Root Locus

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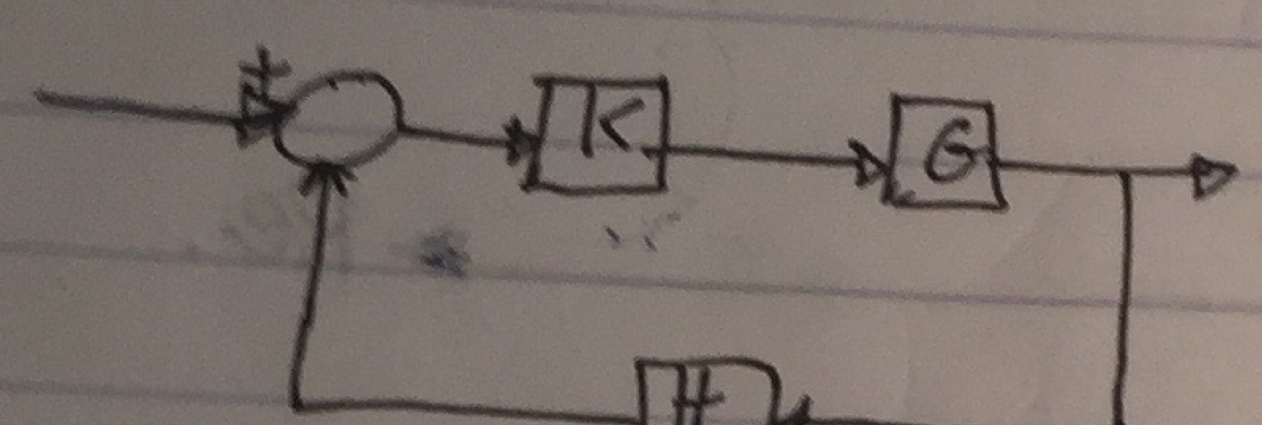
ROOT LOCUS TECHIQUES

Root locus : The set of all possible ~~closed Loop poles~~ that can be obtained by the variation of a proportial gain parameter with the open Loop function of the system.

$1+KGH=0 \quad , P=P(K).$



Root locus :- هي مجال الـ Poles الذي أحصل عليه من الـ closed loop

$1+KGH=0$

$GH=\frac{-1}{K} \quad , \quad \bar{S}\in L \Longleftrightarrow GH(\bar{S})=\frac{-1}{K}$

Pole zero form.

$GH=\frac{\prod_{i=1}^{m}(S+Z_i)}{\prod_{i=1}(S+P_i)}$ → open loop function.

$gh=\frac{\prod(1+\tau_{Z_i}S)}{\prod(1+\tau_{P_i}S)}$ → time constant form.

Test on open loop function:

$GH(\bar{S})=\frac{-1}{K}$

شرط على الزاوية. $\angle GH=\begin{cases}(2\nu+1)\pi , & K>0\\ 2\nu\pi , & K<0\end{cases}$

شرط على القيمة $|GH|=\frac{1}{|K|}$ عندما نحدد النقطة في الـ locus فإننا بإمكاننا إيجاد قيمة K لحتى نصل إلى النقطة.

when $\bar{S}$: one of poles in root locus.

every point satisfy the equation

$GH(\bar{S})=\frac{-1}{K}$

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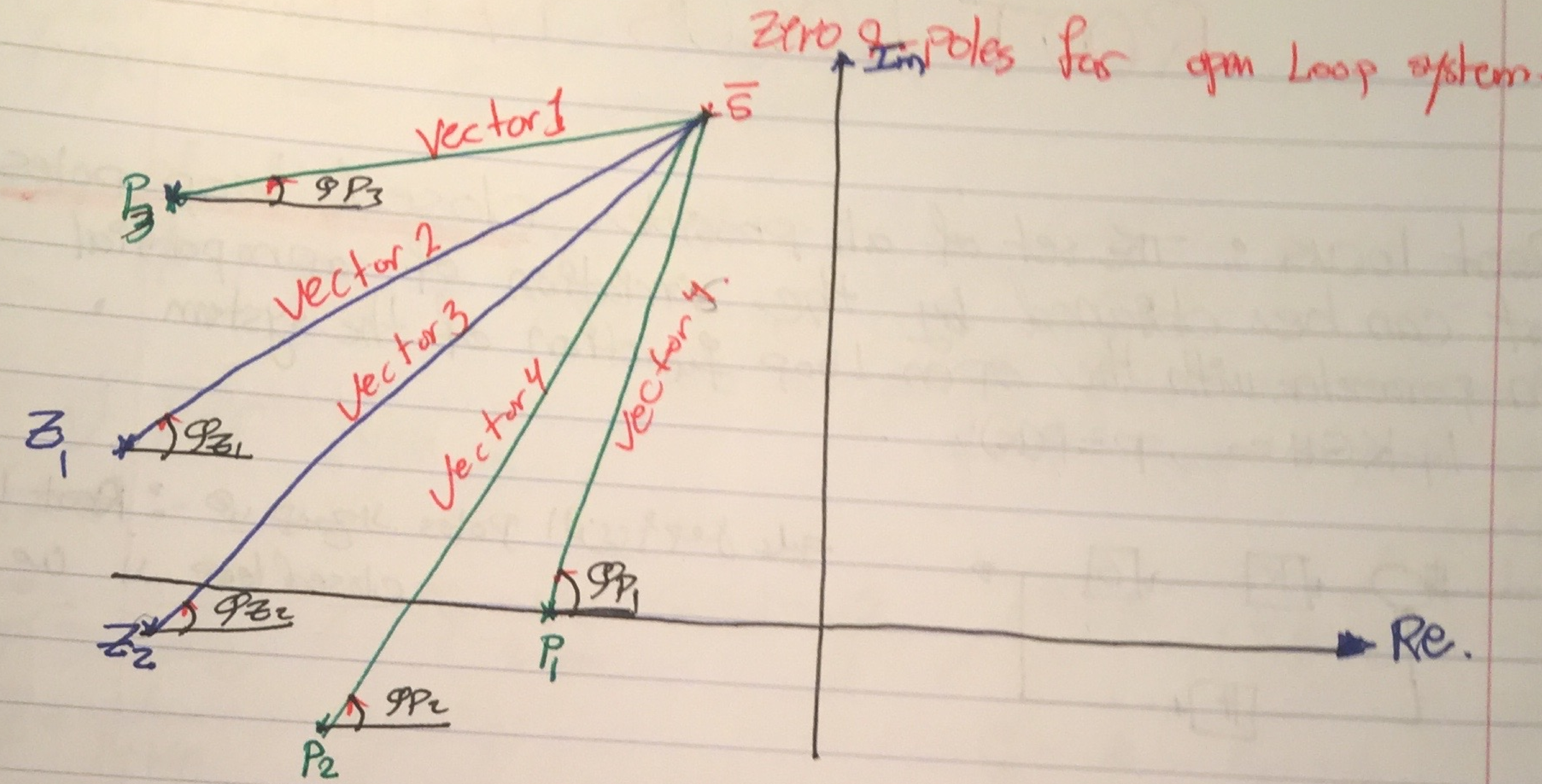


$$G(\bar{s})H(\bar{s}) = \frac{\prod_{i=1}^{m}(s+Z_i)}{\prod_{i=1}^{n}(s+P_i)}$$

زاویه pole ها را [illegible]

با استفاده از [illegible] s

$\tan^{-1}(\varphi_z) - \tan^{-1}(\varphi_p)$

when $K>0$

$$\varphi_{Z_1} + \varphi_{Z_2} - \varphi_{P_1} - \varphi_{P_2} - \varphi_{P_3} = (2\nu+1)\pi$$

?? if yes $\rightarrow \bar{s} \in L$

if No $\rightarrow \bar{s} \notin L$

when $K<0$

$$\varphi_{Z_1} + \varphi_{Z_2} - \varphi_{P_1} - \varphi_{P_2} - \varphi_{P_3} = 2\nu\pi \xrightarrow{\text{if yes}} \bar{s} \in L$$

$$\frac{\prod \rho_{Zi}}{\prod \rho_{Pi}} = \frac{1}{|K|} \Rightarrow |K| = \frac{\prod \rho_{Pi}}{\prod \rho_{Zi}}$$

product for poles vector

product for zero vector

K [illegible]

[illegible]

[illegible] like root locus.

