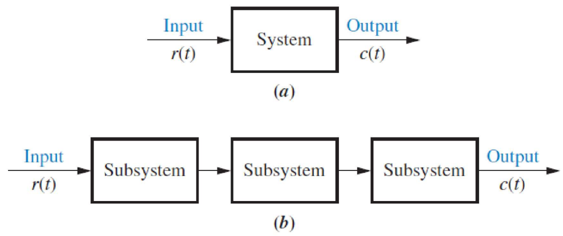
Modeling in the Frequency Domain 2



Note: The input, $r(t)$, stands for *reference input*. The output, $c(t)$, stands for *controlled variable*.

FIGURE 2.1 **a.** Block diagram representation of a system; **b.** block diagram representation of an interconnection of subsystems

2.2 Laplace Transform Review

TABLE 2.1 Laplace transform table

Item no.	$f(t)$		$F(s)$
1.	$\delta(t)$		1
2.	$u(t)$		$\frac{1}{s}$
3.	$tu(t)$	t	$\frac{1}{s^2}$
4.	$t^n u(t)$	t^n	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	e^{-at}	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	sin	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	cos	$\frac{s}{s^2+\omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Chapter 7

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Partial-Fraction Expansion

$$F_1(s) = \frac{s^3+2s^2+6s+7}{s^2+s+5}$$

$$F_1(s) = s+1+\frac{2}{s^2+s+5}$$

Case 1. Roots of the Denominator of *F*(*s*) Are Real and Distinct

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{s+1} + \frac{-2}{s+2}$$

$$f(t) = \left(2e^{-t} - 2e^{-2t}\right) \cancel{u(t)}$$

Case 2. Roots of the Denominator of *F*(*s*) Are Real and Repeated

$$(s+1)(s+2)^2\left[F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}\right], \quad \boxed{A = 2}$$

$$2 = 2(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

$$2 = 2s^2 + 8s + 8 + B(s^2+3s+2) + Cs + C$$

$$2 = \underline{(2+B)}s^2 + \underbrace{(8+3B+C)}_{=\text{zero}}s + \underbrace{(8+C+2B)}_{=2}$$

$$\boxed{B = -2}$$

$$\boxed{C = -2}$$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} - \frac{2}{(s+2)^2}$$

$$\therefore f(t) = 2e^{-t} - 2e^{-2t} - 2te^{-2t}$$

$$t^2 \longleftrightarrow \frac{2!}{s^3}$$

$$\frac{1}{(s+2)^3} \longleftrightarrow \frac{1}{2}t^2 e^{-2t}$$

Case 3. Roots of the Denominator of $F(s)$ Are Complex or Imaginary

$$F(s)=\frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

$$\left(\frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} + \frac{Bs+C}{s^2+2s+5}\right) \times s(s^2+2s+5)$$

$$3 = \frac{3}{5}(s^2+2s+5) + Bs^2 + Cs$$

$$B=-\frac{3}{5}, \quad C=-\frac{6}{5}$$

$$F(s)=\frac{3/5}{s} - \frac{3}{5}\,\frac{s+2}{s^2+2s+5+1-1}$$

$$= \frac{3/5}{s}-\frac{3}{5}\,\frac{s+2}{(s+1)^2+4}$$

$\sin \omega t \qquad \frac{\omega}{s^2+\omega^2}$

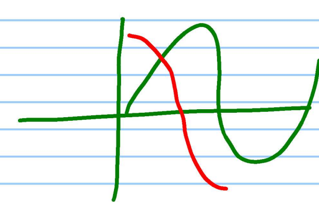
$$= \frac{3/5}{s}-\frac{3}{5}\,\frac{(s+1)+1}{(s+1)^2+(2)^2}$$

$\cos \omega t \qquad \frac{s}{s^2+\omega^2}$

$$= \frac{3/5}{s}-\frac{3}{5}\left[\frac{(s+1)}{(s+1)^2+(2)^2}+\frac{1}{2}\,\frac{2}{(s+1)^2+(2)^2}\right]$$

$$\therefore f(t)=\frac{3}{5}-\frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$\frac{1}{2}\cos\left(2t-\frac{\pi}{2}\right)$$



$$1\angle 0 + \frac{1}{2}\angle -90$$

$$1-\frac{1}{2}j \longrightarrow 1.118\angle -26.56^\circ$$

$$\therefore f(t) = \frac{3}{5} - \frac{3}{5} e^{-t}\left(1.118 \cos(2t - 26.56)\right)$$

$$= \frac{3}{5} - 0.6708\, e^{-t} \cos(2t - 26.56)$$

2.3 The Transfer Function

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}$$

$C(s) \rightarrow$ output

$R(s) \rightarrow$ input

Notice that Eq. (2.53) separates the output, $C(s)$, the input, $R(s)$, and the system, which is the ratio of polynomials in s on the right. We call this ratio, $G(s)$, the *transfer function* and evaluate it with *zero initial conditions*.

$G(s)$

$$R(s) \longrightarrow \boxed{\frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \longrightarrow C(s)$$

FIGURE 2.2 Block diagram of a transfer function

$$C(s) = R(s)\, G(s)$$

2.4 Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

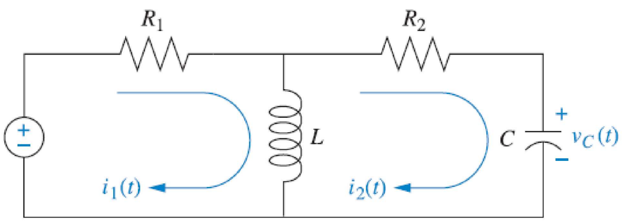
Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

$$s = j\omega$$

Example 2.10

Transfer Function—Multiple Loops

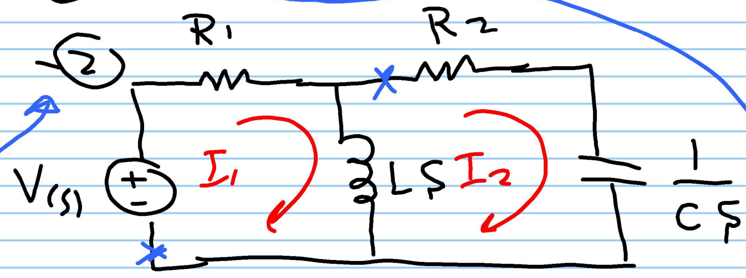
PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.



