Engineering Electromagnetics

Chapter 6:

Capacitance

Capacitance Defined

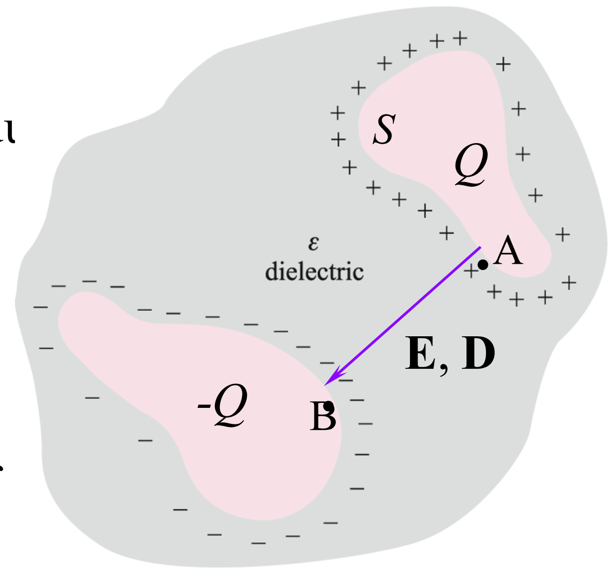
A simple capacitor consists of two oppositely charged conductors surrounded by a uniform dielectric.

An increase in Q by some factor results in an increase in **E** (and in **D**) by the same factor.

where $Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$

Consequently, the potential difference between condu

$$V_0 = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$



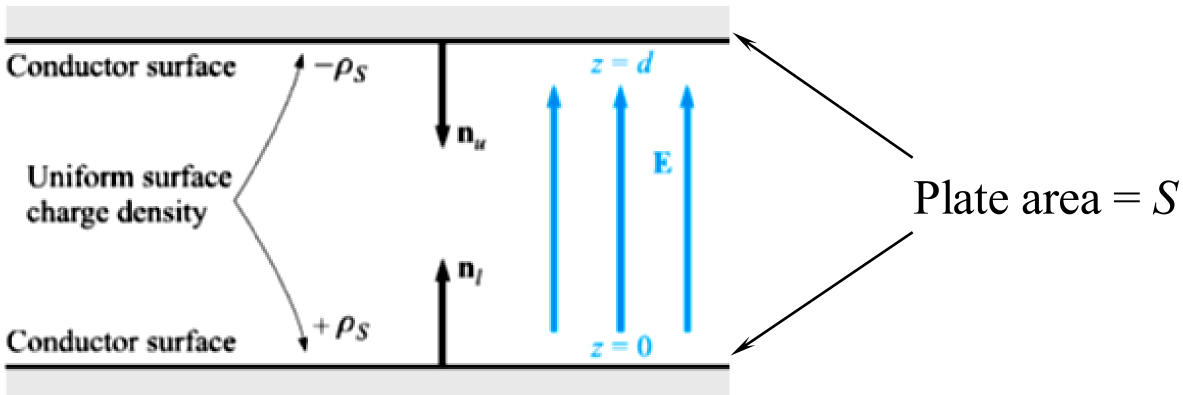
will also increase by the same factor -- so the ratio of Q to V_0 is a constant. We define the *capacitance* of the structure as the ratio of the stored charge to the applied voltage, or

$$C = \frac{Q}{V_0}$$

Units are Coul/V or *Farads*

Parallel Plate Capacitor

The horizontal dimensions are assumed to be much greater than the plate separation, d. Therefore, electric field can be assumed to lie only in the z direction, and potential varies only with z.



Apply the boundary for **D** at the surface of a perfect conductor:

Lower plate: $\mathbf{D}\cdot\mathbf{n}_\ell\big|_{z=0} = \mathbf{D}\cdot\mathbf{a}_z = \rho_s \;\Rightarrow\; \underline{\mathbf{D} = \rho_s\,\mathbf{a}_z}$

