Engineering Electromagnetics

Chapter 3:

Electric Flux Density, Gauss' Law,

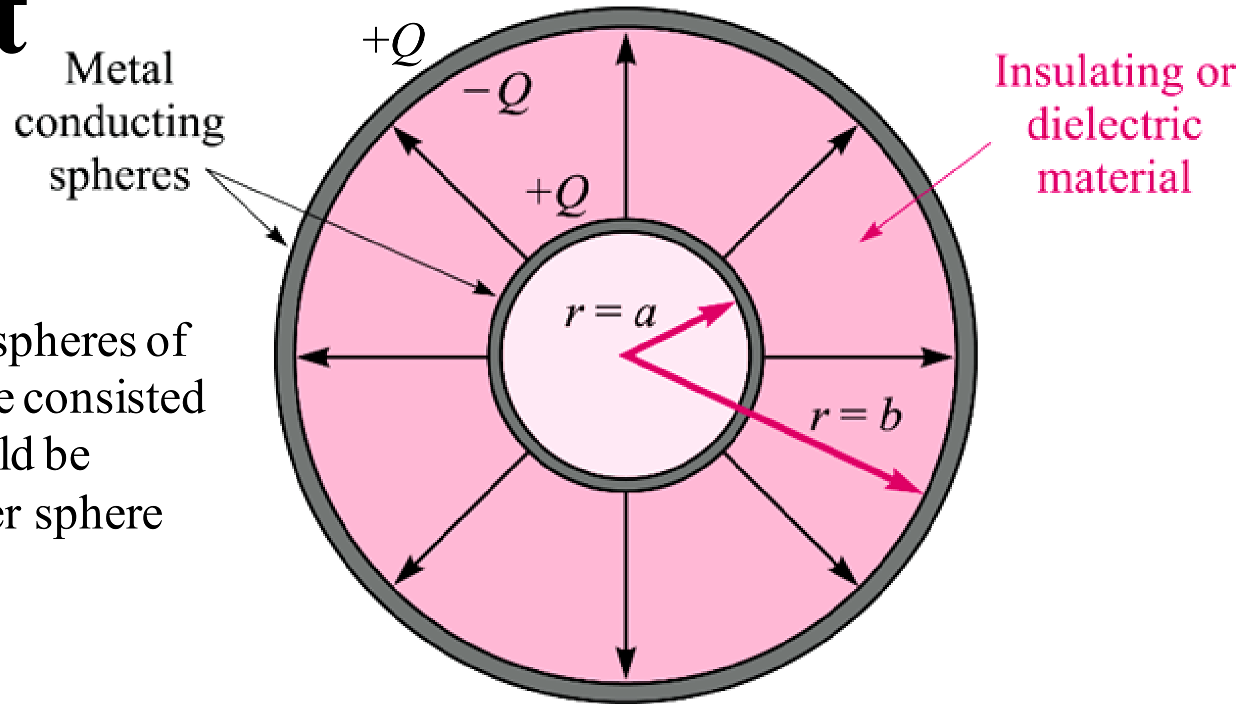
and Divergence

Objectives

- **Electric Flux Density,**
- **Gauss' Law,**
- **Divergence**
- **First Maxwell's Equation**

Faraday Experiment

Conductor in static electric field situations

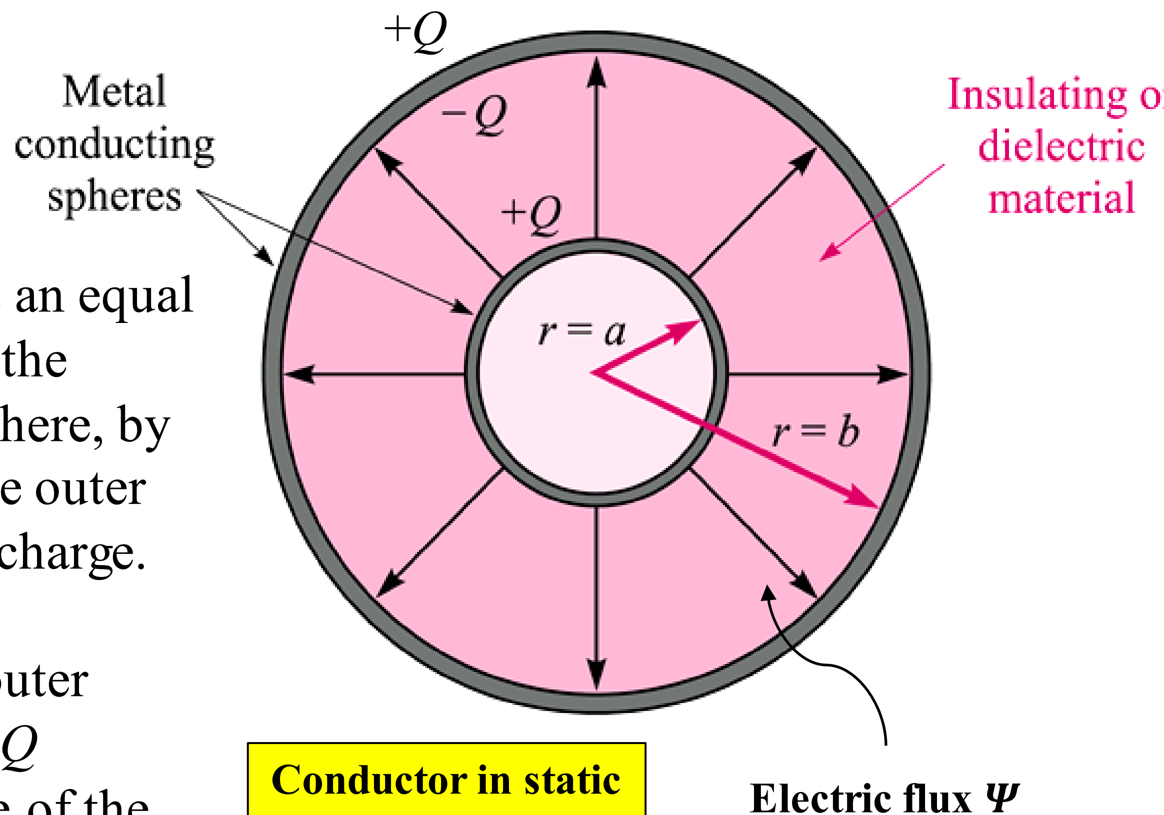


Started with a pair of metal spheres of different sizes; the larger one consisted of two hemispheres that could be assembled around the smaller sphere

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday Apparatus, Before Grounding

Considering an inner sphere of radius **a** and an outer sphere of radius **b**, with charges of **Q** and **−Q**, respectively. The **paths of electric flux *Ψ*** extending from the **inner** sphere to the **outer** sphere are indicated by the **symmetrically** distributed **streamlines** drawn radially from one sphere to the other.