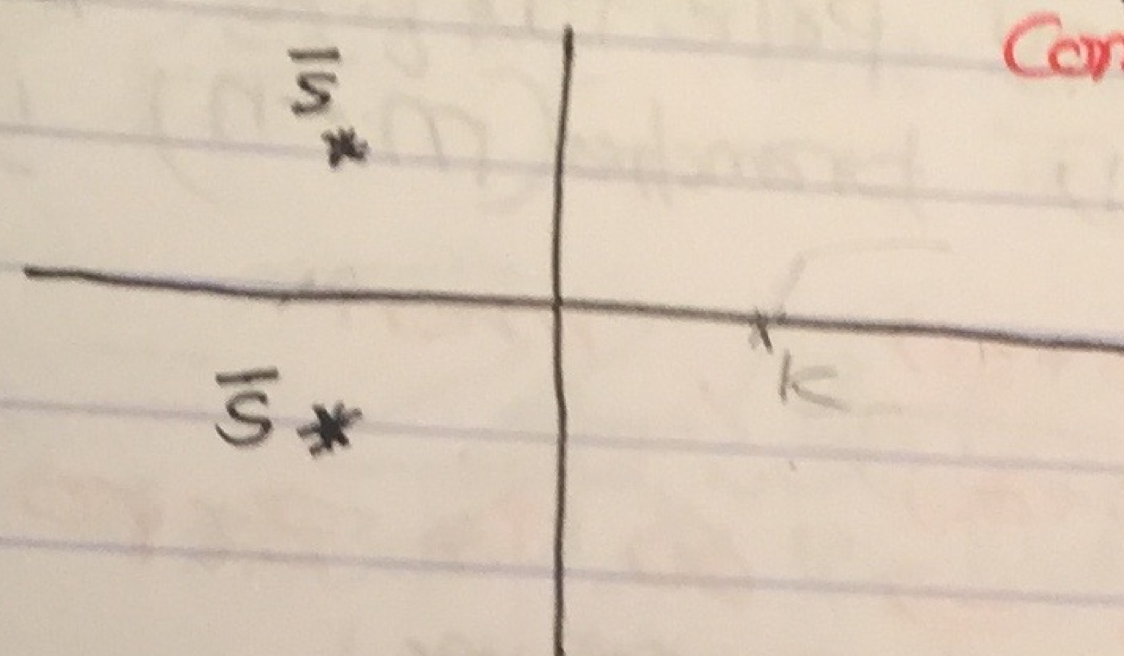
Properties of root Locus.

- The Root Locus has a number of branches equals to the maximume (m,n).

أكبر عدد من ال zero و ال pole هو نفس عدد ال branches.

* The root Locus is symmetric with respect to real axis

لأن في كل pole بيه complex pole and conjugate



* The points of the real axise that belong to the Locus are Twice

at real axis.

الي بيتركوا على يمينهم عدد فردي من (الدول pole و ال zero)

we have a complement on real axis.

1- That Leave an odd number of poles and zero to there right when $K > 0$. $K>0$

2- That Leave an even " " " " " " " " " $K < 0$. $K<0$

* يجب أن نعد ال pole وال zero على اليمين

* each branch departs from apole and terminates at zero at finite or infinite.

$$T(s) = \frac{\prod (s+z_i)}{\prod (s+p_i)}, \quad T(s) = 0 \Rightarrow \text{zeros}$$

m branch ⟶ m zeros at finite

n−m —zeros⟶ at infity asymptotically.

when $m>n$ [m−n ⟶ at infity poles.

ملاحظات على النقطة السابقة:

when $m < n$.

$$T(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+P_i)}$$

عدد ال branches = n

عدد ال zeros = m

- كل branch يخرج من pole فيجب أن ينتهي في ا zero

وباقي $(m-n)$ branches راح ينتهوا في ال ∞

لأن

$$\lim_{s\to\infty} \frac{\prod_{i=1}^{m}(s+z_i)}{\prod^{n}(s+z_i)}, \quad m<n$$

$= 0 \Rightarrow \infty$ is zero for $T(s)$

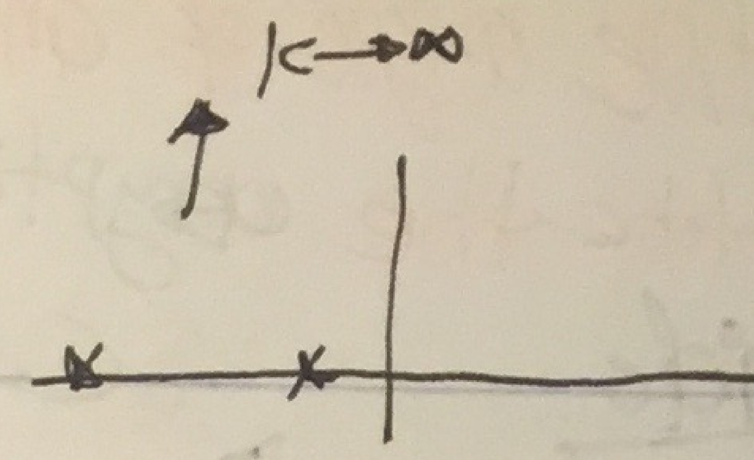
when $m > n$

at ∞ we have $(m-n)$ zero.

$$T(s) = \frac{\prod_{i=1}^{m}(S+Z_i)}{\prod_{i=1}^{n}(S+P_i)}.$$

$$T(s) = \lim_{s\to\infty} \frac{\prod^{m}(s+z_i)}{\prod^{n}(s+P_i)} = \infty \left\{ \text{at } \infty \text{ we have } (m-n) \text{ poles} \right.$$

- When we have Two poles.



الزاوية اللي بينطلق فيها.

- Calculation of departion from a pole P_i & Arraival to a Zero Z_i

for departion { $\varphi_{P_i} = \sum_{k=1} \measuredangle (P_k + P_i) + \sum_{k=1} \measuredangle (Z_k + P_i) = (2\nu+1)\pi$ } for $K > 0$

$\measuredangle P_i = \sum \measuredangle (Z_k + P_i) - \sum \measuredangle (P_k + P_i) + (2\nu+1)\pi$

الزاوية اللي بينطلق فيها.

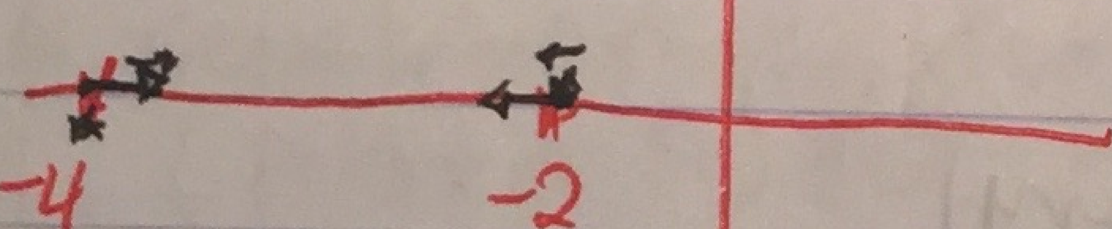
الزوايا بين ال P_i و ال Zeros الموجودة المتقدرة
نحسبها عن طريق $\tan^{-1}\left(\frac{Im}{rel}\right)$

الزوايا بين P_i وال poles نحسبها عن طريق $\tan^{-1}\left(\frac{Im}{rel}\right)$.

for arraival $\varphi_{Z_i} = \sum \measuredangle (P_k + Z_i) - \sum \measuredangle (Z_k + Z_i) + (2\nu+1)\pi$

الزوايا بين ال poles و ال Zero المقصود

الزوايا بين ال Zeros و ال Zero المقصود



$\varphi_{-2} = 0 + 0 + (2\nu+1)\cdot\pi = \pi$ one cycle.

$\varphi_{-4} =$

"Winter is Coming"

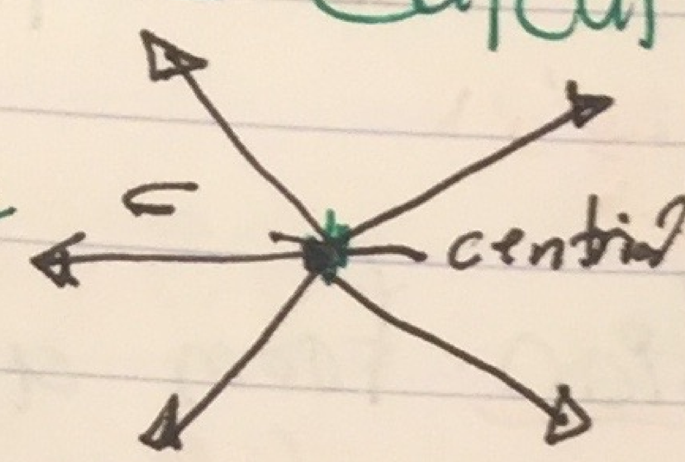
6

- When the angle of arrival at infinity, we want to calculate the assyptitic. (الزاوية التي يصل فيها عند ال ∞)

Centraide

- Calculation the asymptotes. (Direction)

Where the assymptotic cross each other.