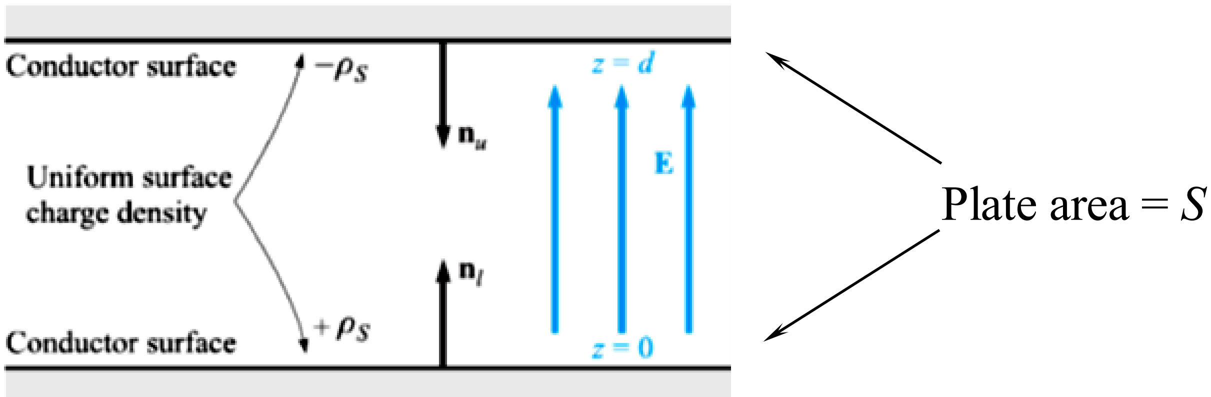
Upper Plate: $\mathbf{D}\cdot\mathbf{n}_u\big|_{z=d} = \mathbf{D}\cdot(-\mathbf{a}_z) = -\rho_s \;\Rightarrow\; \underline{\mathbf{D} = \rho_s\,\mathbf{a}_z}$

Same result either way!

Application of the boundary condition is needed on *only one* surface to obtain the total field between plates.

The electric field between plates is therefore: $$\boxed{\mathbf{E} = \frac{\rho_S}{\epsilon}\mathbf{a}_z}$$

Capacitance of a Parallel Plate Capacitor



Now with $\mathbf{E} = \dfrac{\rho_S}{\epsilon}\mathbf{a}_z$

$$Q = \rho_S S$$

$$V_0 = \frac{\rho_S}{\epsilon} d$$

The voltage between plate can be found through:

$$V_0 = -\int_{\text{upper}}^{\text{lower}} \mathbf{E} \cdot d\mathbf{L} = -\int_d^0 \frac{\rho_S}{\epsilon} dz = \frac{\rho_S}{\epsilon} d$$

Then with $Q = \rho_S S$ we finally obtain

$$C = \frac{Q}{V_0} = \frac{\epsilon S}{d}$$

Stored Energy in a Capacitor

Stored energy is found by integrating the energy density in the electric field over the capacitor volume. We use:

$$W_E = \int_{vol} \frac{1}{2} \mathbf{D} \cdot \mathbf{E} dv$$

or

$$W_E = \frac{1}{2} \int_{\mathrm{vol}} \epsilon E^2 \, dv = \frac{1}{2} \int_S \int_0^d \frac{\epsilon \rho_S^2}{\epsilon^2} dz \, dS = \frac{1}{2} \frac{\rho_S^2}{\epsilon} Sd = \frac{1}{2} \underbrace{\frac{\epsilon S}{d}}_{C} \underbrace{\frac{\rho_S^2 d^2}{\epsilon^2}}_{V_0^2}$$

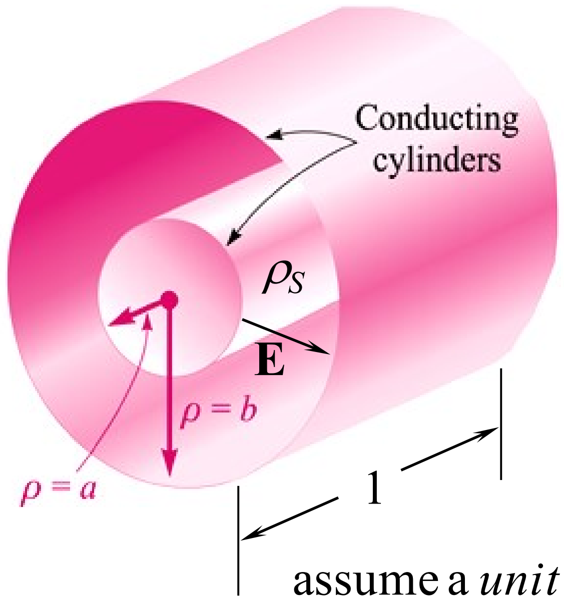
There are three ways of writing the energy:

$$\boxed{W_E = \frac{1}{2} C \, V_0^2 = \frac{1}{2} Q \, V_0 = \frac{1}{2} \frac{Q^2}{C}}$$

