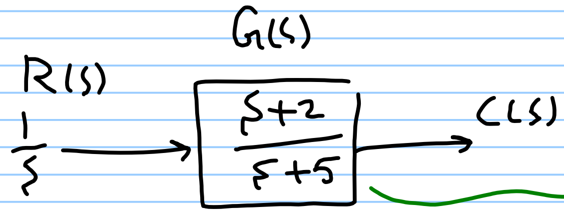


$s = j\omega$

DC gain $= \frac{2}{5}$

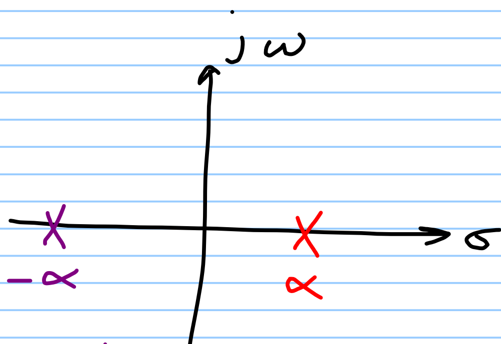
$$C(s) = R(s)\,G(s)$$

$$= \frac{s+2}{s(s+5)} = \frac{\cancel{A}\ \frac{2}{5}}{s} + \frac{\cancel{B}\ +\frac{3}{5}}{s+5}$$

$$C(t) = \underbrace{\frac{2}{5}}_{\text{Forced Response}} + \underbrace{\frac{3}{5}e^{-5t}}_{\text{Natural Response}}$$

Forced Response — INPUT

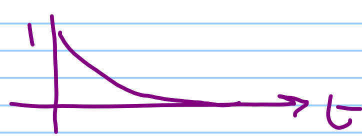
Natural Response — system (R, L, C)



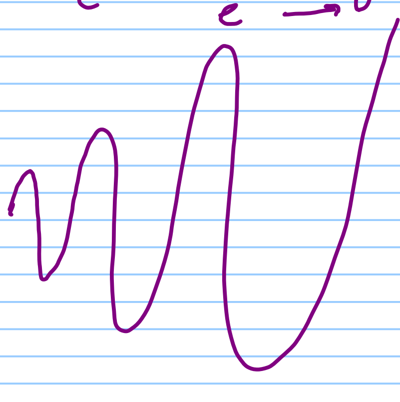
$e^{\alpha t}$ $\quad$ e^{10t}, $t \to \infty$, $e^{10 \cdot 6} \to \infty$ — unstable

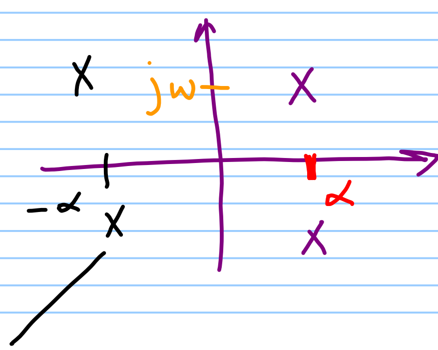
$e^{-\alpha t}$

e^{-10t} $\quad$ $t \to \infty$, $e^{-10t} \to 0$



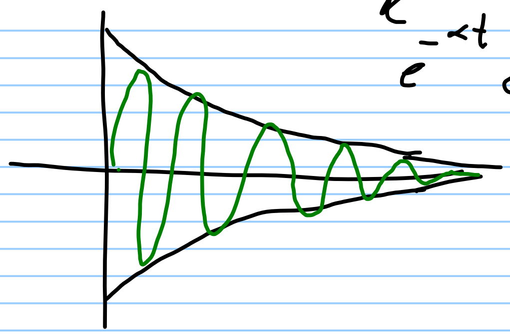
stable

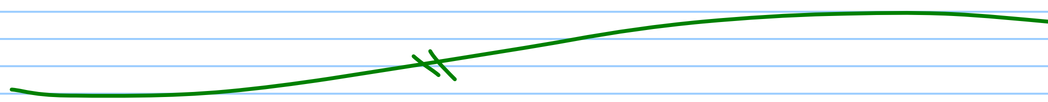




$e^{\alpha t} \cos \omega t$

$e^{-\alpha t}\cos\omega t$

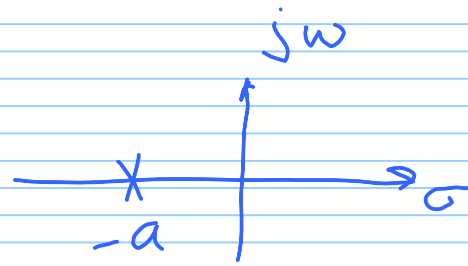




4.3 First-order systems

$G(s) = \frac{a}{s+a}$ $\qquad \frac{10}{s+10} \qquad \frac{5}{s+5} \qquad \frac{1}{s+5}$

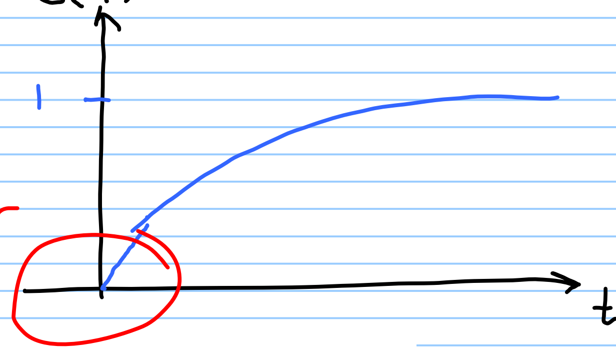
$$\text{DC gain} = \frac{a}{a} = \boxed{1}$$



