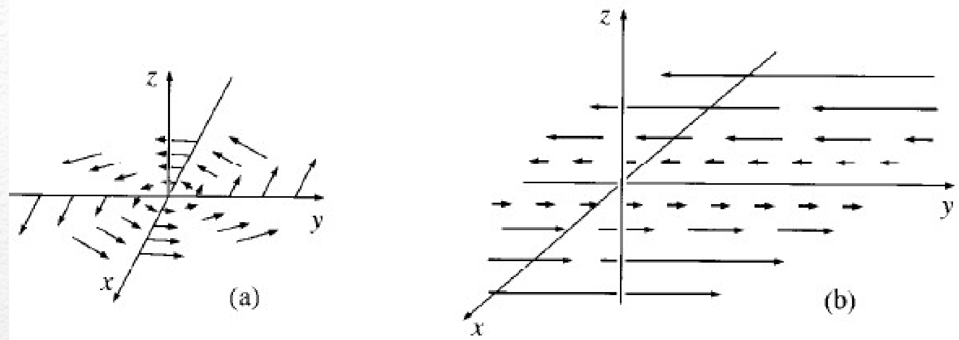
showing that **G** has no source. Hence,

$$\Psi = \int_v (\nabla \cdot G)\, dv = 0$$

Basics of Vector Calculus

Curl of a vector field:

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$



The **curl** of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho\, \mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\, \mathbf{a}_\theta & r \sin\theta\, \mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin\theta\, A_\phi \end{vmatrix}$$

Basics of Vector Calculus

Curl of a vector field:

EXAMPLE 3.8 | Determine the curl of the vector fields

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$\nabla \times \mathbf{P} = \left(\frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y}\right)\mathbf{a}_z$$
$$= (0 - 0)\,\mathbf{a}_x + (x^2y - z)\,\mathbf{a}_y + (0 - x^2z)\,\mathbf{a}_z$$
$$= (x^2y - z)\mathbf{a}_y - x^2z\mathbf{a}_z$$

(b)
$$\nabla \times \mathbf{Q} = \left[\frac{1}{\rho}\frac{\partial Q_z}{\partial \phi} - \frac{\partial Q_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial Q_\rho}{\partial z} - \frac{\partial Q_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho Q_\phi) - \frac{\partial Q_\rho}{\partial \phi}\right]\mathbf{a}_z$$
$$= \left(\frac{-z}{\rho}\sin\phi - \rho^2\right)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(3\rho^2 z - \rho\cos\phi)\mathbf{a}_z$$
$$= -\frac{1}{\rho}(z\sin\phi + \rho^3)\mathbf{a}_\rho + (3\rho z - \cos\phi)\mathbf{a}_z$$

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Basics of Vector Calculus

Curl of a vector field:

$$\text{(c)}\quad \nabla \times \mathbf{T} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(T_\phi \sin\theta) - \frac{\partial}{\partial\phi}T_\theta\right]\mathbf{a}_r$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}T_r - \frac{\partial}{\partial r}(rT_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rT_\theta) - \frac{\partial}{\partial\theta}T_r\right]\mathbf{a}_\phi$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\cos\theta\sin\theta) - \frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\right]\mathbf{a}_r$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{(\cos\theta)}{r^2} - \frac{\partial}{\partial r}(r\cos\theta)\right]\mathbf{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi) - \frac{\partial}{\partial\theta}\frac{(\cos\theta)}{r^2}\right]\mathbf{a}_\phi$$

$$= \frac{1}{r\sin\theta}(\cos 2\theta + r\sin\theta\sin\phi)\mathbf{a}_r + \frac{1}{r}(0 - \cos\theta)\mathbf{a}_\theta$$

$$+ \frac{1}{r}\left(2r\sin\theta\cos\phi + \frac{\sin\theta}{r^2}\right)\mathbf{a}_\phi$$