نقطة الإلتقاء.

$n > m$

إلتقاء ال assyptote.

$$\boxed{\sigma} = \frac{\sum P_i - \sum Z_i}{n-m}$$

when $K > 0$

$$\varphi_\nu = \frac{(2\nu+1)\pi}{n-m}, \quad \nu = 0, 1, \ldots, n-m-1 \quad K > 0.$$

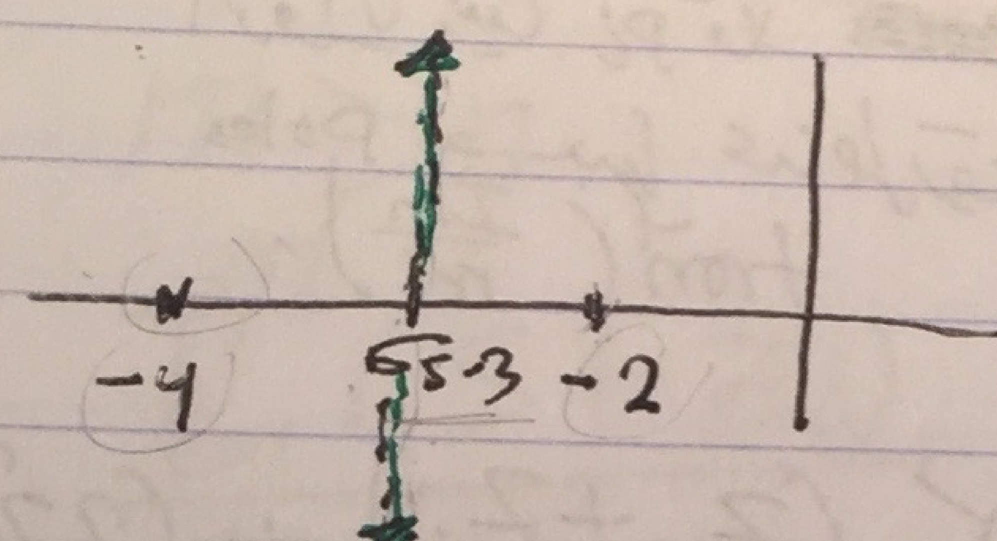
for ~~arrival~~ departion to infinite (not to zero)

(زاوية ال assyptotic على ال infinity.) φ_ν

when $K < 0$

$$\sigma = \frac{2\nu\pi}{n-m}$$

Ex



$$\sigma = \frac{-2-4-(0)}{2} = -3.$$

$$\varphi_0 = \frac{(2\nu+1)\pi}{2} = \frac{\pi}{2}.$$

$$\varphi_1 = -\frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad \text{(By symmetry)}$$

~~Branch~~

To calculate where we have a point branches. (نقطة التفرع) repeated poles.

$$F = 1 + GH = 0$$

$$\frac{dF}{ds} = 0 = \frac{dGH}{ds}$$

$$\frac{d^2F}{ds^2} \quad \vdots \quad \rightarrow \text{repeated solution.}$$

$$\vdots$$

$$\frac{d^{n-1}}{ds^{n-1}} = 0$$

or by.

$$\sum_{i=1}^{n} \frac{1}{(S_d + P_i)} - \sum_{i=1}^{m} \frac{1}{(S_d + Z_i)} = 0$$

pole قيمة ال

$$\frac{1}{s+2} + \frac{1}{s+4} - 0 = 0.$$

The calculation by $1 + GH$.

$(s+4+s+2) = 0 \rightarrow \boxed{s = -3}$ } the point of branches.

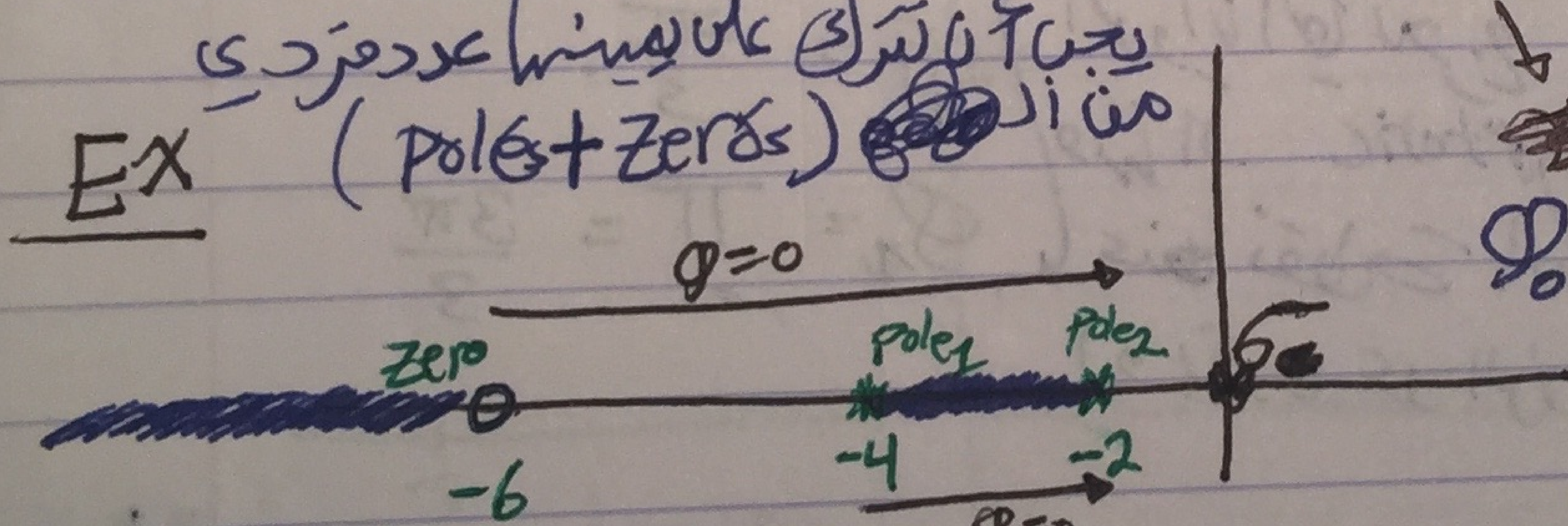
2-solutions اذا كان يوجد
يجب ان ناخذ التي تنتمي للمجال
zeros poles وعدد

- ليس شرطاً أن تكون σ تنتمي لل Locus.
- نقطة ال repeated poles هي نفس (التقاء ال branches)
نقطة ال Centroid (التقاء ال asymptotic) الذي تذهب لها الاسمبتوتيك
ولكن الذي يشترط أن تكون في ال Locouse.
هو S (ال repeated pole)

$\sigma_c \in I$ not necessary.

$S_d \in L$.

Ex يجب أن نترك على يمينها عدد فردي من ال (poles + zeros)



كم زاوية لل
$\ell (0, 1, 2, \ldots (n-m)-1)$.

$\varphi_0 = \frac{(2\ell+1)\pi}{1} = \pi$ (the angle of departion to infinity).

