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Dr.Ashraf Al-Rimawi

Electromagnetic Theory I

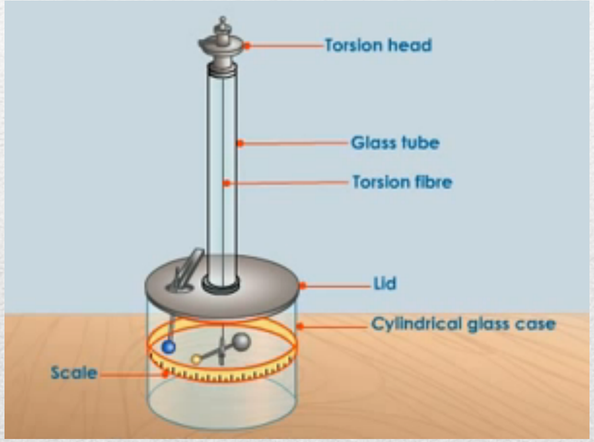
Second Semester, 2017/2018

Electrostatic Fields-Coulomb‘s Law

Electrostatic: charges (the field source) are at rest, they cannot generate magnetic field

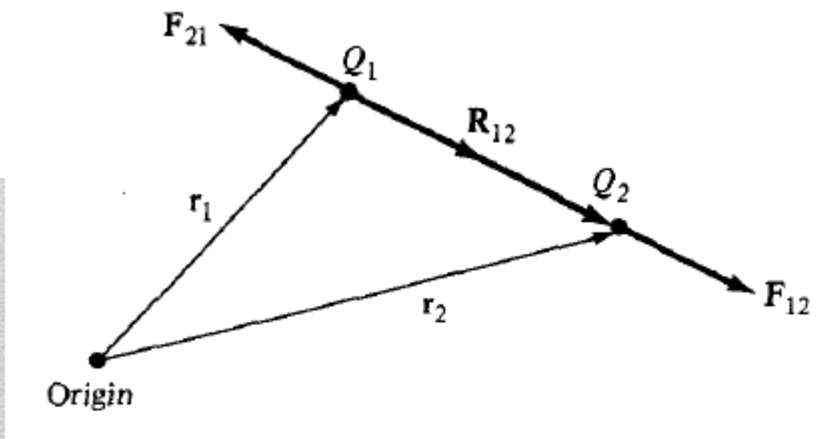
Coloumb‘s experiment using torsion balance 1785:

Coloumb found that when two charges in close vicinity, they excert a force on on the other. This force was proportional to the amount of individual charges and inversly proportional to the square of the displacement between them.



Torsion balance: the device by which Coulomb discovered Coulomb's law

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o |\mathbf{r}_2 - \mathbf{r}_1|^3}$$



Electrostaic Field

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$\mathbf{F} = \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_n)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}}$$

Electric field intensity

The force between the two charges that happens from distance was assumed to happen due to a field of force which is generated from the charge and affects the other, this field was called the electric field and the electric field intensity is defined as:

Electric field intensity is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists. That is,

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q}$$

$$\mathbf{E} = \frac{\mathbf{F}}{Q}$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}$$

$$\mathbf{E} = \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r})_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}$$

Coloumb's law

EXAMPLE 4.1 Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10-nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$\begin{aligned}
\mathbf{F} &= \sum_{k=1,2} \frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2} \frac{QQ_k(\mathbf{r}-\mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r}-\mathbf{r}_k|^3} \\
&= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0,3,1)-(3,2,-1)]}{|(0,3,1)-(3,2,-1)|^3} - \frac{2.10^{-3}[(0,3,1)-(-1,-1,4)]}{|(0,3,1)-(-1,-1,4)|^3}\right\} \\
&= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\frac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right] \\
&= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right] \\
\mathbf{F} &= -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}
\end{aligned}$$

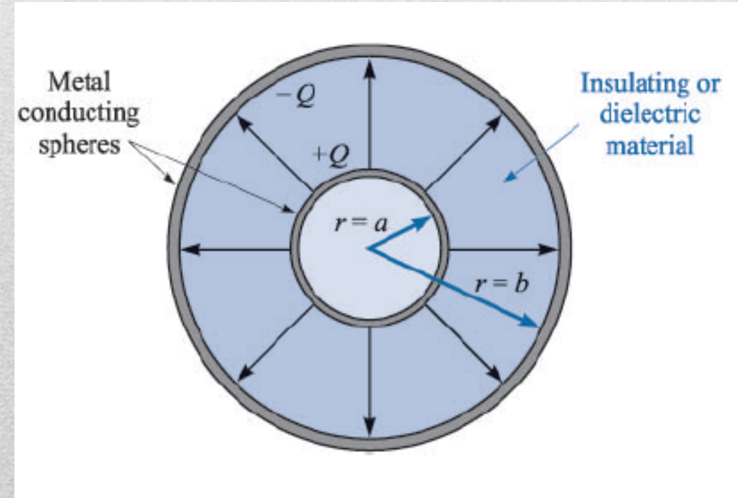
At that point,

$$\begin{aligned}
\mathbf{E} &= \frac{\mathbf{F}}{Q} \\
&= (-6.507, -3.817, 7.506)\cdot\frac{10^{-3}}{10\cdot 10^{-9}} \\
\mathbf{E} &= -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}
\end{aligned}$$

Electric flux and electric field density D

The electric flux and electric flux density: Michael Farady Experiment 1837, using two concentric spheres:

Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux*, or simply *electric flux*.



Electric flux and electric field density D

The electric flux and electric flux density: Michael Farady Experiment 1837, using two concentric spheres:

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

and the electric flux Ψ is measured in coulombs.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D**, which was originally chosen because of the alternate names of *displacement flux density* or *displacement density*. Electric flux density is more descriptive, however, and we shall use the term consistently.

Electric flux and electric field density D

Electric flux density: define a quantity that does not depend on medium, only depend on the free charge (unbound charge). This is obtained by dividing the flux over the area

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad \text{(free space only)}$$

The equation of D is independent from permitivity, which shows that D only depends on charge enclosed and not the dielectric.

An **electric flux line** is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.

Electric flux lines leaves positive charges and enters negative charges.

Electrostatics Postulates in point and integral form

Maxwell's equations:

Electrostatic Postulates in point form:

Charge density is the flow source of electric field

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Electric field is conservative

$$\nabla \times \mathbf{E} = 0.$$

Electrostatic integral form postulates

Maxwell's equations in integral form:

Gauss law (from the postulate and divergence theory):

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

Curl of E integral form(from the postulate and stoke's theory): $\oint \mathbf{E} \cdot d\mathbf{l} = 0,$

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\ell = 0$

Methods of calculating electric field from easiest to most difficult

The main goal is calculating the electric field which is independent from the magnetic field in electrostatics.

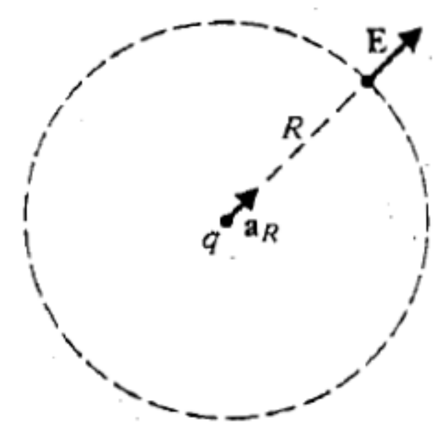
Methods of calculating the eectric field:

1. **Gauss law when symmetry is there which is the easiest way. Always think to solve the problem using Gauss law whenever possible.**
2. **Using the electrostatic potential to be defined later.**
3. **Using coloumbs law (most difficult way)**

Coloumb's law from Gauss law:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$



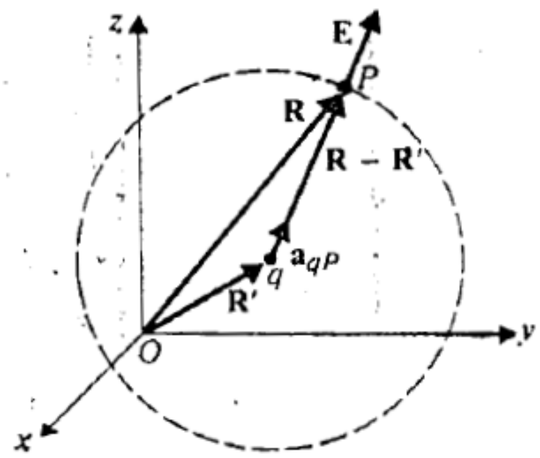
(a) Point charge at the origin.

Q is at the origin

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad \text{(V/m)}.$$

Q is not at the origin

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad \text{(V/m)}.$$



(b) Point charge not at the origin.

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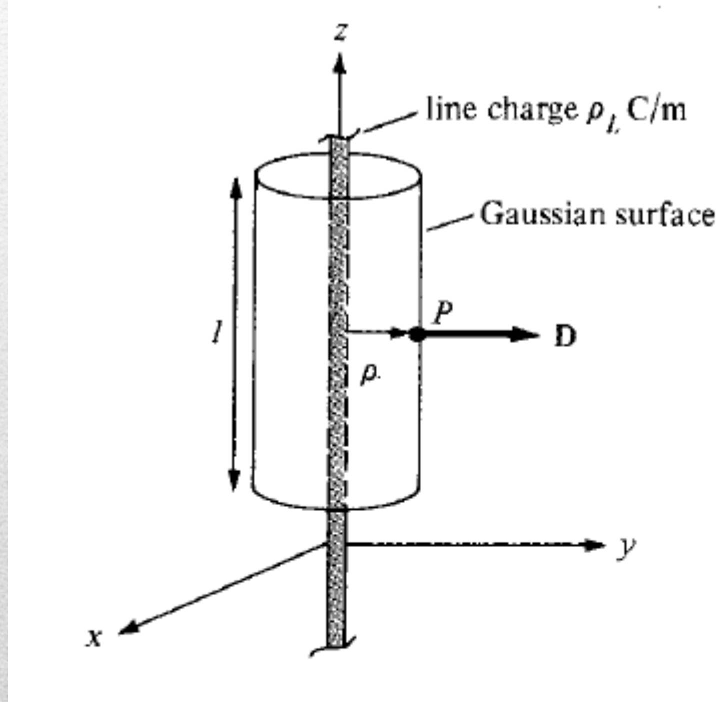
Electric field calculations using Gauss‘ law

Gauss law examples:

Infinite line charge

$$\rho_L \ell = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_\rho \oint dS = D_\rho \, 2\pi\rho\ell$$

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho} \mathbf{a}_\rho$$



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Electric field calculations using Gauss' law

Gauss law examples:

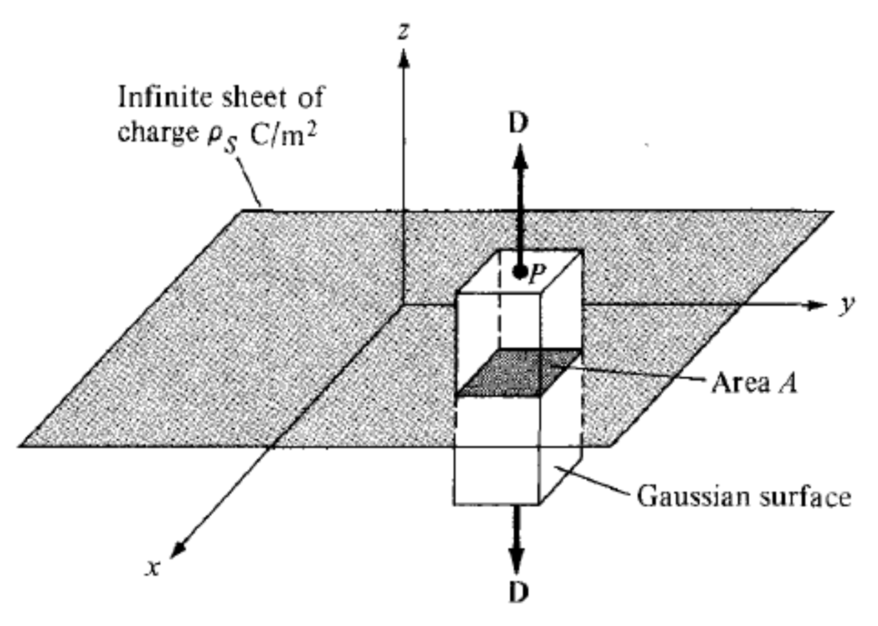
Infinite surface charge

$$\rho_S \int dS = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_z \left[\int_{top} dS + \int_{bottom} dS \right]$$

$$\rho_S A = D_z (A + A)$$

$$\mathbf{D} = \frac{\rho_S}{2} \mathbf{a}_z$$

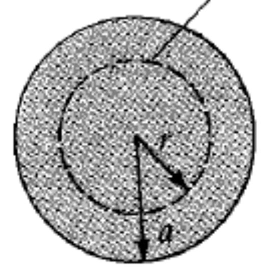
$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{\rho_S}{2\varepsilon_0} \mathbf{a}_z$$



Electric field calculations using Gauss‘ law

Gauss law examples:

Spherical volume charge density



$$Q_{enc} = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_v \frac{4}{3} \pi r^3$$

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r 4\pi r^2$$

Hence, $\Psi = Q_{enc}$ gives

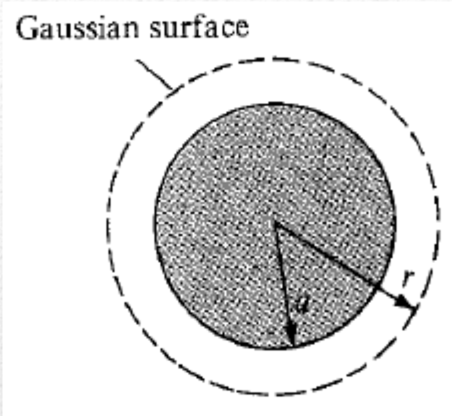
$$D_r 4\pi r^2 = \frac{4\pi r^3}{3} \rho_v$$

$$\mathbf{D} = \frac{r}{3} \rho_v \mathbf{a}_r \qquad 0 < r \leqslant a$$

Electric field calculations using Gauss' law

Gauss law examples:

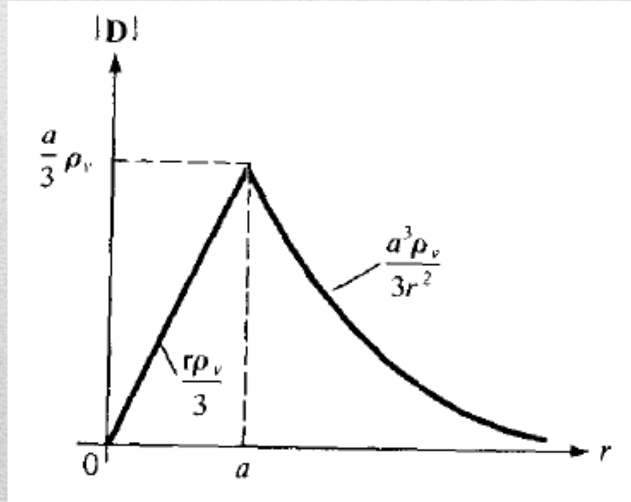
Spherical volume charge density



$$D_r\, 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$$

$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v \mathbf{a}_r \qquad r \geqslant a$$

$$\mathbf{D} = \begin{cases} \dfrac{r}{3}\rho_v\, \mathbf{a}_r & 0 < r \leqslant a \\ \dfrac{a^3}{3r^2}\rho_v\, \mathbf{a}_r & r \geqslant a \end{cases}$$



example:

EXAMPLE 4.8 Given that $\mathbf{D} = z\rho \cos^2\phi \, \mathbf{a}_z$ C/m^2, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \le z \le 2$ m.

Solution:

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_z}{\partial z} = \rho \cos^2 \phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5$ C/m^3. The total charge enclosed by the cylinder can be found in two different ways.

Method 1: This method is based directly on the definition of the total volume charge.

$$Q = \int_v \rho_v \, dv = \int_v \rho \cos^2 \phi \, \rho \, d\phi \, d\rho \, dz$$

$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho = 4(\pi)(1/3)$$

$$= \frac{4\pi}{3} \text{ C}$$

example:

Method 2: Alternatively, we can use Gauss's law.

$$Q = \Psi = \oint \mathbf{D} \cdot d\mathbf{S} = \left[\int_s + \int_t + \int_b \right] \mathbf{D} \cdot d\mathbf{S}$$
$$= \Psi_s + \Psi_t + \Psi_b$$

where Ψ_s, Ψ_t, and Ψ_b are the flux through the sides, the top surface, and the bottom surface of the cylinder, respectively (see Figure 3.17). Since $\mathbf{D}$ does not have component along $\mathbf{a}_\rho$, $\Psi_s = 0$, for Ψ_t, $d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$ so

$$\Psi_t = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2\phi\, \rho\, d\phi\, d\rho \Big|_{z=2} = 2 \int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi\, d\phi$$
$$= 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}$$

and for Ψ_b, $d\mathbf{S} = -\rho\, d\phi\, d\rho\, \mathbf{a}_z$, so

$$\Psi_b = -\int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2\phi\, \rho\, d\phi\, d\rho \Big|_{z=-2} = 2 \int_0^1 \rho^2\, d\rho \int_0^{2\pi} \cos^2\phi\, d\phi$$
$$= \frac{2\pi}{3}$$

Thus

$$Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}\,\text{C}$$

Example superposition

EXAMPLE 4.6 Planes $x = 2$ and $y = -3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate **E** at $(1, 1, -1)$ due to the three charge distributions.

Solution:

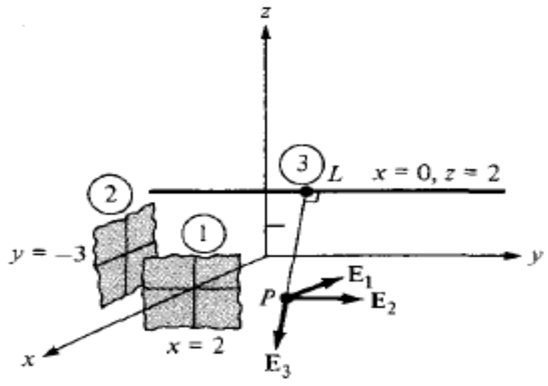
Let

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are, respectively, the contributions to **E** at point $(1, 1, -1)$ due to the infinite sheet 1, infinite sheet 2, and infinite line 3 as shown in Figure 4.10(a). Applying eqs. (4.26) and (4.21) gives

$$\mathbf{E}_1 = \frac{\rho_{S_1}}{2\varepsilon_o}(-\mathbf{a}_x) = -\frac{10 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_x = -180\pi\mathbf{a}_x$$

$$\mathbf{E}_2 = \frac{\rho_{S_2}}{2\varepsilon_o}\mathbf{a}_y = \frac{15 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_y = 270\pi\,\mathbf{a}_y$$



(a)

Example superposition

$$\mathbf{E}_3 = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$$

where $\mathbf{a}_\rho$ (not regular $\mathbf{a}_\rho$ but with a similar meaning) is a unit vector along LP perpendicular to the line charge and ρ is the length LP to be determined from Figure 4.10(b). Figure 4.10(b) results from Figure 4.10(a) if we consider plane $y = 1$ on which $\mathbf{E}_3$ lies. From Figure 4.10(b), the distance vector from L to P is

$$\mathbf{R} = -3\mathbf{a}_z + \mathbf{a}_x$$

$$\rho = |\mathbf{R}| = \sqrt{10}, \qquad \mathbf{a}_\rho = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{1}{\sqrt{10}}\mathbf{a}_x - \frac{3}{\sqrt{10}}\mathbf{a}_z$$

Hence,

$$\mathbf{E}_3 = \frac{10\pi \cdot 10^{-9}}{2\pi \cdot \frac{10^{-9}}{36\pi}} \cdot \frac{1}{10}(\mathbf{a}_x - 3\mathbf{a}_z) = 18\pi(\mathbf{a}_x - 3\mathbf{a}_z)$$

Thus by adding $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, we obtain the total field as

$$\mathbf{E} = -162\pi\mathbf{a}_x + 270\pi\mathbf{a}_y - 54\pi\mathbf{a}_z \text{ V/m}$$

Classification of vector fields

A vector field **A** is said to be **solenoidal** (or **divergenceless**) if $\nabla \cdot \mathbf{A} = 0$.

Such a field has neither source nor sink of flux. From the divergence theorem,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A}\, dv = 0$$

Hence, flux lines of **A** entering any closed surface must also leave it.

In general, the field of curl **F** (for any **F**) is purely solenoidal because $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, Thus, a solenoidal field **A** can always be expressed in terms of another vector **F**; that is,

if $\nabla \cdot \mathbf{A} = 0$

then $\oint_S \mathbf{A} \cdot d\mathbf{S} = 0$ and $\mathbf{F} = \nabla \times \mathbf{A}$

Classification of vector fields

- A vector field **A** is said to be **irrotational** (or **potential**) if $\nabla \times \mathbf{A} = 0$.

- That is, a *curl-free* vector is irrotational.[3] From Stokes's theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l} = 0$$

Thus in an irrotational field **A**, the circulation of **A** around a closed path is identically zero. This implies that the line integral of **A** is independent of the chosen path. Therefore, an irrotational field is also known as a *conservative field.*

In general, the field of gradient V (for any scalar V) is purely irrotational since (see Practice Exercise 3.10)

$$\nabla \times (\nabla V) = 0$$

Classification of vector fields

if $\nabla \times \mathbf{A} = 0$

then $\oint_L \mathbf{A} \cdot d\mathbf{l} = 0$ and $\mathbf{A} = -\nabla V$

For this reason, **A** may be called a *potential* field and *V* the scalar potential of **A**.

Electric potential:

The curl of any gradient was proven to be zero using stokes theorem.

$$\nabla \times (\nabla V) \equiv 0$$

As a result, it was stated that and curl-free (irrotational, conservative) vector field can be expressed as a gradient of a scalar field. Because in electrostatics, the second Maxwell postulate states that the curl of electric field is zero, the electric field can be expressed as a gradient of a scalar field.

The main motivation behind this is that, finding the electric potential is easier as it is a scalar field, then the electric field can be found by the derivation of the electric potential, which is much easier than integration, as will be shown later.

$$\mathbf{E} = -\nabla V$$

Electric potential:

It is defined as the work done to move an electric charge from on point in the electric field to another

$$dW = -\mathbf{F} \cdot d\mathbf{l} = -Q\mathbf{E} \cdot d\mathbf{l}$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving Q from A to B is

$$W = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

the *potential difference* between points A and B

$$V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$$

$$V_{AB} = V_B - V_A$$

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Electric potential:

Note that

1. In determining V_{AB}, A is the initial point while B is the final point.
2. If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
3. V_{AB} is independent of the path taken (to be shown a little later).
4. V_{AB} is measured in joules per coulomb, commonly referred to as volts (V).

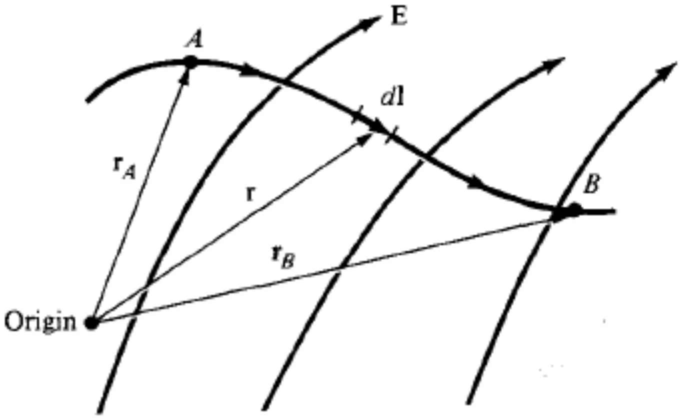


Figure 4.18 Displacement of point charge Q in an electrostatic field **E**.

Electric potential due to a point charge:

The **potential** at any point is the potential difference between that point and a chosen point at which the potential is zero.

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r$$

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \cdot dr\,\mathbf{a}_r = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l}$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon_o r}}$$

For a charge not at the origin:

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|}$$

Electric potential:

Electric potential due to a point charge (superposition):

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}_n|}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|} \qquad \text{(point charges)}$$

A surface whose potential is the same all over is called an equipotential surface. What does this tell us about the direction of electric field?

If the reference is not infinity, the easiest will be to apply indefinite integral and add a constant. The constant is then evaluated from a given reference.

$$V = -\int \mathbf{E} \cdot d\mathbf{l} + C$$

Basics of Vector Calculus

Electric potential Example:

EXAMPLE 4.10 Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

Solution:

Let

$$Q_1 = -4\ \mu C, \qquad Q_2 = 5\ \mu C$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + C_o$$

If $V(\infty) = 0$, $C_o = 0$,

$$|\mathbf{r} - \mathbf{r}_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$V(1, 0, 1) = \frac{10^{-6}}{4\pi \times \dfrac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$

$$= 9 \times 10^3\,(-1.633 + 0.9806)$$

$$= -5.872\ \text{kV}$$

Basics of Vector Calculus

Electric potential Example, reference is not infinity:

EXAMPLE 4.11 A point charge 5 nC is located at $(-3, 4, 0)$ while line $y = 1, z = 1$ carries uniform charge 2 nC/m.

(a) If $V = 0$ V at $O(0, 0, 0)$, find V at $A(5, 0, 1)$.

(b) If $V = 100$ V at $B(1, 2, 1)$, find V at $C(-2, 5, 3)$.

(c) If $V = -5$ V at O, find V_{BC}.

Solution:

Let the potential at any point be

$$V = V_Q + V_L$$

where V_Q and V_L are the contributions to V at that point due to the point charge and the line charge, respectively. For the point charge,

$$V_Q = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{Q}{4\pi\varepsilon_o r^2} \mathbf{a}_r \cdot dr\, \mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon_o r} + C_1$$

Basics of Vector Calculus

Electric potential Example, reference is not infinity:

For the infinite line charge,

$$V_L = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\rho_L}{2\pi\varepsilon_o\rho} \mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho$$

$$= -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + C_2$$

Hence,

$$V = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + \frac{Q}{4\pi\varepsilon_o r} + C$$

where $C = C_1 + C_2$ = constant, ρ is the perpendicular distance from the line $y = 1$, $z = 1$ to the field point, and r is the distance from the point charge to the field point.

(a) If $V = 0$ at $O(0, 0, 0)$, and V at $A(5, 0, 1)$ is to be determined, we must first determine the values of ρ and r at O and A. Finding r is easy; we use eq. (2.31). To find ρ for any point (x, y, z), we utilize the fact that ρ is the perpendicular distance from (x, y, z) to line $y = 1$, $z = 1$, which is parallel to the x-axis. Hence ρ is the distance between (x, y, z) and $(x, 1, 1)$ because the distance vector between the two points is perpendicular to $\mathbf{a}_x$. Thus

$$\rho = |(x, y, z) - (x, 1, 1)| = \sqrt{(y-1)^2 + (z-1)^2}$$

Basics of Vector Calculus

Electric potential Example, reference is not infinity:

Applying this for ρ and eq. (2.31) for r at points O and A, we obtain

$$\begin{aligned}
\rho_O &= |(0,0,0) - (0,1,1)| = \sqrt{2} \\
r_O &= |(0,0,0) - (-3,4,0)| = 5 \\
\rho_A &= |(5,0,1) - (5,1,1)| = 1 \\
r_A &= |(5,0,1) - (-3,4,0)| = 9
\end{aligned}$$

Hence,

$$\begin{aligned}
V_O - V_A &= -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_O}{\rho_A} + \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_O} - \frac{1}{r_A}\right] \\
&= \frac{-2 \cdot 10^{-9}}{2\pi \cdot \dfrac{10^{-9}}{36\pi}} \ln \frac{\sqrt{2}}{1} + \frac{5 \cdot 10^{-9}}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left[\frac{1}{5} - \frac{1}{9}\right] \\
0 - V_A &= -36 \ln \sqrt{2} + 45\left(\frac{1}{5} - \frac{1}{9}\right)
\end{aligned}$$

Basics of Vector Calculus

Electric potential Example, reference is not infinity:

$$V_A = 36 \ln \sqrt{2} - 4 = 8.477 \text{ V}$$

Notice that we have avoided calculating the constant C by subtracting one potential from another and that it does not matter which one is subtracted from which.