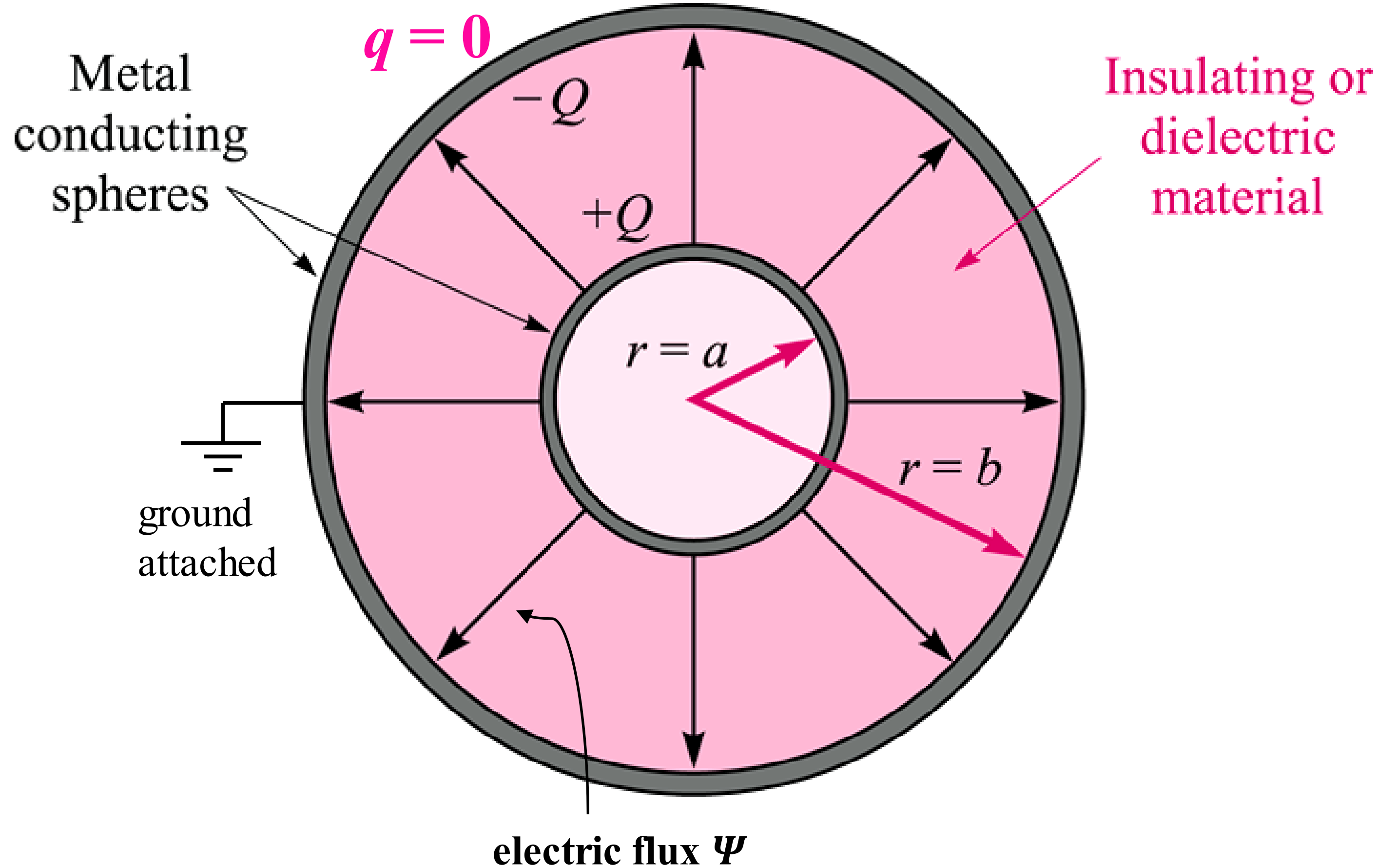


The inner charge, *Q,* induces an equal and opposite charge, *-Q,* on the inside surface of the outer sphere, by attracting free electrons in the outer material toward the positive charge.

This means that before the outer sphere is grounded, charge +*Q* resides on the *outside* surface of the outer conductor.

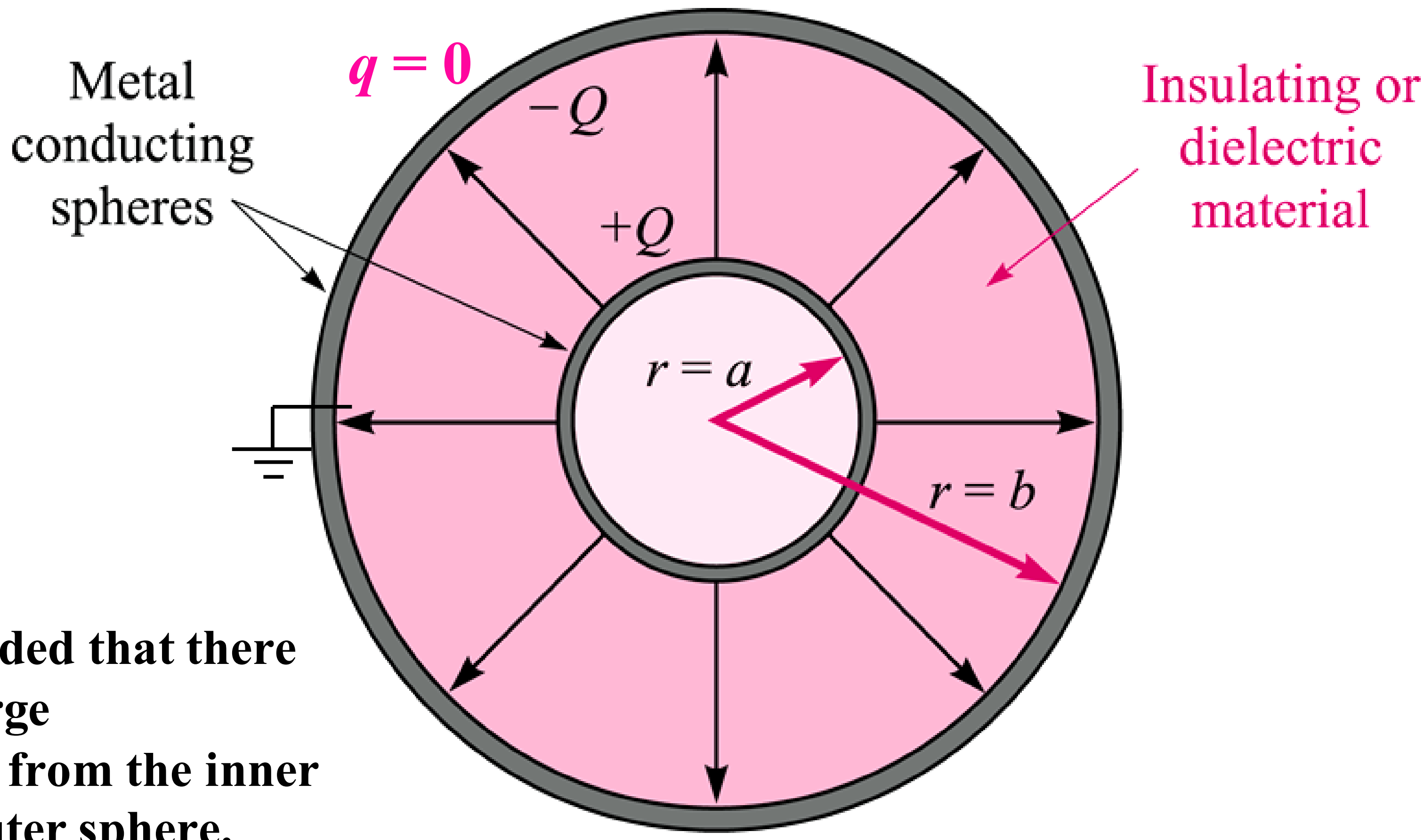
Conductor in static electric field situations

Faraday Apparatus, After Grounding



Attaching the ground connects the outer surface to an unlimited supply of free electrons, which then neutralize the positive charge layer. The net charge on the outer sphere is then the charge on the inner layer, or -Q.