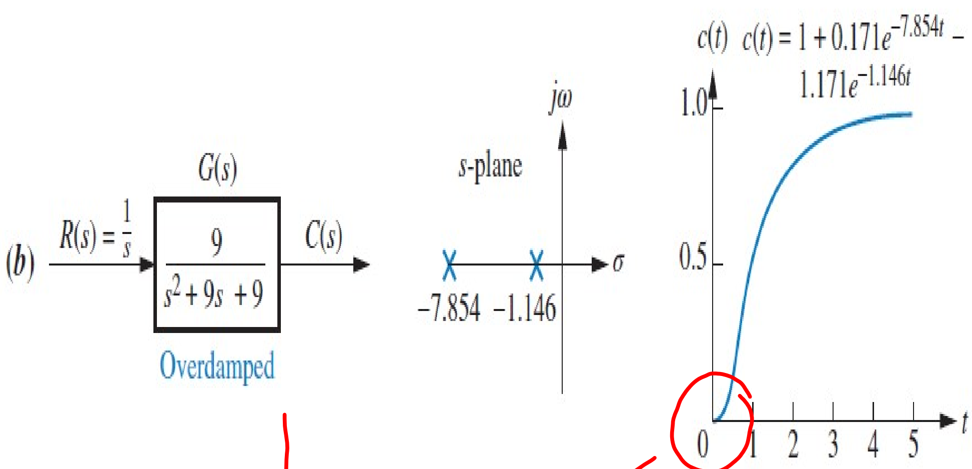
input $\frac{1}{s}$

output $C(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$

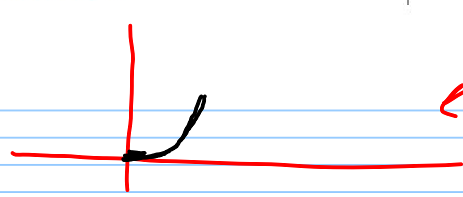
$c(t) = 1 - e^{-at}$



1^{st} order



2^{nd} order



T_s

$$c(t) = 0.98$$

$$1 - e^{-at} = 0.98$$

$$\ln\left(e^{-at} = 0.02\right)$$

$$-at = \ln(0.02)$$

$$-at = -3.912$$

$$t = \frac{3.912}{a} \approx \frac{4}{a} = T_s$$

$$\frac{a}{s+a} \qquad \boxed{\frac{K}{s+a}}$$

X

input $\frac{1}{s}$

1st order

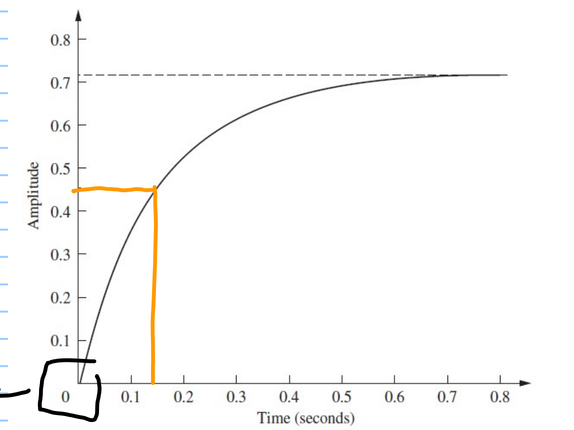


FIGURE: Laboratory results of a system step response test

→ final value (forced response)

$$\boxed{\frac{K}{a} = 0.72}$$

$$\boxed{G(s) = \frac{5.54}{s+7.7}}$$

→ time constant, $\frac{1}{a}$

$$t_{@63\% \text{ of } 0.72} = \frac{1}{a}$$

$$\boxed{K = 5.54}$$

$$t_{@0.45} = \frac{1}{a}$$

$$0.13 = \frac{1}{a}$$

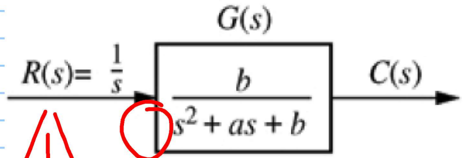
$$\boxed{a = \frac{1}{0.13} = 7.7}$$

0.14

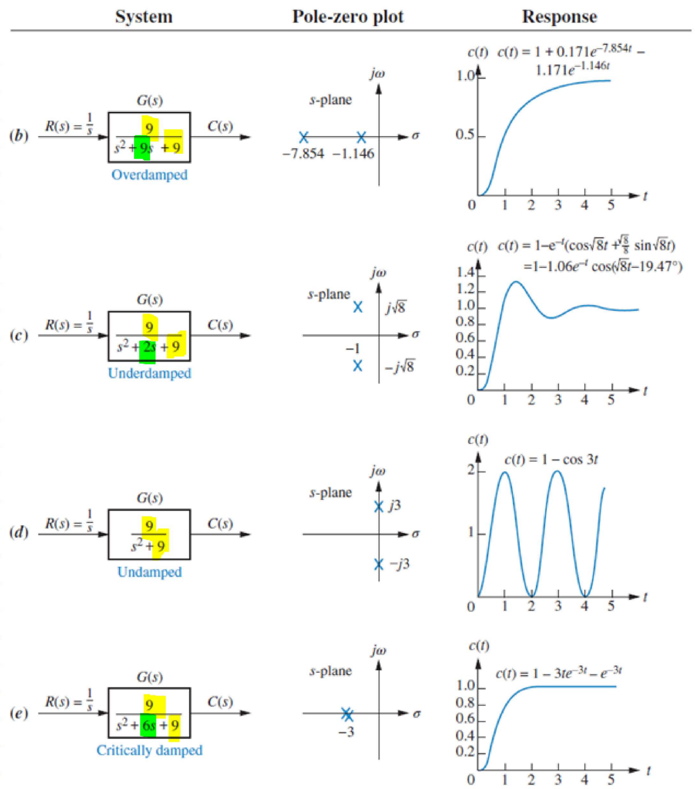
$\boxed{K = 5.14}$

$\boxed{a = 7.14}$

4.4 Second-Order Systems: Introduction



$\triangle = 1$



(b) $C(s) = \dfrac{9}{s(s+7.854)(s+1.146)}$

(c) $C(s) = \dfrac{9}{s(s^2+2s+9)}$

$$\frac{9}{s(s^2+2s+9)} = \frac{1\,\cancel{A}}{s} + \frac{Bs+C}{s^2+2s+9}$$

$$9 = s^2+2s+9 + Bs^2+Cs$$

$\boxed{B=-1}$ $\boxed{C=-2}$

$\cos \quad \dfrac{s}{s^2+\omega^2}$

$\sin \quad \dfrac{\omega}{s^2+\omega^2}$

$$C(s) = \frac{1}{s} - \frac{s+2}{(s^2+2s+1)+9-1} = \frac{1}{s} - \frac{s+\cancel{2}\ {+1+1}}{(s+1)^2+8}$$