(a)

$$(R_1+Ls)I_1(s) - (Ls)I_2(s) = V(s) \quad —(1)$$

$$-(Ls)I_1(s) + \left(R_2+Ls+\frac{1}{Cs}\right)I_2(s) = 0 \quad —(2)$$



$$-V_s + I_1R_1 + Ls(I_1-I_2) = 0$$

$$R_2I_2 + \frac{1}{Cs}I_2 + Ls(I_2-I_1) = 0$$

$$I_2(s) = \frac{\begin{vmatrix} R_1+Ls & V(s) \\ -Ls & 0 \end{vmatrix}}{\begin{vmatrix} R_1+Ls & -Ls \\ -Ls & R_2+Ls+\frac{1}{Cs} \end{vmatrix}}$$

$$I_2(s) = \frac{Ls\,V(s)}{(R_1+Ls)\left(R_2+Ls+\frac{1}{Cs}\right)-(Ls)^2}$$

$$\therefore \frac{I_2(s)}{V(s)} \Rightarrow \quad G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Nodal

$$G(s) = \frac{V_o(s)}{V(s)}, \quad V_o(s) = V_1(s)$$

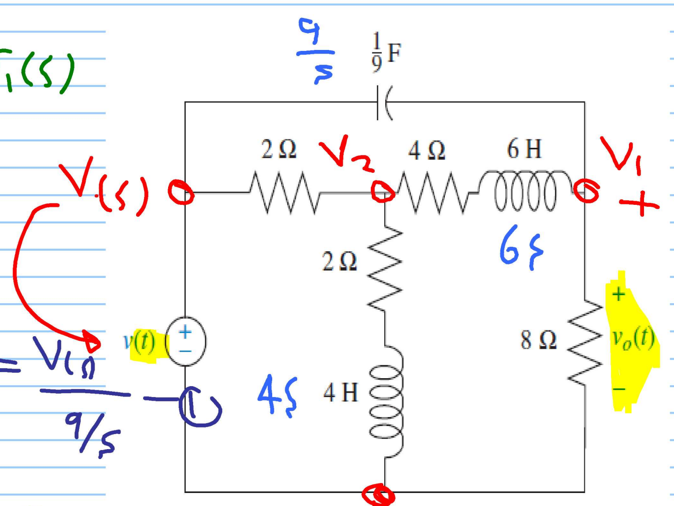


FIGURE P2.6

$$\frac{V_1(s)-V(s)}{9/s} + \frac{V_1(s)-V_2(s)}{4+6s} + \frac{V_1(s)}{8} = 0$$

$$\left(\frac{1}{9/s} + \frac{1}{4+6s} + \frac{1}{8}\right)V_1(s) - \frac{1}{4+6s}V_2(s) = \frac{V(s)}{9/s} \quad —(1)$$

$$\frac{V_2(s)-V(s)}{2} + \frac{V_2(s)}{2+4s} + \frac{V_2(s)-V_1(s)}{4+6s} = 0$$

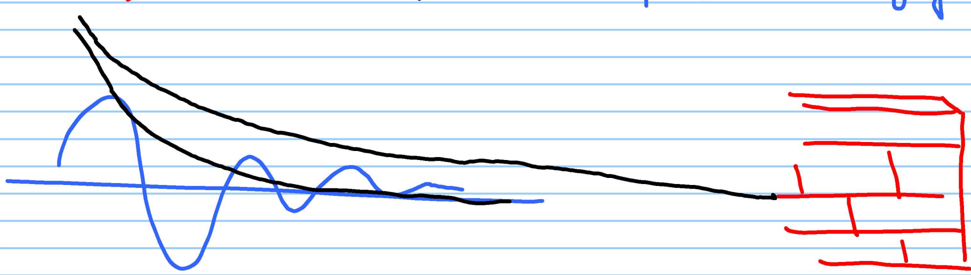
$$-\frac{1}{4+6s}V_1(s)+\left(\frac{1}{2}+\frac{1}{2+4s}+\frac{1}{4+6s}\right)V_2(s)=\frac{V(s)}{2} \quad \text{—} \ (2)$$

2.5 Translational Mechanical System Transfer Functions

→ Mechanical systems → 3 passive linear components

Spring, Mass } → energy storage elements

Viscous damper → dissipates energy.



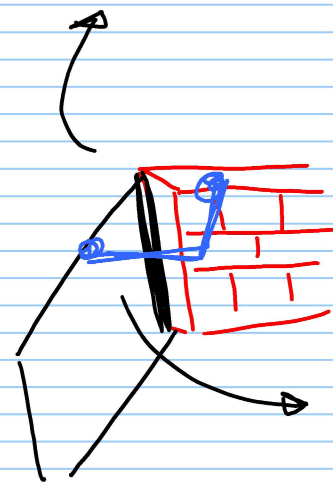


TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity (V–I)	Force-displacement	Impedence $Z_M(s)=F(s)/X(s)$
Spring ($x(t)$, $f(t)$, K)	$f(t)=K\int_0^t v(\tau)d\tau$ $V=\frac{1}{C}\int I\,dt$	$f(t)=Kx(t)$ $F(s)=K\,X(s)$	K
Viscous damper ($x(t)$, $f(t)$, f_v)	$f(t)=f_v v(t)$ $V=RI$	$f(t)=f_v\frac{dx(t)}{dt}$ $F(s)=f_v\,s\,X(s)$	$f_v s$
Mass ($x(t)$, $f(t)$, M)	$f(t)=M\frac{dv(t)}{dt}$ $V=L\,di/dt$	$f(t)=M\frac{d^2x(t)}{dt^2}$ $F(s)=Ms^2X(s)$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

