Engineering Electromagnetics

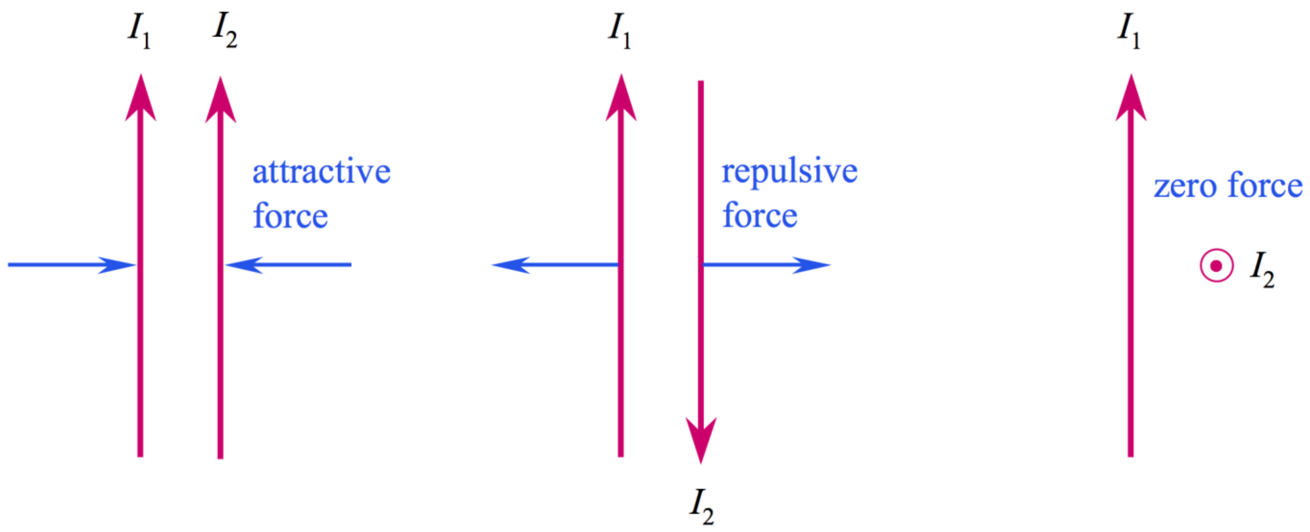
Chapter 7:

The Steady Magnetic Field

Motivating the Magnetic Field Concept: Forces Between Currents

<u>**Magnetic forces arise whenever we have charges in motion**</u>. Forces between current-carrying wires present familiar examples that we can use to determine what a magnetic force field should look like:

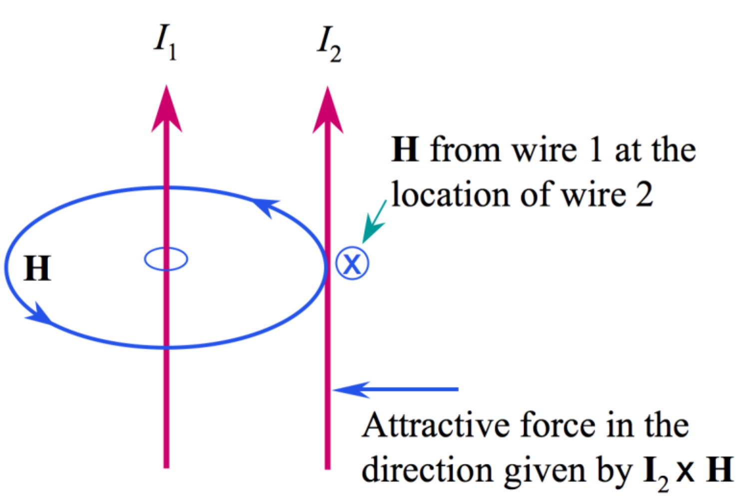
Here are the easily-observed facts:

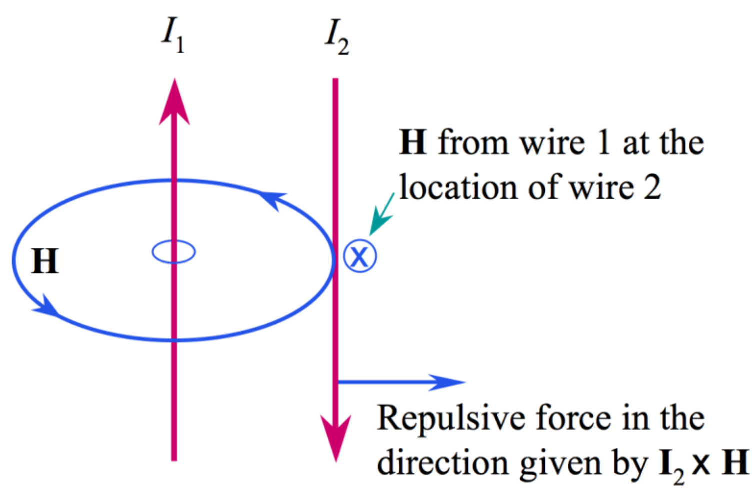


How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

Magnetic Field

The geometry of the magnetic field is set up to correctly model forces between currents that allow for any relative orientation. The magnetic field intensity, **H**, circulates *around* its source, I_1, in a direction most easily determined by the *right-hand rule*: Right thumb in the direction of the current, fingers curl in the direction of **H**





