BIRZEIT UNIVERSITY

FACULTY OF ENGINEERING

DEPARTMENT OF ELECTRICAL ENGINEERING

ENEE 3303

PRINCIPLES OF COMMUNICATION SYSTEMS

LECTURE NOTES

BY

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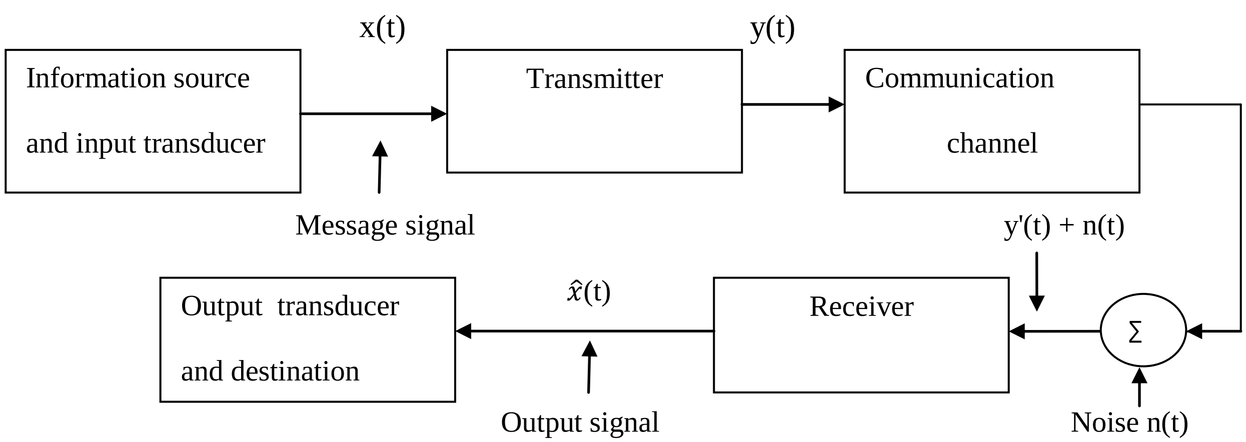
Model of a Communication System

- *Communication* is defined as “exchange of information“.
- *Telecommunication* refers to communication over a distance greater than would normally be possible without artificial aids.
- Telephony is an example of point-to-point communication and normally involves a two – way flow of information.
- Broadcast radio and television : Information is transmitted from one location but is received at many locations using different receivers (point to multi-point communication)
- Model of a communication system :



- The purpose of a communication system is to transmit information – bearing signals from a source located at one point to a user located at another end.
- The input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal*.
- The original message is recreated at the destination using an output transducer.
- The *transmitter* modifies the message signal into a form suitable for transmission over the channel. *Here modulation takes place*.
- The *channel* is the medium over which signal is transmitted, (like free space, an optical fiber, transmission lines, twisted pair of wires…). Here signal is distorted due to

  A. Nonlinearities and/or imperfections in the frequency response of the channel.
  B. Noise and interference are added to the signal during the course of transmission.

- The purpose of the *receiver* is to recreate the original signal x(t) from the degraded version y’(t) + n(t) of the transmitted signal after propagating through channel . *Here, demodulation takes place*.

Classification of Signals

Definition: A signal may be defined as a single valued function of time that conveys information.

Depending on the feature of interest, we may distinguish four different classes of signals:

1. Periodic Signals, Non-periodic Signals:

A *periodic signal* g(t) is a function of time that satisfies the condition $g(t) = g(t+T_0)$, ∀ t.

The smallest value of T_0 that satisfies this condition is called the period of g(t).

Example: A Periodic Signal

The saw-tooth function shown below is an example of a periodic signal.

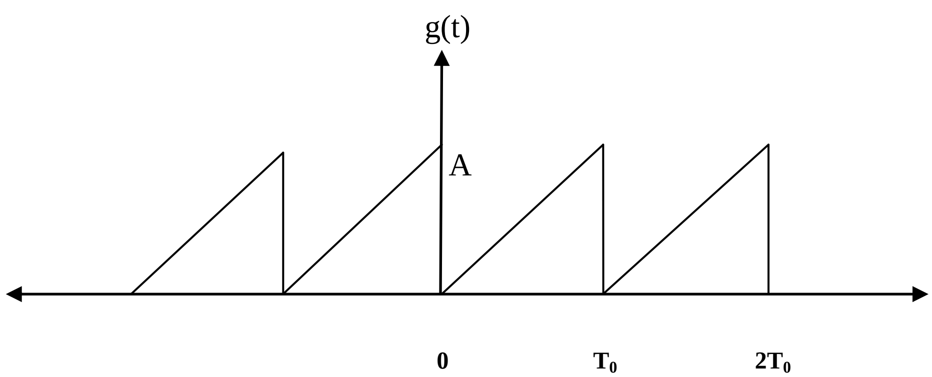


Fig. 1.1: A periodic signal with period T_0

Example: A Non-periodic Signal

The signal

$$g(t) = \begin{cases} A, & 0 \leq t \leq \tau \\ 0, & otherwise \end{cases}$$

is non-periodic, since there does not exist a T_0 for which the condition $g(t) = g(t+T_0)$ is satisfied.

2. Determinstic Signals, Random Signals:

A *deterministic signal* is one about which there is no uncertainty with respect to its value at any time. It is a completely specified function of time .

Example: A Deterministic Signal

$x(t) = Ae^{-at}u(t)$; A and α are constants.

A *random signal* is one about which there is some degree of uncertainty before it actually occurs. (It involves a random variable)

Example : A Random Signal

x(t) = A e^{-at}u(t) ; α is a constant and A is a random variable with the following probability density function (pdf).

$F_A(a)= \begin{cases} 1 & 0 \le a \le 1 \\ 0 & otherwise \end{cases}$

3. Energy Signals, Power Signals:

The *instantaneous power* in a signal g(t) is defined as that power dissipated in a 1-Ω resistor, i.e.,

$$P(t) = |g(t)|^2 .$$

The *average power* is defined as:

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2\, dt$$

The total energy of a signal g(t) is

$$E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2\, dt$$

A signal g(t) is classified as *energy signal* if it has a finite energy, i.e, $0 < E < \infty$

A signal g(t) is classified as *power signal* if it has a finite power, i.e, $0 < P_{av} < \infty$

The average power in a periodic signal is

$$P_{av} = \frac{1}{T_0}\int_{0}^{T_0} |g(t)|^2\, dt \text{ ; } T_0 \text{ is the period .}$$

Usually, periodic signals and random signals are power signals. Both deterministic and non periodic signals are energy signals.

4. Analog Signals, Digital Signals :

An *analog signal* is a continuous time - continuous amplitude function of time .

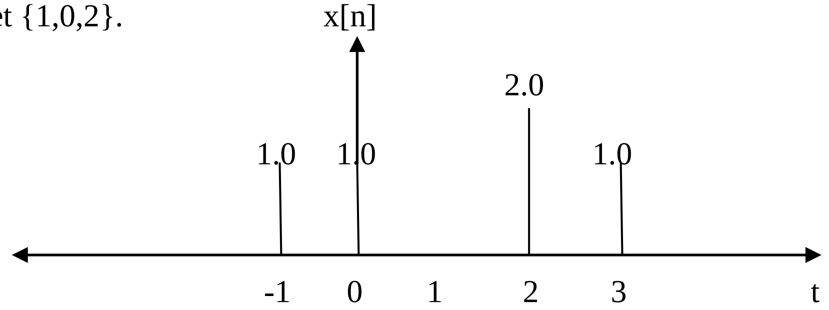
Example:

The sinusoidal signal $x(t) = A\cos 2\pi f t$, $-\infty < t < \infty$, is an example of an analog signal.

A *discrete time- discrete amplitude* (digital) signal is defined only at discrete times. Here, the independent variable takes on only discrete values.

Example:

The sequence x[n] shown below is an examples of a digital signal. The amplitudes are drawn from the finite set {1,0,2}.



More Examples

Example: An Exponential Pulse

Find the energy in the signal g(t) = A $e^{-\propto t}$ u(t).

E= $\int_0^\infty A^2\ e^{-2\propto t}dt = A^2 \frac{-e^{-2\propto t}}{2\propto}\Big|_0^\infty = \frac{A^2}{2\propto}$. Since E is finite, then g(t) is an energy signal.

Example: A rectangular Pulse

Find the energy in the signal:

$$g(t) = \begin{cases} A, & 0 < t < \tau \\ 0, & o.w \end{cases}$$

E= $\int_0^\tau A^2\ dt = A^2\tau$. This signal is an energy since E is finite.

Example: A Periodic Sinusoidal Signal

Find the average power in the signal :

g(t) = A $\cos\omega t$, $-\infty < t < \infty$

Since g(t) is periodic, then :

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} A^2\cos^2\omega t\, dt = \frac{A^2}{T_0}\int_0^{T_0}\left(\frac{1+\cos 2\omega t}{2}\right) = \frac{A^2}{T_0}\cdot\frac{T_0}{2} = \frac{A^2}{2}\ .$$

P_{av} is finite and so g(t) is a power signal.

Example: A Periodic Saw-tooth Signal

Find the average power in the saw-tooth signal g(t) plotted in Fig.1.

g(t) = $\frac{A}{T_0}$ t , $0 \leq t \leq T_0$

$P_{av} = \frac{1}{T_0}\int_0^{T_0} \frac{A^2}{T_0{}^2}\, t^2 dt = \frac{1}{T_0}\frac{A^2}{T_0{}^2}\frac{t^3}{3}\Big|_0^{T_0} = \frac{A^2\, T_0{}^3}{3\, T_0{}^3} = \frac{A^2}{3}$.

Example: The Unit Step Function

Consider the signal: g(t) = A u(t).

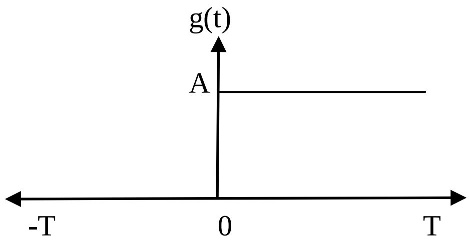


Fig. 1.2

This is a non periodic signal. So let us first try to find its energy:

$$E = \int_0^{\infty} A^2\, dt = \infty .$$

Therefore, g(t) is not an energy signal (E is not finite).

To find the average power, we employ the definition :

$$P_{av} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(t)|^2\, dt ,$$

where 2T is chosen to be a symmetrical interval about the origin, as in Fig. 1.2 above.

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T}\int_0^{T} A^2\, dt = \lim_{T\to\infty} \frac{A^2\, T}{2T} = \frac{A^2}{2} .$$

So, even-though g(t) is non a periodic, it turns out that it is a power signal.

This is an example where the general rule (periodic signals are power signals and energy signals are non periodic signals) fails to hold.

Fourier Series

Let g(t) be a periodic signal with period $T_0 = \frac{1}{f_0}$. The signal g(t) may be expanded in one of three possible Fourier series forms:

The complex form:

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}\, dt$;

C_n : is a complex valued quantity that can be written as:

$C_n = | C_n | e^{j\theta n}$

Discrete Amplitude Spectrum: A plot of $|C_n|$ vs. frequency

Discrete Phase Spectrum: A plot of θ_n vs. frequency

The term at ω_0 is referred to as the fundamental frequency. The term at $2\omega_0$ is referred to as the second harmonic, …

The trigonometric form:

$$g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Where : $a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\, dt$ (dc or average value)

$a_n = \frac{2}{T_0}\int_0^{T_0} g(t) \cos n\omega_0 t\, dt$

$b_n = \frac{2}{T_0}\int_0^{T_0} g(t) \sin n\omega_0 t\, dt$

The polar form :

$$g(t) = c_0 + \sum_{n=1}^{\infty} 2|C_n| \cos(n\omega_0 t + \theta_n)$$

where C_n and θ_n are those terms defined in the complex form.

Remark: The above three forms are equivalent and are representations of the same waveform. If you know one representation, you can easily deduce the other.

Example: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

Solution:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\,dt = A/2$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin\left(\frac{2\pi n}{T_0}\right)t\,dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\sin\left(\frac{2\pi n}{T_0}\right)t\,dt = 0$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos\left(\frac{2\pi n}{T_0}\right)t\,dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} A\cos\left(\frac{2\pi n}{T_0}\right)t\,dt$$

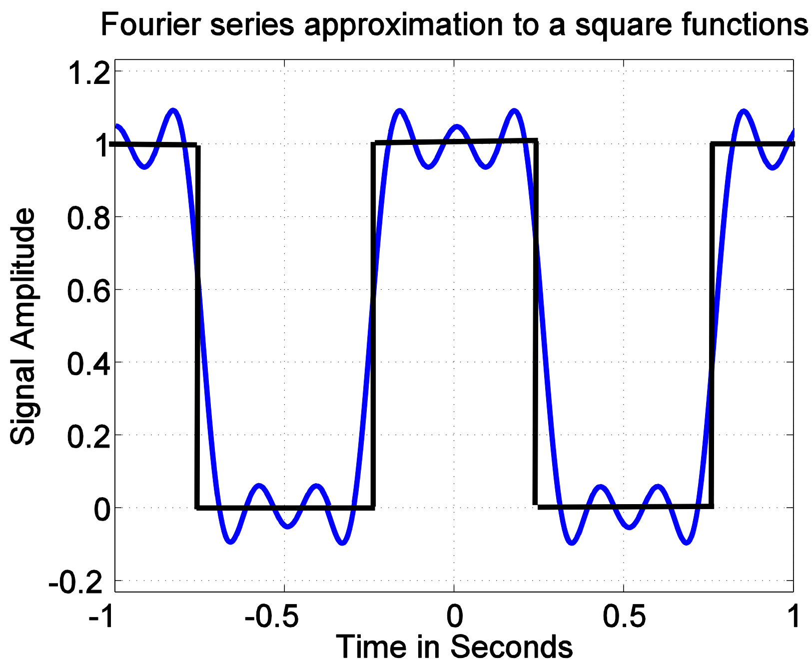
$$a_n = \begin{cases} \frac{2A}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{-2A}{n\pi}, & n = 3, 7, 11, \ldots \\ 0, & n = 2, 4, 6 \ldots \end{cases}$$

The first four terms in the expansion of $g(t)$ are:

$$\tilde{g}(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0)\,t - \frac{1}{3}\cos(2\pi 3 f_0)\,t + \frac{1}{5}\cos(2\pi 5 f_0)\,t\}$$

The function $\tilde{g}(t)$ along with $g(t)$ are plotted in the figure for $-1 \le t \le 1$

assuming $A = 1$ and $f_0 = 1$



Remark: As more terms are added to $\tilde{g}(t)$, $\tilde{g}(t)$ becomes closer to $g(t)$ and in the limit as $n \to \infty$, $\tilde{g}(t)$ becomes equal to $g(t)$ at all points except at the points of discontinuity.

Parseval's Power Theorem

The average power of a periodic signal g(t) is given by:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = |C_n|^2 + 2\sum_{n=1}^{\infty} |C_n|^2$$

$$= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$

Power Spectral Density

A plot of $|C_n|^2$ vs. frequency is called the *power spectral density* (PSD). It portrays the power content of each frequency (spectral) component of a signal. For a periodic signal, the PSD consists of discrete values at multiples of the fundamental frequency.

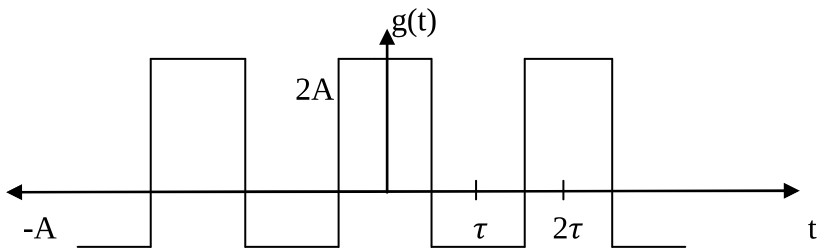
Exercise: Consider again the saw-tooth function defined over one period as $g(t) = t, 0 \le t \le 1$

a. Use matlab to find the dc terms and the first three harmonics (i.e., let n = 3) in the Fourier series expansion

$$\tilde{g}(t) = a_0 + \sum_{n=1}^{3}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

b. Plot $\tilde{g}(t)$ and $g(t)$ versus time for $-1 \le t \le 1$ on the same graph.
c. Find the fraction of the power contained in $\tilde{g}(t)$ to that in $g(t)$.
d. Sketch the power spectral density.

Example : Find the power spectral density of the periodic function g(t) shown in the figure :



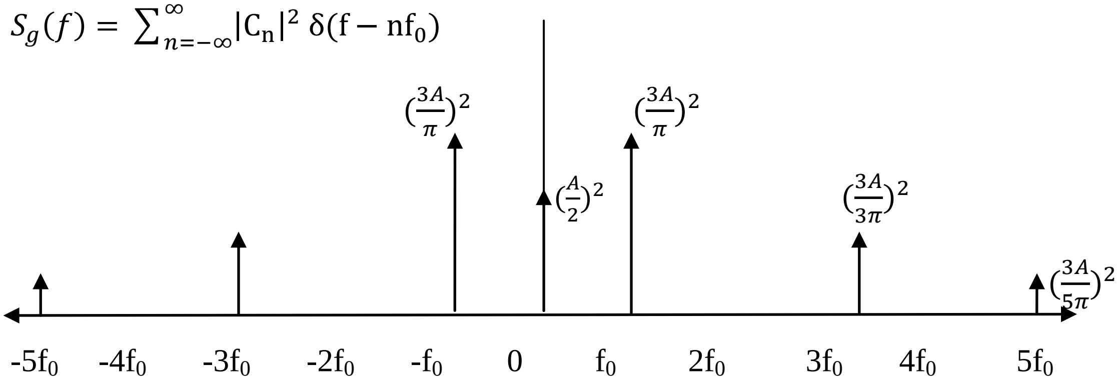
Solution: Here we need to find the complex Fourier series expansion, where the period $T_0 = 2\tau$

$$g(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} g(t)\, e^{-jn\omega_0 t}\, dt$$

$$C_n = \begin{cases} \frac{A}{2}, & n = 0 \\ \frac{3A}{|n|\pi}, & n = \pm1, \pm5, \pm9, \ldots \\ \frac{-3A}{|n|\pi}, & n = \pm3, \pm7, \pm11, \ldots \\ 0, & n = \pm2, \pm4, \ldots \end{cases} \quad \Longrightarrow \quad |C_n|^2 = \begin{cases} (\frac{A}{2})^2, & n = 0 \\ (\frac{3A}{n\pi})^2, & n: odd \\ 0, & n: even \end{cases}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \, \delta(f - nf_0)$$



As can be seen, the power spectral density of this periodic signal is a discrete function in frequency.

Exercise: Verify Parseval's power theorem for this signal, i.e., show that

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = 2.5A^2$$

Fourier Transform

Let g(t) be a non periodic square integrable function of time. That is one for which

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

The Fourier transform of g(t) exists and is defined as:

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} \, dt$$

The time function g(t) can be recovered from $G(f)$ using the inverse Fourier Transform:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} \, df$$

Remarks:

- All energy signals for which $E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$ are Fourier transformable.
- G(f) is a complex function of frequency f, which can be expressed as:
  G(f) = | G(f) | $e^{j\theta(f)}$

where, | G(f) | : is the *continuous amplitude spectrum* of g(t), (even function of f).

$\theta(f)$: is the *continuous phase spectrum* of g(t), (odd function of f).

Rayleigh Energy Theorem :

The energy in a signal g(t) is given by :

$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

The function $|G(f)|^2$ is called the *energy spectral density*. It illustrates the range of frequencies over which the signal energy extends and the frequency bands which are significant in terms of their energy contents. For a non-period signal energy signal, the energy spectral density is a continuous function of f.

A General Form of the Rayleigh Energy Theorem

For two energy functions g(t) and $v(t)$, the following result holds:

$$\int_{-\infty}^{\infty} g(t) v(t)^* \, dt = \int_{-\infty}^{\infty} G(f) V(f)^* \, df$$

Example: **Energy spectral density of the exponential signal**

$$v(t) = \begin{cases} A\,e^{-bt} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$V(f) = \int_0^\infty v(t) e^{-j2\pi ft}\, dt = \int_0^\infty A e^{-bt}\; e^{-j2\pi ft} dt$$

$V(f) = A \int_0^\infty e^{-(b+j2\pi f)t}\; dt$ = A $\frac{e^{-(b+j2\pi f)t}}{-(b+j2\pi f)} \Big|_0^\infty = \frac{A}{b+j2\pi f}$.

$V(f) = \frac{A}{b+j2\pi f} \Rightarrow |V(f)| = \frac{A}{(b^2+(2\pi f)^2)^{1/2}}$

The energy spectral density is: $S_v(f) = |V(f)|^2 = \frac{A^2}{b^2+\omega^2}$

Remark: The signal $v(t)$ is called a *baseband signal* since the signal occupies the low frequency part of the spectrum. That is, the energy in the signal is found around the zero frequency. When the signal is multiplied by a high frequency carrier, the spectrum becomes centered around the carrier and the modulated signal is called a *bandpass signal*.

Exercise : For the exponential pulse verify Rayleigh energy theorem, i.e., show that

$\int_0^\infty |v(t)|^2 dt = 2\int_0^\infty |V(f)|^2 df = \frac{A^2}{2b}$.

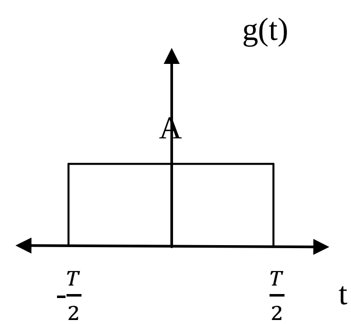
Example: The Rectangular Pulse $g(t) = A rect(\frac{t}{T})$

$G(f) = \int_{-T/2}^{T/2} A e^{-j2\pi ft} dt = \frac{A}{\pi f} \sin \pi f T$

$= \text{A}T\, \frac{\sin \pi f T}{\pi f T} \triangleq AT\; sinc\; Tf$

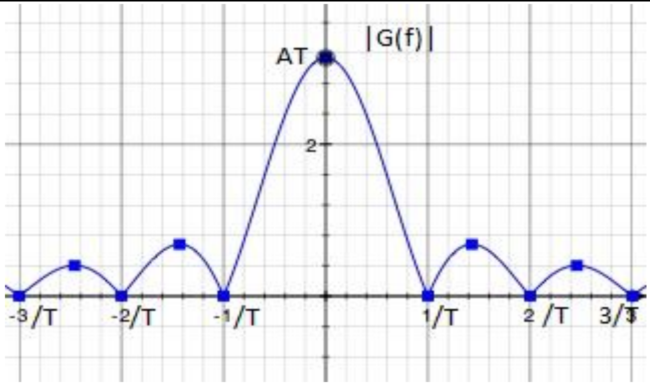
$|G(f)| = AT\; |sinc\; Tf|$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow$ max. of function



$G(f) = 0\; when\; sin\; \pi f T = 0$ or $when\; \pi f T = n\pi$, n=±1, ±2, ±3,…

$fT = n\;, \therefore f = \frac{n}{T}$



<u>Remark:</u> Time duration and bandwidth :

Note that as the signal time duration T *increases,* the first zero crossing at $f = \frac{1}{T}$ decreases, implying that B.W of signal decreases. More on this will be said later when we discuss the time bandwidth product.

Exercise : For the rectangular pulse $g(t) = A rect(\frac{t}{T})$ verify Rayleigh energy theorem, i.e., show that

$\int_0^\infty |g(t)|^2 dt = 2\int_0^\infty |G(f)|^2 df = A^2 T.$

Properties of the Fourier Transform:

1. Linearity (superposition)
   Let $g_1(t) \leftrightarrow G_1$(f) and $g_2(t) \leftrightarrow G_2$(f) , then :
   $c_1 g_1(t)$+ $c_2 g_2(t) \leftrightarrow c_1 G_1(f)$+ $c_2 G_2(f)$; c_1, c_2 are constants
2. Time scaling

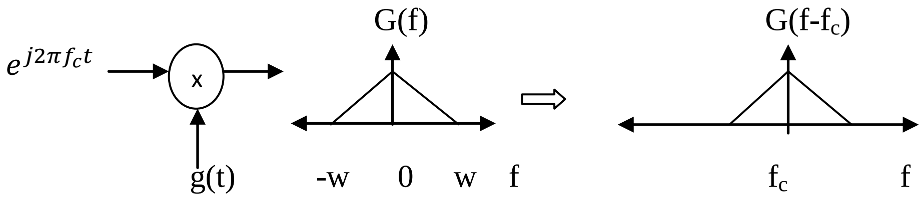
   $$g(at) \leftrightarrow \frac{1}{|a|} G(f/a)$$
3. Duality
   If $g(t) \leftrightarrow G(f)$, then : $G(t) \leftrightarrow g(-f)$
4. Time shifting

   If $g(t) \leftrightarrow G(f)$, then $g(t - t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$

   Delay in time $\Longrightarrow$ phase shift in frequency domain

5. Frequency shifting: If $g(t) \leftrightarrow G(f)$, then :

$g(t)\, e^{j2\pi fct} \leftrightarrow G(f - fc)$; f_c is a real constant



6. Area under $G(f)$: $If\ g(t) \leftrightarrow G(f)$, then:

$$g(0) = \int_{-\infty}^{\infty} G(f)df$$

The value $g(t = 0)$ is equal to the area under its Fourier transform.

7. Area under $g(t)$: If $g(t) \leftrightarrow G(f)$, then:

$$G(0) = \int_{-\infty}^{\infty} g(t)dt$$

The area under a function g(t) is equal to the value of its Fourier transform G(t) at f = 0.

Where $G(0)$ implies the presence of a dc component.

8. Differentiation in the time domain

If $g(t)$ and its derivative $g'(t)$ are Fourier transformable, then :

$$g'(t) \leftrightarrow (j2\pi f)G(f)$$

i.e., differentiation in the time domain ⟹ multiplication by $j2\pi f$ in the frequency domain.

(enhances high frequency components of a signal while attenuates low frequency components)

Also, $\frac{d^n\, g(t)}{dt^n} \leftrightarrow (j2\pi f)^n\, G(f)$

9. Integration in the time domain

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j2\pi f} G(f) ;\ \text{assuming } G(0) = 0.$$

i.e., integration in the time domain ⟹ division by $(j2\pi f)$ in the frequency domain.

(enhancement of low frequency components of the signal).

When $G(0) \neq 0$, the above result becomes :

$$\int_{-\infty}^{t} g(\tau)d\tau \quad \leftrightarrow \quad \frac{1}{j2\pi f} G(f) + \frac{1}{2} G(0)\delta(f).$$

10. Conjugate Functions

For a complex – valued time signal $g(t)$, we have:

$$g^*(t) \quad \leftrightarrow \quad G^*(-f) \quad ;$$

Also, $$g^*(-t) \quad \leftrightarrow \quad G^*(f) \quad ;$$

Therefore, $$\mathrm{Re}\{g(t)\} \quad \leftrightarrow \quad \frac{1}{2} \{ G(f) + G^*(-f)\}$$

$$\mathrm{Im}\{g(t)\} \quad \leftrightarrow \quad \frac{1}{2j} \{ G(f) - G^*(-f)\}$$

11. Multiplication in the time domain

$$g_1(t)\, g_2(t) \quad \leftrightarrow \quad \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)\, d\lambda = G_1(f) * G_2(f)$$

Multiplication of two signals in the time domain is transformed into the convolution of their Fourier transforms in the frequency domain.

12. Convolution in the time domain

$$g_1(t) * g_2(t) \quad \leftrightarrow \quad G_1(f)G_2(f)$$

Convolution of two signals in the time domain is transformed into the multiplication of their Fourier transforms in the frequency domain.

Fourier Transform of Power Signals

For a non-periodic (energy) signal, the Fourier transform exists when

$$\mathrm{E} = \int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$$

So that $G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$.

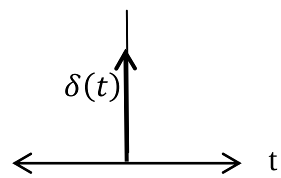
For power signals, the integral $\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$ **does not exist**.

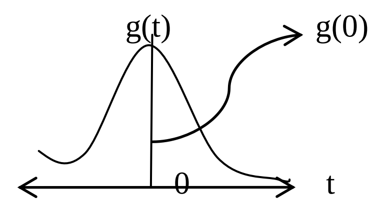
However, one can still find the Fourier transform of power signals by employing the delta function. This function is defined next.

Dirac – Delta Function (impulse function)

This function is defined as

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$





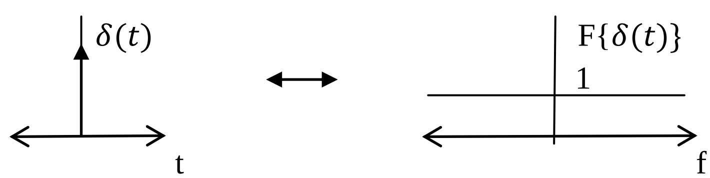
Such that $\int_{-\infty}^{\infty} \delta(t)dt = 1$

and $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$

(Here, g(t) is a continuous function of time).

Some properties of the delta function:

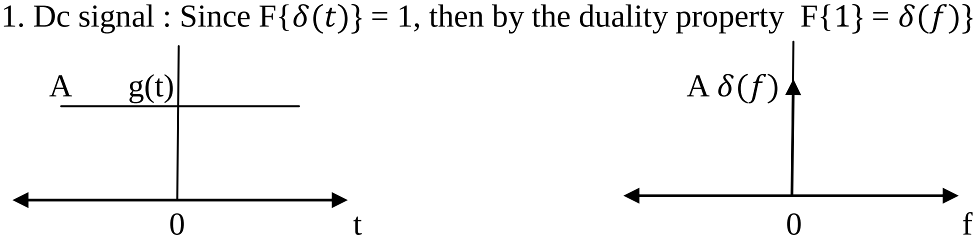
1. $g(t)\delta(t - t_0) = g(t_0)\delta(t - t_0)$; (Multiplication)
2. $\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0)$; (Shifting)
3. $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$
4. $\delta(t) * g(t) = g(t)$
5. $\delta(t) = \frac{du(t)}{dt}$ $\Longrightarrow$ u(t) = $\int_{-\infty}^{t} \delta(t)dt$
6. $\delta(t) = \delta(-t)$
7. Fourier transform : F{$\delta(t)$} = 1



8. 8. F{$\delta(t - t_0)$} $= e^{-j2\pi f t_0}$

Applications of delta functions

1. Dc signal : Since F{$\delta(t)$} = 1, then by the duality property F{1} = $\delta(f)$}



2. Complex exponential function

$F\{A\, e^{j2\pi f_c t}\} = A\, \delta(f - f_c)$

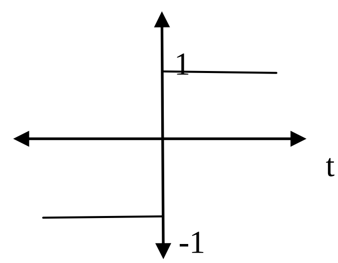
3. Sinusoidal functions

$$F\{\cos 2\pi f_c t\} = \frac{1}{2}\{\delta(f - f_c) + \delta(f + f_c)\}$$

$$F\{\sin 2\pi f_c t\} = \frac{1}{2j}\{\delta(f - f_c) - \delta(f + f_c)\}$$

4. Signum function

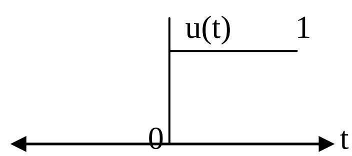
$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$$F\{\text{sgn}(t)\} = \frac{1}{j\pi f}$$

5.Unit Step function :

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}$$



$\text{sgn}(t) = 2u(t) - 1$

$u(t) = \frac{1}{2}\{\text{sgn}(t) + 1\}$

$$F\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)$$

6.Periodic Signals

A periodic signal g(t) is expanded in the complex form as :

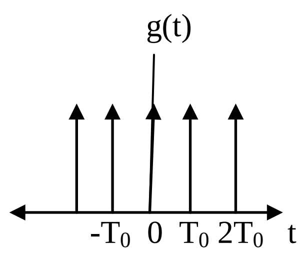
$$g(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

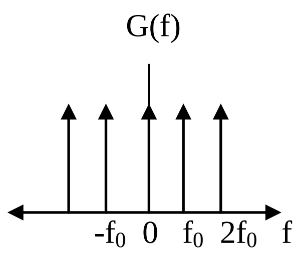
$$F\{g(t)\} = \sum_{n=-\infty}^{\infty} C_c\, \delta(f - nf_0)$$

When $g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$; impulse train in time domain

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0} = f_0$$

$$\Rightarrow F\{g(t)\} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

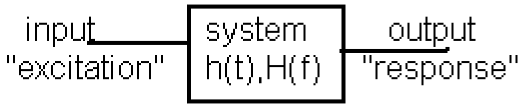




Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain. This sequence will be found useful when the sampling theorem is considered.

Transmission of Signals Through Linear Systems

Definition : A <u>system</u> refers to any physical device that produces an output signal in response to an input signal.



Definition : A system is <u>linear</u> if the principle of superposition applies.

If	$x_1(t)$	produces output	$y_1(t)$
	$x_2(t)$	produces output	$y_2(t)$
then	$a_1x_1(t)+ a_2x_2(t)$	produces an output	$a_1y_1(t)+ a_2y_2(t)$
Also,	a zero input	should produce	a zero output.

Example of linear systems include <u>filters</u> and <u>communication</u> channels .

Definition : A <u>filter</u> refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies.

Definition : A <u>channel</u> refers to a transmission medium that connects the transmitter and receivers of a communication system .

Time domain and frequency domain may be used to evaluate system performance.

Time response :

Definition : The <u>impulse response</u> h(t) is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .

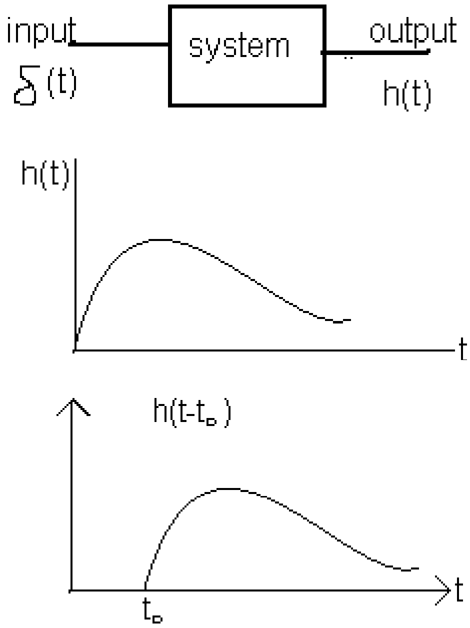
Definition : A system is <u>time-invariant</u> when the shape of the impulse response is the same no matter when the impulse is applied to the system .

$\delta(t) \longrightarrow h(t)$, then $\delta(t-t_d) \longrightarrow h(t-t_d)$

When the input to a linear time-invariant system in a signal x(t) , then the output is given by

$$y(t) = y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda \;; \quad \text{convolution integral}$$



Definition : A system is said to be causal if it doesn't respond before the excitation is applied , i.e. ,

$$h(t) = 0 \qquad t < 0$$

The causal system is physically realizable.

Definition : A system is said to be stable if the output signal is bounded for all bounded input signals .

If $|x(t)| \leq M$; M is the maximum value of the input

then $|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| \, d\tau$

$$= M \int_{-\infty}^{\infty} |h(\tau)| \, d\tau$$

⟹ A necessary and sufficient condition for stability (a bounded output) is

$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$; h(t) is absolutely integrable.

∴ zero initial conditions assumed .

Frequency Response :

Definition : The transfer function of a linear time invariant system is defined as the Fourier transform of the impulse response .

$$H(f) = F\{h(t)\}$$

Since $y(t) = x(t)*h(t)$, then

$$Y(f) = H(f)\, X(f)$$

or $\frac{Y(f)}{X(f)} = H(f)$

The transfer function H(f) is a complex function of frequency, which can be obtained as the ratio of the Fourier transform of the output to that of the input.

$$H(f) = |H(f)| e^{j\,\theta(f)}$$

where

H(f) : amplitude spectrum

θ(f) : phase spectrum.

System Input–Output Energy Spectral Density

Let x(t) be applied to a LTI system , then the Fourier transform of the output is related to the Fourier transform of the input through the relation

$Y(f) = H(f)\, X(f)$

Taking the absolute value and squaring both sides, we get

$|Y(f)|^2 = |H(f)|^2\ |X(f)|^2$

$S_Y(f) = |H(f)|^2\, S_X(f)$

$S_Y(f)$: Output Energy Spectral Density
$S_X(f)$: Input Energy Spectral Density.

Output energy spectral density $= |H(f)|^2$ x Input energy spectral density

The total output energy

$$E_y = \int_{-\infty}^{+\infty} S_Y(f)df$$
$$= \int_{-\infty}^{+\infty} |H(f)|^2\, S_X(f)df.$$

The total input energy is

$$E_x = \int_{-\infty}^{+\infty} S_x(f)df\ .$$

Example: Response of a Filter to a Sinusoidal Input

The signal $x(t) = \cos w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the filter output $y(t)$.

Solution:

We will find the output using the frequency domain approach.

$$Y(f) = H(f)X(f)$$

$$H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta}; \qquad \theta = \tan^{-1}\frac{f}{B}; \qquad \theta_0 = \tan^{-1}\frac{f_0}{B}$$

$$Y(f) = H(f)[\tfrac{1}{2}\delta(f - f_0) + \tfrac{1}{2}\delta(f + f_0)]$$

$$Y(f) = \tfrac{1}{2}H(f_0)\delta(f - f_0) + \tfrac{1}{2}H(-f_0)\delta(f + f_0)$$

$$Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f - f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f + f_0)$$

Taking the inverse Fourier transform, we get

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\frac{1}{2}[e^{j(2\pi f_0 t-\theta_0)} + e^{-j(2\pi f_0 t-\theta_0)}]$$

$$y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\cos(2\pi f_0 t - \theta_0)$$

Note that in the last step we have made use of the Fourier transform pair

$$e^{j2\pi f_c t} \leftrightarrow \delta(f - f_c)$$

Assume, for instance, that $f_0 = B$. Then $\theta_0 = \tan^{-1}\frac{f_0}{B} = \tan^{-1} 1 = 45°$ and the output can be written as:

$$y(t) = \frac{1}{\sqrt{1+1}}\cos(2\pi f_0 t - 45°)$$

$$y(t) = \frac{1}{\sqrt{2}}\cos(2\pi f_0 t - 45°)$$

Exercise: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.

Exercise: Consider the periodic rectangular signal $g(t)$ defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}$$

If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.

Example:

The signal $g(t) = Arect(\frac{t}{T})$ is applied to the filter $(f) = \frac{1}{1+jf/B}$. Find the output energy spectral density.

Solution:

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_Y(f) = \frac{1}{1+(\frac{f}{B})^2}|AT\ |sinc\ Tf||^2$$

Example:

The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output.

Solution:

The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is

$$h(t) = 2\pi B e^{-2\pi Bt} u(t)$$

Using the linearity and time invariance property, the output can be obtained as:

$$y(t) = h(t)u(t) - h(t-1)u(t-1)$$

$$y(t) = 2\pi B[e^{-2\pi Bt}u(t) - e^{-2\pi B(t-1)}u(t-1)]$$

Exercise: The signal $g(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.

Signal Distortion in Transmission

As we have said before, the objective of a communication system is to deliver to the receiver an almost exact copy of what the source sends. However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes attenuation, phase delay, interference from other transmissions, Doppler shift in the carrier frequency, and many other effects. In this introductory discussion we will explain some of the reasons that cause the received signal to be distorted.

a. Linear Distortion

A signal transmission is said to be distortion-less if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.

Condition in the time domain for a distortion-less transmission:

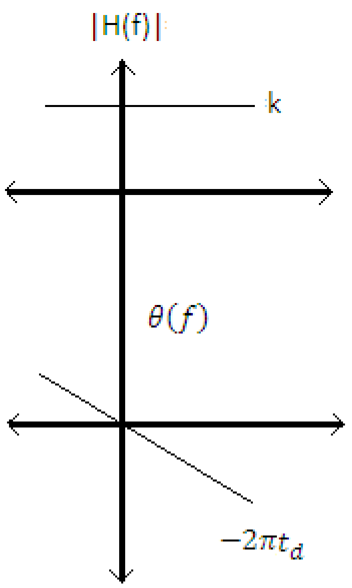
$$y(t) = k\, x(t-t_d)$$

where k: is a constant amplitude scaling

t_d: is a constant time delay

In the frequency domain, the condition for a distortion-less transmission becomes

$$Y(f) = k\, X(f)\, e^{-j2\pi f t_d}$$



or $\quad H(f) = \frac{Y(f)}{X(f)} = k\, e^{-j2\pi f t_d} = ke^{-j\theta(f)}$

That is, for a distortion-less transmission, the transfer function should satisfy two conditions:

1. $|H(f)| = k$; where k is a constant amplitude over the frequency range of interest.
2. $\theta(f) = -2\pi f t_d = -(2\pi t_d)f$; linear phase with negative slope that passes through the origin (or multiples of π).

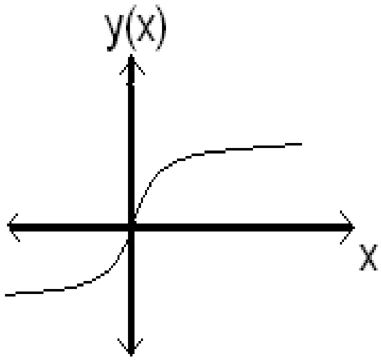
When |H(f)| is not a constant for all frequencies of interest, amplitude distortion results.

When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have phase distortion (or delay distortion).

b. Non Linear Distortion

When a system contains nonlinear elements , it is not described by a transfer function , but by a transfer characteristic of the form

$y(t) = a_1\, x(t) + a_2\, x^2(t) + a_3\, x^3(t) + \ldots$ (time domain)



In the frequency domain ,

$Y(f) = a_1\, X(f) + a_2\, X(f)*X(f) + a_3\, X(f)*X(f)*X(f) + \ldots$

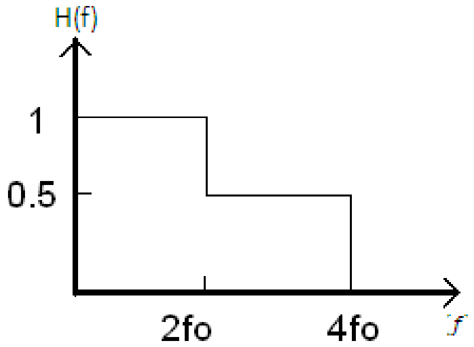
Here, the output contains new frequencies not originally present in the original signal . The nonlinearity produces undesirable frequency component for $|f| \leq w$.