(b) If $V = 100$ at $B(1, 2, 1)$ and V at $C(-2, 5, 3)$ is to be determined, we find

$$\rho_B = |(1, 2, 1) - (1, 1, 1)| = 1$$

$$r_B = |(1, 2, 1) - (-3, 4, 0)| = \sqrt{21}$$

$$\rho_C = |(-2, 5, 3) - (-2, 1, 1)| = \sqrt{20}$$

$$r_C = |(-2, 5, 3) - (-3, 4, 0)| = \sqrt{11}$$

$$V_C - V_B = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_O}{\rho_B} + \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_C} - \frac{1}{r_B}\right]$$

$$V_C - 100 = -36 \ln \frac{\sqrt{20}}{1} + 45 \cdot \left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right]$$

$$= -50.175 \text{ V}$$

Basics of Vector Calculus

Electric potential Example, reference is not infinity:

$$V_C = 49.825 \text{ V}$$

(c) To find the potential difference between two points, we do not need a potential reference if a common reference is assumed.

$$\begin{aligned} V_{BC} &= V_C - V_B = 49.825 - 100 \\ &= -50.175 \text{ V} \end{aligned}$$

Electric potential due to continuous distributions:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0.$$

$$\nabla \times \mathbf{F} = \mathbf{C}.$$

$$\nabla \cdot \mathbf{F} = D,$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_L \frac{\rho_L(\mathbf{r}')dl'}{|\mathbf{r} - \mathbf{r}'|} \quad \text{(line charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_S \frac{\rho_S(\mathbf{r}')dS'}{|\mathbf{r} - \mathbf{r}'|} \quad \text{(surface charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \int_v \frac{\rho_v(\mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|} \quad \text{(volume charge)}$$

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W},$$

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{\mathscr{r}} d\tau',$$

$$\mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{\mathscr{r}} d\tau';$$

Basics of Vector Calculus

Electric potential due to continuous distributions Example:

Example 3–8 Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries a uniform surface charge density ρ_s.

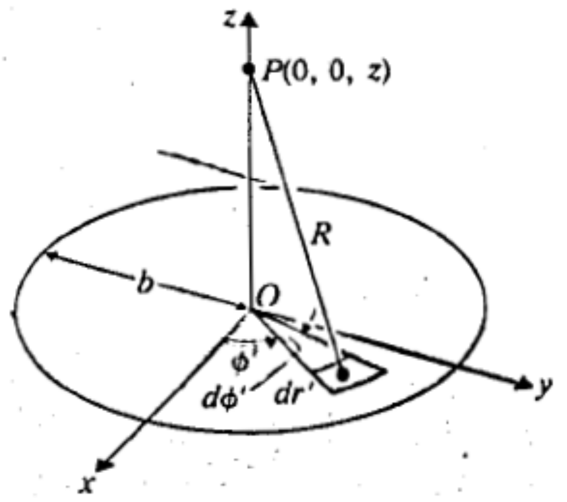


Fig. 3–15 A uniformly charged disk (Example 3–8).

Basics of Vector Calculus

Electric potential due to continuous distributions Example:

$$ds' = r'\,dr'\,d\phi'$$

and

$$R = \sqrt{z^2 + r'^2}$$

The electric potential at the point $P(0, 0, z)$ referring to the point at infinity is

$$V = \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^b \frac{r'}{(z^2 + r'^2)^{1/2}}\, dr'\, d\phi'$$

$$= \frac{\rho_s}{2\epsilon_0} \left[(z^2 + b^2)^{1/2} - |z|\right].$$

Therefore,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z \frac{\partial V}{\partial z}$$

$$= \begin{cases} \mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}\left[1 - z(z^2 + b^2)^{-/12}\right], & z > 0 \\ -\mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}\left[1 + z(z^2 + b^2)^{-1/2}\right], & z < 0. \end{cases}$$

Basics of Vector Calculus

Electric potential due to continuous distributions Example:

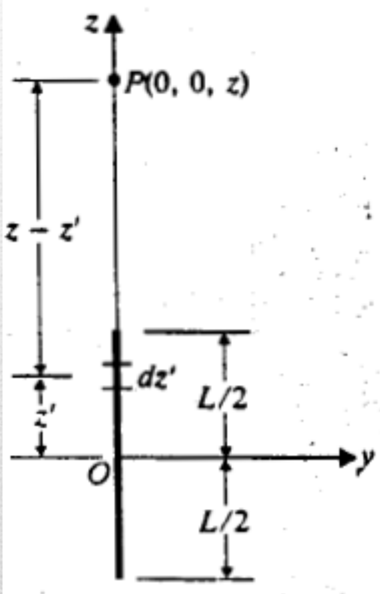
Example 3–9 Obtain a formula for the electric field intensity along the axis of a uniform line charge of length L. The uniform line-charge density is ρ_ℓ.

Solution: For an infinitely long line charge, the **E** field can be determined readily by applying Gauss's law, as in the solution to Example 3–4. However, for a line charge of finite length, as shown in Fig. 3–16, we cannot construct a Gaussian surface over which $\mathbf{E} \cdot d\mathbf{s}$ is constant. Gauss's law is therefore not useful here.

Instead, we use Eq. (3–58) by taking an element of charge $d\ell' = dz'$ at z'. The distance R from the charge element to the point $P(0, 0, z)$ along the axis of the line charge is

$$R = (z - z'), \qquad z > \frac{L}{2}.$$

Here it is extremely important to distinguish the position of the field point (unprimed coordinates) from the position of the source point (primed coordinates). We integrate



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Basics of Vector Calculus

Electric potential due to continuous distributions Example:

over the source region

$$V = \frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{dz'}{z-z'}$$

$$= \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left[\frac{z+(L/2)}{z-(L/2)}\right], \qquad z > \frac{L}{2}.$$

The $\mathbf{E}$ field at P is the negative gradient of V with respect to the unprimed field coordinates. For this problem,

$$\mathbf{E} = -\mathbf{a}_z\frac{dV}{dz} = \mathbf{a}_z\frac{\rho_\ell L}{4\pi\epsilon_0[z^2-(L/2)^2]}, \qquad z > \frac{L}{2}.$$

The preceding two examples illustrate the procedure for determining $\mathbf{E}$ by first finding V when Gauss's law cannot be conveniently applied. However, we emphasize that, *if symmetry conditions exist such that a Gaussian surface can be constructed over which* $\mathbf{E}\cdot d\mathbf{s}$ *is constant, it is always easier to determine* $\mathbf{E}$ *directly*. The potential V, if desired, may be obtained from $\mathbf{E}$ by integration.

Basics of Vector Calculus

Electric potential due to continuous distributions Example:

EXAMPLE 4.12 Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$,

(a) Find the electric flux density **D** at $(2, \pi/2, 0)$.

(b) Calculate the work done in moving a 10-μC charge from point $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$.

Solution:

(a) $\mathbf{D} = \varepsilon_o \mathbf{E}$

But

$$\mathbf{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right]$$
$$= \frac{20}{r^3}\sin\theta\cos\phi\,\mathbf{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\mathbf{a}_\theta + \frac{10}{r^3}\sin\phi\,\mathbf{a}_\phi$$

Basics of Vector Calculus

Electric potential due to continuous distributions Example:

At $(2, \pi/2, 0)$,

$$\mathbf{D} = \varepsilon_o \mathbf{E}\,(r = 2, \theta = \pi/2, \phi = 0) = \varepsilon_o\left(\frac{20}{8}\,\mathbf{a}_r - 0\mathbf{a}_\theta + 0\mathbf{a}_\phi\right)$$

$$= 2.5\varepsilon_o \mathbf{a}_r \text{ C/m}^2 = 22.1\ \mathbf{a}_r \text{ pC/m}^2$$

(b) The work done can be found in two ways, using either **E** or V.

$$W = -Q\int_A^B \mathbf{E} \cdot d\mathbf{l} = QV_{AB}$$

$$= Q(V_B - V_A)$$

$$= 10\left(\frac{10}{16}\sin 90^\circ \cos 60^\circ - \frac{10}{1}\sin 30^\circ \cos 120^\circ\right) \cdot 10^{-6}$$

$$= 10\left(\frac{10}{32} - \frac{-5}{2}\right) \cdot 10^{-6}$$

$= 28.125\ \mu$J as obtained before

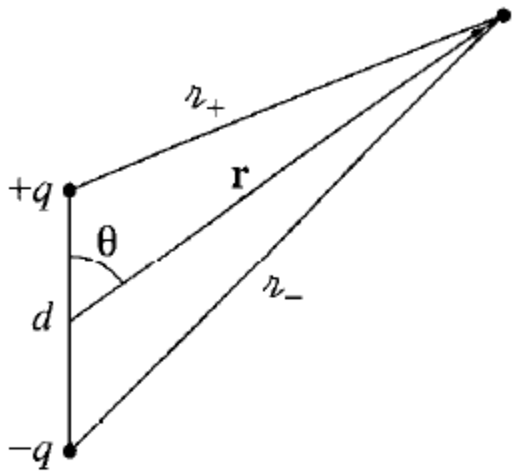
Electric Dipole

The electric dipole: It is important to understand a dipole in order to understand the effect of materials on electric field and electric flux.

An **electric dipole** is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

The electric dipole:

A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.



Solution: Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$ (Fig. 3.26). Then

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right),$$

and (from the law of cosines)

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right).$$

We're interested in the régime $r \gg d$, so the third term is negligible, and the binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right).$$

Thus

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

and hence

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}. \tag{3.90}$$

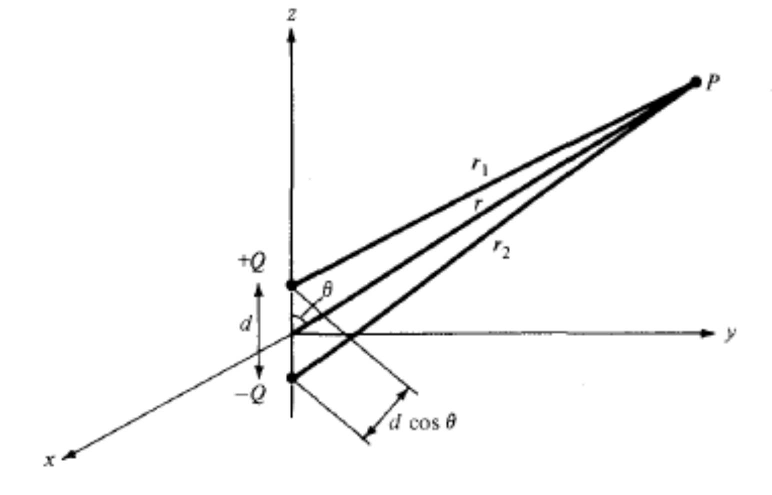
Basics of Vector Calculus

The electric dipole:

Since $d \cos\theta = \mathbf{d} \cdot \mathbf{a}_r$, where $\mathbf{d} = d\mathbf{a}_z$, if we define as the *dipole moment*,

$$\mathbf{p} = Q\mathbf{d}$$

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\varepsilon_o r^2}$$



Note that the dipole moment $\mathbf{p}$ is directed from $-Q$ to $+Q$. If the dipole center is not at the origin but at $\mathbf{r}'$

$$V(\mathbf{r}) = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o |\mathbf{r} - \mathbf{r}'|^3}$$

Basics of Vector Calculus

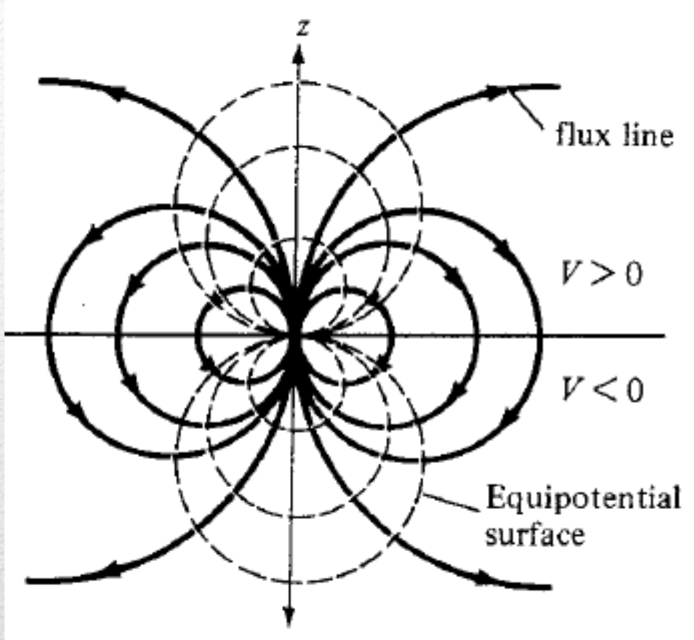
The dipole electric field:

$$\mathbf{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta\right]$$

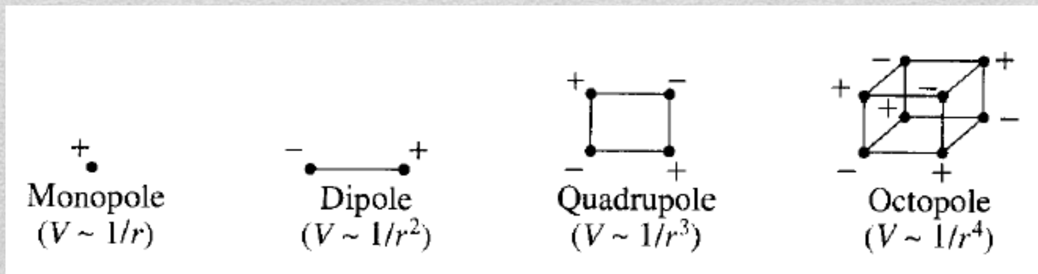
$$= \frac{Qd\cos\theta}{2\pi\varepsilon_o r^3}\mathbf{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_o r^3}\mathbf{a}_\theta$$

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

where $p = |\mathbf{p}| = Qd$.



There also exist:



Basics of Vector Calculus

The electric dipole example:

EXAMPLE 4.13 Two dipoles with dipole moments $-5\mathbf{a}_z$ nC/m and $9\mathbf{a}_z$ nC/m are located at points $(0, 0, -2)$ and $(0, 0, 3)$, respectively. Find the potential at the origin.

Solution:

$$V = \sum_{k=1}^{2} \frac{\mathbf{p}_k \cdot \mathbf{r}_k}{4\pi\varepsilon_o r_k^3}$$

$$= \frac{1}{4\pi\varepsilon_o}\left[\frac{\mathbf{p}_1 \cdot \mathbf{r}_1}{r_1^3} + \frac{\mathbf{p}_2 \cdot \mathbf{r}_2}{r_2^3}\right]$$

where

$$\mathbf{p}_1 = -5\mathbf{a}_z, \quad \mathbf{r}_1 = (0,0,0) - (0,0,-2) = 2\mathbf{a}_z, \quad r_1 = |\mathbf{r}_1| = 2$$

$$\mathbf{p}_2 = 9\mathbf{a}_z, \quad \mathbf{r}_2 = (0,0,0) - (0,0,3) = -3\mathbf{a}_z, \quad r_2 = |\mathbf{r}_2| = 3$$

Hence,

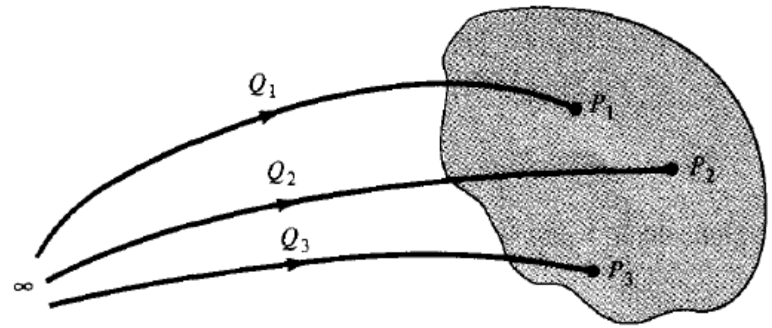
$$V = \frac{1}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left[\frac{-10}{2^3} - \frac{27}{3^3}\right] \cdot 10^{-9}$$

$$= -20.25 \text{ V}$$

Basics of Vector Calculus

Energy density in electrostatic fields: It corresponds to how much energy is needed to assemble the charge distribution together. Also, the energy stored in the system.

The work done in assembling three charges



$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32})$$

If the charges were positioned in reverse order,

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13})$$

$$2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$$
$$= Q_1V_1 + Q_2V_2 + Q_3V_3$$

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k$$

Basics of Vector Calculus

Example (point charges)

EXAMPLE 4.14 Three point charges −1 nC, 4 nC, and 3 nC are located at (0, 0, 0), (0, 0, 1), and (1, 0, 0), respectively. Find the energy in the system.

$$W = \frac{1}{2}\sum_{k=1}^{3} Q_k V_k = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

$$= \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(\sqrt{2})}\right]$$

$$+ \frac{Q_3}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_2}{4\pi\varepsilon_o(\sqrt{2})}\right]$$

$$= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{nJ} = 13.37\text{ nJ}$$

Basics of Vector Calculus

Energy density in continuous charge distributions

$$W_E = \frac{1}{2}\int \rho_L V\, dl \qquad \text{(line charge)}$$

$$W = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int\left(\frac{q^2}{r^4}\right)(r^2 \sin\theta\, dr\, d\theta\, d\phi) = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}\, dr = \infty.$$

$$W_E = \frac{1}{2}\int \rho_S V\, dS \qquad \text{(surface charge)}$$

$$W_E = \frac{1}{2}\int \rho_v V\, dv \qquad \text{(volume charge)}$$

Since $\rho_v = \nabla \cdot \mathbf{D}$

$$W_E = \frac{1}{2}\int_v (\nabla \cdot \mathbf{D})\, V\, dv$$

Using the identity $(\nabla \cdot \mathbf{A})V = \nabla \cdot V\mathbf{A} - \mathbf{A} \cdot \nabla V$

$$W_E = \frac{1}{2}\int_v (\nabla \cdot V\mathbf{D})\, dv - \frac{1}{2}\int_v (\mathbf{D} \cdot \nabla V)\, dv$$

$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2}\int_v (\mathbf{D} \cdot \nabla V)\, dv$$

Basics of Vector Calculus

Energy density in continuous charge distributions

Integrating the whole space gives only the second part of the integral, which means the energy is stored in the field

$$W_E = -\frac{1}{2}\int_v (\mathbf{D}\cdot\nabla V)\,dv = \frac{1}{2}\int_v (\mathbf{D}\cdot\mathbf{E})\,dv$$

and since $\mathbf{E} = -\nabla V$ and $\mathbf{D} = \varepsilon_o\mathbf{E}$

$$\boxed{W_E = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,dv = \frac{1}{2}\int \varepsilon_o E^2\,dv}$$

From this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}\varepsilon_o E^2 = \frac{D^2}{2\varepsilon_o}$$

$$W_E = \int w_E\,dv$$

Basics of Vector Calculus

Energy density in continuous charge distributions example:

EXAMPLE 4.15 A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_o, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

Determine V everywhere and the energy stored in region $r < R$.

Solution:

The **D** field has already been found in Section 4.6D using Gauss's law.

(a) For $r \geqslant R$, $\mathbf{E} = \dfrac{\rho_o R^3}{3\varepsilon_o r^2}\mathbf{a}_r$.

Once **E** is known, V is determined as

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\rho_o R^3}{3\varepsilon_o}\int \frac{1}{r^2}\,dr$$

$$= \frac{\rho_o R^3}{3\varepsilon_o r} + C_1, \qquad r \geqslant R$$

Since $V(r = \infty) = 0$, $C_1 = 0$.

Basics of Vector Calculus

Energy density in continuous charge distributions example:

(b) For $r \leq R$, $\mathbf{E} = \dfrac{\rho_o r}{3\varepsilon_o}\mathbf{a}_r$.

Hence,

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\rho_o}{3\varepsilon_o}\int r\,dr$$
$$= -\frac{\rho_o r^2}{6\varepsilon_o} + C_2$$

From part (a) $V(r = R) = \dfrac{\rho_o R^2}{3\varepsilon_o}$. Hence,

$$\frac{R^2\rho_o}{3\varepsilon_o} = \frac{\rho_o R^2}{6\varepsilon_o} + C_2 \rightarrow C_2 = \frac{R^2\rho_o}{2\varepsilon_o}$$

and

$$V = \frac{\rho_o}{6\varepsilon_o}(3R^2 - r^2)$$

Basics of Vector Calculus

Energy density in continuous charge distributions example:

Thus from parts (a) and (b)

$$V = \begin{cases} \dfrac{\rho_o R^3}{3\varepsilon_o r}, & r \geq R \\ \dfrac{\rho_o}{6\varepsilon_o}(3R^2 - r^2), & r \leq R \end{cases}$$

(c) The energy stored is given by

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\varepsilon_o \int E^2\, dv$$

For $r \leq R$,

$$\mathbf{E} = \frac{\rho_o r}{3\varepsilon_o}\mathbf{a}_r$$

Hence,

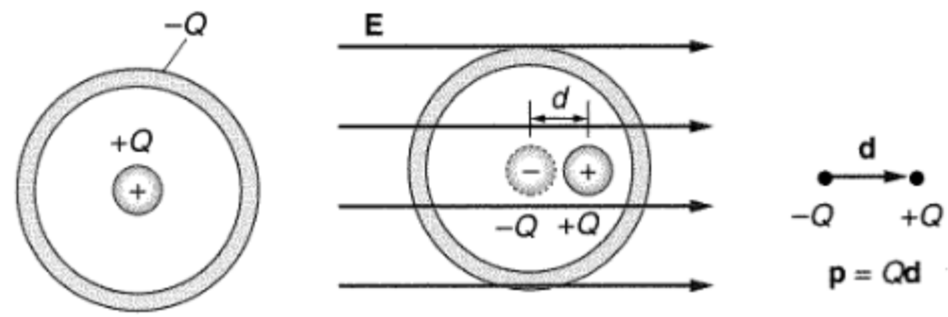
$$W = \frac{1}{2}\varepsilon_o \frac{\rho_o^2}{9\varepsilon_o^2}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \cdot r^2 \sin\theta\, d\phi\, d\theta\, dr$$

$$= \frac{\rho_o^2}{18\varepsilon_o} 4\pi \cdot \frac{r^5}{5}\bigg|_0^R = \frac{2\pi\rho_o^2 R^5}{45\varepsilon_o}\text{ J}$$

Electric field in materials: (polarization and polarization charge, dielectrics)

Dipole moment of a single dipole

$$\mathbf{p} = Q\mathbf{d}$$



More than a dipole in differential volume:

$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N = \sum_{k=1}^{N} Q_k\mathbf{d}_k$$

Polarization density vector: a point function telling the direction and magnitude of the polarization at that point

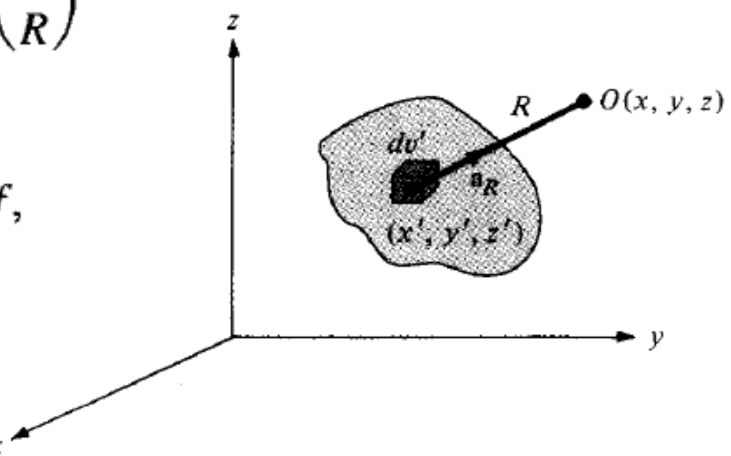
$$\mathbf{P} = \frac{\lim_{\Delta v \to 0} \sum_{k=1}^{N} Q_k\mathbf{d}_k}{\Delta v}$$

Electric field in materials: (polarization and polarization charge, dielectrics)

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R \, dv'}{4\pi\varepsilon_o R^2}$$

$$\nabla' \frac{1}{R} = \frac{\mathbf{a}_R}{R^2}$$

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right)$$



Applying the vector identity $\nabla' \cdot f\mathbf{A} = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f$,

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \nabla' \cdot \frac{\mathbf{P}}{R} - \frac{\nabla' \cdot \mathbf{P}}{R}$$

$$V = \int_{v'} \frac{1}{4\pi\varepsilon_o}\left[\nabla' \cdot \frac{\mathbf{P}}{R} - \frac{1}{R}\nabla' \cdot \mathbf{P}\right] dv'$$

$$V = \int_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{4\pi\varepsilon_o R} \, dS' + \int_{v'} \frac{-\nabla' \cdot \mathbf{P}}{4\pi\varepsilon_o R} \, dv'$$

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$
$$\rho_{pv} = -\nabla \cdot \mathbf{P}$$

Bound surface and volume charge densities

$$\rho_t = \rho_v + \rho_{pv} = \nabla \cdot \varepsilon_o \mathbf{E}$$

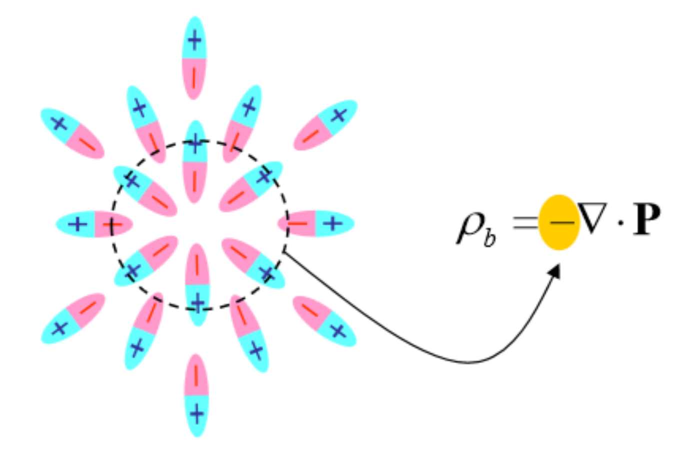
$$\begin{aligned}\rho_v &= \nabla \cdot \varepsilon_o \mathbf{E} - \rho_{pv} \\ &= \nabla \cdot (\varepsilon_o \mathbf{E} + \mathbf{P}) \\ &= \nabla \cdot \mathbf{D}\end{aligned}$$

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

Electric field in materials: (polarization and polarization charge, dielectrics)

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$
$$\rho_{pv} = -\nabla \cdot \mathbf{P}$$

Bound surface and volume charge densities



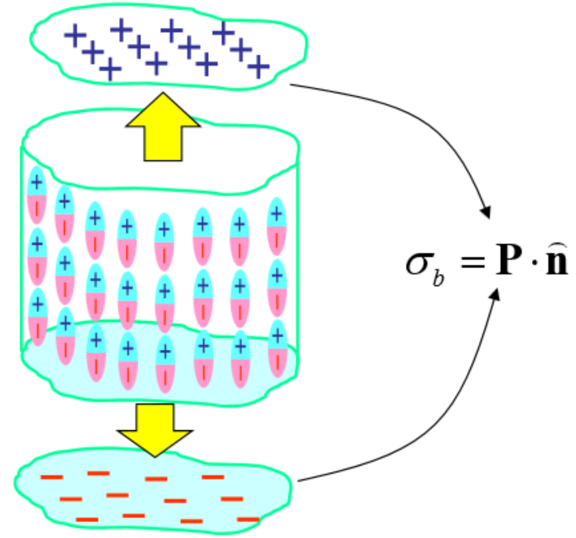
Bound charges are the charges due to polarization. But, free charges are the ones that are not due to polarization

$$Q_b = \oint \mathbf{P} \cdot d\mathbf{S} = \int \rho_{ps}\, dS$$

$$-Q_b = \int_v \rho_{pv}\, dv = -\int_v \nabla \cdot \mathbf{P}\, dv$$

$$\text{Total charge} = \oint_S \rho_{ps}\, dS + \int_v \rho_{pv}\, dv = Q_b - Q_b = 0$$

the total charge of the dielectric material remains zero.