Interpretation of the Faraday Experiment



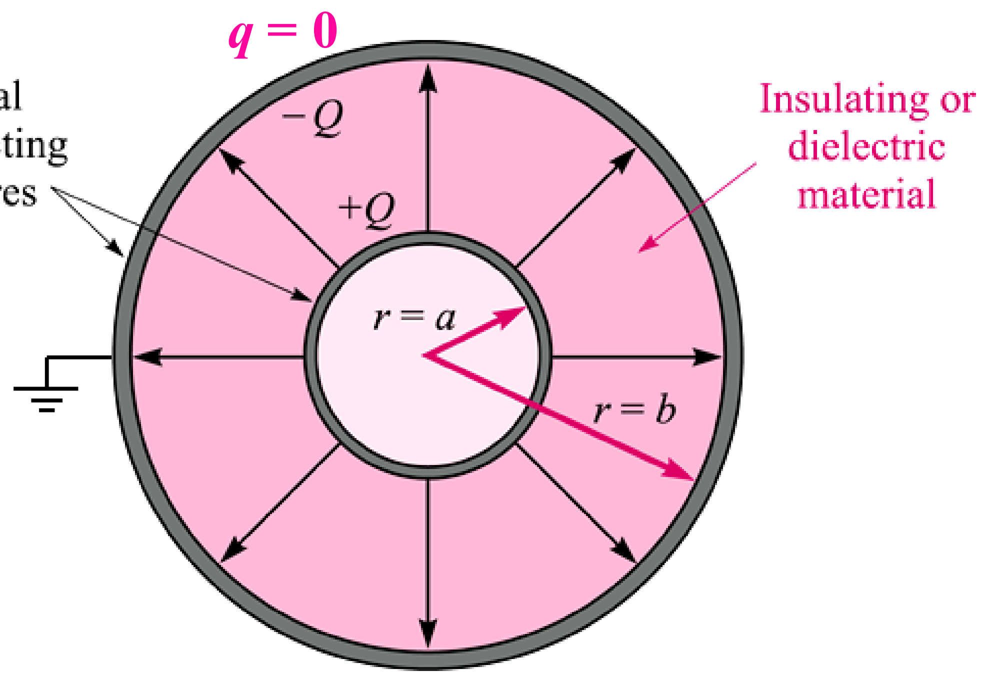
Faraday concluded that there occurred a charge "displacement" from the inner sphere to the outer sphere.

Displacement involves a *flow* or *flux*, Ψ, existing within the dielectric, and whose magnitude is equivalent to the amount of "displaced" charge.

Specifically:

$$\Psi = Q$$

Electric Flux Density



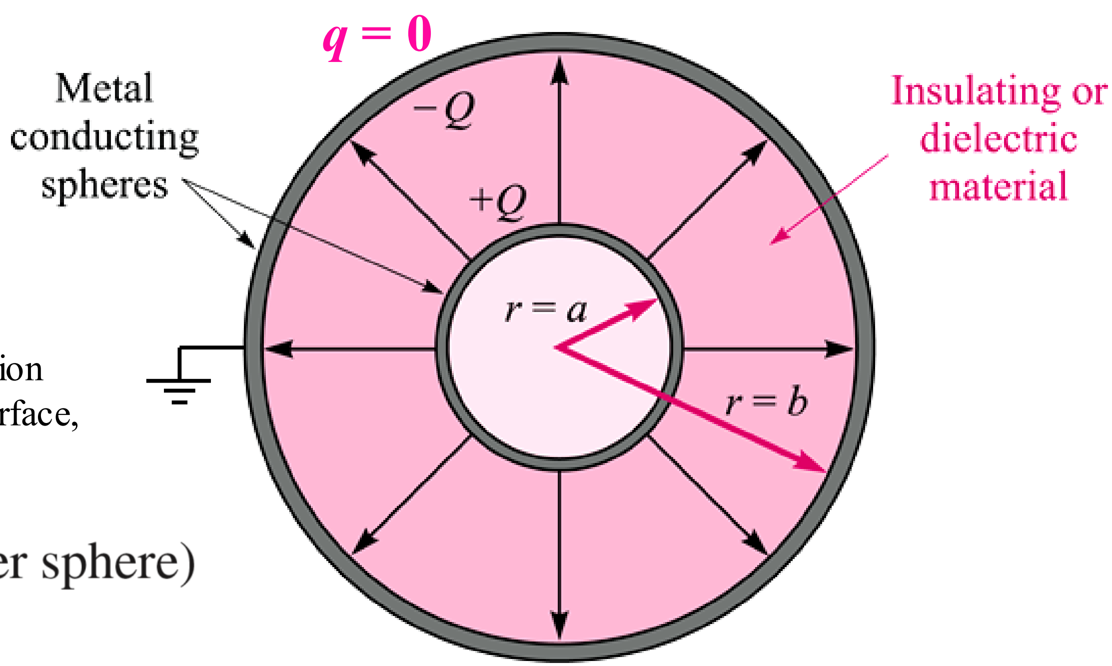
The density of flux at the inner sphere surface is equivalent to the density of charge there (in Coul/m^2)

$$D(r=a) = \frac{\Psi}{4\pi a^2} = \frac{Q}{4\pi a^2}$$

- ***Electric flux density***, measured in coulombs per square meter (sometimes described as "**lines per square meter**," for each line is due to one coulomb),
- ***Electric flux density*** is given the letter ***D***, which was originally chosen because of the alternate names of displacement flux density or displacement density

Vector Field Description of Flux Density

- The electric flux density ***D*** is a **vector field**.
- The direction of ***D*** at a point is the direction of the flux lines at that point, and
- The magnitude of ***D*** is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.



A vector field is established which points in the direction of the "flow" or displacement. In this case, the direction is the outward radial direction in spherical coordinates. At each surface, we would have:

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

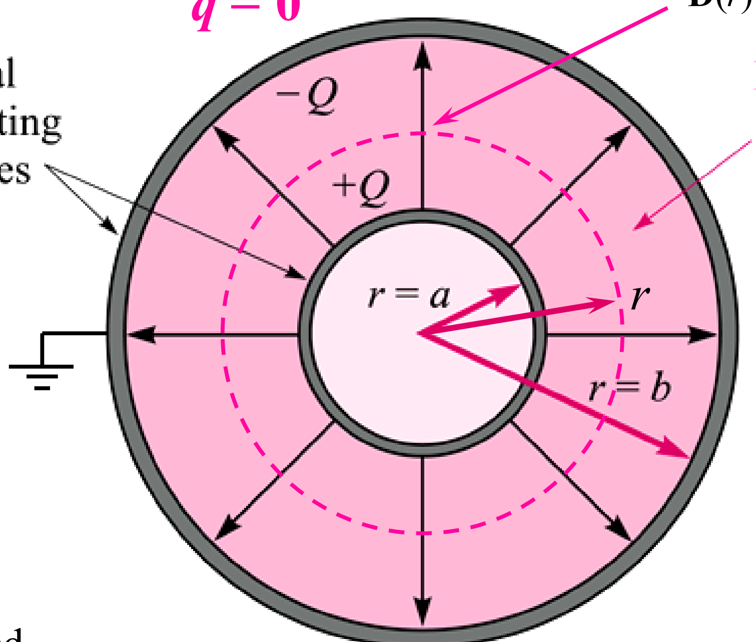
$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