$$= \frac{1}{s} - \left[\frac{(s+1)+1}{(s+1)^2+(\sqrt{8})^2}\right]$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1)^2+(\sqrt{8})^2} - \frac{1}{\sqrt{8}}\,\frac{\sqrt{8}}{(s+1)^2+(\sqrt{8})^2}$$

$$c(t) = 1 - e^{-t}\cos\sqrt{8}t - \frac{1}{\sqrt{8}}e^{-t}\sin\sqrt{8}\,t$$

$$= 1 - e^{-t}\left[\cos\sqrt{8}t - \frac{1}{\sqrt{8}}\sin\sqrt{8}t\right]$$

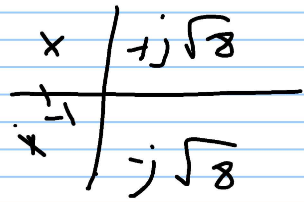
$$= 1 - e^{-t}\left[\cos\sqrt{8}t - \frac{1}{\sqrt{8}}\cos(\sqrt{8}t - 90^\circ)\right]$$

$$c(t) = 1 - 1.06e^{-t}\cos(\sqrt{8}\,t - 19.47^\circ)$$

$$1\angle 0^\circ - \frac{1}{\sqrt{8}}\angle -90^\circ$$

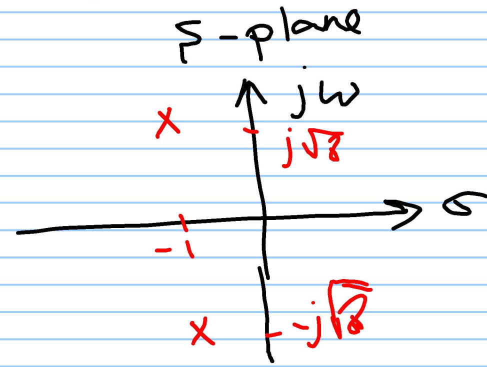
$$1 - j\frac{1}{\sqrt{8}}$$

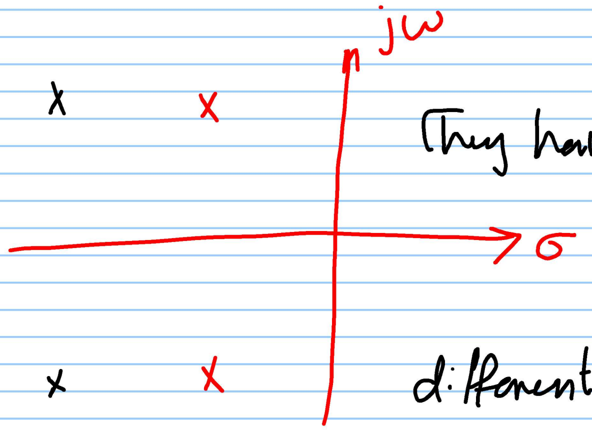
$$1.06\angle -19.47^\circ$$



$$\frac{9}{s^2+2s+9}$$

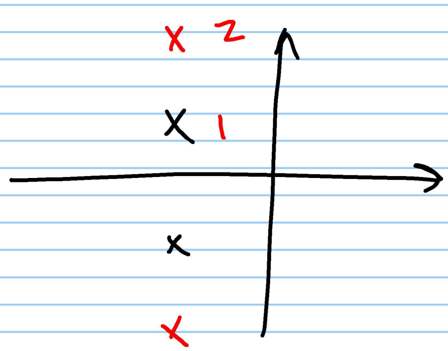
Poles:- $\dfrac{-2 \pm \sqrt{4 - 4\times 9}}{2} = \boxed{-1} \pm j\boxed{\sqrt{8}}$





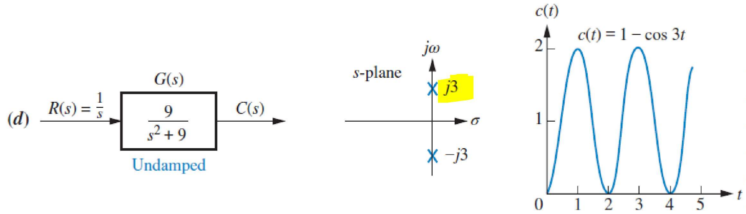
They have the same sinusoidal freq. ω_d

different exponential freq. σ



$\sigma_1 = \sigma_2$

$\omega_{d_1} < \omega_{d_2}$



$$C(s) = \frac{9}{s(s^2+9)}$$

$s_{1,2} = \pm j3$

$\rightarrow$ No real part $\boxed{e^0 = 1}$

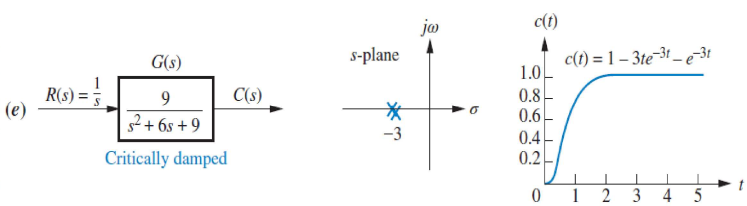
$$C(s) = \frac{1}{s} + \frac{Bs+C}{s^2+9}$$

$$= \frac{1}{s} - \frac{s}{s^2+9}$$

$$9 = s^2 + 9 + Bs^2 + Cs$$

$B = -1, \; C = 0$

$$c(t) = 1 - \cos 3t$$



$s_{1,2} = -3$

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$$

$$\left(\frac{9}{s(s+3)^2} = \frac{1}{s} + \frac{B}{(s+3)^2} + \frac{C}{s+3} \right) s(s+3)^2$$

$$9 = s^2 + 6s + 9 + Bs + C(s^2 + 3s)$$

$1 + C = 0$ $\boxed{C = -1}$

$6 + B + 3C = 0$

$6 + B - 3 = 0$ $\boxed{B = -3}$

$$C(s) = \frac{1}{s} - \frac{3}{(s+3)^2} - \frac{1}{s+3}$$

$\therefore c(t) = 1 - 3e^{-3t}\, t - e^{-3t}$

4.5 The general second-order system

- Natural frequency, ω_n
  - The frequency of oscillation of the system without damping
- Damping ratio, ζ