$$\sum f(t) = \text{Zero}$$

$$Z_M(s) = \frac{F(s)}{X(s)}$$

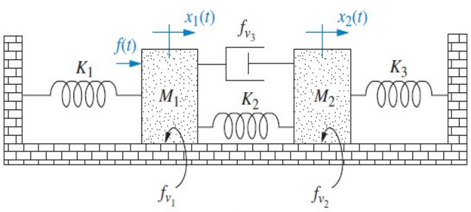
$$F(s) = Z_M(s)\, X(s)$$

[sum of impedances] $X(s)$ = sum of applied forces

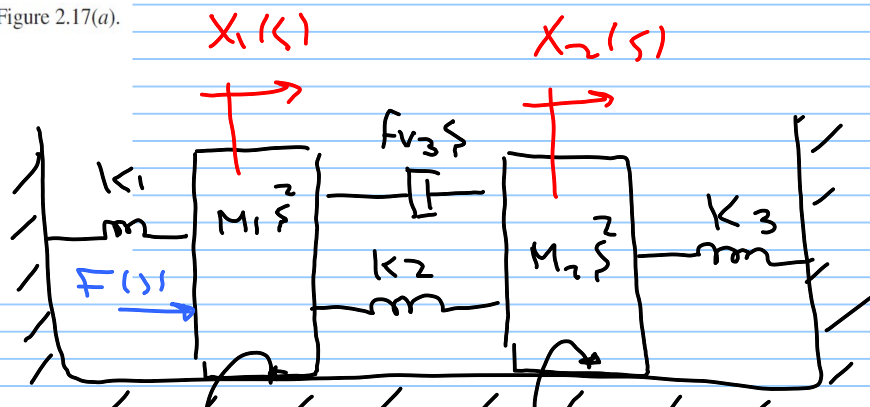
Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).



(a)



$$[M_1 s^2 + (f_{v_1}+f_{v_3})s + (K_1+K_2)]X_1(s) - [f_{v_3}s + K_2]X_2(s) = F(s) \quad — ①$$

$$-[f_{v_3}s + K_2]X_1(s) + [M_2 s^2 + (f_{v_2}+f_{v_3})s + (K_2+K_3)]X_2(s) = 0 \quad — ②$$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{(f_{v_3}s + K_2)}{\Delta}$$

$$\begin{bmatrix}\text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of} \\ \text{applied forces} \\ \text{at } x_1\end{bmatrix} \qquad (2.120a)$$

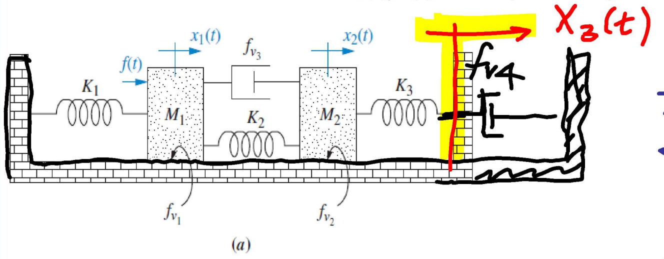
$$-\begin{bmatrix}\text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of} \\ \text{applied forces} \\ \text{at } x_2\end{bmatrix} \qquad (2.120b)$$

Note: # of eq. of motion required is equal to the number of linearly independent motions.

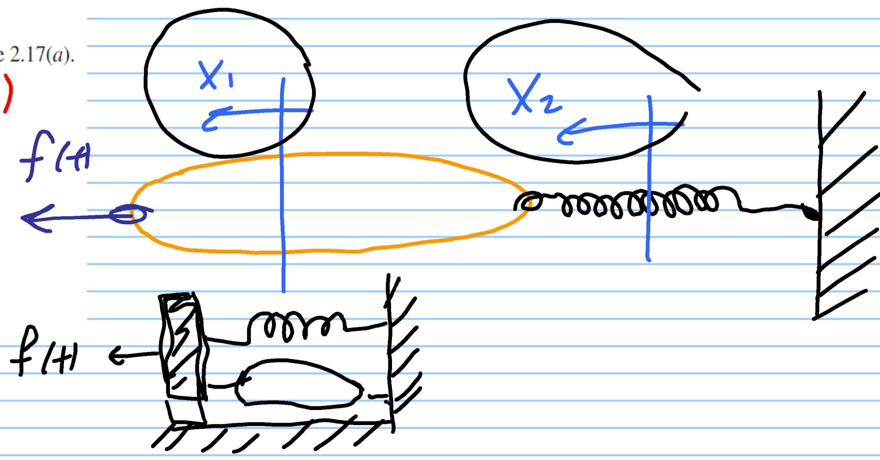
Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).



(a)



3 equations

$$[M_1 s^2 + (f_{v_1}+f_{v_3})s + (K_1+K_2)]X_1(s) - [f_{v_3}s + K_2]X_2(s) + 0\,X_3(s) = F(s) \quad — (1)$$

$$-[f_{v_3}s + K_2]X_1(s) + [M_2 s^2 + (f_{v_2}+f_{v_3})s + (K_2+K_3)]X_2(s) - K_3 X_3(s) = 0 \quad — (2)$$

$$-0\,X_1(s) - K_3 X_2(s) + (f_{v_4}s + K_3)X_3(s) = 0 \quad — (3)$$

Example 2.18

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.

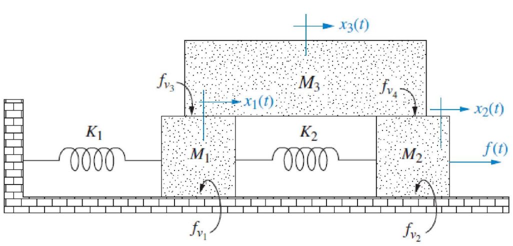


FIGURE 2.20 Three-degrees-of-freedom translational mechanical system

$$[M_1 s^2 + (f_{v_1}+f_{v_3})s + (K_1+K_2)]X_1(s) - K_2X_2(s) - f_{v_3}sX_3(s) = 0 \quad — (1)$$

$$-K_2X_1(s) + [M_2 s^2 + (f_{v_2}+f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s) \quad — (2)$$

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3 s^2 + (f_{v_3}+f_{v_4})s]X_3(s) = 0 \quad — (3)$$

Skill-Assessment Exercise 2.8

PROBLEM: Find the transfer function, $G(s)=X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

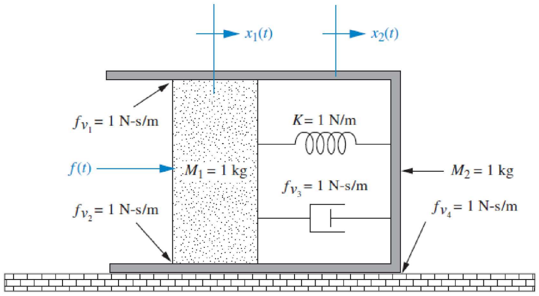


FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

ANSWER: $G(s) = \dfrac{3s+1}{s(s^3+7s^2+5s+1)}$

The complete solution is at www.wiley.com/college/nise.