Radially-Dependent Electric Flux Density

At a general radius r between spheres, we would have:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Expressed in units of Coulombs/m^2, and defined over the range ($a \leq r \leq b$)



Point Charge Fields

If we now let the inner sphere radius reduce to a point, while maintaining the same charge, and let the outer sphere radius approach infinity, we have a point charge. The electric flux density is unchanged, but is defined over all space:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \quad \text{C/m}^2 \quad (0 < r < \infty)$$

We compare this to the electric field intensity in free space:

$$\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r \quad \text{V/m} \quad (0 < r < \infty)$$

..and we see that: $\mathbf{D} = \epsilon_0 \mathbf{E}$ (free space only)

Finding E and D from Charge Distributions

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad \text{(free space only)}$$

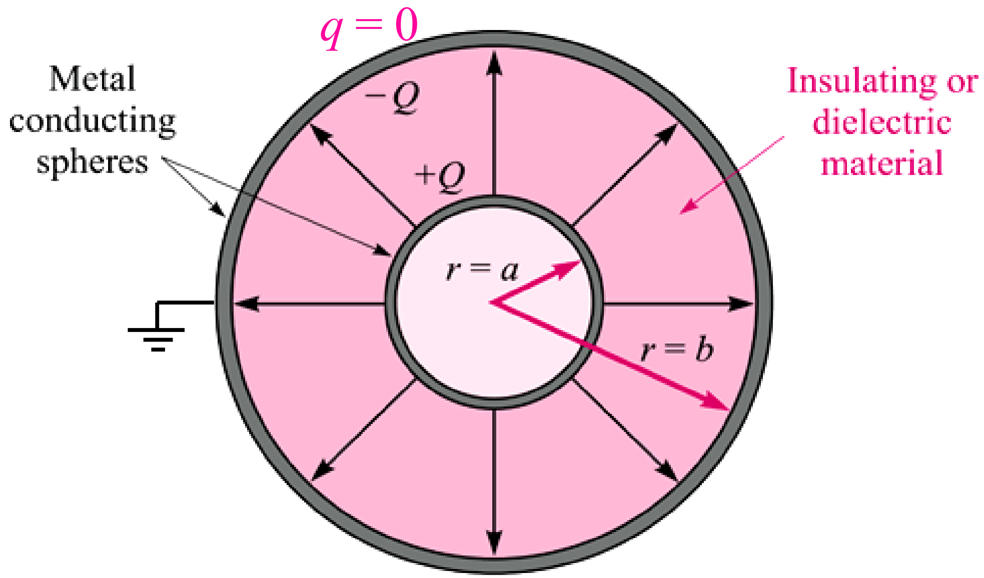
We learned in Chapter 2 that:

$$\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad \text{(free space only)}$$

It now follows that:

$$\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2} \mathbf{a}_R$$

Summary of Faraday's experiments



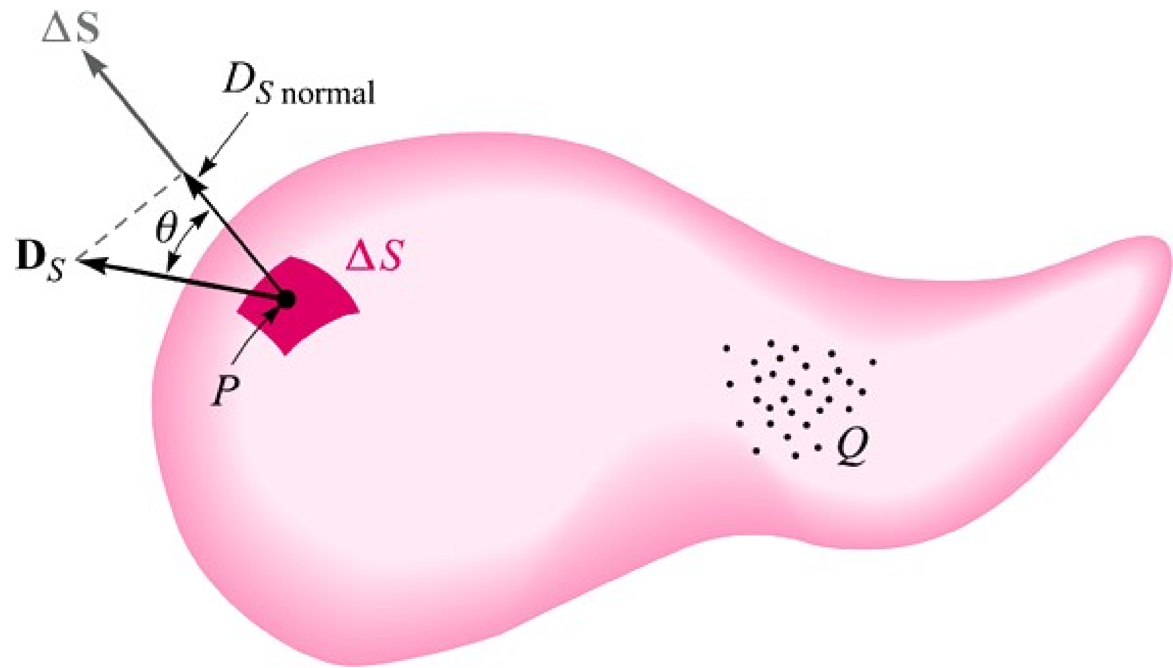
- The electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface.
- This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere.

Gauss' Law

The electric flux passing through any closed surface is

<u>equal</u>

to the total charge enclosed by that surface



- If the total charge is $\boldsymbol{Q}$, then $\boldsymbol{Q}$ coulombs of ***electric flux*** will pass through the enclosing surface.
- At every point on the surface the ***electric-flux-density vector D*** will have some value $\boldsymbol{D_S}$, where the subscript ***S*** merely reminds us that ***D*** must be <u>evaluated at the surface</u>, and
- $\boldsymbol{D_S}$ will in general vary in <u>magnitude</u> and <u>direction</u> <u>from one point on the surface to another</u>.

Development of Gauss' Law

At any point P, consider an incremental element of surface ΔS and let $\boldsymbol{D_S}$ make an angle θ with ΔS. The **flux** $\boldsymbol{\Delta\Psi}$ crossing ΔS is then the product of the normal component of $\boldsymbol{D_S}$ and ΔS

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta\,\Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

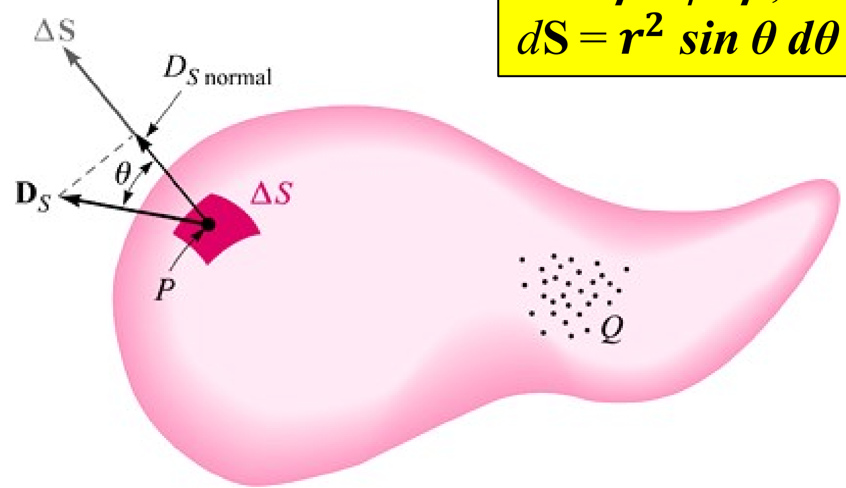
$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