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/s)}} = \frac{1}{2\pi}\frac{\text{Natural period (s)}}{\text{Exponential time constant}}$$

- General TF

$$\boxed{G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

where

$$a = 2\zeta\omega_n, \quad b = \omega_n^2, \quad \zeta = \frac{a}{2\omega_n}, \quad \omega_n = \sqrt{b}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$c(t) \rightarrow$ step response of the general 2^{nd} order system

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\cancel{A}\,1}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

$\boxed{B = -1}$ $\boxed{C = -2\zeta\omega_n}$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$= \frac{1}{s} - \left[\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2} + \frac{\zeta}{\sqrt{1-\zeta^2}}\,\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]$$

$$\boxed{c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$\boxed{1 - j\frac{\zeta}{\sqrt{1-\zeta^2}}}$$

$$\sqrt{1^2 + \frac{\zeta^2}{1-\zeta^2}}$$

$$\sqrt{\frac{1-\cancel{\zeta^2}+\cancel{\zeta^2}}{1-\zeta^2}}$$

$$\frac{1}{\sqrt{1-\zeta^2}}$$

$9 = 2 \times \zeta \times 3$

$s^2 + 2\zeta\omega_n s + \omega_n^2$

$\omega_n = 3$

$\zeta = \frac{9}{6} = 1.5$

$2\zeta \times 3 = 2$

$\omega_n = 3$

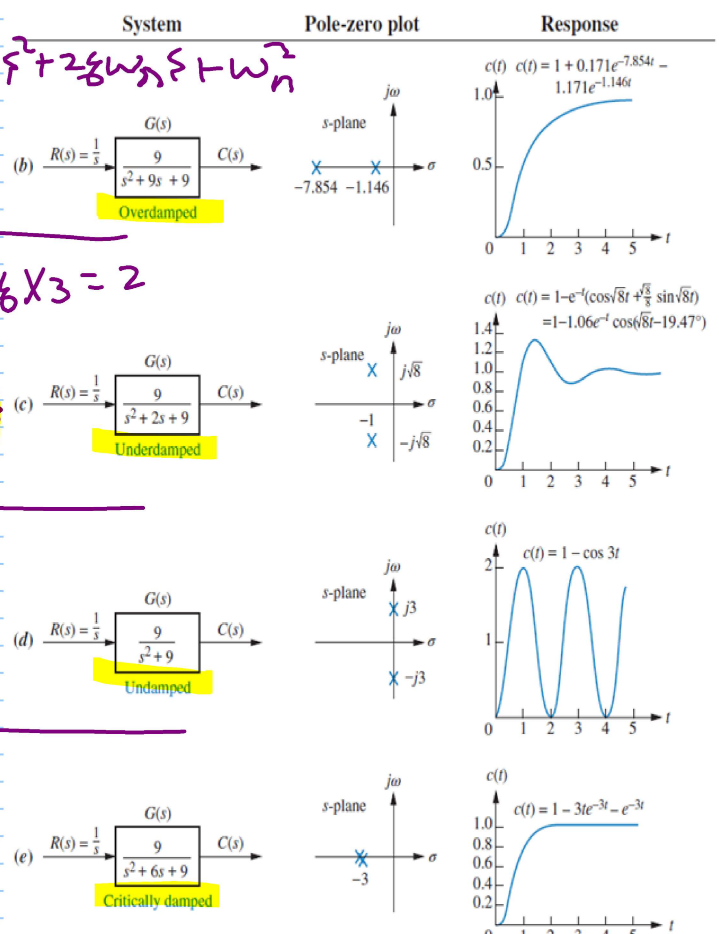
$\zeta = \frac{1}{3} = 0.333$

$\omega_n = 3$

$\zeta = 0$

$\omega_n = 3$

$\zeta = 1$

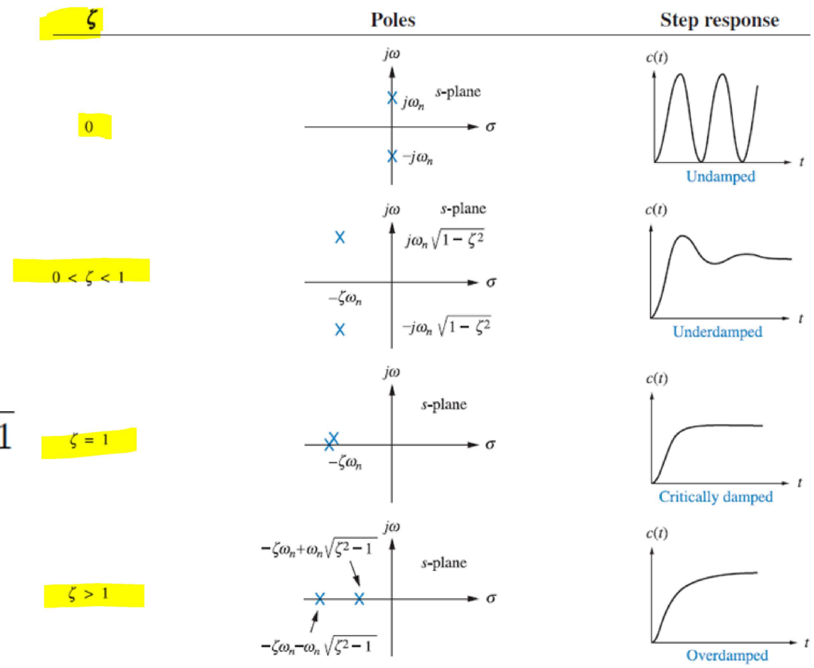


4.5 The general second-order system

Response as a function of ζ

Poles

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

if underdamped $0 < \zeta < 1$

σ_d

$$s_{1,2} = \underbrace{-\zeta\omega_n}_{e^{-(\ )t}} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\cos(\ )t} \qquad \omega_d$$