$$(s^2+3s+1)X_1(s) - (3s+1)X_2(s) = F(s) \quad \text{—} (1)$$

$$-(3s+1)X_1(s) + [s^2+4s+1]X_2(s) = 0 \quad \text{—} (2)$$

$$G(s) = \frac{X_2(s)}{F(s)}, \quad X_2(s) = \frac{\begin{vmatrix} s^2+3s+1 & F(s) \\ -(3s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+3s+1) & -(3s+1) \\ -(3s+1) & (s^2+4s+1) \end{vmatrix}}$$

$$\begin{array}{r} s^4+4s^3+s^2 \\ 3s^3+12s^2+3s \\ s^2+4s+1 \\ \hline \end{array}$$

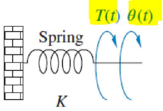
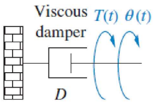
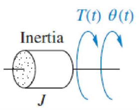
$$9s^2+6s+1$$

$$X_2(s) = \frac{(3s+1)F(s)}{s^4+7s^3+14s^2+7s+1-(3s+1)^2}$$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{3s+1}{s^4+7s^3+5s^2+s}$$

2.6 Rotational Mechanical System Transfer Functions

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad (radians), $\omega(t)$ – rad/s (radians/second), K – N-m/rad (newton-meters/radian), D – N-m-s/rad (newton-meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

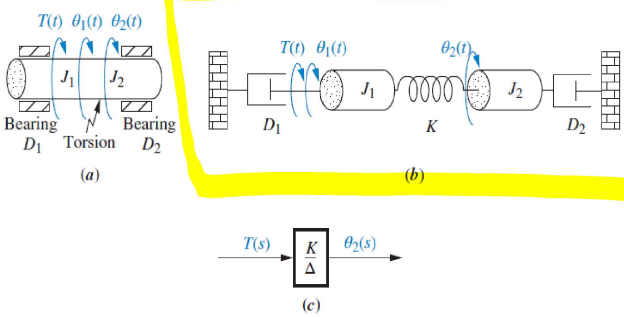


FIGURE 2.22 **a.** Physical system; **b.** schematic; **c.** block diagram

$$(J_1 s^2 + D_1 s + K)\Theta_1(s) - K\,\Theta_2(s) = T(s) \quad —(1)$$

$$-K\,\Theta_1(s) + (J_2 s^2 + D_2 s + K)\Theta_2(s) = 0 \quad —(2)$$

then $G(s) = \dfrac{\Theta_2(s)}{T(s)}$ (Do it)

Example 2.20

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

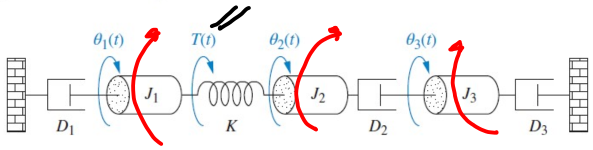


FIGURE 2.25 Three-degrees-of-freedom rotational system

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) - 0\,\theta_3(s) = T(s) \quad \text{— (1)}$$

$$-(K)\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) - D_2 s\,\theta_3(s) = 0 \quad \text{— (2)}$$

$$0\,\theta_1(s) - D_2 s\,\theta_2(s) + (J_3 s^2 + (D_3 + D_2)s + 0)\theta_3(s) = 0 \quad \text{— (3)}$$

Skill-Assessment Exercise 2.9

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure 2.26.

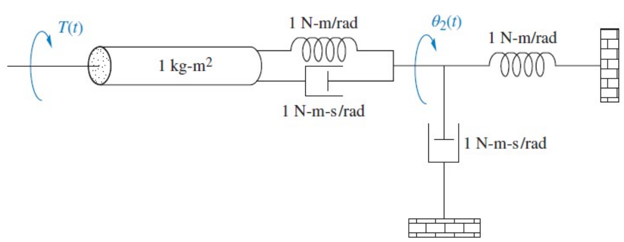
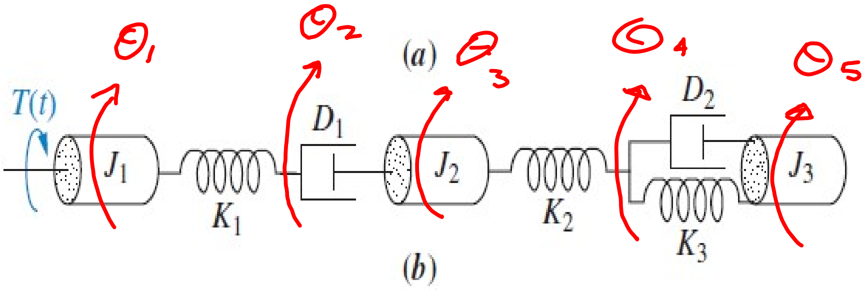


FIGURE 2.26 Rotational mechanical system for Skill-Assessment Exercise 2.9

ANSWER: $G(s) = \dfrac{1}{2s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

H.W.



32. For each of the rotational mechanical systems shown in Figure P2.17, write, but do not solve, the equations of motion. [Section: 2.6]

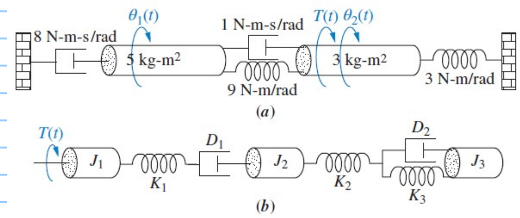


FIGURE P2.17

$$(J_1 s^2 + K_1)\theta_1(s) - K_1\theta_2(s) - 0\,\theta_3(s) - 0\,\theta_4(s) - 0\,\theta_5(s) = T(s) \quad \text{— (1)}$$

$$-K_1\theta_1(s) + (D_1 s + K_1)\theta_2(s) - D_1 s\,\theta_3(s) - 0\,\theta_4(s) - 0\,\theta_5(s) = 0 \quad \text{— (2)}$$

$$0\,\theta_1(s) - D_1 s\,\theta_2(s) + (J_2 s^2 + D_1 s + K_2)\theta_3(s) - K_2\theta_4(s) - 0\,\theta_5(s) = 0 \quad \text{— (3)}$$

$$0\,\theta_1(s) - 0\,\theta_2(s) - K_2\theta_3(s) + (D_2 s + [K_2 + K_3])\theta_4(s) - (D_2 s + K_3)\theta_5(s) = 0 \quad \text{— (4)}$$

$$-(D_2 s + K_3)\theta_4(s) + (J_3 s^2 + D_2 s + K_3)\theta_5(s) = 0 \quad \text{— (5)}$$