* each region have one repeated pole

1^{st} between $(-4 \rightarrow -2)$

2^{nd} between $(-\infty \rightarrow -6)$

$$\sigma_c = \frac{\sum P_i - \sum Z_i}{n-m} = \frac{-2-4+6}{1} = 0$$

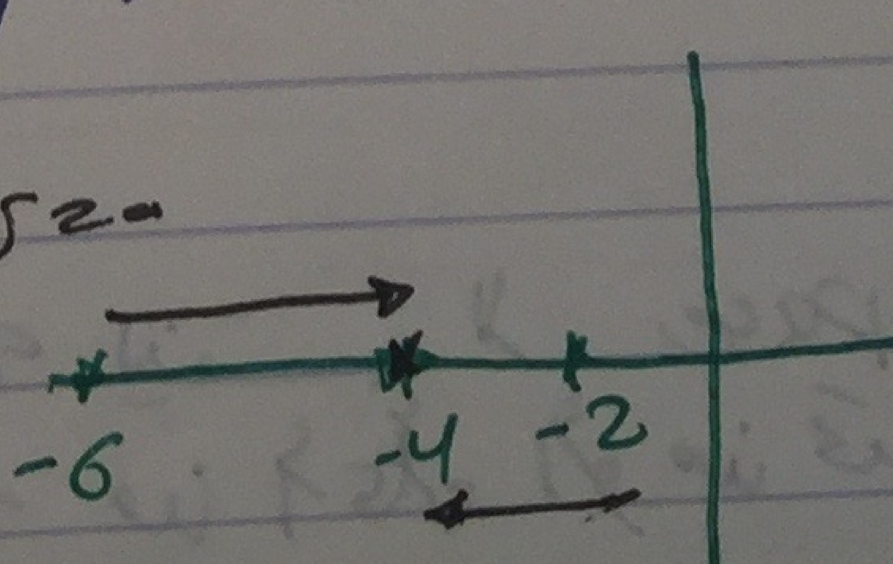
ليس مشكلة, Locouse خارج ال.

$\varphi = 0 - 0 \pm (2\ell+1)\pi = \pi$ (so depaotion at $-2 = \pi$).

-2 pole

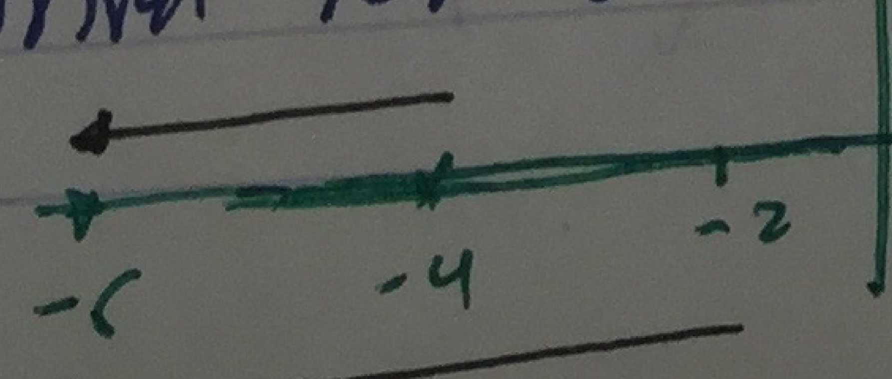
$\varphi_{-4} = 0 - \pi + (2\ell+1)\pi = 0$

the angle of departion at -4 pole



the angle of arrival for -6

$\varphi_{-6} = +\pi + \pi + \pi = 3\pi$



كم طريقة نقدر نحل critical solution
هو نفسه ال repeated pole.

- To find the $\overset{(\text{Double})}{\text{repeated}}$ poles.

$$\frac{1}{s+2}+\frac{1}{s+4}-\frac{1}{s+6}=0$$

$$(s+4)(s+6)+(s+2)(s+6)-(s+4)(s+2)=0$$

$$s^2+12s+28=0$$

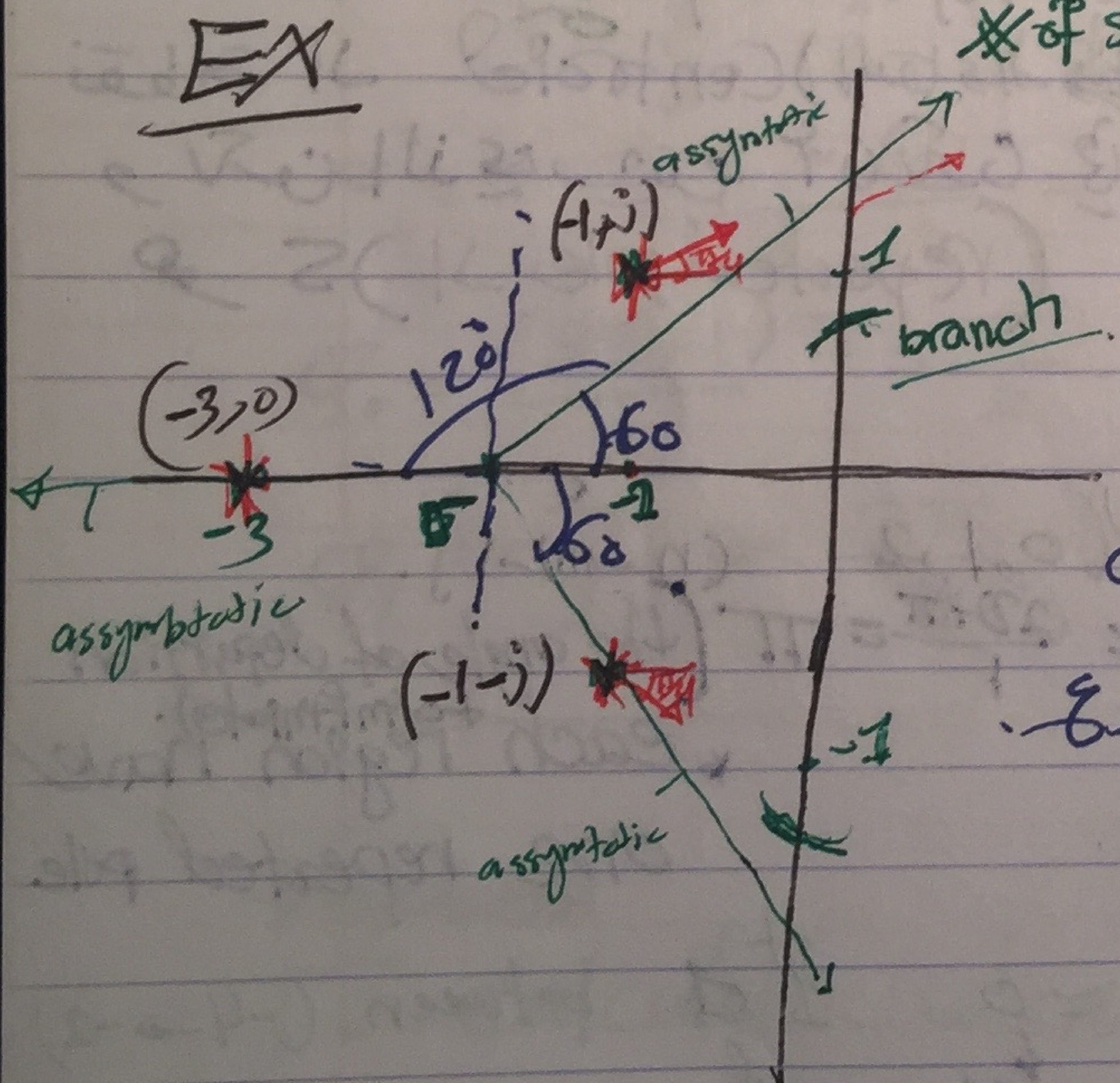
$$s_1=-3.178,\quad s_2=-8.825 \rightarrow \text{repeated pole.}$$

خاصية: عدد ال sector = 2 * # Poles , $2 * P^2 = 4$ = هـ

وبالتالي الزاوية لكل sector = $\frac{2\pi}{\#\text{ of poles}} = \frac{2\pi}{\#\text{ of sector}}$

repeated pole

EX



$$\sigma_c=\frac{-1-1-3}{3-0}=\frac{-5}{3}$$

الزوايا التي تخرج منها ال asyptotic عند نقطة التفرع (ويكون عدد الزوايا)

$$\varphi_0=\frac{\pi}{3},\quad \varphi_1=\frac{-\pi}{3},\quad \varphi_2=\pi=\frac{3\pi}{3}$$

كل ما التقى 2-branches فإن نقطة الخروج من الالتقاء فإنه يقسم ال space لكل sector $2n_0$

- عندما يكون عندنا 2 ـه فإنه يقسم ال space لـ 4-sector
- لعدم تعدد ال branch ← ما أكمل أو من تحت ال asyptotic.

$$1+\frac{K}{(s^2+2s+2)(s+3)}=0 \qquad (1+KGH=0)$$

$(s^2+3s+2)(s+3)+K=0$

$s^3+5s^2+8s+6+K=0$

to Knao where crossing the imaginary axis.