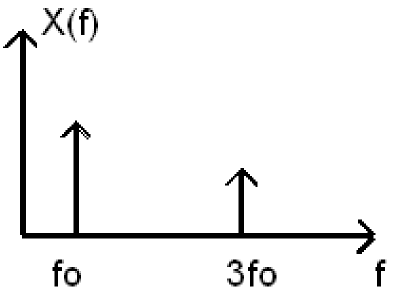
The following examples demonstrate the types of distortion mentioned above.

Example : Amplitude Distortion

Consider the signal $x(t) = \cos w_0 t - \frac{1}{3} \cos 3w_0 t$. If this signal passes through a channel with zero time delay (i.e. , $t_d = 0$) and amplitude spectrum as shown in the figure

a. Find y(t)
b. Is this a distortion-less transmission ?





Solution :

x(t) consists of two frequency components, f_0 and $3f_0$. Upon passing through the channel, each one of them will be scaled by a different factor.

a. $y(t) = \ cos\, w_0 t \ - \frac{1}{2} . \ \frac{1}{3} \ cos\ 3w_0 t$

b. Since y(t) ≠ k x(t), this is not a distortion-less transmission .

Example : Phase Distortion

If x(t) in the previous example is passed through a channel whose amplitude spectrum is a constant k. Each component in x(t) suffers a $\frac{\pi}{2}$ phase shift

a. Find y(t).

b. Is this a distortion-less transmission ?

Solution :

$$x(t) = \cos w_o t - \frac{1}{3}\cos 3w_o t$$

$$y(t) = \text{k} \ \cos(w_o t - \frac{\pi}{2}) \ - \ \frac{1}{3} \ k \ \cos\left(3w_o t - \frac{\pi}{2}\right)$$

$$y(t) = \text{k} \ \cos w_o (t - \frac{\pi}{2w_o}) \ - \ \frac{1}{3} \ k \ \cos\left(3w_o (t - \frac{\pi}{2x3w_o})\right)$$

$$y(t) = \text{k} \ \cos w_o (t - t_{d1}) \ - \ \frac{1}{3} \ k \ \cos(3w_o (t - t_{d2}))$$

Note that $t_{d1} \neq t_{d2}$, i.e., each component in $x(t)$ suffers from a different time delay. Hence this transmission introduces phase (delay) distortion.

Harmonic Distortion

Let the input to a nonlinear system be the single tone signal

$x(t)=\cos 2\pi f_o t$

This signal is applied to a channel with characteristic

$y(t)=a_1x+a_2x^2+a_3x^3$

upon substituting x(t) and arranging terms, we get

$$y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}\, a_2 \cos 4\pi f_0 t + \frac{1}{4}a_3 cos6\pi\, f_0 t$$

Note that the output contains a component proportional to x(t) which is $\left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t$, in addition to a second and a third harmonic term (terms at twice and three times the frequency of the input). These new terms are the result of the nonlinear characteristic and are , therefore, considered harmonic distortion.

Define second harmonic distortion

$$D_2 = \frac{|amplitude\ of\ second\ harmonic\,|}{|amplitude\ of\ fundamental\ term|}$$

$$D_2 = \frac{|\frac{1}{2}a_2\,|}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

In a similar way we can define the third harmonic distortion as:

$$D_2 = \frac{|amplitude\ of\ third\ harmonic\,|}{|amplitude\ of\ fundamental\ term|}$$

Therefore,

$$D_3 = \frac{|\frac{1}{4}a_3\,|}{|\left(a_1 + \frac{3}{4}a_3\right)|}\ x\ 100\%$$

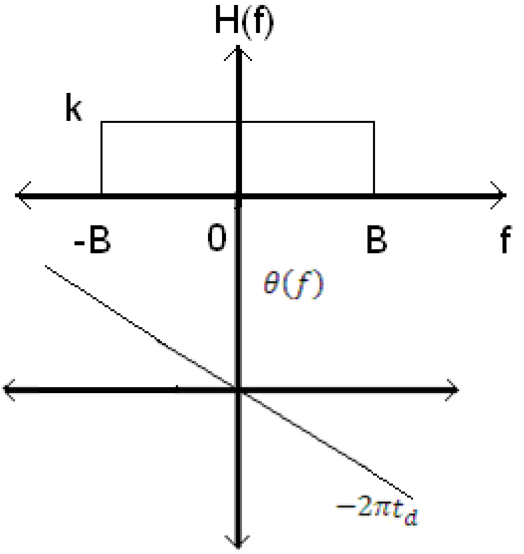
Remark: In the solution above we have made use of the following two identities:

$\mathrm{Cos}^2 x = \frac{1}{2}\{1 + cos2x\}$

$\text{Cos}^3x = \frac{1}{4}\{3cosx + cos3x\}$.

Filters and Filtering

A filter is a frequency selective device . It allows certain frequencies to pass almost without attenuation wile it suppresses other frequencies

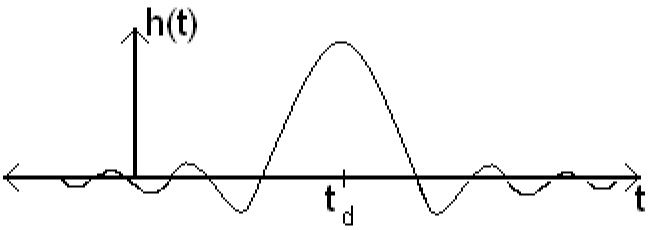


A. <u>Ideal Filter:</u>

<u>Ideal low pass filter :</u>

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$$

$$h(t) = 2Bk\ sinc\ 2B(t - t_d)$$



since h(t) is the response to an impulse applied at t=0 ,and because h(t) has nonzero values for t<0 , the filter is <u>noncausal</u> (physically non realizable)

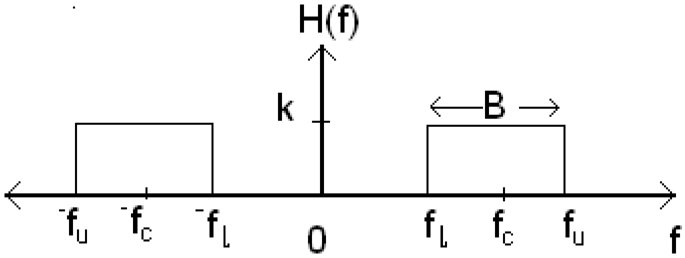
<u>Band Pass Filter</u>

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$$

Filer bandwidth B = $f_u - f_l$

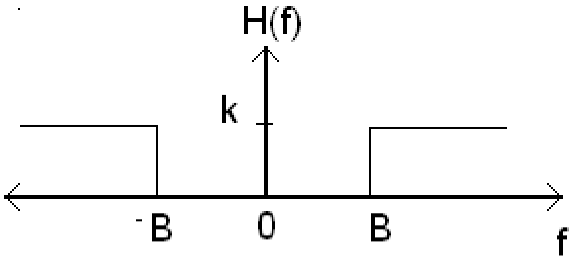
$$f_c = \frac{f_u + f_l}{2}$$

$$h(t) = 2Bk\ sinc\ B(t - t_d)\ \ cos\ w_c(t - t_d)$$



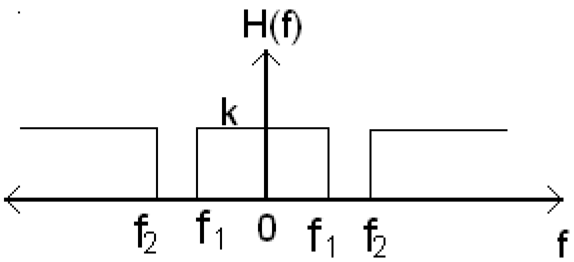
High pass filter :

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$$



Band Rejection or Notch Filter

$$H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}$$



Real Filter:

Here we only consider a Butterworth low pass filter. The transfer function of a low pass Butterworth filter is of the form

$$H(f) = \frac{1}{P_n(\frac{jf}{B})}$$

B is the 3-dB bandwidth of the filter and $P_n(jf/B)$ is a complex polynomial of order n . The family of Butterworth polynomials is defined by the property

$$|P_n\left(\frac{jf}{B}\right)|^2 = 1 + (\frac{f}{B})^{2n}$$

So that

$$|H(f)| = \frac{1}{\sqrt{1+(\frac{f}{B})^{2n}}}$$

The first few polynomials are:

$P_1(x) = 1 + x$

$P_2(x) = 1 + \sqrt{2}x + x^2$

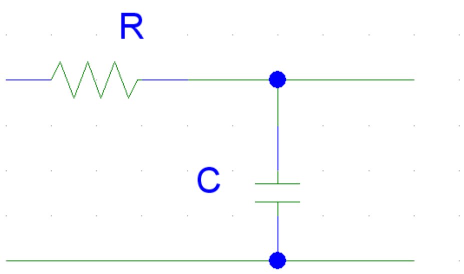
$P_3(x) = (1 + x)(1 + x + x^2)$

<u>A first order LPF :</u>

$$H(f) = \frac{\frac{1}{j2\pi f c}}{R+\frac{1}{j2\pi f c}} = \frac{1}{1+j2\pi fRC}$$

Let $B = \frac{1}{2\pi RC}$

$$H(f) = \frac{1}{1 + jf/B} = \frac{1}{P_1(jf/B)} = \frac{1}{P_1(x)}$$



<u>A Second order LPF :</u>

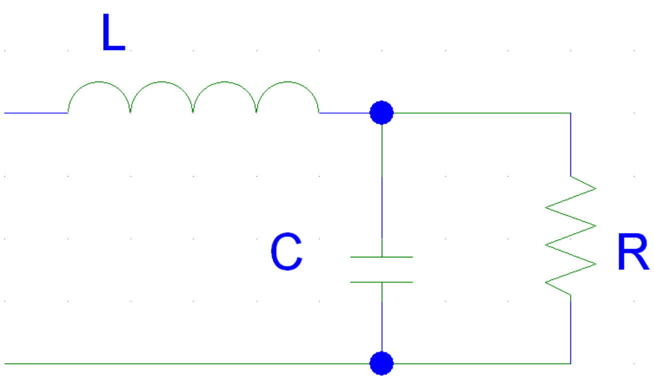
$$H(f) = \frac{1}{1+\frac{jwL}{R}-(2\pi\sqrt{LC}f)^2}$$

$$H(f) = \frac{1}{1 + j\sqrt{2}f/B - (f/B)^2}$$

$where\ \ R = \sqrt{\frac{L}{2C}},\ B = \frac{1}{2\pi\sqrt{LC}}$

$$H(f) = \frac{1}{1 + j\sqrt{2}f/B - (f/B)^2}$$

$$H(f) = \frac{1}{P_2(jf/B)}$$



Hilbert Transform

<u>The quadrature filter :</u> is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°). The transfer function of such a filter is

$$H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

Using the duality property of Fourier transform the impulse response of the filter is

$h(t) = \frac{1}{\pi t}$

The Hilbert transform of a signal g(t) is

$$\hat{g}(t) = \frac{1}{\pi t} * g(t) = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)}\, d\lambda$$

Note that the Hilbert transform of a signal is a function of time. The Fourier transform of $\hat{g}(t)$ is

$$\hat{G}(f) = -j \operatorname{sgn}(f)\, G(f)$$

Hilbert transform can be found by using either the time domain approach or the frequency domain approach depending on the given problem, that is

- Direct convolution in the time domain of g(t) and $\frac{1}{\pi t}$.
- Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform $\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

Some properties of the Hilbert transform

1. A signal g(t) and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density

$$\left|\hat{G}(f)\right|^2 = |-j\, sgn(f) |G(f)||^2 = |-j\, sgn(f)|^2 |G(f)|^2$$
$$= |G(f)|^2$$

The consequences of this property are:

- If a signal g(t) is bandlimited, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and g(t) have the same total energy (or power).
- $\hat{g}(t)$ and g(t) have the same autocorrelation function.

2. A signal g(t) and $\hat{g}(t)$ are orthogonal

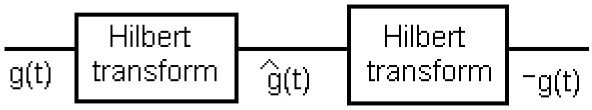
$$\int_{-\infty}^{\infty} g(t)\, \hat{g}(t) dt = 0$$

This property can be verified using the general formula of Rayleigh energy theorem

$$\int_{-\infty}^{\infty} g(t)\, \hat{g}(t) dt = \int_{-\infty}^{\infty} G(f)\, \hat{G}^*(f) df = \int_{-\infty}^{\infty} G(f)\, \{-jsgn(f)\, G(f)\}^* df$$
$$= \int_{-\infty}^{\infty} jsgn(f)\, |G(f)|^2 df = 0$$

The result above follows from the fact that $|G(f)|^2$ is an even function of f while $sgn(f)$ is an odd function of f. Their product is odd. The integration of an odd function over a symmetrical interval is zero.

3. If $\hat{g}(t)$ is a Hilbert transform of g(t) , then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$.



Example on Hilbert Transform

Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

Solution:

Here, we use the convolution in the time domain

$$\hat{g}(t)=\frac{1}{\pi t} * \delta(t)$$

As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,

$$\hat{g}(t)=\frac{1}{\pi t}$$

Example on Hilbert Transform

Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$

Solution :

Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$

$$A\,rect\left(\frac{t}{\tau}\right) \quad \xleftrightarrow{transform} \quad A\tau\, sinc\, f\tau \quad ; \quad when\ \tau = \frac{1}{\pi}$$

$$A\ \mathrm{rect}\left(\frac{t}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad A\,\frac{1}{\pi}\,\frac{\sin \pi f\tau}{\pi f\tau} = \frac{1}{\pi}\frac{\sin f}{f}$$

$$\pi\, rect\left(\frac{t}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad \frac{\sin f}{f}$$

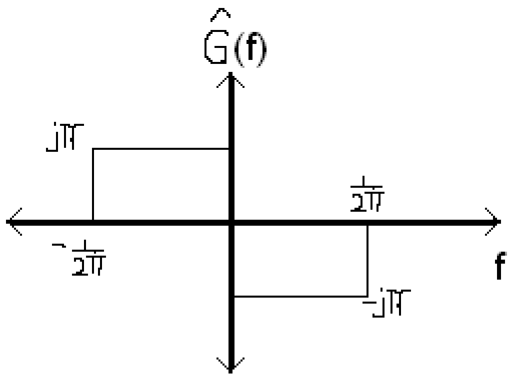
So by the duality property, we get the pair

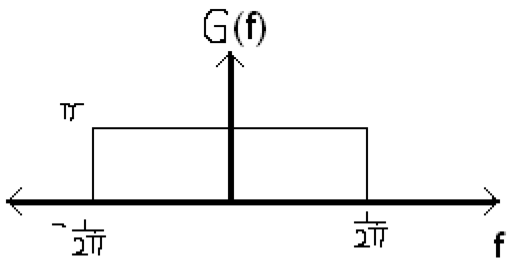
$$\pi\, rect\left(\frac{f}{1/\pi}\right) \quad \xleftrightarrow{transform} \quad \frac{\sin t}{t}$$

i.e. , $G(f) = \pi\, rect(\frac{f}{1/\pi})$, (See the figure below)

$$\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$$

$$\begin{aligned} \hat{g}(t) &= \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi ft}\, df \\ &= \int_{-1/2\pi}^{0} j\pi \quad e^{j2\pi ft}\, df \; - \; \int_{0}^{1/2\pi} j\pi \quad e^{j2\pi ft}\, df \\ &= \frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right) \\ &= \frac{1}{t} - \frac{1}{t}\,\frac{\left(e^{jt}+e^{-jt}\right)}{2} \\ &= \frac{1-\cos t}{t} \end{aligned}$$





Correlation and Spectral Density

Here we consider the relationship between the autocorrelation function and the power spectral density. In this discussion we restrict our attention to real signals. First, we consider power signals and then energy signals.

Definition: The autocorrelation function of a signal g(t) is a measure of similarity between g(t) and a delayed version of g(t).

a. **Autocorrelation function of a power signal**

The autocorrelation function of a power signal g(t) is defined as:

$R_g(\tau) = \langle g(t)g(t-\tau)\rangle$; $\langle(.)\rangle$ means time average.

$$R_g(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(t)g(t-\tau)dt$$

Exercise: Show that for a periodic signal with period T_0, the above definition becomes

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

Exercise: Show that if $g(t)$ is periodic with period T_0, then $R_g(\tau)$ is also periodic with the same period T_0.

Hint: Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0\tau}$. Form the delayed signal $g(t-\tau)$, and then perform the integration over a complete period T_0. You should get the following result:

$$R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau}$$

This formula bears two results

a. $R_g(\tau)$ is periodic with period T_0.
b. The Complex Fourier coefficients D_n of $R_g(\tau)$ are related to the complex Fourier coefficients C_n of $g(t)$ by the relation $D_n = |C_n|^2$.

Properties of R(τ)

- $R_g(0) = \frac{1}{T_0}\int_0^{T_0} g(t)^2 dt$; is the total average signal power.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e. $|R_g(\tau)| \le R_g(0)$.
- If g(t) is periodic with period T_0 , then $R_g(\tau)$ is also periodic with the same period T_0.
- The autocorrelation function of a periodic signal and its power spectral density (represented by a discrete set of impulse functions) are Fourier transform pairs

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

Cross Correlation Function

The cross correlation function of two periodic signals $g_1(t)$ and $g_2(t)$ with period T_0 is defined as;

$$R_{1,2}(\tau) = \frac{1}{T_0}\int_0^{T_0} g_1(t) g_2(t-\tau) dt$$

b- **Autocorrelation function of an energy signal**

When g(t) is an energy signal, $R_g(\tau)$ is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(t) g(t-\tau) dt$$

Properties of R(τ)

- $R_g(0) = \int_{-\infty}^{\infty} g(t)^2 dt$; is the total signal energy.
- $R_g(\tau)$ is an even function of τ, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at $\tau = 0$, i.e. $|R_g(\tau)| \le R_g(0)$.

- The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are Fourier transform pairs, i.e.,

$$S_g(f) = \mathrm{F}\{R_g(\tau)\}$$

$$S_g(f) = \int_{-\infty}^{\infty} R_g(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_g(\tau) = \int_{-\infty}^{\infty} S_g(f) e^{j2\pi f\tau} df.$$

Proof:
The autocorrelation function is defined as:

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda) g(\lambda - \tau) d\lambda$$

In this integral we have replaced t by λ. With this substitution, we can rewrite the integral as

$$R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda) g(-(\tau - \lambda)) d\lambda$$

One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $-g(\tau)$. That is,

$$R_g(\tau) = g(\tau) * g(-\tau)$$

Taking the Fourier transform of both sides, we get

$$F\{R_g(\tau)\} = G(f)G^*(f)$$

Therefore, $S_g(f) = \mathrm{F}\{R_g(\tau)\} = |G(f)|^2$.

Cross Correlation Function

The cross correlation function of two energy signals $g_1(t)$ and $g_2(t)$ is defined as;

$$R_{1,2}(\tau) = \int_{-\infty}^{\infty} g_1(t) g_2(t - \tau) dt$$

Example:

Find the auto-correlation function of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where A and θ are constants.

Solution:

As we know, this is a periodic signal. So, we find $R_g(\tau)$ using the definition

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0}[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$$

$$R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$$

$$R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

Example:

Determine the autocorrelation function of the sinc pulse $g(t) = Asinc2Wt$.

Solution:

Using the duality property of the Fourier transform, you can deduce that

$$G(f) = \frac{A}{2W}rect(\frac{f}{2W})$$

The energy spectral density of $g(t)$ is

$$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$

Taking the inverse Fourier transform, we get the autocorrelation function

$$R_g(\tau) = \frac{A^2}{2W}sinc2Wt$$

Exercise:

a. Find and plot the cross correlation function of the two signals

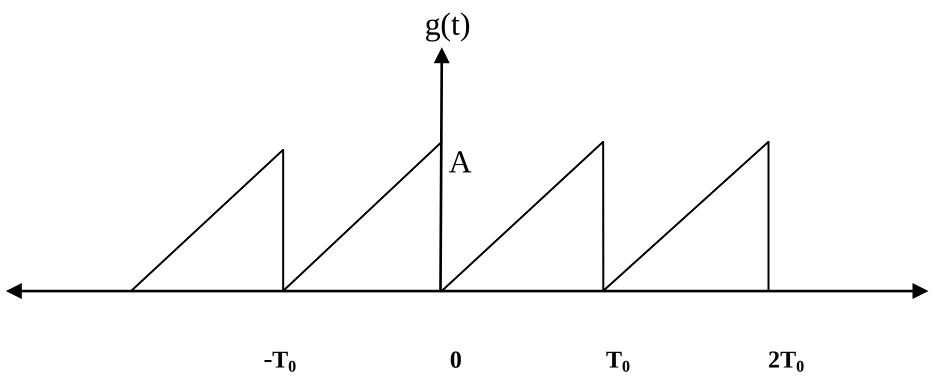
$$g_1(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0, & otherwise \end{cases}$$

$$g_2(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \end{cases}$$

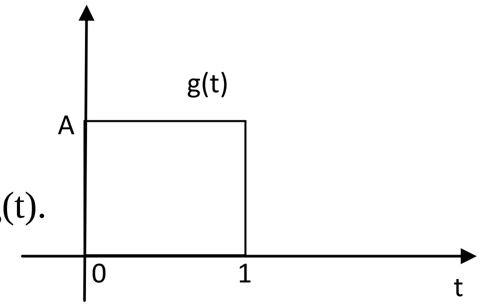
b. Are $g_1(t)$ and $g_2(t)$ orthogonal?

Exercise:

Find and plot the autocorrelation function for the periodic saw-tooth signal shown below:



Example:



Find the autocorrelation function of the rectangular pulse g(t).

Solution:

As we saw earlier, this pulse is an energy signal and , therefore, we can find its $R_g(\tau)$ as:

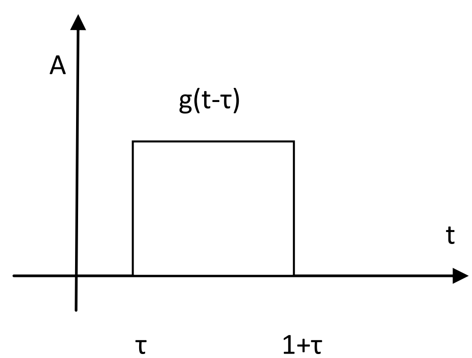
$$R_g(\tau) = \int_{\tau}^{1}(A)(A)dt = A^2\,(1-\tau)\ ;\ 0< \tau <1$$

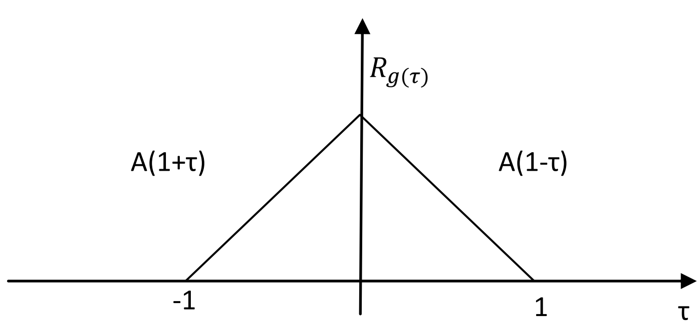
Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for – ve values of τ as:

$$R_g(\tau) = = A^2\,(1+\tau)\ ;\ -1< \tau <0$$

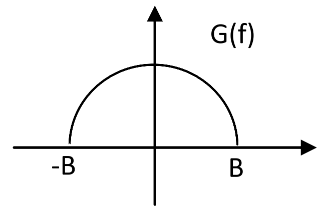
This function is sketched below. Note that that the maximum value occurs at τ = 0 and that g(t) and g(t-τ) become decorrelated for τ = 1 sec, which is the duration of the pulse.

The energy spectral density is $S_g(f)\ = F\{R_g(\tau)\} = A^2 sinc^2 f$



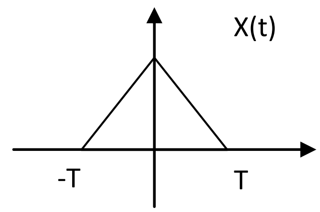


Bandwidth of Signals and Systems:



Def: A signal g(t) is said to be (absolutely) band-limited to BHz if

G(f) = 0 for |f| >B

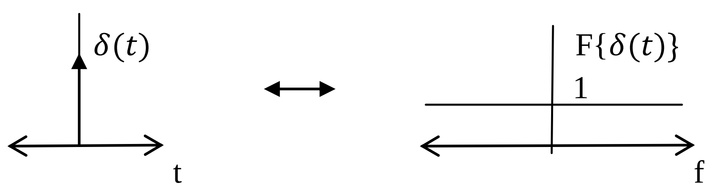


Def: A signal x(t) is said to be (absolutely) time-limited if

x(t) = 0 for |t| >T

Theorem: An absolutely band-limited waveform cannot be absolutely time-limited (theoretically has an infinite time duration) and vice versa.

We have earlier examples that support this theorem. For example, the delta function, which has an almost zero time duration, has a Fourier transform which extends uniformly over all frequencies. Also, a constant value in the time domain has a Fourier transform, which is an impulse in the frequency domain. This is repeated here for convenience.



In general, there is an inverse relationship between the signal bandwidth and the time duration. The bandwidth and the time duration are related through a relation of the form, called the *time bandwidth product*

$$Bandwidth * Time\ Duration \geq constant$$

The value of the constant depends on the way the bandwidth and the time duration of a signal are defined as will be illustrated later (Possible values of the constant = $\frac{1}{2}, \frac{1}{4\pi}$).

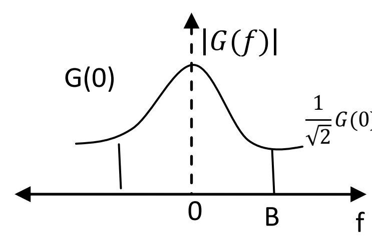
Remarks:

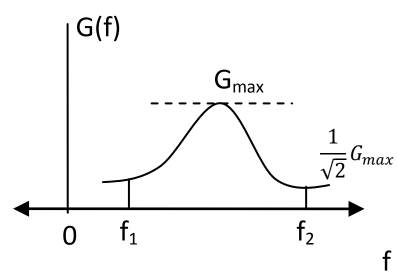
1. The bandwidth of a signal provides a measure of the extent of significant frequency content of the signal.
2. The bandwidth of a signal is taken to be the width of a positive frequency band.
3. For baseband signals or networks , where the spectrum extends from –B to B, the bandwidth is taken to be B Hz.
4. For bandpass signals or systems where the spectrum extends between (f_1, f_2) and ($-f_1$, $-f_2$), the B.W= $f_2 - f_1$.

Some Definitions of Bandwidth:

1- Absolute bandwidth
Here, the Fourier transform of a signal is non zero only within a certain frequency band. If G(f) = 0 for |f| >B , then g(t) is absolutely band-limited to BHz. When $G(f) \neq 0$ for $f_1 < |f| < f_2$, then the absolute bandwidth is $f_2 - f_1$.

2- 3-dB (half power points) bandwidth
The range of frequencies from 0 to some frequency B at which |G(f)| drops to $\frac{1}{\sqrt{2}}$ of its maximum value (for a low pass signal).
As for a band pass signal, the B.W = $f_2 - f_1$





3- The 95 % (energy or power) bandwidth.
Here , the B.W is defined as the band of frequencies where the area under the energy spectral density (or power spectral density) is at least 95% (or 99%) of the total area .

Total Energy = $\int_{-\infty}^{\infty} |G(f)|^2\, df$ = $2\int_{0}^{\infty} |G(f)|^2 df$

$\int_{-B}^{B} |G(f)|^2 df$ = 0.95 $\int_{-\infty}^{\infty} |G(f)|^2\, df$

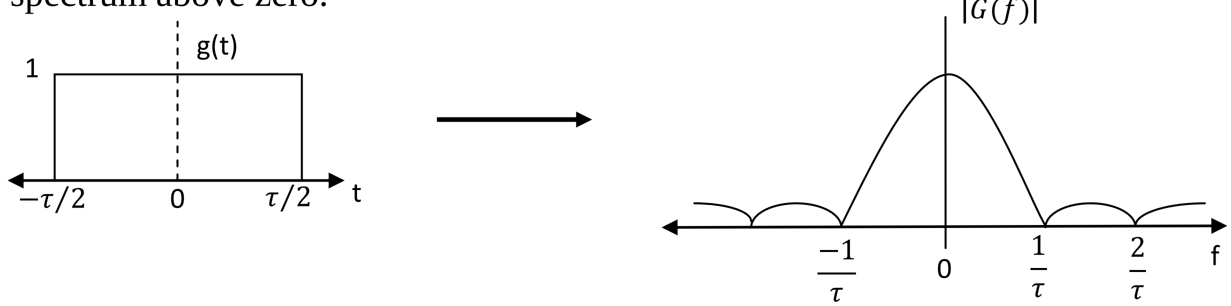
4- Equivalent Rectangular Bandwidth.
It is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequency
Area under rectangle = Area under curve

$|G(0)|^2 * 2B_{eq} = \int_{-\infty}^{\infty} |G(f)|^2 df$

$|G(0)|^2 * 2B_{eq} = 2\int_{0}^{\infty} |G(f)|^2 df$

$B_{eq} = \frac{1}{|G(0)|^2} \int_{0}^{\infty} |G(f)|^2 df$

5- <u>Null – to –null bandwidth</u>**:**
For baseband signals , B.W is the first null in the envelope of the magnitude spectrum above zero.



rect($\frac{t}{\tau}$) → τ sincfτ = τ $\frac{\sin\pi f\tau}{\pi f\tau}$

Zero crossing take place when $\sin\pi f\tau = 0$

$\pi f\tau = n\pi \;\rightarrow\; f = \frac{n}{\tau}$; n = 1,2,………

B.W = $\frac{1}{\tau}$ smaller τ large bandwidth.

For a band pass signal, B.W= $f_2 - f_1$

6- <u>Bounded spectrum bandwidth</u>**:**

Range of frequencies as (0,B) such that outside the band , the power spectral density must be down by say 50 dB below the maximum value

$$-50 \text{ dB} = 10 \log \frac{|G(B)|^2}{|G(0)|^2}$$

7- <u>RMS Bandwidth</u>**:**

$$B_{rms} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 \text{df}}{\int_{-\infty}^{\infty} |G(f)|^2 \text{df}}\right)^{1\backslash 2}$$

The corresponding rms duration of g(t) is

$$T_{rms} = = \left(\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 \text{dt}}{\int_{-\infty}^{\infty} |g(t)|^2 \text{dt}}\right)^{1\backslash 2}$$

(here g(t) is assumed to be centered around the origin).

Remark**:** The time bandwidth product is $T_{rms}\, B_{rms} \geq \frac{1}{4\pi}$

<u>Time – Bandwidth Product :</u>

To illustrate the time – bandwidth product, consider the equivalent rectangular bandwidth defined earlier as

$$B_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 \text{df}}{2|G(0)|^2}$$

Analogous to this definition, we define an equivalent rectangular time duration as :

$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \text{dt}}$$

The time bandwidth product is

$$B_{eq} T_{eq} = \frac{\int_{-\infty}^{\infty} |G(f)|^2 \text{df}}{2|G(0)|^2} \cdot \frac{\left(\int_{-\infty}^{\infty} |g(t)| dt\right)^2}{\int_{-\infty}^{\infty} |g(t)|^2 \text{dt}}$$

Note $\int_{-\infty}^{\infty} |g(t)|^2$ dt = $\int_{-\infty}^{\infty} |G(f)|^2$ df ; Rayleigh energy theorem. Note also that $G(0) = \int_{-\infty}^{\infty} g(t) dt$. Using these relations, we get

$$B_{eq}T_{eq} = \frac{1}{2}\frac{\left(\int_{-\infty}^{\infty}|g(t)|dt\right)^2}{|\int_{-\infty}^{\infty} g(t)\mathrm{d}t|^2}$$

Case 1:

When g(t) is positive for all time t, then |g(t)| = g(t) and $B_{eq}T_{eq}$ becomes

$$B_{eq}T_{eq} = \frac{1}{2}$$

Case 2 :

For a general g(t) that can take on positive as well as negative values, $B_{eq}T_{eq}$ satisfies the inequality

$$B_{eq}T_{eq} \geq \frac{1}{2}$$

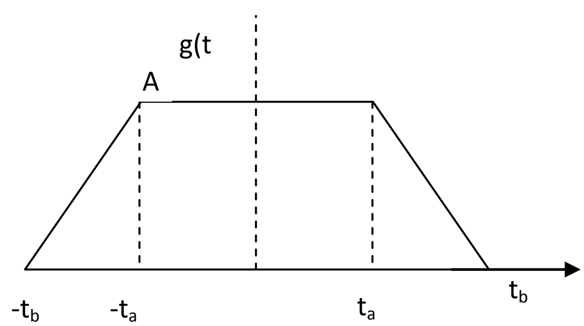
Note : For B_{rms} and T_{rms} , the time – bandwidth satisfies the inequality

$$B_{rms}\ T_{rms} \geq \frac{1}{4\pi}$$

Example : Bandwidth of a trapezoidal signal

Find the equivalent rectangular bandwidth, B_{eq} , for the trapezoidal pulse shown.

Solution :



$$T_{eq} = \frac{\left(\int_{-\infty}^{\infty}|g(t)|dt\right)^2}{\int_{-\infty}^{\infty}|g(t)|^2\mathrm{dt}}$$

$$\int_{-\propto}^{\propto}|g(t)|dt\ = A\ (\ t_a + t_b)$$

$$\int_{-\propto}^{\propto}|g(t)|^2 dt\ = \frac{2A^2}{3}\ (\ 2t_a + t_b)$$

$$T_{eq} = \frac{3}{2}\frac{(t_a+t_b)^2}{(2t_a + t_b)}$$

$B_{eq} = \frac{0.5}{T_{eq}} = \frac{2t_a+t_b}{3(t_a+t_b)^2}$.

Remark: Note that using this method we were able to determine the signal bandwidth without the need to go through the Fourier transform.

Exercise: Use the above method to find the equivalent rectangular bandwidth for the triangular signal $g(t) = tri(\frac{t}{T})$.

Example: Bandwidth of a periodic signal:

Find the bandwidth For the periodic square function define over one period as

$$g(t) = \begin{cases} 2A, & \frac{-T}{4} \leq t \leq \frac{T}{4} \\ -A, & o.w \end{cases}$$

Solution:

The average power, computed using the time average, is

$$P_{av} = \frac{1}{T_0} \int_0^{T_0} |g(t)|^2 \ dt$$

$$= \frac{1}{T_0} [4A^2\tau + A^2\tau] = \frac{5A^2\tau}{2\tau} = \frac{5A^2}{2} = 2.5A^2$$

Also, by using the Parseval's theorem, the average power can be computed as:

$$P_{av} = |C_0|^2 + 2\sum_{n=1}^{\infty} |C_n|^2$$

We recall that the Fourier coefficients for this signal were found in Chapter 1. Using these values we get

$$P_{av} = \left(\frac{A}{2}\right)^2 + 2\sum_{n=1}^{\infty} \frac{(3A)^2}{(n\pi)^2}$$

$$P_{av} = \frac{A^2}{4} + 2A^2 \sum_{n=1}^{\infty} \frac{(3)^2}{(n\pi)^2}$$

Let us take n = 1

$$P_1 = A^2 \left\{0.25 + 2 \ . \ \frac{9}{\pi^2}\right\} = 2.073A^2$$

$$\frac{P_1}{P_{av}} = \frac{2.073A^2}{2.5A^2} = 82.95\%$$

(This is the percentage of the total power that lies in the dc and the fundamental frequency).

For n = 3

$$P_3 = A^2 \left\{0.25 + 2 \left(\frac{3^2}{\pi^2} + \frac{3^2}{3^2\pi^2}\right)\right\} = 2.276A^2$$

$$\frac{P_3}{P_{av}} = \frac{2.276A^2}{2.5A^2} = 91.05\%$$

(Fraction of power in the dc, fundamental and third harmonic terms)

For n = 5

$$P_5 = A^2 \left\{0.25 + 2 \left(\left(\frac{3}{\pi}\right)^2 + \left(\frac{3}{3\pi}\right)^2 + \left(\frac{3}{5\pi}\right)^2\right)\right\} = 2.349A^2$$

$$\frac{P_5}{P_{av}} = \frac{2.349A^2}{2.5A^2} = 93.97\%$$

Here , the 93% power band width is $5f_0$.

Example: Bandwidth of an energy signal .

If the signal $g(t) = Ae^{-\propto t} \ u(t)$ is passed through an ideal LPF with B.W = B Hz,

find the fraction of the signal energy contained in B.

Solution

The Fourier transform of g(t) is:

$$G(f) = \frac{A}{\propto + j2\pi f}$$

The energy in g(t), using the time domain, is

$$E_g = \int_0^{\infty} |g(t)|^2 \, dt = \int_0^{\infty} A^2 e^{-2\propto t} \, dt = \frac{A^2}{2 \propto}$$

Energy contained in the filter output y(t) is

$$E_y = \int_{-B}^{B} |G(f)|^2 \, df = \int_{-B}^{B} \frac{A^2}{(\propto^2 + (2\pi f)^2)} \, df$$

$$E_y = \frac{2A^2}{2\pi \propto} tan^{-1} \frac{2\pi B}{\propto}$$

The ratio of E_y to the total energy is

$$\frac{E_y}{E_g} = \frac{2}{\pi} tan^{-1} \frac{2\pi B}{\propto}$$

The table below shows this ratio for various values of B .

B	$\boldsymbol{(E_y/E_g) \times 100}$
$\frac{\propto}{4}$	63.9
$\frac{\propto}{2}$	80.38
$\propto$	89.95
$2 \propto$	94.94

Thus, the 95% energy bandwidth is $2 \propto$.

Exercise: Find the 98% energy bandwidth.

Pulse Response and Risetime

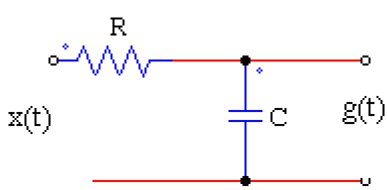
A rectangular pulse contains significant high frequency components. When that pulse is passed through a LPF, the high frequency components will be attenuated resulting in signal distortion.

We need to investigate the relationship that should exist between the pulse bandwidth and the channel bandwidth. This subject is of particular importance, especially, when we study the transmission of data over band-limited channels. In the simplest form, a binary digit 1 may be represented by a pulse $A,\ \ 0 \le t \le T_b$, while binary digit 0 may be represented by the negative pulse $-A,\ \ 0 \le t \le T_b$. So, in order to retrieve the transmitted data, the channel bandwidth must be wide enough to accommodate the transmitted data.

To convey this idea in a simple form, we first consider the response of a first order low pass filter to a unit step function and then to a pulse.

<u>Step response of a first order LPF (channel)</u>

Let x(t) = u(t) be applied to a first order RC circuit. This first order filter is a fair representation of a low pass communication channel.



The system D.E is:

$$x(t) = Ri + g(t) = Rc\,\frac{dg(t)}{dt} + g(t)$$

where g(t) is the channel output.

$$Rc\,\frac{dg(t)}{dt} + g(t) = u(t)$$

The solution to this first order system is

$$g(t) = (1-e^{-t/RC})\ u(t)$$

The 3-db B.W of the channel is

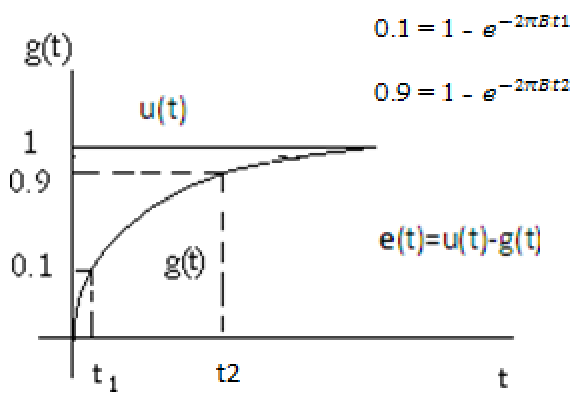
B= $\frac{1}{2\pi Rc}$ (to be derived shortly)

g(t) = (1-$e^{-2\pi Bt}$) u(t)

Define the difference between the input and the output as:

e(t) = u(t) - g(t) = $e^{-2\pi Bt}$

Note that e(t) decreases as B increases. Meaning that as the channel bandwidth increases, the output becomes closer and closer to the input. In the ideal case, when the channel bandwidth becomes infinity, the output becomes a step function. In essence, to reproduce a step function (or a rectangular pulse), a channel with infinite bandwidth is needed.



The Risetime

The Rise time is a measure of the speed of a step response. One common measure is the 10-90 % rise time defined as the time it takes for the output to rise between 10% to 90% of the final (steady state) value (1) when a step function is applied to a LIT system. For the step response g(t) and the first order RC circuit considered above, the rise time can be easily calculated as:

$t_r = t_2 - t_1 \approx \frac{0.35}{B}$

From this result we conclude that: increasing the bandwidth of the channel will decrease the rise time (a faster response).

Exercise: For the system above, verify that the rise time is given as $t_r = \frac{0.35}{B}$

Exercise: Find the 10-90% rise time for a second order low pass filter with 3-dB bandwidth B and transfer function

$$H(f) = \frac{1}{P_2(\frac{jf}{B})}$$

Where $P_2(x) = 1 + \sqrt{2}x + x^2$.
(Hints: You may let B=10, for example, use matlab to find the step response, and then find the rise time).