Defining

$\theta_1(s)$ = rotation of J_1

$\theta_2(s)$ = rotation between K_1 and D_1

$\theta_3(s)$ = rotation of J_3

$\theta_4(s)$ = rotation of right-hand side of K_2

the equations of motion are

$$(J_1 s^2 + K_1)\theta_1(s) - K_1\theta_2(s) = T(s)$$
$$-K_1\theta_1(s) + (D_1 s + K_1)\theta_2(s) - D_1 s\,\theta_3(s) = 0$$
$$-D_1 s\,\theta_2(s) + (J_2 s^2 + D_1 s + K_2)\theta_3(s) - K_2\theta_4(s) = 0$$
$$-K_2\theta_3(s) + (D_2 s + (K_2 + K_3))\theta_4(s) = 0$$

Sol. manual.

2.7 Transfer Functions for Systems with Gears

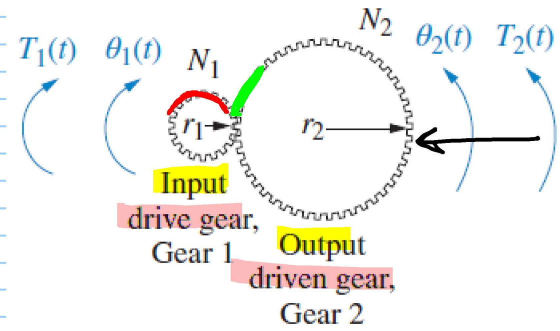


FIGURE 2.27 A gear system

→ The distance traveled along each gear circumference is the same

$$r_1 \theta_1 = r_2 \theta_2$$

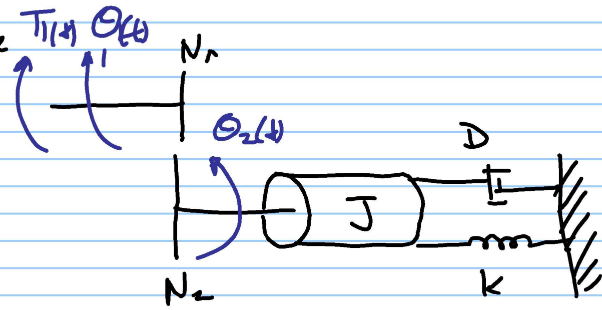
$$\boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$

→ The energy into G_1 = energy out of G_2

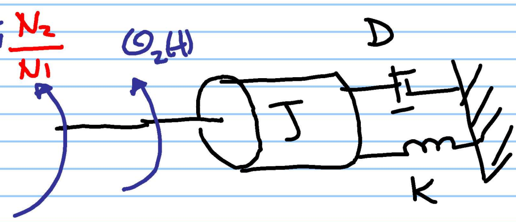
$$T_1 \theta_1 = T_2 \theta_2 \qquad \boxed{\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}}$$

→ what about mechanical impedances driven by gears?

* example



Case (a)



$$\frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

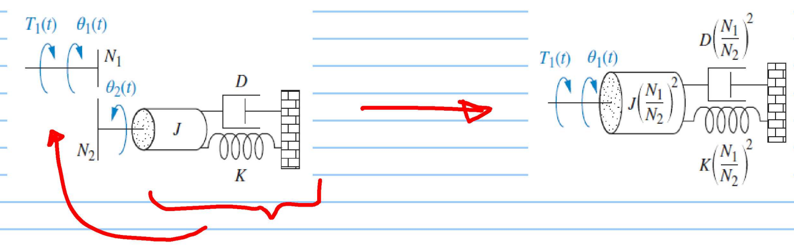
Case (b) The eq. of motion:-

$$(Js^2 + Ds + K)\,\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left((Js^2 + Ds + K)\,\theta_1(s)\frac{N_1}{N_2} = T_1(s)\frac{N_2}{N_1} \right) \times \frac{N_1}{N_2}$$

$$(Js^2 + Ds + K)\,\Theta_1(s)\left(\frac{N_1}{N_2}\right)^2 = T_1(s)$$

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\Theta_1(s) = T_1(s)$$



Generalizing the results, we can make the following statement: *Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio*

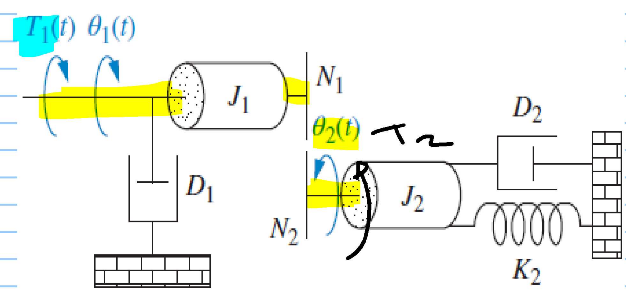
$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

$$\left(\frac{N_{destination}}{N_{source}}\right)^2$$

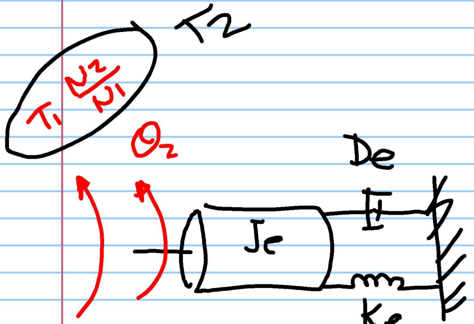
Example 2.21

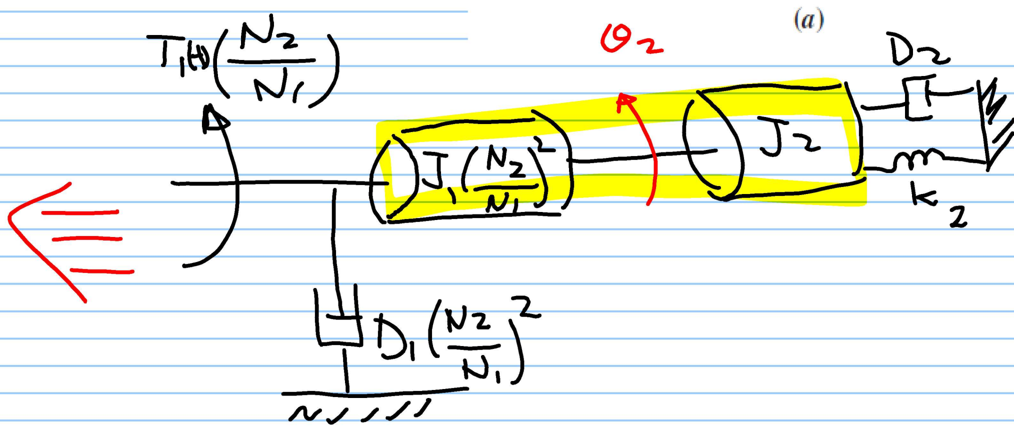
Transfer Function—System with Lossless Gears

PROBLEM: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure 2.30(*a*).