3	1	8	0
2	5	6+K	0
1	34-K	0	
0	6+K	0	

when $K=34$

or $K=-6$ ✗

لأنه K يجب أن تكون

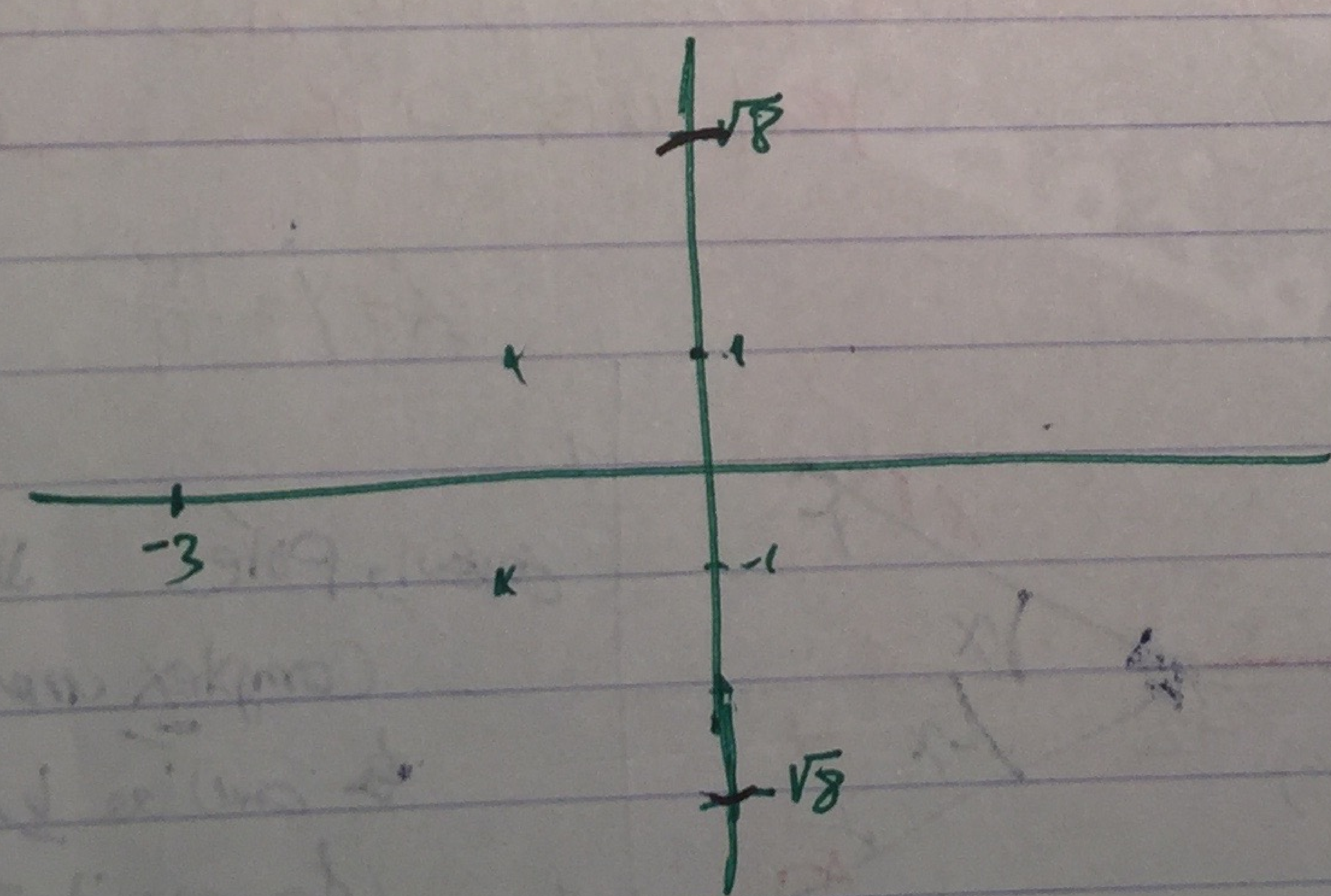
so $K=34$

Auxiliary polynomial

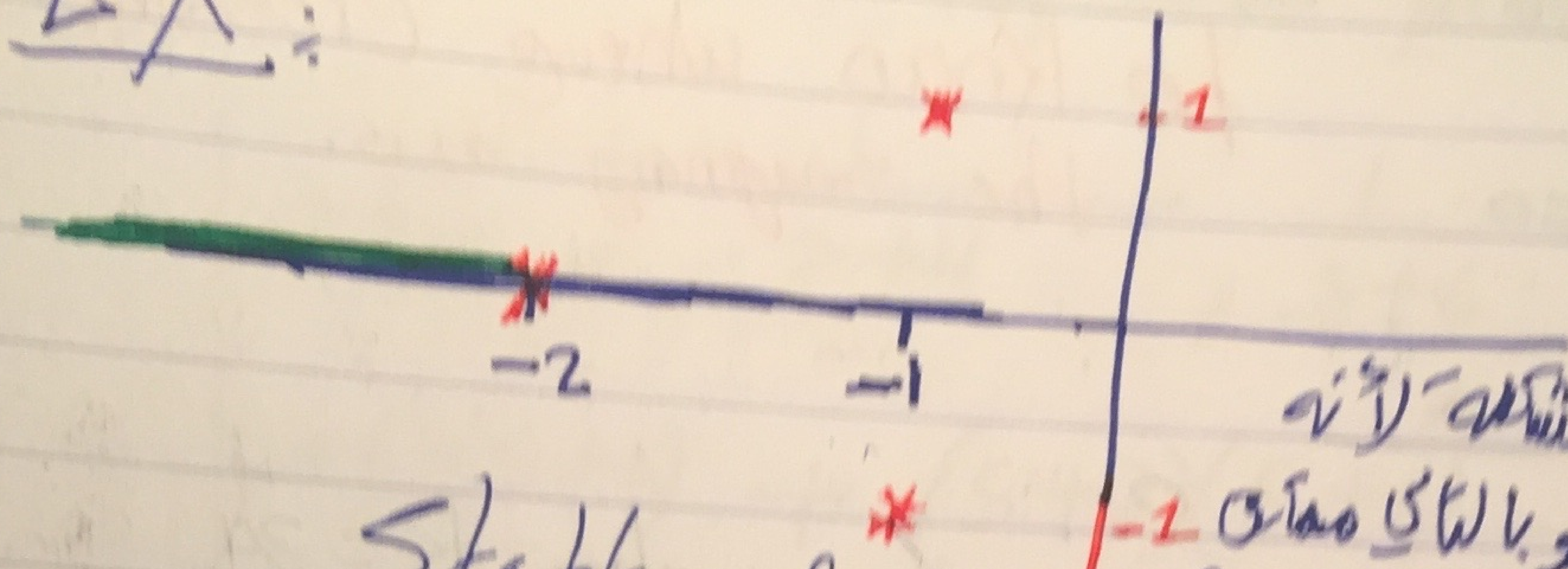
$= 5x^2+6+K = 5x^2+6+34 = 5x^2+40$

$= 5s^2+40$

$s = \mp j\sqrt{8}$



EX :



~~[illegible]~~ $n > m$

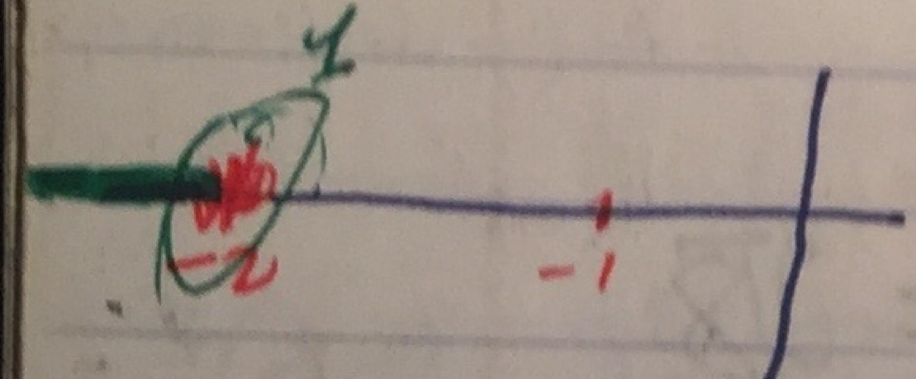
- لو كان ال pole على اليمين ليس مشكلة لأنه هذا ال poles ال openloop و بالتالي ممكن أن تكون ال openloop (unstable) وال closed loop هو يكون Stable

☒ Number of branches = number of poles = 3.
(max between poles and zeros).