Electric field in materials: (polarization and polarization charge, dielectrics)

Linear isotropic and homogenius media:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$$

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e)\, \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

where χ_e, known as the *electric susceptibility* of the material

$$\mathbf{D} = \varepsilon \mathbf{E}$$

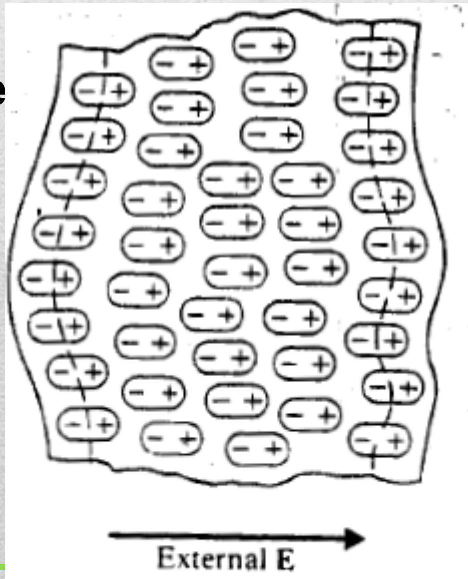
$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$

Linear: D depends linearly on E
Isotropic: D is in the direction of E

Homogenius: permittivity does not depend on space (position of the point)

Permittivity is determined by the ability of a material to polarize in response to the field, and reduce the total electric field inside the material. It permittivity relates to a material's ability to transmit (or "permit") an electric field. It is directly related to electric susceptibility, which is a measure of how easily a dielectric polarizes in response to an electric field.



Electric field in materials: (polarization and polarization charge, dielectrics)

Non-Linear, anisotropic and non-homogenius media:

Non-Linear: permitivity is not, it is constant a function of electric field.
anisotropic: permitivity is different in different directions

Non-homogenius: permittivity depends on space (position of the point) and as a result the polarization is not uniform inside the dielectric

In a homogenious medium the polarization is uniform and the volume bound charges vanishes. As a result, only surface charges exist and they cancel each other. In a non homogenius medium both surface and volume bound charges exist and the total bound charge is zero.

Electric field in materials: (polarization and polarization charge, dielectrics)

A **dielectric material** (in which $\mathbf{D} = \varepsilon\mathbf{E}$ applies) is linear if ε does not change with the applied $\mathbf{E}$ field, homogeneous if ε does not change from point to point, and isotropic if ε does not change with direction.

The **dielectric constant** (or **relative permittivity**) ε_r is the ratio of the permittivity of the dielectric to that of free space.

The **dielectric strength** is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

In a linear homogenious isotropic medium, all the equations derived still applies if the free space permitivity is replaced by the medium permitivity.

Polarization charges example:

EXAMPLE 5.5

A dielectric cube of side L and center at the origin has a radial polarization given by $\mathbf{P} = a\mathbf{r}$, where a is a constant and $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$. Find all bound charge densities and show explicitly that the total bound charge vanishes.

Solution:

For each of the six faces of the cube, there is a surface charge ρ_{ps}. For the face located at $x = L/2$, for example,

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_x \Big|_{x=L/2} = ax \Big|_{x=L/2} = aL/2$$

The total bound surface charge is

$$Q_s = \int \rho_{ps}\, dS = 6 \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \rho_{ps}\, dy\, dz = \frac{6aL}{2} L^2$$
$$= 3aL^3$$

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Polarization charges example:

Ex.

The bound volume charge density is given by

$$\rho_{pv} = -\nabla \cdot \mathbf{P} = -(a + a + a) = -3a$$

and the total bound volume charge is

$$Q_v = \int \rho_{pv}\, dv = -3a \int dv = -3aL^3$$

Hence the total charge is

$$Q_t = Q_s + Q_v = 3aL^3 - 3aL^3 = 0$$

Basics of Vector Calculus

EXAMPLE 5.6

The electric field intensity in polystyrene ($\varepsilon_r = 2.55$) filling the space between the plates of a parallel-plate capacitor is 10 kV/m. The distance between the plates is 1.5 mm. Calculate:

(a) D

(b) P

(c) The surface charge density of free charge on the plates

(d) The surface density of polarization charge

(e) The potential difference between the plates

Solution:

(a) $D = \varepsilon_o \varepsilon_r E = \dfrac{10^{-9}}{36\pi} \cdot (2.55) \cdot 10^4 = 225.4 \text{ nC/m}^2$

(b) $P = \chi_e \varepsilon_o E = (1.55) \cdot \dfrac{10^{-9}}{36\pi} \cdot 10^4 = 137 \text{ nC/m}^2$

(c) $\rho_S = \mathbf{D} \cdot \mathbf{a}_n = D_n = 225.4 \text{ nC/m}^2$

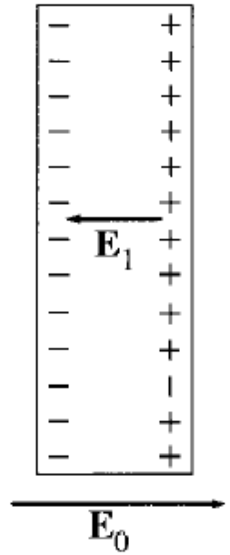
(d) $\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n = P_n = 137 \text{ nC/m}^2$

(e) $V = Ed = 10^4 (1.5 \times 10^{-3}) = 15 \text{ V}$

Electric Field in Conductors:

In an **insulator,** such as glass or rubber, each electron is attached to a particular atom. In a metallic **conductor,** by contrast, one or more electrons per atom are free to roam about at will through the material. (In liquid conductors such as salt water it is *ions* that do the moving.) A *perfect* conductor would be a material containing an *unlimited* supply of completely free charges. In real life there are no perfect conductors, but many substances come amazingly close. From this definition the basic electrostatic properties of ideal conductors immediately follow:

(i) E = 0 inside a conductor.



Basics of Vector Calculus

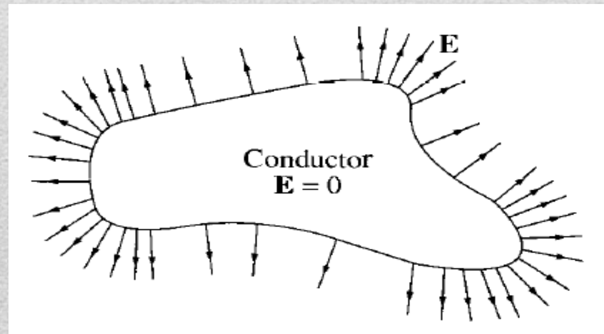
Conductors:

(ii) $\rho = 0$ inside a conductor. This follows from Gauss's law: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. If $\mathbf{E} = 0$, so also is ρ. There is still charge around, but exactly as much plus charge as minus, so the *net* charge density in the interior is zero.

(iii) Any net charge resides on the surface. That's the only other place it *can* be.

(iv) A conductor is an equipotential. For if **a** and **b** are any two points within (or at the surface of) a given conductor, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$, and hence $V(\mathbf{a}) = V(\mathbf{b})$.

(v) E is perpendicular to the surface, just outside a conductor. Otherwise, as in **(i)**, charge will immediately flow around the surface until it kills off the tangential component



Boundary conditions:

So far, we have considered the existence of the electric field in a homogeneous medium. If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called *boundary conditions*. These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. Obviously, the conditions will be dictated by the types of material the media are made of. We shall consider the boundary conditions at an interface separating

- dielectric (ε_{r1}) and dielectric (ε_{r2})
- conductor and dielectric
- conductor and free space

To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} \qquad \oint \mathbf{E} \cdot d\mathbf{l} = 0$$

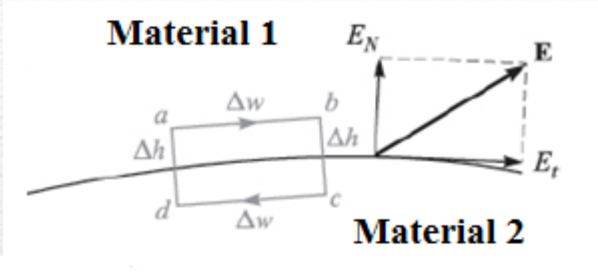
Also we need to decompose the electric field intensity **E** into two orthogonal components:

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_n$$

where $\mathbf{E}_t$ and $\mathbf{E}_n$ are, respectively, the tangential and normal components of **E** to the interface of interest. A similar decomposition can be done for the electric flux density **D**.

Boundary conditions:

For the tangential component:



$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

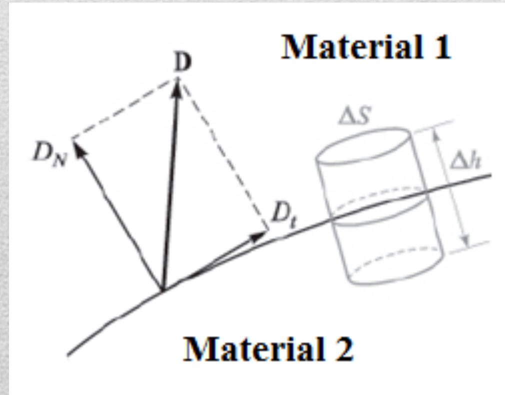
$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

where $E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$. As $\Delta h \rightarrow 0$,

$$\boxed{E_{1t} = E_{2t}}$$

For the normal component:

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$



As Δh goes to zero the walls contribute nothing

$$\Delta Q = \rho_S\,\Delta S = D_{1n}\,\Delta S - D_{2n}\,\Delta S$$

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

Basics of Vector Calculus

Boundary Conditions:

A. Dielectric–Dielectric Boundary Conditions

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

D tangential is discontinuous but E tangential is continuous

$$D_{1n} = D_{2n}$$

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

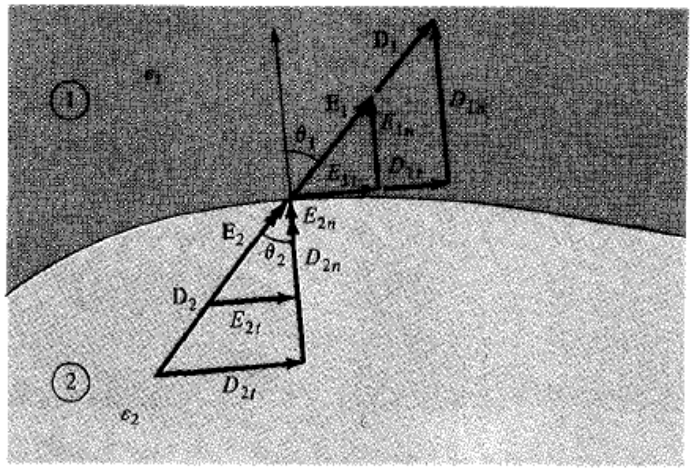
E normal is discontinuous but D normal is continuous
The charge at the interface is assumed to be zero

Law of refraction:

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$$

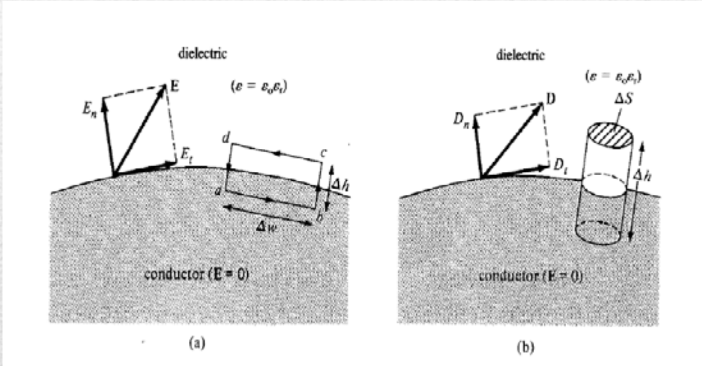
$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$



Basics of Vector Calculus

B. Conductor–Dielectric Boundary Conditions

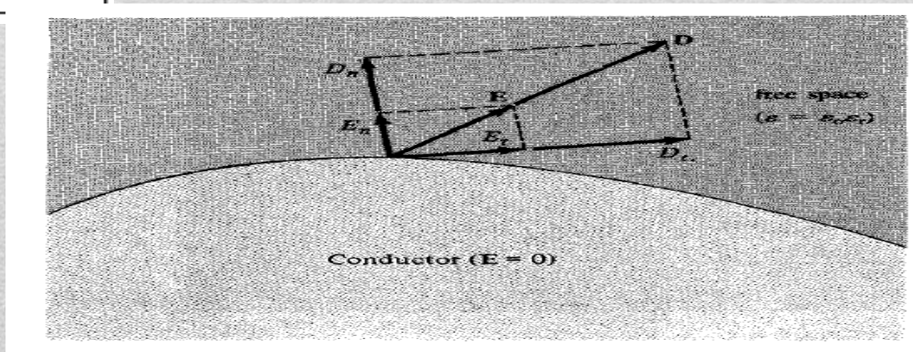
The conductor is assumed to be perfect, meaning there is no charges inside and the field is zero inside



$$D_t = \varepsilon_o \varepsilon_r E_t = 0, \qquad D_n = \varepsilon_o \varepsilon_r E_n = \rho_S$$

C. Conductor–Free Space Boundary Conditions

$$D_t = \varepsilon_o E_t = 0, \qquad D_n = \varepsilon_o E_n = \rho_S$$



Basics of Vector Calculus

EXAMPLE 5.9 Two extensive homogeneous isotropic dielectrics meet on plane $z = 0$. For $z \geq 0$, ε_{r1} = and for $z \leq 0$, $\varepsilon_{r2} = 3$. A uniform electric field $\mathbf{E}_1 = 5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ kV/m exists f $z \geq 0$. Find

(a) $\mathbf{E}_2$ for $z \leq 0$

(b) The angles E_1 and E_2 make with the interface

(c) The energy densities in J/m^3 in both dielectrics

(d) The energy within a cube of side 2 m centered at (3, 4, −5)

Solution:

Let the problem be as illustrated in Figure 5.15.

(a) Since $\mathbf{a}_z$ is normal to the boundary plane, we obtain the normal components as

$$E_{1n} = \mathbf{E}_1 \cdot \mathbf{a}_n = \mathbf{E}_1 \cdot \mathbf{a}_z = 3$$

$$\mathbf{E}_{1n} = 3\mathbf{a}_z$$

$$\mathbf{E}_{2n} = (\mathbf{E}_2 \cdot \mathbf{a}_z)\mathbf{a}_z$$

Also

$$\mathbf{E} = \mathbf{E}_n + \mathbf{E}_t$$

Hence,

$$\mathbf{E}_{1t} = \mathbf{E}_1 - \mathbf{E}_{1n} = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Basics of Vector Calculus

Thus

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Similarly,

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} \rightarrow \varepsilon_{r2}\mathbf{E}_{2n} = \varepsilon_{r1}\mathbf{E}_{1n}$$

or

$$\mathbf{E}_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\mathbf{E}_{1n} = \frac{4}{3}(3\mathbf{a}_z) = 4\mathbf{a}_z$$

Thus

$$\begin{aligned}\mathbf{E}_2 &= \mathbf{E}_{2t} + \mathbf{E}_{2n} \\ &= 5\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z \text{ kV/m}\end{aligned}$$

(b) Let α_1 and α_2 be the angles $\mathbf{E}_1$ and $\mathbf{E}_2$ make with the interface while θ_1 and θ_2 are the angles they make with the normal to the interface as shown in Figures 5.15; that is,

$$\alpha_1 = 90 - \theta_1$$

$$\alpha_2 = 90 - \theta_2$$

Since $E_{1n} = 3$ and $E_{1t} = \sqrt{25 + 4} = \sqrt{29}$

$$\tan\theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{\sqrt{29}}{3} = 1.795 \rightarrow \theta_1 = 60.9°$$

Basics of Vector Calculus

Hence,

$$\alpha_1 = 29.1°$$

Alternatively,

$$\mathbf{E}_1 \cdot \mathbf{a}_n = |\mathbf{E}_1| \cdot 1 \cdot \cos\theta_1$$

or

$$\cos\theta_1 = \frac{3}{\sqrt{38}} = 0.4867 \rightarrow \theta_1 = 60.9°$$

Similarly,

$$E_{2n} = 4 \qquad E_{2t} = E_{1t} = \sqrt{29}$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{29}}{4} = 1.346 \rightarrow \theta_2 = 53.4°$$

Hence,

$$\alpha_2 = 36.6°$$

Basics of Vector Calculus

Note that $\dfrac{\tan \theta_1}{\tan \theta_2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}$ is satisfied.

(c) The energy densities are given by

$$w_{E1} = \frac{1}{2}\varepsilon_1|\mathbf{E}_1|^2 = \frac{1}{2} \cdot 4 \cdot \frac{10^{-9}}{36\pi} \cdot (25 + 4 + 9) \times 10^6$$
$$= 672\ \mu\text{J/m}^3$$

$$w_{E2} = \frac{1}{2}\varepsilon_2|\mathbf{E}_2|^2 = \frac{1}{2} \cdot 3 \cdot \frac{10^{-9}}{36\pi}(25 + 4 + 16) \times 10^6$$
$$= 597\ \mu\text{J/m}^3$$

(d) At the center $(3, 4, -5)$ of the cube of side 2 m, $z = -5 < 0$; that is, the cube is in region 2 with $2 \leq x \leq 4, 3 \leq y \leq 5, -6 \leq z \leq -4$. Hence

$$W_E = \int w_{E2}\, dv = \int_{x=2}^{4}\int_{y=3}^{5}\int_{z=-6}^{-4} w_{E2}\, dz\, dy\, dz = w_{E2}(2)(2)(2)$$
$$= 597 \times 8\mu\text{J} = 4.776\text{ mJ}$$

Example

EXAMPLE 5.10

Region $y \leq 0$ consists of a perfect conductor while region $y \geq 0$ is a dielectric medium ($\varepsilon_{1r} = 2$) as in Figure 5.16. If there is a surface charge of 2 nC/m^2 on the conductor, determine **E** and **D** at

(a) $A(3, -2, 2)$

(b) $B(-4, 1, 5)$

Solution:

(a) Point $A(3, -2, 2)$ is in the conductor since $y = -2 < 0$ at A. Hence,

$$\mathbf{E} = 0 = \mathbf{D}$$

(b) Point $B(-4, 1, 5)$ is in the dielectric medium since $y = 1 > 0$ at B.

$$D_n = \rho_S = 2 \text{ nC/m}^2$$

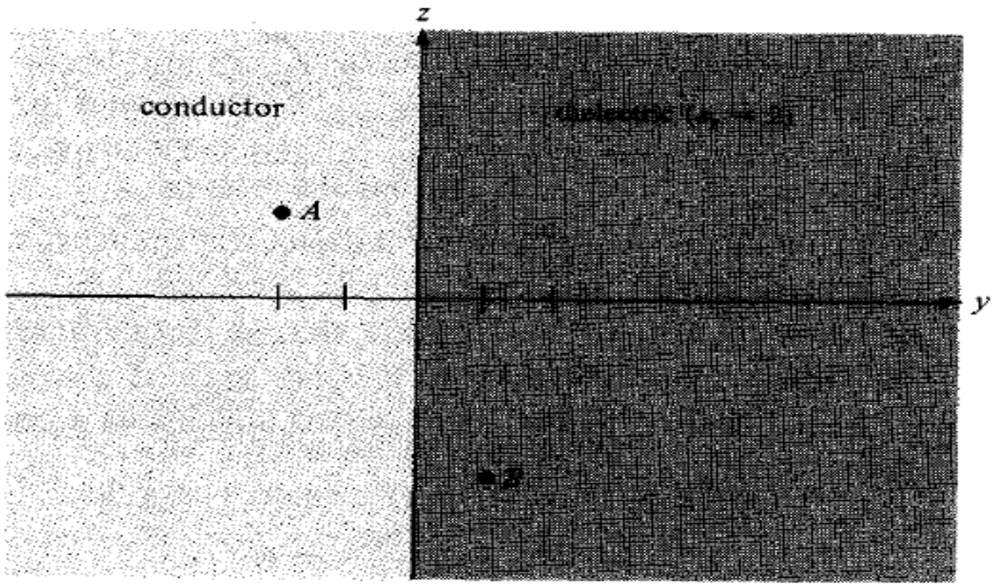
Example

Hence,

$$\mathbf{D} = 2\mathbf{a}_y \text{ nC/m}^2$$

and

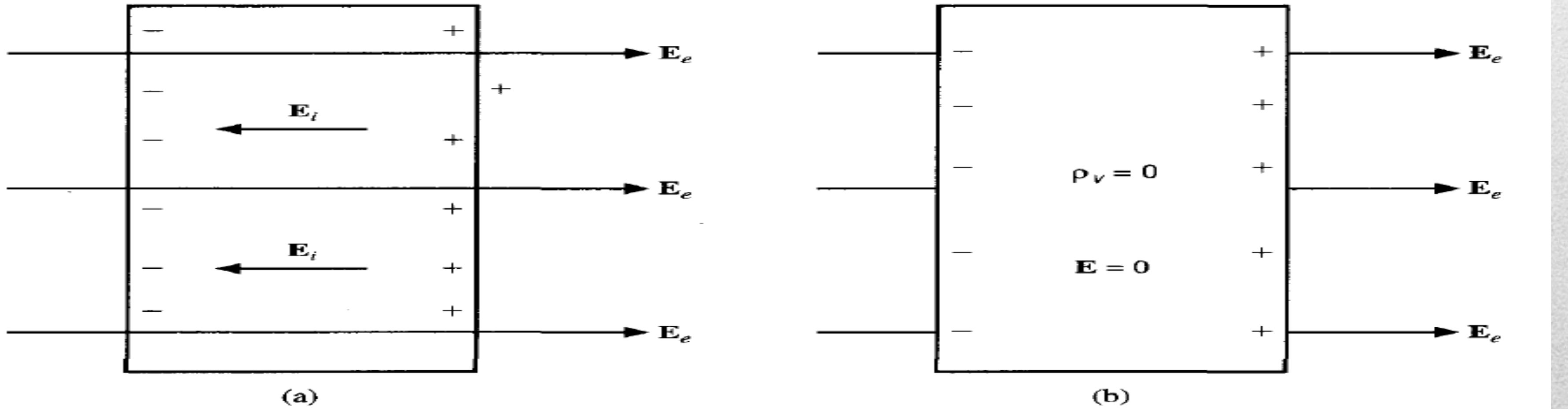
$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o \varepsilon_r} = 2 \times 10^{-9} \times \frac{36\pi}{2} \times 10^9 \, \mathbf{a}_y = 36\pi \mathbf{a}_y$$
$$= 113.1\mathbf{a}_y \text{ V/m}$$



Conductors

A perfect conductor cannot contain an electrostatic field within it.

$$\mathbf{E} = 0, \qquad \rho_v = 0, \qquad V_{ab} = 0 \qquad \text{inside a conductor}$$



Conductors

The electric field applied is uniform and its magnitude is given by

$$E = \frac{V}{\ell}$$

Since the conductor has a uniform cross section,

$$J = \frac{I}{S}$$

Substituting eqs. (5.11) and (5.13) into eq. (5.14) gives

$$\frac{I}{S} = \sigma E = \frac{\sigma V}{\ell}$$

Hence

$$R = \frac{V}{I} = \frac{\ell}{\sigma S}$$

or

$$\boxed{R = \frac{\rho_c \ell}{S}}$$

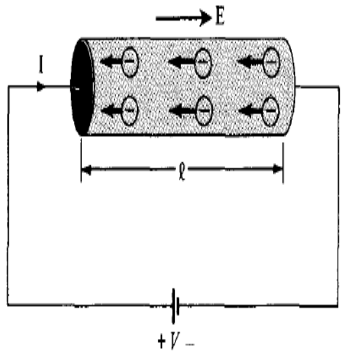


Figure 5.3 A conductor of uniform cross section under an applied **E** field.

where $\rho_c = 1/\sigma$ is the *resistivity* of the material.

Conductors

the resistance of a conductor of nonuniform cross section; that is,

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\int \sigma \mathbf{E} \cdot d\mathbf{S}}$$

Power P (in watts) is defined as the rate of change of energy W (in joules) or force times velocity. Hence,

$$\int \rho_v \, dv \, \mathbf{E} \cdot \mathbf{u} = \int \mathbf{E} \cdot \rho_v \mathbf{u} \, dv$$

or

$$P = \int \mathbf{E} \cdot \mathbf{J} \, dv$$

which is known as *Joule's law.* The power density w_P (in watts/m^3) is given by

$$w_P = \frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} = \sigma \, |\mathbf{E}|^2$$

Conductors

For a conductor with uniform cross section, $dv = dS\,dl$; that is,

$$P = \int_L E\,dl \int_S J\,dS = VI$$

or

$$P = I^2R$$

which is the more common form of Joule's law in electric circuit theory.

Example

EXAMPLE 5.3

A wire of diameter 1 mm and conductivity 5×10^7 S/m has 10^{29} free electrons/m^3 when an electric field of 10 mV/m is applied. Determine

(a) The charge density of free electrons

(b) The current density

(c) The current in the wire

(d) The drift velocity of the electrons. Take the electronic charge as $e = -1.6 \times 10^{-19}$ C.

Solution:

(In this particular problem, convection and conduction currents are the same.)

(a) $\rho_v = ne = (10^{29})(-1.6 \times 10^{-19}) = -1.6 \times 10^{10}$ C/m^3

(b) $J = \sigma E = (5 \times 10^7)(10 \times 10^{-3}) = 500$ kA/m^2

(c) $I = JS = (5 \times 10^5)\left(\frac{\pi d^2}{4}\right) = \frac{5\pi}{4} \cdot 10^{-6} \cdot 10^5 = 0.393$ A

(d) Since $J = \rho_v u$, $u = \frac{J}{\rho_v} = \frac{5 \times 10^5}{1.6 \times 10^{10}} = 3.125 \times 10^{-5}$ m/s.

Example

EXAMPLE 5.4

A lead ($\sigma = 5 \times 10^6$ S/m) bar of square cross section has a hole bored along its length of 4 m so that its cross section becomes that of Figure 5.5. Find the resistance between the square ends.

Solution:

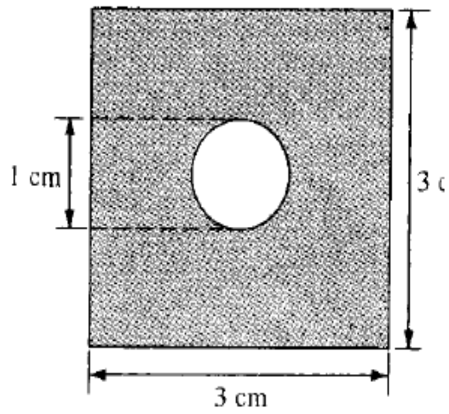
Since the cross section of the bar is uniform, we may apply eq. (5.16); that is,

$$R = \frac{\ell}{\sigma S}$$

where $S = d^2 - \pi r^2 = 3^2 - \pi\left(\frac{1}{2}\right)^2 = 9 - \frac{\pi}{4}\,\text{cm}^2$.

Hence,

$$R = \frac{4}{5 \times 10^6 (9 - \pi/4) \times 10^{-4}} = 974\ \mu\Omega$$



Boundary Value Problem

Poisson's and Laplace's equations:

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \rho_v$$

$$\mathbf{E} = -\nabla V$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

Poisson's equation

A special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge-free region).

$$\boxed{\nabla^2 V = 0}$$

Laplace's equation

Uniqueness Theorem

Uniqueness theorem:

This is the **uniqueness theorem:** If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Proof:

Suppose there were *two* solutions to Laplace's equation:

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0,$$

both of which assume the specified value on the surface. I want to prove that they must be equal. The trick is look at their *difference:*

$$V_3 \equiv V_1 - V_2.$$

This obeys Laplace's equation,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0,$$

and it takes the value *zero* on all boundaries (since V_1 and V_2 are equal there). But Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero everywhere, and hence

$$V_1 = V_2. \quad \text{qed}$$

Derivation

Proof more mathematical treatment:

$$\nabla^2 V_1 = 0, \qquad \nabla^2 V_2 = 0 \tag{6.9a}$$

$$V_1 = V_2 \qquad \text{on the boundary} \tag{6.9b}$$

We consider their difference

$$V_d = V_2 - V_1 \tag{6.10}$$

which obeys

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \tag{6.11a}$$

$$V_d = 0 \qquad \text{on the boundary} \tag{6.11b}$$

according to eq. (6.9). From the divergence theorem.

$$\int_v \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{6.12}$$

We let $\mathbf{A} = V_d \nabla V_d$ and use a vector identity

$$\nabla \cdot \mathbf{A} = \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d$$

But $\nabla^2 V_d = 0$ according to eq. (6.11), so

$$\nabla \cdot \mathbf{A} = \nabla V_d \cdot \nabla V_d \tag{6.13}$$

Boundary-Value Problems

Proof more mathematical treatment:

Substituting eq. (6.13) into eq. (6.12) gives

$$\int_v \nabla V_d \cdot \nabla V_d \, dv = \oint_S V_d \nabla V_d \cdot d\mathbf{S} \qquad (6.14)$$

From eqs. (6.9) and (6.11), it is evident that the right-hand side of eq. (6.14) vanishes.