Example: Coaxial Transmission Line

We previously found using Gauss' Law:

$$\mathbf{E}(\rho) = \frac{a\rho_s}{\epsilon\rho}\,\mathbf{a}_\rho \ \mathrm{V/m} \qquad (a < \rho < b)$$



E = 0 elsewhere, assuming a hollow inner conductor, and equal and opposite charges on the inner and outer conductors.

assume a *unit length* in *z*

The potential difference between conductors is now:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,d\rho = \frac{a\rho_s}{\epsilon}\ln\left(\frac{b}{a}\right)$$

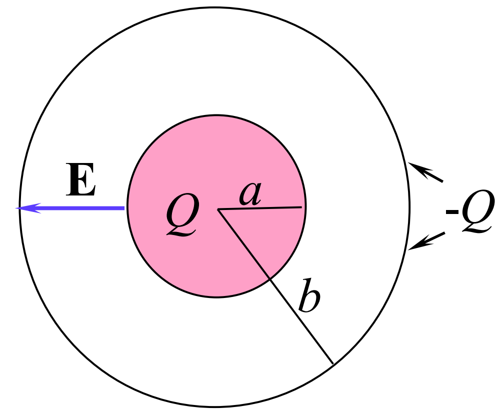
…and the charge on the inner conductor per unit length is:

$$Q = 2\pi a(1)\rho_s$$

Finally: $$C = \frac{Q}{V_0} = \frac{2\pi\epsilon}{\ln(b/a)} \ \mathrm{F/m}$$

Another Example: Spherical Capacitor

Consider two concentric spherical conductors, having radii a and b.
Equal and opposite charges, Q, are on the inner and outer conductors.



Gauss' Law tells us that electric field will exist only in the region between spheres, and will be given by:

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon r^2}\,\mathbf{a}_r$$

The potential difference between inner and outer spheres is then:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{Q}{4\pi\epsilon r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr = \underline{\frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

and the capacitance is:

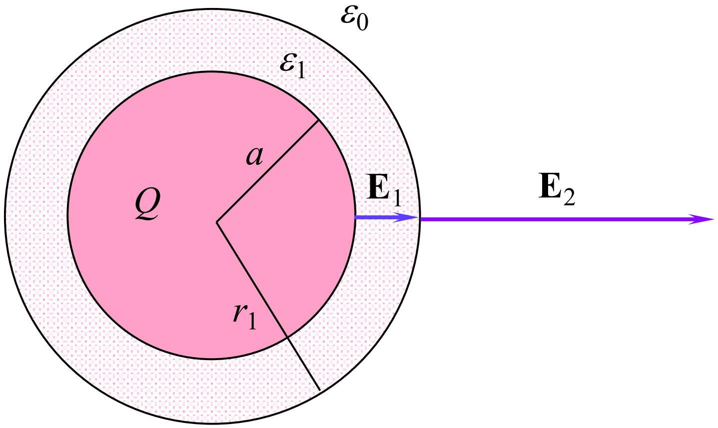
$$\boxed{C = \frac{Q}{V_0} = \frac{4\pi\epsilon}{(1/a) - (1/b)}}$$

Note that as $b \to \infty$ (isolated sphere)

$$\underline{C \to 4\pi\epsilon a}$$

Example: Isolated Sphere with a Dielectric Coating

A conducting sphere of radius a carries charge Q.
A dielectric layer of thickness r_1 - a and of permittivity ε_1 surrounds the conductor. Electric field in the two regions is found from Gauss' Law to be:



$$E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

The potential at the sphere surface is (with zero reference at infinity):