<u>Pulse response</u>

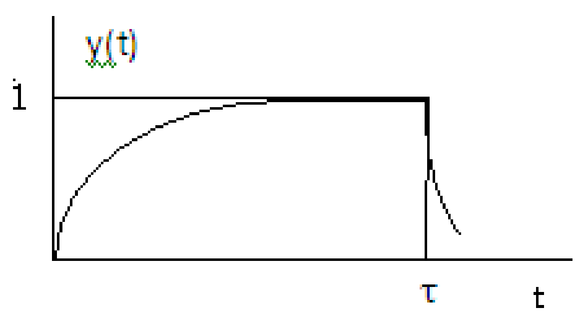
It is the response of the circuit to a pulse of duration τ. For the same circuit let us apply the pulse

$x(t) = u(t) - u(t - \tau)$

Using the linearity and time invariance properties, the output can be obtained from the step response as:

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/RC} & 0 < t < \tau \\ \left(1 - e^{-\frac{\tau}{RC}}\right) . e^{-\frac{t-\tau}{RC}} & t > \tau \end{cases}$$

This is sketched in the figure below.



<u>Bandwidth considerations</u>:

The transfer function of the RC circuit is

$$H(f) = \frac{1/j2\pi fc}{R + 1/j2\pi fc} = \frac{1}{1 + j2\pi fRc}$$

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi f Rc)^2}}$$

Let $B = \frac{1}{2\pi Rc}$; 3-db bandwidth ; $2\pi f Rc = 1$; $f = \frac{1}{2\pi Rc}$

Then , $H(f) = \frac{1}{1+jf/B}$

$$|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^2}}$$

For the rectangular pulse x(t), we have

$$X(f) = sincf\tau$$

The first null frequency of X(f) is an estimate of the bandwidth B_x of x(t), which is of the order of $\approx \frac{1}{\tau}$.

1. When τ is large, such that signal bandwidth $B_x = \frac{1}{\tau} << B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx X(f)$$

and the output resembles the input .There is enough time for x(t) to reach the maximum value .

2. When τ is small, such that signal $B_x = \frac{1}{\tau} >> B$ (channel B.W)

$$Y(f) = X(f)H(f) \approx H(f)$$

The signal suffers a considerable amount of distortion and $Y(f)$ is no longer proportional to X(f).

Band pass Signals and Systems

A signal g(t) is called a *band pass signal* if its Fourier transform G(f) is non-negligible only in a band of frequencies of total extent 2W centered about f_c .

A signal is called *narrowband* if 2W is small compared with f_c.

A band pass signal g(t) represented in the form:

$$g(t) = g_I(t) \cos \omega_c t - g_Q(t) \sin \omega_c t.$$

$g_I(t)$ is a low pass signal of B.W = W Hz called the *in phase component* of g(t) .

$g_Q(t)$ is a low pass signal of B.W = W Hz called the *quadrature component* .

g(t) is a modulated signal in which $g_I(t)$ and $g_Q(t)$ are the low pass signals. Recall the modulation property of the Fourier transform :

$$x(t) \cos \omega_c t \rightarrow \frac{1}{2} (X(f- f_c)+ X(f+ f_c))$$

$$x(t) \sin \omega_c t \rightarrow \frac{1}{j2} (X(f- fc)- X(f+ fc))$$

Define the *complex envelope* of a signal g(t) as:

$$\tilde{g}(t) = g_I(t) + j\, g_Q(t)$$

$\tilde{g}(t)$ is a low pass signal of B.W =W. The signals g(t) and $\tilde{g}(t)$ are related by :

$$g(t) = \mathrm{Re}\{\tilde{g}(t)\, e^{j\omega ct}\}$$

<u>How to get $g_I(t)$ and $g_Q(t)$ from g(t):</u>

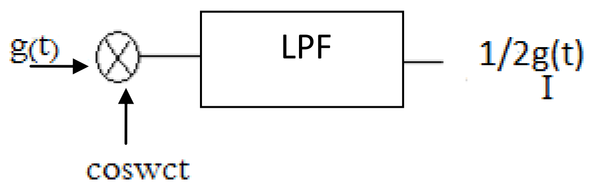
If we multiply g(t) by cos ωct, we get

$$g(t) \cos\omega ct = g_I(t) \cos^2 \omega_c t - g_Q(t) \sin \omega_c t \cos \omega_c t$$

$$= \frac{1}{2} g_I(t) + \frac{1}{2} g_I(t) \cos 2\omega ct - \frac{1}{2} g_Q(t) \sin 2\omega_c t .$$

The first term is the desired low pass signal. The second and third terms are high frequency components centered about 2 f_c.

$$g_I(t) = \text{low pass}\{2g(t) \cos\omega_c t\}$$



Or, in the frequency domain

$$G_I(f)=\begin{cases} G(f-fc\,)+G(f+fc\,) & -w\le f\le w \\ 0 & otherwise \end{cases}$$

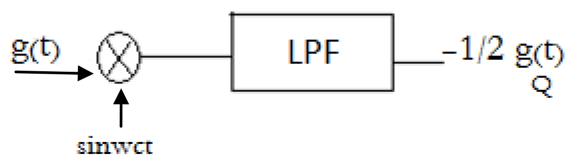
Now if we multiply g(t) by $\sin\omega_c t$, we get

$$g(t)\sin\omega_c t = g_I(t)\sin\omega_c t\cos\omega_C t - g_Q(t)\sin^2\omega c t$$

$$= -\frac{1}{2}g_Q(t)+\frac{1}{2}g_I(t)\sin 2\omega_c t+\frac{1}{2}g_Q(t)\cos 2\omega_c t$$

Again, the first term is a low pass signal, while the second and third are high frequency terms centered about 2 f_c.

$$g_Q(t) = -\text{ low pass}\{2g(t)\sin\omega_c t\}$$



In the frequency domain, this is equivalent to

$$G_Q(t)=\begin{cases} j[G(f-fc)-G(f+fc) & -w\le f\le w \\ 0 & otherwise \end{cases}$$

Band pass systems:

The analysis of band pass systems can be simplified by using the complex envelope concept. Here, results and techniques from low pass systems can be easily applied to band pass systems .

The problem to be addressed is :

The input x(t) is a band pass signal

$$x(t)=x_I(t)\cos\omega_c t - x_Q(t)\sin\omega_c t$$

x(t) is applied to a band pass filter represented as:

$$h(t) = h_I(t)\cos\omega_c t - h_Q(t)\sin\omega_c t$$

The objective is to find the filter output y(t). The output is of course, the convolution of x(t) and h(t) (y(t) = x(t)*h(t)) which can also be expressed as:

$$y(t)=y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

But due to the band-pass nature of the problem, carrying out the direct convolution will be a tedious task. The complex envelope concept simplifies the problem to a very great extent. The procedure is summarized as follows:

a. Form the complex envelope for both the input and the channel:

$$\tilde{x}(t)= x_I(t) + jx_Q(t)$$

$$\tilde{h}(t)= h_I(t) + jh_Q(t)$$

b. Carry out the convolution between $\tilde{x}(t)$ and $\tilde{h}(t)$. Note that both signals are low pass signals and so $\tilde{y}(t)$ is also low pass.

$$2\,\tilde{y}(t)= \tilde{h}(t) * \tilde{x}(t)$$

$$\tilde{y}(t) = y_I(t) + jy_Q(t)$$

c. The band-pass filter output is obtained from the low pass signal $\tilde{y}(t)$ through the relation

$$y(t) =\mathrm{Re}\{\tilde{y}(t)\, e^{jw_c t}\}$$

or the relation

$$y(t)=y_I(t)\cos\omega_c t - y_Q(t)\sin\omega_c t$$

Example :

The rectangular radio frequency (RF) pulse

$$x(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

is applied to a linear filter with impulse response (We will see later that this is a filter matched to x(t), called the *matched filter*).

$$h(t) = x(T - t)$$

Assume that $T = nT_C$; n is an integer, $T_C = \frac{1}{fc}$. Determine the response of the filter and sketch it.

Solution: We follow the three steps outlined above.

$h(t) = A\cos 2\pi f_c(T - t)$

$= A\cos 2\pi f_c T \cos 2\pi f_c t + A\sin 2\pi f_c T \sin 2\pi f_c t$

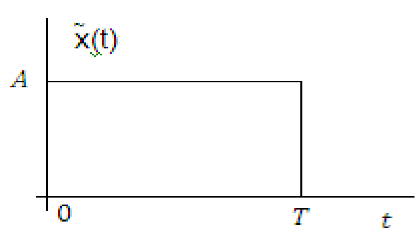
$= \underbrace{A\cos 2\pi\left(\frac{nT_c}{T_c}\right)}_{\cos 2n\pi \equiv 1} \cos 2\pi f_c t + \underbrace{A\sin 2\pi\left(\frac{nT_c}{T_c}\right)}_{\sin 2n\pi \equiv 0} \sin 2\pi f_c t$

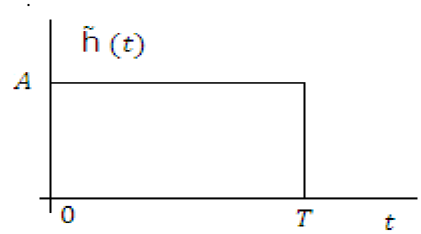
Therefore, $h(t) = \begin{cases} A\cos 2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$

The complex envelopes of x(t) and h(t) are (step a)

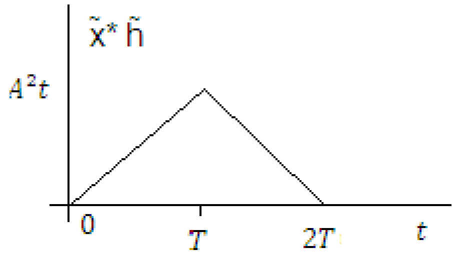
$$\tilde{x}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$

$$\tilde{h}(t) = \begin{cases} A & 0 \le t \le T \\ 0 & o.w \end{cases}$$





$\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)$ is the triangular signal shown in the Figure (step b).

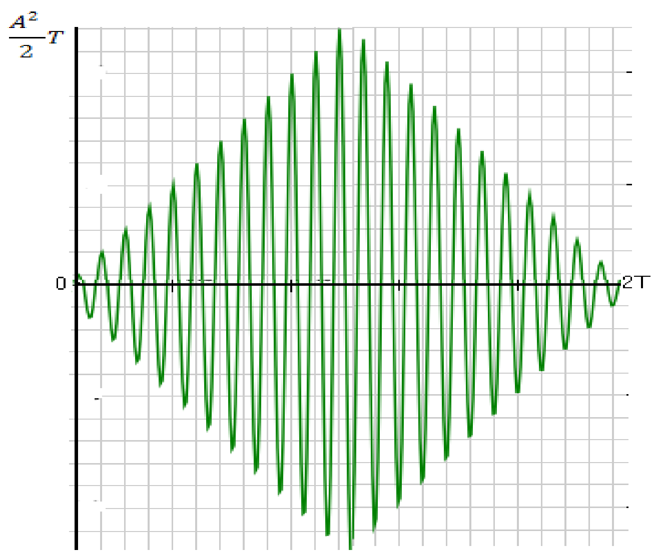


$$2\tilde{y}(t)=\begin{cases} A^2t & 0 \le t \le T \\ A^2(2T-t) & T \le t \le 2T \end{cases}$$

The bandpass signal is obtained as (step c)

$$y(t)=\begin{cases} \frac{A^2}{2} t\, cosw_c t & 0 \le t \le T \\ \frac{A^2}{2}(2T-t)\, cosw_c t & T \le t \le 2T \end{cases}$$

and is sketched as in the figure below.



Exercise

The band-pass signal $x(t) = e^{-\frac{t}{\tau}} \cos(2\pi f_c t)\, u(t)$ is applied to a band-pass filter with impulse response $h(t)$ given as:

$$h(t) = \begin{cases} A\, cos2\pi f_c t & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

Find and sketch the filter output.

Amplitude Modulation Systems

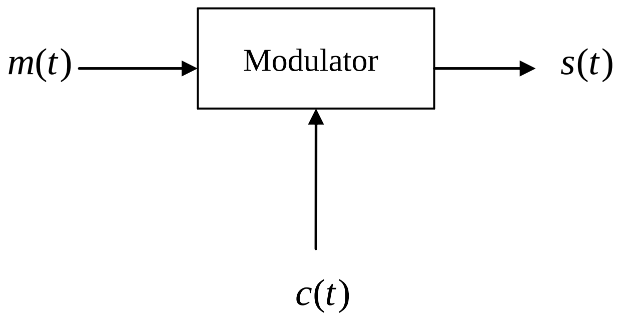
Modulation: is the process by which some characteristic of a carrier $c(t)$ is varied in accordance with a message signal $m(t)$.

Amplitude modulation is defined as the process in which the amplitude of the carrier $c(t)$ is varied linearly with $m(t)$. Four types of amplitude modulation will be considered in this chapter. These are normal amplitude modulation, double sideband suppressed carrier modulation, single sideband modulation, and vestigial sideband modulation.

A common form of the *carrier*, in the case of continuous wave modulation, is a sinusoidal signal of the form

$$c(t) = A_C \cos(2\pi f_C t + \phi)$$

The baseband (message) signal $m(t)$ is referred to as the *modulating signal* and the result of the modulation process is referred to as the *modulated signal* $s(t)$. The following block diagram illustrates the modulation process.



We should point out that modulation is performed at the transmitter and demodulation, which is the process of extracting $m(t)$ from $s(t)$, is performed at the receiver.

Normal Amplitude Modulation

A *normal AM* signal is defined as:

$$s(t) = A_C\big(1 + k_a m(t)\big)\cos 2\pi f_C t$$

where k_a : Amplitude sensitivity (units 1/volt)

Here $s(t)$ can be written in the form:

$$s(t) = A(t)\cos 2\pi f_C t$$

where $A(t) = A_C + A_C k_a\, m(t)$

From which we can observe that $A(t)$ is related linearly to $m(t)$ in a relationship of the form " $y = a + bx$ ". Therefore, the amplitude of $s(t)$ is linearly related to $m(t)$.

The *envelope* of $s(t)$ is defined as

$$|A(t)| = A_C |1 + k_a m(t)|$$

Notice that the envelope of $s(t)$ has the same shape as $m(t)$ provided that:

1. $|1 + k_a m(t)| \geq 0$ or $|k_a m(t)| \leq 1$

Overmodulation results when $|k_a m(t)| > 1$ resulting in envelope distortion.

2. $f_C >> w$, where w is the highest frequency component in $m(t)$.

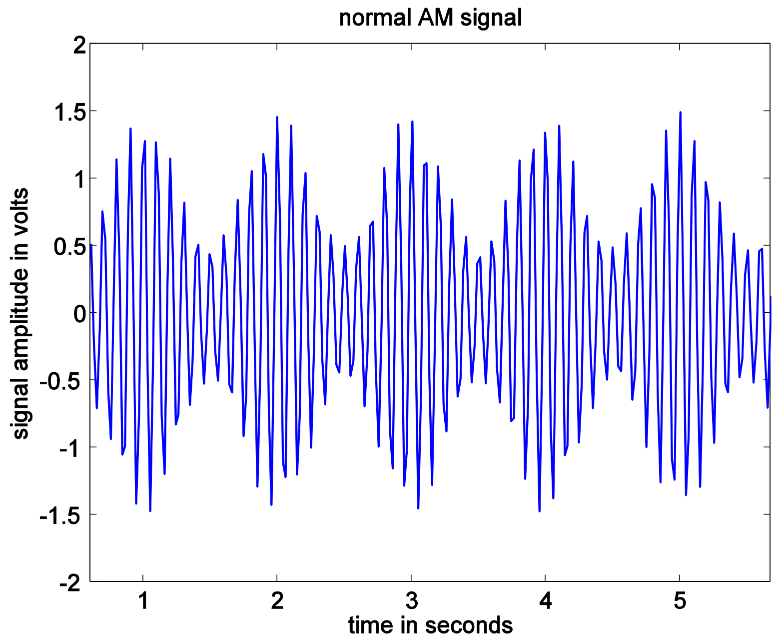
(f_C has to be at least $10w$). This ensures the formation of an envelope that has a shape that resembles the message.

Matlab Demonstration

The figure below shows the normal AM signal $s(t) = (1 + 0.5\cos 2\pi t)\cos 2\pi(10)t$

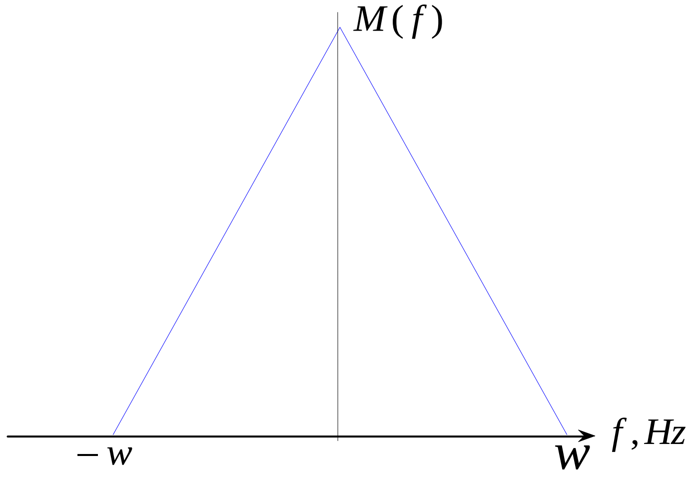
a. Make similar plots for the cases (μ = 0.5, 1, and 1.5)

b. Show the effect of f_C on the envelope. (Take f_C = 4 Hz, and f_C = 25Hz)



Spectrum of the Normal AM Signal

Let the Fourier transform of $m(t)$ be as shown (The B.W of $m(t)$ = w Hz).

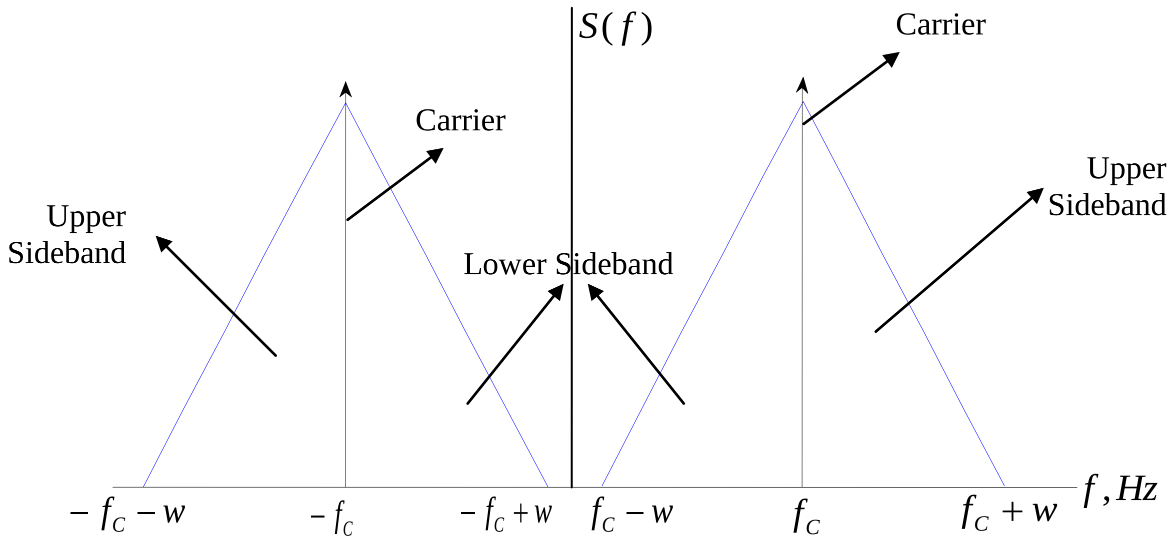


$$s(t) = A_C(1 + k_a m(t))\cos 2\pi f_C t \quad \text{(dc + message)*carrier}$$
$$s(t) = A_C \cos 2\pi f_C t + A_C k_a m(t)\cos 2\pi f_C t \quad \text{(carrier + message*carrier)}$$

Taking the Fourier transform, we get

$$S(f) = \frac{A_C}{2}\delta(f - f_C) + \frac{A_C}{2}\delta(f + f_C) + \frac{A_C k_a}{2}M(f - f_C) + \frac{A_C k_a}{2}M(f + f_C)$$

The spectrum of $s(t)$ is shown below



Remarks

a. $M(f)$ Has been shifted to f_C resulting in a bandpass signal.
b. Two sidebands (upper sideband and lower sideband) and a carrier form the spectrum of $s(t)$.
c. The transmission bandwidth of $s(t)$ is:

$$B.W = (f_C + w) - (f_C - w) = 2w$$

= Twice the message bandwidth.

Power Efficiency

The *power efficiency* of a normal AM signal is defined as:

$$\eta = \frac{power\ \ in\ \ the\ \ sidebands}{power\ in\ the\ sidebands + power\ in\ the\ carrier}$$

Now, we find the power efficiency of the AM signal for the single tone modulating signal $m(t) = A_m \cos(2\pi f_m t)$. If we denote $\mu = A_m k_a$, then $s(t)$ can be expressed as

$$s(t) = A_C(1 + \mu \cos 2\pi f_m t)\cos 2\pi f_C t$$

$$s(t) = A_C \cos 2\pi f_C t + A_C \mu \cos 2\pi f_C t \cos 2\pi f_m t$$

$$s(t) = A_C \cos 2\pi f_C t + \frac{A_C \mu}{2}\cos 2\pi (f_C + f_m)t + \frac{A_C \mu}{2}\cos 2\pi (f_C - f_m)t$$

Carrier — Upper Sideband — Lower Sideband

Power in carrier $= \dfrac{A_C{}^2}{2}$

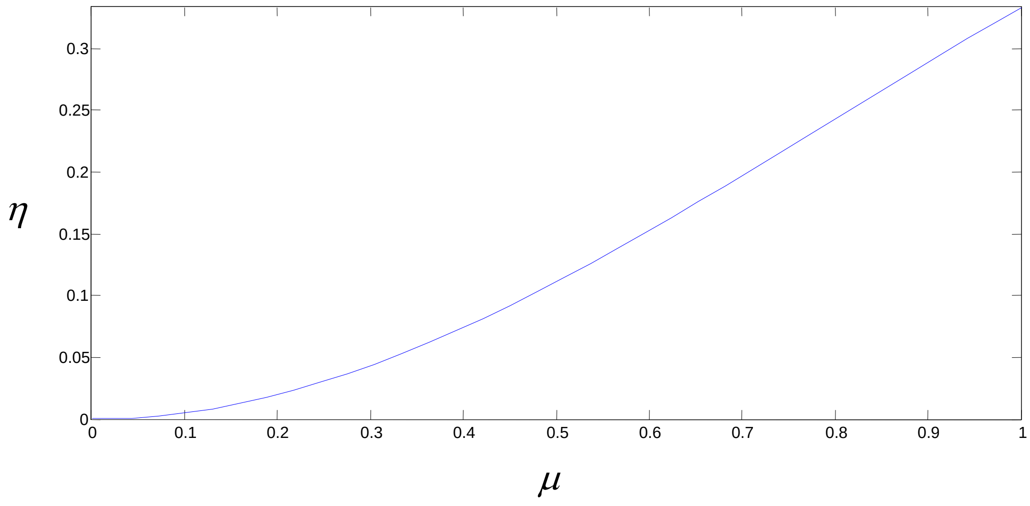
Power in sidebands $= \dfrac{1}{2}\left(\dfrac{A_C \mu}{2}\right)^2 + \dfrac{1}{2}\left(\dfrac{A_C \mu}{2}\right)^2$

$$= \frac{1}{8}A_C{}^2\mu^2 + \frac{1}{8}A_C{}^2\mu^2 = \frac{1}{4}A_C{}^2\mu^2$$

Therefore,

$$\eta = \frac{\frac{1}{4}A_C{}^2\mu^2}{\frac{A_C{}^2}{2} + \frac{1}{4}A_C{}^2\mu^2} = \frac{\mu^2}{2 + \mu^2} \qquad ; \qquad 1 \geq \mu \geq 0$$

The following figure shows the relationship between η and μ



The maximum efficiency occurs when μ = 1 (i.e. for 100% modulation. And even then, η = 1/3). So that 2/3 of the power goes to the carrier.

<u>Remark:</u> Normal AM is not an efficient modulation scheme in terms of the utilization of the transmitted power.

<u>Exercise</u>:

a. Show that for the general AM signal $s(t) = A_C\ \ 1 + k_a m(t)\ \ \cos(2\pi f_c t)$, the power efficiency is given by $\eta = \dfrac{\frac{1}{2} A_C^{\ 2} \left\langle k_a^{\ 2} m(t)^2 \right\rangle}{\frac{A_C^{\ 2}}{2} + \frac{1}{2} A_C^{\ 2} \left\langle k_a^{\ 2} m(t)^2 \right\rangle} = \dfrac{\left\langle k_a^{\ 2} m(t)^2 \right\rangle}{1 + \left\langle k_a^{\ 2} m(t)^2 \right\rangle}$, where $\left\langle k_a^{\ 2} m(t)^2 \right\rangle$ is the average power in $k_a m(t)$

b. Apply the above formula for the single tone modulated signal $s(t) = A_C(1 + \mu \cos 2\pi f_m t)\cos 2\pi f_C t$

<u>AM Modulation Index</u>

Consider the AM signal

$$s(t) = A_C\left(1 + k_a m(t)\right)\cos 2\pi f_C t = A(t)\cos 2\pi f_C t$$

The envelope of $s(t)$ is defined as:

$$|A(t)| = A_C\left|1 + k_a m(t)\right|$$

To avoid distortion, the following condition must hold

$\left|1 + k_a m(t)\right| \geq 0$ or $\left|k_a m(t)\right| \leq 1$

The modulation index of an AM signal is defined as:

$$Modulation\ \ Index\ (M.I) = \frac{|A(t)|_{\max} - |A(t)|_{\min}}{|A(t)|_{\max} + |A(t)|_{\min}}$$

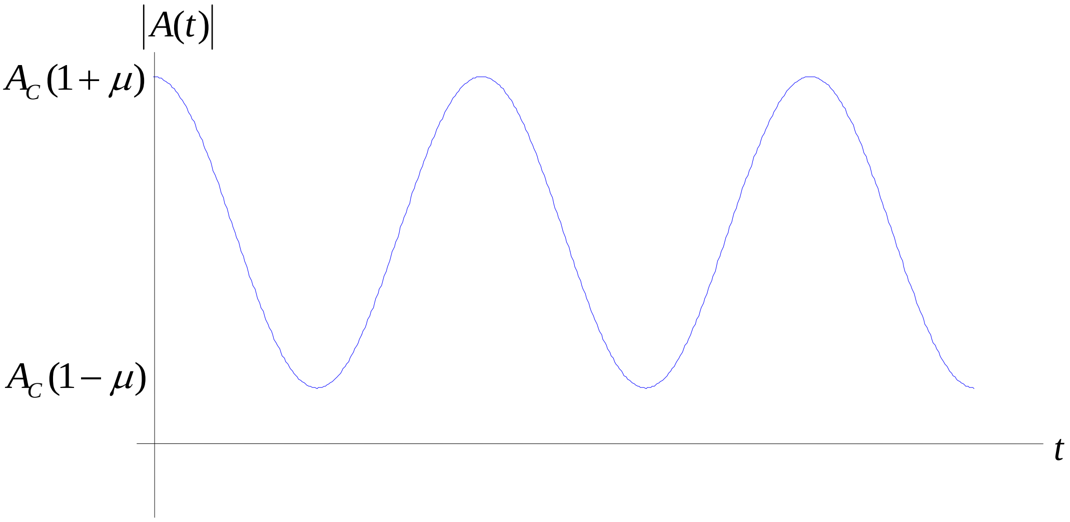
Example: (single tone modulation)

Let $m(t) = A_m \cos 2\pi f_m t$

$$then,\ s(t) = A_C(1 + k_a A_m \cos 2\pi f_m t)\cos 2\pi f_C t$$
$$= A_C(1 + \mu \cos 2\pi f_m t)\cos 2\pi f_C t \qquad where,\ \mu = k_a A_m$$

To avoid distortion $k_a A_m = \mu < 1$

The envelope $|A(t)| = A_C|(1 + \mu \cos 2\pi f_m t)|$ is plotted below



$$|A(t)|_{\max} = A_C(1+\mu), \qquad |A(t)|_{\min} = A_C(1-\mu)$$

$So,$

$$M.I = \frac{A_C(1+\mu) - A_C(1-\mu)}{A_C(1+\mu) + A_C(1-\mu)} = \frac{2A_C\mu}{2A_C} = \mu$$

Therefore, the modulation index is μ.

Overmodulation

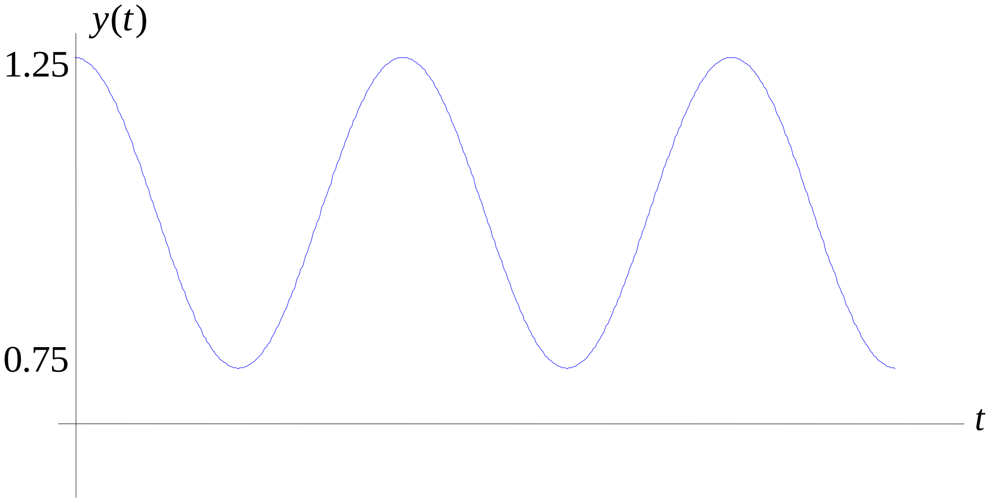
When the modulation index $\mu > 1$, an ideal envelope detector cannot be used to extract $m(t)$ and *distortion* takes place.

Example: Let $s(t) = A_C(1 + \mu \cos 2\pi f_m t)\cos 2\pi f_C t$ be applied to an ideal envelope detector, sketch the demodulated signal for $\mu = 0.25, 1.0,\ and\ 1.25$.

As was mentioned before, the output of the envelope detector is $y(t) = A_C|1 + \mu \cos 2\pi f_m t|$

Case1 : (μ = 0.25)

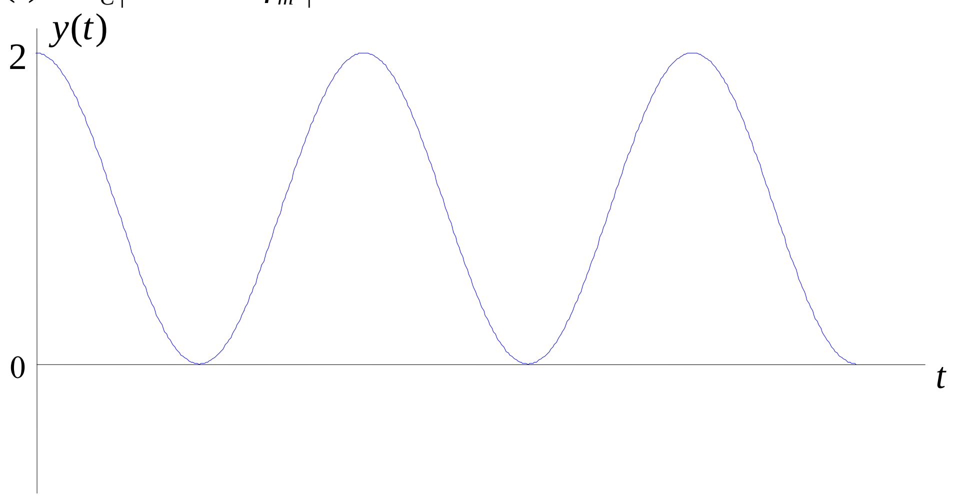
$$y(t) = A_C|1 + 0.25\cos 2\pi f_m t|$$



Here, $m(t)$ can be extracted without distortion.

<u>Case2:</u> (μ = 1.0)

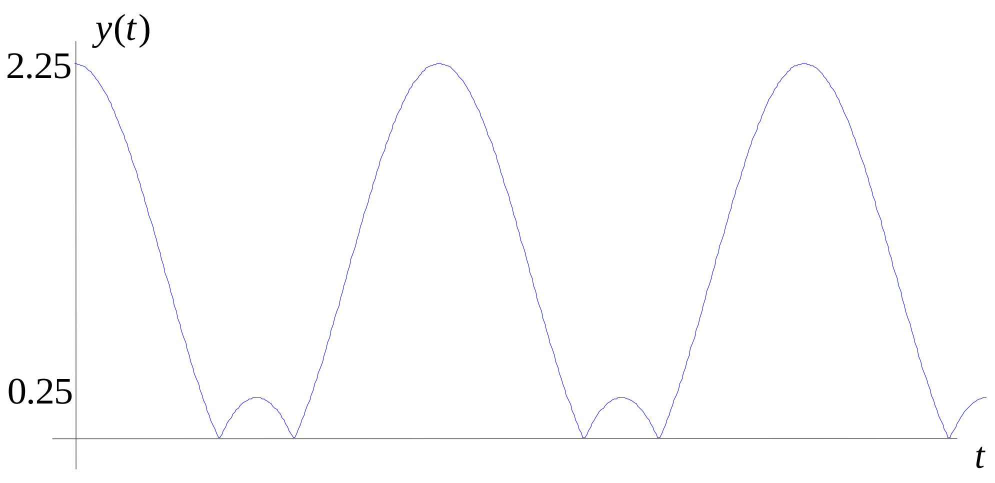
$$y(t) = A_C\left|1 + \cos 2\pi f_m t\right|$$



Here again, $m(t)$ can be extracted without distortion.

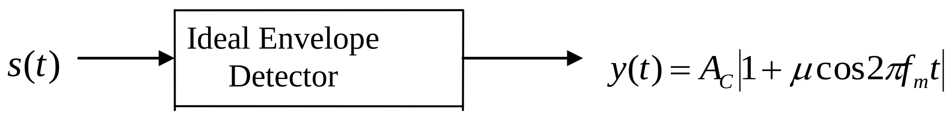
<u>Case3:</u> (μ = 1.25)

$$y(t) = A_C\left|1 + 1.25\cos 2\pi f_m t\right|$$



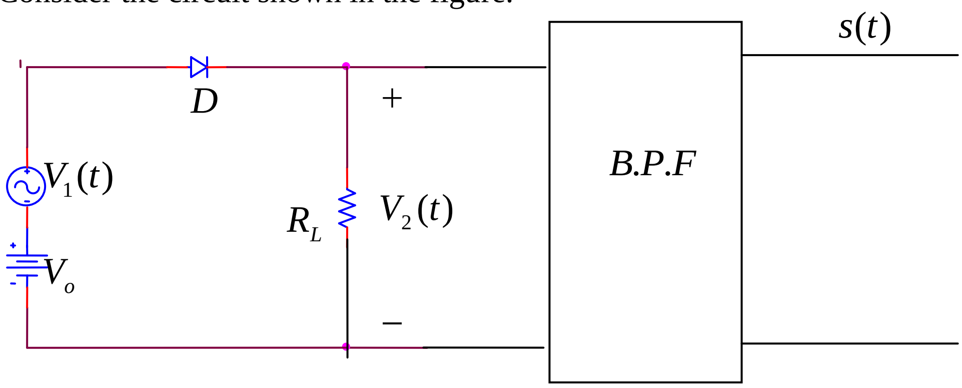
Here, $m(t)$ cannot be recovered without distortion.

Note: the block diagram that illustrates the envelope detector process is shown below



Generation of Normal AM: (Square Law Modulator)

Consider the circuit shown in the figure.



For small variations of $V_1(t)$ around a suitable operating point, $V_2(t)$ can be expressed as:

$$V_2 = \alpha_1 V_1 + \alpha_2 V_1^2 \ ; \qquad \text{Where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

Let $V_1(t) = m(t) + A_C' \cos 2\pi f_C t$

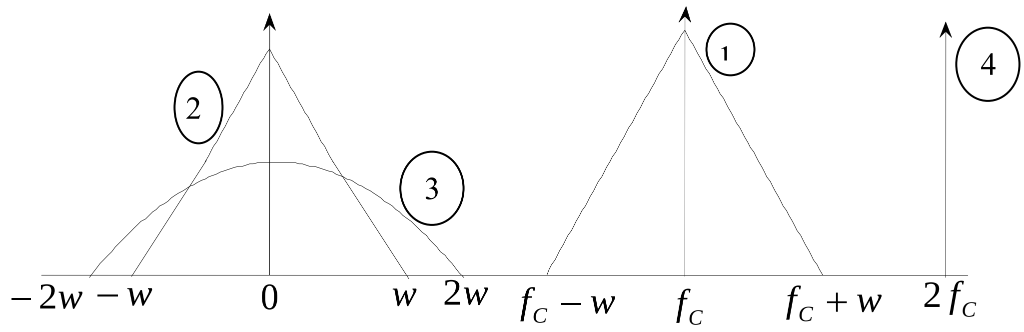
Substituting $V_1(t)$ into the nonlinear characteristics and arranging terms, we get

$$V_2(t) = \alpha_1 A_C'\left[1+\frac{2\alpha_2}{\alpha_1}m(t)\right]\cos 2\pi f_C t + \alpha_1 m(t) + \alpha_2 m(t)^2 + \alpha_2 A_C'^2 \cos^2(2\pi f_C t)$$

$$V_2(t) = (1) + (2) + (3) + (4)$$

The first term is the desired AM signal obtained by passing $V_2(t)$ through a bandpass filter.

$$s(t) = \alpha_1 A_C'\left[1+\frac{2\alpha_2}{\alpha_1}m(t)\right]\cos 2\pi f_C t$$



Note: the numbers shown in above figure represent the number of term in $V_2(f)$.

(1) = The desired normal AM signal

(2) = $M(f)$

(3) = $M(f)*M(f)$

(4) = The cosine square term amounts to a term at $2f_c$ and a DC term.

Limitations of this technique:

a. Variations of $V_1(t)$ should be small to justify the second order approximation of the nonlinear characteristic.

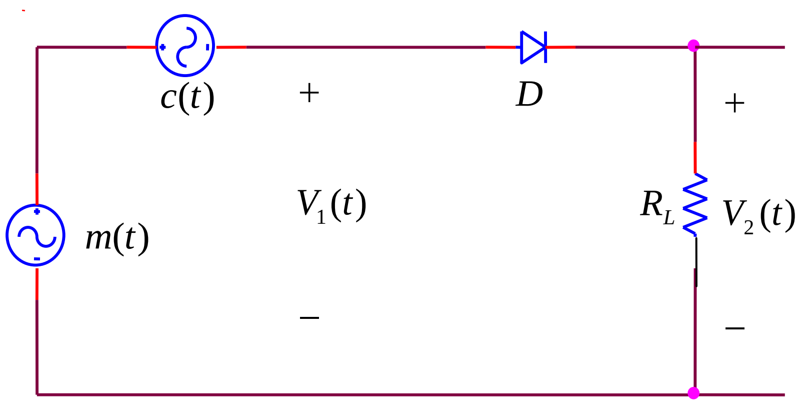
b. The bandwidth of the filter should be such that $f_C - w > 2w \Rightarrow f_C \geq 3w$

When $f_C >> w$, a bandpass filter with reasonable edge could be used.

When f_C is of the order $3w$, a filter with sharp edges should be used.

<u>Generation of Normal AM</u>: (The switching Modulator)

Assume that the carrier $c(t)$ is large in amplitude so that the diode –shown in the figure below- acts like an ideal switch.