Hence:

$$\int_v |\nabla V_d|^2 \, dv = 0$$

Since the integration is always positive.

$$\nabla V_d = 0 \qquad (6.15a)$$

or

$$V_d = V_2 - V_1 = \text{constant everywhere in } v \qquad (6.15b)$$

But eq. (6.15) must be consistent with eq. (6.9b). Hence, $V_d = 0$ or $V_1 = V_2$ everywhere, showing that V_1 and V_2 cannot be different solutions of the same problem.

GENERAL PROCEDURE FOR SOLVING POISSON'S OR LAPLACE'S EQUATION

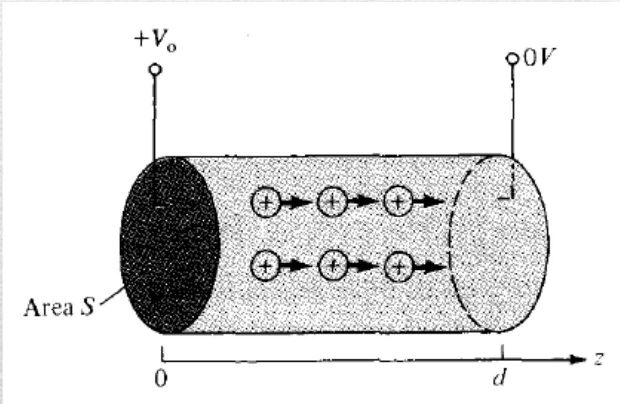
BVP solving approach:

The following general procedure may be taken in solving a given boundary-value problem involving Poisson's or Laplace's equation:

1. Solve Laplace's (if $\rho_v = 0$) or Poisson's (if $\rho_v \neq 0$) equation using either (a) direct integration when V is a function of one variable, or (b) separation of variables if V is a function of more than one variable. The solution at this point is not unique but expressed in terms of unknown integration constants to be determined.
2. Apply the boundary conditions to determine a unique solution for V. Imposing the given boundary conditions makes the solution unique.
3. Having obtained V, find $\mathbf{E}$ using $\mathbf{E} = -\nabla V$ and $\mathbf{D}$ from $\mathbf{D} = \varepsilon\mathbf{E}$.
4. If desired, find the charge Q induced on a conductor using $Q = \int \rho_S \, dS$ where $\rho_S = D_n$ and D_n is the component of $\mathbf{D}$ normal to the conductor. If necessary, the capacitance between two conductors can be found using $C = Q/V$.

EXAMPLE 6.1

Current-carrying components in high-voltage power equipment must be cooled to carry away the heat caused by ohmic losses. A means of pumping is based on the force transmitted to the cooling fluid by charges in an electric field. The electrohydrodynamic (EHD) pumping is modeled in Figure 6.1. The region between the electrodes contains a uniform charge ρ_o, which is generated at the left electrode and collected at the right electrode. Calculate the pressure of the pump if $\rho_o = 25\ \text{mC/m}^3$ and $V_o = 22$ kV.



Solution:

Since $\rho_v \neq 0$, we apply Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

The boundary conditions $V(z = 0) = V_o$ and $V(z = d) = 0$ show that V depends only on z (there is no ρ or ϕ dependence). Hence

$$\frac{d^2V}{dz^2} = \frac{-\rho_o}{\varepsilon}$$

Integrating once gives

$$\frac{dV}{dz} = \frac{-\rho_o z}{\varepsilon} + A$$

Integrating again yields

$$V = -\frac{\rho_o z^2}{2\varepsilon} + Az + B$$

where A and B are integration constants to be determined by applying the boundary conditions. When $z = 0$, $V = V_o$,

$$V_o = -0 + 0 + B \longrightarrow B = V_o$$

When $z = d$, $V = 0$,

$$0 = -\frac{\rho_o d^2}{2\varepsilon} + Ad + V_o$$

or

$$A = \frac{\rho_o d}{2\varepsilon} - \frac{V_o}{d}$$

The electric field is given by

$$\begin{aligned}\mathbf{E} &= -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = \left(\frac{\rho_o z}{\varepsilon} - A\right)\mathbf{a}_z \\ &= \left[\frac{V_o}{d} + \frac{\rho_o}{\varepsilon}\left(z - \frac{d}{2}\right)\right]\mathbf{a}_z\end{aligned}$$

The net force is

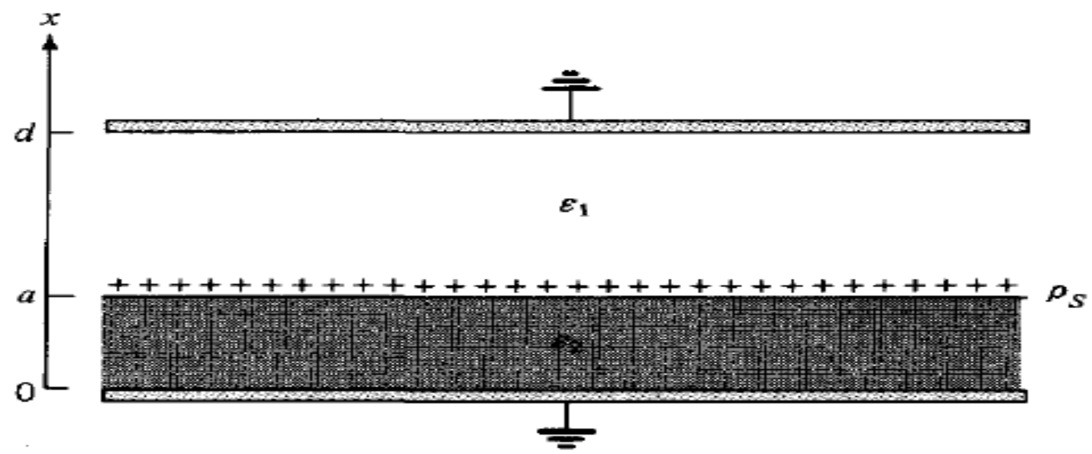
$$\begin{aligned}\mathbf{F} &= \int \rho_v \mathbf{E}\, dv = \rho_o \int dS \int_{z=0}^{d} \mathbf{E}\, dz \\ &= \rho_o S \left[\frac{V_o z}{d} + \frac{\rho_o}{2\varepsilon}(z^2 - dz)\right]\Bigg|_0^d \mathbf{a}_z \\ \mathbf{F} &= \rho_o S V_o \mathbf{a}_z\end{aligned}$$

The force per unit area or pressure is

$$\rho = \frac{F}{S} = \rho_o V_o = 25 \times 10^{-3} \times 22 \times 10^3 = 550 \text{ N/m}^2$$

EXAMPLE 6.2

The xerographic copying machine is an important application of electrostatics. The surface of the photoconductor is initially charged uniformly as in Figure 6.2(a). When light from the document to be copied is focused on the photoconductor, the charges on the lower surface combine with those on the upper surface to neutralize each other. The image is developed by pouring a charged black powder over the surface of the photoconductor. The electric field attracts the charged powder, which is later transferred to paper and melted to form a permanent image. We want to determine the electric field below and above the surface of the photoconductor.



Solution:

Since $\rho_v = 0$ in this case, we apply Laplace's equation. Also the potential depends only on x. Thus

$$\nabla^2 V = \frac{d^2 V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

Let the potentials above and below be V_1 and V_2, respectively.

$$V_1 = A_1 x + B_1, \qquad x > a$$

$$V_2 = A_2 x + B_2, \qquad x < a$$

The boundary conditions at the grounded electrodes are

$$V_1(x = d) = 0 \quad (6.2.2.a)$$

$$V_2(x = 0) = 0 \quad (6.2.2b)$$

At the surface of the photoconductor,

$$V_1(x = a) = V_2(x = a) \quad (6.2.3a)$$

$$D_{1n} - D_{2n} = \rho_S \Big|_{x=a} \quad (6.2.3b)$$

We use the four conditions in eqs. (6.2.2) and (6.2.3) to determine the four unknown constants A_1, A_2, B_1, and B_2. From eqs. (6.2.1) and 6.2.2),

$$0 = A_1 d + B_1 \rightarrow B_1 = -A_1 d \quad (6.2.4a)$$

$$0 = 0 + B_2 \rightarrow B_2 = 0 \quad (6.2.4b)$$

From eqs. (6.2.1) and (6.2.3a),

$$A_1 a + B_1 = A_2 a \quad (6.2.5)$$

To apply eq. (6.2.3b), recall that $\mathbf{D} = \varepsilon \mathbf{E} = -\varepsilon \nabla V$ so that

$$\rho_S = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = -\varepsilon_1 \frac{dV_1}{dx} + \varepsilon_2 \frac{dV_2}{dx}$$

or

$$\rho_S = -\varepsilon_1 A_1 + \varepsilon_2 A_2 \quad (6.2.6)$$

Solving for A_1 and A_2 in eqs. (6.2.4) to (6.2.6), we obtain

$$\mathbf{E}_1 = -A_1 \mathbf{a}_x = \frac{\rho_S \mathbf{a}_x}{\varepsilon_1 \left[1 + \frac{\varepsilon_2}{\varepsilon_1}\frac{d}{a} - \frac{\varepsilon_2}{\varepsilon_1}\right]}$$

$$\mathbf{E}_2 = -A_2 \mathbf{a}_x = \frac{-\rho_S\left(\frac{d}{a} - 1\right)\mathbf{a}_x}{\varepsilon_1 \left[1 + \frac{\varepsilon_2}{\varepsilon_1}\frac{d}{a} - \frac{\varepsilon_2}{\varepsilon_1}\right]}$$

Poisson equation example in one dimension:

EXAMPLE 6.3 Semiinfinite conducting planes $\phi = 0$ and $\phi = \pi/6$ are separated by an infinitesimal insulating gap as in Figure 6.3. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V, calculate V and $\mathbf{E}$ in the region between the planes.

Solution:

As V depends only on ϕ, Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

Since $\rho = 0$ is excluded due to the insulating gap, we can multiply by ρ^2 to obtain

$$\frac{d^2V}{d\phi^2} = 0$$

which is integrated twice to give

$$V = A\phi + B$$

We apply the boundary conditions to determine constants A and B. When $\phi = 0$, $V = 0$,

$$0 = 0 + B \rightarrow B = 0$$

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Poisson equation example in one dimension:

When $\phi = \phi_o$, $V = V_o$,

$$V_o = A\phi_o \rightarrow A = \frac{V_o}{\phi_o}$$

Hence:

$$V = \frac{V_o}{\phi_o}\phi$$

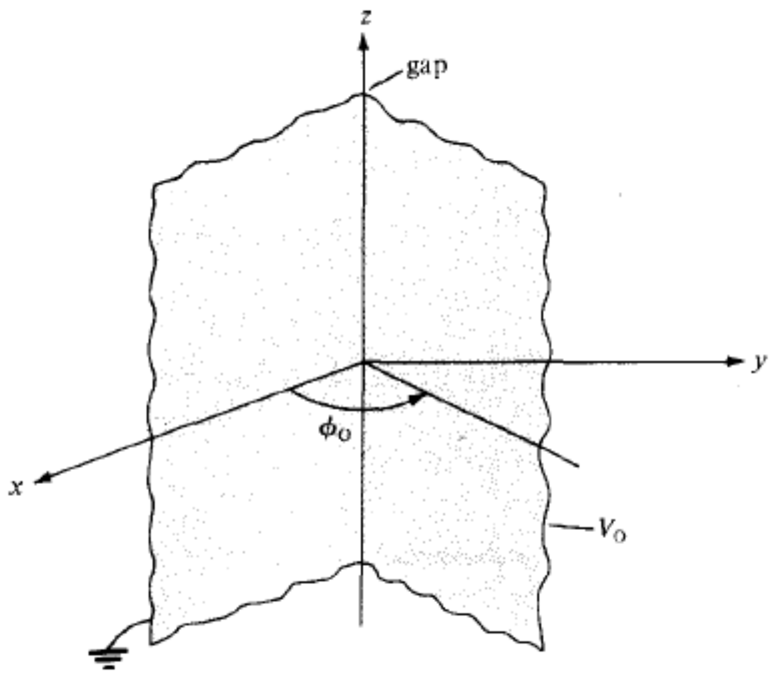
and

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi$$

Substituting $V_o = 100$ and $\phi_o = \pi/6$ gives

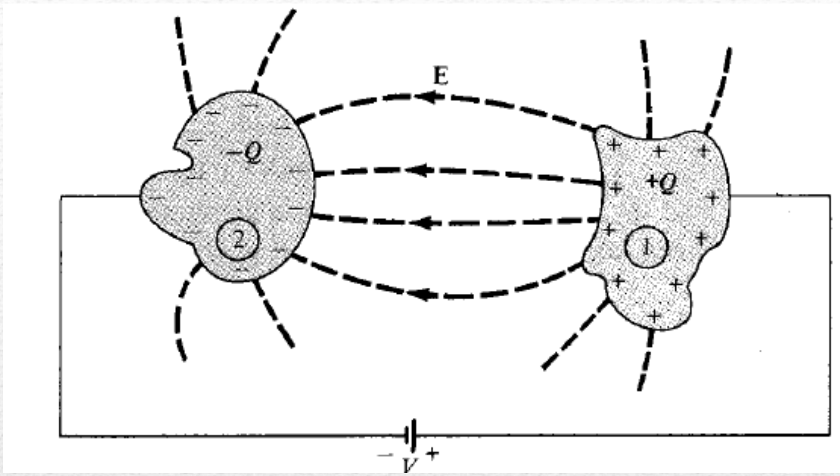
$$V = \frac{600}{\pi}\phi \qquad \text{and} \qquad \mathbf{E} = \frac{600}{\pi\rho}\mathbf{a}_\phi$$

Check: $\nabla^2 V = 0$, $V(\phi = 0) = 0$, $V(\phi = \pi/6) = 100$.



Capacitors and capacitance:

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}.$$



Doubling Q doubles the electric field of the conductor which inturn doubles the potential. The relation between Q and V is linear. The proportionality constant is C (the capacitance)

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}$$

C is measured in **farads** (F)

Capacitors and capacitance:

Two approaches for calculating capacitance:

1. Assuming Q and determining V in terms of Q (involving Gauss's law)
2. Assuming V and determining Q in terms of V (involving solving Laplace's equation)

Approach

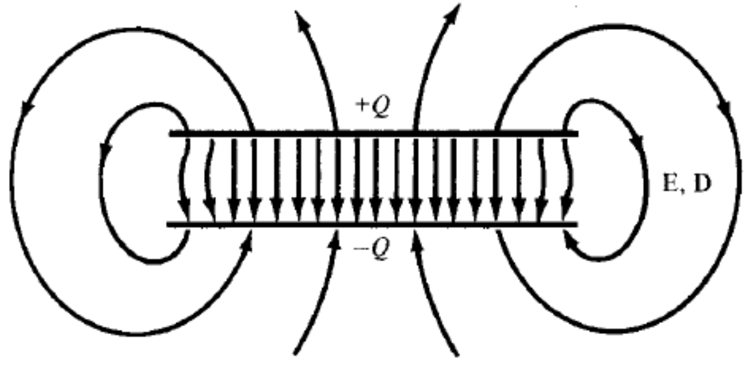
1. Choose a suitable coordinate system.
2. Let the two conducting plates carry charges $+Q$ and $-Q$.
3. Determine $\mathbf{E}$ using Coulomb's or Gauss's law and find V from $V = -\int \mathbf{E} \cdot d\mathbf{l}$. The negative sign may be ignored in this case because we are interested in the absolute value of V.
4. Finally, obtain C from $C = Q/V$.

Capacitors and capacitance:

Capacitance of two parallel plates:

Assume uniformly distributed Q and –Q on the two plates

$$\rho_S = \frac{Q}{S}$$

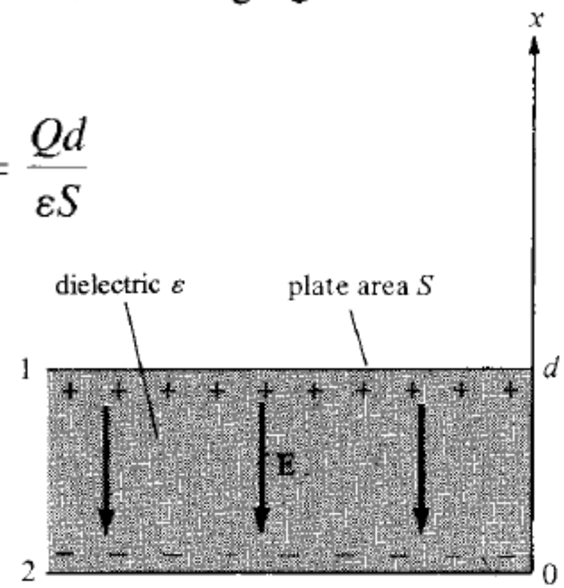


(b)

An ideal parallel-plate capacitor is one in which the plate separation d is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates

$$\mathbf{E} = \frac{\rho_S}{\varepsilon}(-\mathbf{a}_x) = -\frac{Q}{\varepsilon S}\mathbf{a}_x$$

$$V = -\int_2^1 \mathbf{E}\cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\varepsilon S}\mathbf{a}_x\right]\cdot dx\,\mathbf{a}_x = \frac{Qd}{\varepsilon S}$$

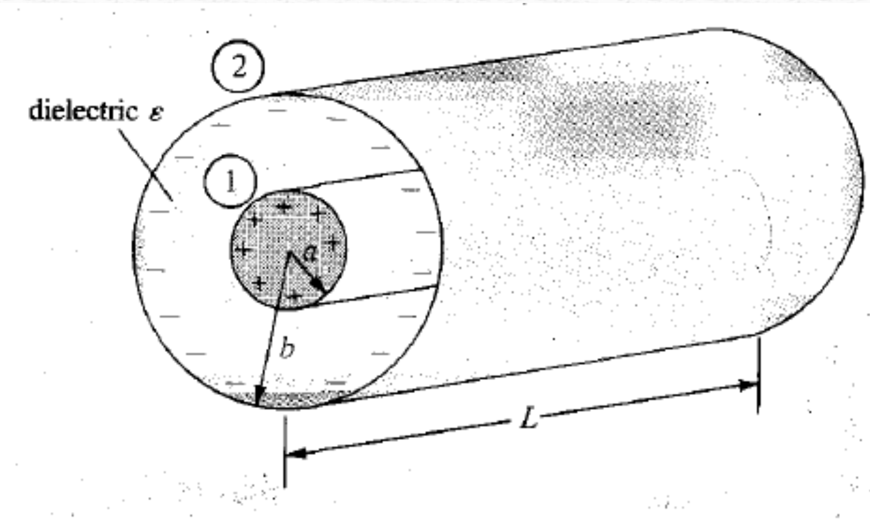


$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$

$$W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}$$

Capacitors and capacitance:

Coaxial capacitance:



$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L$$

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\mathbf{a}_\rho$$

Neglecting flux fringing at the cylinder ends,

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\varepsilon\rho L}\mathbf{a}_\rho\right] \cdot d\rho\, \mathbf{a}_\rho$$

$$= \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}$$

Thus the capacitance of a coaxial cylinder is given by

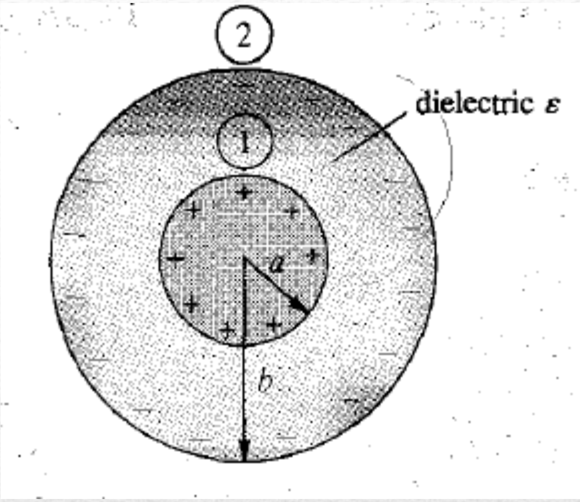
$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

Capacitors and capacitance:

Spherical capacitance:

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r$$



The potential difference between the conductors is

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\right] \cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Thus the capacitance of the spherical capacitor is

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

By letting $b \to \infty$, $C = 4\pi\varepsilon a$, which is the capacitance of a spherical capacitor whose outer plate is infinitely large.

Resistance:

Calculating the resistance of homogenious media:

- **we found that the resistance of a uniform cross section conductor**

$$R = \frac{\ell}{\sigma S} \qquad (\Omega).$$

- **In this the general method for calculating resistance is explained.**

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}}$$

1. Choose a suitable coordinate system.
2. Assume V_o as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine $\mathbf{E}$ from $\mathbf{E} = -\nabla V$ and I from $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$.
4. Finally, obtain R as V_o/I.

In essence, we assume V_o, find I, and determine $R = V_o/I$.

Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3**

A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 6.17. Show that (a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

and (b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma \pi (b^2 - a^2)}$$

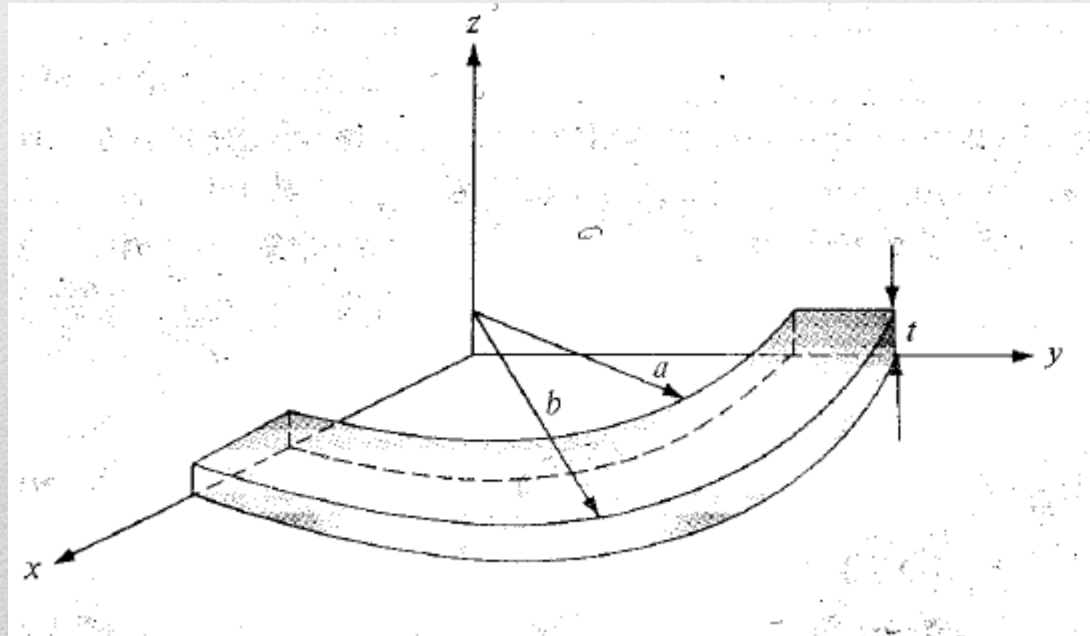
Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3**

(a) Between the vertical curved ends located at $\rho = a$ and $\rho = b$, the bar has a nonuniform cross section

Let a potential difference V_o be maintained between the curved surfaces at $\rho = a$ and $\rho = b$ so that



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Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3**

$V(\rho = a) = 0$ and $V(\rho = b) = V_o$. We solve for V in Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates. Since $V = V(\rho)$,

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

As $\rho = 0$ is excluded, upon multiplying by ρ and integrating once, this becomes

$$\rho\frac{dV}{d\rho} = A$$

or

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating once again yields

$$V = A \ln \rho + B$$

Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3** where A and B are constants of integration to be determined from the boundary conditions.

$$V(\rho = a) = 0 \rightarrow 0 = A \ln a + B \quad \text{or} \quad B = -A \ln a$$

$$V(\rho = b) = V_o \rightarrow V_o = A \ln b + B = A \ln b - A \ln a = A \ln \frac{b}{a} \quad \text{or} \quad A = \frac{V_o}{\ln \frac{b}{a}}$$

Hence,

$$V = A \ln \rho - A \ln a = A \ln \frac{\rho}{a} = \frac{V_o}{\ln \frac{b}{a}} \ln \frac{\rho}{a}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho} \mathbf{a}_\rho = -\frac{A}{\rho} \mathbf{a}_\rho = -\frac{V_o}{\rho \ln \frac{b}{a}} \mathbf{a}_\rho$$

$$\mathbf{J} = \sigma \mathbf{E}, \qquad d\mathbf{S} = -\rho \, d\phi \, dz \, \mathbf{a}_\rho$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/2} \int_{z=0}^{t} \frac{V_o \sigma}{\rho \ln \frac{b}{a}} dz \, \rho \, d\phi = \frac{\pi}{2} \frac{t V_o \sigma}{\ln \frac{b}{a}}$$

$$R = \frac{V_o}{I} = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3**

(b) Let V_o be the potential difference between the two horizontal surfaces so that $V(z = 0) = 0$ and $V(z = t) = V_o$. $V = V(z)$, so Laplace's equation $\nabla^2 V = 0$ becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating twice gives

$$V = Az + B$$

We apply the boundary conditions to determine A and B:

$$V(z = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

$$V(z = t) = V_o \rightarrow V_o = At \qquad \text{or} \qquad A = \frac{V_o}{t}$$

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Resistance:

Calculating the resistance of homogenious media:

- **Ex 6.3**

$$V = \frac{V_o}{t} z$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{V_o}{t}\mathbf{a}_z$$

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma V_o}{t}\mathbf{a}_z, \qquad d\mathbf{S} = -\rho\, d\phi\, d\rho\, \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^{b} \int_{\phi=0}^{\pi/2} \frac{V_o \sigma}{t} \rho\, d\phi\, d\rho$$

$$= \frac{V_o \sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2}\bigg|_a^b = \frac{V_o\, \sigma\, \pi\, (b^2 - a^2)}{4t}$$

$$R' = \frac{V_o}{I} = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Capacitors and capacitance:

Calculating the resistance of a capacitor in homogenious media:

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}$$

$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$

For a parallel-plate capacitor,

$$C = \frac{\varepsilon S}{d}, \qquad R = \frac{d}{\sigma S}$$

For a cylindrical capacitor,

$$C = \frac{2\pi\varepsilon L}{\ln \frac{b}{a}}, \qquad R = \frac{\ln \frac{b}{a}}{2\pi\sigma L}$$

Capacitors and capacitance:

Calculating the resistance of a capacitor in homogenious media:

For a spherical capacitor,

$$C = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}, \qquad R = \frac{\dfrac{1}{a} - \dfrac{1}{b}}{4\pi\sigma}$$

And finally for an isolated spherical conductor,

$$C = 4\pi\varepsilon a, \qquad R = \frac{1}{4\pi\sigma a}$$

Magnetostatic

When a charge is moving beside a current carrying wire, a force is exerted on it. The force is due to a field resulting from the current carrying wire called magnetic field.

- **Charges moving with constant velocity generate steady magnetic field**
- **Charges moving with varying velocity generate dynamic magnetic field(depends on time)**

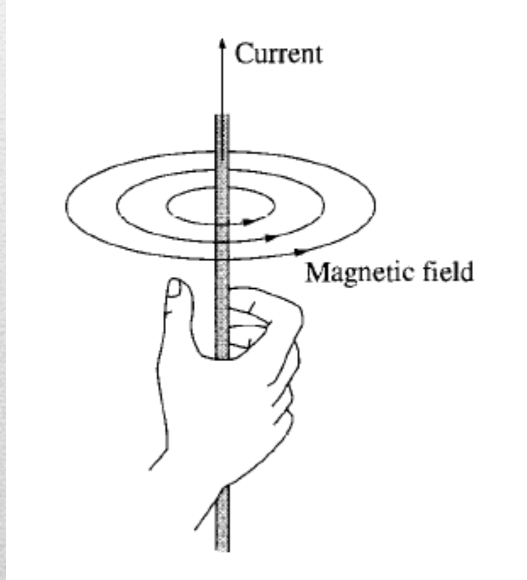
- **In this chapter, we learn how to calculate magnetic field density and intensity using Biot Savart's law, and Ampere's law.**

Magnetostatic

Defining the magnetic field: A force is noticed on a moving charge with velocity u near a current carrying conductor. This force is proportional to q, the velocity component perpendicular to the magnetic field and to the magnetic field(Tesla, weber/m2).