$d\mathbf{S} = \boldsymbol{dx\;dy}$, in rectangular
$d\mathbf{S} = \boldsymbol{\rho\;d\varphi\;d\rho}$, in cylindrical or
$d\mathbf{S} = \boldsymbol{r^2\;\sin\theta\;d\theta\;d\varphi}$ in spherical

We define the differential surface area (a vector) as

$$d\mathbf{S} = \mathbf{n}\,dS$$

where **n** is the unit *outward* normal vector to the surface, and where dS is the area of the differential spot on the surface



Mathematical Statement of Gauss' Law

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

In which the charge can exist in the form of point charges: $Q = \sum Q_m$
or a continuous charge distribution:

Line charge: $Q = \int \rho_L \, dL$

Surface charge: $Q = \int_S \rho_S dS$ (not necessarily a closed surface)

Volume charge: $Q = \int_{\text{vol}} \rho_v \, dv$

For a volume charge, we would have:

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv$$

A mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

Using Gauss' Law to Solve for D Evaluated at a Surface

Knowing Q, we need to solve for **D**, using Gauss' Law:

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

where $d\mathbf{S} = \mathbf{n}\, dS$

The solution is easy if we can choose a surface, S, over which to integrate (Gaussian surface) that satisfies the following two conditions:

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

The integral now simplifies:

$$\oint_S \mathbf{D}_s \cdot d\mathbf{S} = \underbrace{\oint_S D_s\, dS}_{\text{Condition 1}} = \underbrace{D_s \oint_S dS}_{\text{Condition 2}} = Q$$

So that:

$$D_s = \frac{Q}{\oint_S dS}$$

Example: Point Charge Field

Begin with the radial flux density: $\mathbf{D} = \dfrac{Q}{4\pi r^2}\mathbf{a}_r$

and consider a spherical surface of radius a that surrounds the charge, on which:

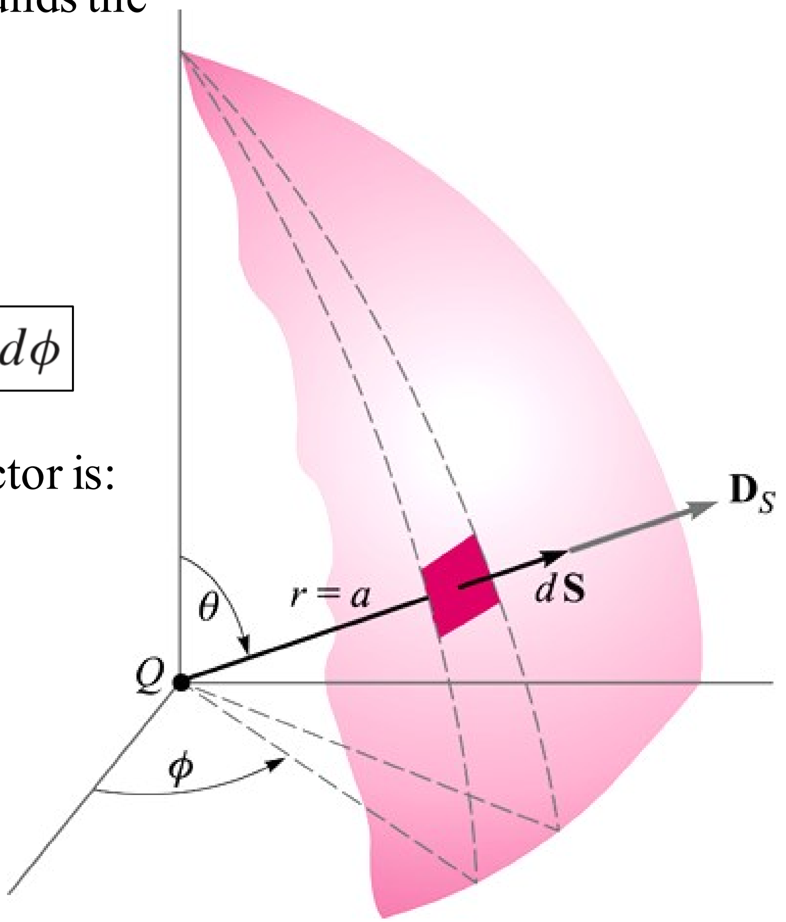
$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

On the surface, the differential area is:

$$\boxed{dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi}$$

and this, combined with the outward unit normal vector is:

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \; \mathbf{a}_r$$



$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r \qquad \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

Point Charge Application (continued)

Now, the integrand becomes:

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

and the integral is set up as:

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \frac{Q}{4\pi} \left(-\cos\theta \right)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = \underline{Q}$$

Another Example: Line Charge Field

The Gaussian surface for an infinite uniform line charge is a right circular cylinder of length L and radius ρ. ***D*** is **<u>constant</u>** in magnitude and everywhere perpendicular to the cylindrical surface; ***D*** is parallel to the end faces.

Consider a line charge of uniform charge density ρ_L on the z axis that extends over the range $-\infty < z < \infty$.

We need to choose an appropriate Gaussian surface, being mindful of these considerations:

1. With which coodinates does the field vary (or of what variables is D a function)?
2. Which components of **D** are present?

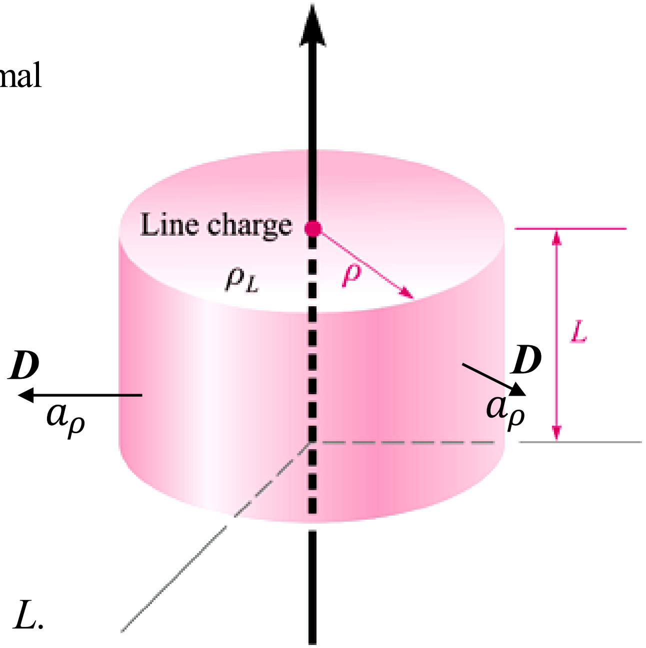
We know from symmetry that the field will be radially-directed (normal to the z axis) in cylindrical coordinates:

$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

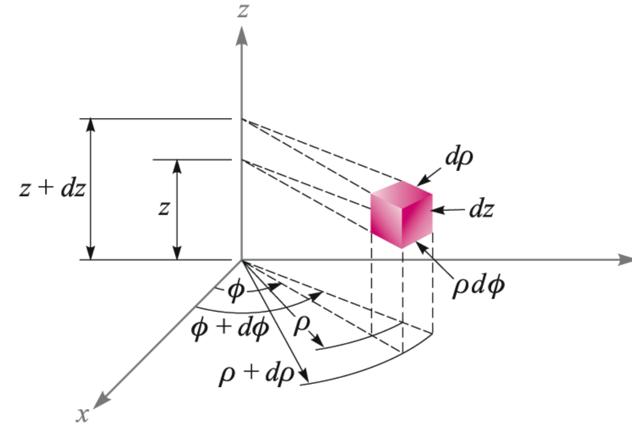
and that the field will vary with radius only:

$$D_\rho = f(\rho)$$

So we choose a cylindrical surface of radius ρ, and of length L.