When $m(t)$ is small compared with $|c(t)|$, then

$$V_2(t) = \begin{cases} m(t) + A_C \cos\omega_C t & ; \quad c(t) > 0 \\ 0 & ; \quad c(t) < 0 \end{cases}$$

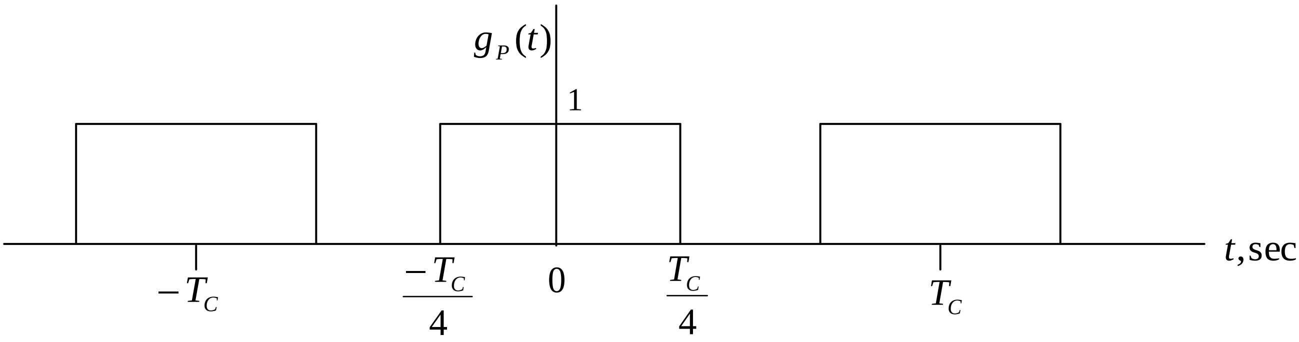
i.e. diode opens and closes at a rate of f_C (times/sec) (frequency of $c(t)$, the carrier)

$$V_2(t) = [A_C \cos\omega_C t + m(t)]g_P(t)$$

Where $g_P(t)$: is the periodic switching function.

$$g_P(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_C t - \frac{1}{3}\cos 3\omega_C t + \frac{1}{5}\cos 5\omega_C t + \ldots\right)$$

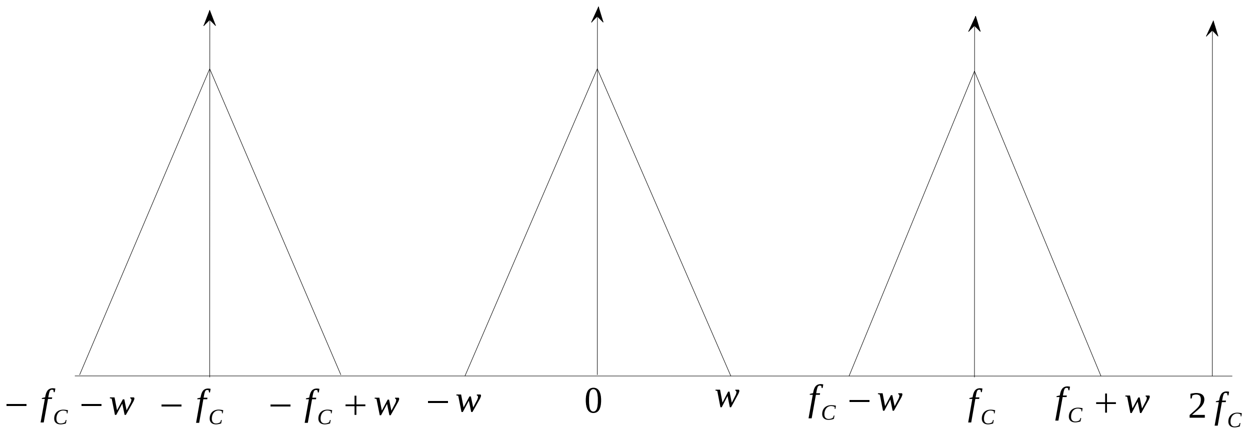
Expanding $g_P(t)$ in a Fourier series, we get



$$V_2(t) = [A_C \cos\omega_C t + m(t)]\left(\frac{1}{2}\right) + \left(\frac{2}{\pi}\cos\omega_C t\right)(A_C \cos\omega_C t + m(t)) - \left(\frac{2}{3\pi}\cos 3\omega_C t\right)$$
$$(m(t) + A_C \cos\omega_C t) + \ldots$$

$\Longrightarrow$

$$V_2(t) = \frac{m(t)}{2} + \frac{A_C}{2}\cos\omega_C t + \frac{2}{\pi}m(t)\cos\omega_C t + \frac{A_C}{\pi} + \frac{A_C}{\pi}\cos 2\omega_C t + \frac{2}{3\pi}m(t)\cos 3\omega_C t$$
$$+ \frac{2}{3\pi}A_C \cos 2\omega_C t + \ldots$$



With a bandpass filter centered at f_C with a bandwidth of $2w$; the filter passes the second term (a carrier) and the third term (a carrier multiplied by the message). we get:

$$s(t) = \frac{A_C}{2}\cos\omega_C t + \frac{2}{\pi}m(t)\cos\omega_C t$$

$$s(t) = \frac{A_C}{2}\left(1 + \frac{4}{\pi A_C}m(t)\right)\cos\omega_C t \quad ; \qquad \text{Desired AM signal.}$$

Modulation Index $= M.I = \dfrac{4}{\pi A_C}\left|m(t)\right|_{\max}$

<u>Demodulation of AM signal</u>: (The Envelope Detector)

The ideal envelope detector is one which responds to the envelope of the signal, but insensitive to phase variation. If

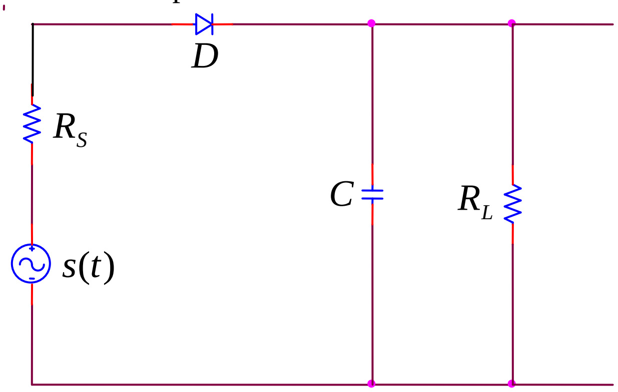
$$s(t) = A_C\left(1 + k_a m(t)\right)\cos 2\pi f_C t$$

Then, the output of the ideal envelope detector is

$$y(t) = A_C\left|1 + k_a m(t)\right|$$

<u>A simple practical envelope detector</u>

It consists of a diode and resistor-capacitor filter.

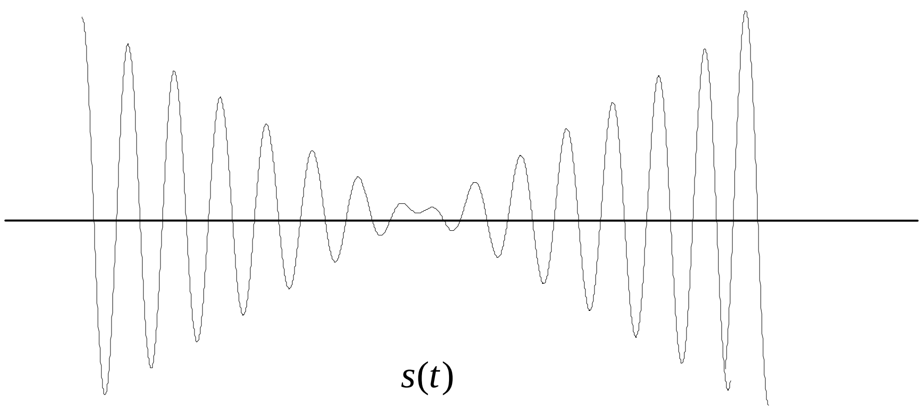


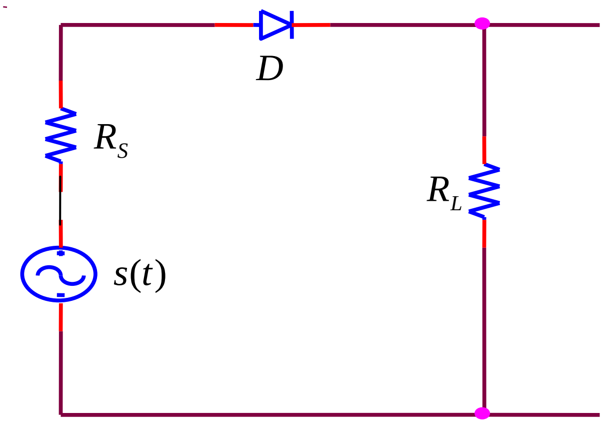
During the positive half cycle of the input, the diode is forward biased and C charges rapidly to the peak value of the input. When $s(t)$ falls below the maximum value, the diode becomes reverse biased and C discharges slowly through R_L. To follow the envelope of $s(t)$, the circuit time constant should be chosen such that :

$$\frac{1}{f_C} << R_L C << \frac{1}{w}$$

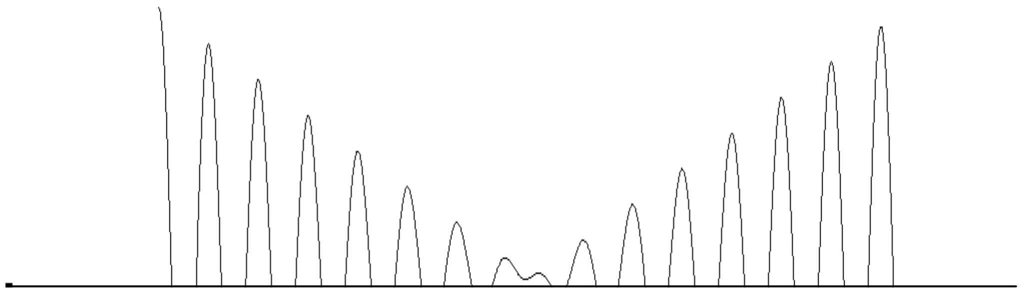
where:

w is the message B.W and f_C is the carrier frequency





Half – Wave Rectifier

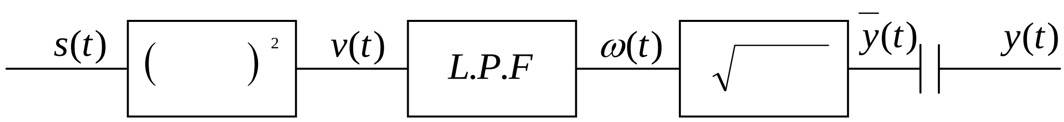


Output of half wave rectifier (without C)

When C is added, the output follows the envelope of $s(t)$. The circuit output (with C connected) follows a curve that connects the tips of the positive half cycles.

Example: (Demodulation of AM signal)

Let $s(t) = (1 + k_a m(t))\cos\omega_C t$ be applied to the scheme shown below, find $y(t)$.



$$v(t) = s(t)^2 = \left[1 + k_a m(t)\right]^2 \cos^2 \omega_C t$$

$$= \frac{1}{2}\left[1 + k_a m(t)\right]^2 + \frac{1}{2}\left[1 + k_a m(t)\right]^2 \cos 2\omega_C t$$

The filter suppresses the second term and passes only the first term, hence

$$\omega(t) = \frac{1}{2}(1 + k_a m(t))^2$$

$$\overline{y}(t) = \sqrt{\omega(t)} = \frac{1}{\sqrt{2}}\left(1 + k_a m(t)\right)$$

$$y(t) = \frac{1}{\sqrt{2}} k_a m(t)$$

Note that the dc term is blocked by capacitor.

Concluding remarks about AM:

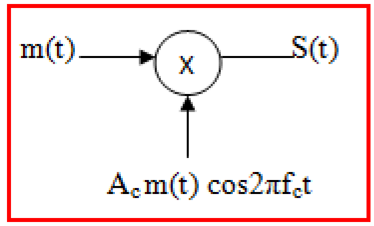
i. Modulation is accomplished using a nonlinear device.
ii. Demodulation is accomplished using a simple envelope detector.
iii. AM is wasteful of power; most power resides in the carrier (not in the sidebands).
iv. The transmission B.W = twice message B.W

Double Sideband Suppressed Carrier Modulation (DSB-SC)

A DSB-SC signals is an AM signals that has the form:

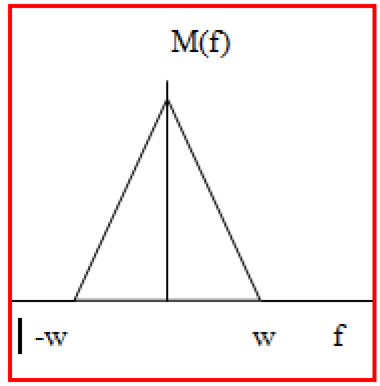
$s(t) = A_c m(t) \cos 2\pi f_c t.$

where $f_c >> w$, w is the baseband signal's band width .



The spectrum of s(t) is:

$S(f)= \frac{Ac}{2}M(f- f_C)+ \frac{Ac}{2}M(f+ f_c)$



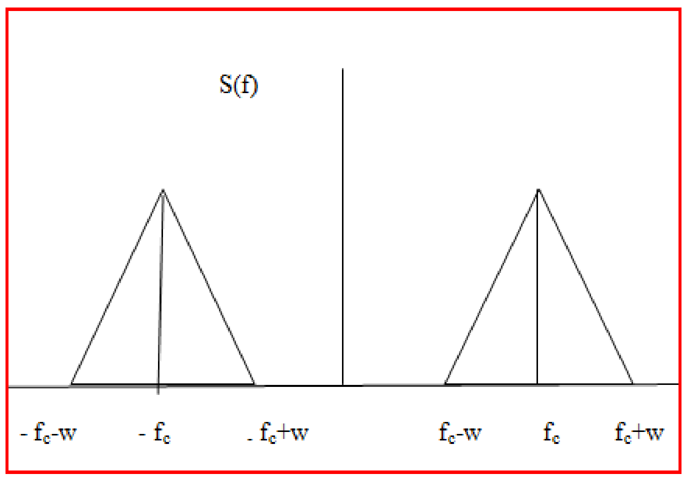
Remarks:

1. No impulses are present in the spectrum at $\pm$ f_c and, hence, no carriers is transmitted.

2. The transmission B.W of s(t)=2w. (same of AM).

3. power efficiency

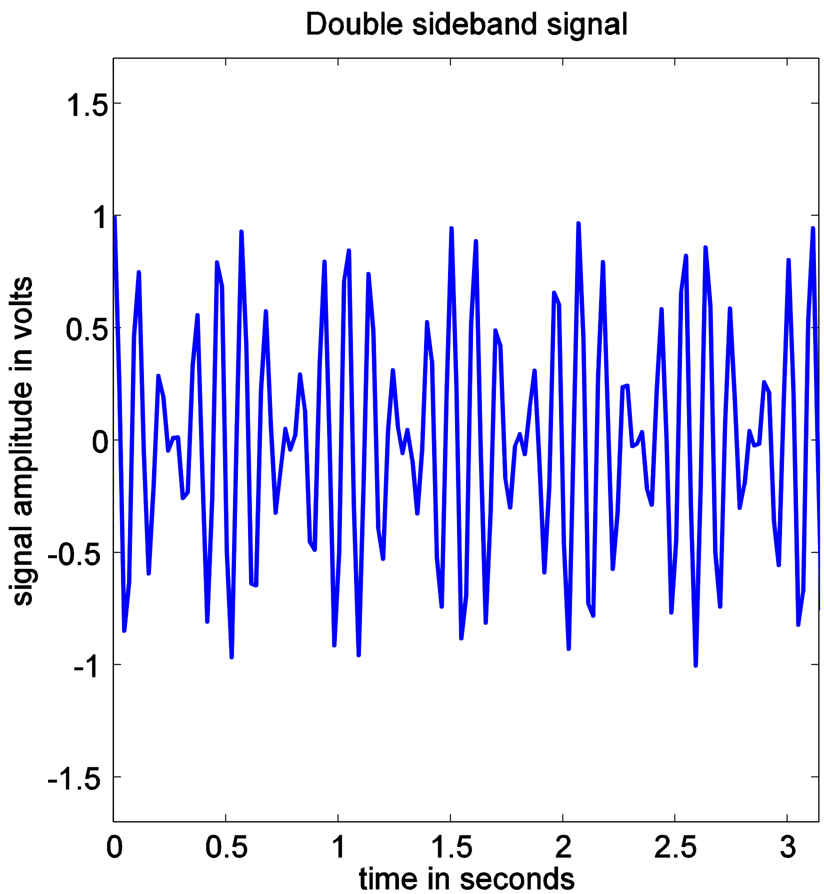
$=\frac{power\ in\ the\ side\ bands}{total\ transmitted\ power}=100\%.$



This is a power efficient modulation scheme.

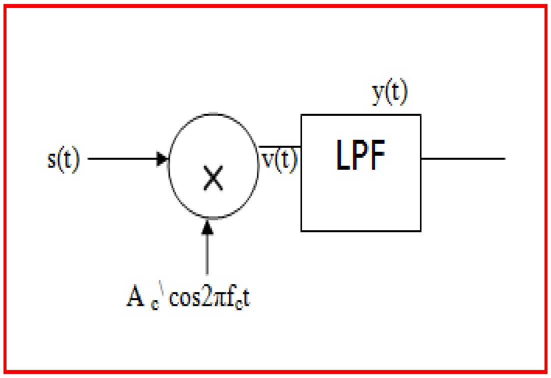
4. Coherent detector is required to extract m(t) from s(t) (will be demonstrated shortly) No envelope detection is used.

5. **Computer simulation**: The next figure shows a DSB-SC signal when $m(t)=\cos 2\pi t$ and $c(t)=\cos 2\pi(10)t$. You can easily see that m(t) cannot be recovered using envelope detection.



Demodulation of a DSB-SC signals

A DSB-SC signal is demodulated using what is known as *coherent demodulation*. This means that the modulated signal s(t) is multiplied by a locally generated signal at the receiver which has the same frequency and phase as the carrier c(t) at transmitting side.



a. **Perfect coherent demodulation**.

Let $c(t)= A_c\cos 2\pi f_c t$, $c^{\backslash}(t)= A_c^{\backslash}\cos(2\pi f_c t)$

Mixing the received signal with the version of the carrier at the receiving side, we get

$$v(t)=s(t)\ A_c^{\backslash}\cos 2\pi f_c t = A_c A_c^{\backslash} m(t)\cos^2 2\pi f_c t$$

$= \frac{A_c A_c^{\backslash}}{2} m(t)[1 + \cos 2\,(2\pi f_c t)]$

$= \frac{A_c A_c^{\backslash}}{2} m(t) + \frac{A_c A_c^{\backslash}}{2} m(t) \cos 2(2\pi f_c t)$

↑ Proportional to m(t) ↑ high frequency signals at $2f_c$ (A DSB-SC)

The high frequency component can be eliminated using the LPF. The output is

$$y(t) = \frac{A_c A_c^{\backslash}}{2} m(t) = Km(t),$$

Therefore, m(t) has been recovered from s(t) without distortion, i.e., a distortion less system.

b. Effect of carrier noncoherence on demodulated signal

Here we consider two cases.

Case 1: A constant phase difference between c(t) and $c^{\backslash}(t)$

Let $c(t) = A_c \cos 2\pi f_c t$, $c^{\backslash}(t) = A_c^{\backslash} \cos(2\pi f_c t + Ø)$

We use the demodulator considered above

$$v(t) = A_c m(t) \cos 2\pi f_c t \,.\, A_c^{\backslash} \cos(2\pi f_c t + Ø)$$

$$= \frac{A_c A_c^{\backslash}}{2} m(t)[\cos(4\pi f_c t + Ø) + \cos Ø]$$

$$= \frac{A_c A_c^{\backslash}}{2} m(t) \cos(4\pi f_c t + Ø) + \frac{A_c A_c^{\backslash}}{2} m(t) \cos Ø$$

↑ high frequency term ↑ low frequency term

The output of the low pass filter is:

$$y(t) = \frac{A_c A_c^{\backslash}}{2} m(t) \cos Ø$$

For $0 < Ø < \frac{\pi}{2}$, $0 < \cos Ø < 1$, and y(t) suffer from an attenuation due to Ø.

However, for $Ø = \frac{\pi}{2}$, $\cos Ø = 0$ and $y(t) = 0$, signal disappears. The disappearance of a message component at the demodulator output is called *quadrature null effect.*

This highlights the importance of maintaining synchronism between the transmitting and receiving carrier signals $c^\backslash(t)$ and c(t).

Case 2: Constant frequency difference between c(t) and $c^\backslash(t)$

Let $c(t)= A_c\cos 2\pi f_c t$, $C^\backslash(t)= A_c^\backslash \cos(2\pi f_c+\Delta f)t$

In an analysis similar to case a, we get

$$v(t)= A_c m(t)\cos 2\pi f_c t \, . \, A_c^\backslash \cos(2\pi f_c+\Delta f)t$$

$$=\frac{Ac\ A\ c\backslash}{2} m(t)[\cos(4\pi f_c t+2\pi\Delta ft)+\cos 2\pi\Delta ft]$$

After low pass filtering,

$$y(t)= \frac{Ac\ A\ c\backslash}{2} m(t)\cos 2\pi\Delta ft$$

So the demodulated signals appears as if double side band modulated on a carrier with magnitude Δf. As can be observed, this is not a distortionless transmission.

Example: Let m(t)=cos2π(1000)t and let Δf=100HZ.

From the analysis in case 2 above,

$$y(t)= \frac{Ac\ A\ c\backslash}{2} \cos 2\pi(1000)t \cos 2\pi(100)t$$

$$=\frac{Ac\ A\ c\backslash}{4}[\cos 2\pi(1100)t+\cos 2\pi(900)t]$$

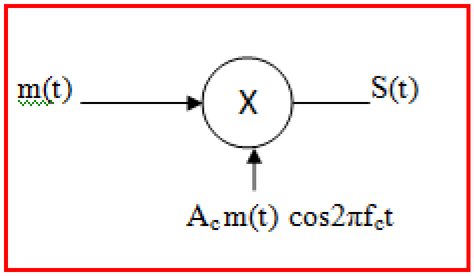
The original message was a signal with a frequency of =1000Hz, while the output consists of a signal two frequencies at f_1=1100Hz and f_2=900Hz.

⇨ Distortion

Exercise: Use Matlab to plot both m(t) and y(t) and see the distortion caused by the lack of synchronization between the transmitting and receiving oscillators.

<u>Generation of DSB-SC</u>

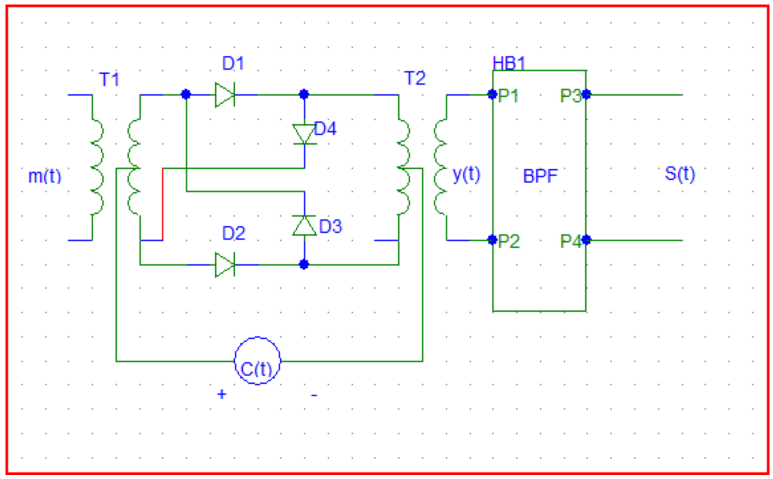
a. **Product modulator** : It multiplies the message signal m(t) with the carrier c(t). This technique is usually applicable when low power levels are possible and over a limited carrier frequency range.



b .**Ring modulator**:

consider the scheme shown in the figure.

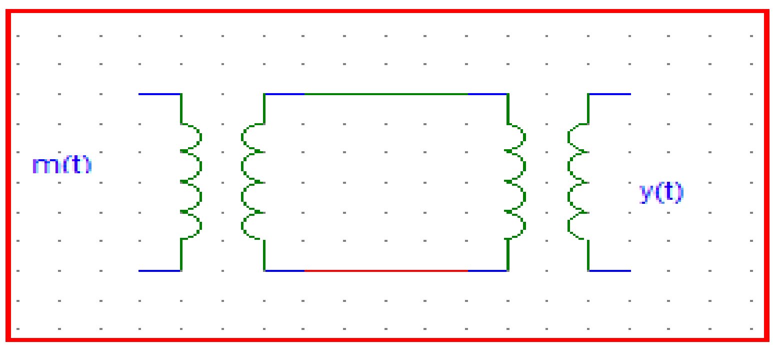
Let c(t)>> m(t). Here the carrier c(t) control the behavior of the diodes .



During the positive half cycle of c(t), c(t) > 0, and D1 and D2 are ON while D3 and D4 are OFF.

y(t) = m(t)

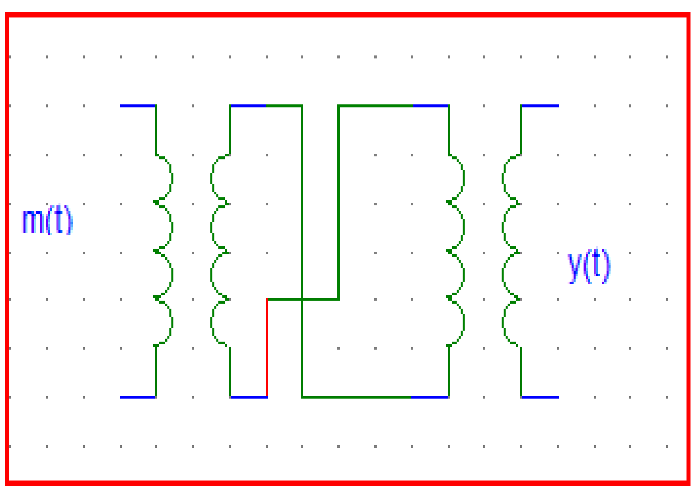
and the circuit appears like this



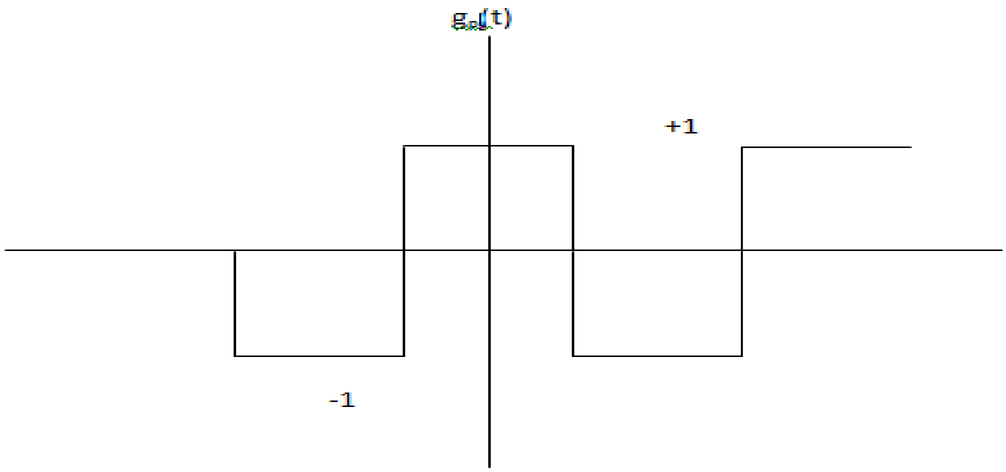
During the negative half cycle of c(t), c(t) < 0 and D3 and D4 are ON while and D1 and D2 are OFF

y(t) = -m(t)

and the circuit appears like this



So m(t) is multiplied by +1 during the +ve half cycle of c(t) and m(t) multiplied by -1 during the -ve half cycle of c(t). Mathematically, y(t) hehaves as if multiplied by the switching function $g_p(t)$ where $g_p(t)$ is the square periodic function with period $T_c=\frac{1}{fc}$, where f_c the period of c(t). By expanding $g_p(t)$ in a Fourier series, we get



$$y(t) = m(t)[\frac{4}{\pi}\cos 2\pi f_c t - \frac{4}{3\pi}\cos 3(2\pi f_c t) + \frac{4}{5\pi}\cos 5(2\pi f_c t)]$$

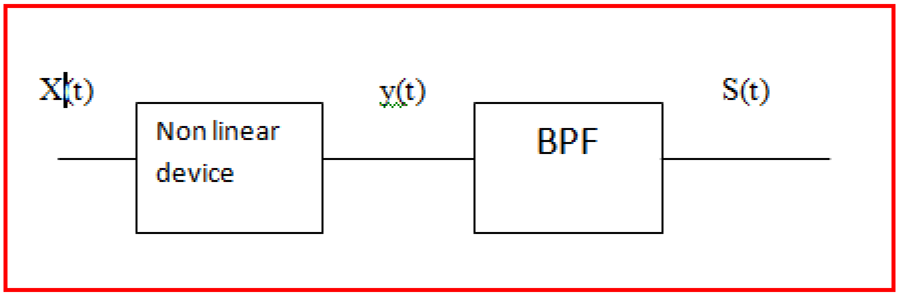
$$= m(t)\frac{4}{\pi}\cos 2\pi f_c t - m(t)\frac{4}{3\pi}\cos 3(2\pi f_c t) + m(t)\frac{4}{5\pi}\cos 5(2\pi f_c t)$$

When y(t) passes through the BPF, the only component that appears at the output is the desired DSB-SC signal, which is

$$s(t) = m(t)\frac{4}{\pi}\cos 2\pi f_c t$$

C. **Nonlinear characteristic**

Consider the scheme shown in the figure



Let the non linear characteristic be of the form

$$y(t)=a_0x(t)+a_1x^3(t)$$

Let x(t)= $A\cos 2\pi f_c t+m(t)$, (m(t) is the message signals)

$$y= a_0(A\cos2\pi f_c t+m(t))+a_1 (A\cos2\pi f_c t+m(t))^3$$

$$= a_0 A\cos2\pi f_c t+ a_0 m(t)+ a_1 A^3\cos^3 2\pi f_c t+ a_1 m(t)^3+3 a_1 A^2 m(t)\cos^2 2\pi f_c t$$

$$+3A a_1\cos2\pi f_c t$$

After some algebraic manipulations, a DSB-SC term appear in x(t) along with other undesirable terms. The band pass filter will admit the desired signal, which is

$$s(t)=\frac{3(A)^2 a1}{2} m(t) \cos(2)2\pi f_c t,$$

Note that the carrier frequency=$2f_c$ in this case.

Carriers recovery for coherent demodulation

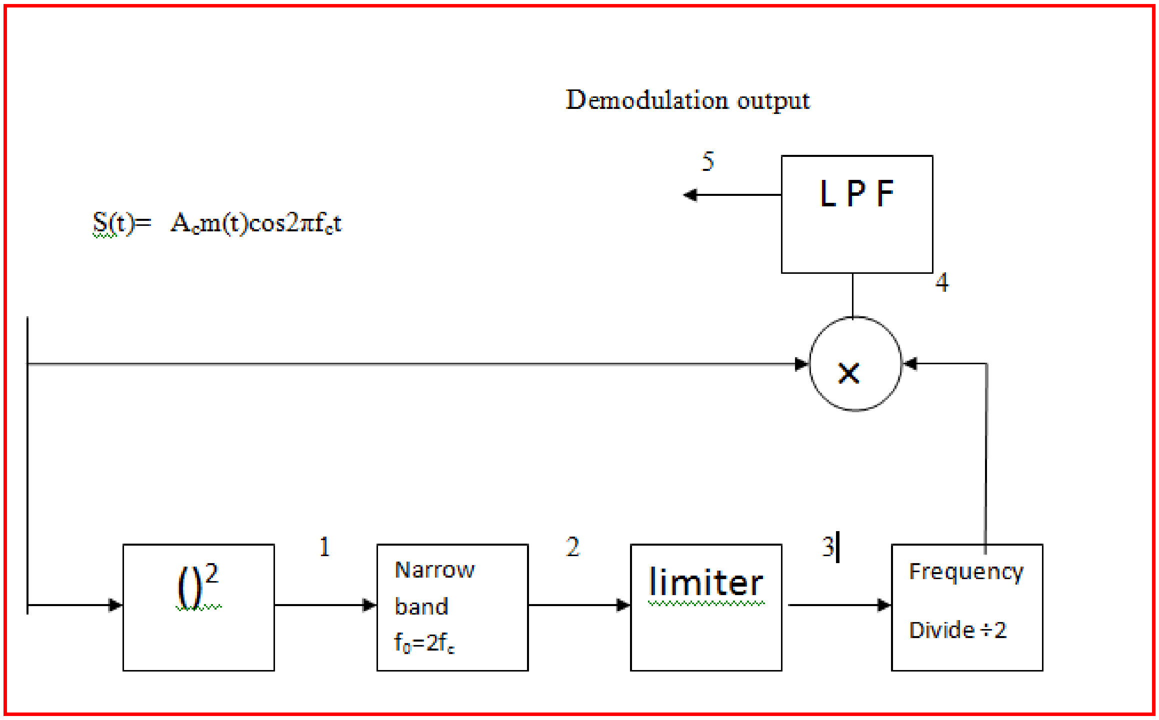
We consider briefly two circuits that are used to extract the carriers f_c from the incoming DSB-SC signal. we recall that demodulation of DSB-SC signal requires the availability of a signal with the same frequency and phase as the carrier c(t) at the transmitter

a. **Squaring loop :**

The basic elements of squaring loop are shown in the figure below. The incoming signal is the DSB-SC signal:

$$s(t)=A_c m(t)\cos2\pi f_c t.$$

In the figure, we mark five signals that appear at the output of the five blocks. In summary these signals are:



1- $(A_c m(t)\cos 2\pi f_c t)^2 = (\frac{Ac}{2} m(t))^2 (1+\cos 2w_c t)$

$= (\frac{Ac}{2} m(t))^2 + (\frac{Ac}{2} m(t))^2 (\cos 2w_c t)$

↑ Low pass term ↑ band pass around $2f_c$

2- $(\frac{Ac}{2} m(t))^2 (\cos 2w_c t) = K\cos 2w_c t$ (when the BPF is of narrow B.W)

3- $K\cos 2w_c t$ (The limiter removes any variation in amplitude but keeps the frequency unchanged). The frequency divider produces a signal $K\cos w_c t$.

4- $(A_c m(t)\cos 2\pi f_c t)\, K\cos w_c t$

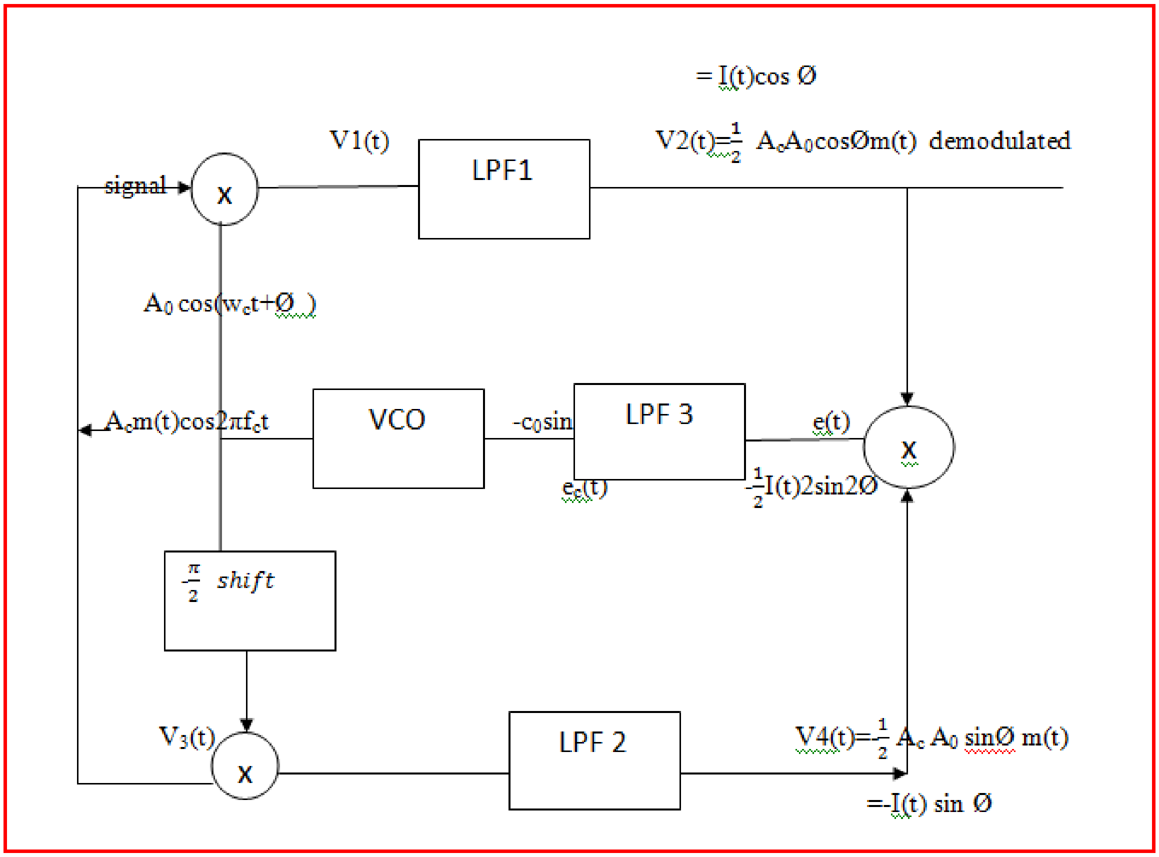
$= A_c K m(t)\cos^2 w_c t$

$\frac{A_c(K)}{2} m(t) + \frac{A_c(K)}{2} m(t) \cos 2w_c t$

5- $A_c \frac{(K)}{2} m(t)$

Therefore, demodulation has been achieved, even though the receiver does not have a copy of the carrier, but was able to generate its own version of the carrier via this loop.

Costas Loop:



The VCO: is an oscillator that produces a signal whose frequency is proportional to the voltage $e_c(t)$.

When $e_c(t) = 0$, the frequency of the oscillator is called the free running frequency. Let this frequency = f_c (the incoming carrier frequency)

When there is a phase difference Ø, we have

$$V_1(t)= A_cA_0m(t)\cos(w_ct)\cos(w_ct+ Ø)$$

$$V_2(t)=\frac{AcA0}{2}m(t)\cos Ø$$

$$V_4(t)=\text{Low pass }\{A_0A_cm(t)\cos(w_ct)\sin(w_ct+ Ø)\}$$

$$V_4(t)= \frac{AcA0}{2}m(t)\sin Ø \text{ after LPF 2}$$

$$e(t)= \frac{AcA0}{2} m(t)^2 (\sin Ø)(\cos Ø)$$

$$= \frac{AcA0}{24} m(t)^2 \sin 2Ø$$

When the B.W of LPF3 is very narrow, the output can be approximated as:

$$e_c(t)= c_0 \sin 2Ø$$

This is the feedback signal that is applied to the VCO. Ideally, when Ø=0, $e_c(t)=0$ and VCO frequency (and phase) are equal to the frequency of the input signal s(t).

If the phase difference Ø between the incoming s(t) and the VCO output increases, then $e_c(t)$ increases forcing the frequency of the VCO to decrease so that it remains in synchronism with the input phase. (Recall that the frequency of the VCO decreases if its input voltage increases; the slope of the VCO characteristics is negative).

Single Sideband Modulation

In this type of modulation, only one of the two sidebands of DSB-SC is retained while the other sideband is suppressed. This mean that B.W of the SSB signal is one half that of DSB-SC. The saving in the bandwidth comes at the expense of increasing modulation complexity.