Lorentz force:

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)},$$



the total force in the presence of both electric and magnetic field

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad \text{(N)},$$

Magnetostatic

Sources of magnetostatic fields are steady currents:

To take into account that the magnetic fields depend not only on current but also the length of the wire, current element is defined:

JdV: current element in general
Idl : thin wire J constant with area
KdS : a thin plate, J does not depend on thickness

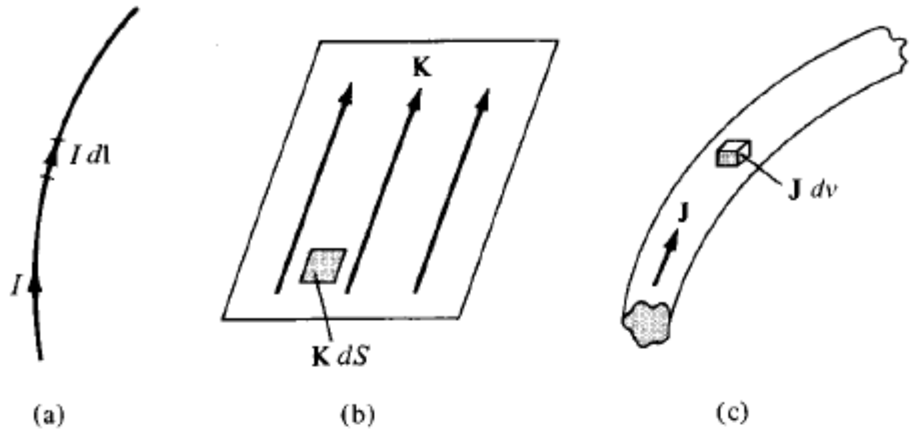


Figure 7.4 Current distributions: (a) line current, (b) surface current, (c) volume current.

Biot-Savart's Law

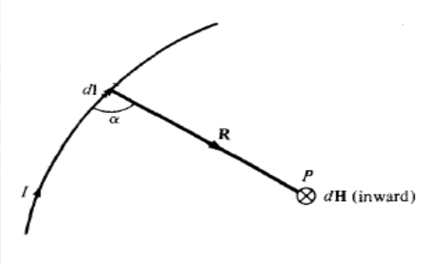
Biot–Savart's law states that the magnetic field intensity dH produced at a point P, as shown in Figure 7.1, by the differential current element $I\,dl$ is proportional to the product $I\,dl$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

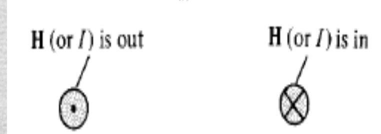
That is,

$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$

or

$$dH = \frac{kI\,dl \sin\alpha}{R^2}$$





where k is the constant of proportionality. In SI units, $k = 1/4\pi$,

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

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Magnetostatic

The magnetic field intensity H: as we did in the electrostatic, we defined a material independent vector called D, the electric field density.We define H the magnetic field intensity, a quantity which does not depend on medium.

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}.$$

Biot savart law in terms of H:

$$\mathbf{H} = \int_L \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(line current)}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\,dS \times \mathbf{a}_R}{4\pi R^2} \quad \text{(surface current)}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\,dv \times \mathbf{a}_R}{4\pi R^2} \quad \text{(volume current)}$$

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Magnetostatic

Biot Savart's law example:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

But $d\mathbf{l} = dz\,\mathbf{a}_z$ and $\mathbf{R} = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, so

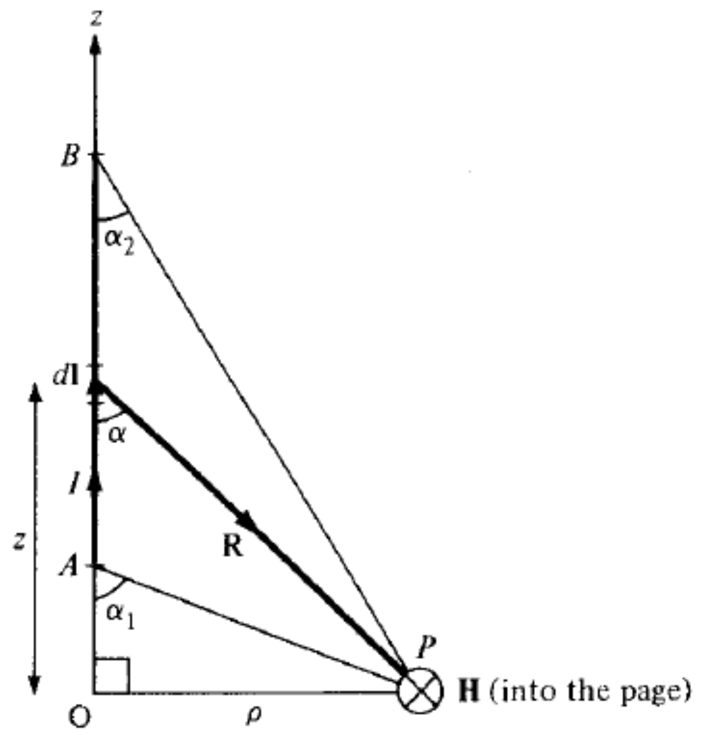
$$d\mathbf{l} \times \mathbf{R} = \rho\,dz\,\mathbf{a}_\phi$$

$$\mathbf{H} = \int \frac{I\rho\,dz}{4\pi[\rho^2 + z^2]^{3/2}}\,\mathbf{a}_\phi$$

Letting $z = \rho\cot\alpha$, $dz = -\rho\,\mathrm{cosec}^2\,\alpha\,d\alpha$

$$\mathbf{H} = -\frac{1}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\,\mathrm{cosec}^2\,\alpha\,d\alpha}{\rho^3\,\mathrm{cosec}^3\,\alpha}\,\mathbf{a}_\phi$$

$$= -\frac{I}{4\pi\rho}\,\mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha\,d\alpha$$



$$\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi$$

when the conductor is *infinite* in length. $\alpha_1 = 180°$, $\alpha_2 = 0°$

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi$$

$$\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho$$

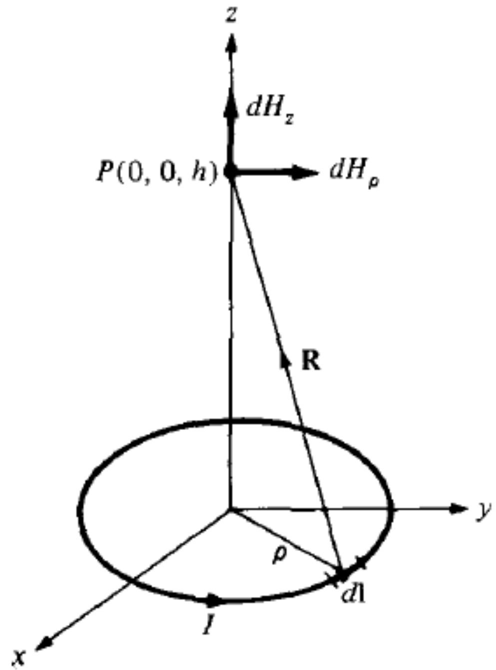
Magnetostatic

Biot Savart's law example 2:

A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\, d\mathbf{l}$ is given by Biot–Savart's law:



$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho \mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

Magnetostatic

Biot Savart's law example 2:

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$). Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z\, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\, d\phi\, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\ A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\, \mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

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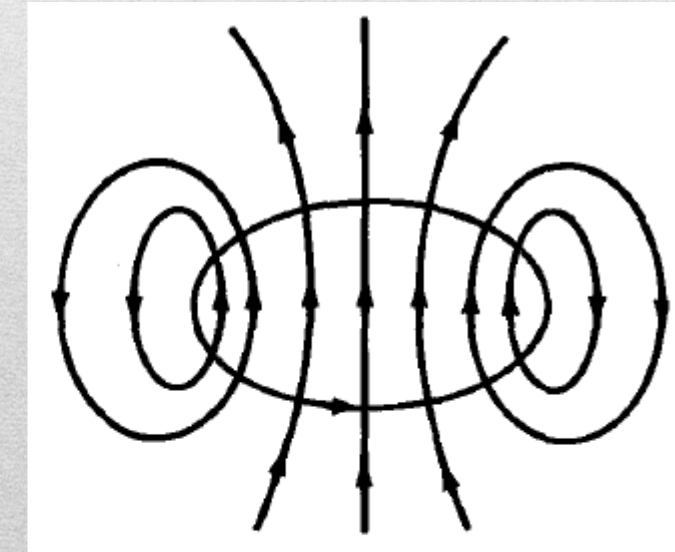
Magnetostatic

Biot Savart's law example 2:

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$



Ampere's Law

Ampere's circuit law states that the line integral of the tangential component of **H** around a *closed* path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of **H** equals I_{enc}; that is,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$$

By applying Stoke's theorem to the left-hand side, we obtain

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

But

$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Comparing the surface integrals

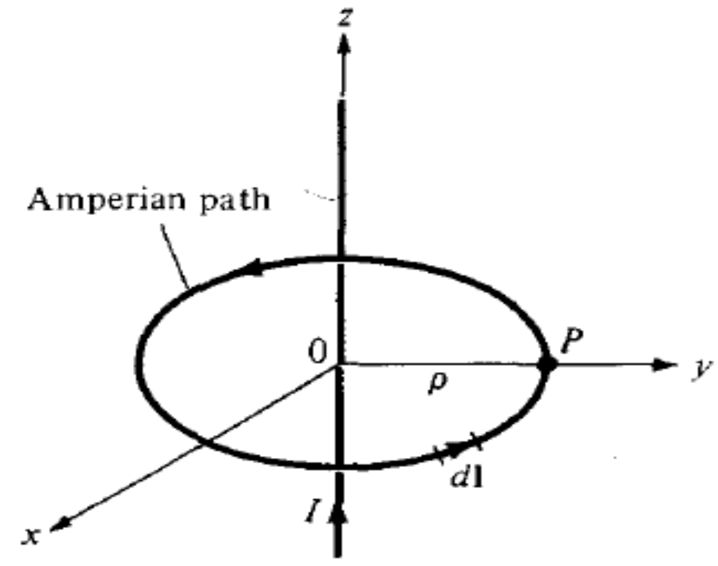
$$\nabla \times \mathbf{H} = \mathbf{J}$$

Applications of Ampere's LAw

A. Infinite Line Current

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$

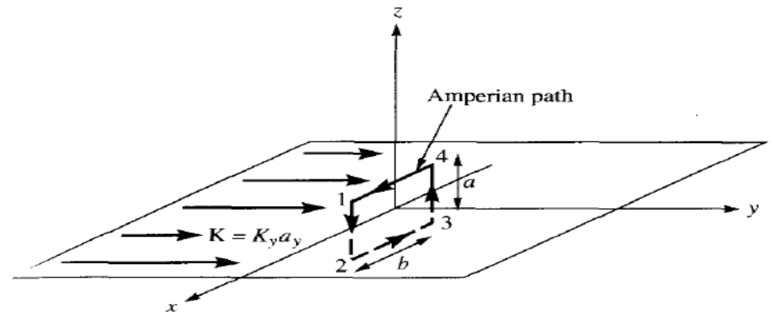
$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$



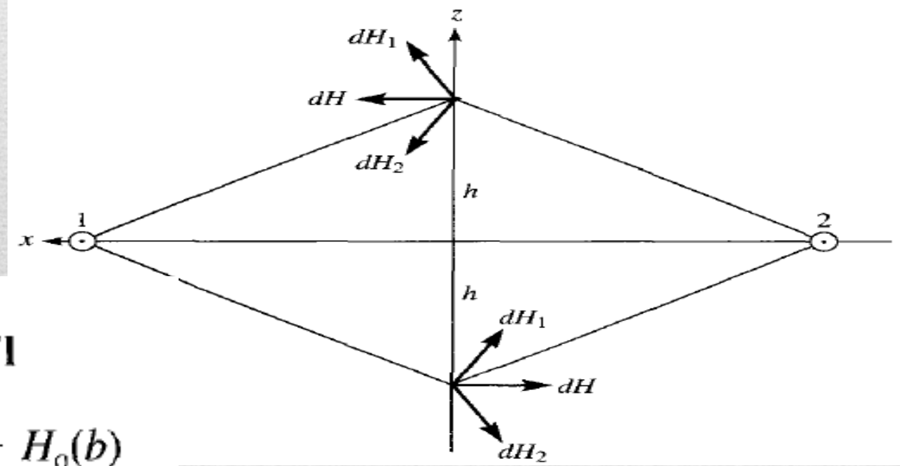
Applications of Ampere's LAw

B. Infinite Sheet of Current

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b$$



$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases}$$



$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l}$$
$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b)$$
$$= 2H_o b$$

Applications of Ampere's LAw

$$H_o = \frac{1}{2} K_y.$$

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

In general, for an infinite sheet of current density **K** A/m,

$$\boxed{\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n}$$

where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

Since the current is uniformly distributed over the cross section,

$$\mathbf{J} = \frac{I}{\pi a^2}\,\mathbf{a}_z, \qquad d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{S} = \frac{I}{\pi a^2} \iint \rho\, d\phi\, d\rho = \frac{I}{\pi a^2}\,\pi\rho^2 = \frac{I\rho^2}{a^2}$$

Hence eq. (7.24) becomes

$$H_\phi \int dl = H_\phi\, 2\pi\rho = \frac{I\rho^2}{a^2}$$

or

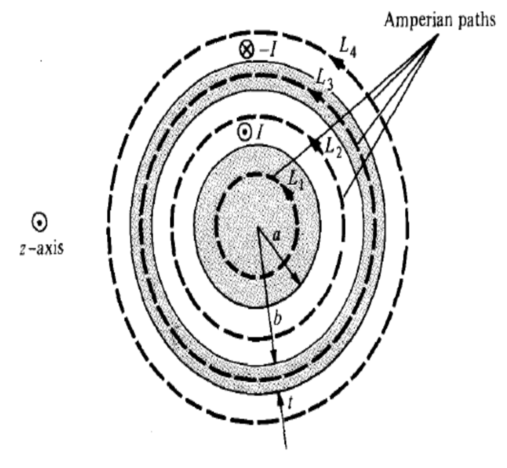
$$H_\phi = \frac{I\rho}{2\pi a^2}$$

Applications of Ampere's LAw

C. Infinitely Long Coaxial Transmission Line

perian path for each of the four possible regions: $0 \le \rho \le a$, $a \le \rho \le b$, $b \le \rho \le b + t$, and $\rho \ge b + t$.

For region $0 \le \rho \le a$, we apply Ampere's law to path L_1, giving



For region $0 \le \rho \le a$, we apply Ampere's law to path L_1, giving

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$$

Since the current is uniformly distributed over the cross section,

$$\mathbf{J} = \frac{I}{\pi a^2}\mathbf{a}_z, \qquad d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{S} = \frac{I}{\pi a^2}\iint \rho \, d\phi \, d\rho = \frac{I}{\pi a^2}\pi\rho^2 = \frac{I\rho^2}{a^2}$$

Applications of Ampere's LAw

$$H_\phi \int dl = H_\phi 2\pi\rho = \frac{I\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2}$$

For region $a \leq \rho \leq b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{enc} = I$$

$$H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

For region $a \leq \rho \leq b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{enc} = I$$

$$H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

For region $b \leq \rho \leq b + t$, we use path L_3, getting

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_\phi \cdot 2\pi\phi = I_{enc}$$

where

$$I_{enc} = I + \int \mathbf{J} \cdot d\mathbf{S}$$

and $\mathbf{J}$ in this case is the current density (current per unit area) of the outer conductor and is along $-\mathbf{a}_z$, that is,

$$\mathbf{J} = -\frac{I}{\pi[(b + t)^2 - b^2]} \mathbf{a}_z$$

Thus

$$I_{enc} = I - \frac{I}{\pi[(b + t)^2 - t^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Substituting this in eq. (7.27a), we have

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

For region $\rho \geq b + t$, we use path L_4, getting

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Applications of Ampere's LAw

$$\oint_{L_4} \mathbf{H} \cdot d\mathbf{l} = I - I = 0$$

or

$$H_\phi = 0$$

Putting eqs. (7.25) to (7.28) together gives

$$\mathbf{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi, & 0 \leq \rho \leq a \\ \dfrac{I}{2\pi\rho}\,\mathbf{a}_\phi, & a \leq \rho \leq b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\mathbf{a}_\phi, & b \leq \rho \leq b + t \\ 0, & \rho \geq b + t \end{cases}$$

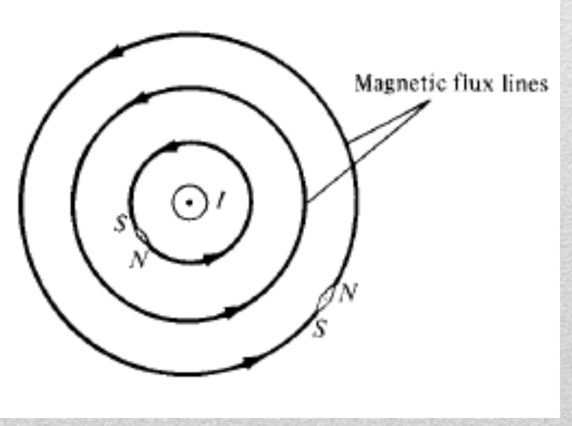
MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux density B

$$\mathbf{B} = \mu_o \mathbf{H}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys/meter (H/m) and has the value of

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$



MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

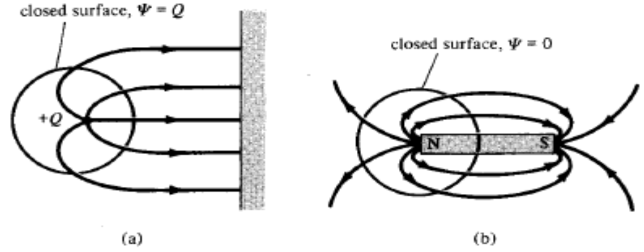


Figure 7.17 Flux leaving a closed surface due to: **(a)** isolated electric charge $\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$, **(b)** magnetic charge, $\Psi = \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$.

where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers/square meter (Wb/m^2) or teslas.

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed; that is, $\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$. Thus it is possible to have an isolated electric charge as shown in Figure 7.17(a), which also reveals that electric flux lines are not necessarily closed. Unlike electric flux lines, magnetic flux lines always close upon themselves as in Figure 7.17(b). This is due to the fact that *it is not possible to have isolated magnetic poles (or magnetic charges)*. For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles as illustrated in Figure 7.18. We find it impossible to separate the north pole from the south pole.

MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

An **isolated magnetic** charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is,

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

This equation is referred to as the *law of conservation of magnetic flux* or *Gauss's law for magnetostatic fields* just as $\oint \mathbf{D} \cdot d\mathbf{S} = Q$ is Gauss's law for electrostatic fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq. (7.33), we obtain

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \, dv = 0$$

or

$$\nabla \cdot \mathbf{B} = 0$$

MAXWELL'S EQUATIONS FOR STATIC EM FIELDS

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be (a) due to a moving charged particle in a **B** field, (b) on a current element in an external **B** field, or (c) between two current elements.

A. Force on a Charged Particle

According to our discussion in Chapter 4, the electric force $\mathbf{F}_e$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity **E** as

$$\mathbf{F}_e = Q\mathbf{E}$$

This shows that if Q is positive, $\mathbf{F}_e$ and **E** have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \tag{8.2}$$

This clearly shows that $\mathbf{F}_m$ is perpendicular to both **u** and **B**.

FORCES DUE TO MAGNETIC FIELDS

TABLE 8.1 Force on a Charged Particle

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

B. Force on a Current Element

To determine the force on a current element $I\,d\mathbf{l}$ of a current-carrying conductor due to the magnetic field $\mathbf{B}$, We have to follow :

FORCES DUE TO MAGNETIC FIELDS

$$\mathbf{J} = \rho_v \mathbf{u} \tag{8.5}$$

From eq. (7.5), we recall the relationship between current elements:

$$I\, d\mathbf{l} = \mathbf{K}\, dS = \mathbf{J}\, dv \tag{8.6}$$

Combining eqs. (8.5) and (8.6) yields

$$I\, d\mathbf{l} = \rho_v \mathbf{u}\, dv = dQ\, \mathbf{u}$$

Alternatively, $I\, d\mathbf{l} = \dfrac{dQ}{dt}\, d\mathbf{l} = dQ\, \dfrac{d\mathbf{l}}{dt} = dQ\, \mathbf{u}$

Hence,

$$\boxed{I\, d\mathbf{l} = dQ\, \mathbf{u}} \tag{8.7}$$

This shows that an elemental charge dQ moving with velocity $\mathbf{u}$ (thereby producing convection current element $dQ\, \mathbf{u}$) is equivalent to a conduction current element $I\, d\mathbf{l}$. Thus the force on a current element $I\, d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is found from eq. (8.2) by merely replacing $Q\mathbf{u}$ by $I\, d\mathbf{l}$; that is,

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B} \tag{8.8}$$

If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\boxed{\mathbf{F} = \oint_L I\, d\mathbf{l} \times \mathbf{B}} \tag{8.9}$$

or a volume current element $\mathbf{J}\, dv$, we simply make use of eq. (8.6) so that eq. (8.8) becomes

$$d\mathbf{F} = \mathbf{K}\, dS \times \mathbf{B} \quad \text{or} \quad d\mathbf{F} = \mathbf{J}\, dv \times \mathbf{B} \tag{8.8a}$$

while eq. (8.9) becomes

$$\mathbf{F} = \int_S \mathbf{K}\, dS \times \mathbf{B} \quad \text{or} \quad \mathbf{F} = \int_v \mathbf{J}\, dv \times \mathbf{B} \tag{8.9a}$$

From eq. (8.8)

The magnetic field B is defined as the force per unit current element.

Magnetostatics

Magnetic force: Lorentz force equation(total force) on moving charge:

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Applications:

ammeters, voltmeters, galvanometers, cyclotrons, motors, and magnetohydrodynamic generators.

Magnetic force does no work because it is normal to velocity. It can only change the direction of velocity, not its magnitude (kinetic energy)

Magnetic force on a current element:

$$\mathbf{J} = \rho_v \mathbf{u}$$

$$I\,d\mathbf{l} = \mathbf{K}\,dS = \mathbf{J}\,dv$$

$$I\,d\mathbf{l} = \rho_v \mathbf{u}\,dv = dQ\,\mathbf{u}$$

$$I\,d\mathbf{l} = dQ\,\mathbf{u}$$

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}$$

$$\mathbf{F} = \oint_L I\,d\mathbf{l} \times \mathbf{B}$$

Magnetostatics

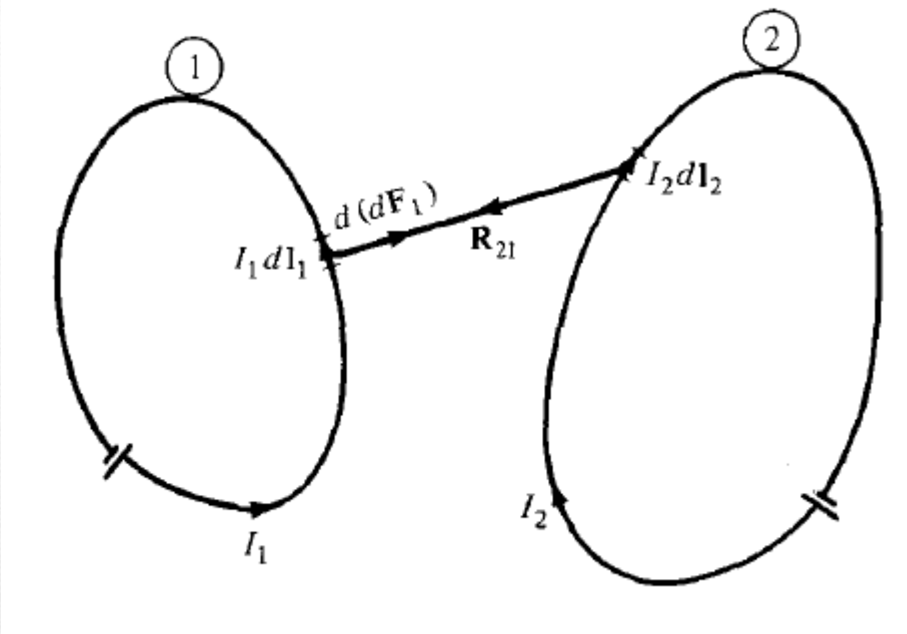
Magnetic force between two current elements:

$$d(d\mathbf{F}_1) = I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2$$

$$d\mathbf{B}_2 = \frac{\mu_o I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2}$$

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2}$$

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2}$$



Example

EXAMPLE 8.4 A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure 8.4(a). Show that the force experienced by the loop is given by

$$\mathbf{F} = -\frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right]\mathbf{a}_\rho \text{ N}$$

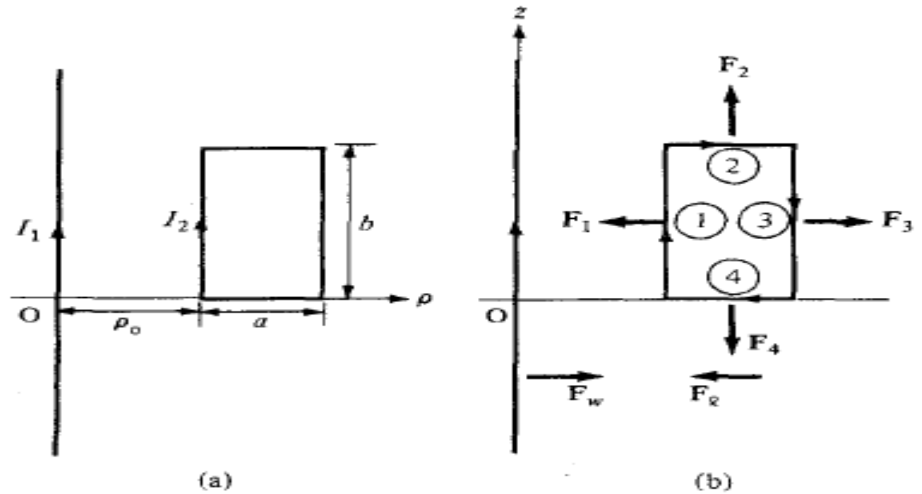


Figure 8.4 For Example 8.4: **(a)** rectangular loop inside the field produced by an infinitely long wire, **(b)** forces acting on the loop and wire.

Example

Solution:

Let the force on the loop be

$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1$$

where $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ are, respectively, the forces exerted on sides of the loop labeled 1, 2, 3, and 4 in Figure 8.4(b). Due to the infinitely long wire

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi$$

Hence,

$$\mathbf{F}_1 = I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=0}^{b} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi$$

$$= -\frac{\mu_o I_1 I_2 b}{2\pi\rho_o}\,\mathbf{a}_\rho \qquad \text{(attractive)}$$

$\mathbf{F}_1$ is attractive because it is directed toward the long wire; that is, $\mathbf{F}_1$ is along $-\mathbf{a}_\rho$ due to the fact that loop side 1 and the long wire carry currents along the same direction. Similarly,

$$\mathbf{F}_3 = I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=b}^{0} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi(\rho_o + a)}\,\mathbf{a}_\phi$$

$$= \frac{\mu_o I_1 I_2 b}{2\pi(\rho_o + a)}\,\mathbf{a}_\rho \qquad \text{(repulsive)}$$

$$\mathbf{F}_2 = I_2 \int_{\rho=\rho_o}^{\rho_o+a} d\rho\,\mathbf{a}_\rho \times \frac{\mu_o I_1\,\mathbf{a}_\phi}{2\pi\rho}$$

$$= \frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\,\mathbf{a}_z \qquad \text{(parallel)}$$

Example

$$\mathbf{F}_4 = I_2 \int_{\rho=\rho_o+a}^{\rho_o} d\rho\, \mathbf{a}_\rho \times \frac{\mu_o I_1\, \mathbf{a}_\phi}{2\pi\rho}$$

$$= -\frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\, \mathbf{a}_z \qquad \text{(parallel)}$$

The total force $\mathbf{F}_\ell$ on the loop is the sum of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$; that is,

$$\mathbf{F}_\ell = \frac{\mu_o I_1 I_2 b}{2\pi} \left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a} \right] (-\mathbf{a}_\rho)$$

which is an attractive force trying to draw the loop toward the wire. The force $\mathbf{F}_w$ on the wire, by Newton's third law, is $-\mathbf{F}_\ell$; see Figure 8.4(b).