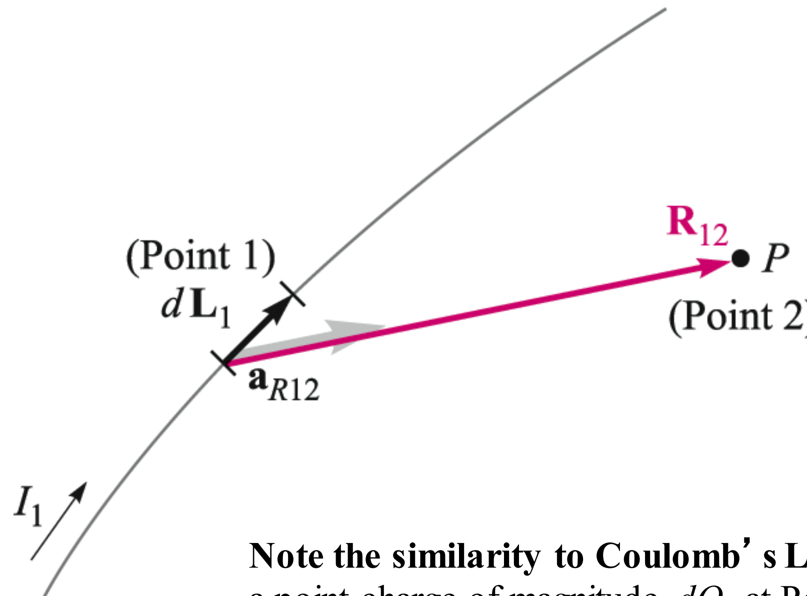
Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, H, arising from a "point source" current element of differential length *d*L.

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

The units of **H** are [A/m]

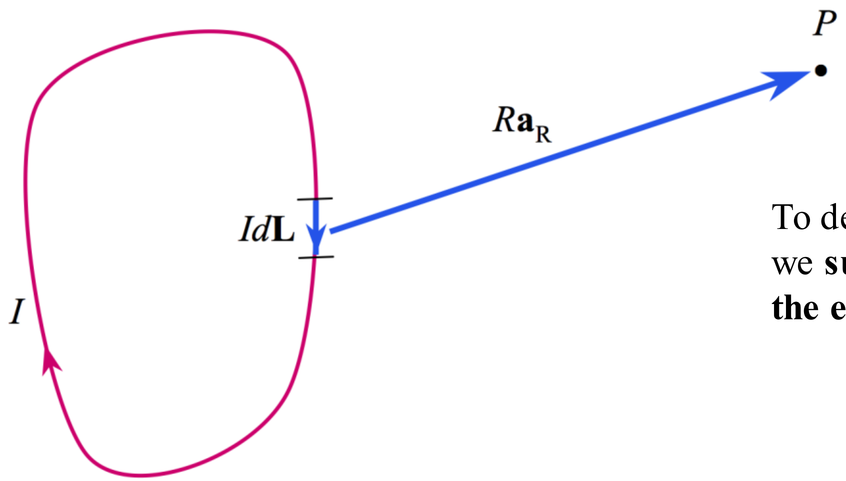
Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page. This is a formal statement of the right-hand rule



Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

Magnetic Field Arising From a Circulating Current



At point *P*, the magnetic field associated with the differential current element $Id\mathbf{L}$ is

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

To determine the total field arising from the closed circuit path, we **sum the contributions from the current elements that make up the entire loop**, or

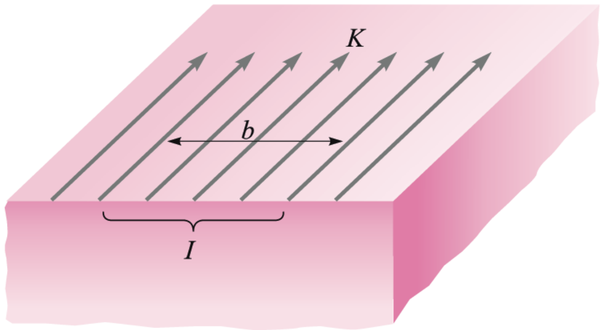
$$\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

The contribution to the field at *P* from any portion of the current will be just the above integral evalated over just that portion.

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width b is

$$I = Kb$$



..and so the differential current quantity that appears in the Biot-Savart law becomes:

$$I\,d\mathbf{L} = \mathbf{K}\,dS$$

The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

$$\mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R\,dS}{4\pi R^2}$$

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R\,dv}{4\pi R^2}$$

Example of the Biot-Savart Law

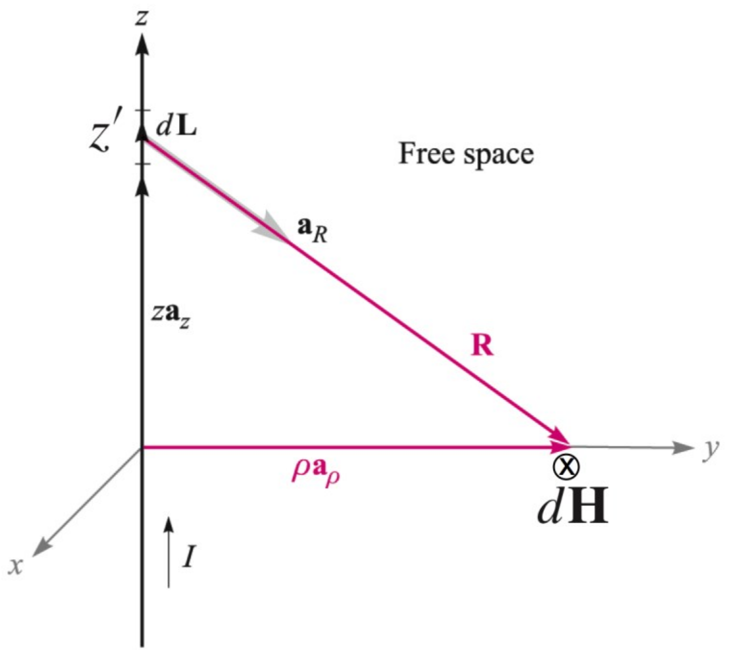
In this example, we evaluate the magnetic field intensity on the *y* axis (equivalently in the *xy* plane) arising from a filament current of infinite length in on the *z* axis.

Using the drawing, we identify:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho - z' \mathbf{a}_z$$

and so..

$$\mathbf{a}_R = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$



so that:

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

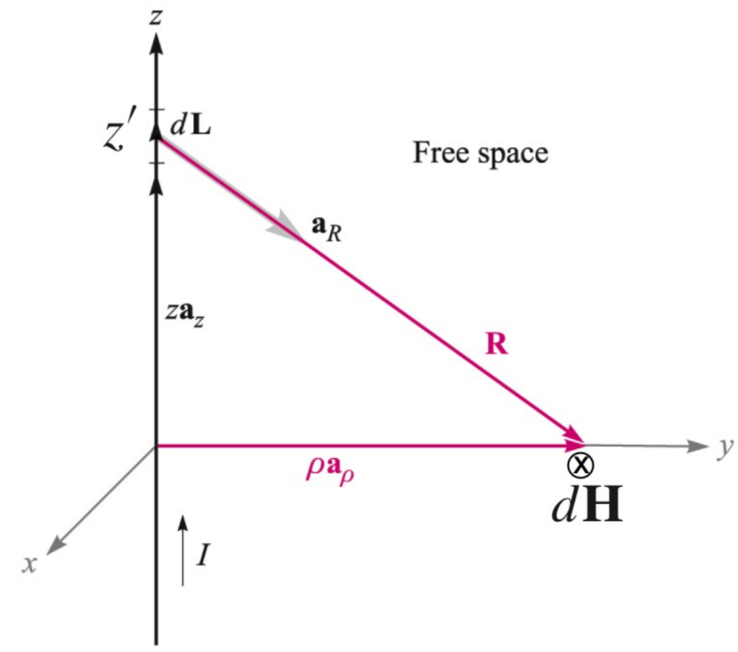
Example: continued

We now have: $$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Integrate this over the entire wire:

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



..after carrying out the cross product

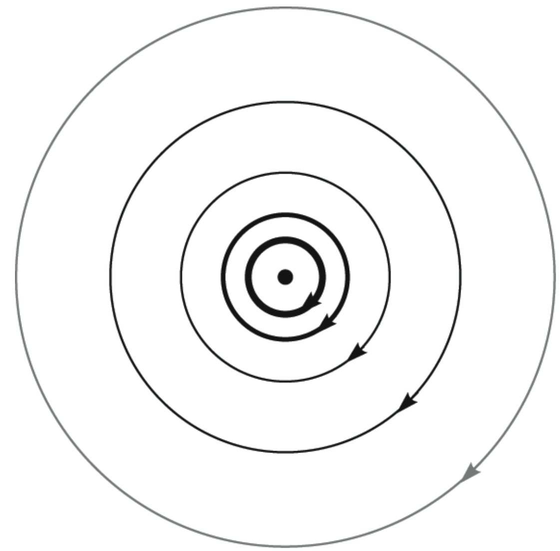
Example: concluded

Evaluating the integral:

we have: $$\mathbf{H} = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{\rho\, dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}}\Bigg|_{-\infty}^{\infty}$$

finally: $$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi}$$

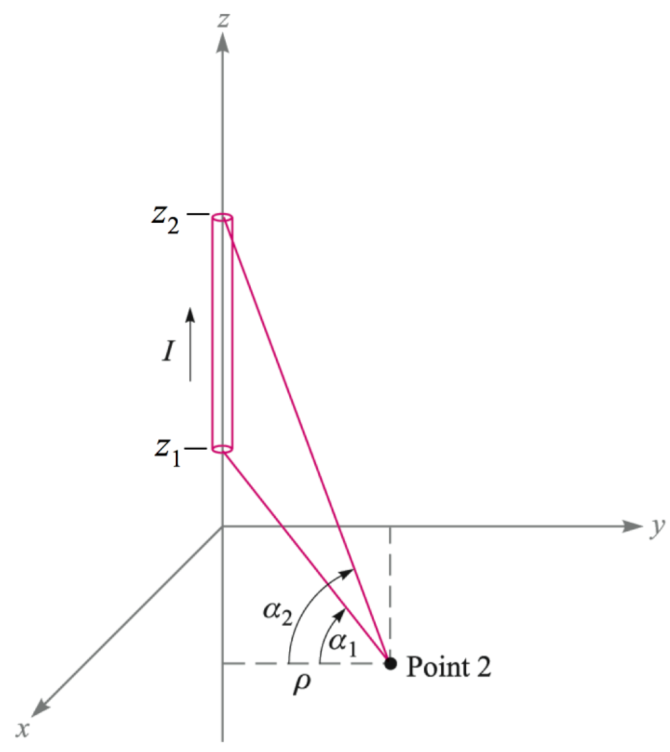


Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the z axis

Field Arising from a Finite Current Segment

In this case, the field is to be found in the *xy* plane at Point 2.
The Biot-Savart integral is taken over the wire length:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$