The time-domain representation of a SSB signal is

$$s(t) = A_c m(t) \cos w_c t \pm A_c \hat{m}(t) \sin w_c t$$

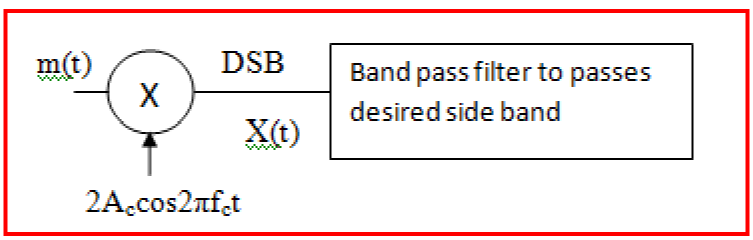
$\hat{m}(t)$: Hilbert transform of m(t)

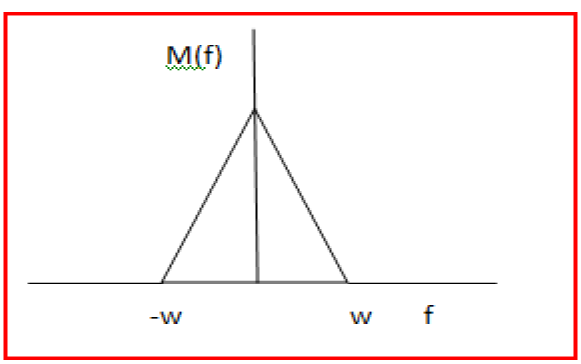
-sign: upper sideband is retained

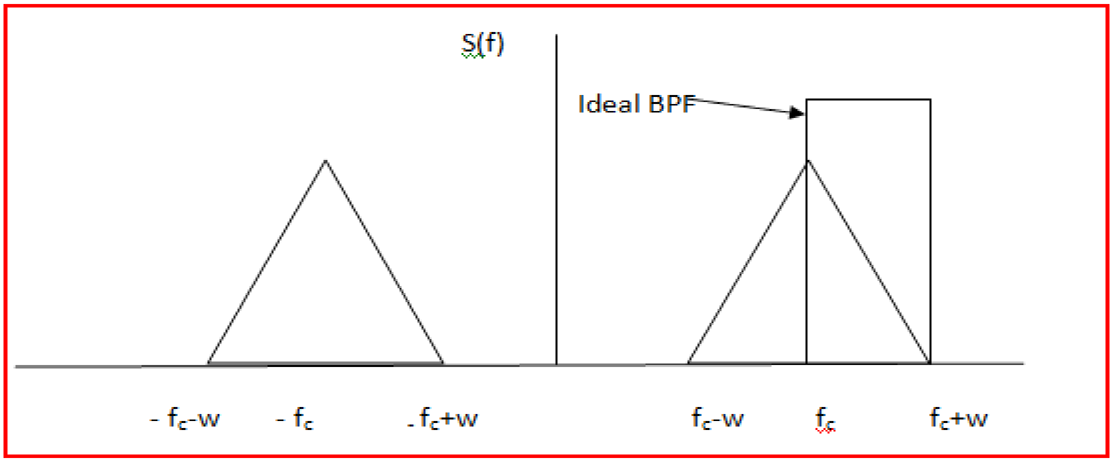
+sign: lower sideband is retained

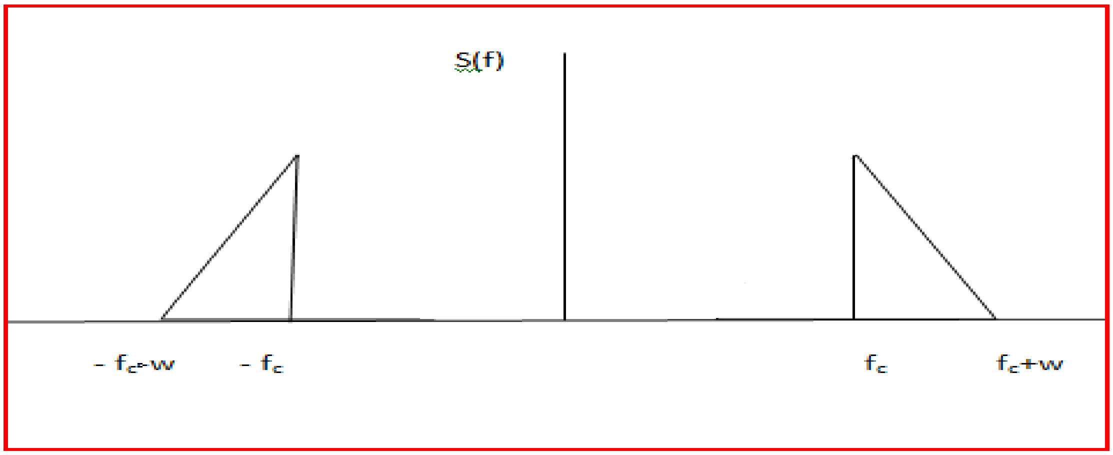
<u>Generation of SSB: Filtering Method (Frequency Discrimination Method)</u>

A DSB-SC signal $X(t) = 2A_c m(t) \cos w_c t$ is generated first. A band pass filter with appropriate B.W and center frequency is used to pass the desired side band only



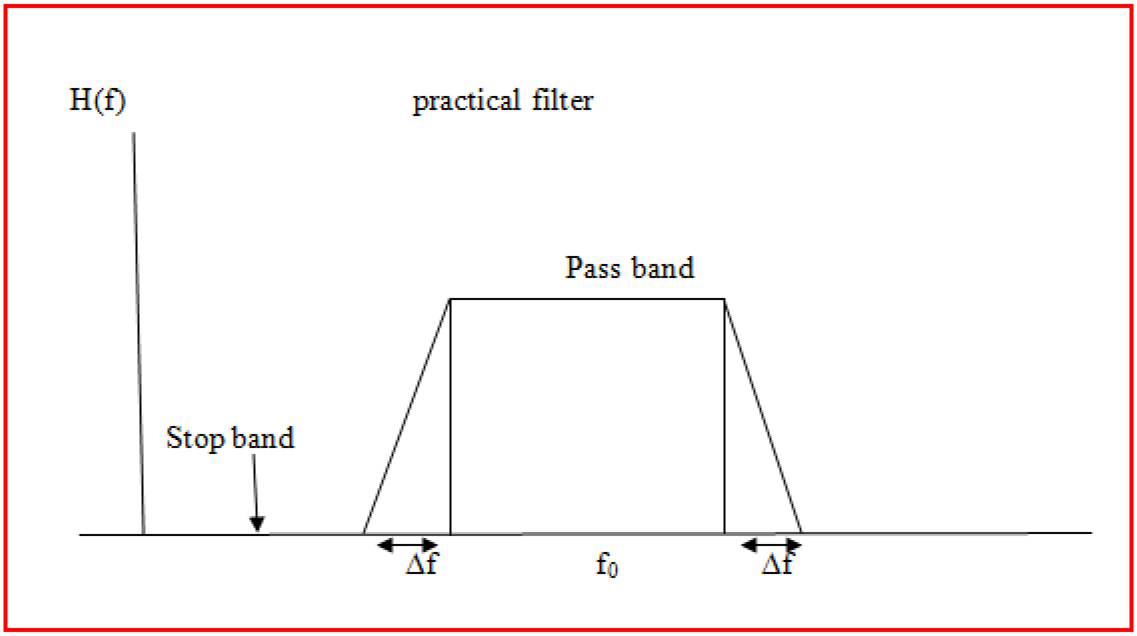






The band pass filter must satisfy two conditions:

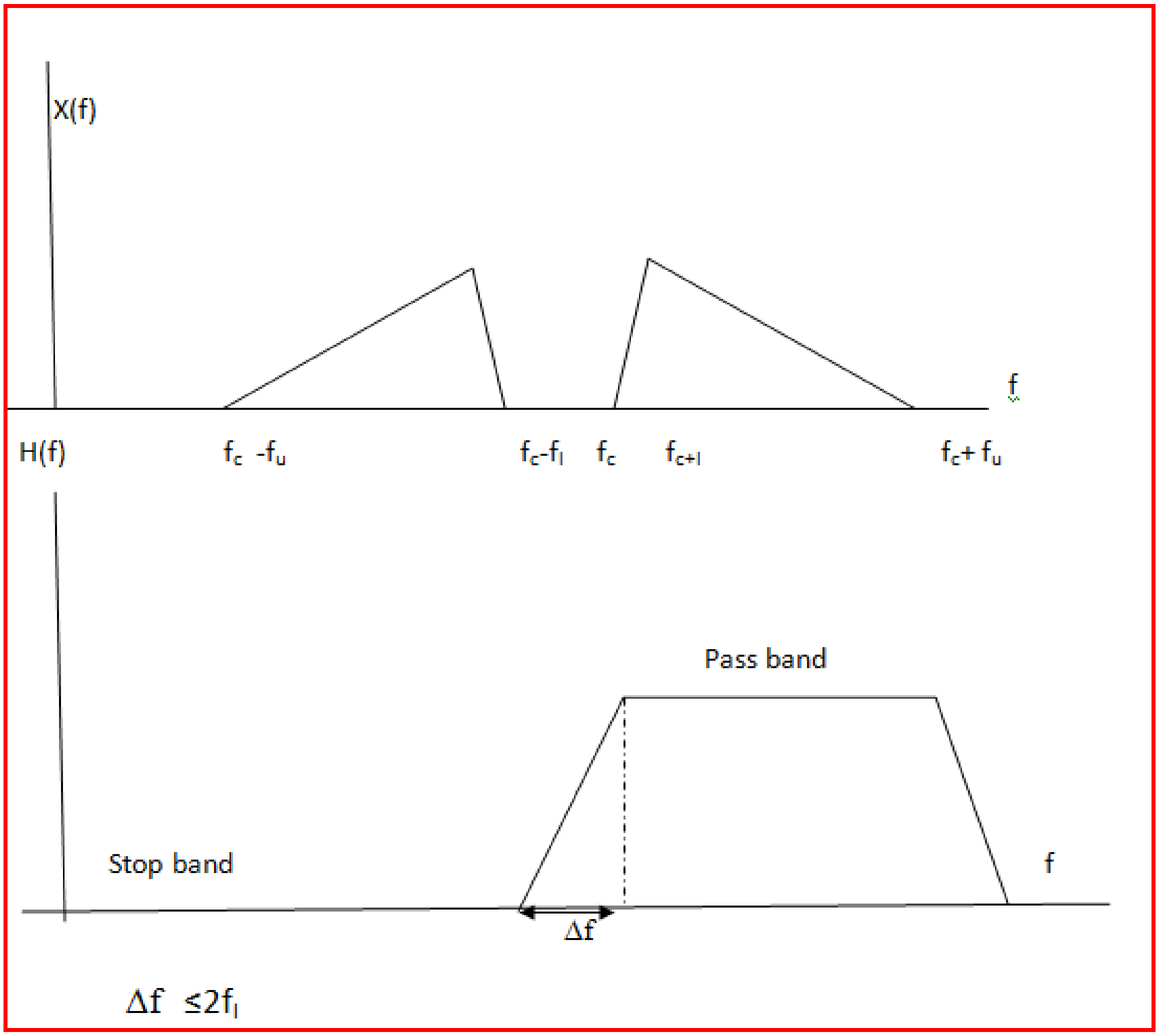
a. The pass band of the filter must occupy the same frequency range as the desired sideband.
b. The width of the transition band of the filter separating the pass band and the stop band must be at least 1% of the center frequency of the filter. i.e., $0.01f_0 \leq \Delta f$. This is sort of a rule of thumb for realizable filters on the relationship between the transition band and the center frequency.



Two remarks should be considered when generating a SSB signal.

1. Ideal filter do not exist in practice meaning that a complete elimination of the undesired side band is not possible. The consequence of this is that either part of the undesired side band is passed or the desired one will be highly attenuated. SSB modulation is suitable for signals with low frequency components that are not rich in terms of their power content.

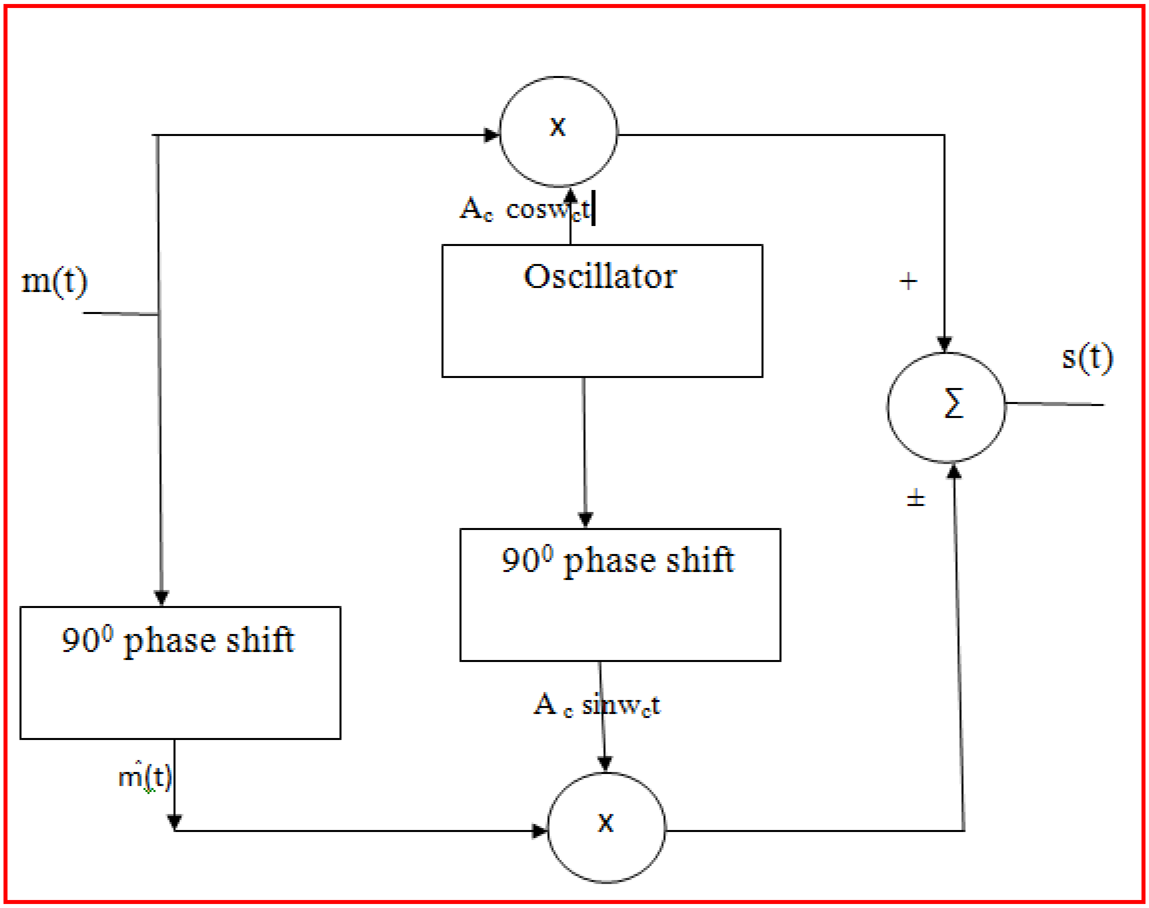
2. The width of the transition band of the filter should be at most twice the lowest frequency components of the message signal so that a reasonable separation of the two side band is possible.



Generation of SSB Signal: Phase Discrimination Method .

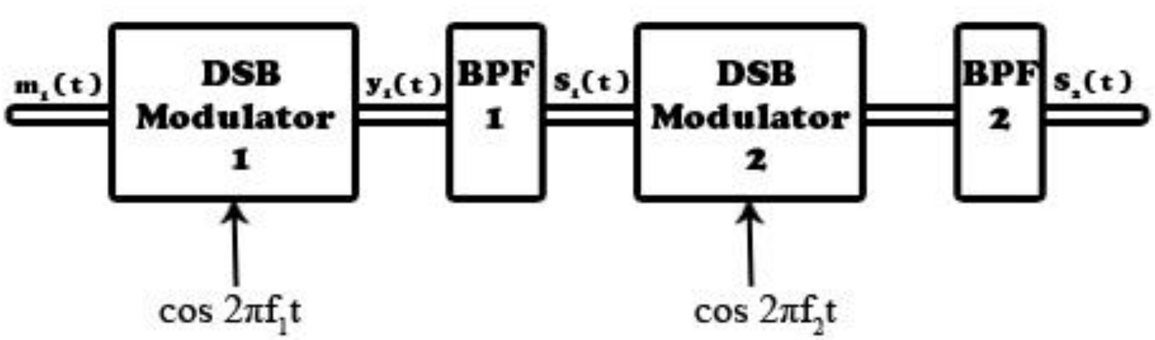
The method is based on the time –domain representation of the SSB signal

$$s(t) = A_c m(t) \cos w_c t \pm A_c \hat{m}(t) \sin w_c t$$



Two- stage generation of SSB signal

When the conditions on the filter cannot be met in a single-stage SSB system, a two-stage scheme is used instead where less stringent conditions on the filters can be imposed. The block diagram illustrates this procedure.



$m_1(t)$ is the base band signal with a gap in its spectrum extending over $(0, f_l)$.

$y_1(t)$: is a DSB-SC signal on a carrier frequency f_1.

BPF_1 selects the upper side band of $y_1(t)$. The parameters of the filter are f_{01} (center frequency) and the transition band length Δf_1.

We must maintain that

$\Delta f_1 \geq 0.01\ f_{01}$ and $\Delta f_1 \leq 2\ f_I$

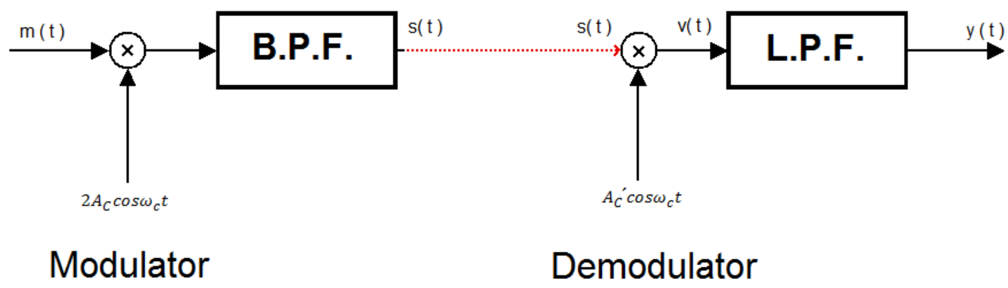
$s_1(t)$ is a single side band signal. The frequency gap of this signal extends over $(0, f_1 + f_L)$. The second modulator views this signal as the baseband signal to be modulated on a carrier with frequency f_2.

The second modulator generates a DSB signal. The second BPF with center frequency f_{02} and transition band Δf_2 selects the upper side band. Again, we maintain that

$\Delta f_2 \geq 0.01\ f_{02}$ and $\Delta f_2 \leq 2\ (f_1 + f_I)$

Demodulation of SSB: Time-Domain Analysis

A SSB signal can be demodulated using coherent demodulation (oscillator at the receiver should have the same frequency and phase as those of transmitter carrier) as shown in the figure:



Let the received signal be the upper single sideband

$$s(t) = A_C m(t) cos\omega_c t - A_C \hat{m}(t) sin\omega_c t$$

At the receiver this is mixed with the carrier signal. The result is

$$v(t) = s(t) A_C' cos\omega_c t$$

$$= A_C' [A_C m(t) cos\omega_c t - A_C \hat{m}(t) sin\omega_c t] cos\omega_c t$$

$$= A_C A_C' m(t) cos2\omega_c t - A_C A_C' \hat{m}(t)\, sin\omega_c t\, cos\omega_c t$$

$$= \frac{A_C A_C'}{2} m(t) + \frac{A_C A_C'}{2} m(t) cos2\omega_c t - \frac{A_C A_C'}{2} \hat{m}(t)\, sin2\omega_c t$$

The low pass filter admits only the first terms. The output is:

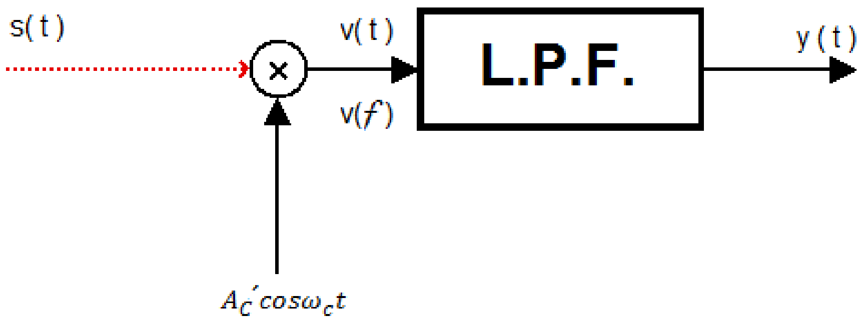
$$y(t) = \frac{A_C A_C'}{2} m(t)$$

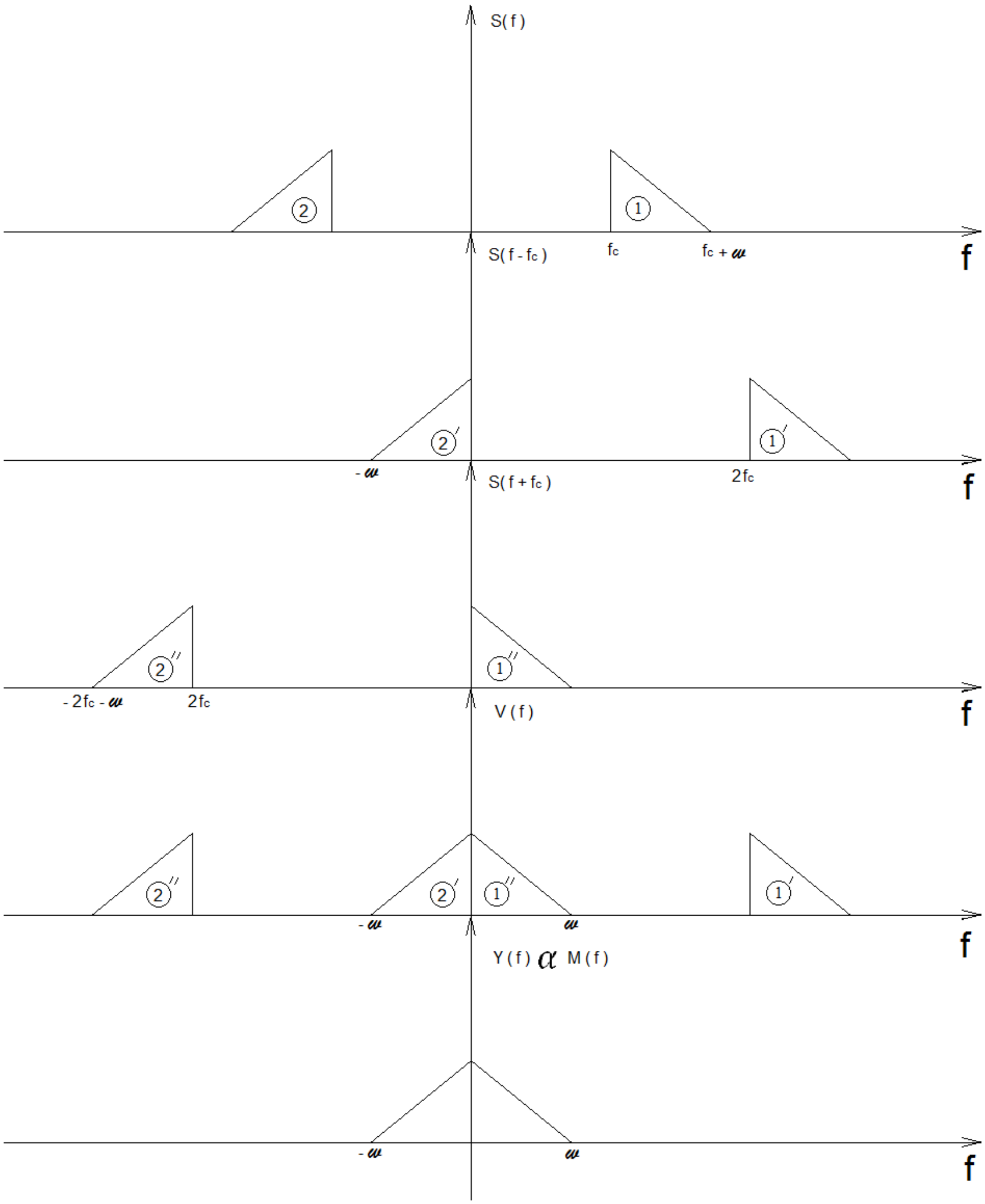
The following steps demonstrate the demodulation process viewed in the frequency domain .

$$V(f) = \frac{A_C'}{2} S(f - f_C) + \frac{A_C'}{2} S(f + f_C)$$

$$Y(f) = Low\ pass \left\{ \frac{A_C'}{2} S(f - f_C) + \frac{A_C'}{2} S(f + f_C) \right\}$$

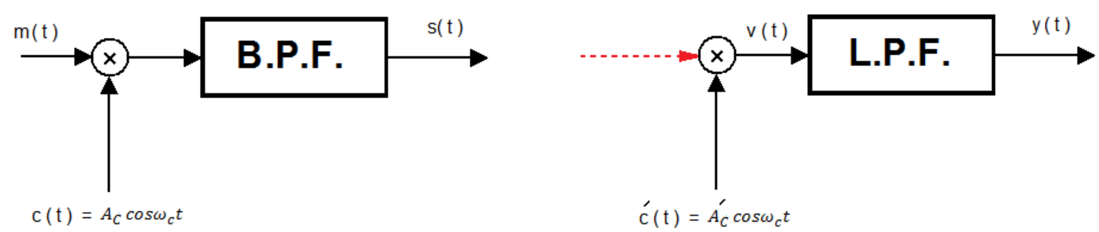
Demodulation of SSB signal : Why one side band is enough ?: A frequency-domain perspective





Demodulation of SSB : Coherent Demodulation

a. **perfect coherent**



when $c(t) = A_c \cos\omega_c t$, $\acute{c}(t) = \acute{A}_c \cos\omega_c t$,

we have perfect coherence and

$$y(t) = \frac{A_c \acute{A}_c}{2} m(t)$$

as was derived earlier

b. **Constant phase difference**

The local oscillator takes the form

$$\acute{c}(t) = \acute{A}_c \cos(\omega_c t + \emptyset);$$

$$v(t) = [A_c m(t) \cos\omega_c t - A_c \hat{m}(t) \sin\omega_c t] \acute{A}_c \cos(\omega_c t + \emptyset)$$

$$= A_c \acute{A}_c m(t) \cos\omega_c t \cos(\omega_c t + \emptyset) - A_c \acute{A}_c \hat{m}(t) \sin\omega_c t \cos(\omega_c t + \emptyset)$$

$$= \frac{A_c \acute{A}_c}{2} m(t) \cos(2\omega_c t + \emptyset) + \frac{A_c \acute{A}_c}{2} m(t) \cos(\emptyset)$$

$$- \frac{A_c \acute{A}_c}{2} \hat{m}(t) \cos(2\omega_c t + \emptyset) - \frac{A_c \acute{A}_c}{2} \hat{m}(t) \sin(\emptyset)$$

➔ $$y(t) = \frac{A_c \acute{A}_c}{2} m(t) \cos(\emptyset) - \frac{A_c \acute{A}_c}{2} \hat{m}(t) \sin(\emptyset)$$

Note that there is a distortion due to the appearance of the Hilbert transform of the message signal at the output.

c. $\acute{c}(t) = \acute{A}_c \cos\ 2\pi(f_c + \Delta f)t$; Constant frequency shift

$$v(t) = [A_c m(t) \cos\omega_c t - A_c \hat{m}(t) \sin\omega_c t] \acute{A}_c \cos\ 2\pi(f_c + \Delta f)t$$

$$= \frac{A_c \acute{A}_c}{2} m(t)[\cos(2\omega_c + \Delta\omega)t + \cos 2\pi\Delta f t]$$

$$- \frac{A_c \acute{A}_c}{2} \widehat{m}(t)[\sin(2\omega_c + \Delta\omega)t + \sin 2\pi\Delta f t]$$

➔ $$y(t) = \frac{A_c \acute{\acute{A}}_c}{2} m(t) \cos 2\pi\Delta f t + \frac{A_c \acute{A}_c}{2} \widehat{m}(t) \sin 2\pi\Delta f t$$

Once again we have distortion and m(t) appears as if single sideband modulated on a carrier frequency = Δ f.

Example :

Let m(t) = $cos2\pi(1000)t$, $\Delta f =$ 100Hz and let s (t) be an upper sideband signal . Then ,

$$y(t) = \frac{A_c \acute{A}_c}{2} cos2\pi(1000)t\ cos2\pi(100)t + \frac{A_c \acute{A}_c}{2} sin2\pi(1000)t\ \ sin2\pi(100)t$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$y(t) = \cos 2\pi(900)t \neq \cos 2\pi(1000)t$$

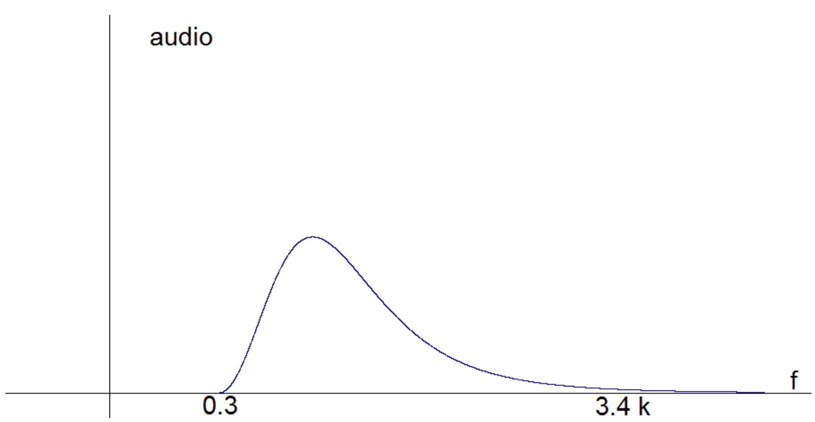
→ Distortion

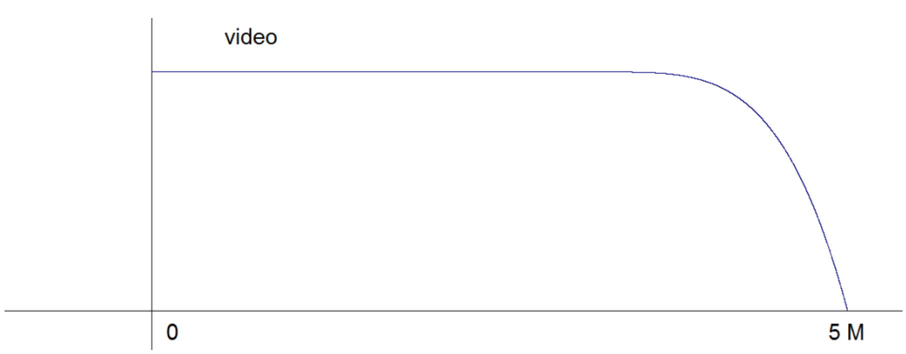
So, a message component with f = 1000Hz appears as a 900Hz component at the demodulator output.

Again , distortion results as a result of failing to synchronize the transmitter and receiver carrier frequencies.

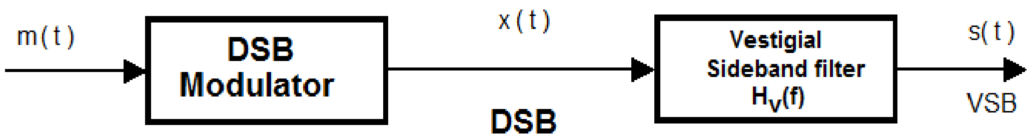
Vestigial sideband (VSB) modulation :

- This type of modulation finds applications in the transmission of video signal .
- Unlike the audio signal , the video signal is rich in low frequency components around the zero frequency .
- The B.W of a video signal is about 5MHz.
- If a video signal is to be transmitted using DSB, it requires a 10 MHz B.W ; too large .
- If a video signal is to be transmitted using SSB (B.W = 5MHz) distortion will results due to the inability to suppress one of the sidebands completely using practical filters.
- A compromise between DSB and SSB was proposed called vestigial sideband modulation .
- Here, a DSB-SC signal is first generated The DSB is applied to a band pass filter (called a *vestigial filter*) that has an asymmetrical frequency response about (-+fc).
- The filter allows one of the sidebands to pass almost without attenuation , while a trace or a vestige of the second sideband is allowed to pass (most of the second sideband is attenuated)
- A typical spectral density of an audio and a video signal is shown below.





Generation of a VSB : Filtering method

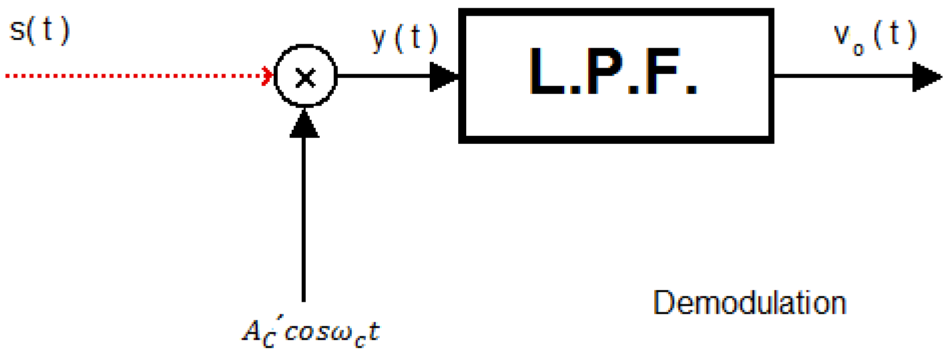


Let $H_v(f)$ be the transfer function of the vestigial filter. We need to find a condition on the characteristic of the filter such that the demodulated signal at the receiver is proportional to the message signal. Now we proceed to find such a condition.

$x(t) = A_c m(t) \cos \omega_c t$;   A DSB-SC signal

$S(f) = X(f) H_v(f)$;   The Fourier transform of the filter output.

$= \frac{A_c}{2}\{M(f-f_c) + M(f+f_c)\}H_v(f)$;   VSB signal



The objective is to specify a condition on $H_v(f)$ such that $V_0(t)$ is an exact replica of m(t) .

$$y(t) = A'_c s(t) \cos \omega_c t$$

$$Y(f) = \frac{A'_c}{2}\{S(f-f_c) + S(f+f_c)\}$$

$$= \frac{A_c A'_c}{4}\{M(f-2f_c) + M(f)\}H_v(f-f_c)$$

$$+ \frac{A_c A'_c}{4}\{M(f+2f_c) + M(f)\}H_v(f+f_c)$$

The LPF will eliminate the high frequency component and retains only the low frequency terms.

$$Vo(f) = \frac{A_c\acute{A}_c}{4}\{H_v(f - f_c) + H_v(f + f_c)\}M(f)$$

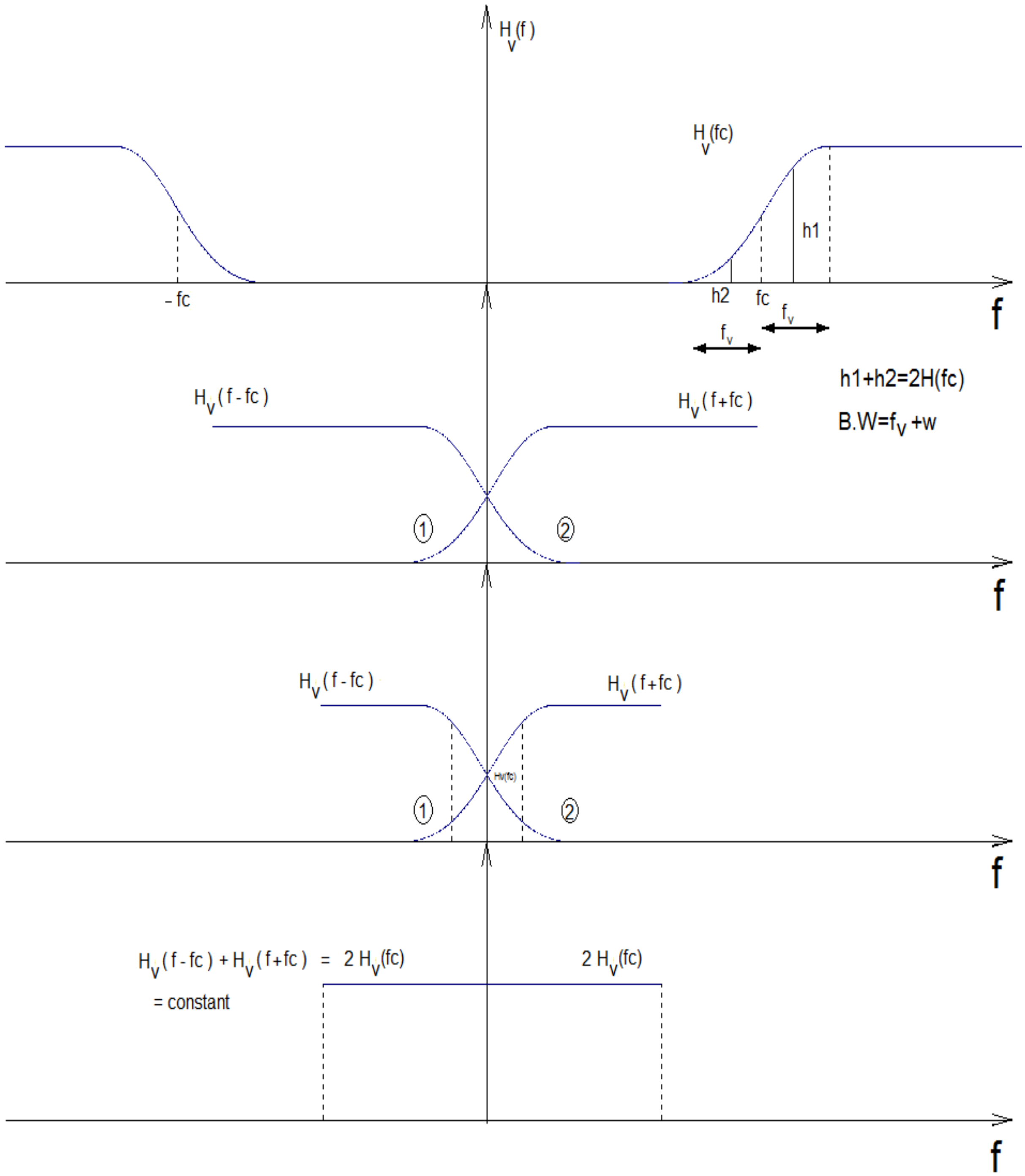
In order for $V_0(f)$ to be proportional to M(f) , we require that

$$H_v(f - f_c) + H_v(f + f_c) = constant = 2H_v(f_c)$$

When this condition is imposed on the filter, the output becomes

$$V_o(f) = \frac{A_c\acute{A}_c}{2}H_v(f_c)M(f)$$

$$v_o(t) = \frac{A_c\acute{A}_c}{2}H_v(f_c)m(t)$$



Two remarks :

1. B.W = $W + f_v$; f_v is the size of the vestige .
2. VSB can be demodulated using coherent demodulation .

Generation of VSB: phase discrimination method

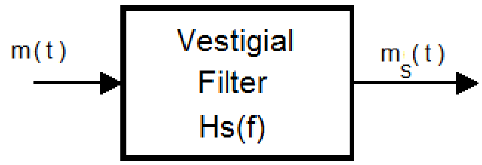
The time-domain representation of a VSB signal is

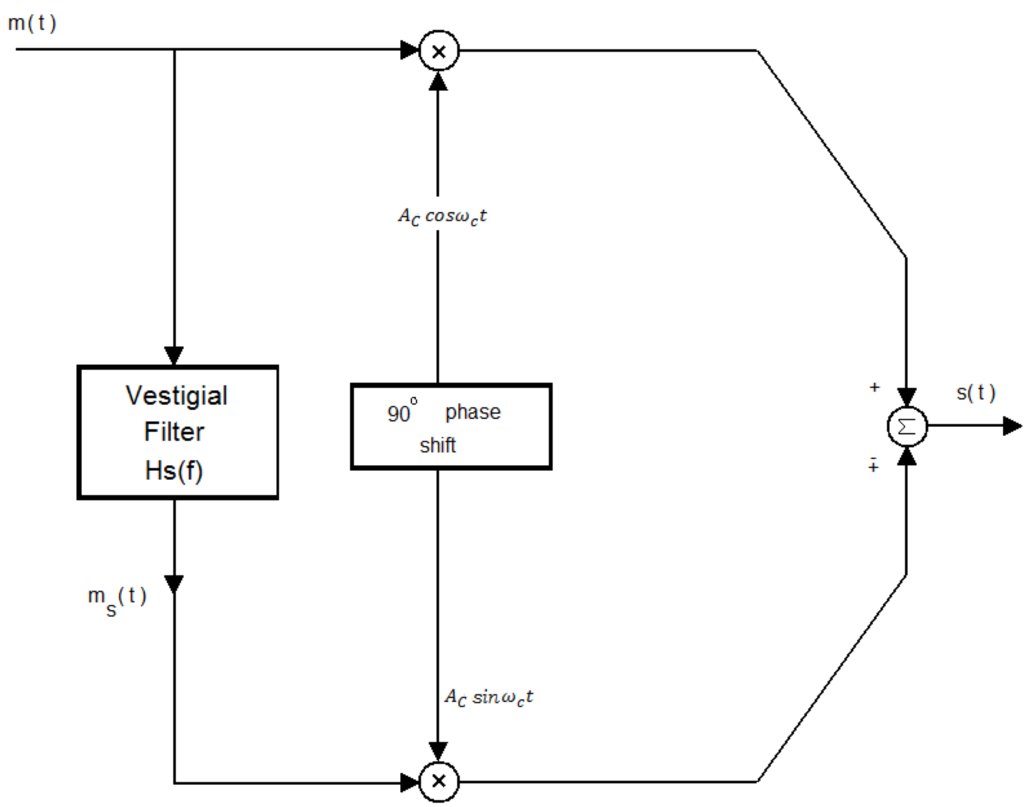
$$s(t) = A_c m(t) \cos \omega_c t \mp A_c m_s(t) \sin \omega_c t$$

Where $m_s(t)$ is the response of a vestigial filter (in the base band spectrum) to the message m(t). Using the time-domain representation ,

the following scheme may be used

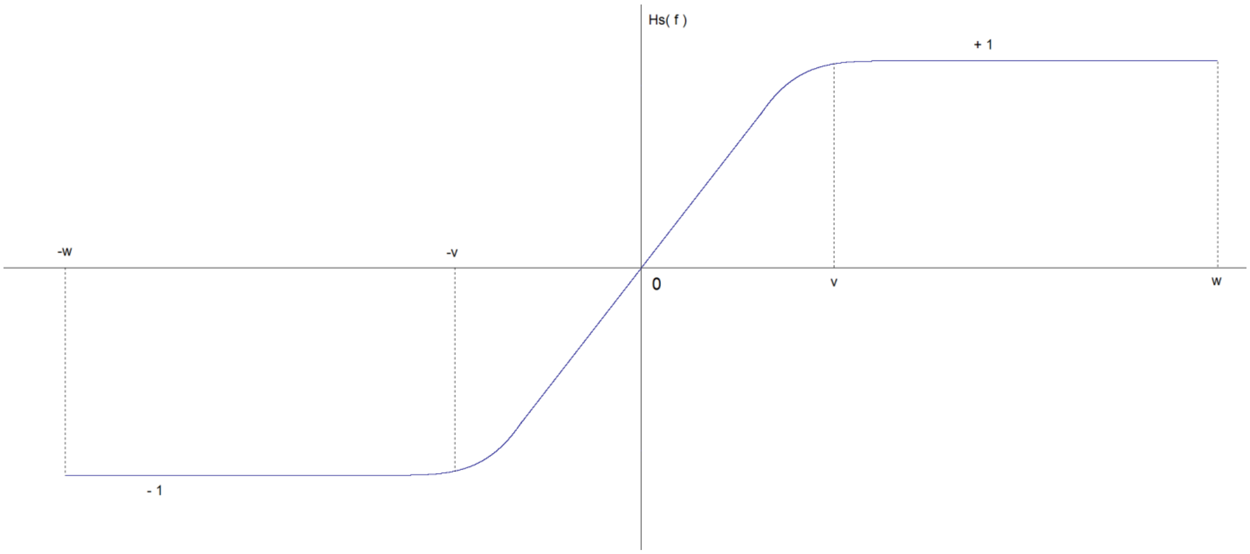
to generate a VSB signal





The – sign means that most of the upper sideband is admitted

+ sign means that most of the lower sideband is admitted

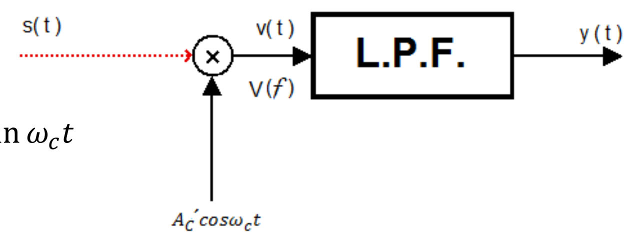


The transfer function HS(f) of the low pass filter is related to the band pass characteristic by:

$$H_s(f) = Low\ pass\ \{H_v(f+f_c) - H_v(f-f_c)\}$$

Coherent Detection of VSB : Time Domain Analysis

Let the received VSB signal be given as:



$$s(t) = A_c m(t)\cos\omega_c t - A_c m_s(t)\sin\omega_c t$$

This signal is mixed with a version of the transmitted carrier of the same phase and frequency.

$$v(t) = s(t)\acute{A}_c\cos 2\pi f_c t$$

$$= A_c\acute{A}_c[m(t)\cos\omega_c t - m_s(t)\sin\omega_c t]\cos\omega_c t$$

$$= A_c\acute{A}_c m(t)\cos^2\omega_c t - A_c\acute{A}_c m_s(t)\sin\omega_c t\cos\omega_c t$$

$$= \frac{A_c\acute{A}_c}{2}m(t) + \frac{A_c\acute{A}_c}{2}m(t)\cos 2\omega_c t - \frac{A_c\acute{A}_c}{2}m_s(t)\sin 2\omega_c t$$

The low pass filter admits only the low pass component, which is nothing but a scaled version of the message signal.

$$y(t) = \frac{A_c \acute{A}_c}{2} m(t)$$

Envelope Detection of VSB + Carrier :

This type of modulation takes the form :

$$s(t) = carrier + VSB$$

$$s(t) = A_c \cos \omega_c t + A_c \beta m(t) \cos \omega_c t \mp A_c \beta m_s(t) \sin \omega_c t$$

β is a scaling factor chosen to minimize envelope distortion. The addition of the carrier is meant to simplify the demodulation of the video signal in practical TV systems and avoids the complexity of coherent demodulation. It is also less expensive since a simple envelope detector, of the type described in demodulating a normal AM signal, can be used.

$$s(t) = \big(A_c + A_c \beta m(t)\big) \cos \omega_c t \mp A_c \beta m_s(t) \sin \omega_c t$$

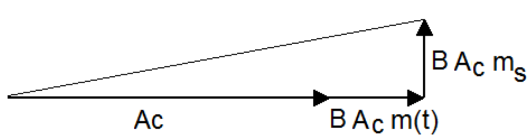
$$s(t) = \sqrt{\big(A_c + A_c \beta m(t)\big)^2 + {A_c}^2 \beta^2 {m_s}^2(t)} \ \cos(\omega_c t + \emptyset)$$

If s(t) is applied to an envelope detector (which is insensitive to phase variations), the output is

$$y(t) = \sqrt{(A_c\big(1 + \beta m(t)\big)^2 + {A_c}^2 \beta^2 {m_s}^2(t)}$$

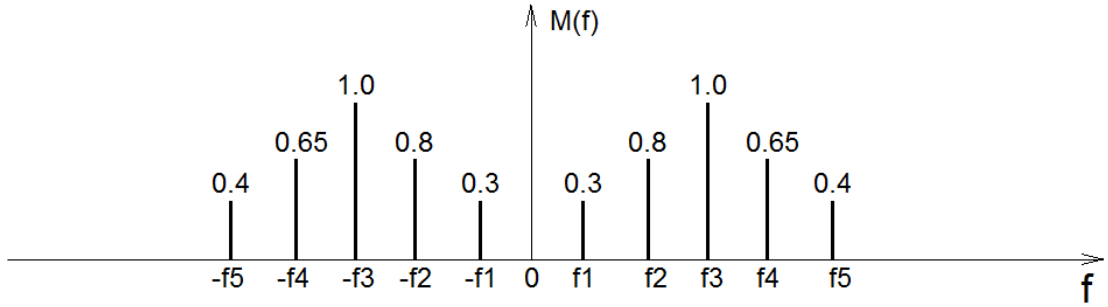
If $A_c \gg \beta m(t)$, then

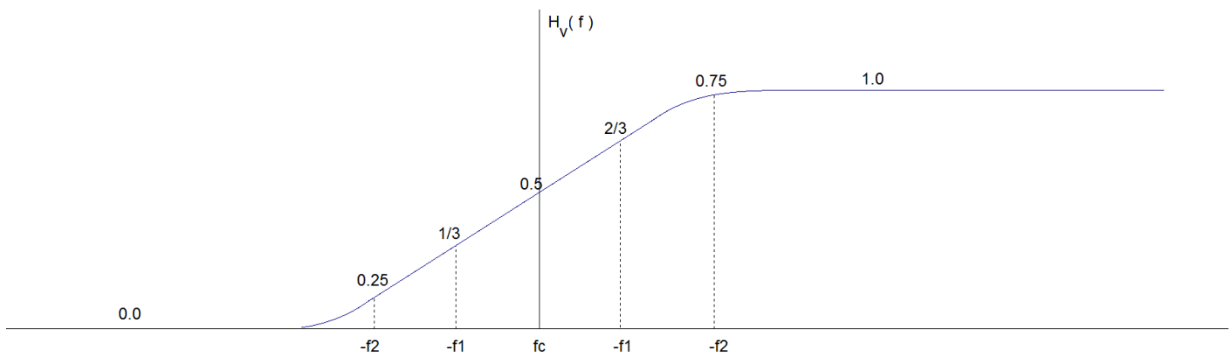
$$y(t) \cong A_c(1 + \beta m(t))$$

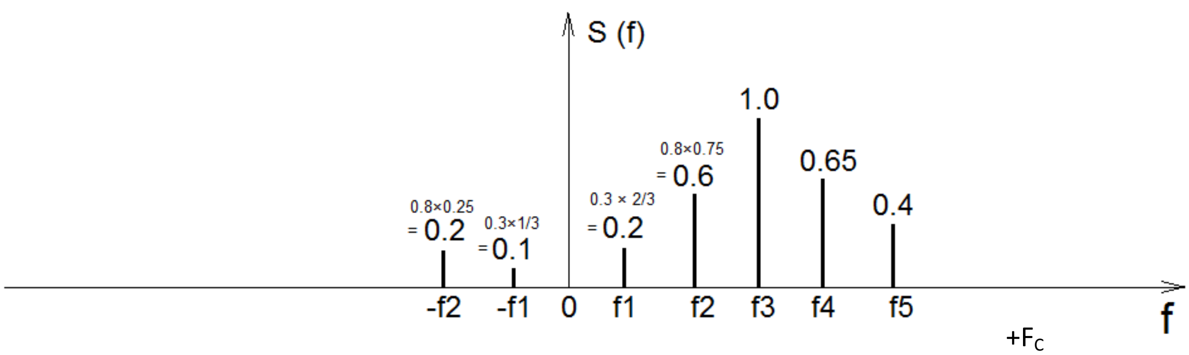


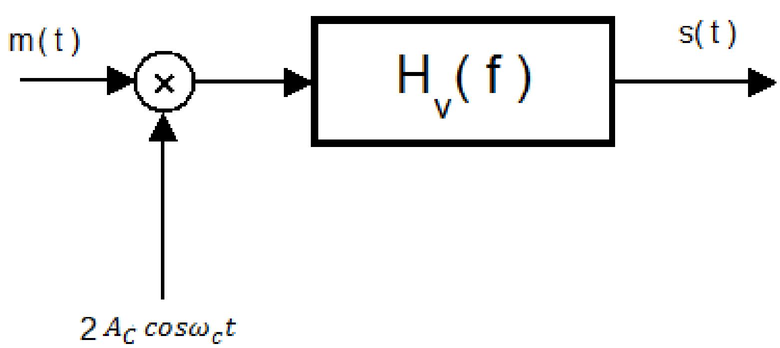
Hence , m(t) can be demodulated, almost without distortion, using simple envelope detection techniques if the above condition is satisfied .