Line Charge Field (continued)



We apply Gauss's law,

$$Q = \oint_{cyl} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{sides} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$$

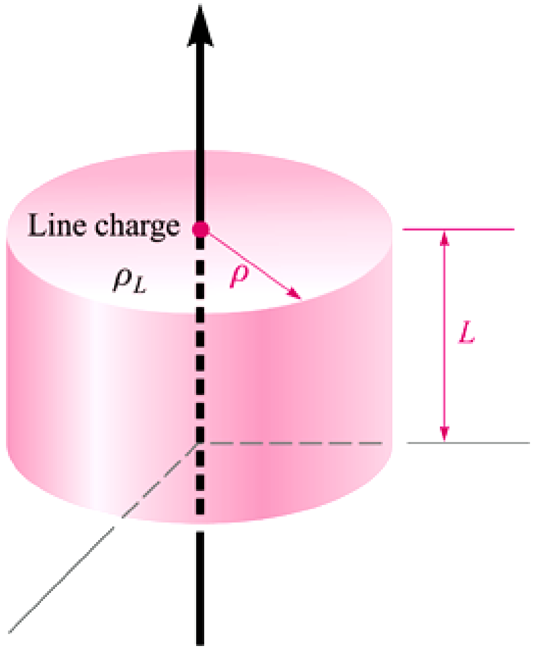
$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L$$

and obtain

$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is

$$Q = \rho_L L$$



Giving:

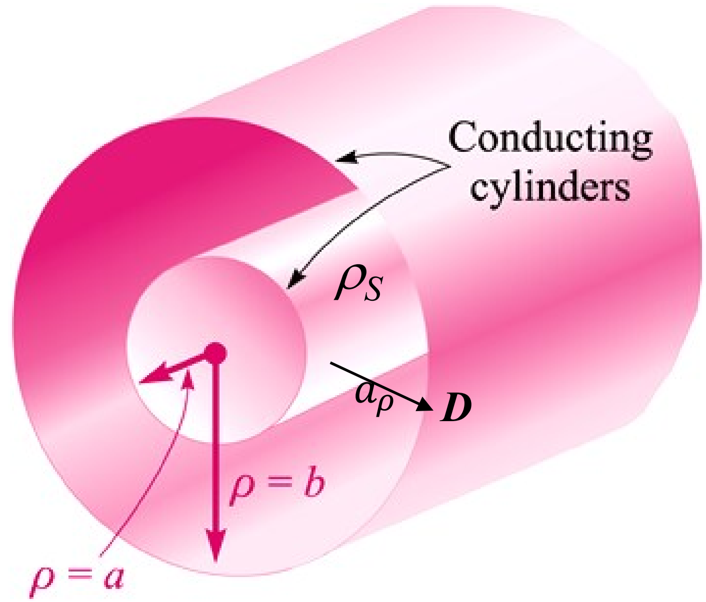
$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$\mathbf{D} = \epsilon_0 \mathbf{E}$ (free space only)

So that finally: $$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Another Example: Coaxial Transmission Line

We have two concentric cylinders, with the z axis down their centers. Surface charge of density ρ_S exists on the outer surface of the inner cylinder.



A ρ-directed field is expected, and this should vary only with ρ (like a line charge). We therefore choose a cylindrical Gaussian surface of length L and of radius ρ, where $a < \rho < b$.

The left hand side of Gauss' Law is written:
(Focusing on the electric flux at the **outer** surface of inner cylinder)

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_0^L \int_0^{2\pi} D_S\, \mathbf{a}_\rho \cdot \underbrace{\mathbf{a}_\rho\, \rho\, d\phi\, dz}_{d\mathbf{S}} = 2\pi \rho D_S L = Q$$

…and the right hand side becomes: (focusing on the charge inside the inner cylinder)

$$Q = \int_{\rho=a} \rho_S\, dS = \int_0^L \int_0^{2\pi} \rho_S\, a\, d\phi\, dz = 2\pi a L \rho_S$$

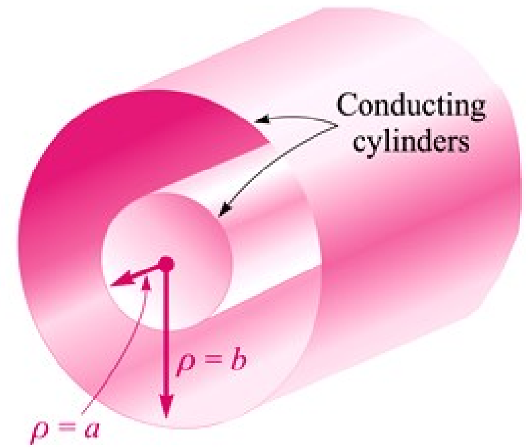
Coaxial Transmission Line (continued)

We may now solve for the flux density:

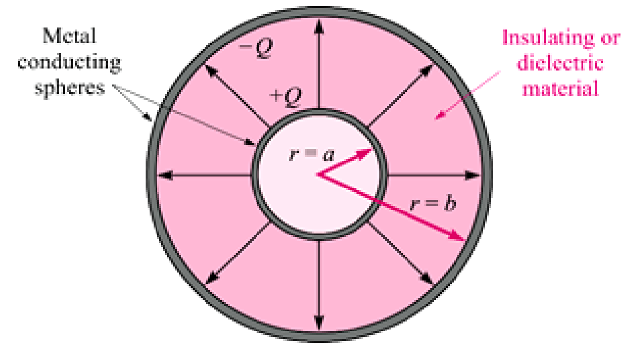
$$\mathbf{D}(\rho) = \frac{Q}{2\pi\rho L}\mathbf{a}_\rho = \frac{a\rho_S}{\rho}\mathbf{a}_\rho$$

and the electric field intensity becomes:

$$\mathbf{E} = \frac{a\rho_S}{\epsilon_0\rho}\mathbf{a}_\rho \ \mathrm{V/m} \qquad (a < \rho < b)$$