..after a few additional steps, we find:

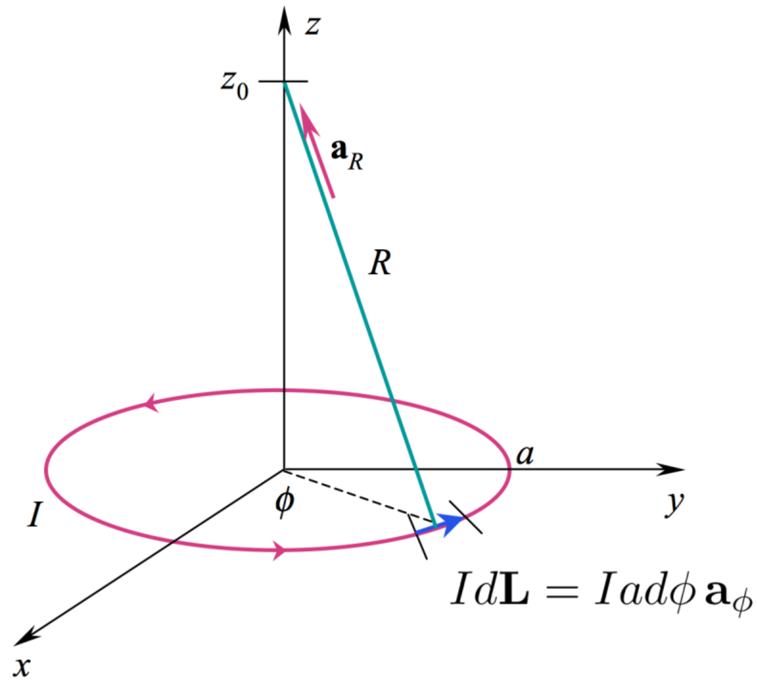
$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

Another Example: Magnetic Field from a Current Loop

Consider a circular current loop of radius *a* in the *x*-*y* plane, which carries steady current *I*. We wish to find the magnetic field strength anywhere on the *z* axis.

We will use the Biot-Savart Law:

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$



where: $$I d\mathbf{L} = I a d\phi \, \mathbf{a}_\phi$$

$$R = \sqrt{a^2 + z_0^2}$$

$$\mathbf{a}_R = \frac{z_0 \, \mathbf{a}_z - a \, \mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}}$$

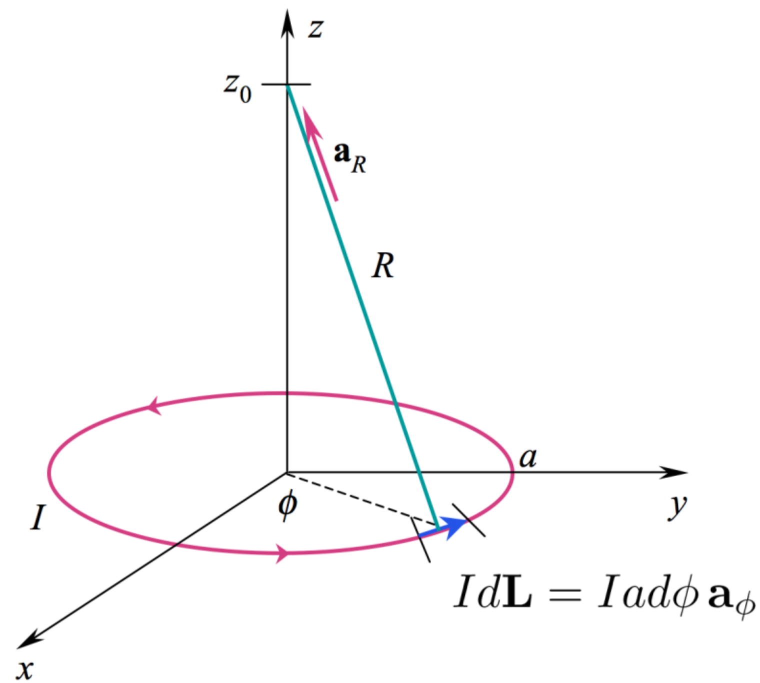
Example: Continued

Substituting the previous expressions, the Biot-Savart Law becomes:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z_0\,\mathbf{a}_z - a\,\mathbf{a}_\rho)}{4\pi(a^2 + z_0^2)^{3/2}}$$

carry out the cross products to find:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,(z_0\,\mathbf{a}_\rho + a\,\mathbf{a}_z)}{4\pi(a^2 + z_0^2)^{3/2}}$$



but we must include the angle dependence in the radial unit vector:

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$

with this substitution, the radial component will integrate to zero, meaning that all radial components will cancel on the *z* axis.