(*a*)





$$J_e = J_2 + J_1\left(\frac{N_2}{N_1}\right)^2$$

$$D_e = D_2 + D_1\left(\frac{N_2}{N_1}\right)^2$$

$$K_e = K_2$$

$$(J_e s^2 + D_e s + K_e)\,\Theta_2(s) = T_1(s)\,\frac{N_2}{N_1}$$

$$\boxed{G(s) = \frac{\Theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}}$$

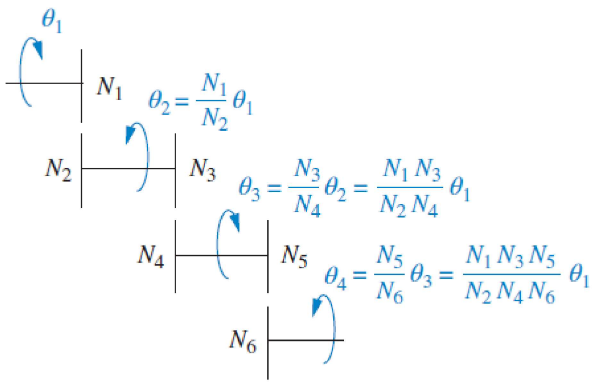


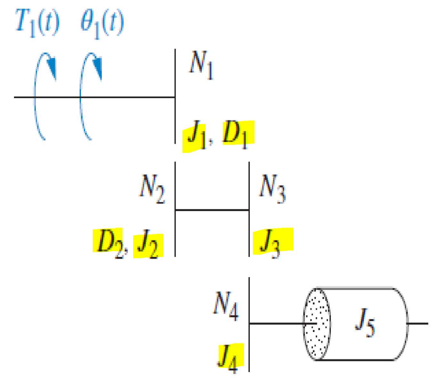
FIGURE 2.31 Gear train

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

Example 2.22

Transfer Function—Gears with Loss

PROBLEM: Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure 2.32(a).



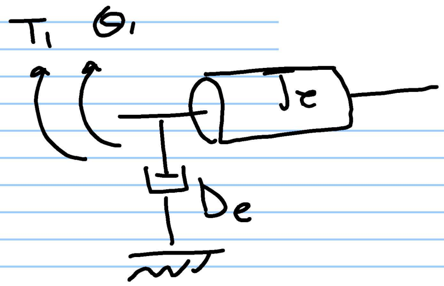
(a)

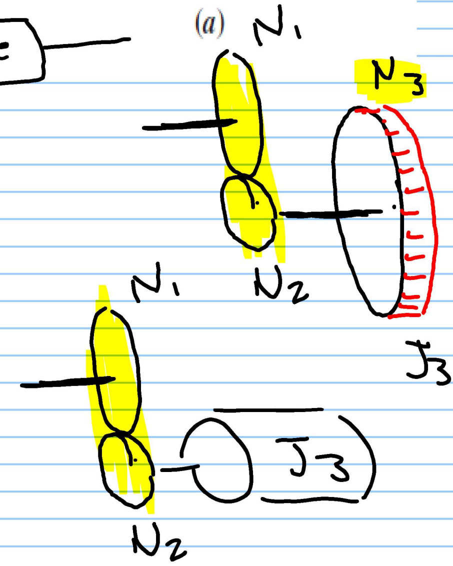
$$J_e = J_1 + (J_2+J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4+J_5)\left(\frac{N_3}{N_4}\,\frac{N_1}{N_2}\right)^2$$

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

$$(J_e s^2 + D_e s)\,\theta_1(s) = T_1(s)$$

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{s(J_e s + D_e)}$$





Skill-Assessment Exercise 2.10

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system with gears shown in Figure 2.33.

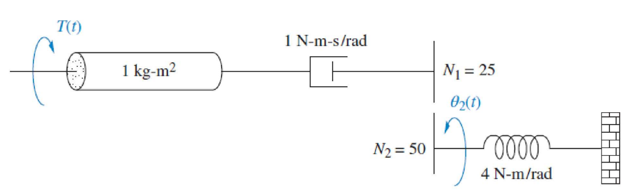


FIGURE 2.33 Rotational mechanical system with gears for Skill-Assessment Exercise 2.10

ANSWER: $G(s) = \dfrac{1/2}{s^2+s+1}$

The complete solution is at www.wiley.com/college/nise.

$$J_e = 150 + 3 \times \left(\frac{50}{5}\right)^2 + 100 \times \left(\frac{5}{25} \cdot \frac{50}{5}\right)^2$$

$$D_e = 500\left(\frac{50}{25}\right)^2$$

$$K_e = 300 + 3 \times \left(\frac{50}{5}\right)^2$$

$$(J_e s^2 + D_e s + k_e)\,\theta_2(s) = T(s) \times \frac{50}{5}$$

37. Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure P2.22. [Section: 2.7]