Coaxial Transmission Line: Exterior Field

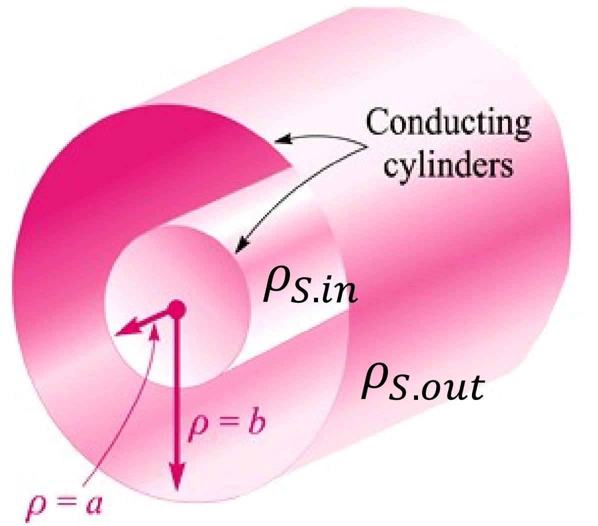


Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$



If a Gaussian cylindrical surface is drawn *outside* ($\rho > b$), a total charge of *zero* is enclosed, leading to the conclusion that

$$0 = D_S 2\pi \rho L \qquad (\rho > b)$$

or: $D_S = 0 \qquad (\rho > b)$

Example

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the **E** and **D** fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

$$\rho_{S,\text{inner cyl}} = \frac{Q_{\text{inner cyl}}}{2\pi a L} = \frac{30 \times 10^{-9}}{2\pi(10^{-3})(0.5)} = 9.55\ \mu\text{C/m}^2$$

The negative charge density on the inner surface of the outer cylinder is

$$\rho_{S,\text{outer cyl}} = \frac{Q_{\text{outer cyl}}}{2\pi b L} = \frac{-30 \times 10^{-9}}{2\pi(4 \times 10^{-3})(0.5)} = -2.39\ \mu\text{C/m}^2$$

The internal fields may therefore be calculated easily:

$$D_\rho = \frac{a\rho_S}{\rho} = \frac{10^{-3}(9.55 \times 10^{-6})}{\rho} = \frac{9.55}{\rho}\ \text{nC/m}^2$$

and

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{9.55 \times 10^{-9}}{8.854 \times 10^{-12}\rho} = \frac{1079}{\rho}\ \text{V/m}$$

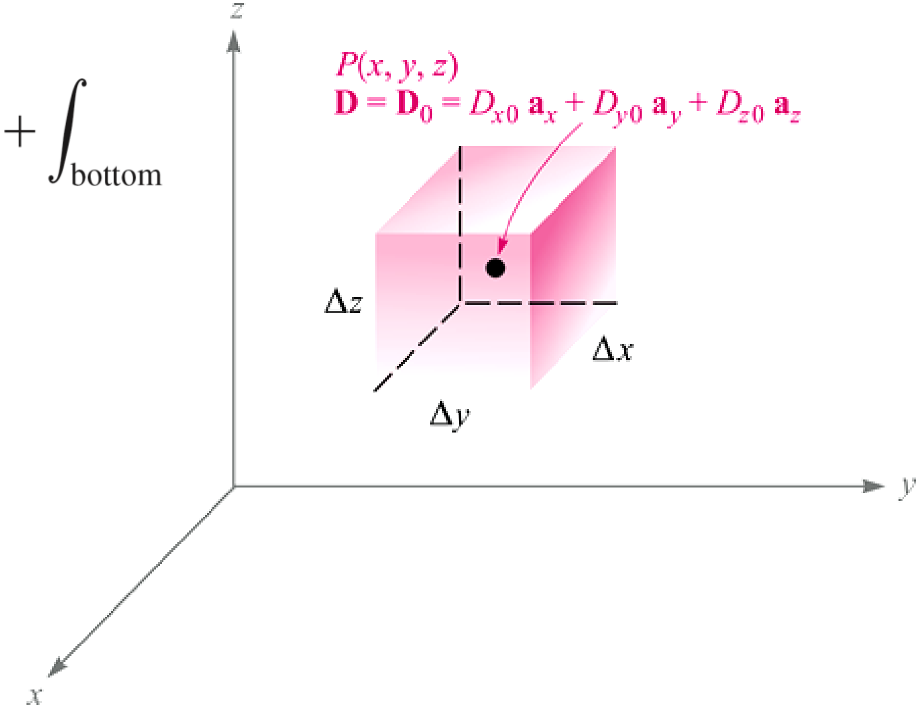
Both of these expressions apply to the region where $1 < \rho < 4$ mm. For $\rho < 1$ mm or $\rho > 4$ mm, **E** and **D** are zero.

Electric Flux Within a Differential Volume Element (For non-symmetrical closed surfaces)

The value of **D** at the point P may be expressed in rectangular components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face (**which are very small surfaces so *D* can be constant on them**)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$



Taking the front surface, for example, we have:

$$\int_{\text{front}} \doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}}$$

$$\doteq \mathbf{D}_{\text{front}} \cdot \Delta y\, \Delta z\, \mathbf{a}_x$$

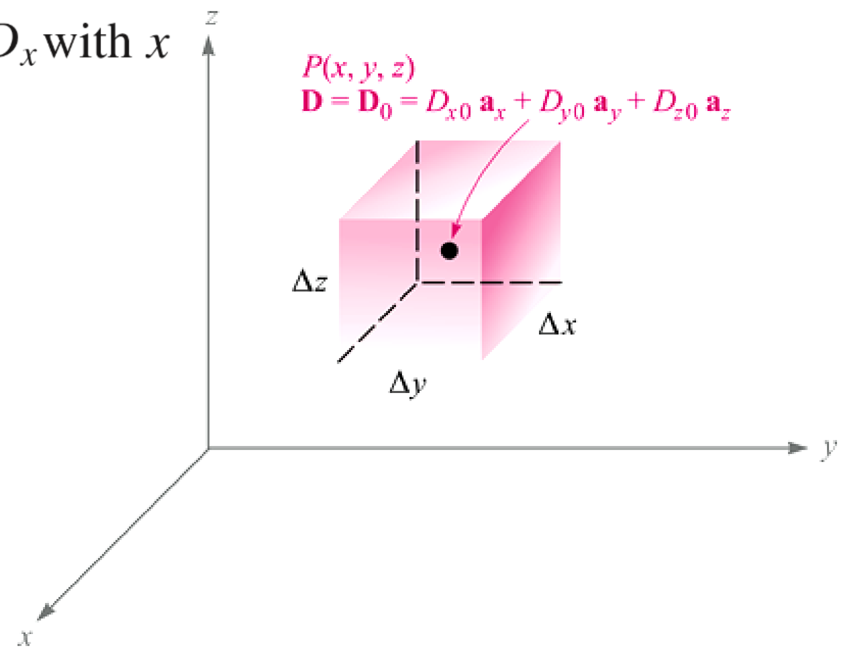
$$\doteq D_{x,\text{front}} \Delta y\, \Delta z$$

Electric Flux Within a Differential Volume Element

Notice that D_{x0} is the value of D_x at $\boldsymbol{P}$, and the distance of front face from $\boldsymbol{P}$ is $\frac{\boldsymbol{\Delta x}}{\mathbf{2}}$

$$D_{x,\text{front}} \doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$\doteq D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$



We now have: $$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

and in a similar manner:

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

minus sign because D_{x0} is *inward* flux through the back surface.

Therefore: $$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\Delta x\,\Delta y\,\Delta z$$

Charge Within a Differential Volume Element

Now, by a similar process, we find that:

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \; \Delta y \; \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \; \Delta y \; \Delta z$$

All results are assembled to yield an **approximation** which becomes better as Δv becomes smaller:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \underbrace{\Delta x \; \Delta y \; \Delta z}_{\Delta v} = Q \quad \text{(by Gauss' Law)}$$

where Q is the charge enclosed within the very small volume Δv

Example on Differential Volume Element

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z \quad = Q$$

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y \, \mathbf{a}_x - e^{-x} \cos y \, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta\nu$. If $\Delta\nu$ is 10^{-9} m^3, then we have enclosed about 2 nC.