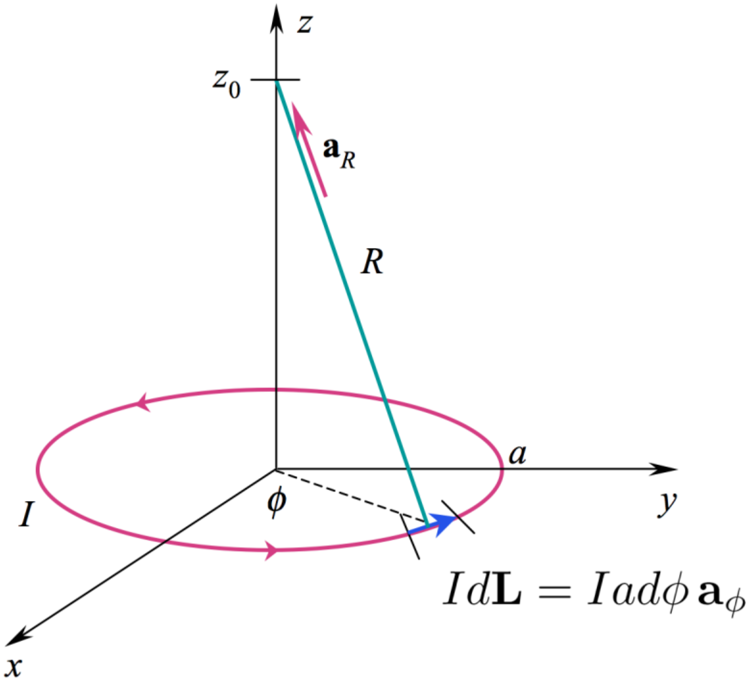
Example: Continued

Now, only the z component remains, and the integral evaluates easily:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

Note the form of the numerator: the product of the current and the loop area. We define this as the *magnetic moment*:

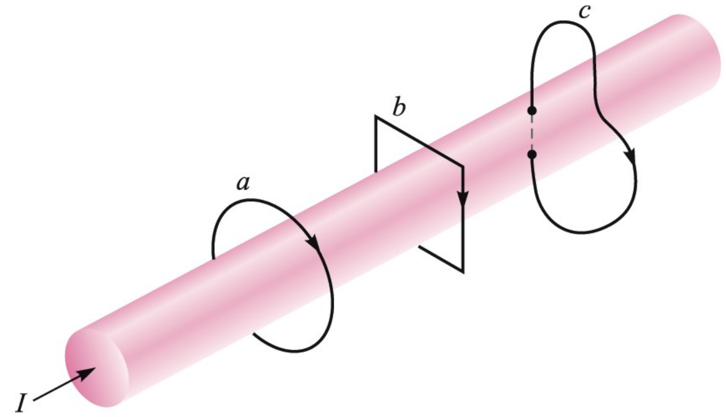
$$\mathbf{m} = I(\pi a^2)\,\mathbf{a}_z$$



Ampere's Circuital Law

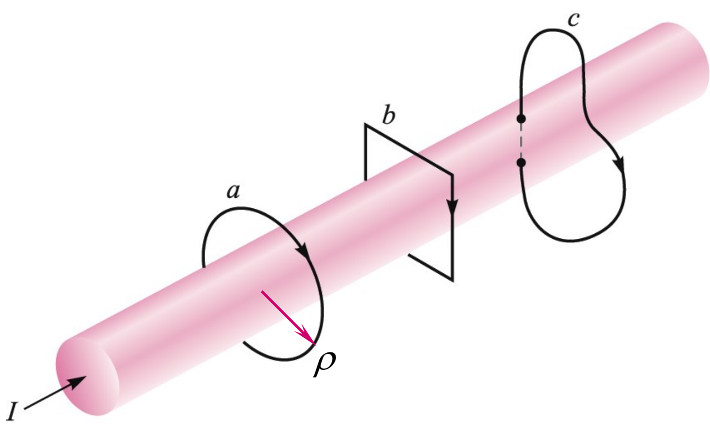
Ampere's Circuital Law states that the line integral of H about *any closed path* is exactly equal to the direct current enclosed by that path.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



In the figure at right, the integral of **H** about closed paths *a* and *b* gives the total current *I*, while the integral over path *c* gives only that portion of the current that lies within *c*

Ampere's Law Applied to a Long Wire



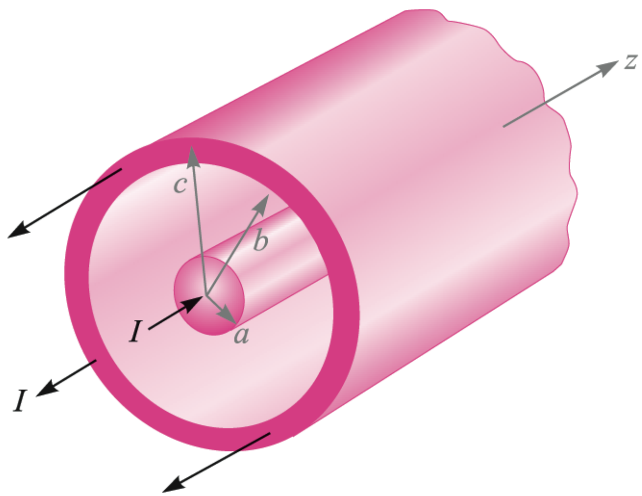
Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (z) axis.

Choosing path a, and integrating **H** around the circle of radius ρ gives the enclosed current, I:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

so that: $$H_\phi = \frac{I}{2\pi\rho}$$ as before.