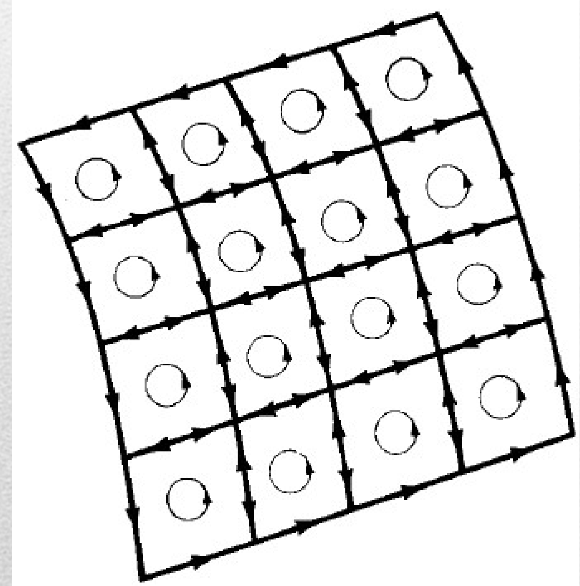
$$= \left(\frac{\cos 2\theta}{r\sin\theta} + \sin\phi\right)\mathbf{a}_r - \frac{\cos\theta}{r}\mathbf{a}_\theta + \left(2\cos\phi + \frac{1}{r^3}\right)\sin\theta\,\mathbf{a}_\phi$$

Basics of Vector Calculus

Stokes‘ theorem for the Curl of a vector field:

Stokes's theorem states that the circulation of a vector field **A** around a (closed) path *L* is equal to the surface integral of the curl of **A** over the open surface *S* bounded by *L* (see Figure 3.20) provided that **A** and $\nabla \times \mathbf{A}$ are continuous on *S*.

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$



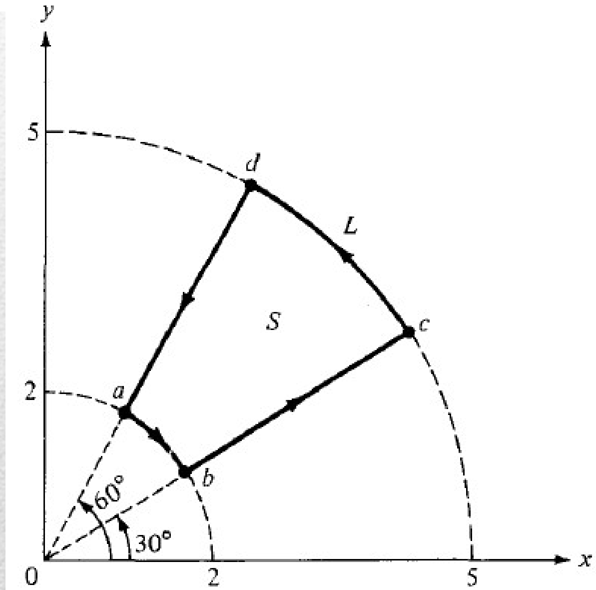
Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

Basics of Vector Calculus

Stokes' theorem example:

EXAMPLE 3.9 If $\mathbf{A} = \rho \cos\phi\, \mathbf{a}_\rho + \sin\phi\, \mathbf{a}_\phi$, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the path shown in Figure 3.22. Confirm this using Stokes's theorem.



Solution:

Let

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] \mathbf{A} \cdot d\mathbf{l}$$

where path L has been divided into segments ab, bc, cd, and da as in Figure 3.22.

