Engineering Electromagnetics

Chapter 5:

Conductors and Dielectrics

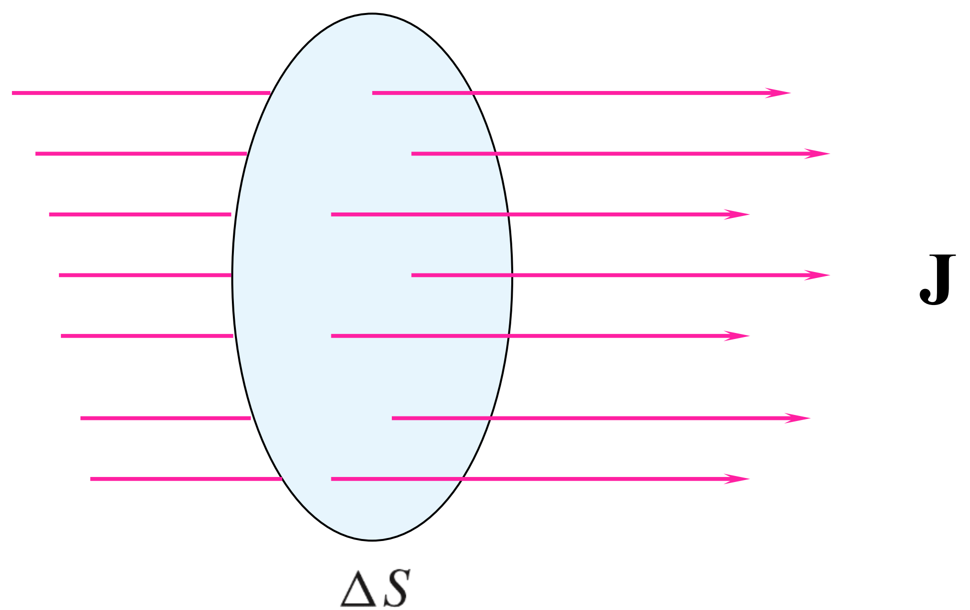
Current and Current Density

Electric charges in motion constitute a current.

Current is a flux quantity and is defined as: $I = \frac{dQ}{dt}$

Current density, J, it is a vector quantity, measured in Amps/m^2 , yields current in Ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second.

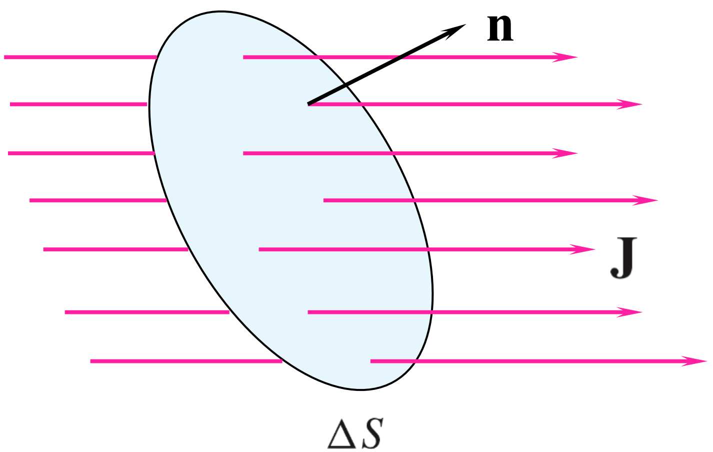
The direction of *J* is normal to the surface, and so: $\Delta I = J_N \Delta S$



Current Density as a Vector Field

In reality, the direction of current flow may not be normal to the surface in question, so we treat current density as a **vector**, and write the incremental flux through the small surface in the usual way:

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$ where $\Delta\mathbf{S} = \mathbf{n}\, \mathrm{d}a$



Then, the current through a large surface is found through the flux integral:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Relation of **Current** to Charge Velocity

Consider a charge ΔQ, occupying volume Δv, moving in the positive x direction at velocity v_x

In terms of the **volume charge density**, we may write:

$$\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$$

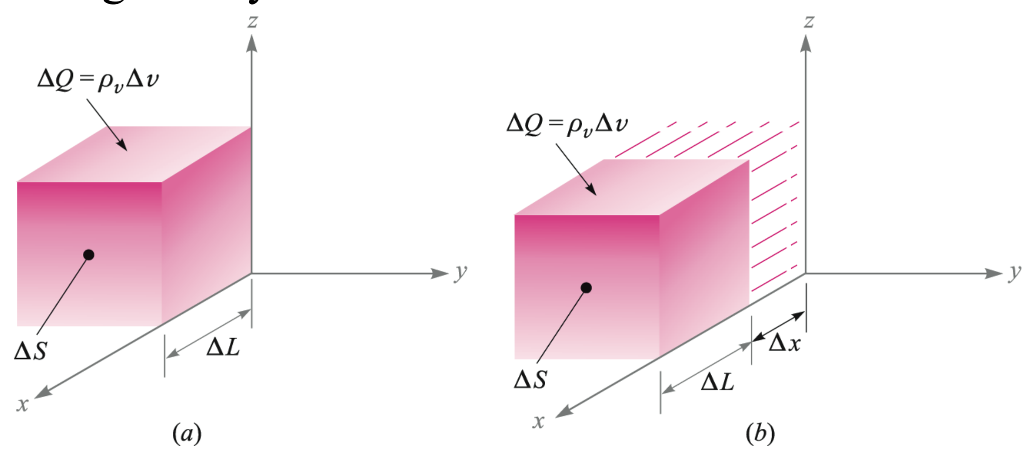
Suppose that **in time Δt,** the charge moves through a distance $\Delta x = \Delta L = v_x \, \Delta t$

Then $\quad \Delta Q = \rho_v \, \Delta S \, \Delta x$

The motion of the charge represents a current given by:

$$\boxed{\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}}$$

or $\quad \boxed{\Delta I = \rho_v \, \Delta S \, v_x}$



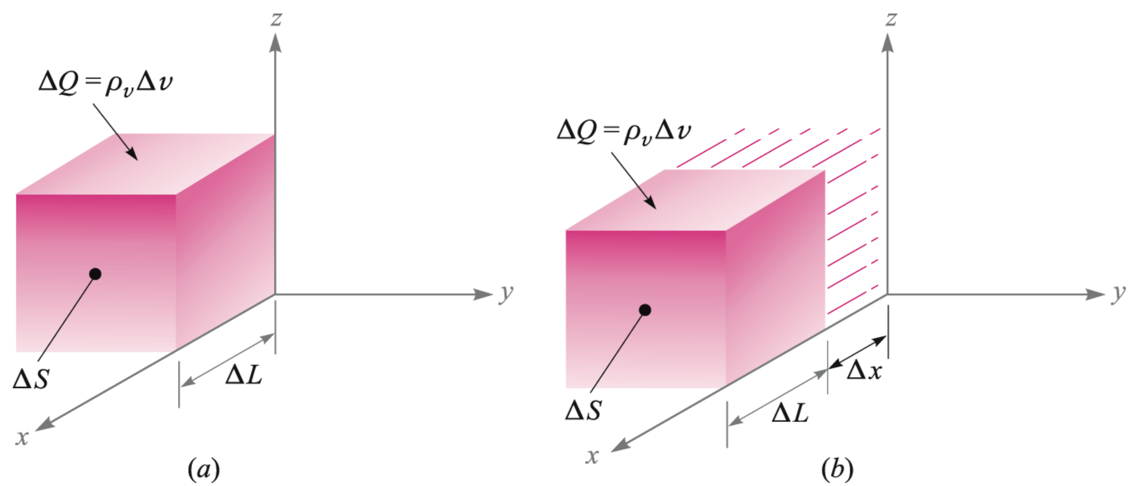
Relation of **Current Density** to Charge Velocity

We now have $\Delta I = \rho_v \, \Delta S \, v_x$

The current density is then: $$J_x = \frac{\Delta Q}{\Delta S} = \rho_v v_x$$

So that in general:

$$\mathbf{J} = \rho_v \mathbf{v}$$

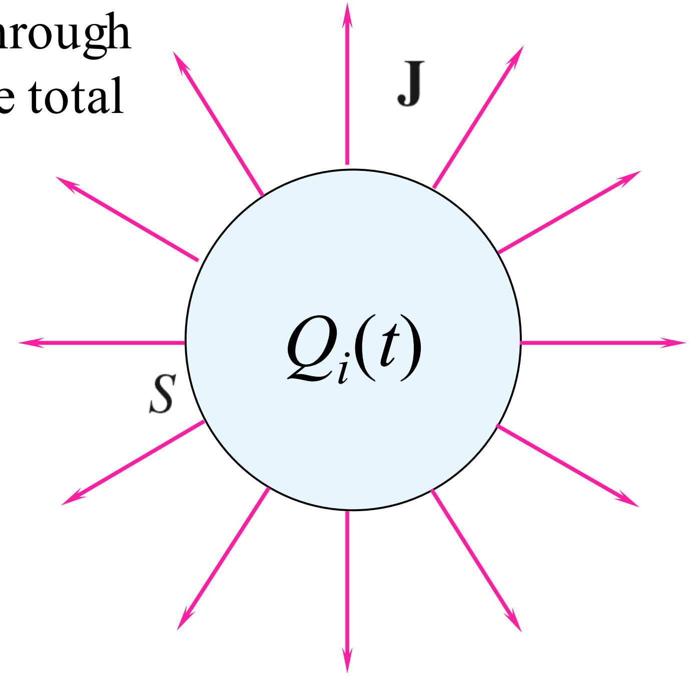


The principle of conservation of charge states simply that:

charges can be neither created nor destroyed, although equal amounts of positive and negative charge may be simultaneously created, obtained by separation, or lost by recombination.