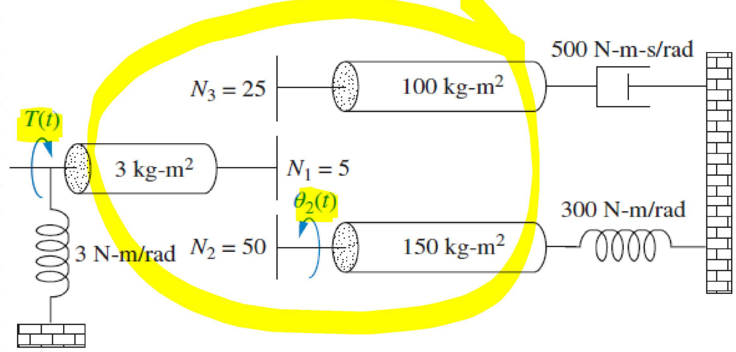
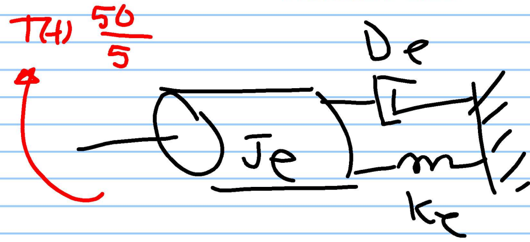
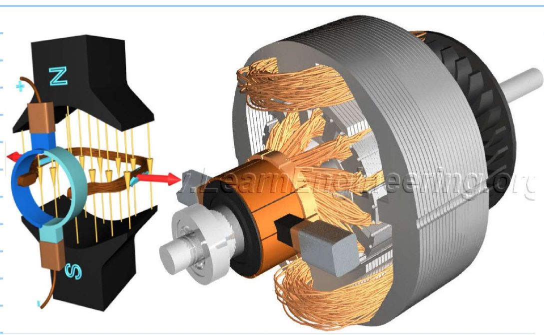


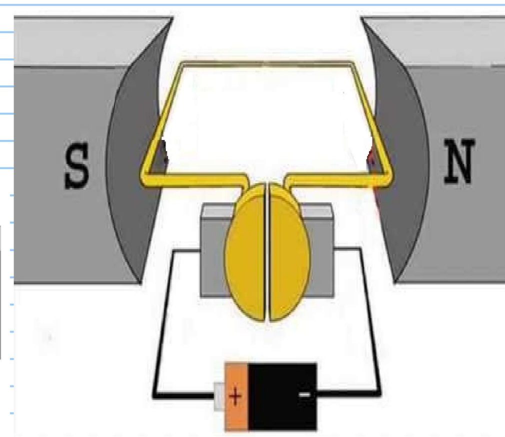
FIGURE P2.22

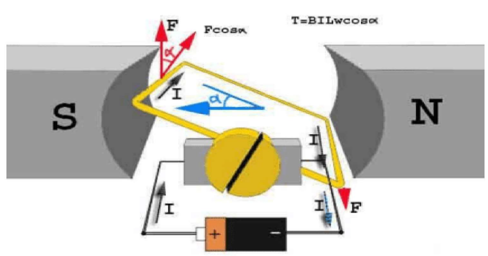


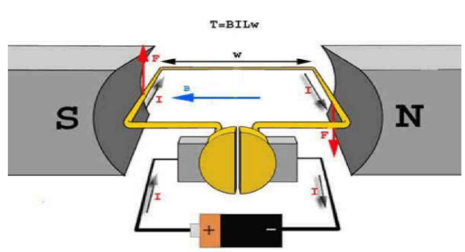
2.8 Electromechanical System Transfer Functions

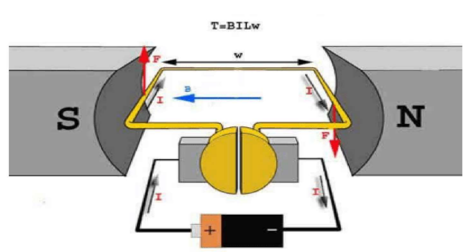
$$E_a(s) \longrightarrow \boxed{G(s)} \longrightarrow \theta_m(s)$$











$F = B\,l\,i_a(t)$ (N)

mag. field strength

$e = B\,l\,v$ (N)

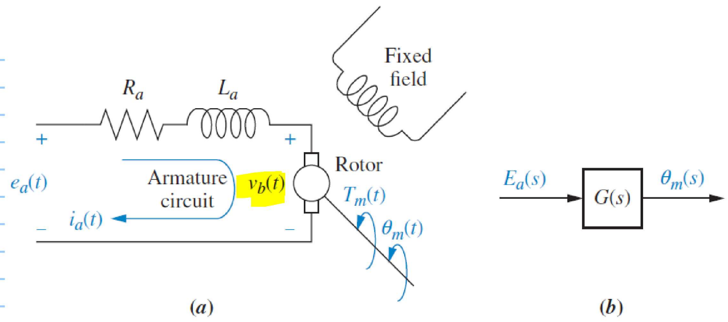
speed

$*$ back emf $\propto$ speed

$$\boxed{v_b(t) = K_b \frac{d}{dt}\theta_m(t)}$$

$\mathcal{L}$

$$\boxed{V_b(s) = K_b\, s\, \theta_m(s)} \quad \text{— (A)}$$



DC motor: **a.** schematic; **b.** block diagram

$v_b(t)$: back electromotive force (back emf)

$\frac{d}{dt}\theta_m(t) = \omega_m(t)$

$*$ developed torque $\propto$ armature current

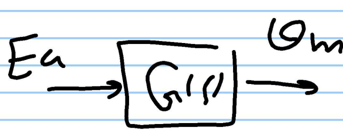
$$T_m(s) = K_t\, I_a(s)$$

$$\boxed{I_a(s) = \frac{1}{K_t} T_m(s)} \quad \text{— (B)}$$

$*$ KVL

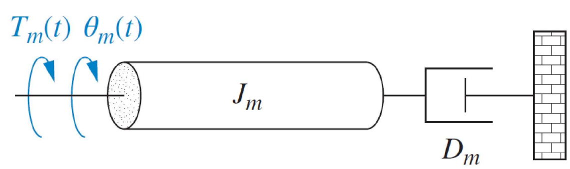
$$R_a I_a(s) + L_a s\, I_a(s) + V_b(s) = E_a(s) \quad \text{— (1)}$$

(A) & (B) $\rightarrow$ (1)



$$(R_a + L_a s)\frac{1}{K_t}\underline{T_m(s)} + K_b s\, \theta_m(s) = E_a(s) \quad \text{— (2)}$$

in terms of $\theta_m(s)$



Typical equivalent mechanical loading on a motor

Note

$J_m = J_a + J_L$ reflected to armature

$D_m = D_a + D_L$ " " "

$$T_m(s) = (J_m s^2 + D_m s)\,\theta_m(s) \longrightarrow \text{sub in (2)}$$

$$\frac{(R_a + sL_a)(J_m s^2 + D_m s)}{K_t}\theta_m(s) + K_b s\, \theta_m(s) = E_a(s)$$

assume $L_a \ll R_a$ (usual for DC motors)