Magnetostatics

Magnetic Torque in uniform field:

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}$$

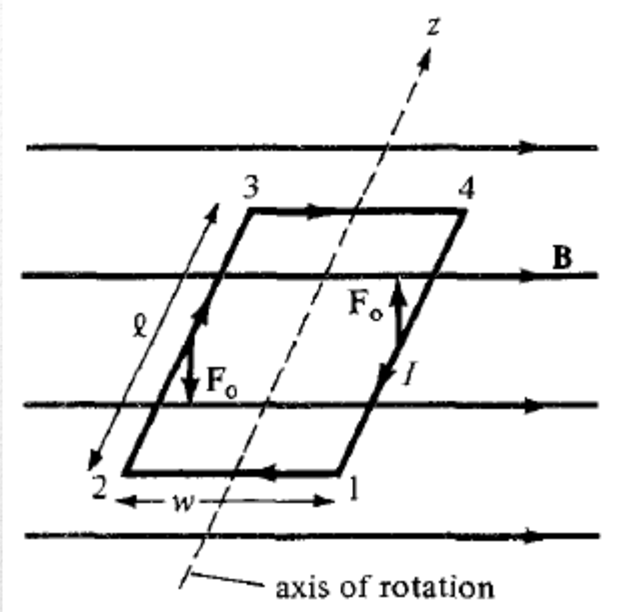
$$|\mathbf{T}| = |\mathbf{F}_o|\, w \sin\alpha$$

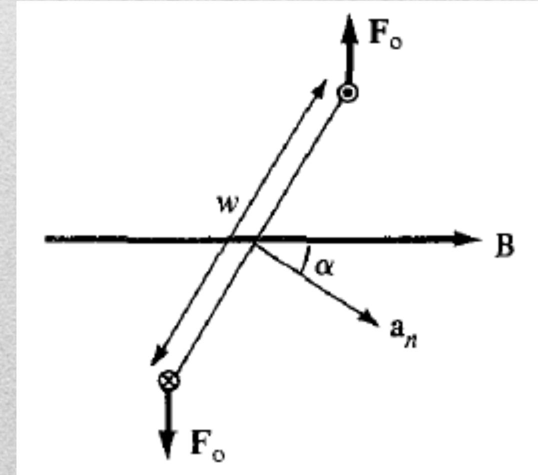
$$T = BI\ell w \sin\alpha$$

$$T = BIS \sin\alpha$$

$$\mathbf{m} = IS\mathbf{a}_n$$

$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$





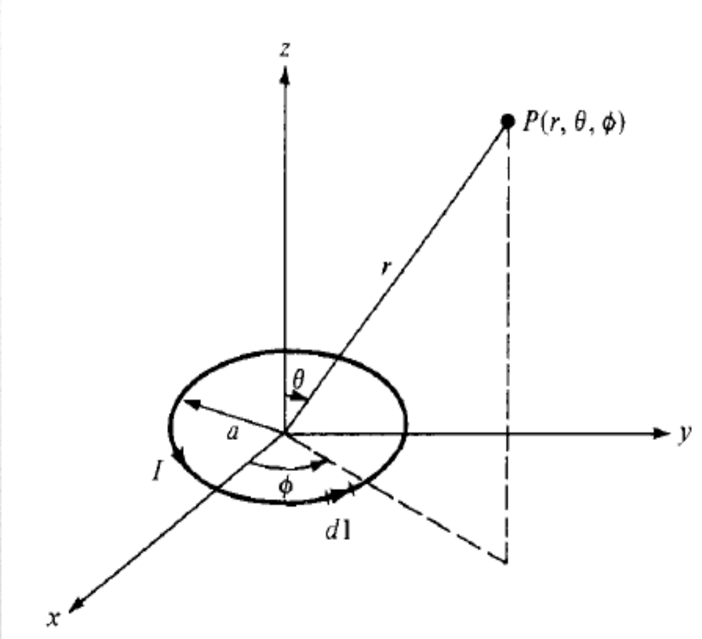
A Magnetic Dipole

Magnetic dipole: it is a small loop of current

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}}{r}$$

$$\mathbf{A} = \frac{\mu_0 I \pi a^2 \sin\theta \, \mathbf{a}_\phi}{4\pi r^2}$$

$$\mathbf{A} = \frac{\mu_0 \, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2}$$



where $\mathbf{m} = I\pi a^2 \mathbf{a}_z$, the magnetic moment of the loop, and $\mathbf{a}_z \times \mathbf{a}_r = \sin\theta \, \mathbf{a}_\phi$. We determine the magnetic flux density $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$ as

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta)$$

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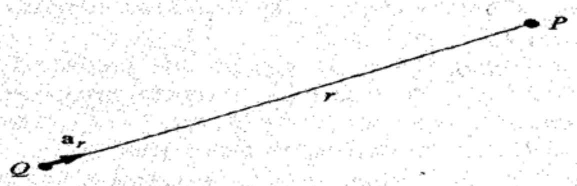
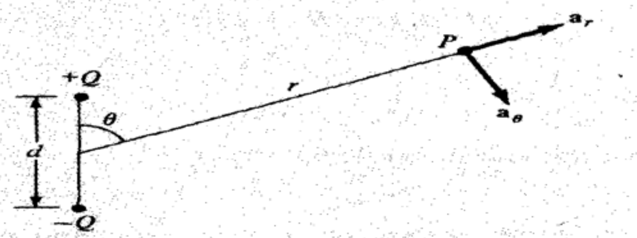
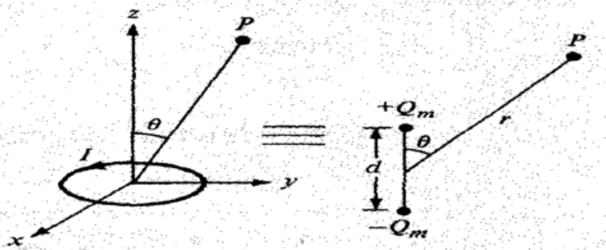
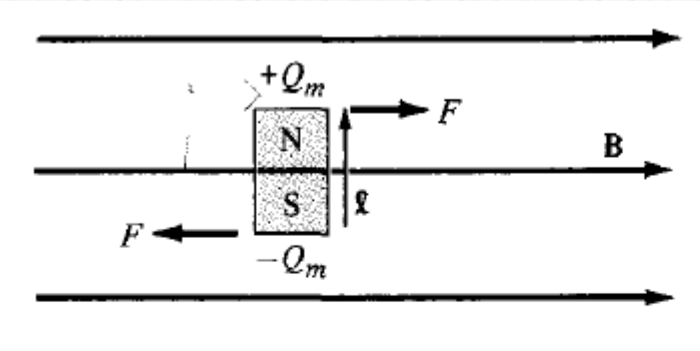
A Magnetic Dipole

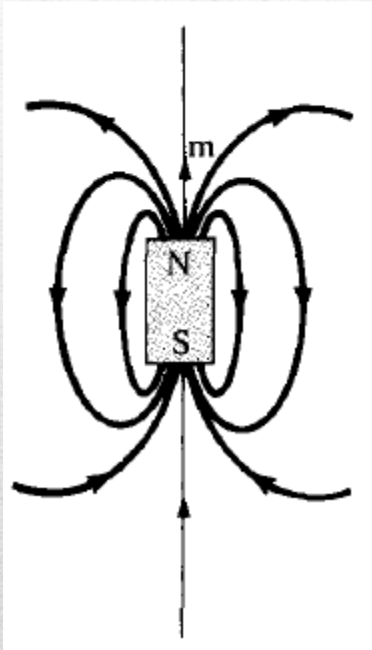
TABLE 8.2 Comparison between Electric and Magnetic Monopoles and Dipoles

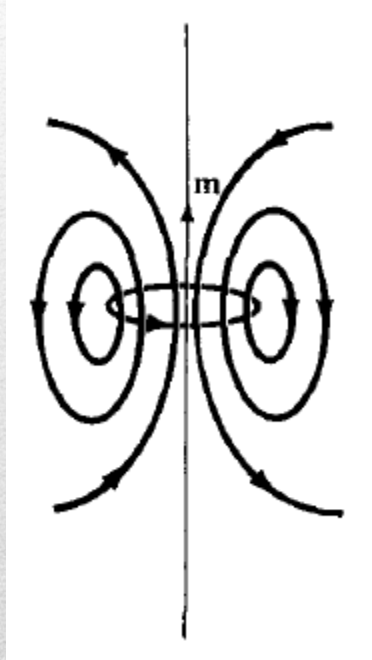
Electric	Magnetic
$V = \dfrac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \dfrac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$  Monopole (point charge)	Does not exist  Monopole (point charge)
$V = \dfrac{Q\cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \dfrac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$  Dipole (two point charge)	$\mathbf{A} = \dfrac{\mu_o m\sin\theta\,\mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \dfrac{\mu_o m}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$  Dipole (small current loop or bar magnet)

Magnetostatics

Magnetic dipole moment: The ability to rotate a current loop







$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m \boldsymbol{\ell} \times \mathbf{B}$$

$$\mathbf{F} = Q_m \mathbf{B}$$

$$T = Q_m \ell B = ISB$$

$$Q_m \ell = IS$$

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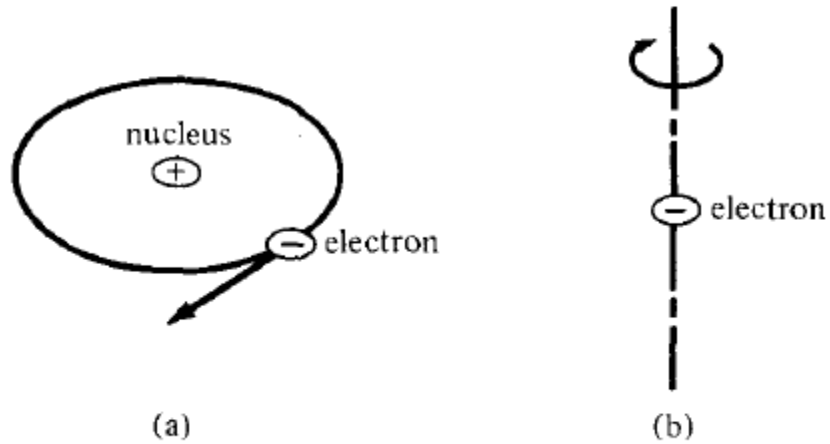
MAGNETIZATION IN MATERIALS

Magnetitization and magnetic field in materials

Orbiting electrons either around nucleus or around them selves produce internal magnetic dipoles which in turn generate magnetic field. On macroscopic level and without and external magnetic field applied to the material, this field average is zero.



The **magnetization M** (in amperes/meter) is the magnetic dipole moment per unit volume.

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v}$$

A medium for which **M** is not zero everywhere is said to be magnetized

Magnetostatics

Magnetitization and magnetic field in materials

$$dA = \frac{\mu_o \mathbf{M} \times \mathbf{a}_R}{4\pi R^2} dv' = \frac{\mu_o \mathbf{M} \times \mathbf{R}}{4\pi R^3} dv'$$

$$\frac{\mathbf{R}}{R^3} = \nabla' \frac{1}{R}$$

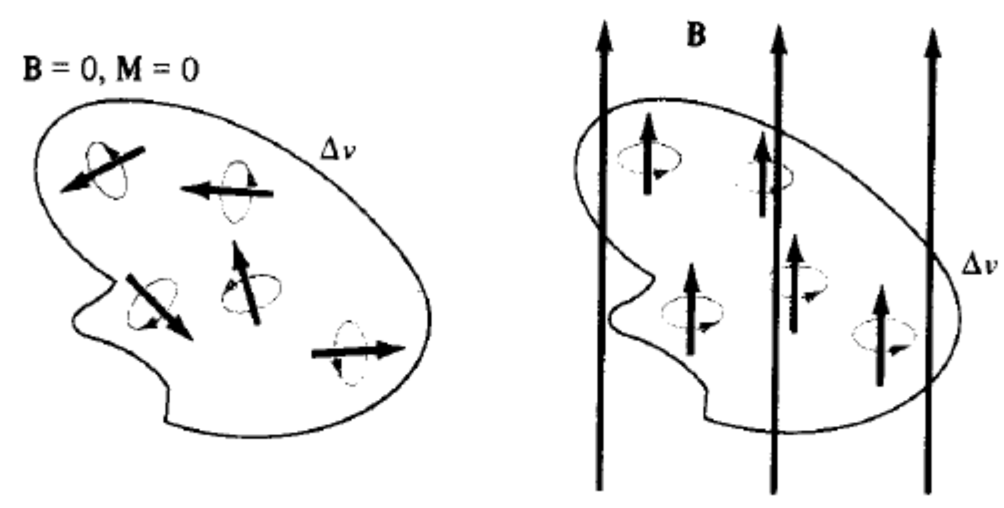
$$\mathbf{A} = \frac{\mu_o}{4\pi} \int \mathbf{M} \times \nabla' \frac{1}{R} dv'$$

$$\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R}$$

$$\int_{v'} \nabla' \times \mathbf{F} \, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{S}$$

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}_n}{R} dS'$$

$$= \frac{\mu_o}{4\pi} \int_{v'} \frac{\mathbf{J}_b \, dv'}{R} + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{K}_b \, dS'}{R}$$



$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n$$

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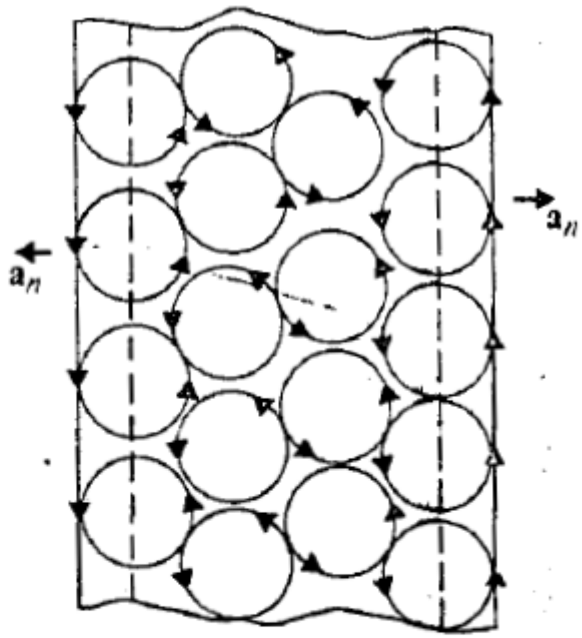
Magnetostatics

Magnetitization and magnetic field in materials

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J}_m = \mathbf{J} + \nabla \times \mathbf{M}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}.$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad (\text{A/m}).$$



For linear materials:

$$\mathbf{M} = \chi_m \mathbf{H}$$

where χ_m is a dimensionless quantity (ratio of M to H) called *magnetic susceptibility*

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

relative permeability of the medium

The quantity $\mu = \mu_0\mu_r$ is called the *permeability* of the material

CLASSIFICATION OF MAGNETIC MATERIALS

In general, we may use the magnetic susceptibility χ_m or the relative permeability μ_r to classify materials in terms of their magnetic property or behavior. A material is said to be *nonmagnetic* if $\chi_m = 0$ (or $\mu_r = 1$); it is magnetic otherwise. Free space, air, and materials with $\chi_m = 0$ (or $\mu_r \approx 1$) are regarded as nonmagnetic.

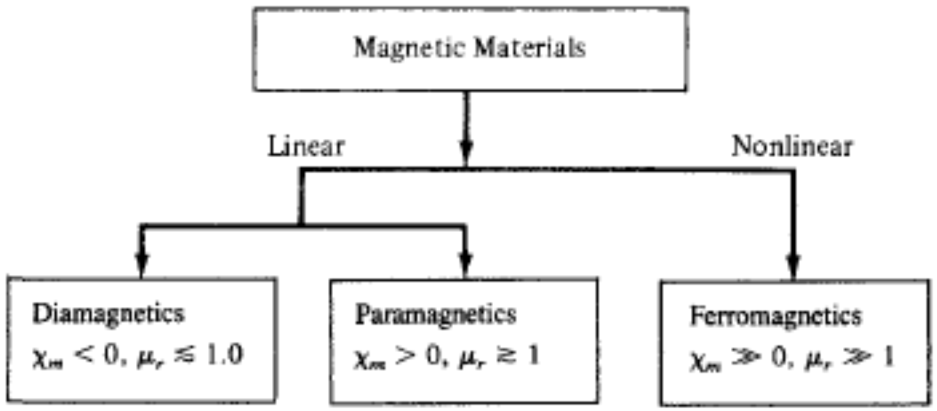


Figure 8.13 Classification of magnetic materials.

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CLASSIFICATION OF MAGNETIC MATERIALS

EXAMPLE 8.7 Region $0 \le z \le 2$ m is occupied by an infinite slab of permeable material ($\mu_r = 2.5$). If $\mathbf{B} = 10y\mathbf{a}_x - 5x\mathbf{a}_y$ mWb/m^2 within the slab, determine: (a) $\mathbf{J}$, (b) $\mathbf{J}_b$, (c) $\mathbf{M}$, (d) $\mathbf{K}_b$ on $z = 0$.

Solution:

(a) By definition,

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times \frac{\mathbf{B}}{\mu_o \mu_r} = \frac{1}{4\pi \times 10^{-7}(2.5)}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{a}_z$$

$$= \frac{10^6}{\pi}(-5 - 10)10^{-3}\mathbf{a}_z = -4.775\mathbf{a}_z \text{ kA/m}^2$$

(b) $\mathbf{J}_b = \chi_m \mathbf{J} = (\mu_r - 1)\mathbf{J} = 1.5(-4.775\mathbf{a}_z) \cdot 10^3$
$= -7.163\mathbf{a}_z$ kA/m^2

(c) $$\mathbf{M} = \chi_m \mathbf{H} = \chi_m \frac{\mathbf{B}}{\mu_o \mu_r} = \frac{1.5(10y\mathbf{a}_x - 5x\mathbf{a}_y) \cdot 10^{-3}}{4\pi \times 10^{-7}(2.5)}$$
$$= 4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y \text{ kA/m}$$

(d) $\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n$. Since $z = 0$ is the lower side of the slab occupying $0 \le z \le 2$, $\mathbf{a}_n = -\mathbf{a}_z$. Hence,

$$\mathbf{K}_b = (4.775y\mathbf{a}_x - 2.387x\mathbf{a}_y) \times (-\mathbf{a}_z)$$
$$= 2.387x\mathbf{a}_x + 4.775y\mathbf{a}_y \text{ kA/m}$$

MAGNETIC BOUNDARY CONDITIONS

We make use of Gauss's law for magnetic fields

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

and Ampere's circuit law

$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$

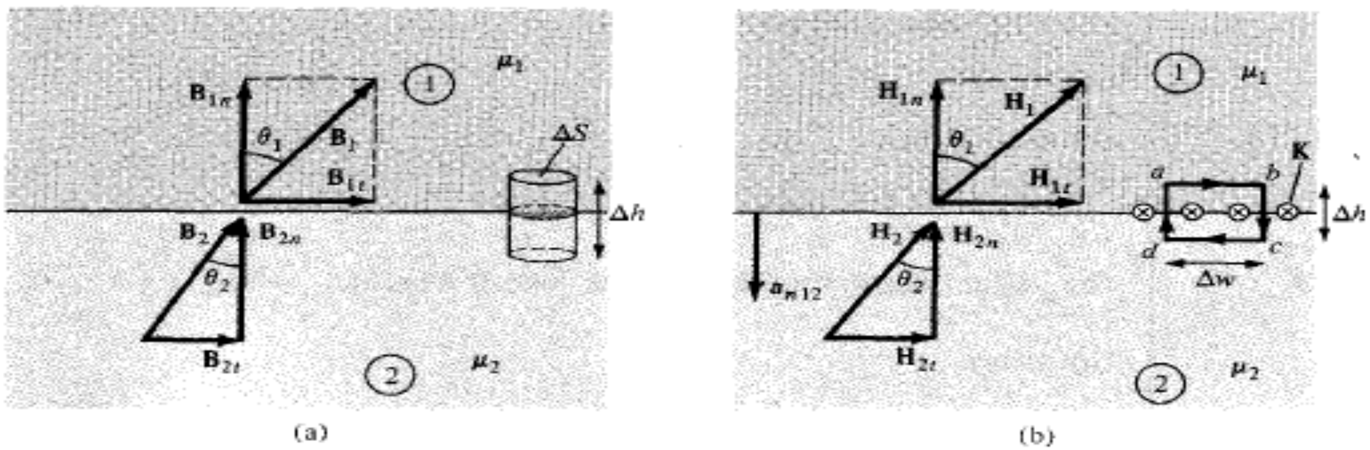

Figure 8.16 Boundary conditions between two magnetic media: **(a)** for **B**, **(b)** for **H**.

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MAGNETIC BOUNDARY CONDITIONS

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0$$

Thus

$$\boxed{\mathbf{B}_{1n} = \mathbf{B}_{2n}} \qquad \text{or} \qquad \mu_1 \mathbf{H}_{1n} = \mu_2 \mathbf{H}_{2n}$$

$$K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} - H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, eq. (8.42) leads to

$$H_{1t} - H_{2t} = K$$

where surface current K on the boundary is assumed normal to the path.

MAGNETIC BOUNDARY CONDITIONS

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K$$

In the general case,

$$\boxed{(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}}$$

If the boundary is free of current or the media are not conductors (for K is free current density), $K = 0$

$$\boxed{\mathbf{H}_{1t} = \mathbf{H}_{2t}} \quad \text{or} \quad \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2}$$

If the fields make an angle θ with the normal to the interface,

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2$$

And

$$\frac{B_1}{\mu_1} \sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin\theta_2$$

Then

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

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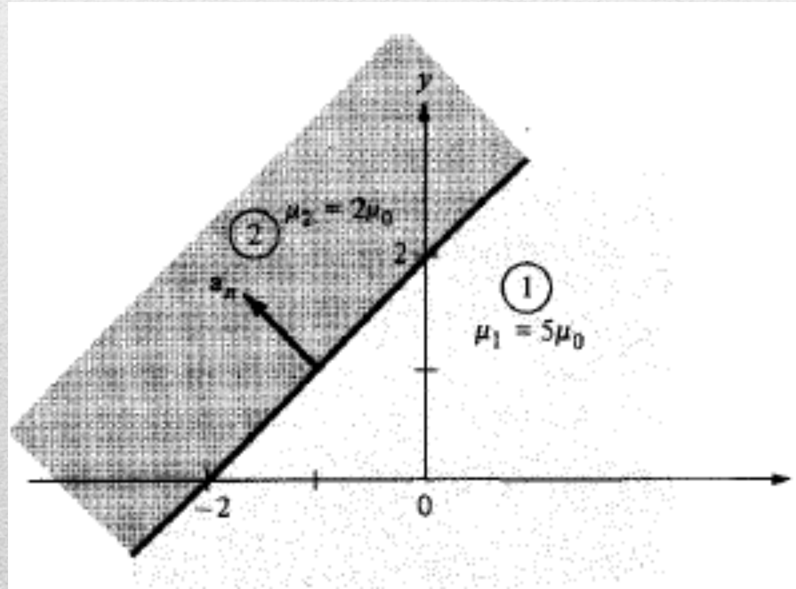
MAGNETIC BOUNDARY CONDITIONS

EXAMPLE 8.8

Given that $\mathbf{H}_1 = -2\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$ A/m in region $y - x - 2 \leq 0$ where $\mu_1 = 5\mu_o$, calculate

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$

(b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in region $y - x - 2 \geq 0$ where $\mu_2 = 2\mu_o$



MAGNETIC BOUNDARY CONDITIONS

Solution:

Since $y - x - 2 = 0$ is a plane, $y - x \leq 2$ or $y \leq x + 2$ is region 1 in Figure 8.17. A point in this region may be used to confirm this. For example, the origin (0, 0) is in this region since $0 - 0 - 2 < 0$. If we let the surface of the plane be described by $f(x, y) = y - x - 2$, a unit vector normal to the plane is given by

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_y - \mathbf{a}_x}{\sqrt{2}}$$

(a)
$$\mathbf{M}_1 = \chi_{m1}\mathbf{H}_1 = (\mu_{r1} - 1)\,\mathbf{H}_1 = (5 - 1)(-2, 6, 4)$$
$$= -8\mathbf{a}_x + 24\mathbf{a}_y + 16\mathbf{a}_z \text{ A/m}$$

$$\mathbf{B}_1 = \mu_1\mathbf{H}_1 = \mu_o\mu_{r1}\mathbf{H}_1 = 4\pi \times 10^{-7}(5)(-2, 6, 4)$$
$$= -12.57\mathbf{a}_x + 37.7\mathbf{a}_y + 25.13\mathbf{a}_z\ \mu\text{Wb/m}^2$$

(b)
$$\mathbf{H}_{1n} = (\mathbf{H}_1 \cdot \mathbf{a}_n)\mathbf{a}_n = \left[(-2, 6, 4) \cdot \frac{(-1, 1, 0)}{\sqrt{2}}\right]\frac{(-1, 1, 0)}{\sqrt{2}}$$
$$= -4\mathbf{a}_x + 4\mathbf{a}_y$$

But

$$\mathbf{H}_1 = \mathbf{H}_{1n} + \mathbf{H}_{1t}$$

Hence,

$$\mathbf{H}_{1t} = \mathbf{H}_1 - \mathbf{H}_{1n} = (-2, 6, 4) - (-4, 4, 0)$$
$$= 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$$

MAGNETIC BOUNDARY CONDITIONS

Using the boundary conditions, we have

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$$

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \rightarrow \mu_2\mathbf{H}_{2n} = \mu_1\mathbf{H}_{1n}$$

or

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-4\mathbf{a}_x + 4\mathbf{a}_y) = -10\mathbf{a}_x + 10\mathbf{a}_y$$

Thus

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = -8\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z \text{ A/m}$$

and

$$\mathbf{B}_2 = \mu_2\mathbf{H}_2 = \mu_o\mu_{r2}\mathbf{H}_2 = (4\pi \times 10^{-7})(2)(-8, 12, 4)$$
$$= -20.11\mathbf{a}_x + 30.16\mathbf{a}_y + 10.05\mathbf{a}_z \ \mu\text{Wb/m}^2$$

MAGNETIC BOUNDARY CONDITIONS

EXAMPLE 8.9 The xy-plane serves as the interface between two different media. Medium 1 ($z < 0$) is filled with a material whose $\mu_r = 6$, and medium 2 ($z > 0$) is filled with a material whose $\mu_r = 4$. If the interface carries current $(1/\mu_0)\,\mathbf{a}_y$ mA/m, and $\mathbf{B}_2 = 5\mathbf{a}_x + 8\mathbf{a}_z$ mWb/m², find $\mathbf{H}_1$ and $\mathbf{B}_1$.

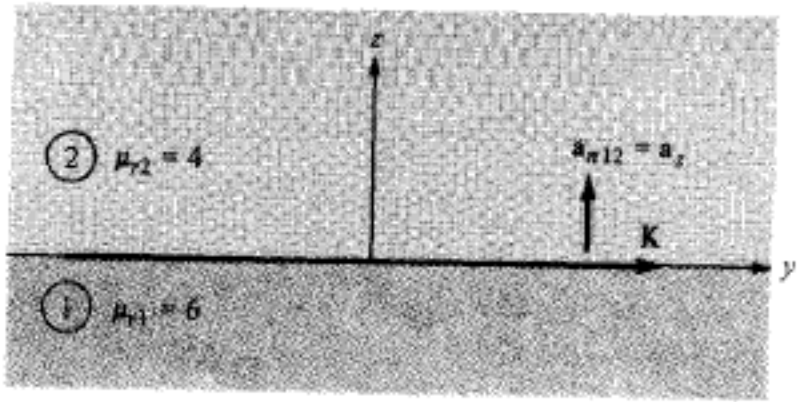


Figure 8.18 For Example 8.9.

MAGNETIC BOUNDARY CONDITIONS

Solution:

In the previous example $\mathbf{K} = 0$, so eq. (8.46) was appropriate. In this example, however, $\mathbf{K} \neq 0$, and we must resort to eq. (8.45) in addition to eq. (8.41). Consider the problem as illustrated in Figure 8.18. Let $\mathbf{B}_1 = (B_x, B_y, B_z)$ in mWb/m^2.