Coaxial Transmission Line

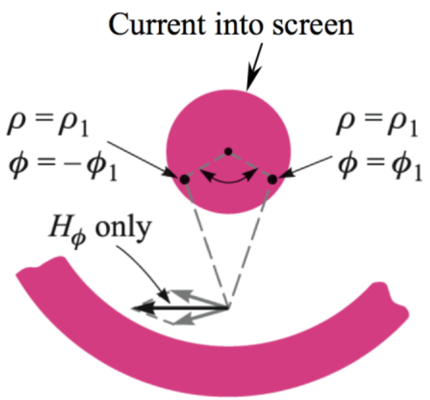


In the coax line, we have two concentric *solid* conductors that carry equal and opposite currents, *I*.

The line is assumed to be infinitely long, and the circular symmetry suggests that **H** will be entirely ϕ - directed, and will vary only with radius ρ.

Our objective is to find the magnetic field for all values of ρ

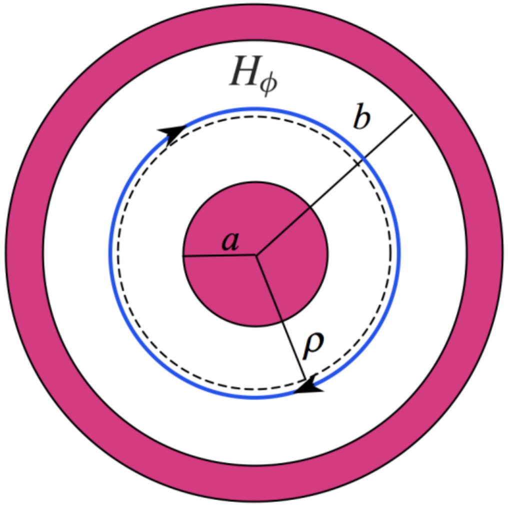
Field Between Conductors



The inner conductor can be thought of as made up of a bundle of filament currents, each of which produces the field of a long wire.

The field between conductors is thus found to be the same as that of filament conductor on the *z* axis that carries current, *I*. Specifically:

$$H_\phi = \frac{I}{2\pi\rho} \qquad a < \rho < b$$

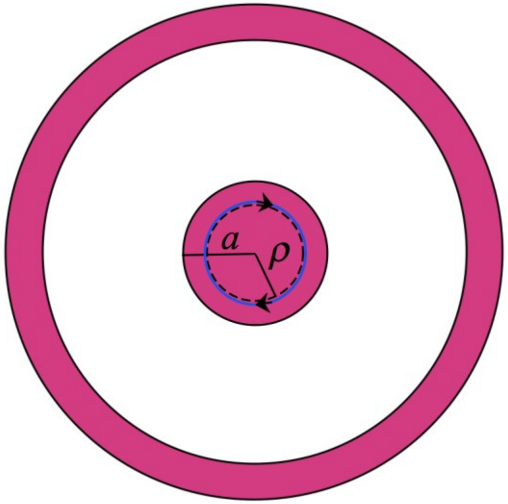


Field Within the Inner Conductor

With current uniformly distributed inside the conductors, the **H** can be assumed circular everywhere.

Inside the inner conductor, and at radius ρ, we again have:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi 2\pi\rho$$



But now, the current enclosed is $I_{\text{encl}} = I \dfrac{\rho^2}{a^2}$

so that $2\pi\rho H_\phi = I \dfrac{\rho^2}{a^2}$ or finally: $H_\phi = \dfrac{I\rho}{2\pi a^2} \quad (\rho < a)$

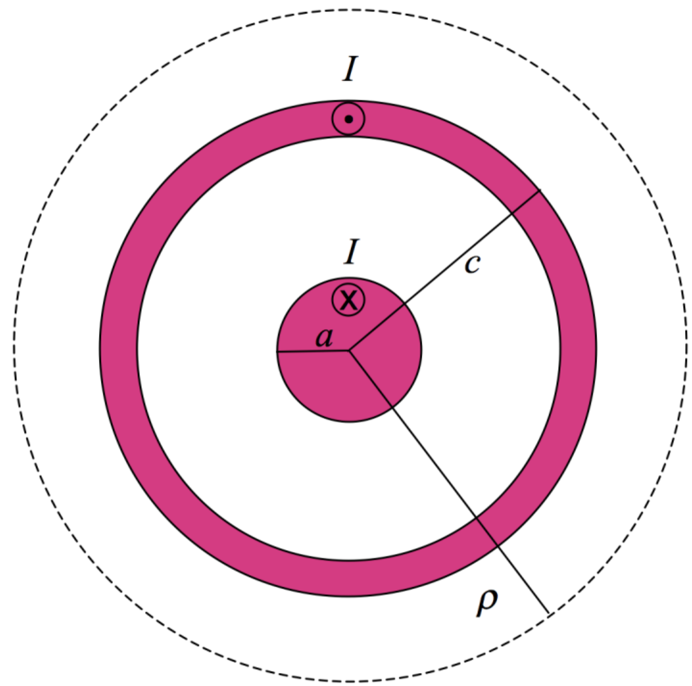
Field Outside Both Conducors

Outside the transmission line, where $\rho > c$, no current is enclosed by the integration path, and so