Continuity of Current

Suppose that charge Q_i is escaping from a volume through closed surface S, to form current density **J**. Then the total current is:



$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{\text{vol}} \rho_\nu \, dv$$

where the minus sign is needed to produce ***positive* outward flux,** while the interior charge is *decreasing* with time.

We now apply the **divergence** theorem:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv$$

so that $$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = -\frac{d}{dt}\int_{\text{vol}} \rho_\nu \, dv$$

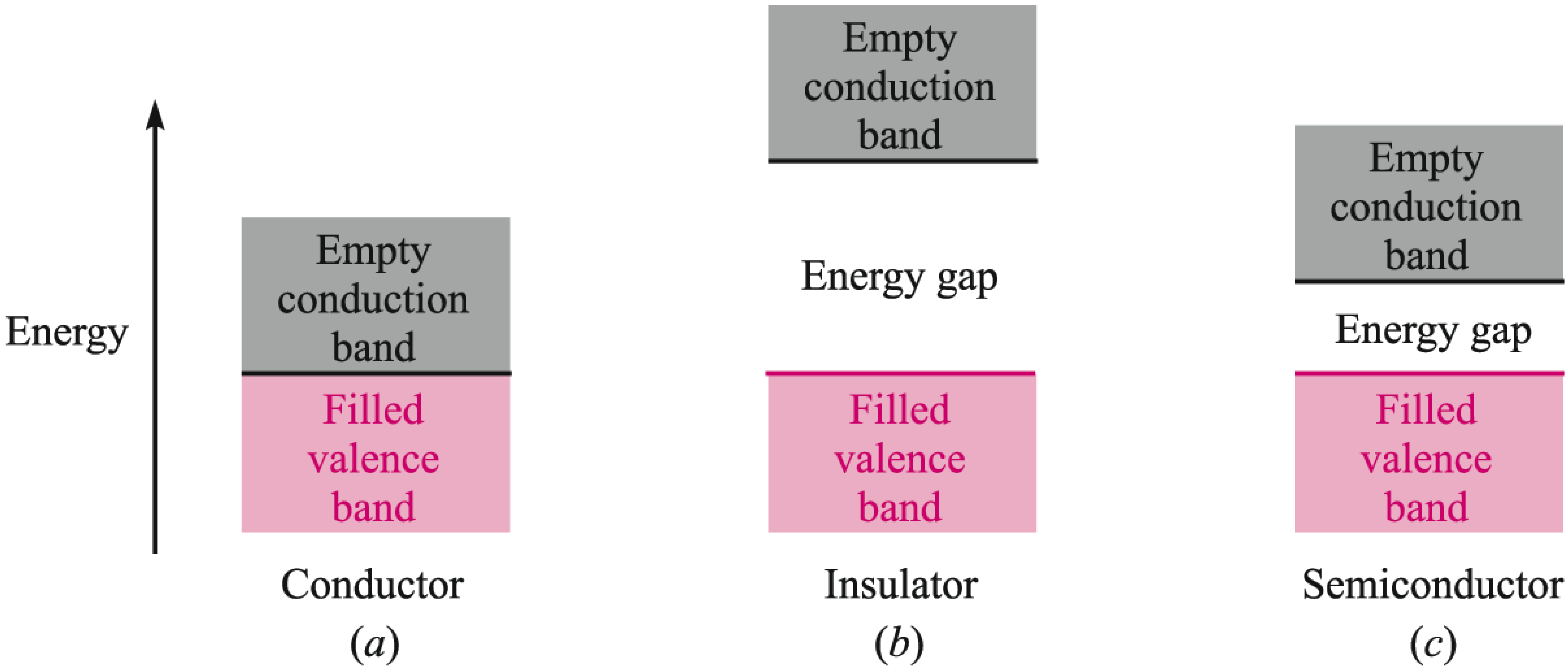
or $$\int_{\text{vol}} (\nabla \cdot \mathbf{J})\, dv = \int_{\text{vol}} -\frac{\partial \rho_\nu}{\partial t}\, dv$$

The integrands of the last expression must be equal, leading to the

Equation of Continuity

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_\nu}{\partial t}$$

Energy Band Structure in Three Material Types



a) Conductors exhibit no energy gap between valence and conduction bands so electrons move freely

b) Insulators show large energy gaps, requiring large amounts of energy to lift electrons into the conduction band. When this occurs, the dielectric breaks down.

c) Semiconductors have a relatively small energy gap, so modest amounts of energy (applied through heat, light, or an electric field) may lift electrons from valence to conduction bands.