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} = 8\mathbf{a}_z \rightarrow B_z = 8 \tag{8.8.1}$$

But

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \text{ mA/m} \tag{8.8.2}$$

and

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \text{ mA/m} \tag{8.8.3}$$

Having found the normal components, we can find the tangential components using

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

or

$$\mathbf{H}_1 \times \mathbf{a}_{n12} = \mathbf{H}_2 \times \mathbf{a}_{n12} + \mathbf{K} \tag{8.8.4}$$

Substituting eqs. (8.8.2) and (8.8.3) into eq. (8.8.4) gives

$$\frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \times \mathbf{a}_z = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \times \mathbf{a}_z + \frac{1}{\mu_o}\mathbf{a}_y$$

MAGNETIC BOUNDARY CONDITIONS

Equating components yields

$$B_y = 0, \qquad \frac{-B_x}{6} = \frac{-5}{4} + 1 \qquad \text{or} \qquad B_x = \frac{6}{4} = 1.5 \tag{8.8.5}$$

From eqs. (8.8.1) and (8.8.5),

$$\mathbf{B}_1 = 1.5\mathbf{a}_x + 8\mathbf{a}_z \text{ mWb/m}^2$$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_o}(0.25\mathbf{a}_x + 1.33\mathbf{a}_z) \text{ mA/m}$$

and

$$\mathbf{H}_2 = \frac{1}{\mu_o}(1.25\mathbf{a}_x + 2\mathbf{a}_z) \text{ mA/m}$$

Note that H_{1x} is $(1/\mu_o)$ mA/m less than H_{2x} due to the current sheet and also that $B_{1n} = B_{2n}$.

Magnetostatics

Inductors and inductance

A circuit (or closed conducting path) carrying current I produces a magnetic field $\mathbf{B}$ which causes a flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$ to pass through each turn of the circuit as shown in Figure 8.19. If the circuit has N identical turns, we define the *flux linkage* λ as

$$\lambda = N\Psi \tag{8.50}$$

Also, if the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it; that is,

$$\lambda \propto I$$

or

$$\lambda = LI \tag{8.51}$$

where L is a constant of proportionality called the *inductance* of the circuit. The inductance L is a property of the physical arrangement of the circuit. A circuit or part of a circuit that has inductance is called an *inductor*. From eqs. (8.50) and (8.51), we may define inductance L of an inductor as the ratio of the magnetic flux linkage λ to the current I through the inductor; that is,

$$\boxed{L = \frac{\lambda}{I} = \frac{N\Psi}{I}} \tag{8.52}$$

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Magnetostatics

Inductors and inductance

The unit of inductance is the henry (H) which is the same as webers/ampere. Since the henry is a fairly large unit, inductances are usually expressed in millihenrys (mH).

The inductance defined by eq. (8.52) is commonly referred to as *self-inductance* since the linkages are produced by the inductor itself. Like capacitances, we may regard inductance as a measure of how much magnetic energy is stored in an inductor. The magnetic energy (in joules) stored in an inductor is expressed in circuit theory as:

$$W_m = \frac{1}{2}LI^2 \tag{8.53}$$

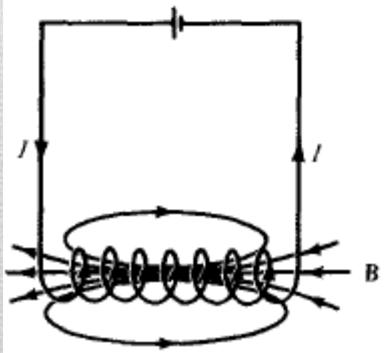

Figure 8.19 Magnetic field **B** produced by a circuit.

Magnetostatics

Inductors and inductance

or

$$L = \frac{2W_m}{I^2} \tag{8.54}$$

Thus the self-inductance of a circuit may be defined or calculated from energy considerations.

If instead of having a single circuit we have two circuits carrying current I_1 and I_2 as shown in Figure 8.20, a magnetic interaction exists between the circuits. Four component fluxes Ψ_{11}, Ψ_{12}, Ψ_{21}, and Ψ_{22} are produced. The flux Ψ_{12}, for example, is the flux passing through circuit 1 due to current I_2 in circuit 2. If $\mathbf{B}_2$ in the field due to I_2 and S_1 is the area of circuit 1, then

$$\Psi_{12} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{S} \tag{8.55}$$

We define the *mutual inductance* M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1\Psi_{12}$ on circuit 1 to current I_2, that is,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\Psi_{12}}{I_2} \tag{8.56}$$

Similarly, the mutual inductance M_{21} is defined as the flux linkages of circuit 2 per unit current I_1; that is,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\Psi_{21}}{I_1} \tag{8.57a}$$

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Magnetostatics

Inductors and inductance

It can be shown by using energy concepts that if the medium surrounding the circuits is linear (i.e., in the absence of ferromagnetic material),

$$M_{12} = M_{21} \quad (8.57b)$$

The mutual inductance M_{12} or M_{21} is expressed in henrys and should not be confused with the magnetization vector $\mathbf{M}$ expressed in amperes/meter.

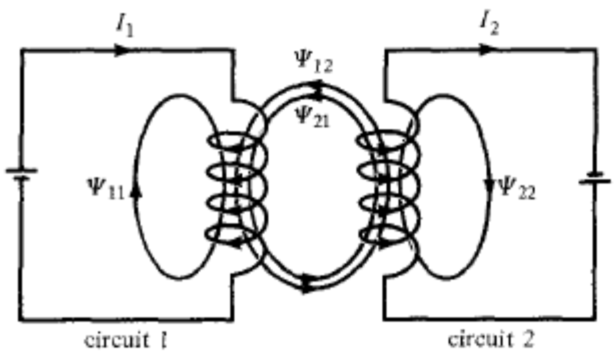

Figure 8.20 Magnetic interaction between two circuits.

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Magnetostatics

Inductors and inductance

We define the self-inductance of circuits 1 and 2, respectively, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Psi_1}{I_1} \tag{8.58}$$

and

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Psi_2}{I_2} \tag{8.59}$$

where $\Psi_1 = \Psi_{11} + \Psi_{12}$ and $\Psi_2 = \Psi_{21} + \Psi_{22}$. The total energy in the magnetic field is the sum of the energies due to L_1, L_2, and M_{12} (or M_{21}); that is,

$$\begin{aligned} W_m &= W_1 + W_2 + W_{12} \\ &= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2 \end{aligned} \tag{8.60}$$

The positive sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuits strengthen each other. If the currents flow such that their magnetic fields oppose each other, the negative sign is taken.

Magnetostatics

Inductors and inductance

As mentioned earlier, an inductor is a conductor arranged in a shape appropriate to store magnetic energy. Typical examples of inductors are toroids, solenoids, coaxial transmission lines, and parallel-wire transmission lines. The inductance of each of these inductors can be determined by following a procedure similar to that taken in determining the capacitance of a capacitor. For a given inductor, we find the self-inductance L by taking these steps:

1. Choose a suitable coordinate system.
2. Let the inductor carry current I.
3. Determine $\mathbf{B}$ from Biot–Savart's law (or from Ampere's law if symmetry exists) and calculate Ψ from $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$.
4. Finally find L from $L = \dfrac{\lambda}{I} = \dfrac{N\Psi}{I}$.

The mutual inductance between two circuits may be calculated by taking a similar procedure.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the *internal inductance* L_{in} while that produced by the flux external to it is called *external inductance* L_{ext}. The total inductance L is

$$L = L_{in} + L_{ext} \tag{8.61}$$

$$\boxed{L_{ext}C = \mu\varepsilon}$$

Magnetostatics

Inductors and inductance

Calculate the self-inductance per unit length of an infinitely long solenoid.

Solution:

We recall from Example 7.4 that for an infinitely long solenoid, the magnetic flux inside the solenoid per unit length is

$$B = \mu H = \mu I n$$

where $n = N/\ell$ = number of turns per unit length. If S is the cross-sectional area of the solenoid, the total flux through the cross section is

$$\Psi = BS = \mu I n S$$

Since this flux is only for a unit length of the solenoid, the linkage per unit length is

$$\lambda' = \frac{\lambda}{\ell} = n\Psi = \mu n^2 I S$$

and thus the inductance per unit length is

$$L' = \frac{L}{\ell} = \frac{\lambda'}{I} = \mu n^2 S$$

$$\boxed{L' = \mu n^2 S} \qquad \text{H/m}$$

Magnetostatics

Inductors and inductance

EXAMPLE 8.11 Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

Solution:

The self-inductance of the inductor can be found in two different ways: by taking the four steps given in Section 8.8 or by using eqs. (8.54) and (8.66).

Method 1: Consider the cross section of the cable as shown in Figure 8.22. We recall from eq. (7.29) that by applying Ampere's circuit law, we obtained for region 1 $(0 \leq \rho \leq a)$,

$$\mathbf{B}_1 = \frac{\mu I \rho}{2\pi a^2}\,\mathbf{a}_\phi$$

and for region 2 $(a \leq \rho \leq b)$,

$$\mathbf{B}_2 = \frac{\mu I}{2\pi \rho}\,\mathbf{a}_\phi$$

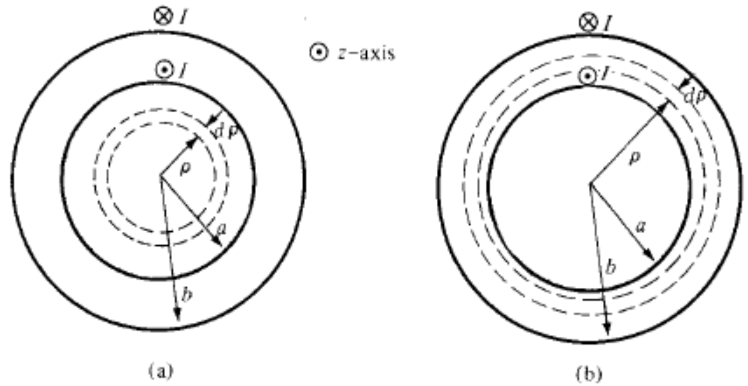


Figure 8.22 Cross section of the coaxial cable: **(a)** for region 1, $0 < \rho < a$, **(b)** for region 2, $a < \rho < b$; for Example 8.11.

Magnetostatics

Inductors and inductance

We first find the internal inductance L_{in} by considering the flux linkages due to the inner conductor. From Figure 8.22(a), the flux leaving a differential shell of thickness $d\rho$ is

$$d\Psi_1 = B_1\, d\rho\, dz = \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz$$

The flux linkage is $d\Psi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area, that is,

$$d\lambda_1 = d\Psi_1 \cdot \frac{I_{enc}}{I} = d\Psi_1 \cdot \frac{\pi\rho^2}{\pi a^2}$$

because I is uniformly distributed over the cross section for d.c. excitation. Thus, the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu I \rho\, d\rho\, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

For length ℓ of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3\, d\rho\, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{in} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi} \qquad (8.11.1)$$

The internal inductance per unit length, given by

$$\boxed{L'_{in} = \frac{L_{in}}{\ell} = \frac{\mu}{8\pi}} \qquad \text{H/m} \qquad (8.11.2)$$

Magnetostatics

Inductors and inductance

We now determine the external inductance L_{ext} by considering the flux linkage between the inner and the outer conductor as in Figure 8.22(b). For a differential shell of thickness $d\rho$,

$$d\Psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz$$

In this case, the total current I is enclosed within the path enclosing the flux. Hence,

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I\, d\rho\, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{ext} = \frac{\lambda_2}{I} = \frac{\mu\ell}{2\pi} \ln \frac{b}{a}$$

Thus

$$L = L_{in} + L_{ext} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln \frac{b}{a}\right]$$

or the inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln \frac{b}{a}\right]} \qquad \text{H/m}$$

Magnetostatics

Inductors and inductance

Method 2: It is easier to use eqs. (8.54) and (8.66) to determine L, that is,

$$W_m = \frac{1}{2} L I^2 \qquad \text{or} \qquad L = \frac{2W_m}{I^2}$$

where

$$W_m = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\,dv = \int \frac{B^2}{2\mu}\,dv$$

Hence

$$L_{\text{in}} = \frac{2}{I^2}\int \frac{B_1^2}{2\mu}\,dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}\,\rho\,d\rho\,d\phi\,dz$$

$$= \frac{\mu}{4\pi^2 a^4}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_0^{a} \rho^3\,d\rho = \frac{\mu\ell}{8\pi}$$

$$L_{\text{ext}} = \frac{2}{I^2}\int \frac{B_2^2}{2\mu}\,dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2}{4\pi^2 \rho^2}\,\rho\,d\rho\,d\phi\,dz$$

$$= \frac{\mu}{4\pi^2}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^{b} \frac{d\rho}{\rho} = \frac{\mu\ell}{2\pi}\ln\frac{b}{a}$$

and

$$L = L_{\text{in}} + L_{\text{ext}} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

as obtained previously.

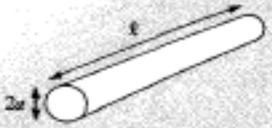
Magnetostatics

Inductors and inductance

TABLE 8.3 A Collection of Formulas for Inductance of Common Elements

1. Wire

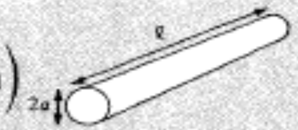
$$L = \frac{\mu_o \ell}{8\pi}$$



2. Hollow cylinder

$$L = \frac{\mu_o \ell}{2\pi}\left(\ln \frac{2\ell}{a} - 1\right)$$

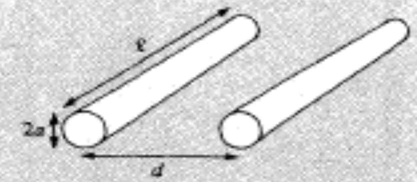
$\ell \gg a$



3. Parallel wires

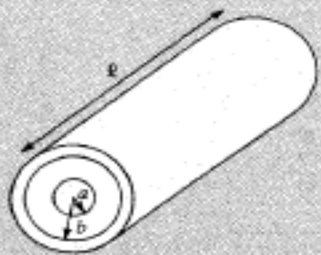
$$L = \frac{\mu_o \ell}{\pi} \ln \frac{d}{a}$$

$\ell \gg d,\ d \gg a$



4. Coaxial conductor

$$L = \frac{\mu_o \ell}{\pi} \ln \frac{b}{a}$$



5. Circular loop

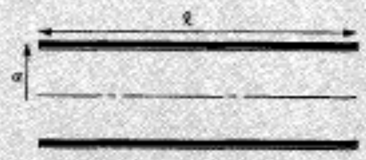
$$L = \frac{\mu_o \ell}{2\pi}\left(\ln \frac{4\ell}{d} - 2.45\right)$$

$\ell = 2\pi\rho_o,\ \rho_o \gg d$

6. Solenoid

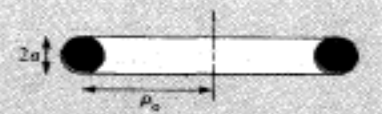
$$L = \frac{\mu_o N^2 S}{\ell}$$

$\ell \gg a$



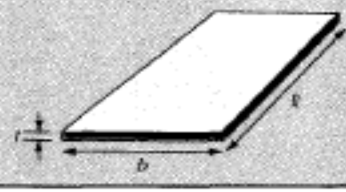
7. Torus (of circular cross section)

$$L = \mu_o N^2 [\rho_o - \sqrt{\rho_o^2 - a^2}]$$



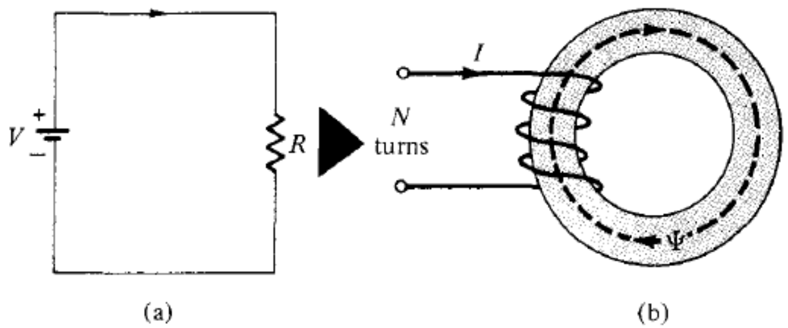
8. Sheet

$$L = \mu_o 2\ell \left(\ln \frac{2\ell}{b + t} + 0.5\right)$$



Magnetostatics

Magnetic circuit



for n magnetic circuit elements in series

$$\Psi_1 = \Psi_2 = \Psi_3 = \cdots = \Psi_n$$

and

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n$$

For n magnetic circuit elements in parallel,

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \cdots + \Psi_n$$

and

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \cdots = \mathcal{F}_n$$

TABLE 8.4 Analogy between Electric and Magnetic Circuits

Electric	Magnetic
Conductivity σ	Permeability μ
Field intensity E	Field intensity H
Current $I = \int \mathbf{J} \cdot d\mathbf{S}$	Magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
Current density $J = \dfrac{I}{S} = \sigma E$	Flux density $B = \dfrac{\Psi}{S} = \mu H$
Electromotive force (emf) V	Magnetomotive force (mmf) $\mathcal{F}$
Resistance R	Reluctance $\mathcal{R}$
Conductance $G = \dfrac{1}{R}$	Permeance $\mathcal{P} = \dfrac{1}{\mathcal{R}}$
Ohm's law $R = \dfrac{V}{I} = \dfrac{\ell}{\sigma S}$ or $V = E\ell = IR$	Ohm's law $\mathcal{R} = \dfrac{\mathcal{F}}{\Psi} = \dfrac{\ell}{\mu S}$ or $\mathcal{F} = H\ell = \Psi\mathcal{R} = NI$
Kirchoff's laws: $\Sigma I = 0$, $\Sigma V - \Sigma RI = 0$	Kirchhoff's laws: $\Sigma \Psi = 0$, $\Sigma \mathcal{F} - \Sigma \mathcal{R}\Psi = 0$

Example

EXAMPLE 8.14 The toroidal core of Figure 8.26(a) has $\rho_o = 10$ cm and a circular cross section with $a = 1$ cm. If the core is made of steel ($\mu = 1000\ \mu_o$) and has a coil with 200 turns, calculate the amount of current that will produce a flux of 0.5 mWb in the core.

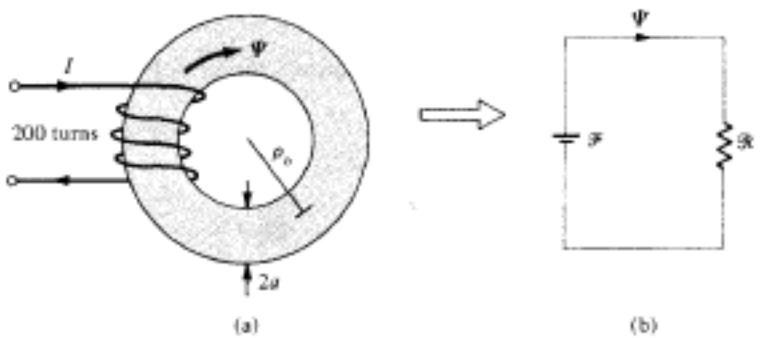


Figure 8.26 **(a)** Toroidal core of Example 8.14; **(b)** its equivalent electric circuit analog.

Example

Solution:

This problem can be solved in two different ways: using the magnetic field approach (direct), or using the electric circuit analog (indirect).

Method 1: Since ρ_o is large compared with a, from Example 7.6,

$$B = \frac{\mu NI}{\ell} = \frac{\mu_o \mu_r NI}{2\pi\rho_o}$$

Hence,

$$\Psi = BS = \frac{\mu_o \mu_r NI\, \pi a^2}{2\pi\rho_o}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = \frac{2(10 \times 10^{-2})(0.5 \times 10^{-3})}{4\pi \times 10^{-7}(1000)(200)(1 \times 10^{-4})}$$

$$= \frac{100}{8\pi} = 3.979 \text{ A}$$

Method 2: The toroidal core in Figure 8.26(a) is analogous to the electric circuit of Figure 8.26(b). From the circuit and Table 8.4.

$$\mathcal{F} = NI = \Psi \mathcal{R} = \Psi \frac{\ell}{\mu S} = \Psi \frac{2\pi\rho_o}{\mu_o \mu_r \pi a^2}$$

or

$$I = \frac{2\rho_o \Psi}{\mu_o \mu_r N a^2} = 3.979 \ A$$

as obtained previously.

Example

EXAMPLE 8.15 In the magnetic circuit of Figure 8.27, calculate the current in the coil that will produce a magnetic flux density of 1.5 Wb/m^2 in the air gap assuming that $\mu = 50\mu_o$ and that all branches have the same cross-sectional area of 10 cm^2.

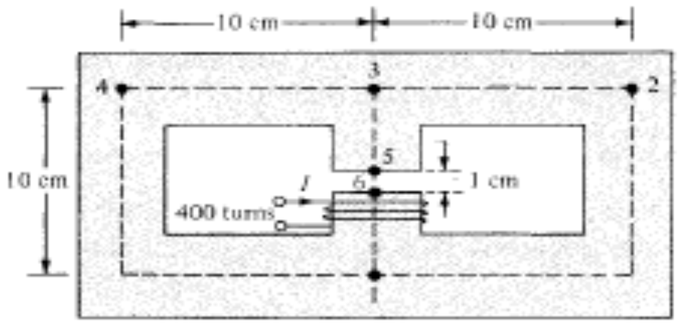


Figure 8.27 Magnetic circuit of Example 8.15.

Example

Solution:

The magnetic circuit of Figure 8.27 is analogous to the electric circuit of Figure 8.28. In Figure 8.27, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_a$ are the reluctances in paths 143, 123, 35 and 16, and 56 (air gap), respectively. Thus

$$\mathcal{R}_1 = \mathcal{R}_2 = \frac{\ell}{\mu_o \mu_r S} = \frac{30 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{3 \times 10^8}{20\pi}$$

$$\mathcal{R}_3 = \frac{9 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{0.9 \times 10^8}{20\pi}$$

$$\mathcal{R}_a = \frac{1 \times 10^{-2}}{(4\pi \times 10^{-7})(1)(10 \times 10^{-4})} = \frac{5 \times 10^8}{20\pi}$$

We combine $\mathcal{R}_1$ and $\mathcal{R}_2$ as resistors in parallel. Hence,

$$\mathcal{R}_1 \| \mathcal{R}_2 = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{\mathcal{R}_1}{2} = \frac{1.5 \times 10^8}{20\pi}$$

The total reluctance is

$$\mathcal{R}_T = \mathcal{R}_a + \mathcal{R}_3 + \mathcal{R}_1 \| \mathcal{R}_2 = \frac{7.4 \times 10^8}{20\pi}$$

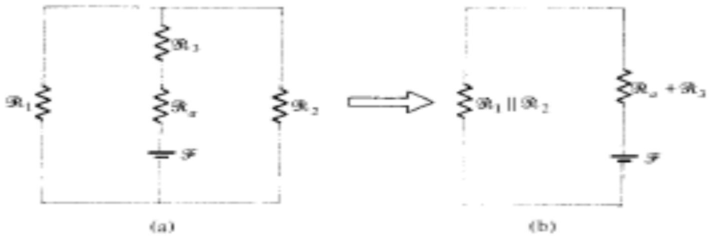


Figure 8.28 Electric circuit analog of the magnetic circuit in Figure 8.27.

Example

Solution:

The magnetic circuit of Figure 8.27 is analogous to the electric circuit of Figure 8.28. In Figure 8.27, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_a$ are the reluctances in paths 143, 123, 35 and 16, and 56 (air gap), respectively. Thus

$$\mathcal{R}_1 = \mathcal{R}_2 = \frac{\ell}{\mu_o \mu_r S} = \frac{30 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{3 \times 10^8}{20\pi}$$

$$\mathcal{R}_3 = \frac{9 \times 10^{-2}}{(4\pi \times 10^{-7})(50)(10 \times 10^{-4})} = \frac{0.9 \times 10^8}{20\pi}$$

$$\mathcal{R}_a = \frac{1 \times 10^{-2}}{(4\pi \times 10^{-7})(1)(10 \times 10^{-4})} = \frac{5 \times 10^8}{20\pi}$$

We combine $\mathcal{R}_1$ and $\mathcal{R}_2$ as resistors in parallel. Hence,

$$\mathcal{R}_1 \| \mathcal{R}_2 = \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{\mathcal{R}_1}{2} = \frac{1.5 \times 10^8}{20\pi}$$

The total reluctance is

$$\mathcal{R}_T = \mathcal{R}_a + \mathcal{R}_3 + \mathcal{R}_1 \| \mathcal{R}_2 = \frac{7.4 \times 10^8}{20\pi}$$

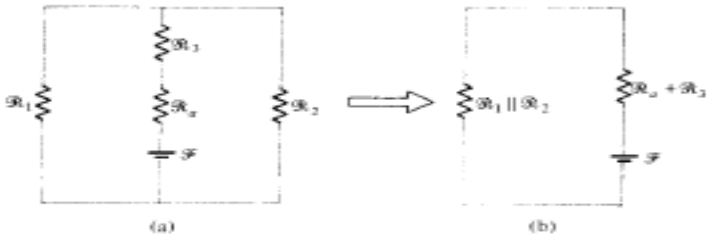


Figure 8.28 Electric circuit analog of the magnetic circuit in Figure 8.27.

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Example

The mmf is

$$\mathcal{F} = NI = \Psi_a R_T$$

But $\Psi_a = \Psi = B_a S$. Hence

$$I = \frac{B_a S \mathcal{R}_T}{N} = \frac{1.5 \times 10 \times 10^{-4} \times 7.4 \times 10^8}{400 \times 20\pi}$$

$$= 44.16 \text{ A}$$

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Electrodynamics

Electrostatics and Magnetostatics vs Electrodynamics

$$\nabla \times \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{D} = \rho$$

electrostatic

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$

magnetostatic

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0.$$

electrodynamic

Using Helmholtz theory in electrostatics electric field can be computed
Using Helmholtz theory in magnetostatics magnetic field can be computed

In electrostatics and magneto staticsElectric field and magnetic field can be computed independently from each other

In electrodynamics where a varying current density and charge density with time exist, the magnetic and electric field are coupled which leads to electromagnetic fields.

Electrodynamics

Electrodynamics: The quantities that we aim to compute are E,D,H,B, each has three components=> 12 equation are needed to solve for the variables.