4.6 Underdamped second-order systems

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t} \cos\left(\omega_n \sqrt{1-\xi^2}\, t - \phi\right)$$

$$\phi = \tan^{-1} \frac{\xi}{\sqrt{1-\xi^2}}$$

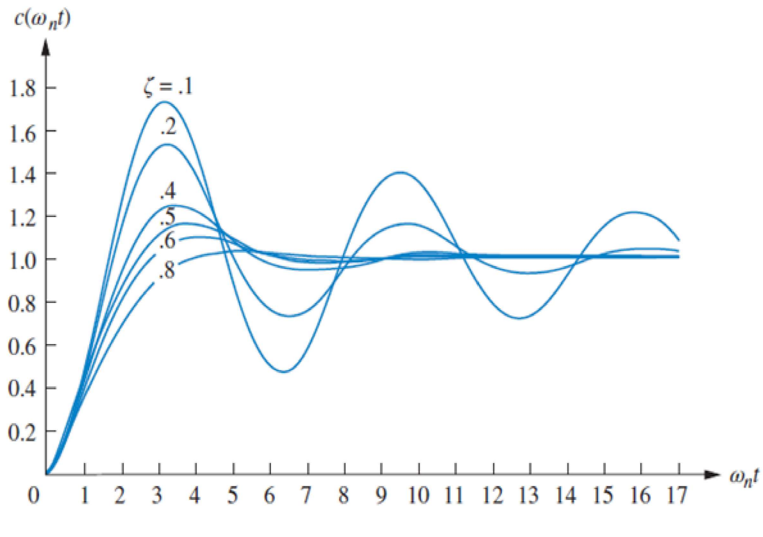
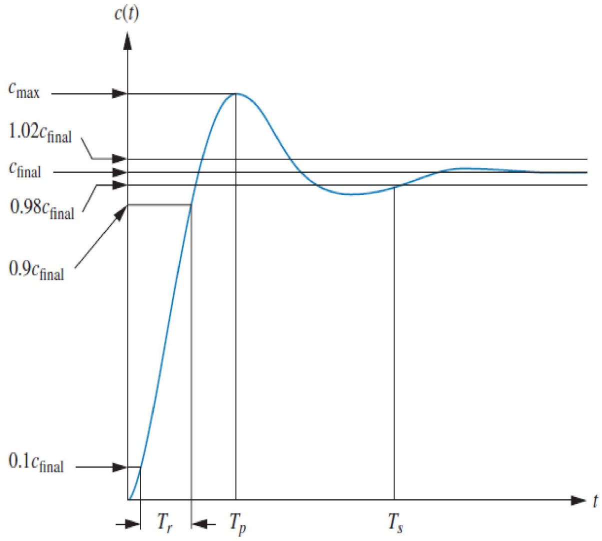


Figure: 2nd-order underdamped responses for damping ratio values



- **Rise time, T_r:** Time required for the waveform to go from 0.1 of the final value to 0.9 of the final value
- **Peak time, T_p:** Time required to reach the first, or maximum, peak
- **Overshoot, %OS:** The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady state value
- **Settling time, T_s:** Time required for the transient's damped oscillations to reach and stay within ±2% of the steady state value

Evaluation of T_p peak time

$$\mathcal{L}\left[\frac{d}{dt}c(t)=0\right] = s\,C(s)$$

$$= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\xi^2\omega_n^2+\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{\omega_n^2}{(s+\xi\omega_n)^2+\left(\omega_n\sqrt{1-\xi^2}\right)^2}$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}}\frac{\omega_n\sqrt{1-\xi^2}}{(s+\xi\omega_n)^2+\left(\omega_n\sqrt{1-\xi^2}\right)^2}$$

$$\therefore \frac{d}{dt}c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\underbrace{\omega_n\sqrt{1-\xi^2}}\right)t$$

$$\frac{d}{dt}c(t)=0$$

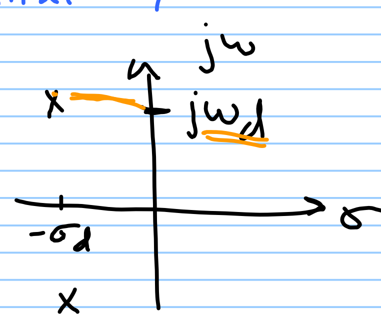
$$\omega_n\sqrt{1-\xi^2}\,t = n\pi$$

$$t = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$$

$n=0 \rightarrow \underline{t=0}$
initial slope $=0$

$\boxed{n=1}$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} = \frac{\pi}{\omega_d}$$



%OS

$$\%OS = \frac{C_{max} - C_{final}}{C_{final}} \times 100$$

$\rightarrow C_{final} = 1$

$\rightarrow C_{max} = C(T_p)$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_n\sqrt{1-\xi^2})t + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_n\sqrt{1-\xi^2})t\right]$$

$$C_{max} = 1 - e^{-\xi \cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\xi^2}}}\left[\underbrace{\cos(\pi)}_{-1} + \frac{\xi}{\sqrt{1-\xi^2}}\underbrace{\sin(\pi)}_{0}\right]$$

$$C_{max} = 1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\boxed{\%OS = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100}$$

function of ξ only!!!