Electron Flow in Conductors

In the conductor the valence electrons, or conduction, or free electrons, move under the influence of an electric field. The applied force on an electron of charge $Q = -e$ will be

$$\mathbf{F} = -e\mathbf{E}$$

When forced, the electron accelerates to an equilibrium velocity, known as the *drift velocity*:

$$\mathbf{v}_d = -\mu_e \mathbf{E}$$

where μ_e is the electron *mobility,* expressed in units of m^2/V-s. The drift velocity is used to find the current density through:

$$\mathbf{J} = \rho_v \mathbf{v}_d = -\rho_e \mu_e \mathbf{E} = \sigma \mathbf{E}$$

from which we identify the *conductivity* for the case of electron flow :

$$\sigma = -\rho_e \mu_e \quad \text{S/m}$$

One siemens (1 S) is defined as one ampere per volt.

The expression:

$$\mathbf{J} = \sigma \mathbf{E}$$

is Ohm's Law in point form

In a semiconductor, we have hole current as well, and

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

3.82 × 107 for aluminum,
5.80 × 107 for copper, and
6.17 × 107 for silver

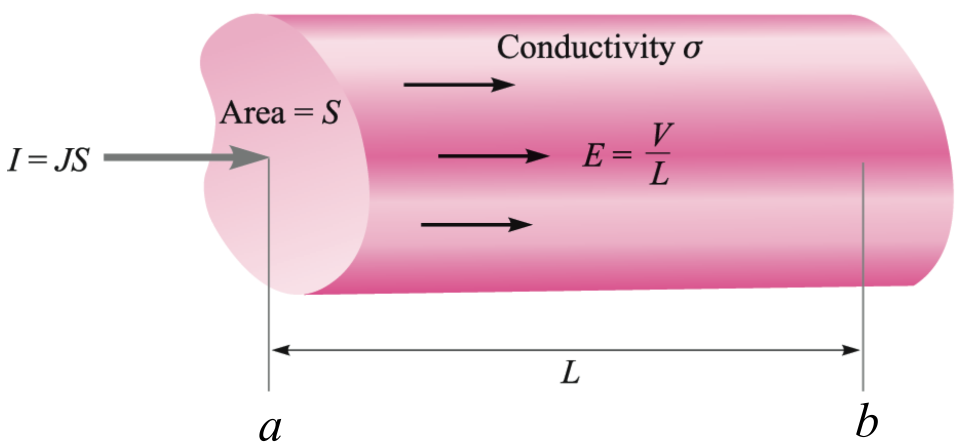
Resistance

Consider the cylindrical conductor shown here, with voltage V applied across the ends. Current flows down the length, and is assumed to be uniformly distributed over the cross-section, S.

First, we can write the voltage and current in the cylinder in terms of field quantities:

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba} = \mathbf{E} \cdot \mathbf{L}_{ab} \Longrightarrow V = EL$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS \Longrightarrow J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L} \Longrightarrow V = \frac{L}{\sigma S} I$$



Using Ohm's Law:

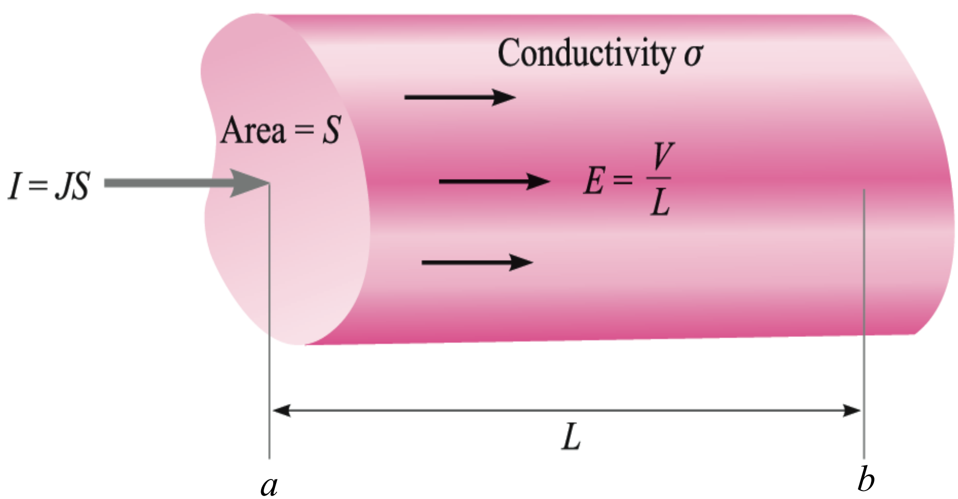
$$V = IR$$

We find the resistance of the cylinder:

$$R = \frac{L}{\sigma S}$$

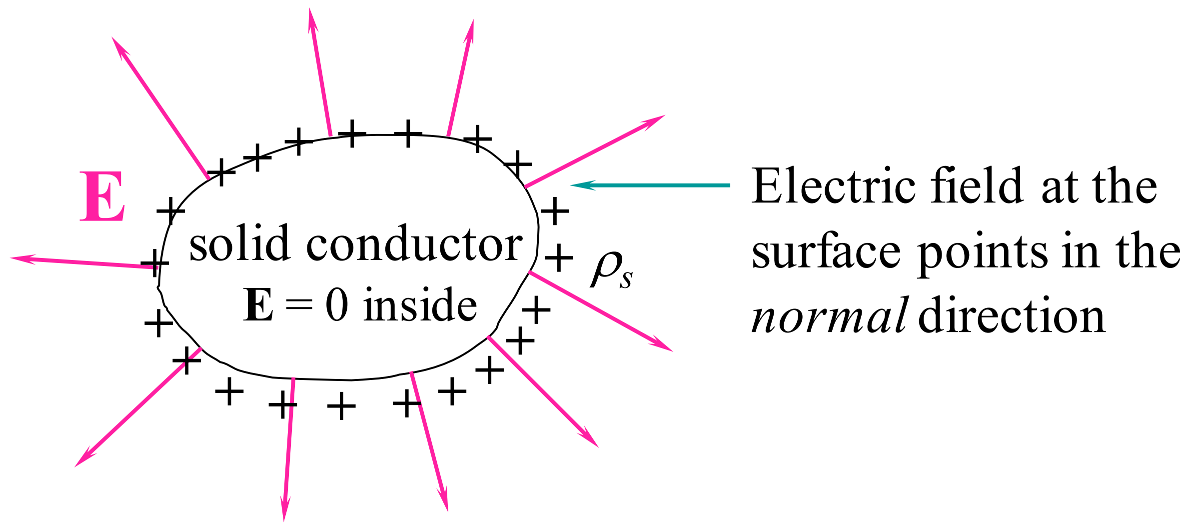
General Expression for Resistance

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}}$$



Electrostatic Properties of Conductors

Consider a conductor, on which *excess* charge has been placed



1. **Charge can exist only on the surface as a surface charge density, ρ_s -- *not* in the interior.**

2. **Electric field *cannot* exist in the interior, nor can it possess a tangential component at the surface.**

3. **The surface of a conductor is an *equipotential*.**

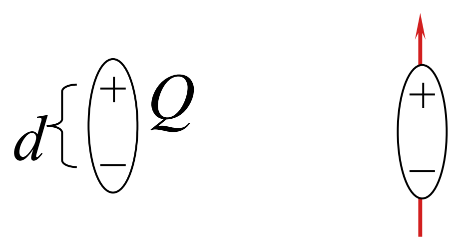
Electric Dipole and Dipole Moment

In dielectric,

1.Charges are held in position (bound), and ideally **there are no free charges that can move and form a current.**

2.Atoms and molecules may be polar (**having separated positive and negative charges**), or may be polarized by the application of an electric field.