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0$$

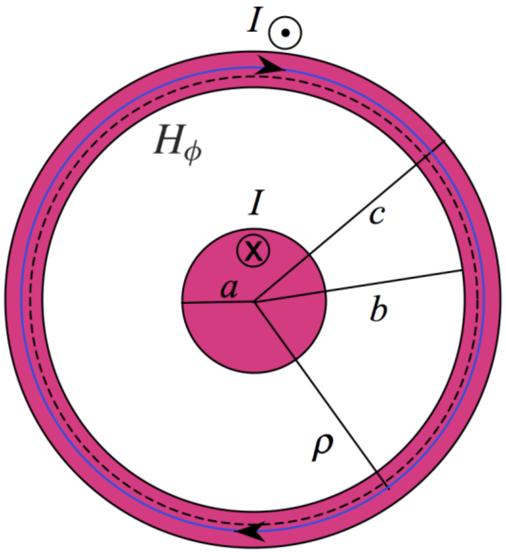
As the current is uniformly distributed, and since we have circular symmetry, the field would have to be constant over the circular integration path, and so it must be true that:

$$H_\phi = 0 \quad (\rho > c)$$



Field Inside the Outer Conductor

Inside the outer conductor, the enclosed current consists of that within the inner conductor plus that portion of the outer conductor current existing at radii less than ρ



Ampere's Circuital Law becomes

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

..and so finally:

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)$$

Curl of a Vector Field

In general, the curl of a vector field is another field that is normal to the original field.

The curl component in the direction N, normal to the plane of the integration loop is:

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} = \mathbf{J}$$

where ΔS_N is the planar area enclosed by the closed line integral.

The direction of N is taken using the right-hand convention: With fingers of the right hand oriented in the direction of the path integral, the thumb points in the direction of the normal (or curl).

Curl in Rectangular Coordinates

Assembling the results of the rectangular loop integration exercise, we find the vector field that comprises curl **H**:

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

An easy way to calculate this is to evaluate the following determinant:

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

which we see is equivalent to the cross product of the del operator with the field:

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

Curl in Other Coordinate Systems

…a little more complicated!

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi + \left(\frac{1}{\rho} \frac{\partial (\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z \quad \text{(cylindrical)}$$

$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta} \left(\frac{\partial (H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r} \right) \mathbf{a}_\theta + \frac{1}{r} \left(\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi \quad \text{(spherical)}$$

Look these up as needed….

Another Maxwell Equation

It has just been demonstrated that:

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}$$

.....which is in fact one of Maxwell's equations for static fields:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

This is Ampere's Circuital Law in point form.

….and Another Maxwell Equation

We already know that for a *static* electric field:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

This means that: $$\nabla \times \mathbf{E} = 0$$ (applies to a static electric field)

Recall the condition for a conservative field: that is, its closed path integral is zero everywhere.

Therefore, a field is conservative if it has *zero curl* at all points over which the field is defined.

Obtaining Ampere's Circuital Law in Integral Form, using Stokes' Theorem

Begin with the point form of Ampere's Law for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

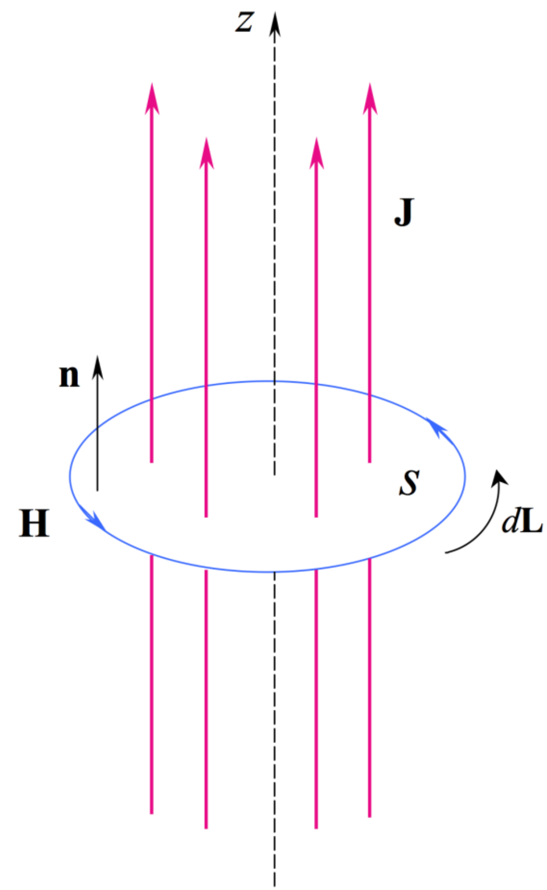
Integrate both sides over surface S:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

..in which the far right hand side is found from the left hand side using Stokes' Theorem. The closed path integral is taken around the perimeter of S. Again, note that we use the right-hand convention in choosing the direction of the path integral.

The center expression is just the net current through surface S, so we are left with the integral form of Ampere's Law:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



Magnetic Flux and Flux Density

We are already familiar with the concept of electric flux:

$$\Psi = \int_s \mathbf{D} \cdot d\mathbf{S} \quad \text{Coulombs}$$

in which the electric flux density in free space is: $\mathbf{D} = \epsilon_0 \mathbf{E} \quad \mathrm{C/m^2}$

and where the free space permittivity is $\epsilon_0 = 8.854 \times 10^{-12} \ \mathrm{F/m}$

In a similar way, we can define the *magnetic flux* in units of Webers (Wb):

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{S} \quad \text{Webers}$$

in which the *magnetic flux density* (or *magnetic* induction) in free space is: $\mathbf{B} = \mu_0 \mathbf{H} \ \ \mathrm{Wb/m^2}$

and where the free space *permeability* is $\mu_0 = 4\pi \times 10^{-7} \ \mathrm{H/m}$

This is a *defined* quantity, having to do with the definition of the ampere (we will explore this later).