Example: A VSB is generated from the DSB-SC signal $2m(t)\cos\omega_c t$. M(f) and H_v(f) are shown below . Find the spectrum of the transmitted signal s(t).









Baseband Signal :

The input signal consists of five frequency components. It is represented as:

$$m(t) = 0.6\cos 2\pi f_1 t + 1.6\cos 2\pi f_2 t$$
$$+2\cos 2\pi f_3 t + 1.3\cos 2\pi f_4 t + 0.8\cos 2\pi f_5 t$$

Transmitted signal :

The spectrum of the transmitted signal is:

$$S(f) = H_v(f)M(f - f_c) + H_v(f)M(f + f_c)$$

If we perform the multiplication in the frequency domain and take the inverse Fourier transform, we get the time domain representation of the transmitted signal.

$$s(t) = 0.4\cos 2\pi(f_c - f_2)t + 0.2\cos 2\pi(f_c + f_1)t$$
$$+0.4\cos 2\pi(f_c + f_1)t + 1.2\cos 2\pi(f_c + f_2)t + 2\cos 2\pi(f_c + f_3)t$$
$$+ 1.3\ \cos 2\pi(f_c + f_4)t + 0.8\ \cos 2\pi(f_c + f_5)t$$

Frequency and Phase Modulation

To generate an angle modulated signal, the amplitude of the modulated carrier is held constant and either the phase or the time derivative of the phase is varied linearly with the message signal m(t).

The expression for an angle modulated signal is:

$$s(t) = A_c \cos\big(\omega_c t + \Theta\theta(t)\big) ,$$ ω_c is the modulated carrier frequency.

The instantaneous frequency of s(t) is :

$$f_i(t)= \frac{1}{2\pi} \times \frac{d}{dt}\big(\omega_c t + \theta(t)\big) = f_c + \frac{1}{2\pi} \times \frac{d\theta(t)}{dt}$$

For **phase modulation**, the phase is directly proportional to the modulating signal :

$$\theta(t) = k_p m(t),$$ k_p is the phase sensitivity measured in rad/volt.

The peak phase deviation is

$$\Delta\theta \quad = k_p \times \max\ \big(m(t)\big).$$

For **frequency modulation**, the frequency deviation of the carrier is proportional to the modulating signal:

$$\frac{1}{2\pi} \times \frac{d\theta(t)}{dt} = k_f m(t) \implies f_i = f_c + k_f m(t).$$

The frequency deviation from the un-modulated carrier is

$$f_i(t) - f_c = \frac{1}{2\pi}\frac{d\theta}{dt}$$

The peak frequency deviation is

$$\Delta f = \max\ \{\frac{1}{2\pi} \times \frac{d\theta}{dt}\}.$$

The time domain representation of a phase modulated signal is :

$$s(t) = A_c \cos\left(\omega_c t + k_p m(t)\right).$$

The time domain representation of a frequency modulated signal is :

$$s(t) = A_c \cos\left(\omega_c t + 2\pi k_f \int_{-\infty}^{t} m(\alpha) d\alpha\right).$$

where $\theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\alpha) d\alpha$

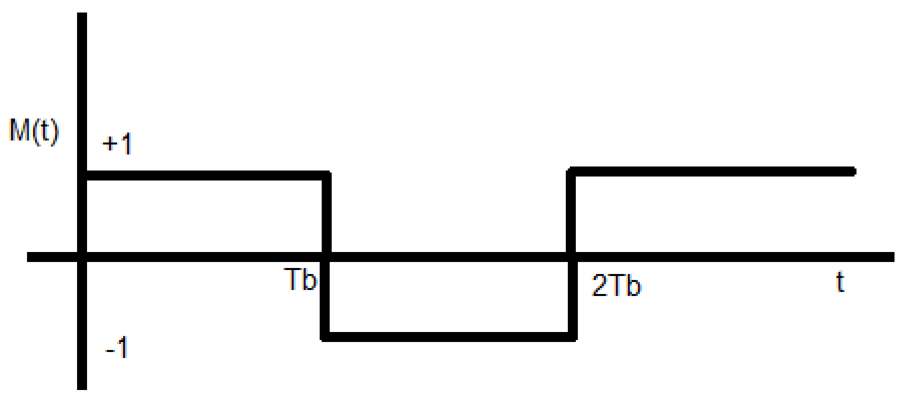
The average power in $s(t)$, for frequency modulation (FM) or phase modulation (PM) is:

$$p_{ava} = \frac{(A_c)^2}{2} = constant.$$

<u>Example</u>: Frequency Shift Keying.

The periodic square signal m(t), shown below, frequency modulates the carrier $c(t) = A_c \cos(2\pi 100t)$ to produce the signal $s(t) = A_c \cos\left((2\pi 100t) + 2\pi k_f \int m(\alpha) d\alpha\right)$ where k_f =10 HZ/V.

a. Find and plot the instantaneous frequency $f_i(t)$.
b. Find and sketch s(t).



Solution:

a) The instantaneous frequency is

$f_i = f_c + k_f \times m(t)$

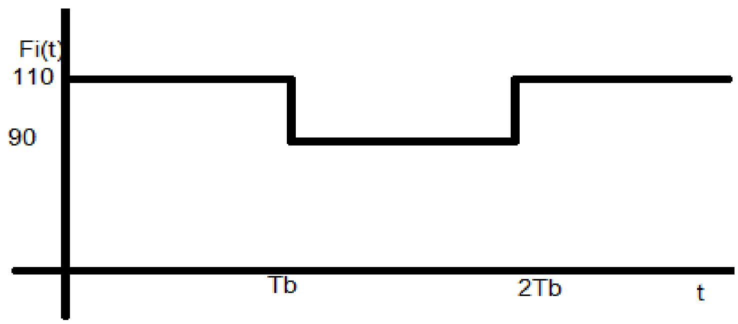
$f_i =$ { 100+10 =110 Hz   when m(t) =+1.

$f_i =$ { 100 -10 = 90Hz   when m(t) =-1.

For $0 < t \leq T_b$, $f_i = 110$ Hz

$For\ T_b \leq t \leq 2T_b\ f_i = 90$ Hz

The instantaneous frequency hops between the two values 110 Hz and 90 Hz as shown below



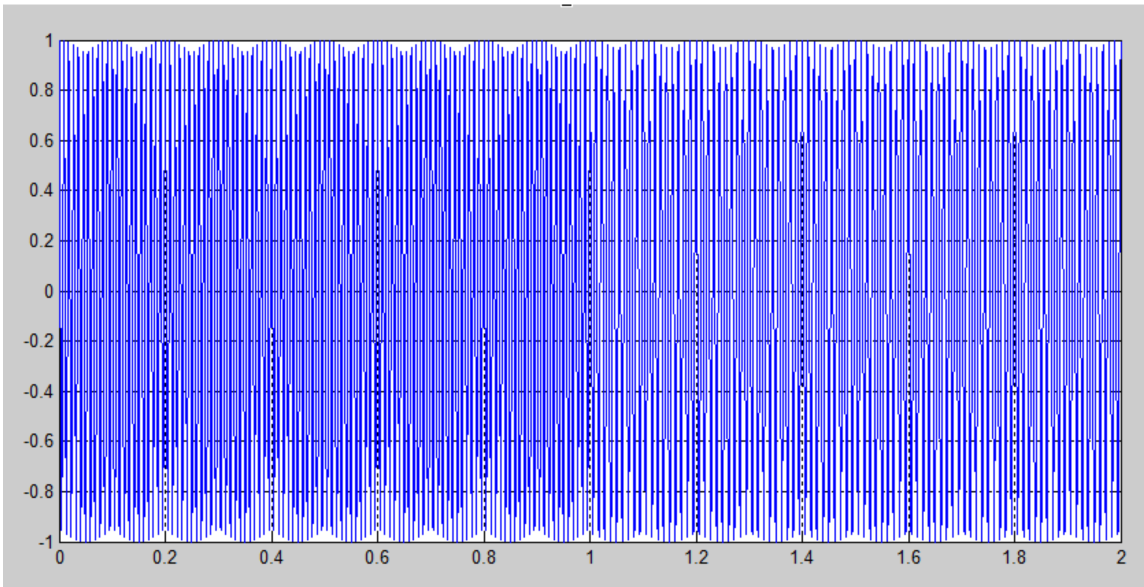
In digital transmission, we will see that a binary (1) may be represent by a signal of frequency f_1 for $0 \leq t \leq T_b$ and a binary (0) by a signal of frequency f_2 for $0 \leq t \leq T_b$.

b) The two signal possible to transmitted signal are :

$s(t) = A\cos(2\pi(110)t\ ), \quad when\ m(t) = +1$

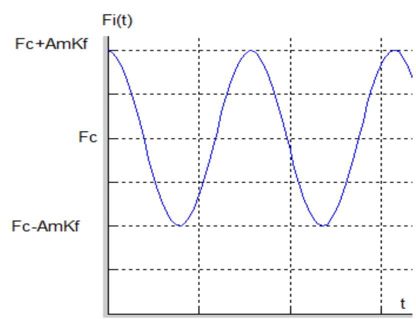
$s(t) = A\cos(2\pi(90)t\ \ ), \quad when\ m(t) = -1$

For $T_b = 1\ sec$, the transmitted signal may look like as in the figure.



Single Tone Frequency Modulation:

Assume that the message $m(t) = A_m \cos \omega_m t$.



The instantaneous frequency is:

$$f_i = f_c + k_f m(t) = f_c + A_m k_f \cos 2\pi f_m t.$$

This frequency is plotted in the figure.

The peak frequency deviation (from the un-modulated carrier) is :

$$\Delta f = k_f A_m.$$

The FM signal is:

$$s(t) = A_c \cos\ (\omega_c t + \beta \sin 2\pi f_m t).$$

Where β is the FM modulation index:

$$\beta = \frac{k_f A_m}{f_m} = \frac{peak\ frequency\ deviation}{message\ bandwidth} = \frac{\Delta f}{f_m}$$

Spectrum of a Single-Tone FM Signal

The objective is to find a meaningful definition of the bandwidth of an FM signal:

Let $m(t) = A_m \cos 2\pi f_m t$ be the message signal, then the FM signal is:

$$s(t) = A_c \cos(\ 2\pi f_c t + \beta \sin 2\pi f_m t\ )$$

Where $\beta = \frac{\Delta f}{f_m} = \frac{peak\ frequency\ deviation}{message\ bandwidth}$ is the modulation index.

⟹ Recall that:

$$s(t) = A_c \cos(\ 2\pi f_c t + \beta \sin 2\pi f_m t\ )$$

which can be rewritten as:

$$s(t) = Re\left\{e^{j(2\pi f_c t + \beta \sin 2\pi f_m t)}\right\}$$

$$= Re\{e^{j(2\pi f_c t)} \times e^{j(\beta \sin 2\pi f_m t)}\}$$

Remember that: $e^{j\theta} = cos\theta + jsin\ \theta$ and that $cos\theta = Re\{e^{j\theta}\}$

The function [$\beta \sin 2\pi f_m t$] is "sinusoidal" and periodic with $T_m = \frac{1}{f_m}$. Therefore, $e^{j(\beta \sin 2\pi f_m t)}$ is also periodic with $T_m = \frac{1}{f_m}$.

As we know, a periodic function g(t) can be expanded into a complex Fourier series as:

$$g(t) = \sum_{-\infty}^{\infty} C_n\, e^{jn\omega_m t} \quad where\ \ C_n = \frac{1}{T_m}\int_0^{T_m} g(t)\, e^{-j\omega_m n t}\, dt\ .$$

⟹ If we let $g(t) = e^{j(\beta \sin 2\pi f_m t)}$

then, $C_n = \frac{1}{T_m}\int_0^{T_m} e^{j(\beta \sin 2\pi f_m t)} \times e^{-j\omega_m n t}\, dt$

It out that $\Longrightarrow$ $C_n = J_n(\beta)$.

Where $J_n(\beta)$ is the Bessel function of the first kind of order n.

Hence, $g(t) = \sum_{-\infty}^{\infty} J_n(\beta)\, e^{jn\omega_m t}$

Substituting into $s(t)$, we get:

$$\Longrightarrow s(t) = A_c\, Re\{e^{j(2\pi f_c t)} \times \sum_{-\infty}^{\infty} J_n(\beta)\, e^{jn\omega_m t}\}$$

$$= A_c\, Re\{\sum_{-\infty}^{\infty} J_n(\beta) \times e^{j2\pi(f_c + nf_m)t}\}$$

$$= A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + nf_m)t)$$

Finally, the FM signal can be represented as

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + nf_m)t)$$

<u>Bessel Functions</u>:

The Bessel equation of order n is:

$$x^2 \frac{dy^2}{dx_2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

This is a second order differential equation with variable coefficient. We can solve it by the power series method, for example:

Let $y = \sum_{n=0}^{\infty} C_n\, x^n$, $\frac{dy}{dx} = \sum_{n=1}^{\infty} n\, C_n\, x^{n-1}$, $\frac{dy^2}{dx_2} = \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2}$.

Substituting $y, \frac{dy}{dx}$ and $\frac{dy^2}{dx_2}$ into the differential equation and equating terms of equal power results in:

$$y = \sum_{m=0}^{\infty} \frac{(-1)^m \times (\frac{1}{2}x)^{n+2m}}{m!\,(n+m)!}$$

The solution for each value of n (see the D.E where n appears) is $J_n(x)$, the Bessel function of the first kind of order n.

Some Properties of $J_n(x)$:

1- $J_n(x) = (-1)^n J_{-n}(x)$.

2- $J_n(x) = (-1)^n J_n(-x)$.

3- Recurrence formula $\Longrightarrow$ $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$.

4- For small values of x: $\Longrightarrow$ $J_n(x) \cong \frac{x^n}{2^n\, n!}$

Therefore, $J_0(x) \cong 1$

$J_1(x) \cong \frac{x}{2}$

$J_n(x) \cong 0 \;\; for\; n > 1$.

5- For large value of x:

$J_n(x) \cong \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{n\pi}{2})$, $J_n(x)$ behaves like a sine function with progressively decreasing amplitude.

6- For real x and fixed, $J_n(x) \longrightarrow 0 \; as\; n \longrightarrow \infty$.

7- $\sum_{-\infty}^{\infty} (J_n(x))^2 = 1$, $for\; all\; x$.

The Bessel Functions Table for various values of β and n.

- Table 1, notice that x = β.

					$J_n(x)$				
n\x	0.5	1	2	3	4	6	8	10	12
0	0.9385	0.7652	0.2239	−0.2601	−0.3971	0.1506	0.1717	−0.2459	0.0477
1	0.2423	0.4401	0.5767	0.3391	−0.0660	−0.2767	0.2346	0.0435	−0.2234
2	0.0306	0.1149	0.3528	0.4861	0.3641	−0.2429	−0.1130	0.2546	−0.0849
3	0.0026	0.0196	0.1289	0.3091	0.4302	0.1148	−0.2911	0.0584	0.1951
4	0.0002	0.0025	0.0340	0.1320	0.2811	0.3576	−0.1054	−0.2196	0.1825
5	—	0.0002	0.0070	0.0430	0.1321	0.3621	0.1858	−0.2341	−0.0735
6		—	0.0012	0.0114	0.0491	0.2458	0.3376	−0.0145	−0.2437
7			0.0002	0.0025	0.0152	0.1296	0.3206	0.2167	−0.1703
8			—	0.0005	0.0040	0.0565	0.2235	0.3179	0.0451
9				0.0001	0.0009	0.0212	0.1263	0.2919	0.2304
10				—	0.0002	0.0070	0.0608	0.2075	0.3005
11					—	0.0020	0.0256	0.1231	0.2704
12						0.0005	0.0096	0.0634	0.1953
13						0.0001	0.0033	0.0290	0.1201
14						—	0.0010	0.0120	0.0650

Table 2.

n	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 8$	$\beta = 10$
0	0.997	0.990	0.938	0.765	0.224	−0.178	0.172	−0.246
1	0.050	0.100	0.242	0.440	0.577	−0.328	0.235	0.043
2	0.001	0.005	0.031	0.115	0.353	0.047	−0.113	0.255
3				0.020	0.129	0.365	−0.291	0.058
4				0.002	0.034	0.391	−0.105	−0.220
5					0.007	0.261	0.186	−0.234
6					0.001	0.131	0.338	−0.014
7						0.053	0.321	0.217
8						0.018	0.223	0.318
9						0.006	0.126	0.292
10						0.001	0.061	0.207
11							0.026	0.123
12							0.010	0.063
13							0.003	0.029
14							0.001	0.012
15								0.004
16								0.001

The FM Signal Series Representation

We saw earlier that a single tone FM signal can be represented in a Fourier series as :

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta) \times \cos(2\pi(f_c + nf_m)t)$$

The first few terms in this expansion are:

$$s(t) = A_c \{ J_0(\beta)\cos(2\pi f_c t) + J_1(\beta)\cos 2\pi(f_c + f_m)t + J_{-1}(\beta)\cos 2\pi(f_c - f_m)t \ + \\ J_2(\beta)\cos 2\pi(f_c + 2f_m)t \ + J_{-2}(\beta)\cos 2\pi(f_c - 2f_m)t + \cdots..$$

The FM signal consists of infinite number of spectral components concentrated around f_c. Therefore, the theoretical bandwidth of the signal is infinity. That is to say, if we need to recover the FM signal without any distortion, all spectral components must be accommodated. This means that a channel with infinite bandwidth is needed. This is of course not practical since the frequency spectrum is shared by many users.

In the following discussion we need to truncate the series so that say 99% of the total average power is contained within a certain bandwidth. But first let us find the total average power using the series approach.

<u>The total average power in s(t)</u>

Note that s(t) consists of an infinite number of Fourier terms, and the power in s(t) will be equal the power in the respective Fourier components .

Any term in s (t) takes the form: $A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$

The average power in this term is: $\frac{(A_c)^2 (J_n(\beta))^2}{2}$

Hence the total power in s(t) is:

$$< S^2(t) > \ = \ \frac{A_C{}^2\, J_0{}^2(\beta)}{2} + \frac{A_C{}^2 J_1{}^2(\beta)}{2} + \frac{A_C{}^2 J_{-1}{}^2(\beta)}{2} + \frac{A_C{}^2 J_2{}^2(\beta)}{2} + \frac{A_C{}^2 J_{-2}{}^2(\beta)}{2} + \cdots$$

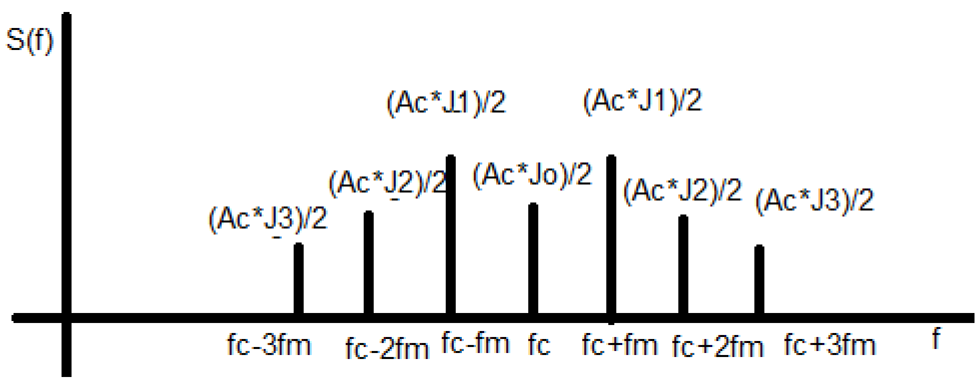
$$= \frac{A_C{}^2}{2} \{ J_0{}^2(\beta) + J_1{}^2(\beta) + J_{-1}{}^2(\beta) + J_2{}^2(\beta) + J_{-2}{}^2(\beta) + \cdots.. \}$$

$= \ \frac{A_C{}^2}{2} \{ \sum_{n=-\infty}^{\infty} J_n{}^2(\beta) \}$, where $\sum_{n=-\infty}^{\infty} J_n{}^2(\beta) \ = 1$, (A property of Bessel Functions)

The average power becomes

$$< S^2(t) > = \frac{A_C{}^2}{2}.$$

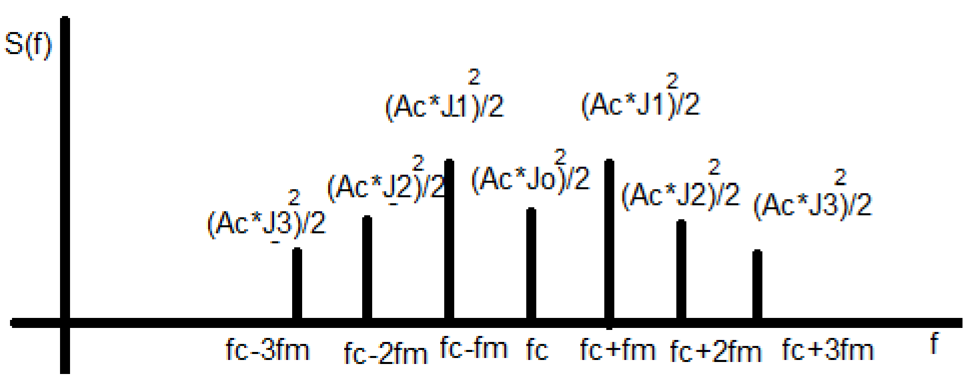
Spectrum of an Fm Signal



Fourier transform of $s(t) = A_c \cos\ (\omega_c t + \beta \sin 2\pi f_m t)$. (only + ve frequencies shown)

Note that in the figure above as f_m *decreases*, the spectral lines become closely concentrated about f_c .

The power spectral density, which is a plot of $|C_n|^2$ versus f, is shown below:



Example:

Plot the FM spectrum and find the 99% power bandwidth when $\beta = 1$ and $\beta = 0.2$

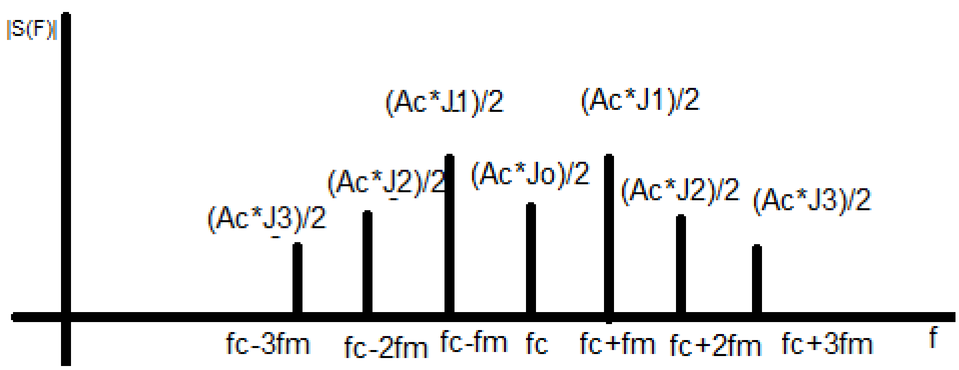
Solution:

$$s(t) = A_c \sum_{-\infty}^{\infty} J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

Case a: $\boldsymbol{\beta = 1}$

For $\beta = 1$, there are five significant Bessel terms (but we may not need all of them to achieve the 99% power bandwidth)

$J_0(1)$ = 0.7652, $J_1(1)$ = 0.4401, $J_2(1)$ = 0.1149, $J_3(1)$ = 0.01956, $J_4(1)$ = 0.002477



The power in s(t) is $< S^2(t) > = \frac{A_c^2}{2}$

Let us try to find the average power in the terms at f_c , $f_c + f_m$, $f_c - f_m$, $f_c + 2f_m$, $f_c - 2f_m$

The average power in these five components can be calculated as:

1. f_c : $\frac{A_c^2 J_0^2(\beta)}{2}$
2. $f_c + f_m$: $\frac{A_c^2 J_1^2(\beta)}{2}$
3. $f_c - f_m$: $\frac{A_c^2 J_{-1}^2(\beta)}{2}$
4. $f_c + 2f_m$: $\frac{A_c^2 J_2^2(\beta)}{2}$

5. $f_c - 2f_m$: $\frac{A_C{}^2 J_{-2}{}^2(\beta)}{2}$

The average power in the five spectral components is the sum:

$$= \frac{A^2{}_c}{2}\left[J_0{}^2(1) + 2J_1{}^2(1) + 2J_2{}^2(1)\right]$$
$$= \frac{A^2{}_c}{2}\ [(0.7652)^2 + 2 * (0.4401)^2 + (0.1149)^2] = \ 0.9993\ \frac{A^2{}_c}{2}$$

So, these terms have 99.9 % of the total power.

Therefore, the 99.9 % power bandwidth is

B.W= $(f_c + 2f_m) - (f_c - 2f_m) = 4f_m$

Case b: $\boldsymbol{\beta = 0.2}$

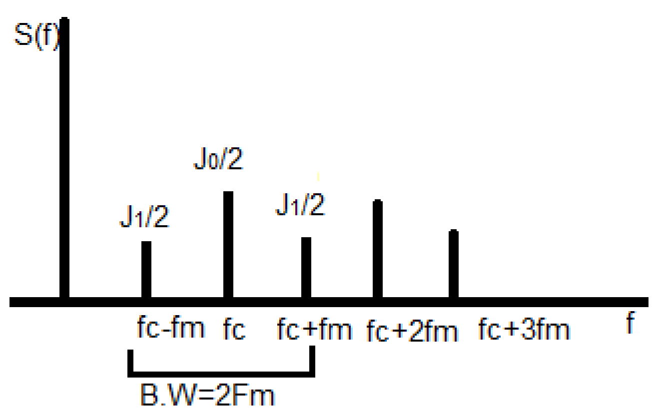
For $\beta = 0.2,\ J_0(0.2) = 0.99,\ J_1(0.2) = 0.0995,\ J_2(0.2) = 0.00498335$

The power in the carrier and the two sidebands (at f_c , $f_c + f_m$, $f_c - f_m$) is

$$P = \frac{A^2{}_c}{2}\left[J_0{}^2(0.2) + 2J_1{}^2(0.2)\right]$$

$$P = \frac{A^2{}_c}{2}[0.9999]$$

Therefore,
$99.99\%\ of\ the\ total\ power\ is\ found\ in\ the\ carrier\ and\ two\ side\ bands$



The 99% bandwidth is

$$\text{B.W} = (f_c + f_m) - (f_c - f_m) = 2f_m$$

Remark:

We observe that the spectrum of an FM signal when $\beta \ll 1$ (called narrow band FM) is "similar" to the spectrum of a normal AM signal, in the sense that it consists of a carrier and two sidebands. The bandwidth of both signals is $2f_m$.

Carson's Rule

A 98% power B.W of an FM signal is estimated using Carson's rule:

$$B_T = 2(\beta + 1)f_m$$

Generation of an FM Signal

First: Generation of a Narrowband FM Signal

Consider an angel modulated signal:

$$s(t) = A_c \cos(2\pi f_c t + \theta(t))$$

When $s(t)$ is an FM signal, $\theta(t) = 2\pi k_f \int m(t)dt$

$s(t)$ can be expanded as:

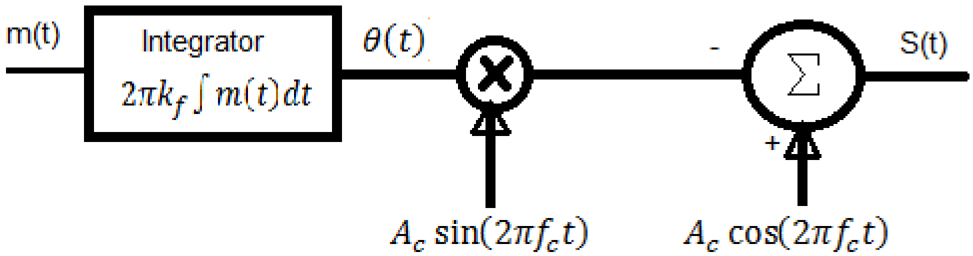
$$s(t) = A_c \cos(2\pi f_c t) \cos\big(\theta(t)\big) - A_c \sin(2\pi f_c t) \sin\big(\theta(t)\big)$$

When $|\theta(t)| \ll 1$, $cos\ \theta \cong 1$, $\sin(\theta) \cong \theta$ and $s(t)$, termed narrowband, can be approximated as:

$$s(t) \cong A_c \cos(2\pi f_c t) - A_c \theta \sin(2\pi f_c t)$$

This expression can serve as the basis for the generation of a narrowband FM or PM signals.

To Generate a narrowband FM, consider the block diagram below:



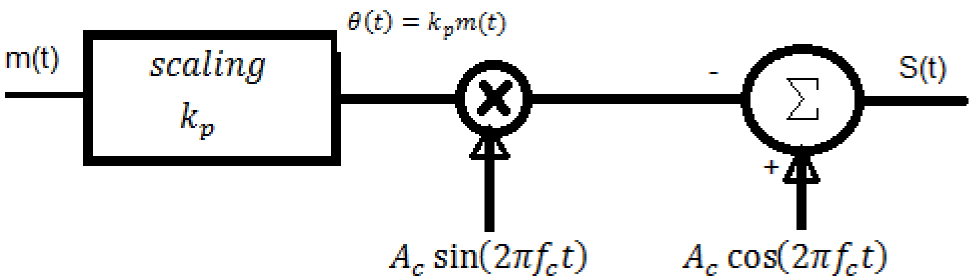
When $m(t) = A_m \cos(2\pi f_m t)$

$$\theta(t) = \beta \sin(2\pi f_m t)$$

And the modulated signal takes the form

$$s(t) = A_c \cos(2\pi f_c t) - A_c \beta \sin(2\pi f_m t) \sin(2\pi f_c t)$$

To generate a narrow band PM signal, we can use the scheme:



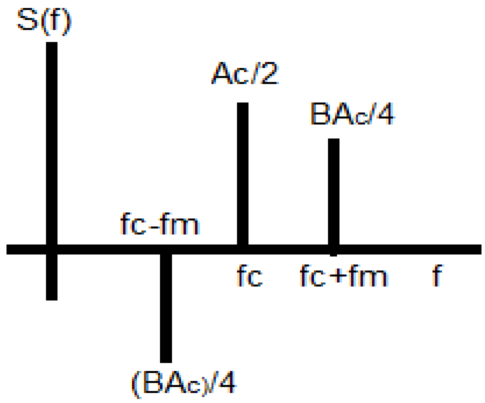
Spectrum of a single- tone NBFM:

For an FM signal, $\theta(t) = \beta\sin(2\pi f_m t)$

$$s(t) = A_c\cos(2\pi f_c t) - A_c\beta\sin(2\pi f_m t)\sin(2\pi f_c t)$$

$$s(t) = A_c\cos(2\pi f_c t) - \frac{A_c\beta}{2}[\cos(2\pi(f_c - f_m)t) - \cos(2\pi(f_c + f_m)t)]$$

The spectrum of $s(t)$ is shown below:



The spectrum consists of a component at the carrier frequency f_c , and at the two sidebands $f_c + f_m$ and $f_c - f_m$). Note the negative sign at the lower sideband. The bandwidth of this signal is $2f_m$.

Now consider the normal AM signal with sinusoidal modulation.

$$s(t)_{AM} = A_c\cos(2\pi f_c t) + A_c A_m\cos(2\pi f_m t)\cos(2\pi f_c t)$$

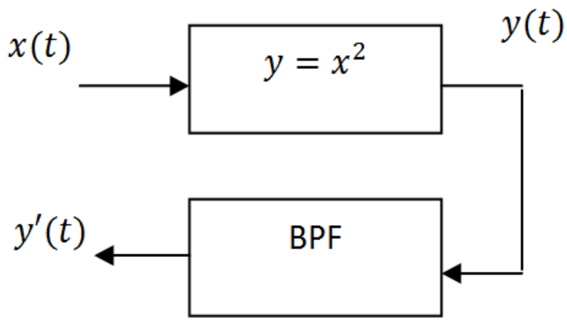
It can be represented as

$$s(t) = A_c\cos(2\pi f_c t) - \frac{A_c A_m}{2}[\cos(2\pi(f_c - f_m)t) + \cos(2\pi(f_c + f_m)t)]$$

As we recall this signal consists of a term at the carrier and two terms at $f_c + f_m$ and $f_c - f_m$.

Frequency multiplier

It is a device for which the frequency of the output signal is an integer multiple of the frequency of the input signal. It is primarily a nonlinear characteristic followed by a band pass filter. Now we illustrate the operation of this device.



The Square law device:

Let the input be an FM signal of the form:

$$x(t) = A_c \cos(2\pi f'_c + \beta' \sin 2\pi f_m t)$$

$$y(t) = A_c \cos(\phi)$$

The output of the square law characteristic is:

$$y(t) = x(t)^2 = A_c^2 cos^2(\phi) = \frac{A_c^2}{2}[1 + \cos(2\phi)] = \frac{A_c^2}{2} + \frac{A_c^2}{2}\cos(2\phi)$$
$$= \frac{A_c^2}{2} + \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$

The bandpass filter

If $y(t)$ is passed through a BPF of center frequency $2f_c$, then the DC term will be suppressed and the filter output is:

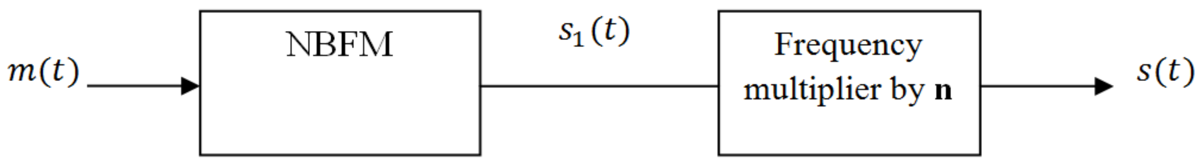
$$y'(t) = \frac{A_c^2}{2} cos\left[2\pi\left(2f'_c\right) + 2\beta' \sin(2\pi f_m t)\right]$$

$$y'(t) = \frac{A_c^2}{2} cos[2\pi(f_c) + \beta \sin(2\pi f_m t)]$$

As can be seen from this result, the output is a signal with twice the frequency of the input signal and a modulation index twice that of the input. To get frequency multiplication higher than two, a cascade of units, similar to what was described above, can be formed with the number of stages that achieve the desired frequency.

Indirect Method for Generating a Wideband FM:

A wideband FM can be generated indirectly using the block diagram below (Armstrong Method). First a narrowband FM is generated, and then the wideband FM is obtained by using frequency multiplication. Next, we analyze the operation of this modulator.



If $m(t) = A_m \cos 2\pi f_m t$ is the baseband signal, then

$$s_1(t) = A_c \cos(2\pi f_c' t + \beta' \sin 2\pi f_m t) \;\; ; \;\; \beta' = \frac{k_f A_m}{f_m}$$

is a narrowband FM with $\beta' \ll 1$. The frequency of $s_1(t)$ is $f'_i = f_c' + k_f A_m \cos 2\pi f_m t$.

Multiplying f_i by n , we get the frequency of $s(t)$ as $f_i = nf_c' + nk_f A_m cos 2\pi f_m t$. This result in

$$s(t) = A_c \cos[2\pi(nf_c')t + n\beta' \sin 2\pi f_m t]$$

$$= A_c \cos[2\pi f_c t + \beta \sin 2\pi f_m t]$$

Where $\beta = n\beta'$ is the desired modulation index of WBFM

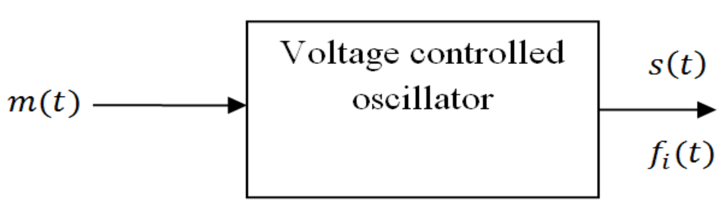
$f_c = nf_c'$ is the desired carrier frequency of WBFM

<u>Direct method for generating FM signal:</u>

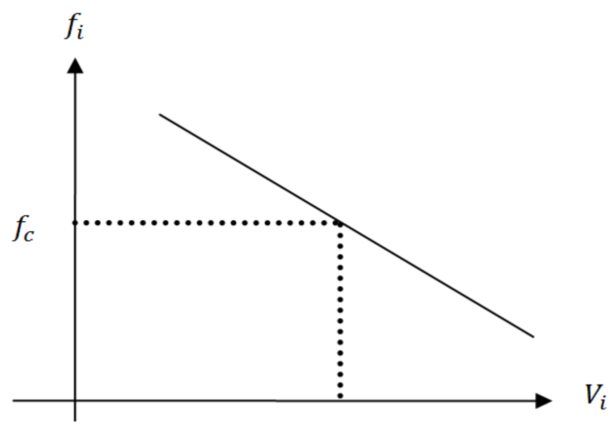
In a direct FM system, the instantaneous frequency of the carrier is varied in accordance with a message signal by means of a voltage controlled oscillator (VCO). The voltage – frequency characteristic of a VCO is given by

$$f_i = f_c - k\, m(t)$$

and is plotted in the figure below.