dipole moment



$$\mathbf{p} = Qd\,\mathbf{a}_x$$

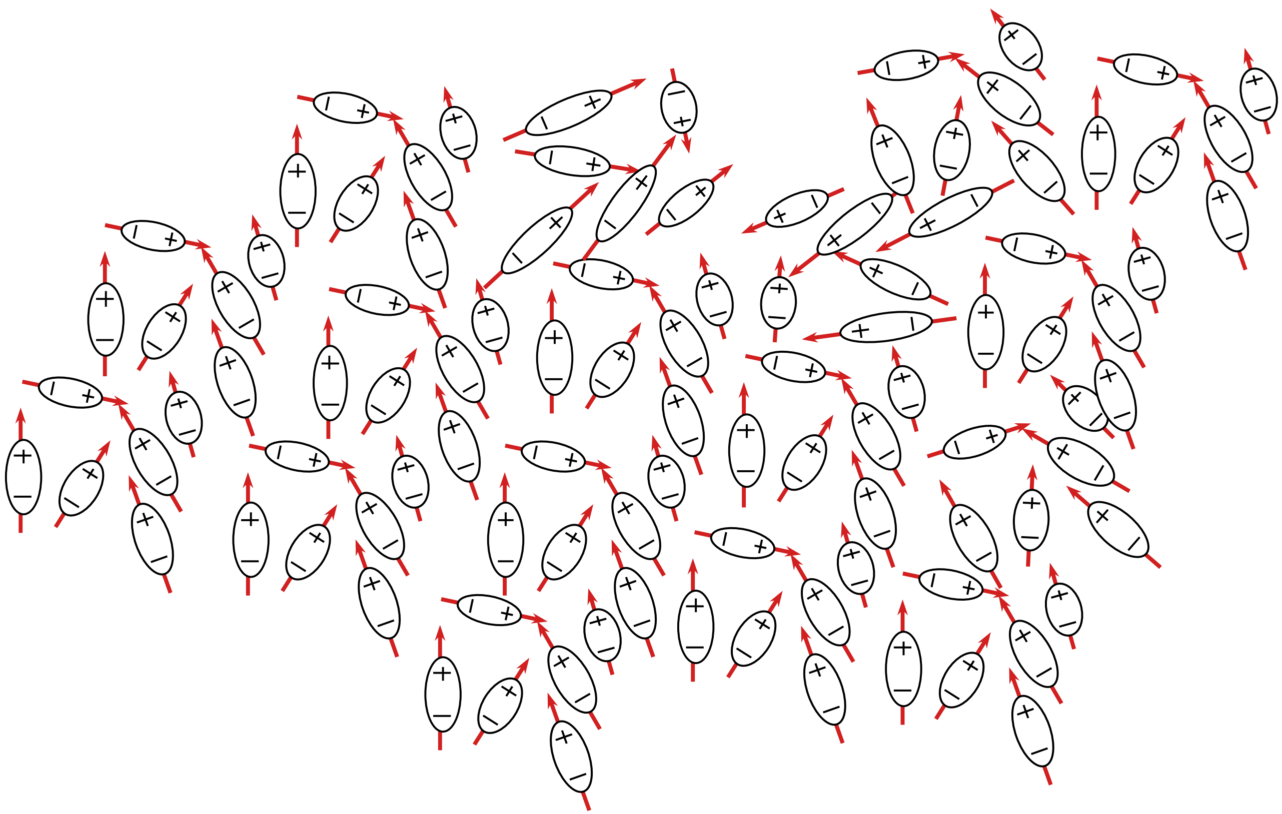
Consider such a polarized atom or molecule, which possesses a *dipole moment*, **p**, defined as the charge magnitude present, Q, times the positive and negative charge separation, d.

Dipole moment is a vector that points from the negative to the positive charge.