☒ The root Locus is symmetric with respect to the real axis.

☒ points of the real axis that belong to the Locus

$$S \in \,]-\infty, -2[$$

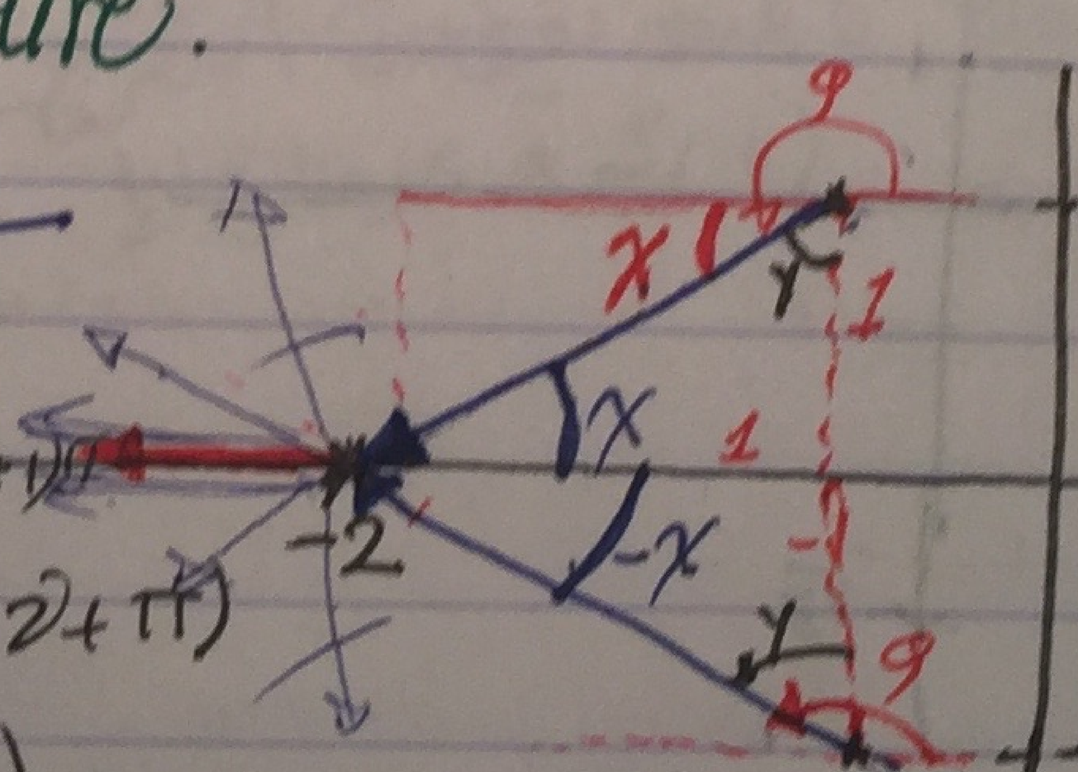


- ال complex لا تقدم (للتستيل) لأنهم دائماً يلغوا بعضهم حيث لا نطلع عليهم , حيث تقفوا ال poles و ال zeros اعلى وحدة على ال real axis

☒ Angles of departure.

$$x = y = \frac{\pi}{4}$$

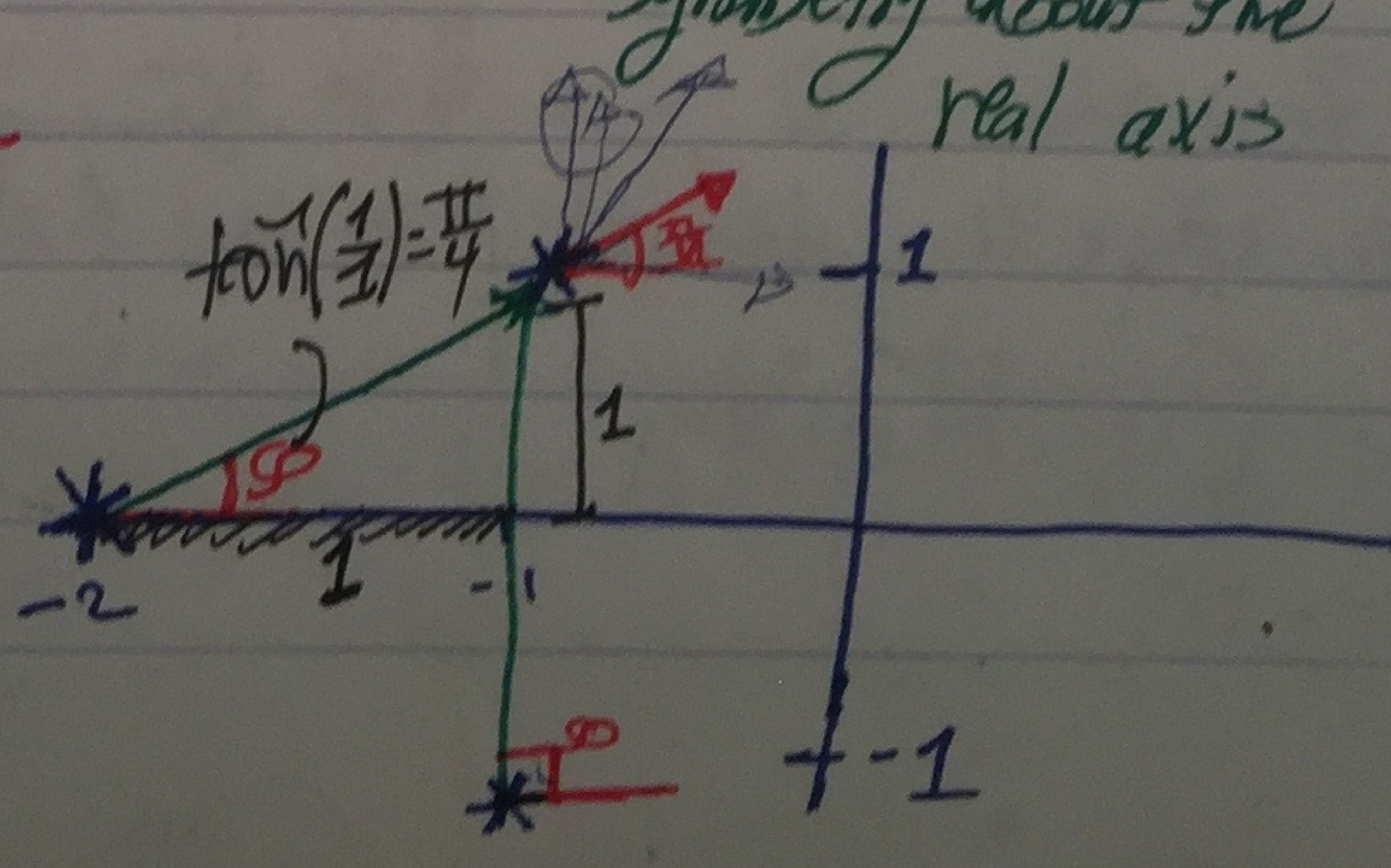
$$\varphi_{-2} = x - x + (2\nu+1) = \pi$$
$$= (90 + y) + (180 + x) + (2\nu+1)\pi$$
$$= 90 + 45 + 180 + 45 + (2\nu + \pi)$$
$$= 135 + 225 + (2\nu + \pi)$$
$$= 135 - 135 + (2\nu + \pi) \overset{\nu=0}{=} \pi$$



الزوايا لل poles والzeros ال complex and conju تعطي زوايا معاكسة بالنسبة لنقطة على ال real axis

لأنه يوجد symmetry about the real axis

$$\varphi_{-1+j} = -\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + (2\nu+1)\pi$$
$$= -\frac{3\pi}{4} + \pi$$
$$= \frac{\pi}{4}$$



$$\varphi_{-1-j} = -\frac{\pi}{4} \text{ by symmetry.}$$

☒ Angles of arraival at infinity (the angle of the Assymptotic)