$$\boxed{\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}}$$

allows to solve for ξ for a given %OS

→ Settling time T_s

time it takes for the amplitude of the decaying sinusoid $= 0.02$

$$\frac{1}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} = 0.02$$

$$e^{-\xi\omega_n T_s} = 0.02\sqrt{1-\xi^2}$$

$$-\xi\omega_n T_s = \ln\left(0.02\sqrt{1-\xi^2}\right)$$

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n}$$

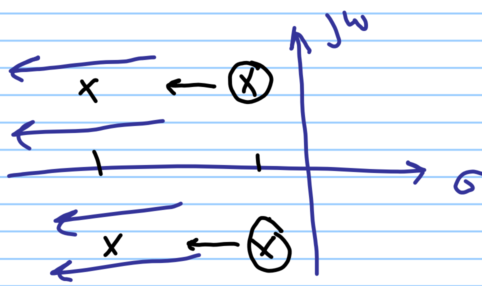
ξ: 0.001, 0.99

+3.91
4.7
5.8

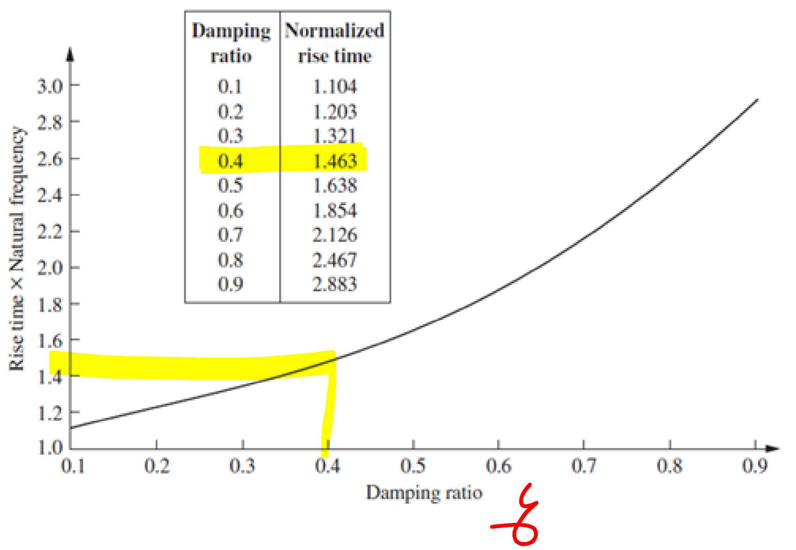
$$\boxed{T_s \cong \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}}$$

$$s_{1,2} = \sigma_d \pm j\omega_d$$

$$\boxed{T_p = \frac{\pi}{\omega_d}}$$



Rise Time T_r ⟶ from tables



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

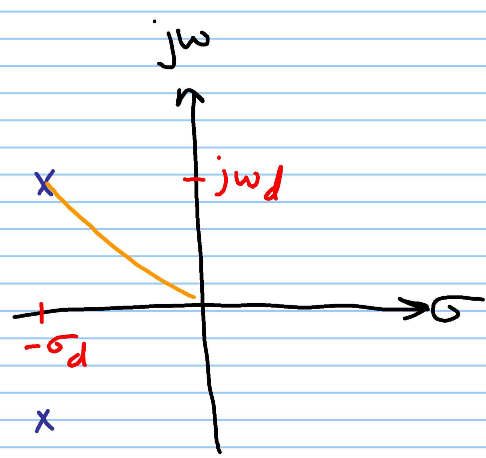
Figure: Normalized T_r vs. ζ for a 2nd-order underdamped response

$\zeta = 0.4$, $\therefore \boxed{T_r \times \omega_n = 1.463}$

$$T_r = \frac{1.463}{\omega_n} \quad (\text{when } \zeta = 0.4)$$

Location of poles

$$\zeta = \frac{\sigma_d}{\omega_n}$$



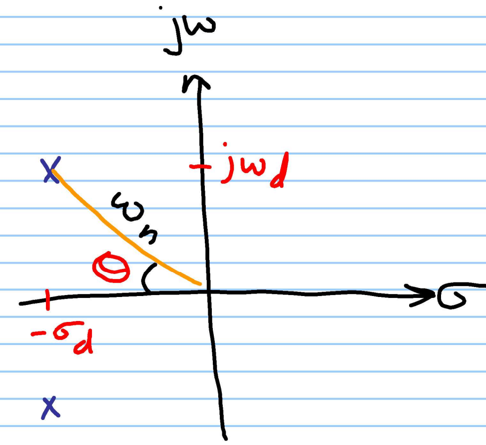
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\sigma_d \pm j\omega_d$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} , \quad \%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} , \quad T_r \text{ tables}$$

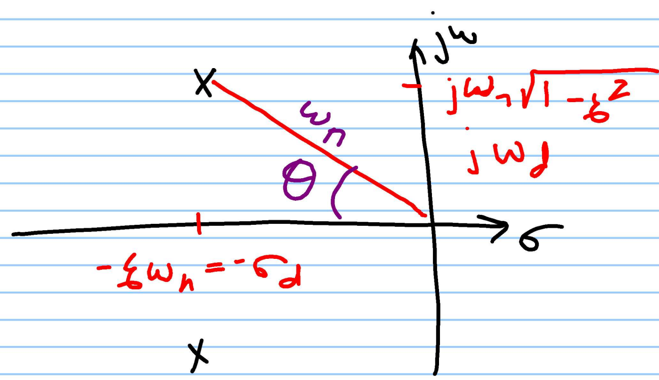
$$\sqrt{\sigma_d^2+\omega_d^2}$$



$$\sqrt{(\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

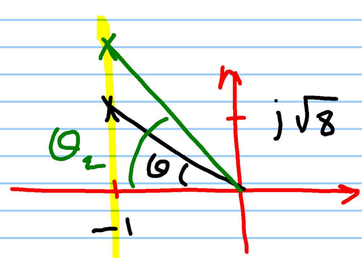
$$\sqrt{\zeta^2\omega_n^2+\omega_n^2(1-\zeta^2)}$$

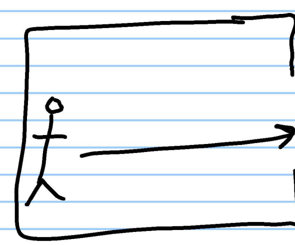
$$\sqrt{\cancel{\zeta^2\omega_n^2}+\omega_n^2-\cancel{\omega_n^2\zeta^2}}=\omega_n$$

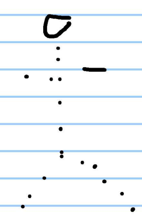


$$\boxed{\boxed{\zeta=\cos\theta}}$$

$\hookrightarrow \cos\theta = \dfrac{|-\zeta\omega_n|}{\omega_n} = \zeta$

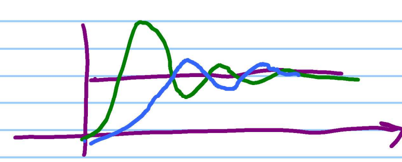






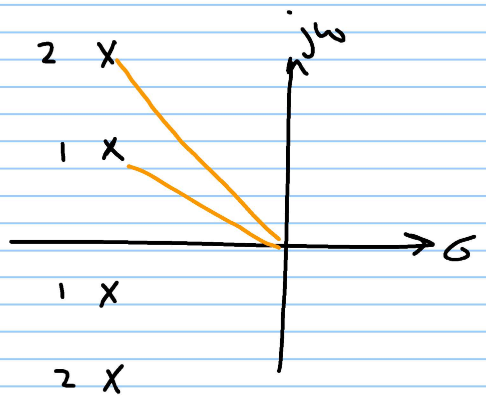
$$\zeta=\cos\theta=\frac{1}{\omega_n}=\frac{1}{\sqrt{1+8}}=\frac{1}{\sqrt{9}}=\frac{1}{3}$$