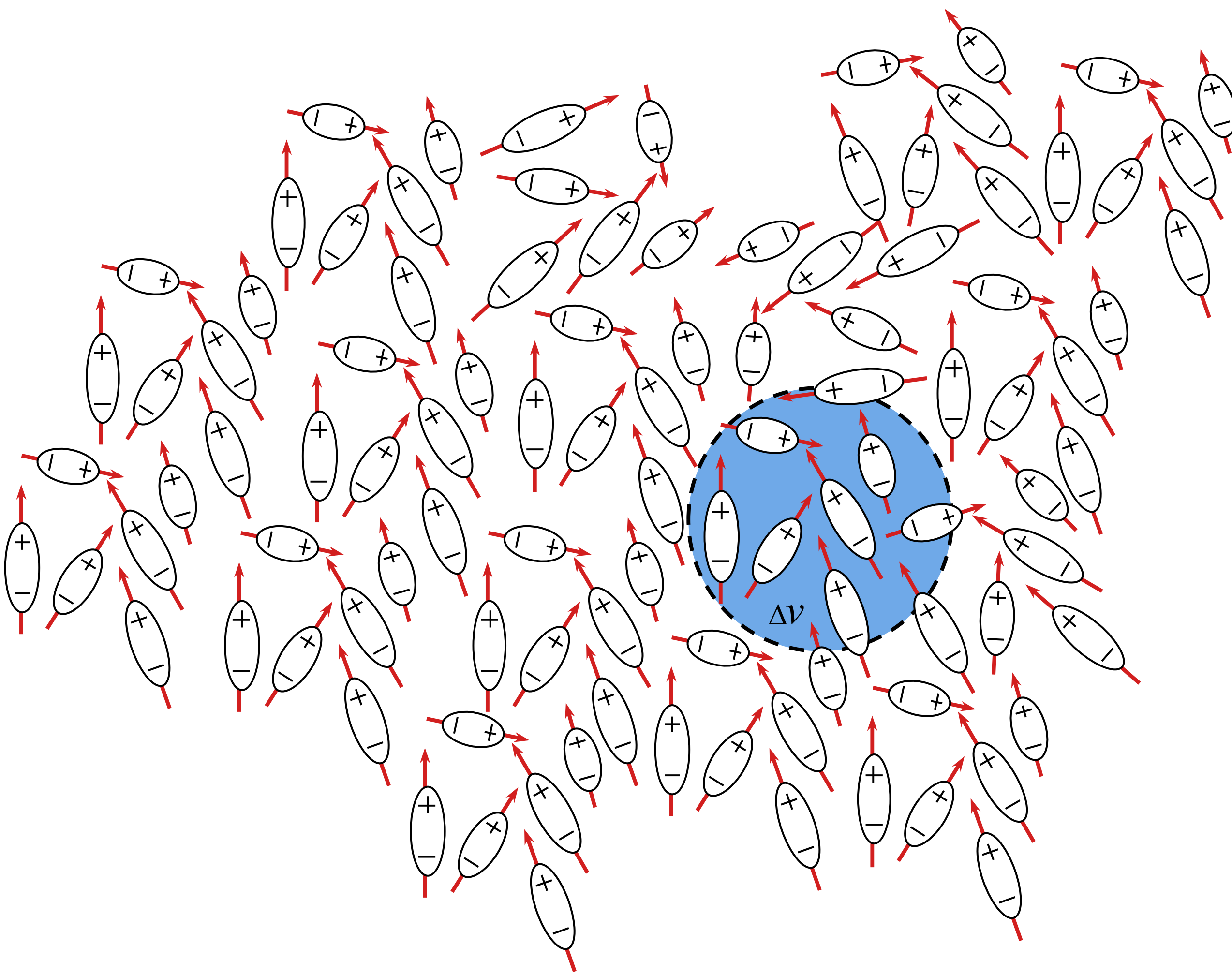
Model of a Dielectric

A dielectric can be modeled as **<u>group of bound charges in free space</u>**, associated with the atoms and molecules that make up the material.

Some of these may have dipole moments, others not. In some materials (such as liquids), dipole moments are in random directions.



Polarization Field



The number of dipoles is expressed as a density, *n* dipoles per unit volume.

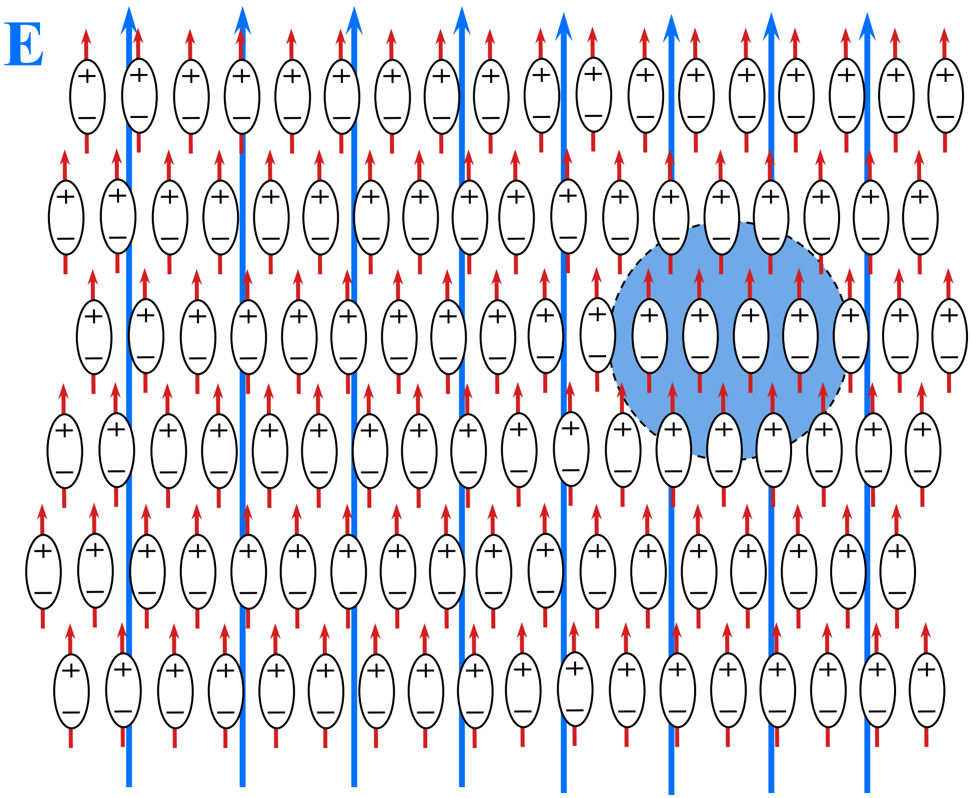
The *Polarization Field* of the medium is defined as:

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

[dipole moment/vol]
or
[C/m²]

Polarization Field (with Electric Field Applied)

Introducing an electric field may increase the charge separation in each dipole, and possibly re-orient dipoles so that there is some aggregate alignment, as shown here. The effect is small, and is greatly exaggerated here!