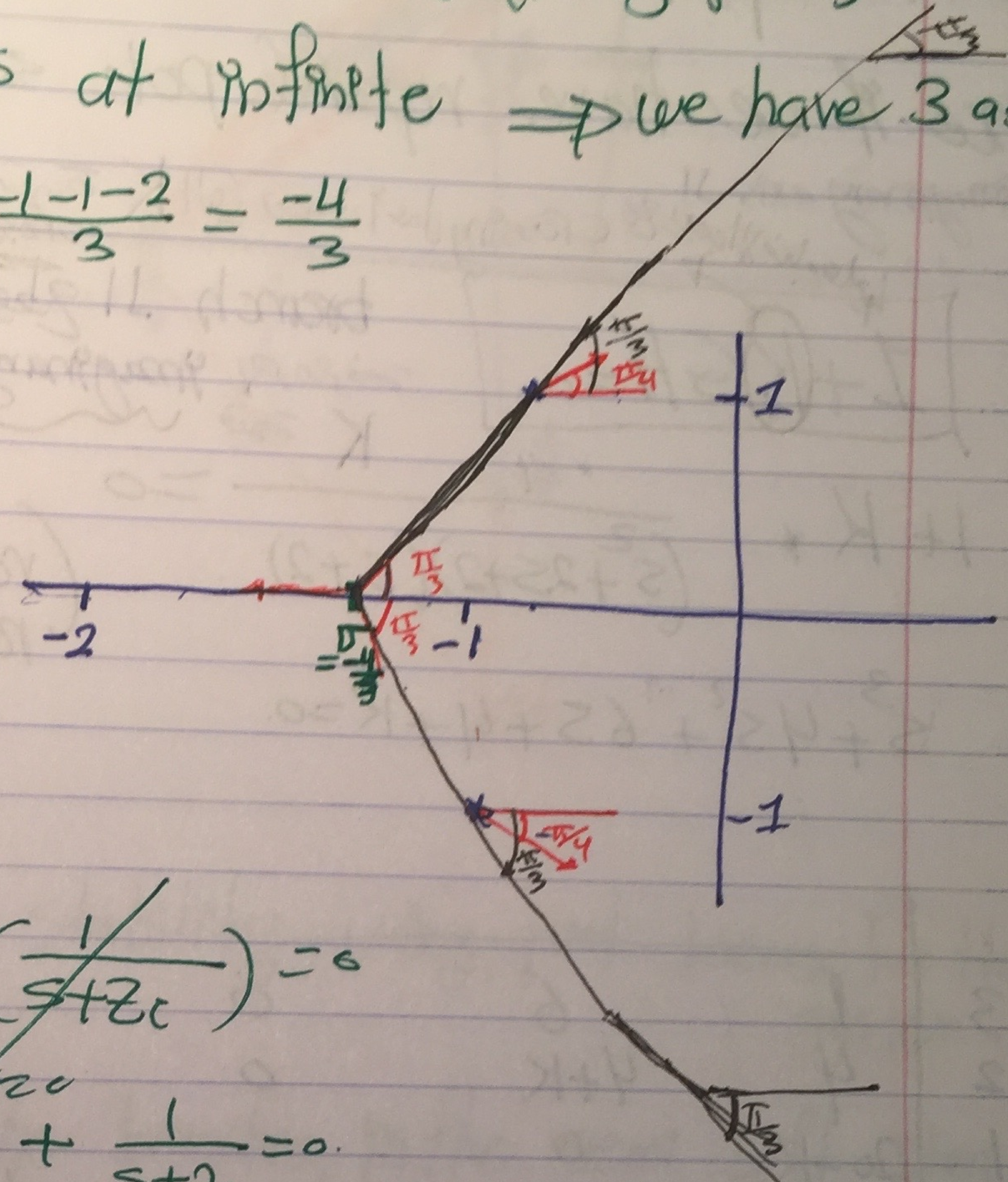
we have 3 zeros at infinite $\Rightarrow$ we have 3 assyptotic

Centroide $\quad \sigma_c = \frac{-1-1-2}{3} = \frac{-4}{3}$

$\theta_0 = \frac{\pi}{3}$

$\theta_{-1} \overset{i=-1}{=} -\frac{\pi}{3}$

$\theta_1 = \pi$



☒ repeated poles.

$$\sum\left(\frac{1}{s+p_i}\right) \overset{\text{ناقص}}{=} \cancel{\sum\left(\frac{1}{s+z_i}\right)} = 0$$

$=0$

$$\frac{1}{(s+1+j)} + \frac{1}{(s+1-j)} + \frac{1}{s+2} = 0.$$

$$(s+1-j)(s+2) + (s+1+j)(s+2) + s^2+2s+2 = 0.$$

$$(s^2+3s+2) + (s^2+3s+2) + s^2+2s+2 = 0$$

$$3s^2 + 8s + 6 = 0$$

$$\Delta = -8 < 0 \quad (\text{NO solution})$$

as expected from geometry.

- ممكن أن يلتقوا في نقطة Complex ولكن هنا لا يمكن أن يلتقوا في 3
نقطة لأن إذا التقوا فإنه يجب أن تكون ال order = 5
ولكن هنا ال order = 3

2-Complex pole, 2-Conject
1-real pole.

So No repeated poles for Closed Loop.

since if we have repeated pole say how much sector

ال imaginary axis مع بعض الاشياء عليها

أطلا نستفيد ليحرزة cross قيمة K اللي رجعلني أحصل crossing

$\boxed{*}$ $\boxed{1 + K\,GH = 0}$

و قسا ب ٢ ين يقطع ال branch

ال imaginary axis بزعزعزة قيمة K

(row of zeros in routh Hurwitz table)

$$1 + K * \frac{1}{(s^2+2s+2)(s+2)} = 0$$

$$s^3 + 4s^2 + 6s + 4 + K = 0$$

3	1	6	0
2	4	4+K	0
1	20-K	0	
0	4+K	0	

for $K > 0$.

$20 - K = 0 \Rightarrow K = 20$

$4 + K = 0 \Rightarrow K = -4$

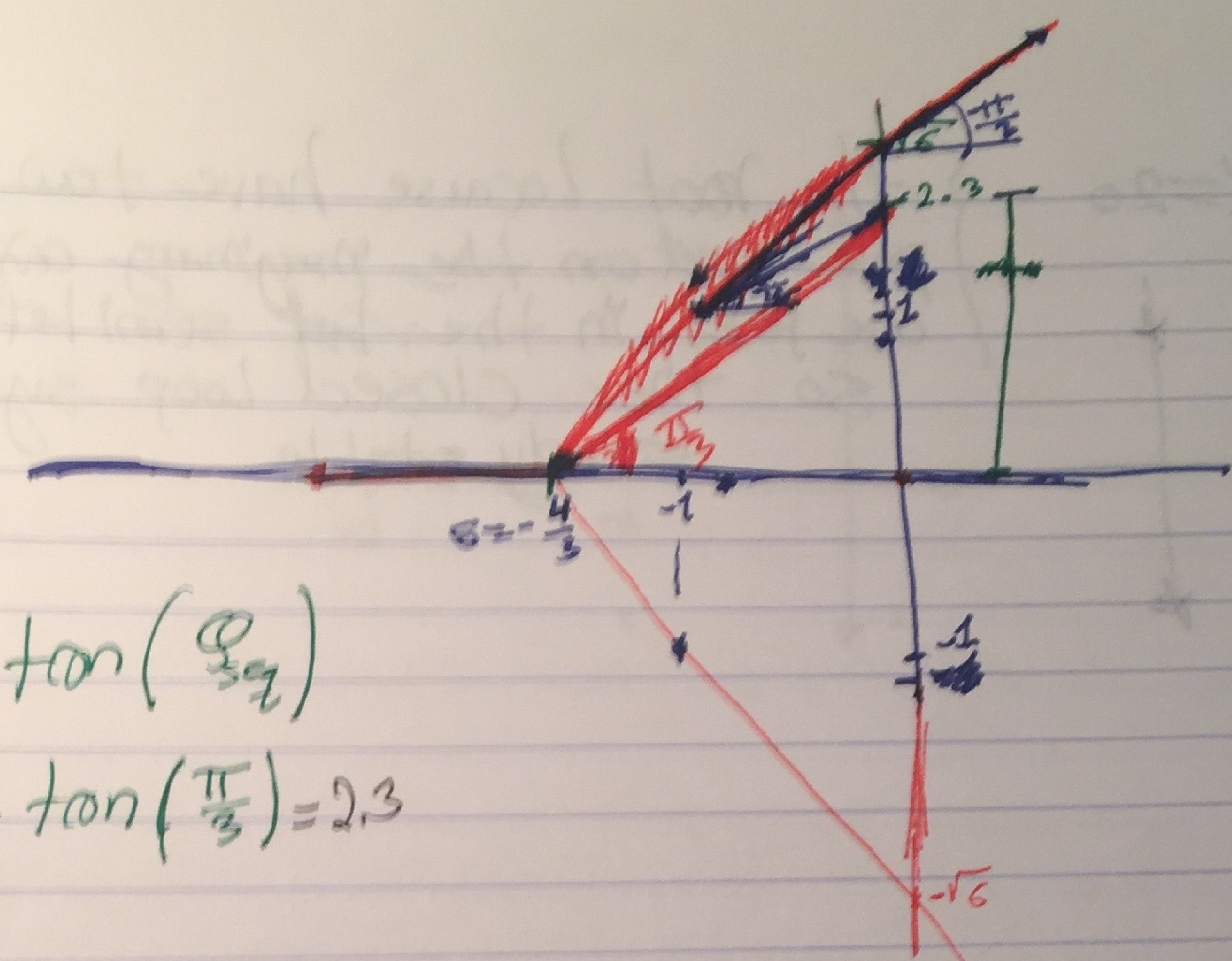
so we have.