A realization of the CVO may be obtained by considering an oscillator (like the Hartley oscillator) shown below in which a varactor ((voltage variable capacitor) is used. The capacitance of the varactor varies in response to variations in the message signal. The variation is linear when the variation in the message is too small.



The frequency of the oscillator is

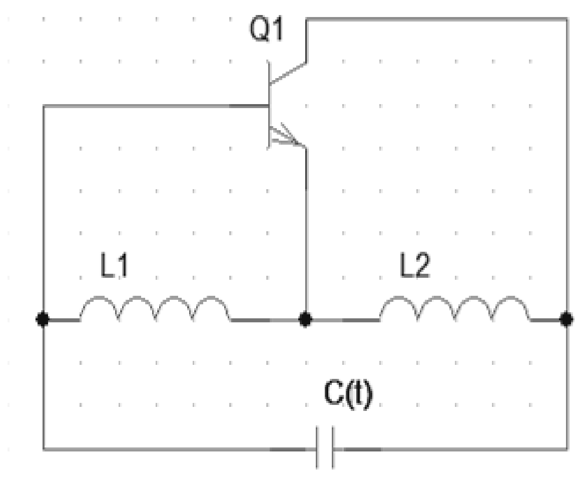
$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1 + L_2)C(t)}}$$

Let $C(t) = C_0 - k\, m(t)$

k: is a constant,

When m(t) = 0, $C(t) = C_0$, and the unmodulated frequency of oscillation is

$$f_c = \frac{1}{\sqrt{(L_1+L_2)C_0}}$$



Hartley Oscillator

When m(t) has a finite value, the frequency of oscillation is

$$f_i(t) = \frac{1}{2\pi\sqrt{(L_1 + L_2)\left(C_0 - k\, m(t)\right)}}$$

$$= \frac{1}{2\pi} \frac{1}{\sqrt{(L_1 + L_2)C_0}} \frac{1}{\sqrt{(1 - \frac{k\, m(t)}{C_0})}}$$

$$= f_c\left(1 - \frac{k\,m(t)}{C_0}\right)^{-1/2} \qquad , \qquad [(1+x)^n \cong 1 + nx]$$

When $\frac{k\,m(t)}{C_0} \ll 1$, we can make the approximation

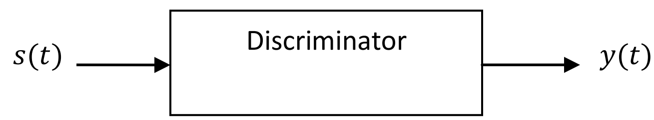
$$f_i(t) = f_c\left(1 + \frac{k\,m(t)}{2C_0}\right) = f_c + C_f\,m(t)$$

Here it is clear that the instantaneous frequency varies linearly with the message signal.

Demodulation of the FM signal:

An FM signal may be demodulated by means of what is called a *discriminator*.

Let $s(t) = A_c\cos\big(\omega_c t + \theta(t)\big)$ be an angle modulated signal. The output of an ideal discriminator is defined as:

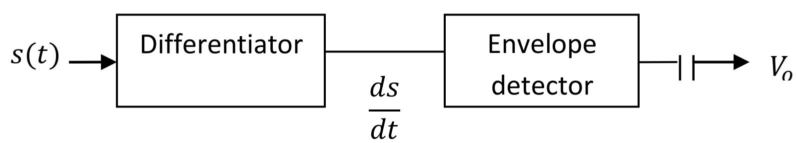


$$y(t) = \frac{1}{2\pi}\,k_D\,\frac{d\theta}{dt}$$

When $\theta = 2\pi k_f \int_{-\infty}^{t} m(\alpha)d\alpha$, then $\frac{d\theta}{dt} = 2\pi k_f\,m(t)$ and y(t) becomes

$$y(t) = k_D k_f\,m(t)$$

One practical realization of a discriminator is a differentiator followed by an envelope detector. The operation of this discriminator can be explained as follows:



Let $s(t) = A_c\cos\big(\omega_c t + \theta(t)\big)$

$$\frac{ds(t)}{dt} = -A_c\left(\omega_c + \frac{d\theta}{dt}\right)\sin\big(\omega_c t + \theta(t)\big)$$

The output of the envelope detector is $A_c\left|\left(\omega_c + \frac{d\theta}{dt}\right)\right|$

The capacitor blocks the DC term and so output is:

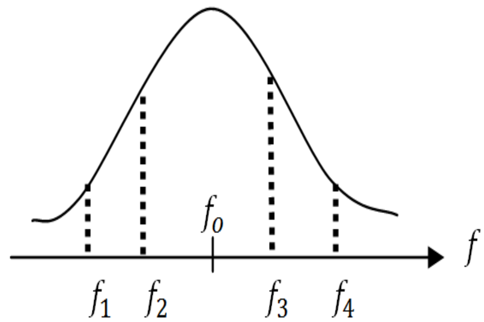
$$V_0 = A_c \frac{d\theta}{dt} = 2\pi k_f \, A_c m(t)$$

We know what an envelope detector is when we considered amplitude demodulation. Now we explain how differentiation is accomplished.

From the properties of Fourier transform we know that if $F\{g(t)\} = G(f)$, then

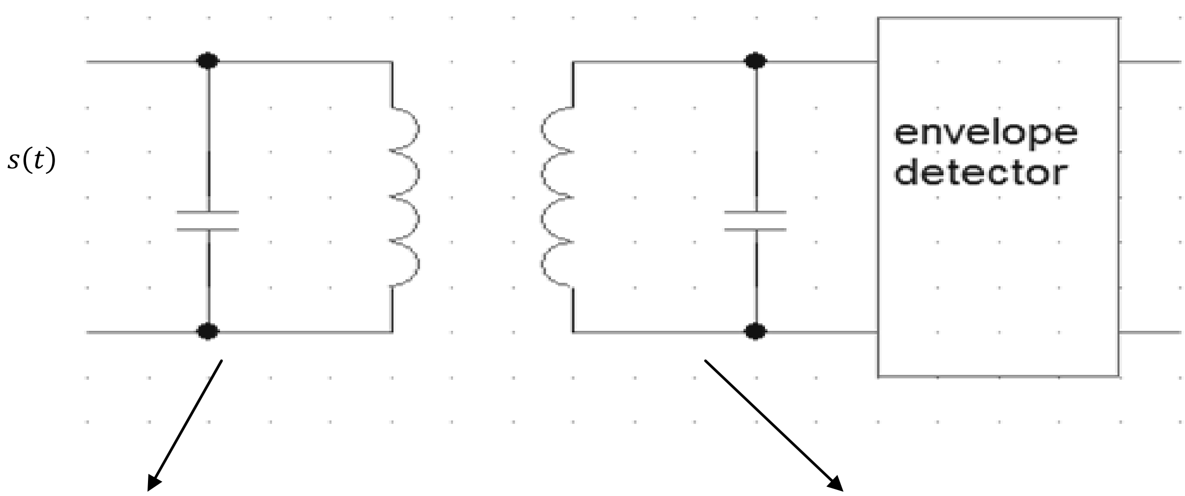
$$F\left\{\frac{dg(t)}{dt}\right\} = j2\pi f \; G(f)$$

This means that multiplication by $j2\pi f$ in the frequency domain amounts to differentiating the signal in the time-domain. Hence, we need a circuit whose frequency response is linear in f to perform time differentiation. A circuit that performs this task is a tuned circuit, provided that the signal frequency falls within the linear part of the characteristic, i.e., between either (f_1, f_2) or (f_3, f_4).



A balanced FM detector called *balanced discriminator* is such a circuit.

Tuned circuit demodulator



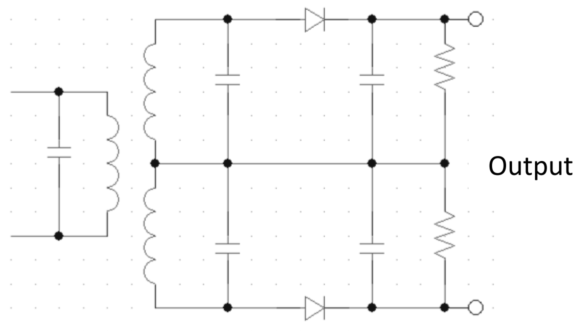
Primary circuit tuned to carrier freq. f_c Secondary circuit tuned to $f_o > f_c$

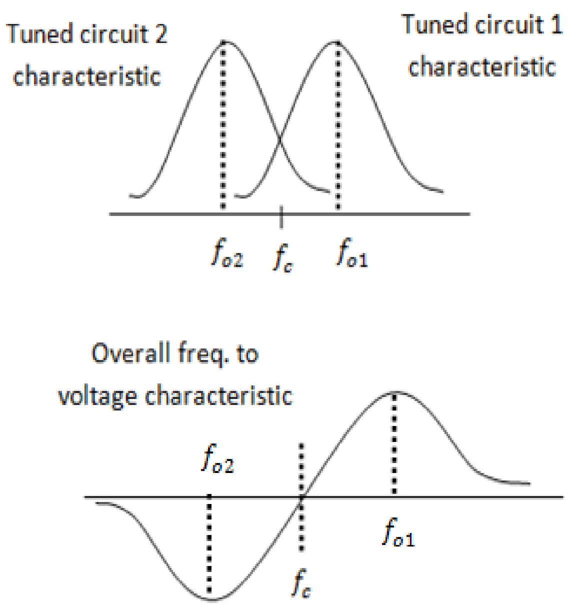
To extend the dynamic range of the differentiating circuit, two tuned circuits with center frequencies f_{o1} and f_{o2} are used as will illustrated next.

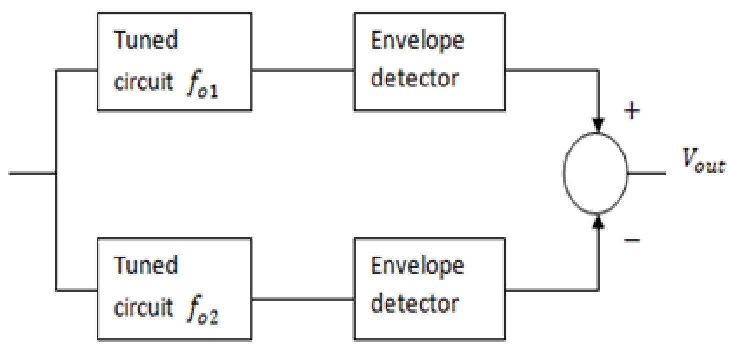
Balanced slope detector:

Two tuned circuits are tuned to two different frequencies $f_{o_1} > f_{o_2}$. The primary circuit is tuned to f_c .

- This circuit has wider width of linear frequency response.
- No DC blocking is necessary.

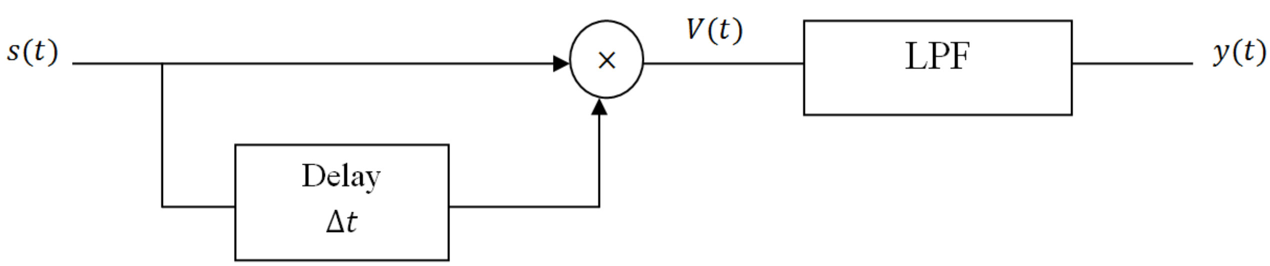






Phase shift discriminator:

The quadrature detector: This demodulator converts frequency variations into phase variation and detecting the phase changes. The block diagram of the demodulator is shown below



Let $s(t) = A_c \cos\big(2\pi f_c t + \varphi(t)\big)$; $\varphi(t) = 2\pi K_f \int_0^t m(\alpha)\, d\alpha$

$$s(t - \Delta t) = A_c \cos[2\pi f_c (t - \Delta t) + \varphi(t - \Delta t)]$$

$$= A_c \cos[2\pi f_c t - 2\pi f_c \Delta t + \varphi(t - \Delta t)]$$

The delay Δt is chosen such that $2\pi f_c \Delta t = {}^{\pi}/_{2}$

Hence,

$$s(t - \Delta t) = A_c \cos\left[2\pi f_c t - \frac{\pi}{2} + \varphi(t - \Delta t)\right]$$

$$= A_c \sin[2\pi f_c t + \varphi(t - \Delta t)]$$

$$V(t) = s(t)s(t - \Delta t)$$

$$= A_c^2 \sin[2\pi f_c t + \varphi(t - \Delta t)]\, cos[2\pi f_c t + \varphi(t)]$$

$$= \frac{A_c^2}{2} \sin[2\pi(2f_c)t + \varphi(t) + \varphi(t - \Delta t)] + \frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t - \Delta t)]$$

The high frequency component is suppressed by the LPF. What remains is the second term

$$\frac{A_c^2}{2} \sin[\varphi(t) - \varphi(t - \Delta t)] \cong \frac{A_c^2}{2} [\varphi(t) - \varphi(t - \Delta t)]$$

Where Δt is small to justify the approximation $\sin(x) \cong x$

Hence,

$$y(t) = \frac{A_c^2}{2} [\varphi(t) - \varphi(t - \Delta t)]$$

$$y(t) = \frac{A_c^2}{2} \;.\Delta t. \quad \frac{\varphi(t) - \varphi(t - \Delta t)}{\Delta t}$$

The second term is the derivative $\frac{d\varphi(t)}{dt}$ **.** The output then becomes

$$y(t) = \frac{A_c^2}{2} \;.\Delta t. \quad \frac{d\varphi}{dt}$$

But $\varphi(t) = 2\pi K_f \int_0^t m(\alpha)\, d\alpha$ and $\frac{d}{dt}\varphi(t) = 2\pi K_f m(t)$

$$y(t) = \frac{A_c^2}{2}\, \Delta t\, .2\pi K_f\, m(t)$$

$$y(t) = K\, m(t)$$

$y(t)$ is proportional to $m(t)$. It performs demodulation.

Transfer function of the delay:

From Fourier transform properties

$$g(t) \rightarrow G(f)$$

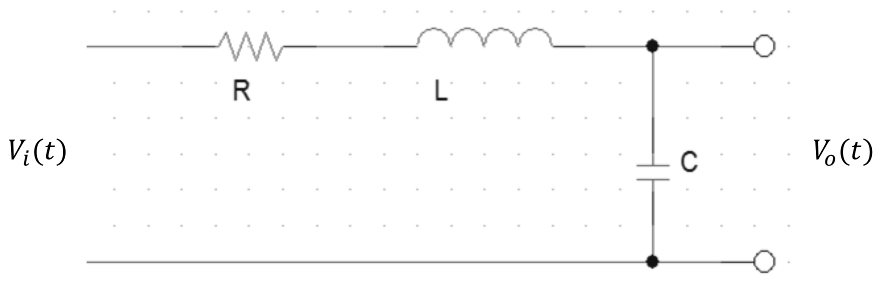
$$g(t-\Delta t) \rightarrow G(f)e^{-j2\pi f\Delta t}$$

The transfer function of the time delay is

$$H(f) = e^{-j2\pi f\Delta t}$$

Therefore, a circuit whose phase characteristic is linear in f can provide time delay of the type that we need.

A circuit with linear phase characteristic is the network shown



If $f_o = \frac{1}{2\pi\sqrt{LC}}$, $f_b = \frac{R}{2\pi L}$

then it can be shown that arg $(H(f))$ for this circuit is

$$\arg\big(H(f)\big) = -\frac{\pi}{2} - \frac{2Q}{f_o}(f - f_c) \;,\; Q = \frac{f_o}{f_b}$$

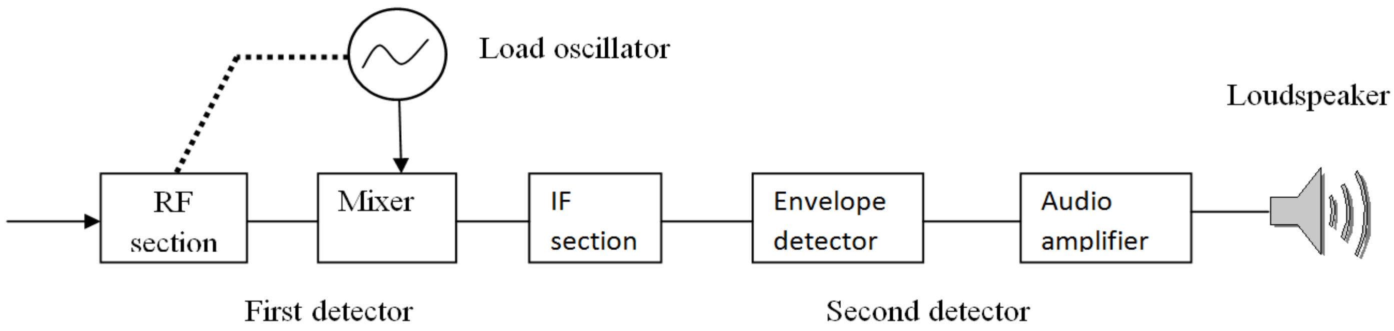
Remark: To perform time differentiation, we searched for a circuit whose amplitude spectrum varies linearly with frequency, while to perform time delay, we searched for a circuit with a linear phase spectrum.

The Super heterodyne Receiver:

Practically, all radio and TV receivers are made of the super heterodyne type. The receiver performs the following functions :

- Carrier frequency tuning: The purpose of which is to select the desired signal.
- Filtering: the desired signal is to be separated from other modulated signals.
- Amplification: to compensate for the loss of signal power incurred in the course of transmission.

The description of the receiver is summarized as follows:



- The incoming signal is picked up by the antenna and amplified in the RF section that is tuned to the carrier frequency of the incoming signal.
- The incoming RF section is down converted to a fixed intermediate frequency (IF). $f_{IF} = f_{lo} - f_{RF}$
- The IF section provides most of the amplification and selectivity in the receiver. The IF bandwidth corresponds to that required for the particular type of modulation.
- The IF output is applied to a demodulator, the purpose of which is to recover the baseband signal.
- The final operation in the receiver is the power amplification of the recovered signal.
- The basic difference between AM and FM super heterodyne lies in the use of an FM demodulator such as a discriminator (differentiator followed envelope detector)

Quadrature Carrier Multiplexing (QAM)

Quadrature Carrier Multiplexing: Modulation

This scheme enables two DSB-SC modulated signals to occupy the same transmission B.W and yet allows for the separation of the message signals at the receiver.

$m_1(t)$ and $m_2(t)$ are low pass signals each with a B.W = W Hz .

The composite signal is:

$$s(t) = A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t$$

$$s(t) = s_1(t) + s_2(t)$$

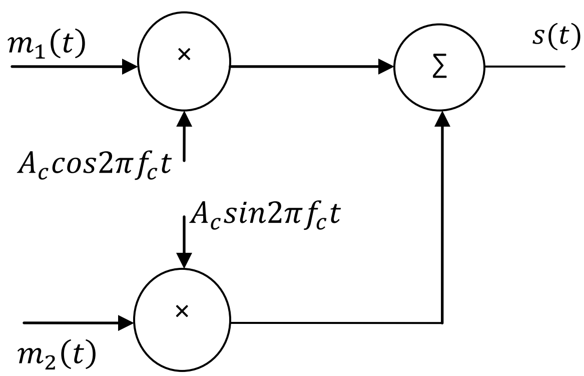
where $s_1(t)$ and $s_2(t)$ are both DSB-SC signals.

B.W of $s_1(t)$ = 2W

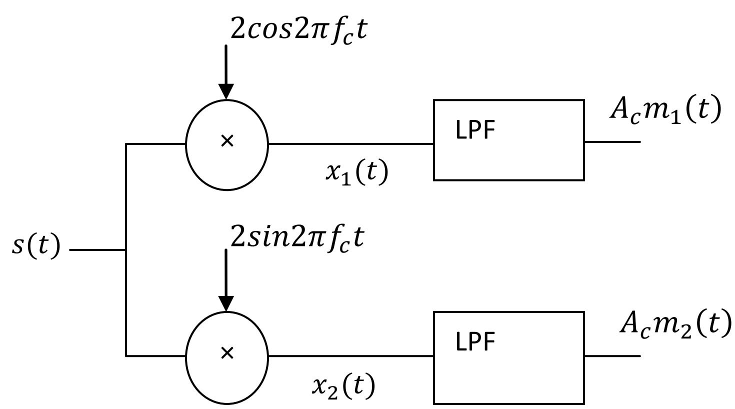
B.W of $s_2(t)$ = 2W

B.W of $s(t)$ = 2W

This method provides bandwidth conservation. That is, two DSB-SC signals are transmitted within the bandwidth of one DSB-SC signal. Therefore, this multiplexing technique provides bandwidth reduction by one half.



Modulation



Demodulation

<u>Quadrature Carrier Multiplexing: Demodulation</u>

Given $s(t)$, the objective is to recover $m_1(t)$ and $m_2(t)$ from $s(t)$. Consider first the in-phase channel

$$x_1(t) = 2cos2\pi f_c t \; s(t)$$

$$= 2\cos 2\pi f_c t \; (A_c m_1(t) cos2\pi f_c t + A_c m_2(t) sin2\pi f_c t)$$

$$= 2A_c m_1(t) cos^2 2\pi f_c t + 2A_c m_2(t) sin\omega_c t \; cos\omega_c t$$

$$= 2A_c m_1(t) \left(\frac{1+cos2\omega_c t}{2}\right) + A_c m_2(t) sin2\omega_c t$$

$$= A_c m_1(t) + A_c m_1(t) cos2\omega_c t + A_c m_2(t) sin2\omega_c t$$

After low pass filtering, the output of the in-phase channel is

$$y_1(t) = A_c m_1(t).$$

Likewise, it can be shown that

$$y_2(t) = A_c m_2(t) \; .$$

Note: Synchronization is a problem. That is to recover the message signals it is important that the two carrier signals (the sine and the cosine functions) at the receiver should have the same phase and frequency as the signals at the transmitting side. A phase error or a frequency error will result is an interference type of distortion. That is, A component of $m_2(t)$ will appear in the in-phase channel in addition to the desired signal $m_1(t)$ and a component of $m_1(t)$ will appear at the quadrature output.

Frequency Division Multiplexing:

A number of independent signals can be combined into a composite signal suitable for transmission over a common channel. The signals must be kept apart so that they do not interfere with each other and thus they can be separated at the receiving end.

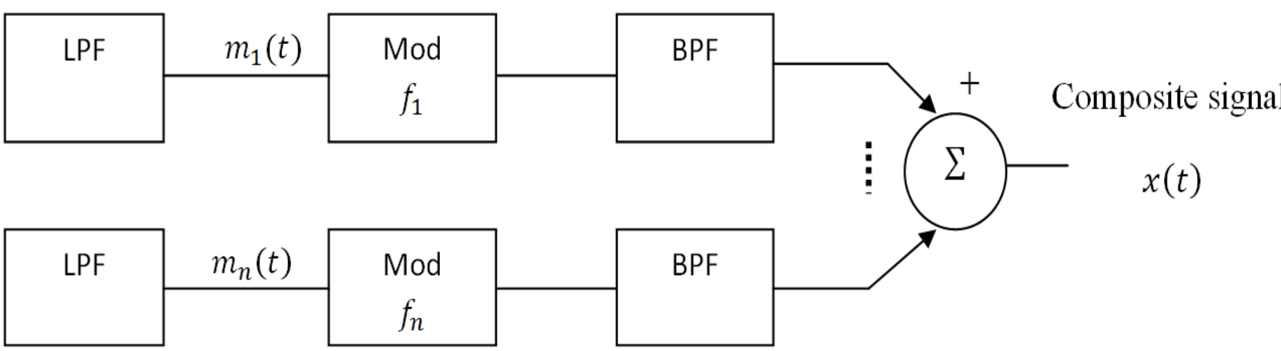


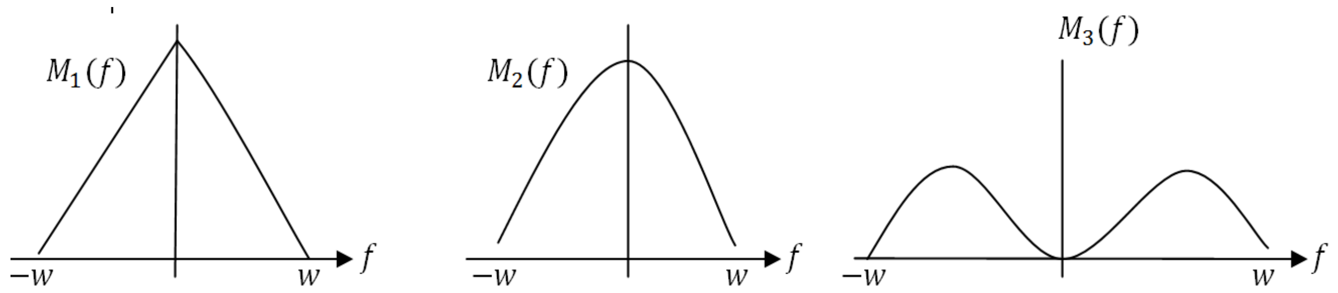
Illustration of FDM

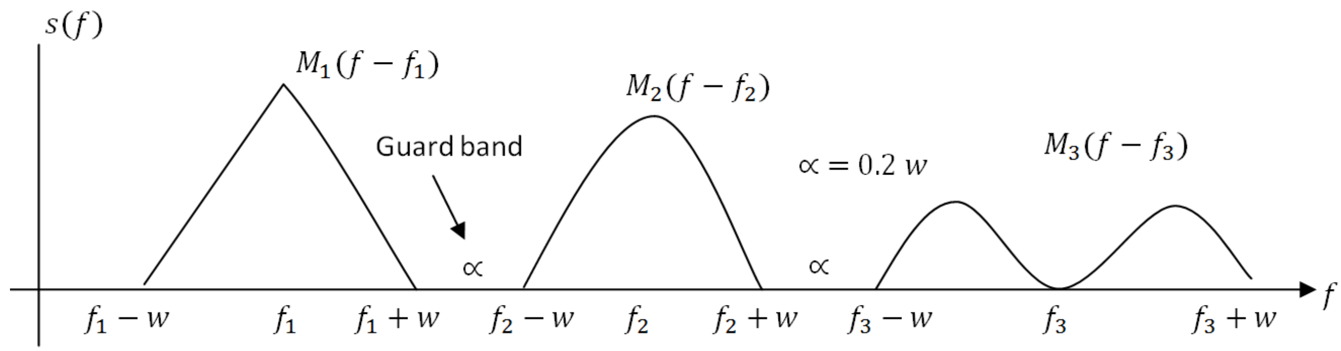
Let $m_1, m_2 \; and \; m_3$ be three baseband message signals each with a B.W = w.

The composite modulated signal $s(t)$ is

$$s(t) = A_{c_1} m_1(t) cos2\pi f_1 t + A_{c_2} m_2(t) cos2\pi f_2 t + A_{c_3} m_3(t) cos2\pi f_3 t$$
$$= s_1(t) + s_2(t) + s_3(t)$$

$s_1, s_2 \; and \; s_3$ are DSB-SC signals with carrier frequencies $f_1, \; f_2 \; and \; f_3$, respectively. If the spectrum of $m_1(t), \; m_2(t) \; and \; m_3(t)$ are as shown, the spectrum of $s(t)$ can be found as shown below.



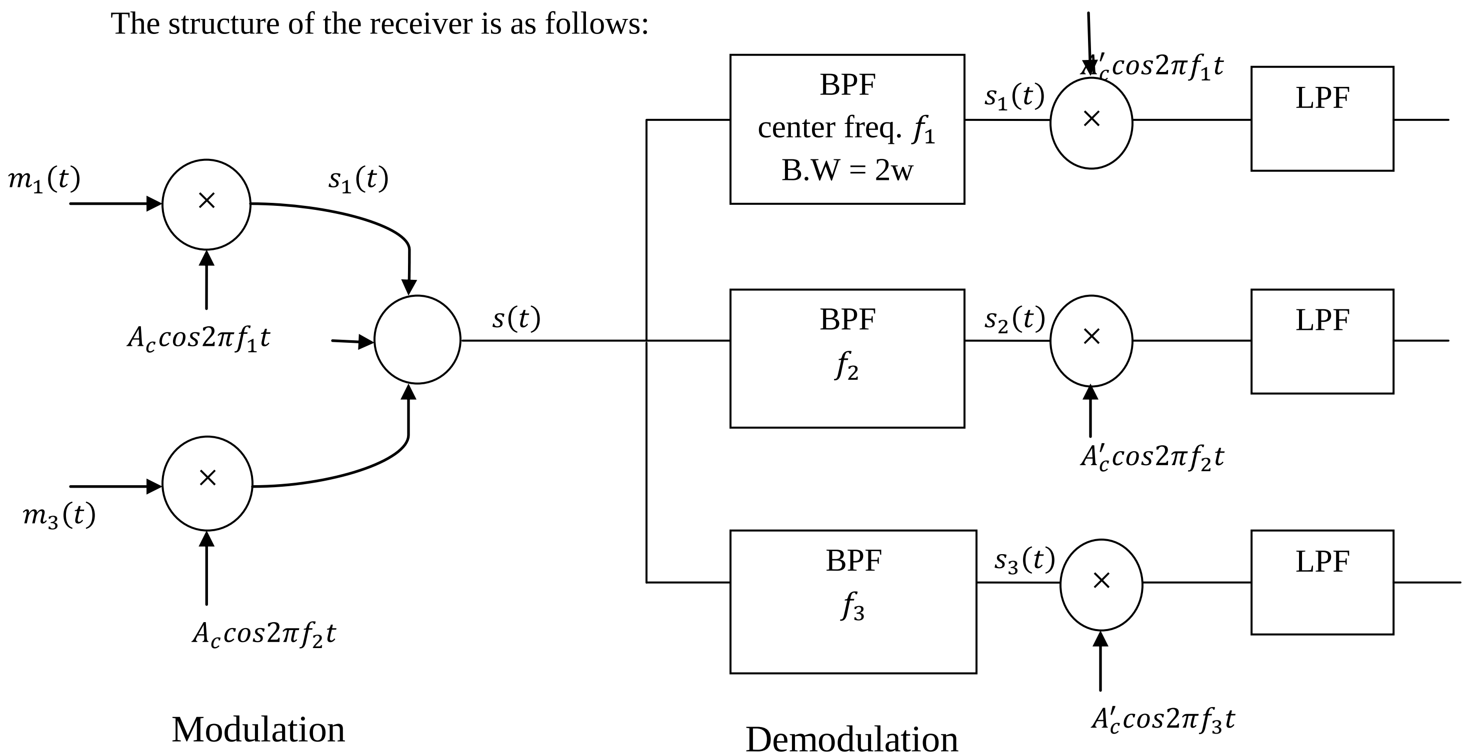


To prevent interference we demand that

$$f_2 - w \geq f_1 + w \quad or \quad f_2 - f_1 \geq 2w$$

$$f_3 - w \geq f_2 + w \quad or \quad f_3 - f_2 \geq 2w$$

The structure of the receiver is as follows:



Random Processes

A random process $X(t)$ is defined as an ensemble of time functions together with a probability rule that assigns a probability to any meaningful event associated with an observation of one of the sample functions of the random process.

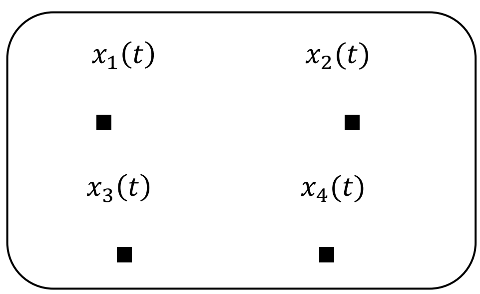
Consider the following experiment: An oscillator produces a waveform of the form $A_m\, cos(w_m t + \theta)$; where θ is a discrete R.V with a probability mass function

$$P(\theta = 0) = 0.2 \qquad P\left(\theta = \frac{\pi}{2}\right) = 0.2$$

$$P(\theta = \pi) = 0.3 \qquad P\left(\theta = \frac{3\pi}{2}\right) = 0.3$$

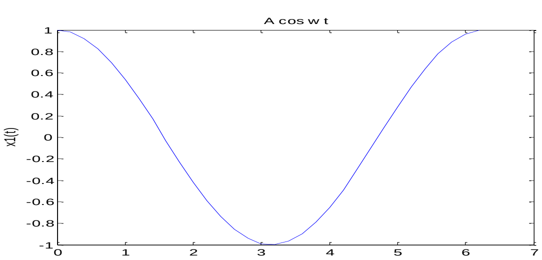
Here the sample space of the experiment consists of four time functions:

$$x_1(t) = A_m \cos(\, w_m t \,) \qquad P(x_1(t)) = 0.2$$
$$x_2(t) = A_m \cos\left(w_m\, t + \frac{\pi}{2}\right) \qquad P(x_2(t)) = 0.2$$
$$x_3(t) = A_m \cos(\, w_m t + \pi\,) \qquad P(x_3(t)) = 0.3$$
$$x_4(t) = A_m \cos(\, w_m t + \frac{3\pi}{2}) \qquad P(x_4(t)) = 0.3$$

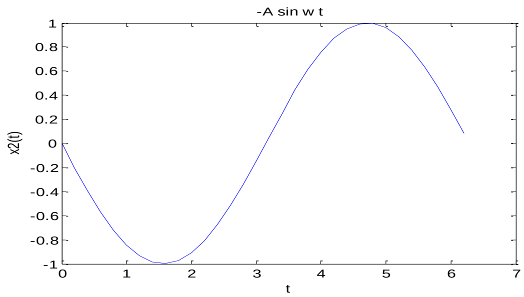


Plot four signals:

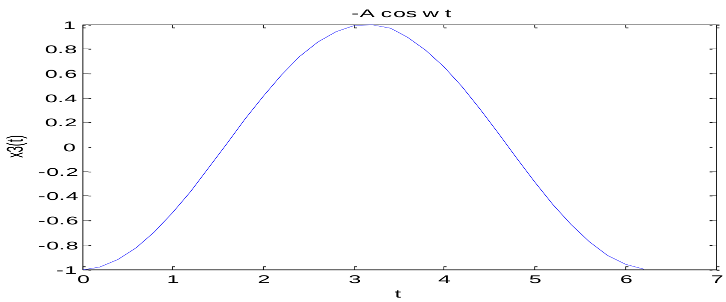
$$\boldsymbol{x_1(t) = A\, cos(2\pi f_m t)}$$



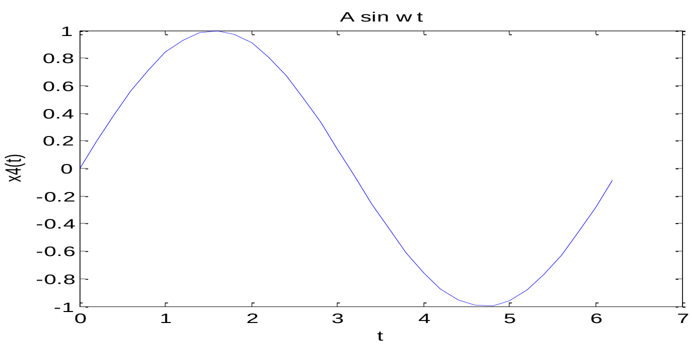
$$x_2\,(t) = -A\,\sin(2\pi f_m t)$$



$$x_3(t) = -A\,cos(2\pi f_m t)$$



$$x_4(t) = +A\,sin(2\pi f_m t)$$



Each realization of the experiment is called a *sample function* $x(t)$. The sample space (ensemble) composed of functions is called *a random or stochastic process* denoted by $X(t)$. The value assumed by a random process at a particular time is a random variable with a certain probability density function.

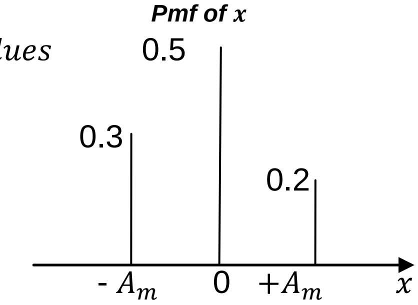
For the example above, $X(0)$ $assumes\ three\ values$

P $\{X(0) = A_m\} = 0.2$

P $\{X(0) = -A_m\} = 0.3$

$P\{X(0) = 0\} = 0.2 + 0.3 = 0.5$

(Corresponding to $= \frac{\pi}{2}, \frac{3\pi}{2}$)



Pmf of X at t = 0.

$P(X = 0)$ = 0.2+0.3 = 0.5

$P(X = +A_m) = 0.2$

$P(x = -A_m) = 0.3$

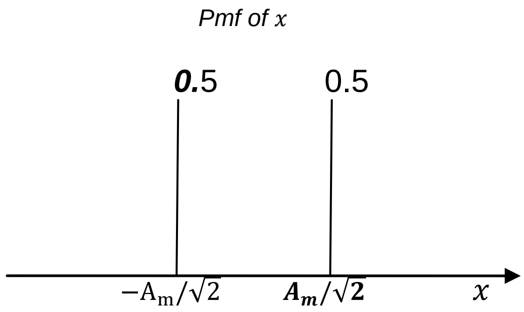
The mean value of the random variable X is

$$E(X) = -A_m \times 0.3 + 0 \times 0.5 + 0.2 \times A_m$$
$$E(X) = -0.3A_m + 0.2A_m = -0.1A_m$$

Pmf of X at $w_m t_n = \pi/4$

	$X(\mathbf{w_m\ t = \frac{\pi}{4}})$	Prob.
Possible values:	$A_m \cos\left(\frac{\pi}{4} + 0\right) = +A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = -A_m/\sqrt{2}$	0.2
	$A_m \cos\left(\frac{\pi}{4} + \pi\right) = -A_m/\sqrt{2}$	0.3
	$A_m \cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) = +A_m/\sqrt{2}$	0.3

The Pmf of X at $\mathrm{w_m\ t} = \frac{\pi}{4}$ is sketched here.



$$E\{X\left(w_m t = \frac{\pi}{4}\right) = 0$$

⇛ Process is not stationary [mean at $t = 0$ is not the same as the mean at $w_m\, t = \frac{\pi}{4}$]

In general, $X(t) = A_m \cos(w_m\, t + \theta\,)$

$$E\{X(t)\} = \sum X(t, \theta_i)\, P(\theta = \theta_i)$$

$$= A_m \cos w_m\, t \times 0.2 + 0.2 \times A_m \cos\left(w_m\, t + \frac{\pi}{2}\right) + 0.3 \cos(w_m\, t + \pi\,) + 0.3 \times A_m \cos(w_m\, t + \frac{3\pi}{2}).$$

Mean is not a constant (function of time).

⇛ **Process is non stationary.**

Stationarity of a random process:

The mean of a process $X(t)$ is defined as the expectation of the r.v obtained by observing the process at some time t as

$$\mu_x(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_x(x) dx$$

$f_x(x)\ is\ the\ first\ order\ pdf\ of\ the\ process\ X(t).$

The autocorrelation function of the process $X(t)$ is defined as the expectation of the product of two r.v $X(t_1)\ and\ X(t_2)$ obtained by observing the process $X(t)$ at times t_1 and t_2.

$$R_x\ (t_1, t_2) = E\{X(t_1)X(t_2)\} = \iint_{-\infty}^{\infty} x_1\ x_2\ f(x_1, x_2)dx_1\ dx_2$$

$f(x_1, x_2\ )$ is the second order (joint pdf) of x_1 and x_2.

A random process is said to be wide sense stationary or (stationary) when the following two conditions hold:

1) $E\{X(t)\} = \mu_x$ = constant for all t

2) $R_x(t_1, t_2) = E\{X(t_1)X(t_2)\} = R_x(t_2 - t_1\ )$
   i.e., $R_x\ \ is$ a function of the time difference and not on the absolute values of t_1 and t_2. i.e.,

   $R_x(\tau) = E\{X(t)X(t+\tau)\};\ where\ \tau = t_2 - t_1$

<u>Properties of the autocorrelation function of a stationary process:</u>

1) $Rx(0) = E\{X^2(t)\}$; the mean square value (second moment of x) { total power in $X(t)$ }

2) $R_x(\tau) = R_x(-\tau); R_x(\tau)\ is\ an\ even\ function\ of\ \tau.$

3) $\mathrm{R_x(\tau)}$attains its maximum value at $\tau = 0$

$$|\mathrm{R_x(\tau)}| \le \mathrm{R_x(0)}$$

Proof:

Consider the quadratic quantity

$[\ X(t) \pm X(t+\tau)]^2 \ge 0$

Taking the expectation of both sides, and then expanding, we get

$$E\{[X(t) \pm X(t+\tau)]^2\} \geq 0$$

$$E\{\,X(t)^2\} + E\{\,X(t+\tau)^2\} \pm 2E\{\,X(t)X(t+\tau)\} \geq 0$$

But, $E\{\,X(t)^2\} = R_x(0)$ and $R_x(0) = E\{\,X(t+\tau)^2\}$ as well. Combining these results, we get

$$-R_x(0) < \ R_x(\tau) < R_x(0)$$

4) If the sample function are periodic with period T_0, then the autocorrelation function R_x is periodic with period T_0.

5) If the sample functions have a deterministic average value (dc) term A, then $x(t)$ can be represented as $x(t) = A + g(t)$;.
$g(t)\ is\ a\ zero-mean\ process\ with\ \ R_x(\tau) = A^2 + R_g(\tau).$

6) If the sample function are non periodic, then

$$\lim_{\tau\to\infty} R_x\,(\tau) = E\{g(t)\}^2$$

Here, as $\tau \to \infty, x(t)\ and\ x(t+\tau)$ become independent and so
$\lim_{\tau\to\infty} R_x(\tau) = \lim E\{x(t)x(t+\tau)\} = E\{x(t)\}^2$

Decorrelation Time : The decorrelation time τ_0 of the a stationary process X(t) of zero mean is taken as the time taken for the magnitude of the autocorrelation function $R_x(\tau)$ to decrease say 1% of its maximum value $R_x(0)$.