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right] = V_0$$

and the capacitance is:

$$C = \frac{4\pi}{\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}}$$

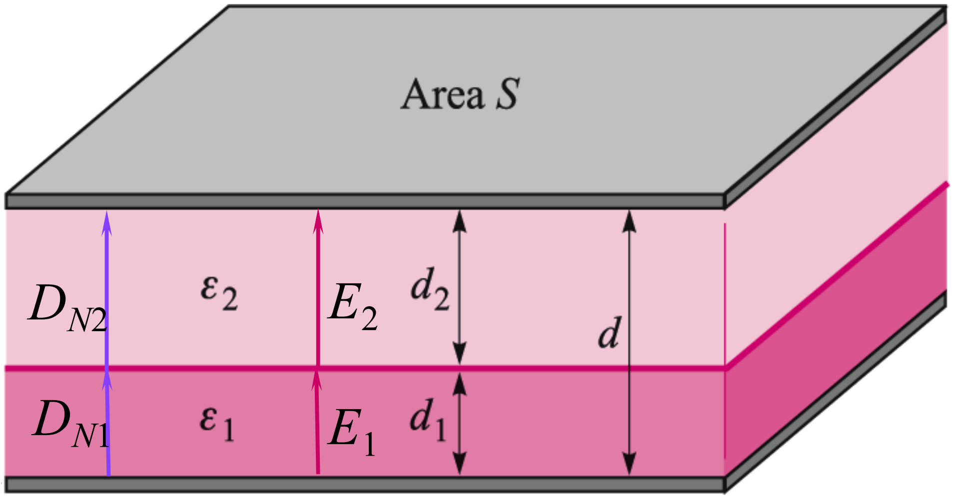
Capacitor with a Two-Layer Dielectric

In this case, we use the fact that **D normal** to an interface between two dielectrics will be continuous across the boundary, assuming no surface charge is present there:

Thus $D_{N1} = D_{N2}$ and therefore: $\epsilon_1 E_1 = \epsilon_2 E_2$

The potential difference between bottom and top plates will be: $V_0 = E_1 d_1 + E_2 d_2$

..which leads to: $E_1 = \dfrac{V_0}{d_1 + d_2(\epsilon_1/\epsilon_2)}$



The surface charge density on the bottom plate:

$$\rho_{S1} = D_1 = \epsilon_1 E_1 = \frac{V_0}{\dfrac{d_1}{\epsilon_1} + \dfrac{d_2}{\epsilon_2}}$$

The capacitance is then:

$$C = \frac{Q}{V_0} = \frac{\rho_S S}{V_0} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

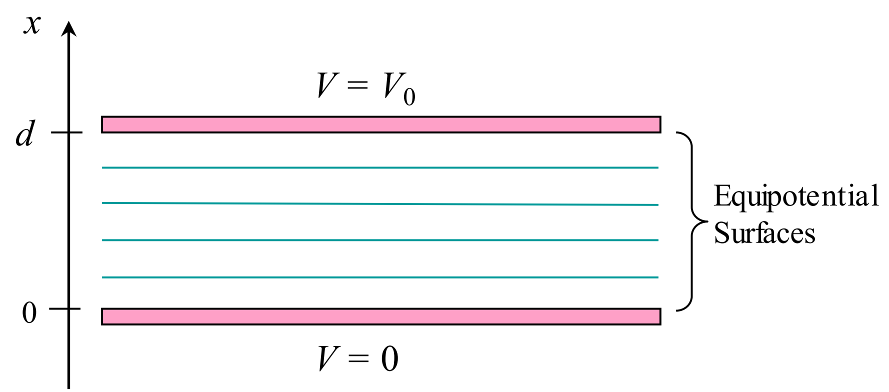
Poisson's and Laplace's Equations

These equations allow one to find the potential field in a region, in which values of potential or electric field are known at its boundaries.

Start with Maxwell's first equation: $\nabla \cdot \mathbf{D} = \rho_\nu$

where $\mathbf{D} = \epsilon \mathbf{E}$

and $\mathbf{E} = -\nabla V$



so that $\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_\nu$

or finally: $$\nabla \cdot \nabla V = -\frac{\rho_\nu}{\epsilon}$$

Derivation (continued)

Recall the divergence as expressed in rectangular coordinates:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

…and the gradient:

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

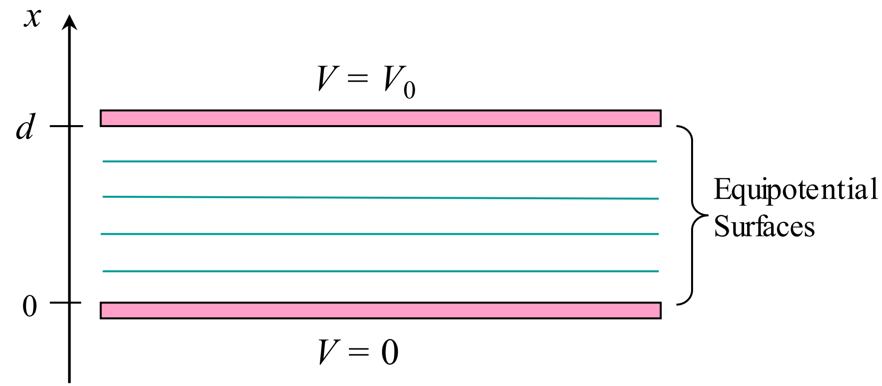
then:

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"). It is known as the Laplacian operator

Statement of Poisson's and Laplace's Equations



we already have: $\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$

which becomes:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}$$

This is Poisson's equation, as stated in rectangular coordinates.

In the event that there is zero volume charge density, the right-hand-side becomes zero, and we obtain

Laplace's equation:

$$\nabla^2 V = 0$$

The Laplacian Operator in the Three Coordinate Systems

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(rectangular)}$$

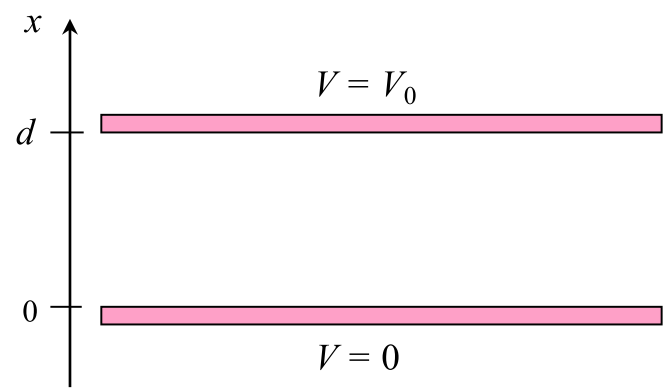
(Laplace's equation)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

Application on Laplace and Poisson equations: Parallel Plate Capacitor

In this case, the plate separation, d, is assumed much less than the smallest plate dimension. Therefore V can be assumed to vary only with x.



Laplace's equation reduces to:

$$\frac{d^2V}{dx^2} = 0$$

Integrating once, obtain: $$\frac{dV}{dx} = A$$

Integrate a second time to get: $$V = Ax + B$$

Boundary conditions:

1. $V = 0$ at $x = 0$
2. $V = V_0$ at $x = d$

where A and B are integration constants that are to be evaluated subject to the boundary conditions.

Interior Potential Field

We now have: $V = Ax + B$

Apply boundary condition 1:

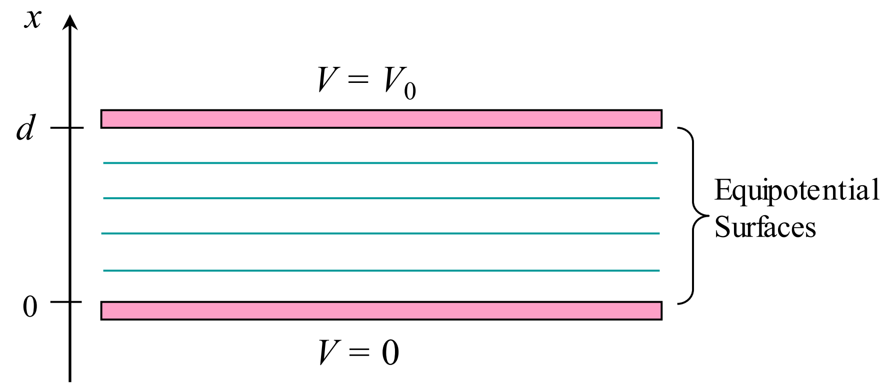
$$0 = A(0) + B \quad \Longrightarrow \quad B = 0$$

Apply boundary condition 2:

$$V_0 = Ad \quad \Longrightarrow \quad A = \frac{V_0}{d}$$

Finally: $$V = \frac{V_0 x}{d}$$

This function is pictured by the equipotential surfaces shown in the capacitor, in which there is a constant voltage difference between adjacent surfaces



Boundary conditions:

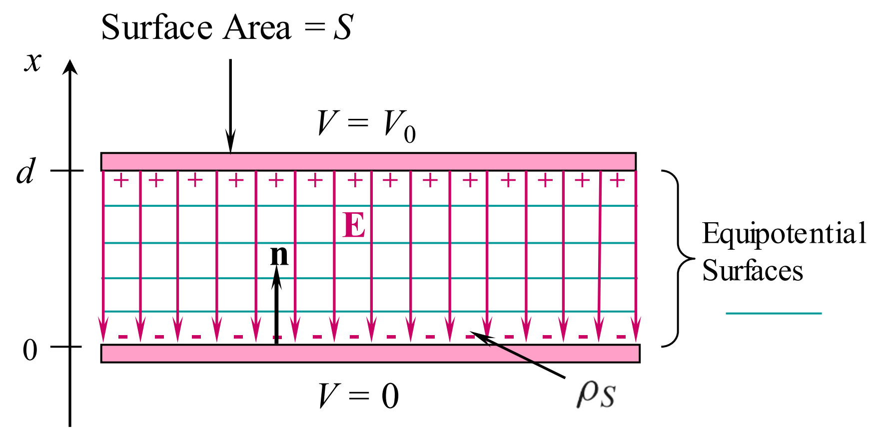
1. $V = 0$ at $x = 0$
2. $V = V_0$ at $x = d$

Finding the Electric Field, Charge, and Capacitance

Start with: $V = V_0 \dfrac{x}{d}$

Then: $\mathbf{E} = -\nabla V = -\dfrac{V_0}{d}\mathbf{a}_x$

From which: $\mathbf{D} = -\epsilon \dfrac{V_0}{d}\mathbf{a}_x$



At the lower plate surface: $\mathbf{D}_S = \mathbf{D}\big|_{x=0} = -\epsilon \dfrac{V_0}{d}\mathbf{a}_x$ and $\mathbf{n} = \mathbf{a}_x$

Then use: $\mathbf{D}\cdot\mathbf{n}\big|_s = \rho_s$ to find $D_N = -\epsilon \dfrac{V_0}{d} = \rho_S$

The charge on the lower plate is now: $Q = \displaystyle\int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$ and the capacitance is

$$C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}$$

Another Example: Coaxial Transmission Line

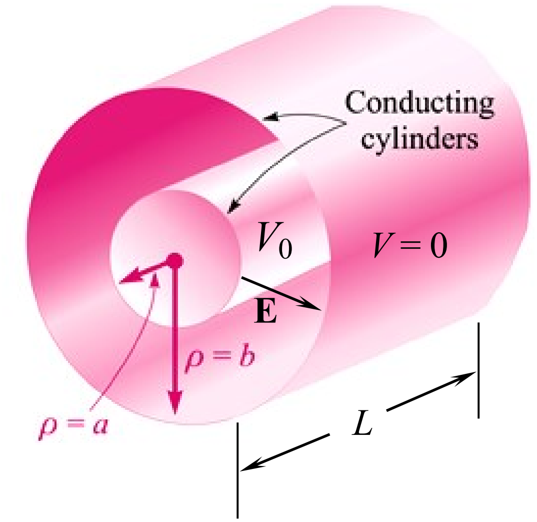
As V is assumed to vary with radius only, Laplace's equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad \text{(not valid at } \rho = 0\text{)}$$

Our goal is to evaluate the potential function in the region ($a < \rho < b$)

Integrate once: $$\rho\frac{dV}{d\rho} = A$$

.. and a second time: $$V = A\ln\rho + B$$

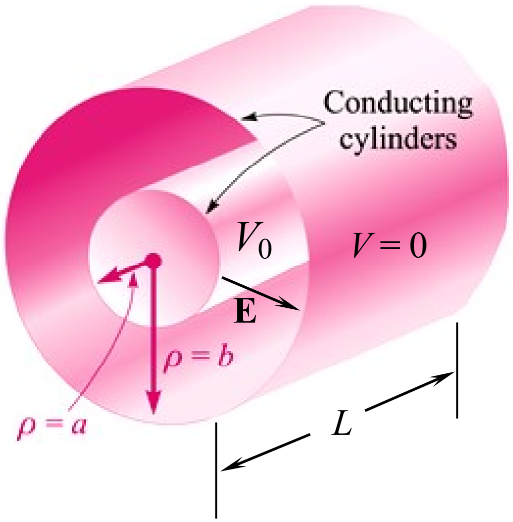


Boundary conditions:

1. $V = 0$ at $\rho = b$
2. $V = V_0$ at $\rho = a$

Coaxial Line: continued

Now have: $V = A \ln \rho + B$



Boundary conditions:

1. $V = 0$ at $\rho = b$
2. $V = V_0$ at $\rho = a$

Apply boundary condition 1:

$$0 = A \ln(b) + B \quad \Rightarrow \quad B = -A \ln(b)$$

Apply boundary condition 2:

$$V_0 = A \ln(a) - A \ln(b) = A \ln(a/b) \quad \Rightarrow \quad A = -\frac{V_0}{\ln(b/a)}$$

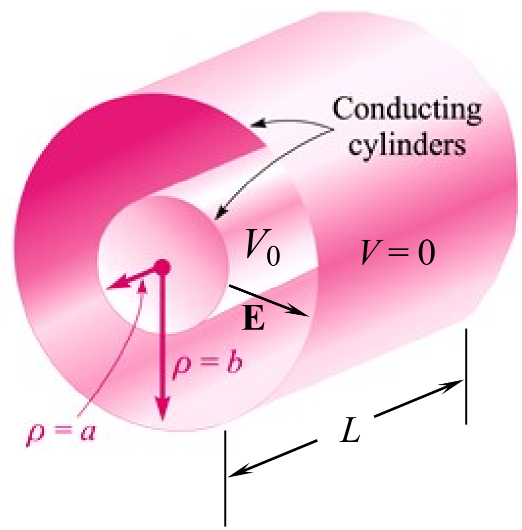
Putting it all together:

$$V(\rho) = -\frac{V_0}{\ln(b/a)} \left[\ln(\rho) - \ln(b)\right] = \boxed{V_0 \frac{\ln(b/\rho)}{\ln(b/a)}}$$

Coaxial Line Capacitance

We have found the potential field between conductors:

$$V(\rho) = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}$$



Then:

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\,\mathbf{a}_\rho = \frac{V_0}{\rho}\frac{1}{\ln(b/a)}\,\mathbf{a}_\rho$$

The charge density on the inner conductor is:

$$\rho_s = \mathbf{D}\cdot\mathbf{a}_\rho\Big|_{\rho=a} = \frac{\epsilon V_0}{a}\frac{1}{\ln(b/a)}\ \mathrm{C/m^2}$$

The total charge on the inner conductor is:

$$Q = \int_S \rho_s\, da = 2\pi a L\,\rho_s = \frac{2\pi\epsilon L V_0}{\ln(b/a)}\ \mathrm{C}$$

…and the capacitance is finally:

$$C = \frac{Q}{V_0} = \frac{2\pi\epsilon L}{\ln(b/a)}\ \mathrm{F}$$

6.2. Let $S = 100\,\text{mm}^2$, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:

$$C = \frac{\epsilon_r \epsilon_0 A}{d} = \frac{12\epsilon_0(100 \times 10^{-6})}{3 \times 10^{-3}} = 0.4\epsilon_0 = \underline{3.54 \text{ pf}}$$

b) After connecting a 6 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: First,

$$E = V_0/d = 6/(3 \times 10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{then} \quad D = \epsilon_r \epsilon_0 E = 2.4 \times 10^4 \epsilon_0 = \underline{0.21\ \mu\text{C/m}^2}$$

The charge in this case is

$$Q = \mathbf{D} \cdot \mathbf{n}|_s = DA = 0.21 \times (100 \times 10^{-6}) = 0.21 \times 10^{-4}\ \mu\text{C} = \underline{21 \text{ pC}}$$

Finally, $W_e = (1/2)QV_0 = 0.5(21)(6) = \underline{63 \text{ pJ}}$.

c) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, re-calculate E, D, Q, and the energy stored in the capacitor.

$$E = V_0/d = 6/(3 \times 10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{as before.} \quad D = \epsilon_0 E = 2000\epsilon_0 = \underline{17.7 \text{ nC/m}^2}$$

The charge is now $Q = DA = 17.7 \times (100 \times 10^{-6})$ nC $= \underline{1.8 \text{ pC}}$.

Finally, $W_e = (1/2)QV_0 = 0.5(1.8)(6) = \underline{5.4 \text{ pJ}}$.