The four Maxwell's equation are not all independent the divergence of both vector fields can be derived from the curl of the vector fields. As a result the two curl equations represent 6 equations each of them. The constitutive equations complete the 12 equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} \qquad \mathbf{D} = \epsilon \mathbf{E}$$

To reach to the final generalized from of Maxwell's equations two generalizations to the curl equations must be applied:
1)Farady's law
2) Displacement Current Density

Electrodynamics

Farady's law and electromagnetic induction: It is experimental law in which an electric field is induced in a loop when a time varying magnetic field is linking it.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Point form Farady's law for stationary circuit

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{E} \cdot d\ell = -\frac{d}{dt} \int_S \mathbf{B} \cdot ds$$

Integral form of Farady's law for stationary circuit

$$V_{emf} = -\frac{d\Psi}{dt}$$

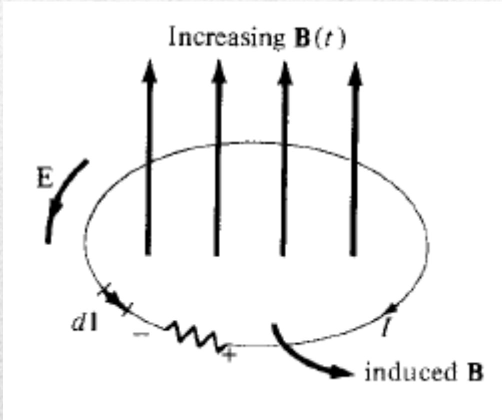
$$\mathscr{V} = \oint_C \mathbf{E} \cdot d\ell = \text{emf induced in circuit with contour } C$$

Electrodynamics

Motional and Transformer EMF:

1. Transformer EMF: static circuit in a time varying magnetic field

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$



2. Motional EMF(flux-cutting emf): moving circuit in a static magnetic field

We define the *motional electric field* $\mathbf{E}_m$ as

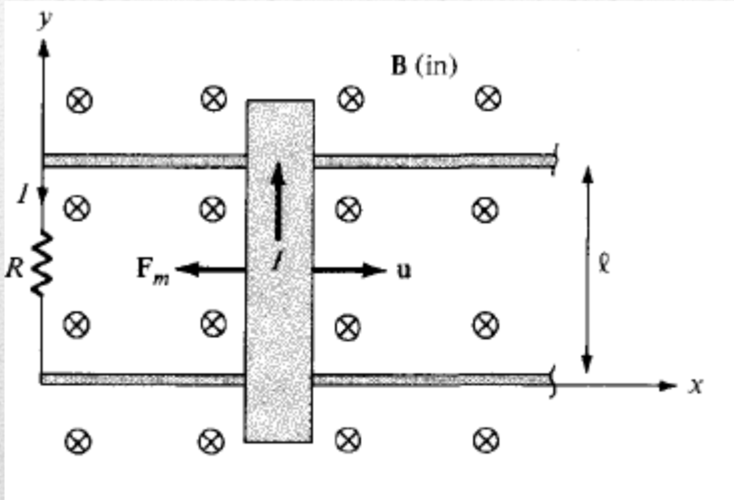
$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

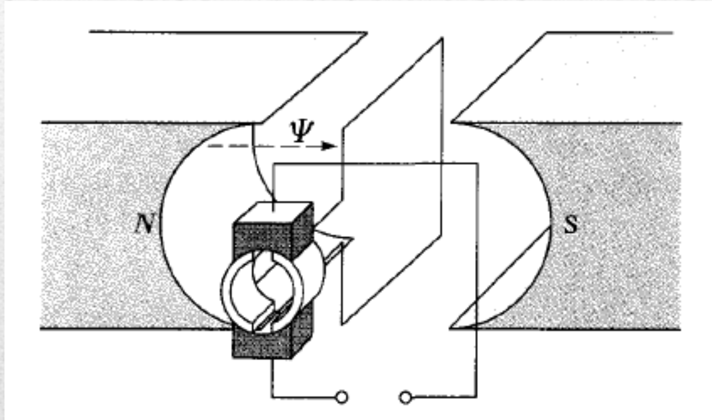
$$V_{emf} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

Electrodynamics

Motional and Transformer EMF:

2. Motional EMF: moving circuit in a static magnetic field





$$V_{emf} = uB\ell$$

Electrodynamics

Motional and Transformer EMF:

3. Both: moving circuit in a time varying magnetic field

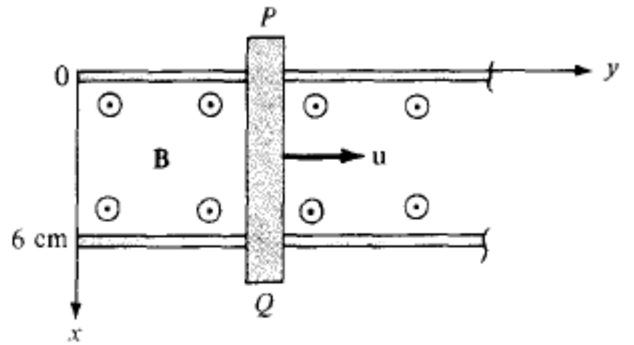
$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

EXAMPLE 9.1 A conducting bar can slide freely over two conducting rails as shown in Figure 9.6. Calculate the induced voltage in the bar

(a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4 \cos 10^6 t\, \mathbf{a}_z$ mWb/m^2

(b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z$ mWb/m^2

(c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4 \cos (10^6 t - y)\, \mathbf{a}_z$ mWb/m^2



Electrodynamics

Motional and Transformer EMF:

Solution:

(a) In this case, we have transformer emf given by

$$V_{emf} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^6) \sin 10^6 t \, dx \, dy$$
$$= 4(10^3)(0.08)(0.06) \sin 10^6 t$$
$$= 19.2 \sin 10^6 t \text{ V}$$

The polarity of the induced voltage (according to Lenz's law) is such that point P on the bar is at lower potential than Q when $\mathbf{B}$ is increasing.

(b) This is the case of motional emf:

$$V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^{0} (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x$$
$$= -uB\ell = -20(4.10^{-3})(0.06)$$
$$= -4.8 \text{ mV}$$

(c) Both transformer emf and motional emf are present in this case. This problem can be solved in two ways.

Electrodynamics

Motional and Transformer EMF:

Method 1: Using eq. (9.15)

$$V_{emf} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad (9.1.1)$$

$$= \int_{x=0}^{0.06} \int_{0}^{y} 4.10^{-3}(10^6) \sin(10^6 t - y')dy'\,dx$$

$$+ \int_{0.06}^{0} [20\mathbf{a}_y \times 4.10^{-3} \cos(10^6 t - y)\mathbf{a}_z] \cdot dx\,\mathbf{a}_x$$

$$= 240 \cos(10^6 t - y')\Big|_0^y - 80(10^{-3})(0.06) \cos(10^6 t - y)$$

$$= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos(10^6 t - y)$$

$$\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t \qquad (9.1.2)$$

because the motional emf is negligible compared with the transformer emf. Using trigonometric identity

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$V_{emf} = 480 \sin\left(10^6 t - \frac{y}{2}\right) \sin \frac{y}{2}\ \mathrm{V} \qquad (9.1.3)$$

Electrodynamics

Motional and Transformer EMF:

Method 2: Alternatively we can apply eq. (9.4), namely,

$$V_{emf} = -\frac{\partial \Psi}{\partial t} \quad (9.1.4)$$

where

$$\begin{aligned}\Psi &= \int \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{y=0}^{y} \int_{x=0}^{0.06} 4\cos(10^6 t - y)\, dx\, dy \\ &= -4(0.06)\sin(10^6 t - y)\Big|_{y=0}^{y} \\ &= -0.24\sin(10^6 t - y) + 0.24\sin 10^6 t \text{ mWb}\end{aligned}$$

But

$$\frac{dy}{dt} = u \rightarrow y = ut = 20t$$

Hence,

$$\Psi = -0.24\sin(10^6 t - 20t) + 0.24\sin 10^6 t \text{ mWb}$$

$$\begin{aligned}V_{emf} &= -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20)\cos(10^6 t - 20t) - 0.24(10^6)\cos 10^6 t \text{ mV} \\ &\simeq 240\cos(10^6 t - y) - 240\cos 10^6 t \text{ V} \quad (9.1.5)\end{aligned}$$

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Electrodynamics

EXAMPLE 9.2 The loop shown in Figure 9.7 is inside a uniform magnetic field $\mathbf{B} = 50\, \mathbf{a}_x$ mWb/m^2. If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop lies in the yz-plane at time $t = 0$, find

(a) The induced emf at $t = 1$ ms

(b) The induced current at $t = 3$ ms

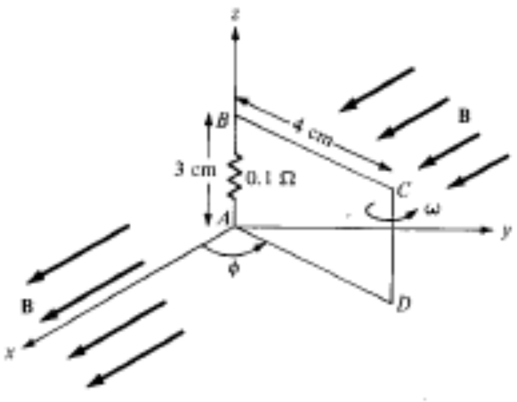

Figure 9.7 For Example 9.2; polarity is for increasing emf.

Electrodynamics

Solution:

(a) Since the **B** field is time invariant, the induced emf is motional, that is,

$$V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

where

$$d\mathbf{l} = d\mathbf{l}_{DC} = dz\,\mathbf{a}_z, \qquad \mathbf{u} = \frac{d\mathbf{l}'}{dt} = \frac{\rho\, d\phi}{dt}\,\mathbf{a}_\phi = \rho\omega\mathbf{a}_\phi$$

$$\rho = AD = 4 \text{ cm}, \qquad \omega = 2\pi f = 100\pi$$

As **u** and $d\mathbf{l}$ are in cylindrical coordinates, we transform **B** into cylindrical coordinates using eq. (2.9):

$$\mathbf{B} = B_o\mathbf{a}_x = B_o(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)$$

where $B_o = 0.05$. Hence,

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\omega & 0 \\ B_o\cos\phi & -B_o\sin\phi & 0 \end{vmatrix} = -\rho\omega B_o\cos\phi\,\mathbf{a}_z$$

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and

$$(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\rho\omega B_o \cos\phi \, dz = -0.04(100\pi)(0.05) \cos\phi \, dz$$
$$= -0.2\pi \cos\phi \, dz$$

$$V_{emf} = \int_{z=0}^{0.03} -0.2\pi \cos\phi \, dz = -6\pi \cos\phi \text{ mV}$$

To determine ϕ, recall that

$$\omega = \frac{d\phi}{dt} \rightarrow \phi = \omega t + C_o$$

where C_o is an integration constant. At $t = 0$, $\phi = \pi/2$ because the loop is in the yz-plane at that time, $C_o = \pi/2$. Hence,

$$\phi = \omega t + \frac{\pi}{2}$$

and

$$V_{emf} = -6\pi \cos\left(\omega t + \frac{\pi}{2}\right) = 6\pi \sin(100\pi t) \text{ mV}$$

$$\text{At } t = 1 \text{ ms}, V_{emf} = 6\pi \sin(0.1\pi) = 5.825 \text{ mV}$$

(b) The current induced is

$$i = \frac{V_{emf}}{R} = 60\pi \sin(100\pi t) \text{ mA}$$

At $t = 3$ ms,

$$i = 60\pi \sin(0.3\pi) \text{ mA} = 0.1525 \text{ A}$$

Electrodynamics

DISPLACEMENT CURRENT

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{9.17}$$

But the divergence of the curl of any vector field is identically zero
Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \tag{9.20}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \tag{9.21}$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{9.22a}$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}} \tag{9.22b}$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{9.23}$$

Electrodynamics

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \qquad (9.24)$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates. This example, shown in Figure 9.10, serves to illustrate the need for the displacement current. Applying an unmodified form of Ampere's circuit law to a closed path L shown in Figure 9.10(a) gives

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{enc} = I \qquad (9.25)$$

EXAMPLE 9.4

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin $10^3 t$ V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_o$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

which is the same as the conduction current, given by

$$I_c = \frac{dQ}{dt} = S\frac{d\rho_s}{dt} = S\frac{dD}{dt} = \varepsilon S\frac{dE}{dt} = \frac{\varepsilon S}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$I_d = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot 10^3 \times 50 \cos 10^3 t$$
$$= 147.4 \cos 10^3 t \text{ nA}$$

Electrodynamics

MAXWELL'S EQUATIONS IN FINAL FORMS

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampere's circuit law

Electrodynamics

MAXWELL'S EQUATIONS IN FINAL FORMS

is associated with Maxwell's equations. Also the equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{9.29}$$

is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_o\mathbf{E} + \mathbf{P} \tag{9.30a}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{9.30b}$$

$$\mathbf{J} = \sigma\mathbf{E} + \rho_v\mathbf{u} \tag{9.30c}$$

hold for time-varying fields. Consequently, the boundary conditions

$$E_{1t} = E_{2t} \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_{n12} = 0 \tag{9.31a}$$

$$H_{1t} - H_{2t} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \tag{9.31b}$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = \rho_s \tag{9.31c}$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_{n12} = 0 \tag{9.31d}$$

remain valid for time-varying fields. However, for a perfect conductor ($\sigma \simeq \infty$) in a time-varying field,

$$\mathbf{E} = 0, \qquad \mathbf{H} = 0, \qquad \mathbf{J} = 0 \tag{9.32}$$

and hence,

$$\mathbf{B}_n = 0, \qquad \mathbf{E}_t = 0 \tag{9.33}$$

For a perfect dielectric ($\sigma = 0$), eq. (9.31) holds except that $\mathbf{K} = 0$. Though eqs. (9.28) to (9.33) are not Maxwell's equations, they are associated with them.

Electrodynamics

TIME-VARYING POTENTIALS

For static EM fields, we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v \, dv}{4\pi\varepsilon R} \tag{9.40}$$

and the magnetic vector potential as

$$\mathbf{A} = \int_v \frac{\mu J \, dv}{4\pi R} \tag{9.41}$$

We would like to examine what happens to these potentials when the fields are time varying. Recall that **A** was defined from the fact that $\nabla \cdot \mathbf{B} = 0$, which still holds for time-varying fields. Hence the relation

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{9.42}$$

holds for time-varying situations. Combining Faraday's law in eq. (9.8) with eq. (9.42) gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \tag{9.43a}$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{9.43b}$$

Since the curl of the gradient of a scalar field is identically zero (see Practice Exercise 3.10), the solution to eq. (9.43b) is

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{9.44}$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{9.45}$$

From eqs. (9.42) and (9.45), we can determine the vector fields **B** and **E** provided that the potentials **A** and V are known. However, we still need to find some expressions for **A** and V similar to those in eqs. (9.40) and (9.41) that are suitable for time-varying fields.

From Table 9.1 or eq. (9.38) we know that $\nabla \cdot \mathbf{D} = \rho_v$ is valid for time-varying conditions. By taking the divergence of eq. (9.45) and making use of eqs. (9.37) and (9.38), we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$$

Electrodynamics

TIME-VARYING POTENTIALS

or

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\varepsilon} \tag{9.46}$$

Taking the curl of eq. (9.42) and incorporating eqs. (9.23) and (9.45) results in

$$\begin{aligned}\nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J} + \varepsilon\mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \mu \mathbf{J} - \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}\end{aligned} \tag{9.47}$$

where $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ have been assumed. By applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{9.48}$$

to eq. (9.47),

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{9.49}$$

A vector field is uniquely defined when its curl and divergence are specified. The curl of **A** has been specified by eq. (9.42); for reasons that will be obvious shortly, we may choose the divergence of **A** as

$$\boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t}} \tag{9.50}$$

This choice relates **A** and V and it is called the *Lorentz condition for potentials*. We had this in mind when we chose $\nabla \cdot \mathbf{A} = 0$ for magnetostatic fields in eq. (7.59). By imposing the Lorentz condition of eq. (9.50), eqs. (9.46) and (9.49), respectively, become

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}} \tag{9.51}$$

and

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}} \tag{9.52}$$

Electrodynamics

TIME-VARYING POTENTIALS

the solutions, or the integral forms of eqs. (6.4) and (7.60), it can be shown that the solutions[5] to eqs. (9.51) and (9.52) are

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi\varepsilon R} \tag{9.53}$$

and

$$\mathbf{A} = \int_v \frac{\mu[\mathbf{J}]\, dv}{4\pi R} \tag{9.54}$$

The term $[\rho_v]$ (or $[\mathbf{J}]$) means that the time t in $\rho_v(x, y, z, t)$ [or $\mathbf{J}(x, y, z, t)$] is replaced by the *retarded time* t' given by

$$t' = t - \frac{R}{u} \tag{9.55}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$ and

$$u = \frac{1}{\sqrt{\mu\varepsilon}} \tag{9.56}$$

is the velocity of wave propagation. In free space, $u = c = 3 \times 10^8$ m/s is the speed of light in a vacuum. Potentials V and $\mathbf{A}$ in eqs. (9.53) and (9.54) are, respectively, called the *retarded electric scalar potential* and the *retarded magnetic vector potential*. Given ρ_v and $\mathbf{J}$, V and $\mathbf{A}$ can be determined using eqs. (9.53) and (9.54); from V and $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ can be determined using eqs. (9.45) and (9.42), respectively.

Electrodynamics

Wave Propagation

In general, **waves** are means of transporting energy or information.

our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

A **wave** is a function of both space and time.

In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0$$

Wave Propagation

Its solutions are of the form

$$E^- = f(z - ut)$$

$$E^+ = g(z + ut)$$

or

$$E = f(z - ut) + g(z + ut)$$

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes

$$\frac{d^2E_s}{dz^2} + \beta^2 E_s = 0 \qquad (10.3)$$

where $\beta = \omega/u$ and E_s is the phasor form of E. The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = Ae^{j(\omega t - \beta z)} \qquad (10.4a)$$

$$E^- = Be^{j(\omega t + \beta z)} \qquad (10.4b)$$

Wave Propagation

Its solutions are of the form

$$E^- = f(z - ut)$$

$$E^+ = g(z + ut)$$

or

$$E = f(z - ut) + g(z + ut)$$

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes

$$\frac{d^2E_s}{dz^2} + \beta^2 E_s = 0 \qquad (10.3)$$

where $\beta = \omega/u$ and E_s is the phasor form of E. The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = Ae^{j(\omega t - \beta z)} \qquad (10.4a)$$

$$E^- = Be^{j(\omega t + \beta z)} \qquad (10.4b)$$

Wave Propagation

the possible solutions to eq. (10.3) are

$$E^{+} = Ae^{j(\omega t-\beta z)} \tag{10.4a}$$

$$E^{-} = Be^{j(\omega t+\beta z)} \tag{10.4b}$$

and

$$E = Ae^{j(\omega t-\beta z)} + Be^{j(\omega t+\beta z)} \tag{10.4c}$$

where A and B are real constants.

Wave propagation using Laplace theorem

$$\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0 \qquad (10.17)$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \qquad (10.18)$$

and γ is called the *propagation constant* (in per meter) of the medium. By a similar procedure, it can be shown that for the **H** field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \qquad (10.19)$$

$$\gamma = \alpha + j\beta \qquad (10.20)$$

We obtain α and β from eqs. (10.18) and (10.20) by noting that

$$-\text{Re}\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon \qquad (10.21)$$

and

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \qquad (10.22)$$

From eqs. (10.21) and (10.22), we obtain

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \qquad (10.23)$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \qquad (10.24)$$

10.4 PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$. It is a special case of that in Section 10.3 except that

$$\boxed{\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o\varepsilon_r, \qquad \mu = \mu_o\mu_r} \tag{10.42}$$

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon} \tag{10.43a}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.43b}$$

Also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^\circ \tag{10.44}$$

and thus **E** and **H** are in time phase with each other.

10.5 PLANE WAVES IN FREE SPACE

This is a special case of what we considered in Section 10.3. In this case,

$$\boxed{\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o} \tag{10.45}$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ε by ε_o and μ by μ_o in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} \tag{10.46a}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.46b}$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

10.6 PLANE WAVES IN GOOD CONDUCTORS

This is another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$; that is,

$$\boxed{\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r} \tag{10.50}$$

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \tag{10.51a}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.51b}$$

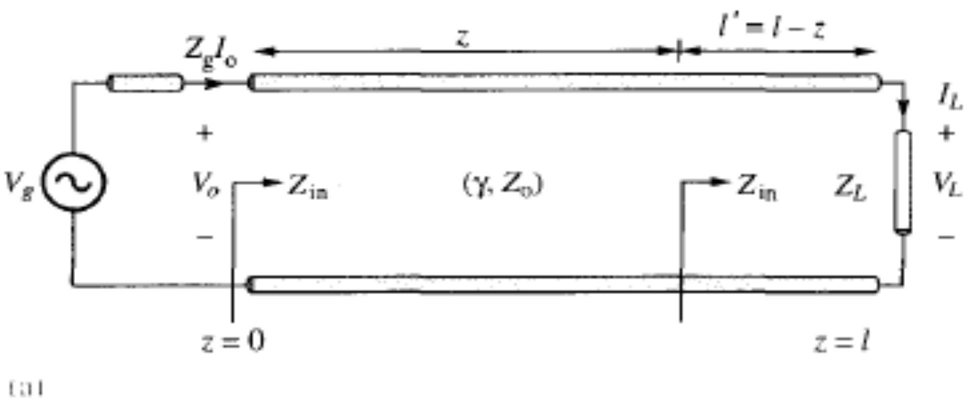
Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ \tag{10.52}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\,\mathbf{a}_x \tag{10.53a}$$

TRANSMISSION LINES



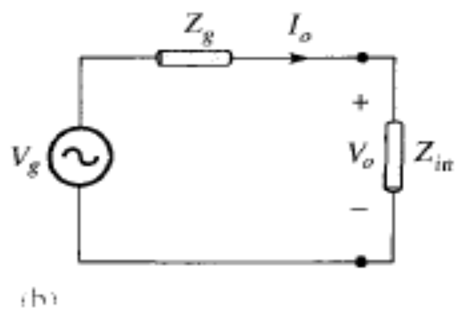


Figure 11.6 **(a)** Input impedance due to a line terminated by a load; **(b)** equivalent circuit for finding V_o and I_o in terms of Z_{in} at the input.

A. Shorted Line ($Z_L = 0$)

For this case, eq. (11.34) becomes

$$Z_{sc} = Z_{in}\Big|_{Z_L=0} = jZ_o \tan \beta\ell \qquad (11.41a)$$

Also,

$$\Gamma_L = -1, \qquad s = \infty \qquad (11.41b)$$

We notice from eq. (11.41a) that Z_{in} is a pure reactance, which could be capacitive or inductive depending on the value of ℓ. The variation of Z_{in} with ℓ is shown in Figure 11.8(a).

B. Open-Circuited Line ($Z_L = \infty$)

In this case, eq. (11.34) becomes

$$Z_{oc} = \lim_{Z_L \to \infty} Z_{in} = \frac{Z_o}{j \tan \beta\ell} = -jZ_o \cot \beta\ell \qquad (11.42a)$$

and

$$\Gamma_L = 1, \qquad s = \infty \qquad (11.42b)$$

The variation of Z_{in} with ℓ is shown in Figure 11.8(b). Notice from eqs. (11.41a) and (11.42a) that

$$Z_{sc}Z_{oc} = Z_o^2 \qquad (11.43)$$

C. Matched Line ($Z_L = Z_o$)

This is the most desired case from the practical point of view. For this case, eq. (11.34) reduces to

$$Z_{in} = Z_o \tag{11.44a}$$

and

$$\Gamma_L = 0, \qquad s = 1 \tag{11.44b}$$

that is, $V_o^- = 0$, the whole wave is transmitted and there is no reflection. The incident power is fully absorbed by the load. Thus maximum power transfer is possible when a transmission line is matched to the load.

TRANSMISSION LINES

EXAMPLE 11.4

A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

TRANSMISSION LINES

Solution:

This problem will be solved with and without using the Smith chart.

Method 1: (Without the Smith chart)

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40} = 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2 \times 10^6)(30)}{0.6\,(3 \times 10^8)} = \frac{2\pi}{3} = 120^\circ$$

Note that $\beta\ell$ is the electrical length of the line.

$$\begin{aligned} Z_{in} &= Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right] \\ &= \frac{50\,(60 + j40 + j50 \tan 120^\circ)}{[50 + j(60 + j40) \tan 120^\circ]} \\ &= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\underline{/3.22^\circ} \\ &= 23.97 + j1.35\ \Omega \end{aligned}$$

TRANSMISSION LINES

Method 2: (Using the Smith chart).

(a) Calculate the normalized load impedance

$$z_L = \frac{Z_L}{Z_o} = \frac{60 + j40}{50}$$
$$= 1.2 + j0.8$$

Locate z_L on the Smith chart of Figure 11.15 at point P where the $r = 1.2$ circle and the $x = 0.8$ circle meet. To get Γ at z_L, extend OP to meet the $r = 0$ circle at Q and measure OP and OQ. Since OQ corresponds to $|\Gamma| = 1$, then at P,

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.2 \text{ cm}}{9.1 \text{ cm}} = 0.3516$$

TRANSMISSION LINES

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TRANSMISSION LINES

Note that $OP = 3.2$ cm and $OQ = 9.1$ cm were taken from the Smith chart used by the author; the Smith chart in Figure 11.15 is reduced but the ratio of OP/OQ remains the same.

Angle θ_Γ is read directly on the chart as the angle between OS and OP; that is

$$\theta_\Gamma = \text{angle } POS = 56°$$

Thus

$$\Gamma = 0.3516 \underline{/56°}$$

(b) To obtain the standing wave ratio s, draw a circle with radius OP and center at O. This is the constant s or $|\Gamma|$ circle. Locate point S where the s-circle meets the Γ_r-axis.

[This is easily shown by setting $\Gamma_i = 0$ in eq. (11.49a).] The value of r at this point is s; that is

$$s = r \text{ (for } r \geq 1)$$
$$= 2.1$$

(c) To obtain Z_{in}, first express ℓ in terms of λ or in degrees.

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{2 \times 10^6} = 90 \text{ m}$$

$$\ell = 30 \text{ m} = \frac{30}{90}\lambda = \frac{\lambda}{3} \rightarrow \frac{720°}{3} = 240°$$

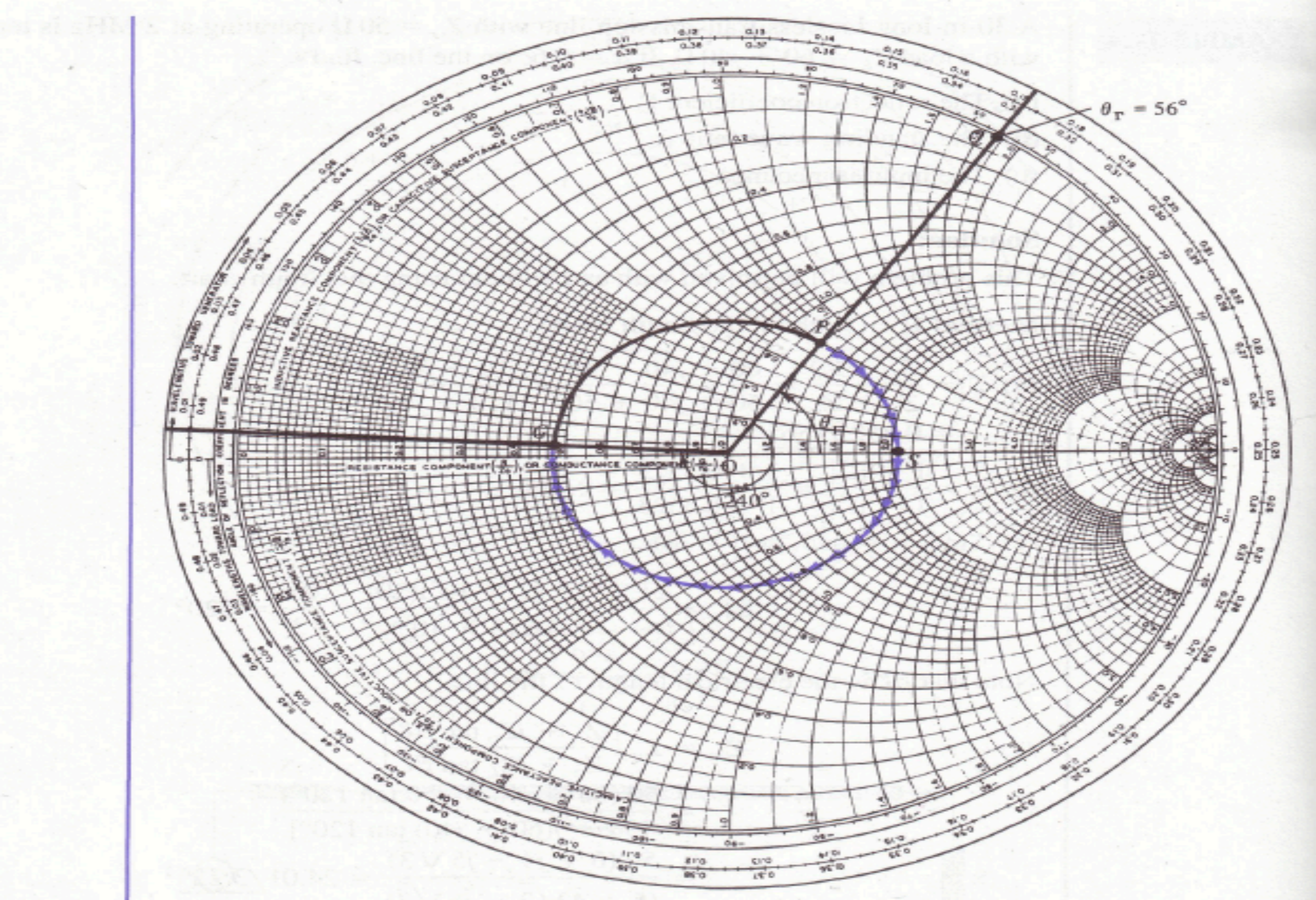
Since λ corresponds to an angular movement of 720° on the chart, the length of the line corresponds to an angular movement of 240°. That means we move toward the generator (or away from the load, in the clockwise direction) 240° on the s-circle from point P to point G. At G, we obtain

$$z_{in} = 0.47 + j0.035$$

Hence

$$Z_{in} = Z_o z_{in} = 50(0.47 + j0.035) = 23.5 + j1.75\ \Omega$$

Although the results obtained using the Smith chart are only approximate, for engineering purposes they are close enough to the exact ones obtained in Method 1.


θr = 56°
P
Q
G
S
O
θr
240°
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)