Along ab, $\rho = 2$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_a^b \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=60^\circ}^{30^\circ} \rho \sin\phi\, d\phi = 2(-\cos\phi) \Big|_{60^\circ}^{30^\circ} = -(\sqrt{3} - 1)$$

Basics of Vector Calculus

Stokes' theorem example:

Along bc, $\phi = 30°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho = \cos 30° \frac{\rho^2}{2}\bigg|_2^5 = \frac{21\sqrt{3}}{4}$$

Along cd, $\rho = 5$ and $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$. Hence,

$$\int_c^d \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=30°}^{60°} \rho \sin\phi \, d\phi = 5(-\cos\phi)\bigg|_{30°}^{60°} = \frac{5}{2}(\sqrt{3} - 1)$$

Along da, $\phi = 60°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_d^a \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho \cos\phi \, d\rho = \cos 60° \frac{\rho^2}{2}\bigg|_5^2 = -\frac{21}{4}$$

Putting all these together results in

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = -\sqrt{3} + 1 + \frac{21\sqrt{3}}{4} + \frac{5\sqrt{3}}{2} - \frac{5}{2} - \frac{21}{4}$$

$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

Basics of Vector Calculus

Stokes' theorem example:

Using Stokes's theorem (because L is a closed path)

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

But $d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$ and

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \mathbf{a}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \mathbf{a}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right]$$

$$= (0 - 0)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(1 + \rho) \sin \phi \, \mathbf{a}_z$$

Hence:

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\phi=30^\circ}^{60^\circ} \int_{\rho=2}^{5} \frac{1}{\rho}(1 + \rho) \sin \phi \, \rho \, d\rho \, d\phi$$

$$= \int_{30^\circ}^{60^\circ} \sin \phi \, d\phi \int_2^5 (1 + \rho) d\rho$$

$$= -\cos \phi \, \Big|_{30^\circ}^{60^\circ} \left(\rho + \frac{\rho^2}{2} \right) \Big|_2^5$$

$$= \frac{27}{4} (\sqrt{3} - 1) = 4.941$$

Basics of Vector Calculus

Second order derivatives:

1. The curl of the gradient of any scalar field is zero

$$\nabla \times (\nabla V) \equiv 0$$

Use Stokes theory to prove it

2. The divergence of the curl of and vector field is zero

$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$$

This also means that any solenoidal field can be wirtten as the curl of another vector field $\mathbf{B} = \nabla \times \mathbf{A}$

Use divergence theory to prove it

3. The divergence of gradient (the Laplacian) $\nabla^2 \varphi$

$$\nabla^2 \varphi = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \varphi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \varphi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \varphi}{\partial u_3}\right)\right)$$

Spherical Coordinates

$$\begin{aligned}\nabla^2 \varphi &= \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \varphi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial}{\partial \lambda}\left(\frac{1}{\sin\theta}\frac{\partial \varphi}{\partial \lambda}\right)\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial \varphi}{\partial \lambda}.\end{aligned}$$

Basics of Vector Calculus

Second order derivatives:

In cylindrical polars, the Laplacian operator is

$$\nabla^2\varphi = \frac{1}{R}\left(\frac{\partial}{\partial R}\left(R\frac{\partial\varphi}{\partial R}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{R}\frac{\partial\varphi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(R\frac{\partial\varphi}{\partial z}\right)\right)$$
$$= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\varphi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} .$$

4. Gradient of divergence

$\nabla(\nabla\cdot\mathbf{v})$ **Seldom occurs in physical applications**

5. The curl of curl

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}.$$

So curl-of-curl gives nothing new;

Really, then, there are just two kinds of second derivatives: the Laplacian (which is of fundamental importance) and the gradient-of-divergence (which we seldom encounter).

Basics of Vector Calculus

Classification of fields:

1. Solenoidal and irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} = 0.$$

Example: A static electric field in a charge-free region.

2. Solenoidal but not irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

Example: A steady magnetic field in a current-carrying conductor.

3. Irrotational but not solenoidal if

$$\nabla \times \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{F} \neq 0.$$

Example: A static electric field in a charged region.

4. Neither solenoidal nor irrotational if

$$\nabla \cdot \mathbf{F} \neq 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

Example: An electric field in a charged medium with a time-varying magnetic field.

An irrotational field is also called conservative field, because it is path independent