The effect is to increase **P**.

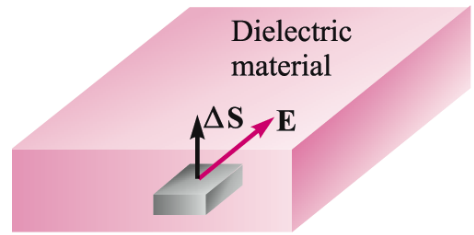
$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

$$= n\mathbf{p}$$

if all dipoles are identical

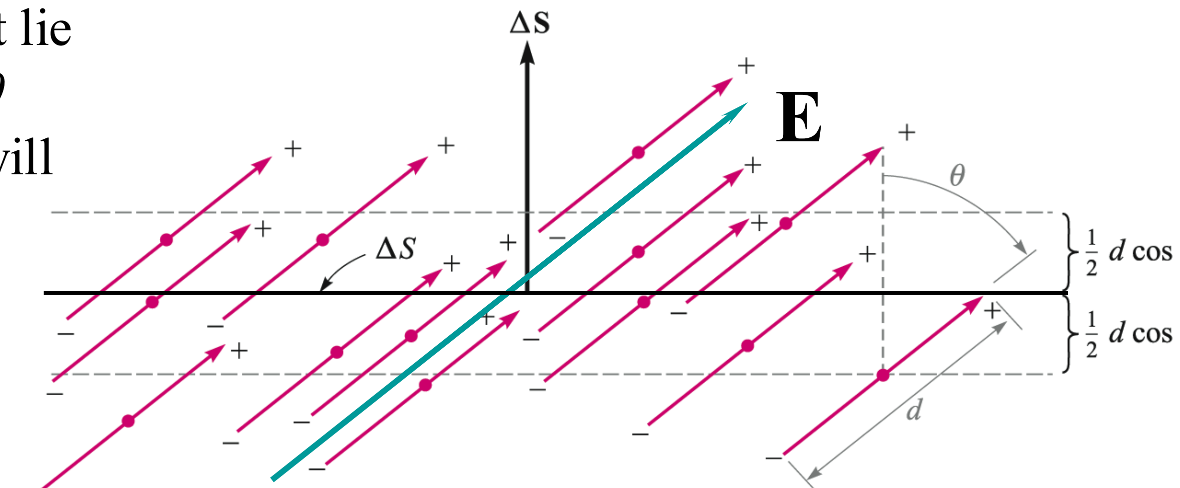
Migration of Bound Charge

Consider an electric field applied at an angle θ to a surface normal as shown. The resulting separation of bound charges (or re-orientation) leads to positive bound charge crossing upward through surface of area ΔS, while negative bound charge crosses downward through the surface.



(a)

Dipole centers (red dots) that lie within the range (1/2) $d \cos\theta$ above or below the surface will transfer charge across the surface.



(b)

Exercises

5.2. Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)\,\text{A/m}^2$, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

At $y = 0$, $\mathbf{J}(x, 0) = -10^4 x\mathbf{a}_y$, so that the current through the plane becomes

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_0^1 \int_0^2 -10^4 x\,\mathbf{a}_y \cdot (-\mathbf{a}_y)\,dx\,dz = \underline{2 \times 10^{-4}}\ \text{A}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2 + 4}\,\mathbf{a}_r \ \text{A/m}^2$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} \frac{400\sin\theta}{(.8)^2+4}(.8)^2 \sin\theta\, d\theta\, d\phi = \frac{400(.8)^2 2\pi}{4.64}\int_{.1\pi}^{.3\pi} \sin^2\, d\theta$$
$$= 346.5\int_{.1\pi}^{.3\pi} \frac{1}{2}[1-\cos(2\theta)]\,d\theta = \underline{77.4\ \text{A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta\, d\theta\, d\phi = 1.46\,\text{m}^2$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\,\mathbf{a}_r = \underline{53.0\,\mathbf{a}_r\ \text{A/m}^2}$.

5.4. If volume charge density is given as $\rho_v = (\cos\omega t)/r^2$ C/m^3 in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

We use the continuity equation (5), along with the assumption of no angular variation to write

$$\nabla \cdot \mathbf{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_r\right) = -\frac{\partial \rho_v}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\cos\omega t}{r^2}\right) = \frac{\omega \sin\omega t}{r^2}$$

So we may now solve

$$\frac{\partial}{\partial r}\left(r^2 J_r\right) = \omega \sin\omega t$$

by direct integration to obtain:

$$\mathbf{J} = J_r\,\mathbf{a}_r = \frac{\omega \sin\omega t}{r}\,\mathbf{a}_r \text{ A/m}^2$$

where the integration constant is set to zero because a steady current will not be created by a time-varying charge density.