A Result we Recall from ENEE 331: If θ is a r.v with pdf $f_\theta(\theta)$ and $Y = g(\theta)$, then $E\{Y\} = \int g(\theta)f(\theta)\,d\theta$

$$E\{g(\theta)\} = \int g(\theta)f_\theta(\theta)\,d\theta$$

Example: A sinusoidal signal with random phase

$$\text{Let } X(t) = A\,cos(2\pi f_c t + \theta)$$

A, f_c are constants, θ is a continuous r.v uniformly distributed over $(-\pi, \pi)$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & o.w \end{cases}$$

The mean value of $X(t)$ is

$$E\{\,X(t)\,\} = \int_{-\pi}^{\pi} \underbrace{Acos(2\pi f_c\, t + \theta)}_{g(\theta)} \cdot \underbrace{\frac{1}{2\pi}}_{f(\theta)}\, d\theta = 0$$

Which is a constant (independent of time). The autocorrelation function is:

$$R_x(\tau) = E\{X(t)X(t+\tau)\}$$

$$= \int_{-\pi}^{\pi} \overbrace{Acos(2\pi f_c\, t + \theta) \cdot Acos[2\pi f_c(t+\tau) + \theta]}^{g(\theta)} \cdot \overbrace{1/2\pi}^{f(\theta)}\, d\theta$$

$$= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \{cos2\pi f_c\, \tau + \cos(2\pi(2) f_c t + 2\pi f_c \tau + 2\theta)\}\, d\theta$$

We can easily recognize that the second integral is zero, leaving only the first term. Hence, $R_x(\tau)$ becomes

$$R_x(\tau) = \frac{A^2}{2\pi} \cdot \frac{\cos 2\pi f_c \tau}{2} \cdot 2\pi = \frac{A^2}{2} cos2\pi f_c \tau$$

Note that:

- The mean value is a constant and $R_x(\tau)$ is a function of τ. These are the two conditions necessary for the process to be stationary. So $X(t)$ is a stationary process.
- The process $X(t)$ is periodic with period $T_c = \frac{1}{f_c}$ .The autocorrelation function $R_x(\tau) = \frac{A^2}{2} \cos 2\pi f_c \tau$ is also periodic with period $T_c = \frac{1}{f_c}$.

Exercise: show that the first order pdf of X is

$$f_x(x) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - x^2}} & -A < x < A \\ 0 & o.w \end{cases}$$

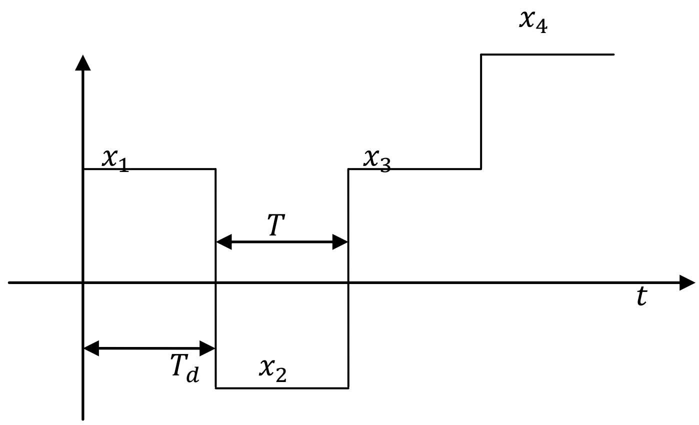
Exercise: Let X and Y be two independent Gaussian random variables each with mean zero and variance σ^2. Define the random process

$$Z(t) = Xcos2\pi f_c t + Ysin2\pi f_c t$$

a. Find the mean and variance of $Z(t)$.
b. Find the autocorrelation function $R_Z(\tau)$.
c. Is this process stationary?

Example: Random digital signal

The figure shows a random sample $x(t)$ of a process $X(t)$ consisting of a random sequence X_1X_2 ... of pulses each with m possible amplitudes (symbols) $a_1, a_2, \dots, a_m$ within each signaling interval T. The possible symbols occur with probabilities $P_1, P_2, \dots, P_m$.

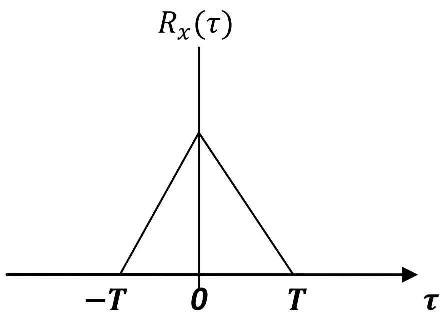


- The time delay T_d is a continuous r.v uniformly distributed over $0 < T_d < T$; where T is the symbol duration $f\ (t_d) = \begin{cases} \frac{1}{T} & 0 < t_d < T \\ 0 & o.w \end{cases}$
- The amplitudes in different intervals are independent.
- The mean value of the process is
$$E(X(t)) = a_1P_1 + aP_2 + \cdots\cdots\cdots a_mP_m$$
Without loss of generality, let $E(X(t)) = 0$

The variance of the process is
$$Var(X(t)) = \sigma^2 = a_1^2P_1 + a_2^2P_2 + \cdots\cdots\cdots + a_m^2P_m$$
- The average power of the process is also $R_x(0) = \sigma^2$.
- The autocorrelation function of the process is

$$R_x(\tau) = \sigma^2 tri(\frac{\tau}{T}).$$

This function is sketched below



Example: Random Binary Signal (also known as polar non return to zero)

Here, the possible symbols of X(t) in each signaling time interval T are:

$+A$	with probability	½	for	$0 \leq t \leq T$
$-A$	with probability	½	for	$0 \leq t \leq T$

Find the mean, variance and autocorrelation function of X(t).

Solution

The mean value is $E\big(X(t)\big) = +A \times \frac{1}{2} - A \times \frac{1}{2} = 0$

The variance is $\sigma^2 = A^2 \times \frac{1}{2} + (-A)^2 \times \frac{1}{2} = A^2$

Therefore, $R_x(\tau) = A^2 tri(\frac{\tau}{T})$

$$R_x(\tau) = \begin{cases} A^2\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

Exercise: Unipolar non return to zero signaling

Let the transmitted symbols of Z(t) in each signaling time interval T be:

$+A$	with probability	½	for	$0 \leq t \leq T$
0	with probability	½	for	$0 \leq t \leq T$

a. Show that Z(t) is related to the polar NZR in the previous example by $Z(t) = (X(t) + A)/2$

b. Find the mean and variance of Z(t).

c. Show that $R_Z(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4}\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ \frac{A^2}{4} & |\tau| > T \end{cases}$

Exercise: Polar non return to zero signaling with non-equal symbol probabilities
Let the possible symbols of X(t) in each signaling time interval T be:

$+A$	with probability	p	for	$0 \le t \le T$
$-A$	with probability	(1-p)	for	$0 \le t \le T$

a. Find the mean and variance of X(t).
b. Find the autocorrelation function of X(t).

Exercise: M-ary pulse amplitude signal
Let the possible symbols of X(t) in each signaling time interval T be ($-3A$, $-A$, $+A$, $+3A$) with equal probabilities

a. Find the mean and variance of X(t).
b. Find the autocorrelation function of X(t).

Ergodic processes:

Given a sample function $x(t)$ of a random process $X(t)$, we define the following two time averages:

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

$$\langle X(t)X(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

Here, $2T$ is an observation interval.

<u>Def:</u> A random process is said to be ergodic if statistical properties can be determined from a sample function representing one realization of the process.

Statistical average = Time Average.

The two quantities of interest are the mean value and the autocorrelation function. For an ergodic process, they can be computed using time average as:

$E\{X(t)\} = \langle x(t) \rangle = constant.$
$R_x(\tau) = E\{X(t)X(t+\tau)\} = \langle x(t)\, x(t+\tau) \rangle$ → function of τ.

Remark: An ergodic process in stationary, but a stationary process is not necessarily ergodic.

Example: consider again the process $X(t) = Acos(2\pi f_c\, t + \theta)$, θ is uniformly distributed over (-π < θ < π).

The two time averages are calculated as follows:

$$\langle x(t)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c\, t + \theta)dt = 0 \;; \qquad T_c = 1/f_c$$

$$\langle x(t)x(t+\tau)\rangle = \frac{1}{T_c}\int_0^{T_c} Acos(2\pi f_c t + \theta)\cdot Acos(2\pi f_c t + 2\pi f_c\, \tau + \theta)dt$$

$$= \frac{A^2}{2T}\int_0^{T_c} \cos(2\pi f_c\, t + 2\pi f_c \tau + 2\theta) + cos2\pi f_c \tau\, dt$$

$$= \frac{A^2}{2} cos2\pi f_c \tau$$

These are the same values found in the previous example.

⇒ process is ergodic.

Power spectral Density and Autocorrelation Function:

Consider a stationary random process $X(t)$ that is ergodic. Consider a truncated segment of $x(t)$ defined over the observation interval $-T < t < T$. Let X_{2T} be the truncated signal:

$$x_{2T}(t) = \begin{cases} x(t) & -T < t < T \\ 0 & o.w \end{cases}$$

The Fourier transform of x_{2T} is:

$$X_{2T}(f) = \int_{-T}^{T} x(t)e^{-j2\pi f t}dt$$

The energy spectral density of $X_{2T}(t)$ is $|X_{2T}(f)|^2$. Since $x(t)_{2T}$ is only one realization of a random process, then we need to find its mean value $E\{|X_{2T}(f)|^2\}$. Dividing this by the observation interval 2T, and letting T becomes very large, we get the power spectral density of the whole process, averaged over all sample functions and over all time.

The power spectral density, of a stationary process, may then be defined as:

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} E\{|X_{2T}(f)|^2\}$$

The Wiener –Khintchine Thorem:

The power spectral density $S_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary random process $X(t)$ form a Fourier transform pairs:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$
$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

Properties of the power spectral Density:

1. The zero frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function ;

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau$$

2. The mean squared value (the total signal power) of a stationary process equals the area under the power spectral density curve.

$$E\{X(t)^2\} = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

3. The power spectral density of a stationary process is always nonnegative ; i.e., $S_X(f) \geq 0$ for all f.

4. The power spectral density of a real-valued random process is an even function of f.

$$S_X(f) = S_X(-f)$$

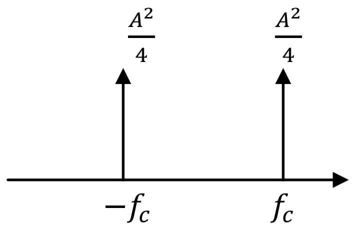
Example: Sinusoidal signal with random phase (revisited)

Find the power spectral density of the random process $X(t) = A\cos(2\pi f_c t + \theta)$;

θ is a uniform r.v over $(-\pi, \pi)$.

For this process, we found earlier that $R_X(\tau) = \frac{A^2}{2} cos 2\pi f_c \tau$

Since $S_X(f) = F\{ R_X(\tau) \}$, then



$$S_X(f) = \frac{A^2}{4}\{ \delta(f - f_c) + \delta(f + f_c) \}$$

The total average power is obtained by integrating $S_X(f)$ over all frequencies.

$$\int_{-\infty}^{\infty} S_X(f) df = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f - f_c) + \delta(f + f_c)] df$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

$$= R_X(0)$$

Example: Random Binary Signal (revisited)

The autocorrelation function of the random binary signal was found to be

$$R_X(\tau) = \begin{cases} A^2\left(1 - \left|\frac{\tau}{T}\right|\right) & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

The power spectral density is the Fourier transform of $R_X(\tau)$ which is

$$S_X(f) = F\{ R_X(\tau)\} = A^2 T \, sinc^2 fT$$

Exercise: For the random binary signal X(t), find the total signal power.

Exercise: Unipolar non return to zero signaling (revisited)

Let the transmitted symbols of X(t) in each signaling time interval T be:

$+A$	with probability	½	for	$0 \le t \le T$
0	with probability	½	for	$0 \le t \le T$

a. Find the power spectral density of X(t).
b. Find the null to null bandwidth of X(t).

Example: Random Binary Signal (Revisited)

Here, the possible symbols of X(t) in each signaling time interval T are represented by a pulse + g(t) and – g(t):

$$\begin{array}{llll} +g(t) & \text{with probability } ½ & \text{for} & 0 < t < T \\ -g(t) & \text{with probability } ½ & \text{for} & 0 < t < T \end{array}$$

The power spectral density for this signal is

$$S_X(f) = F\{ R_X(\tau)\} = \frac{1}{T}|G(f)|^2$$

Where $G(f)$ if the Fourier transform of g(t). As we will see later, the transmission of digital data by means of signals with opposite polarity is called *antipodal signaling*.

Exercise: Manchester Coding

Let g(t) in the previous example be given by

$$g(t) = \begin{cases} A, & T/2 \le t < T \\ -A, & T/2 \le t < T \end{cases}$$

Find the power spectral density of the transmitted signal.

Example: Mixing of a random process with a sinusoidal signal.

A random process $X(t)$ with an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$ is mixed with a sinusoidal function $\cos(2\pi f_c t + \theta)$; θ is a r.v uniformly distribution over $(0, 2\pi)$ to form a new process

$$Y(t) = X(t)\cos(2\pi f_c t + \theta)$$

Find $R_Y(\tau)$ and $S_Y(f)$

Solution

We first find $R_Y(\tau)$

$$R_Y(\tau) = E\{ Y(t)Y(t+\tau)\}$$

$$= E\{X(t)\cos(2\pi f_c t + \theta) \cdot X(t+\tau)\cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$$

When $X(t)$ and θ are independent, then

$$= E\{X(t)\, X(t+T)\}E\{\cos(2\pi f_c t + \theta) \cdot \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\}$$

$$= R_X(\tau)E\{\frac{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) + \cos 2\pi f_c \tau}{2}\}$$

$$R_Y(\tau) = \frac{R_X(\tau)}{2} \cdot cos 2\pi f_c \tau$$

The power spectral density is

$$S_Y(f) = \frac{1}{4}\{S_X(f - f_c) + S_X(f + f_c)\}$$

Which is quite similar to the modulation property of the Fourier transform.

Exercise: Binary Phase Shift Keying

Consider again the random binary signal m(t) which assumes the values $+1$ and -1 in each signaling time interval T as:

$+1$	with probability	½	for	$0 < t < T$
-1	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_c t + \theta)$$

where θ is a random variable uniformly distribution over $(0, 2\pi)$ and independent of m(t),

a. Find the null to null bandwidth of $m(t)$
b. Find the autocorrelation of $Y(t)$
c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Exercise: Binary Amplitude Shift Keying

Consider again the unipolar NRZ signal m(t) which assumes the values $+1$ and 0 in each signaling time interval T as:

$+1$	with probability	½	for	$0 < t < T$
0	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_c t + \theta)$$

where θ is a r.v uniformly distribution over $(0, 2\pi)$ and independent of m(t)

a. Find the null to null bandwidth of $m(t)$
b. Find the autocorrelation function of $Y(t)$.
c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Exercise: Binary Frequency Shift Keying

Consider again the random binary signal m(t) which assumes the values $+1$ and 0 in each signaling time interval T as:

$+1$	with probability	½	for	$0 < t < T$
0	with probability	½	for	$0 < t < T$

A new modulated signal $Y(t)$ is generated from X(t) as:

$$Y(t) = Am(t)\cos(2\pi f_1 t + \theta_1) + A\acute{m}(t)\cos(2\pi f_2 t + \theta_2)$$

where θ_1 and θ_2 are independent random variables uniformly distribution over $(0, 2\pi)$ and $\acute{m}(t) = (1 - m(t))$

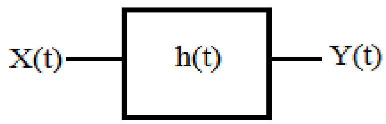
a. Show that $\acute{m}(t)\ and\ m(t)$ have the same autocorrelation function.
b. Show that

$$R_Y(\tau) = \frac{A^2}{2} R_m(\tau) cos2\pi f_1 \tau + \frac{A^2}{2} R_m(\tau) cos2\pi f_2 \tau$$

c. Find and sketched the power spectral density of $Y(t)$
d. Find the null to null bandwidth of $Y(t)$.

Transmission of a Random Process Through a LTI Filter

Suppose that a stationary process X(t) is applied to a LTI filter of impulse response h(t) producing a new process Y(t) at the filter output.



The input Signal X (t) is a stationary process characterized by an auto-correlation function $R_x(\tau)$ and a power spectral density S_x (f).

Mean Value of Y(t)

Y (t) is related to X(t) through the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\lambda) X(t-\lambda) d\lambda$$

The mean value of Y(t) is

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda) E\{X(t-\lambda)\} d\lambda$$

Since X (t) is a stationary process, then $E\{X(t-\lambda)\} = \mu_x$, a constant, so

$$E\{Y(t)\} = \int_{-\infty}^{\infty} h(\lambda)\ \mu_x\ d\lambda = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda$$

$$\boxed{\mu_y = \mu_x \int_{-\infty}^{\infty} h(\lambda) d\lambda = \mu_x H(0)}$$

where, H(0) is the value of the transfer function evaluated at f = 0.

Autocorrelation Function of Y(t)

The autocorrelation function of Y(t) can be evaluated as:

$$R_Y(t,u) = E\{Y(t).\ Y(u)\} \qquad ;\ u = t + \tau$$

$$= E\{\int_{-\infty}^{\infty} h(\lambda_1) X(t-\lambda_1) d\lambda_1 \, . \int_{-\infty}^{\infty} h(\lambda_2) X(u-\lambda_2) d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) E\{X(t-\lambda_1) X(u-\lambda_2)\} d\lambda_1 d\lambda_2$$

$$= \iint_{-\infty}^{\infty} h(\lambda_1) h(\lambda_2) R_X\,[(t-\lambda_1) - (u-\lambda_2]\ d\lambda_1 \lambda_2$$

$R_X\,[(t-\lambda_1)-(u-\lambda_2] = R_X\,[(t-\lambda_1-u+\lambda_2] = R_X\,[(\tau-\lambda_1-\lambda_2]$

Where, $\tau = t - u$. With this, $R_X(t, u)$ becomes

$$R_Y(\tau) = \iint_{-\infty}^{\infty} h\,(\lambda_1) h\,(\lambda_2)\, R_X\,(\tau - \lambda_1 + \lambda_2) d\lambda_1\, d\lambda_2$$

Which can be expressed in a compact form as:

$$R_y\,(\tau) = h\,(\tau) * h(-\tau) * R_x(\tau)$$

Mean Square Value of Y(t)

Setting $\tau = 0$ in the expressions for $R_X\,(\tau)$, we get

$$E\,\{\,Y(t)^2\} = R_y\,(0) = \iint_{-\infty}^{\infty} h\,(\lambda_1) h\,(\lambda_2)\, R_x\,(\lambda_2 - \lambda_1)\, d\lambda_1\, d\lambda_2$$

Power Spectral density S_y (f) of Y(t)

The power spectral density of Y(t) is related to the autocorrelation function through the relations

$$S_y(f) = F\{\,R_y(\tau)\} = F\,\{\,h\,(\tau) * h(-\tau) * R_x(\tau)\,\}$$

$$= H(f)\,.\,H^*(f)\,.\,S_x(f)$$

$$S_Y(f) = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df$$

Total Input and Output Power

The total input and output powers can be found as the total area under the power spectral density curve.

$$E\{X(t)^2\} = \int_{-\infty}^{\infty} S_X(f) df = R_x\,(0)$$

$$E\{Y(t)^2\} = \int_{-\infty}^{\infty} S_Y(f) df = R_y\,(0)$$

The Gaussian Random Process

A random variable X is said to be Gaussian if its probability density function is :

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \, e^{-(x-\mu_x)^2/2\sigma_x^2}$$

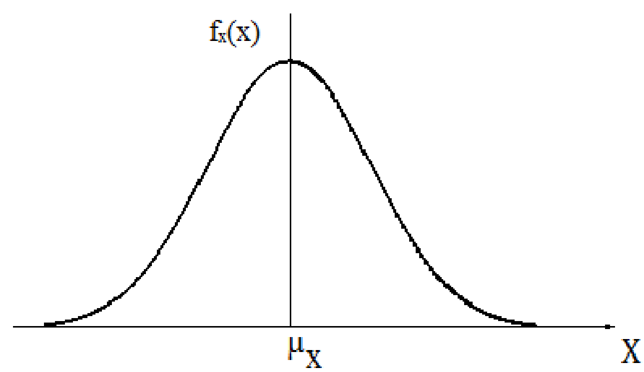
where,

μ_X = E(x) is the mean value of X

$\sigma_x^2 = E\{(X-\mu_x)^2\}$ is the variance of X.

Defenition

A random process X(t) is said to Gaussian if the random variables X_1, X_2,., X_n (obtained by observing the process at times $t_1, t_2, \ldots, t_n$) have a jointly Gaussian probability density function for all possible values of n and all times $t_1, t_2, \ldots, t_n$.



Two Virtues of the Gaussian Process

First : The process has many properties that make analytic results possible (easy to handle mathematically).

Second: The random process produced by physical phenomena is often such that a Gaussian model is appropriate . The use of a Gaussian model to describe the physical phenomena is usually confirmed by experiments.

The Central Limit Theorem

Let $X_1, X_2, ., X_n$ be a set of independent and identically distributed (iid) random variables such that $E(X_i) = \mu_x$ and Var $(X_i) = \sigma_x^2$. Define the random variable

$$U = \frac{\sum_1^n X_i}{n}$$

The probability distribution of U approaches a Gaussian distribution with mean μ_x and variance $\sigma_x{}^2$/n in the limit as n → ∞ .

The theorem provides a justification for using a Gaussian process as a model for a large number of physical phenomena in which the observed random variable at a particular instant of time, is a result of a large number of individual events.

Properties of the Gaussian Process

1- If a Gaussian process X(t) is applied to a stable linear filter, then the random process Y(t) at the output of the filter is also Gaussian .

To see that we consider the convolution integral relating Y(t) to X(t)

$$Y(t) = \int_{-\infty}^{\infty} X(\lambda)h\,(t - \lambda)d\lambda$$

Which comes from the approximation

$$Y(t) = \sum X(\lambda_i)h\,(t - \lambda_i)$$

Note that Y(t) is a linear combination of Gaussian random variables, and so Y(t) is Gaussian for any value of t (any linear operation on X(t) produces another Gaussian process).

2- Consider the set of random variables $X(t_1)$, $X(t_2)$,…., $X(t_n)$, obtained by observing a random process X(t) at times t_1 , t_2 , … , t_n . If the process X(t) is Gaussian , then this set of random variables is jointly Gaussian for any n. The joint pdf is completely determined by specifying

the mean vector

$$\mu = [\ \mu_1 , \mu_2 , \dots \mu_n]^T$$

and the covariance matrix

$$\Sigma = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}, \; c_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\}, \; i, j = 1, \ldots, n$$

The joint pdf of the n random variables is

$$f(X_1, \ldots, X_n) = \frac{1}{(2\pi)^{n/2} \sqrt{\det \Sigma}} e^{-0.5 (X-\mu)^T \Sigma^{-1} (X-\mu)}$$

3- If a Gaussian process is stationary in the wide sense, then it is also stationary in the strict sense (this follows from property 2 above).

4- If the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ obtained by sampling a Gaussian process X(t) at times $t_1, t_2, \ldots, t_n$ are uncorrelated, that is

$$E\{(X_i - \mu_i)(X_j - \mu_j)\} = 0; \qquad i \neq j$$

then these random variables are statistically independent. Here the covariance matrix is diagonal.

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \ldots & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

The joint pdf becomes a product of the marginal pdf’s.

$$f_x(x) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_i-\mu_i)^2/2\sigma_i^2}$$

A diagonal covariance matrix is a necessary and sufficient condition for statistical independence .

Noise in Communication Systems

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signals in communication system and over which we have no control .

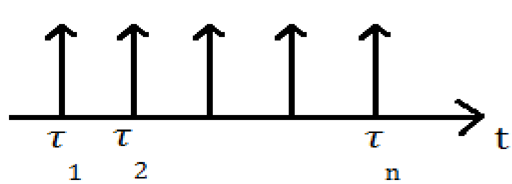
The noise may be external (man-made noise, galactic noise) or internal (arising from spontaneous fluctuation of current or voltage in electronic devices), example of which are shot noise and thermal noise .

Shot Noise

The noise arises in electronic devices such as diodes, transistors and photo-detector circuits, due to the discrete nature of current flow in these devices. Remember that current is a result of the flow of electrons, which have a discrete nature. The Poisson distribution is often used to model this type of noise. Using this model the number of electrons emitted in an interval of length T is a random variable with the pdf

$$P(X=x) = e^{-\lambda T} \frac{(\lambda T)^x}{x!} \quad , x = 0, 1, 2, \ldots$$

λ: average number of electrons emitted /unit time (rate of emission).



Thermal Noise :-

Voltages and currents that exist in a network due to the random motion of electrons in conductors is referred to as thermal noise (Johnson's noise).

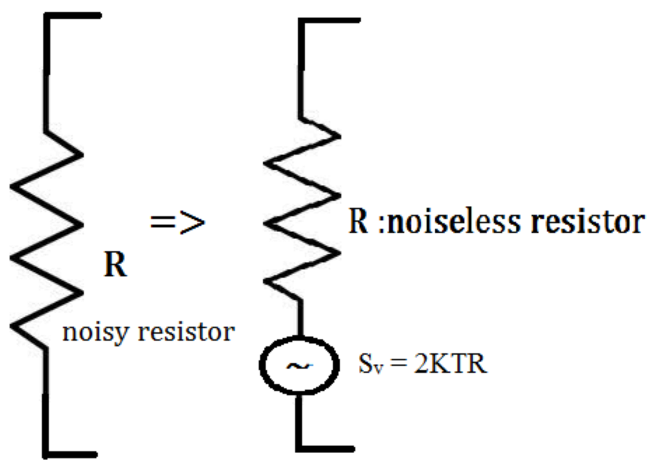
Quantum mechanics shows that the power spectral density of the thermal noise associated with a resistor with a resistance R is given by

$$S_v(f) = \frac{2Rh|f|}{e^{\frac{h|f|}{kT}} - 1} \qquad V^2/Hz$$

K: Boltsman constant ($1.38 * 10^{-23}$ J/degree)

h : Planck constant ($6.62 * 10^{-34}$ Joules-sec)

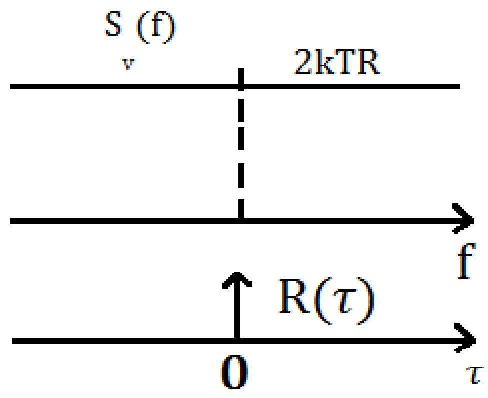
T: degree in Kelvin



For frequencies up to 10^{12} =1000 GHz, this power spectral density is almost constant having the value

$$S_v(f) = 2kTR \quad V^2/Hz .$$

The thermal noise voltage in a zero –mean Gaussian random process.



$E\{V(t)\}=0;$ zero mean.

$S_v(f) = 2kTR;$ constant power spectral density.

The autocorrelation function corresponding to this constant power spectral density is:

$R_v(\tau) = 2kTR\ \delta(\tau);$

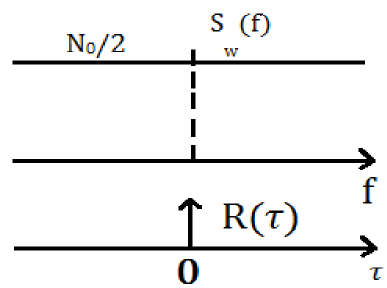
This result shows that two random variables V_{t1} , V_{t2} taken at times t_1 and t_2 are statistically independent for any value of $\tau = t_2 - t_1$ $\tau > 0$.

White Noise

White noise is one whose power spectral density is constant over all frequencies. The power spectral density and autocorrelation function for this type of noise are:

$S_w(f) = N_0/2$

$R_w(\tau) = N_0/2\ \delta(\tau)$



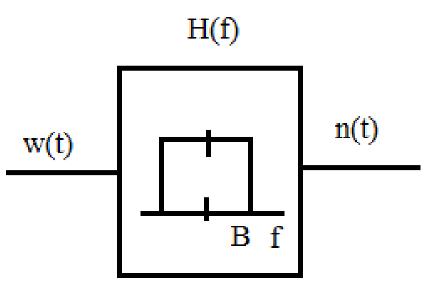
This is the type of noise (model) that we will use in the analysis of communication systems.

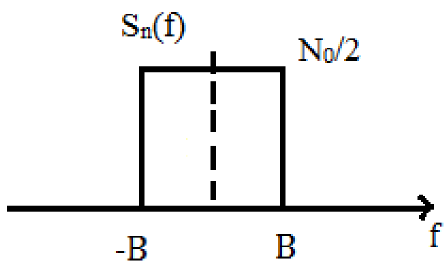
The assumption made is that this noise is additive white Gaussian (AWGN). If s(t) is the transmitted signal and r(t) is the received signal, then

r(t) = s(t) + w(t); additive channel noise .

Filtered White Noise

Assume that a white Gaussian noise w(t) of zero mean and psd = $N_0/2$ is applied to an ideal LPF of B.W = B. Let n(t) denote the filtered noise, then





$S_n(f) = |H(f)|^2 S_w(f)$

$S_n(f)=\begin{cases} N_0/2 & ,-B < f < B \\ 0 & ,\text{o.w} \end{cases}$; Output psd

$R_n(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{(j2\pi f\tau)} df$

$R_n(\tau) = N_0 B \text{ sinc } 2B\tau$; Output autocorrelation function

$E\{n(t)\} = 0$; zero mean noise

$E\{n(t)^2\} = \int_{-B}^{B} S_n(f)df = \frac{N_0}{2}(2B) = N_0 B$; Total output noise power

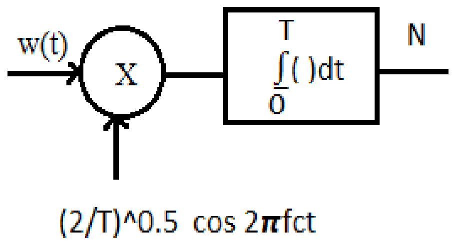
The pdf of the filtered noise at any particular time t is

$f_n(n) = \frac{1}{\sqrt{2\pi(N_0 B)}} e^{\frac{-n^2}{2(N_0 B)}}$, $-\infty < n < \infty$

Remark: note that the pdf is not a function of time indicating that this filtered Gaussian process in stationary in the strict sense.

Correlation of White Noise with a Sinusoidal Signal

Let w(t) be a white Gaussian noise with zero mean. This noise is multiplied by a sinusoidal basis function and integrated over an interval of duration T to produce the scalar N. This scheme is repeatedly used in the coherent demodulation of digital signals. The interval T corresponds to one symbol interval and f_c is the frequency of the carrier. The carrier period and the symbol period are related by $T=nT_c$ where n is an integer. We wish to study the properties of N.



Mathematically, the correlation process is represented as:

$$N = \int_0^T w(t)\sqrt{\frac{2}{T}}\cos 2\pi f_c t \; dt$$

The mean value of N is:

$$E\{N\} = \int_0^T E\{w(t)\}\sqrt{\frac{2}{T}}\cos 2\pi f_c t \; dt = 0.$$

The variance of N is:

$$E\{N^2\} = \frac{2}{T}\iint_0^T E\{w(t_1)w(t_2)\} cos 2\pi f_c t_1 \;\; \cos 2\pi f_c t_2 \; dt_1 \, dt_2$$

$$E\{w(t_1)w(t_2)\} = R_w(t_2 - t_1) = N_0/2 \; \delta(t_2 - t_1)$$

$$\Rightarrow E\{ N^2 \} = \frac{2}{T}\frac{No}{2} \int_0^T \left(\int_0^T \delta(t_2 - t_1) \cos 2\pi f_c \, t_1 \, dt_1\right) \cos 2\pi f_c t_2 dt_2$$

$$= \frac{2}{T}\frac{No}{2}\int_0^T \cos^2 2\pi f_c t_2 \, dt_2; \qquad T=nT_c$$

The last step comes by virtue of the sifting property of the delta function. By performing the integration, we get

$$E\{N^2\} = \frac{No}{2} = \sigma^2$$

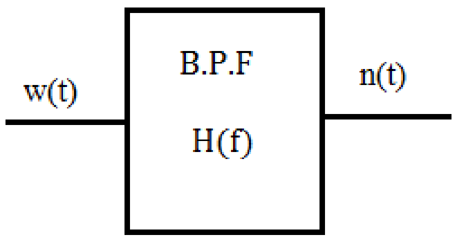
=> N is a zero mean Gaussian r.v with variance $\sigma^2 = \frac{No}{2}$. Its pdf can be written as

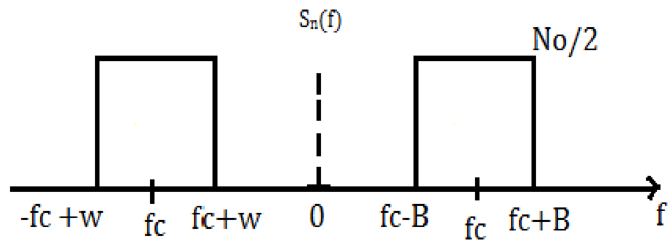
$$f_N(n) = \frac{1}{\sqrt{2\pi(\frac{N_o}{2})}} \; e^{\frac{-n^2}{2.(N_0/2)}}$$

$$f_N(n) = \frac{1}{\sqrt{\pi N_o}} \; e^{\frac{-n^2}{No}}$$

Narrow-band Noise

Now let the white Gaussian noise w(t) of psd $S_w(f) = N_0 / 2$ be applied to an ideal band pass filter with center frequency f_c and bandwidth 2B.





The noise is described as narrow band when $2B \ll f_c$. The analysis is similar to that done for the LPF and the results are summarized as follows:

$S_n(f) = N_0/2$ for $f_c - B < |f| < f_c + B$; Output psd

$E\{n(t)\} = 0$; zero mean noise

$E\{n(t)^2\} = 2 \cdot \frac{N_o}{2} \cdot 2B = N_o(2B) = \sigma^2$; Total output power

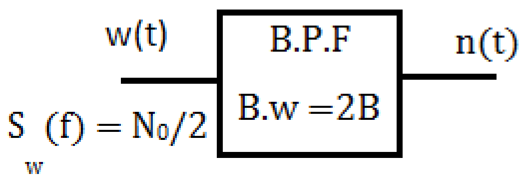
$f_N(n) = \frac{1}{\sqrt{2\pi(2BN_0)}} e^{\frac{-n^2}{2(2BN_0)}}$, $-\infty \le n \le \infty$; output noise pdf.

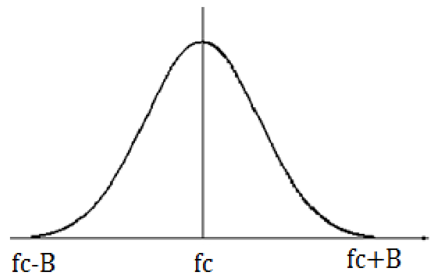
$R_n(0) = E\{n(t)^2\} = \int_{-\infty}^{\infty} S_n(f)\,df$; Mean square value.

Narrow-band Noise: In-phase and Quadrature Representation

Let w(t) be applied to a bandpass filter of B.w = 2B centered at f_c to produce a narrow band noise n(t).

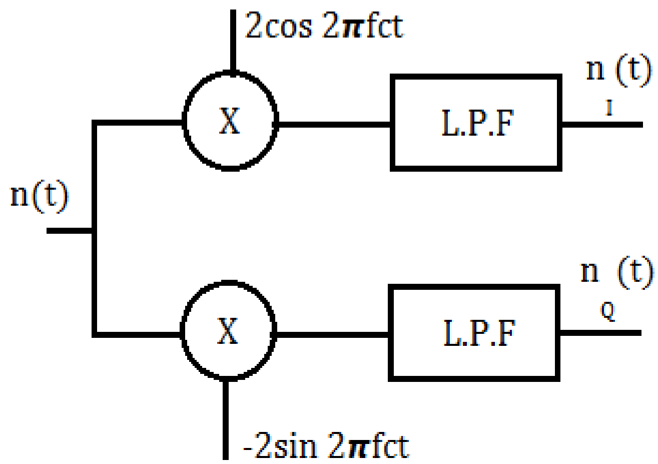
The narrow band noise n(t) can be represented in terms of an in-phase $n_I(t)$ and a quadrature component $n_Q(t)$ as:





$$n(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t$$

The in-phase and quadrature components $n_I(t)$ and $n_Q(t)$ can be recovered from n(t) as demonstrated in the block diagram.



$n_I(t) = Lp\{2n(t)\cos 2\pi f_c t\}$; in-phase noise component

$S_{NI}(f) = Lp\{S_n(f-f_c)+S_n(f+f_c)\}$; in-phase noise psd.

$n_Q(t) = -\ Lp\{2n(t)\sin 2\pi f_c t\}$; quadrature noise component

$S_{NI}(f) = S_{NQ}(f)$; both components have the same psd

Finally, $n_I(t)$ and $n_Q(t)$ can be retrieved from n(t) as:

$$n_I(t) = n(t)\cos 2\pi f_c t + \widehat{n(t)} sin 2\pi f_c t$$

$$n_I(t) = \widehat{n(t)}\cos 2\pi f_c t - n(t) sin 2\pi f_c t$$

Properties of the Noise Components

- The in-phase component $n_I(t)$ and the quadrature component $n_Q(t)$ of narrow band noise n(t) have zero mean .
- If the narrow band noise n(t) is Gaussian, then $n_I(t)$ and $n_Q(t)$ are jointly Gaussian .
- If n(t) is wide sense stationary, then $n_I(t)$ and $n_Q(t)$ are jointly wide sense stationary .
- Both $n_I(t)$ and $n_Q(t)$ have the same power spectral density

$$S_{NI}(f) = S_{NQ}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B < f < B \\ 0 & , o.w \end{cases}$$

- $n_I(t)$, $n_Q(t)$ and n(t) have the same variance

$$E\{n(t)^2\} = E\{n_I(t)^2\} = E\{n_Q(t)^2\} = \sigma^2$$

- The cross-spectral densities of $n_I(t)$ and $n_Q(t)$ are imaginary

$$S_{NINQ}(f) = - S_{NQNI}(f) = \begin{cases} j[S_N(f + f_c) - S_N(f - f_c)], & -B < f < B \\ 0 & , o.w \end{cases}$$

- If n(t) is Gaussian with zero mean and a power spectral density $S_n(f)$ that is symmetric about f_c, then $n_I(t)$ and $n_Q(t)$ are statistically independent. The joint pdf of $n_I(t)$ and $n_Q(t)$ is the product of the marginal pdf's

$$f(n_I, n_Q) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_I^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-n_Q^2}{2\sigma^2}}$$

(i.e, when the cross spectral density = 0 ∀ f, then n_I and n_Q are independent)

Polar Representation of Narrow-band Noise

Let n(t) be a narrow band zero-mean, white Gaussian noise with a symmetric psd about some center frequency f_c .

$$n(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t.$$

Because $S_n(f)$ is symmetric, it follows that $n_I(t)$ and $n_Q(t)$, observed at a fixed time t, are independent Gaussian r.v with zero mean and variance σ^2 .

n(t) can also be represented as

$$n(t) = R(t) \cos (2\pi f_c t + \emptyset(t))$$

where the envelope R(t) and the phase ∅(t) are given as:

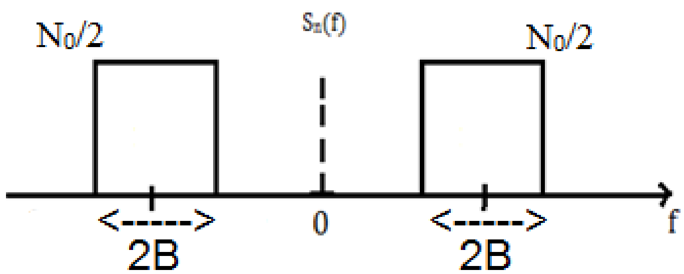
$$R(t) = [n_I(t)^2 + n_Q(t)^2]^{1/2}$$

$$\emptyset(t) = \tan^{-1} (n_Q(t) / n_I(t))$$

It can be shown (Go back to your ENEE 331 lecture notes and go over the proof) that R and ∅ are independent random variables with pdf 's

$$f_\Phi (\Phi) = \begin{cases} \frac{1}{2\pi} & , 0 \le \emptyset \le 2\pi \\ 0 & , \text{o.w} \end{cases} ; \qquad \text{(Uniform pdf)}$$

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp[-(r^2/2\sigma^2)] , & r \ge 0 \\ 0 & , \text{o.w} \end{cases} ; \qquad \text{(Rayleigh distribution)}$$



If $S_n(f)$ has the psd shown then , $\sigma^2 = 2 (N_0/2)(2B) = 2N_0 B$ and the pdf of R is as given above .

A Test Sine Signal Plus Narrow-band Noise

If a test sine signal $A\cos 2\pi f_c t$ is added to the narrow band filtered noise, then the signal plus noise can be expressed (using the in-phase and quadrature representation of the noise) as:

$$X(t) = A_c \cos 2\pi f_c t + n_I(t)\cos 2\pi f_c t - n_Q(t)\sin 2\pi f_c t$$

$$X(t) = (A_c + n_I(t))\cos 2\pi f_c t - n_Q(t)\sin 2\pi f_c t$$

The noise components N_I and N_Q are independent zero mean Gaussian r.v (psd of n(t) is symmetric) each with variance σ^2. X(t) can also be represented in polar form as:

$$X(t) = R(t)\cos(2\pi f_c t + \emptyset(t))$$

Where

$$R(t) = \sqrt{(A_c + n_I(t))^2 + n_Q(t)^2}$$

$$\emptyset(t) = tan^{-1}\frac{n_Q(t)}{(A_c + n_I(t))}$$

It can be shown that the pdf of R is

$f_R(r) = \{\frac{r}{\sigma 2}$ Exp -[$(r^2 + A^2) / 2\sigma^2$] I_o (Ar/σ^2); (Rician distribution)

I_o (.) is the modifies Bessel function of the first kind of zero order, and

$\sigma^2 = E\{n(t)^2\} = E\{n_i(t)^2\} = E\{n_q(t)^2\}$.

The Rician distribution arises in the study of the performance of some digital communication applications like the noncoherent demodulation of ASK and FSK.

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