A Key Property of **B**

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

$$\Psi_{net} = \oint_s \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

If the same were to be done with magnetic flux density, we would find:

$$\Phi_{net} = \oint_s \mathbf{B} \cdot d\mathbf{S} = 0$$

The implication is that (for our purposes) there are no magnetic charges -- specifically, *no point sources of magnetic field exist*.

i.e. magnetic field lines always close on themselves.

Another Maxwell Equation

We may rewrite the closed surface integral of **B** using the divergence theorem, in which the right hand integral is taken over the volume surrounded by the closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\ dv = 0$$

Because the result is zero, it follows that

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

This result is known as Gauss' Law for the magnetic field in point form.

Maxwell's Equations for Static Fields

We have now completed the derivation of Maxwell's equations for no time variation. In point form, these are:

$$\nabla \cdot \mathbf{D} = \rho_v$$ Gauss' Law for the electric field

$$\nabla \times \mathbf{E} = 0$$ Conservative property of the static electric field

$$\nabla \times \mathbf{H} = \mathbf{J}$$ Ampere's Circuital Law

$$\nabla \cdot \mathbf{B} = 0$$ Gauss' Law for the Magnetic Field

where, in free space:

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

Significant changes in the above four equations will occur when the fields are allowed to vary with time, as we'll see later.

Maxwell's Equations in Large Scale Form

The divergence theorem and Stokes' theorem can be applied to the previous four point form equations to yield the integral form of Maxwell's equations for static fields:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v \, dv$$

Gauss' Law for the electric field

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Conservative property of the static electric field

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

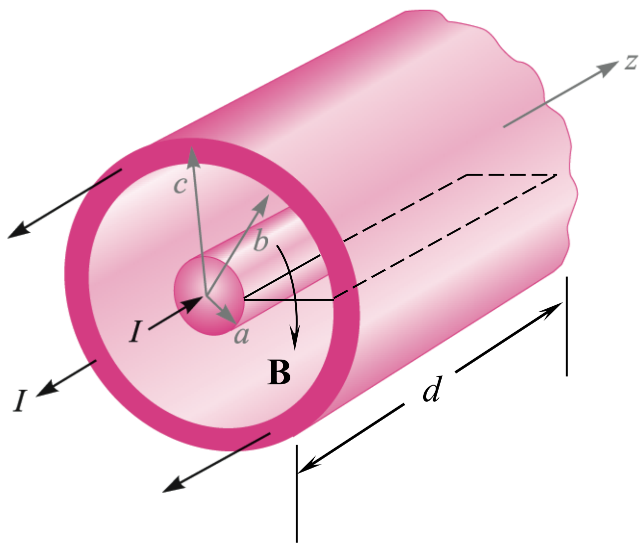
Ampere's Circuital Law

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' Law for the magnetic field

Example: Magnetic Flux Within a Coaxial Line

Consider a length d of coax, as shown here. The magnetic field strength between conductors is:



$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and so: $$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux is now the integral of **B** over the flat surface between radii a and b, and of length d along z:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

The result is: $$\Phi = \frac{\mu_0 I d}{2\pi} \ln\frac{b}{a}$$

The coax line thus "stores" this amount of magnetic flux in the region between conductors. This will have importance when we discuss inductance in a later lecture.

Exercises

7.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad \Rightarrow \quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

7.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density $\mathbf{J}$: This is found through the curl of $\mathbf{H}$, which simplifies to a single term, since $\mathbf{H}$ varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z\ \text{A/m}^2}$$

b) Integrate $\mathbf{J}$ over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\!\!\int \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 60\rho\mathbf{a}_z \cdot \rho\, d\rho\, d\phi\mathbf{a}_z = \underline{40\pi\ \text{A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2\,\mathbf{a}_\phi\big|_{\rho=1}\cdot(1)d\phi\mathbf{a}_\phi = \int_0^{2\pi} 20\,d\phi = \underline{40\pi\ \text{A}}$$

7.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2 \times 10^{-3})^2}\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z \text{ A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad \Rightarrow \quad \mathbf{H} = \frac{\rho J}{2}\mathbf{a}_\phi = \underline{7.96 \times 10^6 \rho\,\mathbf{a}_\phi \text{ A/m}}$$

Then $\mathbf{B} = \mu_0\mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6)\rho\mathbf{a}_\phi = \underline{10\rho\,\mathbf{a}_\phi \text{ Wb/m}^2}$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part *b*, we find,

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1.59 \times 10^7 \rho^2}{2}\right)\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z \text{ A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1}{\pi\rho}\,\mathbf{a}_\phi \text{ A/m}}$$

Now $\mathbf{B} = \mu_0\mathbf{H} = \underline{\mu_0/(\pi\rho)\,\mathbf{a}_\phi \text{ Wb/m}^2}$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{1}{\pi\rho}\right)\mathbf{a}_z = \underline{0} \quad \text{(as expected)}$$