The Del Operator

The del operator is a *vector differential operator*, and is defined as:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

Note that:

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div } \mathbf{D}$$

Divergence and Maxwell's First Equation

The divergence of the vector flux density **A** *is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

Mathematically, this is:

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

Applying divergence function, we have:

$$\text{div } \mathbf{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

and when the vector field is the electric flux density:

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v = \text{div } \mathbf{D}$$

Maxwell's first equation

Maxwell's first equation (or the point form of Gauss' Law)

$$\text{div}\,\mathbf{D} = \nabla \cdot \mathbf{D} = \rho_v$$

This is the first of Maxwell's four equations, and it states that

The electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

This equation is called the point form of Gauss's law. Gauss's law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point.

Divergence Theorem

We now have Maxwell's first equation (or the point form of Gauss' Law) which states:

$$\mathrm{div}\,\mathbf{D} = \nabla \cdot \mathbf{D} = \rho_v$$

and Gauss's Law in large-scale form reads:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\mathrm{vol}} \rho_v dv = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv$$

leading to the Divergence Theorem:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv$$

Divergence merely tells us
how much flux is leaving (or entering) a small volume
on a per-unit-volume basis; no direction is associated with it.

Divergence Theorem
(Gauss Divergence Theorem)

Gives us fast and easier ways to:

- Find relationship between surface and volume integrations.
- Find the locations of the charges' source or sink.
- Explain the rate of change of flux or field.
- Find out how much flux is leaving a source or entering a sink point.

Divergence Expressions in the Three Coordinate Systems

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \qquad \text{(rectangular)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r \sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

Sample Questions

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

3.2. An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2.

The flux density is $\mathbf{D} = \epsilon_0 \mathbf{E} = 5z^2\,\mathbf{a}_z$. As $\mathbf{D}$ is z-directed only, it will intersect only the top and bottom surfaces (both parallel to the x-y plane). From Gauss' law, the charge in the cube is equal to the net outward flux of $\mathbf{D}$, which in this case is

$$Q_{encl} = \oint \mathbf{D} \cdot \mathbf{n}\, da = \int_{-2}^{2}\int_{-2}^{2} 5(2)^2 \mathbf{a}_z \cdot \mathbf{a}_z\, dx\, dy + \int_{-2}^{2}\int_{-2}^{2} 5(-2)^2 \mathbf{a}_z \cdot (-\mathbf{a}_z)\, dx\, dy = \underline{0}$$

where the first and second integrals on the far right are over the top and bottom surfaces respectively.

$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

3.4. An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin. Using Gauss' law, we set up the integral in free space over the sphere surface, whose outward unit normal is $\mathbf{a}_r$:

$$Q = \oint \epsilon_0 \mathbf{E} \cdot \mathbf{n}\, da = \int_0^{2\pi} \int_0^{\pi} 5z^3\, \mathbf{a}_z \cdot \mathbf{a}_r\, (3)^2 \sin\theta\, d\theta\, d\phi$$

where in this case $z = 3\cos\theta$ and (in all cases) $\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$. These are substituted to yield

$$Q = 2\pi \int_0^{\pi} 5(3)^5 \cos^4\theta \sin\theta d\theta = -2\pi(5)(3)^5 \left(\frac{1}{5}\right) \cos^5\theta \Big|_0^{2\pi} = \underline{972\pi}$$

3.10. An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, $\mathbf{E}$, both inside and outside the cylinder.

Inside, we note from symmetry that $\mathbf{D}$ will be radially-directed, in the manner of a line charge field. So we apply Gauss' law to a cylindrical surface of radius ρ, concentric with the charge distribution, having unit length in z, and where $\rho < b$. The outward normal to the surface is $\mathbf{a}_\rho$.

$$\oint \mathbf{D}\cdot\mathbf{n}\,da = \int_0^1\int_0^{2\pi} D_\rho\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,\rho\,d\phi\,dz = Q_{encl} = \int_0^1\int_0^{2\pi}\int_0^\rho a(\rho')^2\rho'\,d\rho'\,d\phi\,dz$$

in which the dummy variable ρ' must be used in the far-right integral because the upper radial limit is ρ. D_ρ is constant over the surface and can be factored outside the integral. Evaluating both integrals leads to

$$2\pi(1)\rho D_\rho = 2\pi a\left(\frac{1}{4}\right)\rho^4 \;\Rightarrow\; D_\rho = \frac{a\rho^3}{4} \quad\text{or}\quad \underline{\mathbf{E}_{in} = \frac{a\rho^3}{4\epsilon_0}\mathbf{a}_\rho\ (\rho < b)}$$

To find the field outside the cylinder, we apply Gauss' law to a cylinder of radius $\rho > b$. The setup now changes only by the upper radius limit for the charge integral, which is now the charge radius, b:

$$\oint \mathbf{D}\cdot\mathbf{n}\,da = \int_0^1\int_0^{2\pi} D_\rho\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,\rho\,d\phi\,dz = Q_{encl} = \int_0^1\int_0^{2\pi}\int_0^b a\rho^2\rho\,d\rho\,d\phi\,dz$$

where the dummy variable is no longer needed. Evaluating as before, the result is

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v\,dv = \int_{\text{vol}} \nabla\cdot\mathbf{D}\,dv$$

$$D_\rho = \frac{ab^4}{4\rho} \quad\text{or}\quad \underline{\mathbf{E}_{out} = \frac{ab^4}{4\epsilon_0\rho}\mathbf{a}_\rho\ (\rho > b)}$$

3.13. Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{s0}, respectively.

a) Find **D** at $r = 1$, 3 and 5 m: Noting that the charges are spherically-symmetric, we ascertain that **D** will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{80 \times 10^{-9}}{r^2} \text{ C/m}^2$$

So $D_r(r = 3) = \underline{8.9 \times 10^{-9}\,\text{C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) + 4\pi(4)^2(-4 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{16 \times 10^{-9}}{r^2}$$

So $D_r(r = 5) = \underline{6.4 \times 10^{-10}\,\text{C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16 \times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9) \times 10^{-9}\,\text{C/m}^2}$.

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

3.16. An electric flux density is given by $\mathbf{D} = D_0\,\mathbf{a}_\rho$, where D_0 is a given constant.

a) What charge density generates this field? Charge density is found by taking the divergence: With radial **D** only, we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho D_0) = \frac{D_0}{\rho}\ \text{C/m}^3$$

b) For the specified field, what total charge is contained within a cylinder of radius a and height b, where the cylinder axis is the z axis? We can either integrate the charge density over the specified volume, or integrate **D** over the surface that contains the specified volume:

$$Q = \int_0^b \int_0^{2\pi} \int_0^a \frac{D_0}{\rho}\,\rho\,d\rho\,d\phi\,dz = \int_0^b \int_0^{2\pi} D_0\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,a\,d\phi\,dz = \underline{2\pi a b D_0}\ \text{C}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \qquad \text{(spherical)}$$

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3\,\mathbf{a}_r \ \text{C/m}^2$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6)\ \text{C/m}^3$$

which we evaluate at $r = 4$ to find $\rho_v(r=4) = \underline{17.50\ \text{C/m}^3}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r\ \text{C/m}^2}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part b, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi\ \text{C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi\ \text{C}}$.