$\theta_2 > \theta_1$



$\cos\theta_2 < \cos\theta_1$

$\zeta_2 < \zeta_1 \Rightarrow \%OS_2 > \%OS_1$



$\hookrightarrow \sigma_{d_1} = \sigma_{d_2}$

$\therefore T_{s_1} = T_{s_2}$

$\hookrightarrow \omega_{d_1} < \omega_{d_2}$

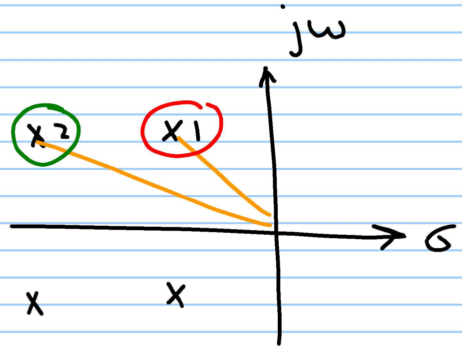
$T_{p_1} > T_{p_2}$

$\hookrightarrow \theta_1 < \theta_2$

$\cos\theta_1 > \cos\theta_2$

$\xi_1 > \xi_2$

$\%OS_1 < \%OS_2$



$\hookrightarrow \omega_{d_1} = \omega_{d_2}$

$T_{p_1} = T_{p_2}$

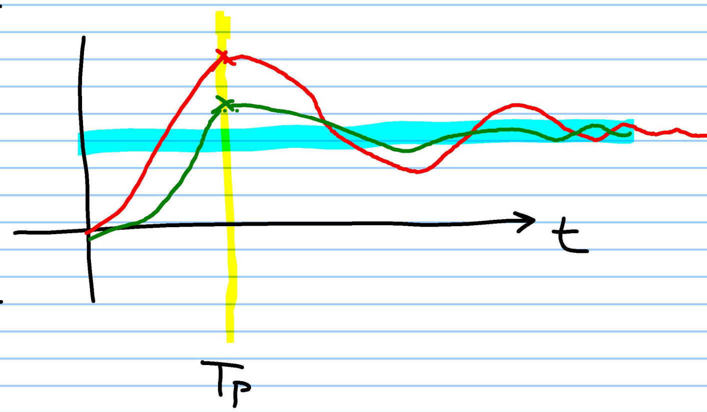
$\hookrightarrow \sigma_{d_1} < \sigma_{d_2}$

$T_{s_1} > T_{s_2}$

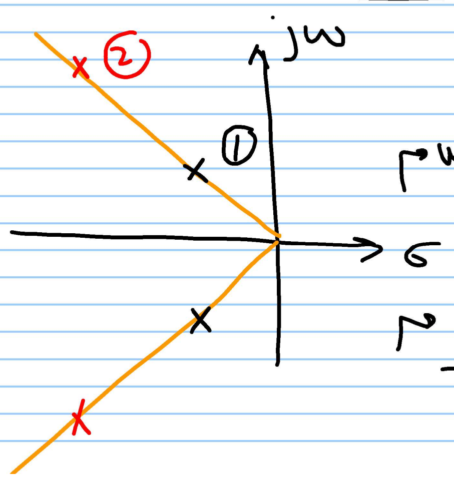
$\hookrightarrow \theta_1 > \theta_2$

$\xi_1 < \xi_2$

$\%OS_1 > \%OS_2$



$\xi = \cos\theta$



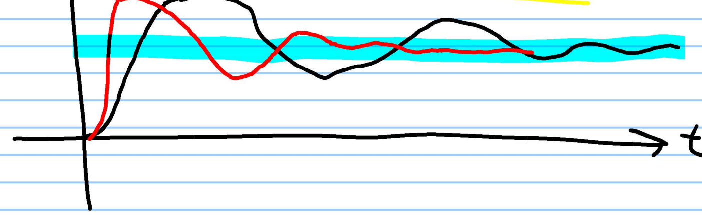
$\theta_1 = \theta_2 \rightarrow \xi_1 = \xi_2 \rightarrow \%OS_1 = \%OS_2$