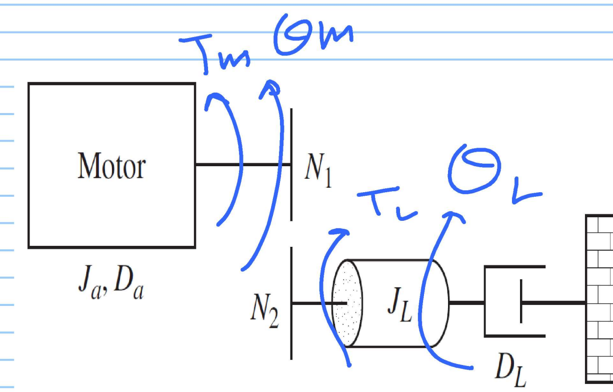
$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\,\Theta_m(s) = E_a(s)$$

$$\boxed{\frac{\Theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s\left[s+\frac{1}{J_m}\left(D_m+\frac{K_t K_b}{R_a}\right)\right]} = \frac{K}{s(s+\alpha)}}$$

How to evaluate mechanical & electrical constants?

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2$$

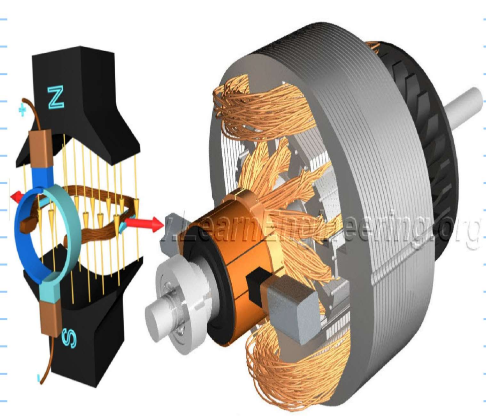
$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$



DC motor driving a rotational mechanical load

* $(K_t \;\&\; K_b)$?

can be obtained through dynamometer test



$$\boxed{\frac{\omega_m}{E_a}}$$

Θ_m $\qquad$ ω_m ✓

$$\omega_m(t) = \frac{d}{dt}\Theta_m(t)$$

$$\omega_m(s) = s\,\Theta_m(s)$$

Dynamometer measures T & ω of the motor under the condition of a const. applied voltage e_a

from 2 $\qquad L_a = \text{zero}$

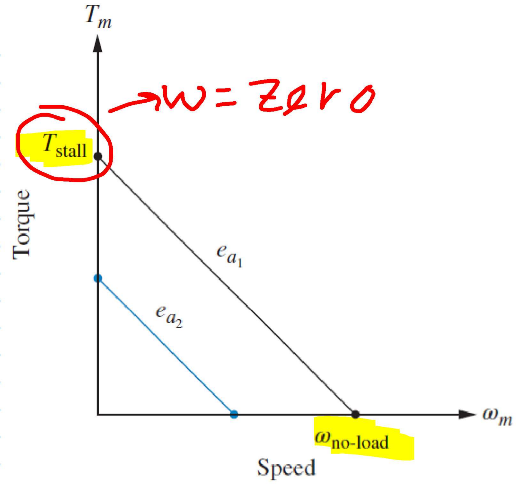
$$(R_a + \cancel{L_a s})\frac{1}{K_t}T_m(s) + K_b \underbrace{s\,\Theta_m(s)} = E_a(s)$$

$$\frac{R_a}{K_t} T_m(t) + K_b\, \omega_m(t) = e_a$$

→ under the S.S. condition

$$\frac{R_a}{K_t} T_m + K_b\, \omega_m = e_a$$

$$\boxed{T_m = -\frac{K_b K_t}{R_a}\,\omega_m + \frac{K_t}{R_a} e_a}$$



Torque-speed curves with an armature voltage, e_a, as a parameter

$$T_{stall} = \frac{K_t}{R_a} e_a$$

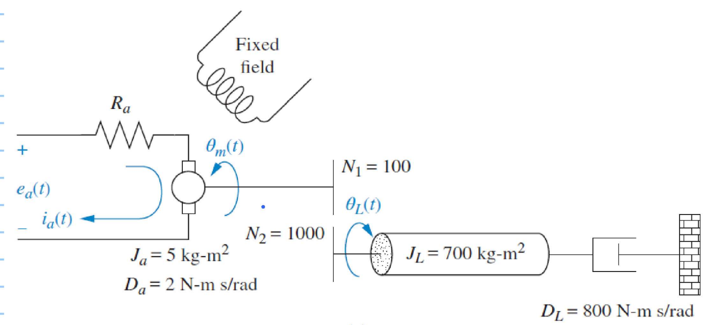
$$\boxed{\frac{K_t}{R_a} = \frac{T_{stall}}{e_a}} \quad *$$

$$\omega_{nl} = \frac{e_a}{K_b}$$

$$\boxed{K_b = \frac{e_a}{\omega_{nl}}} \quad **$$

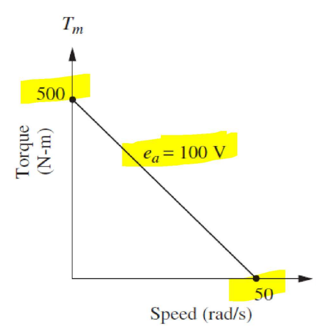
Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.



(a)

$$\begin{cases} J_m = 5 + 700\left(\frac{100}{1000}\right)^2 = 12 \\ D_m = 2 + 800\left(\frac{100}{1000}\right)^2 = 10 \end{cases}$$



(b)

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5$$

$$K_b = \frac{e_a}{\omega_{nL}} = \frac{100}{50} = 2$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K_t/R_a J_m}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{0.417}{s(s+1.667)}$$

$$\Theta_m(s) \longleftrightarrow \Theta_L(s)$$

$$\frac{\Theta_m(s)}{\Theta_L(s)} = \frac{N_2}{N_1} = 10 \Rightarrow \Theta_m(s) = 10\,\Theta_L(s)$$

$$\frac{10\,\Theta_L(s)}{E_a(s)} = \frac{0.417}{s(s+1.667)}$$

$$\boxed{\frac{\Theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}}$$

$\omega \rightarrow$ rad/sec.

$n \longrightarrow$ rpm

$1 \text{ rpm} \longleftrightarrow \frac{2\pi}{60}$ rad/sec.

$1 \text{ rad/s} \longleftrightarrow \frac{60}{2\pi}$ rpm