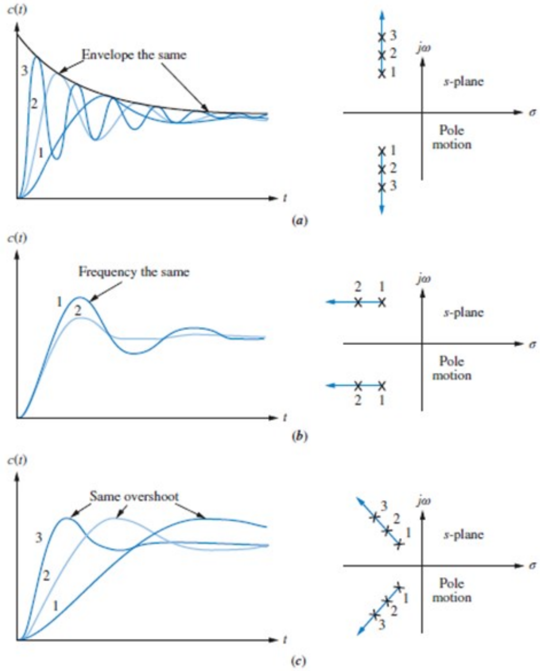
$\omega_{d_1} < \omega_{d_2}$

$T_{p_1} > T_{p_2}$

$\sigma_{d_1} < \sigma_{d_2}$

$T_{s_1} > T_{s_2}$





Finding T_p, %OS, and T_s from Pole Location

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

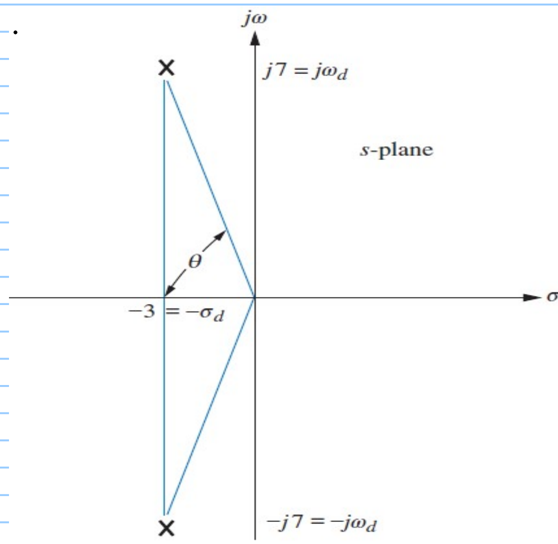


FIGURE 4.20 Pole plot for Example 4.6

$\sigma_d = 3$

$\omega_d = 7$

$\zeta = \cos\theta = \cos\left(\tan^{-1}\frac{7}{3}\right)$

$= 0.394$

$\omega_n = \sqrt{3^2 + 7^2} = 7.616$

$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449$ sec.

$T_s = \frac{4}{\sigma_d} = \frac{4}{3}$ sec.

$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 26\%$

Example 4.7

Transient Response Through Component Design

Design
D

PROBLEM: Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.

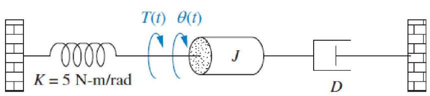


FIGURE 4.21 Rotational mechanical system for Example 4.7

$$(Js^2 + Ds + 5)\,\Theta(s) = T(s)$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$G(s) = \frac{1}{Js^2 + Ds + 5}$$

$$\cancel{\omega_n = \sqrt{5}}$$

$$= \frac{1/J}{s^2 + \frac{D}{J}s + \frac{5}{J}}$$

$$\omega_n = \sqrt{\frac{5}{J}} \quad , \quad 2\zeta\omega_n = \frac{D}{J}$$

$$\rightarrow T_s = 2 = \frac{4}{\zeta\omega_n} \Rightarrow \boxed{\zeta\omega_n = 2}$$

$$\boxed{\frac{D}{J} = 4}$$

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

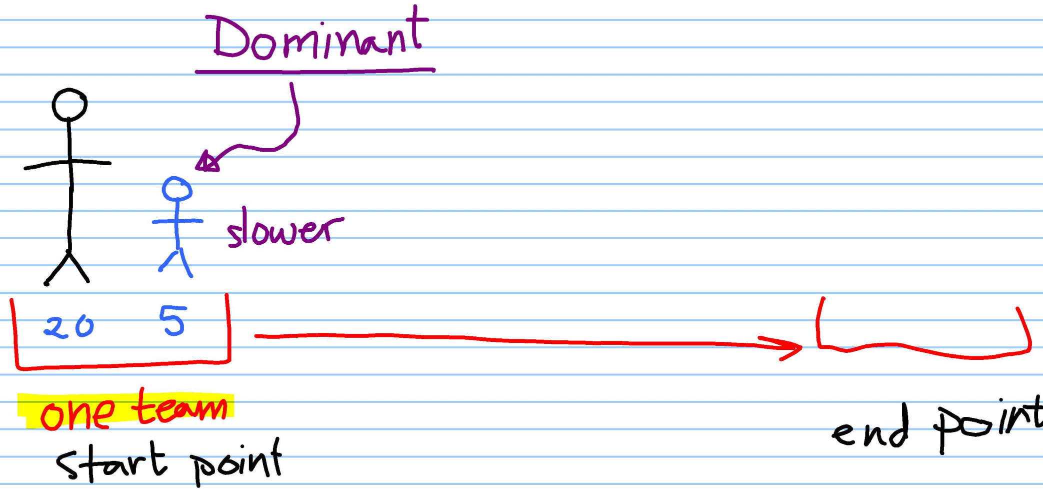
$$20 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

$$\ln\left(\frac{20}{100} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}\right)$$

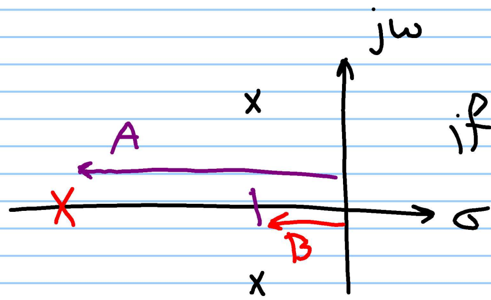
$$-1.61 = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \Rightarrow \boxed{\zeta = 0.456}$$

$$\rightarrow \zeta\omega_n = 2 \quad \therefore \boxed{\omega_n = \frac{2}{\zeta} = 4.386 \text{ rad/sec.}}$$

$$\rightarrow \omega_n = \sqrt{\frac{5}{J}} \quad \boxed{J = 0.26\ \text{kg}\cdot\text{m}^2} \ \& \ \boxed{D = 4J = 1.04\ \text{N m s/rad}}$$



4.7 System response with additional poles



if $\frac{A}{B} > 5$

then the 2^{nd}-order approximation is valid

Example 4.8

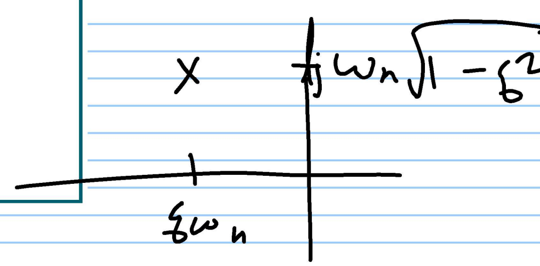
Comparing Responses of Three-Pole Systems

PROBLEM: Find the step response of each of the transfer functions shown in Eqs. (4.62) through (4.64) and compare them.

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542} \tag{4.62}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)} \quad \checkmark \tag{4.63}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)} \quad \times \tag{4.64}$$



$T_2 / \ \frac{A}{B} = \frac{10}{2} = 5$

$T_3 / \ \frac{A}{B} = \frac{3}{2} = 1.5$