$$A(s) = 4s^2 + (4+K) = 4s^2 + 24$$

$$s_{1,2} = \mp\sqrt{6}.$$

$\boxed{*}$ Intersection of the asymptotic with the imaginary axis.

summetry حسب النقطة نقطة ال

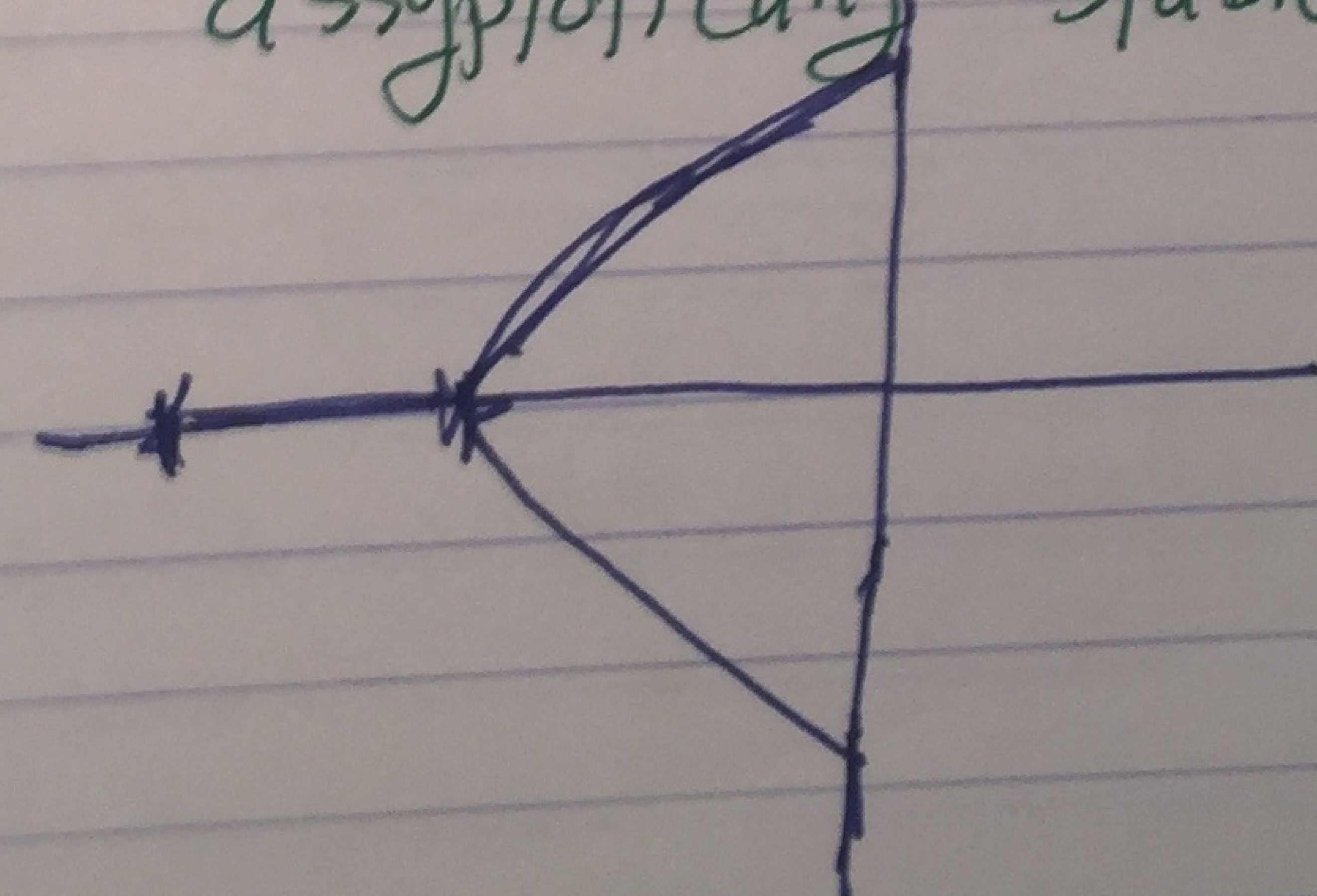


$d = \sigma_c \tan\left(\frac{\varphi}{q}\right)$

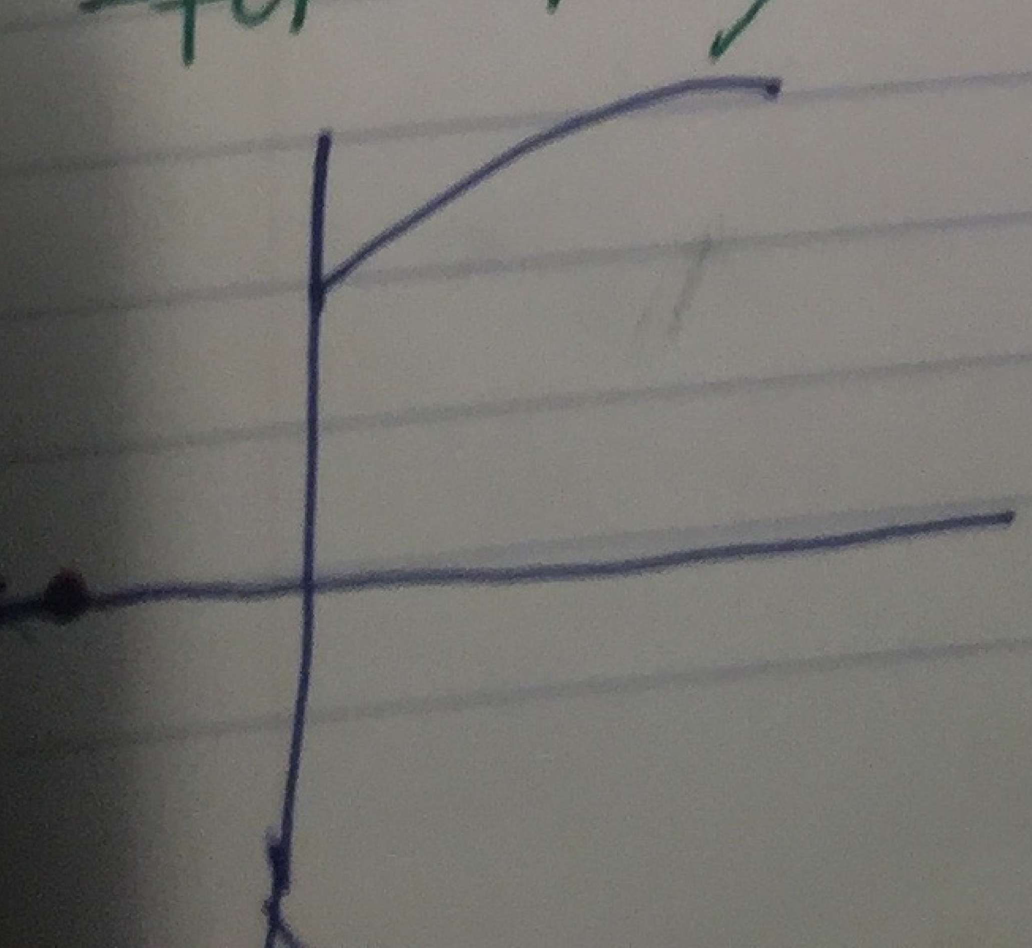
$= \frac{4}{3} \tan\left(\frac{\pi}{3}\right) = 2.3$

☒ Discuss the stability, using root Locus for the system.

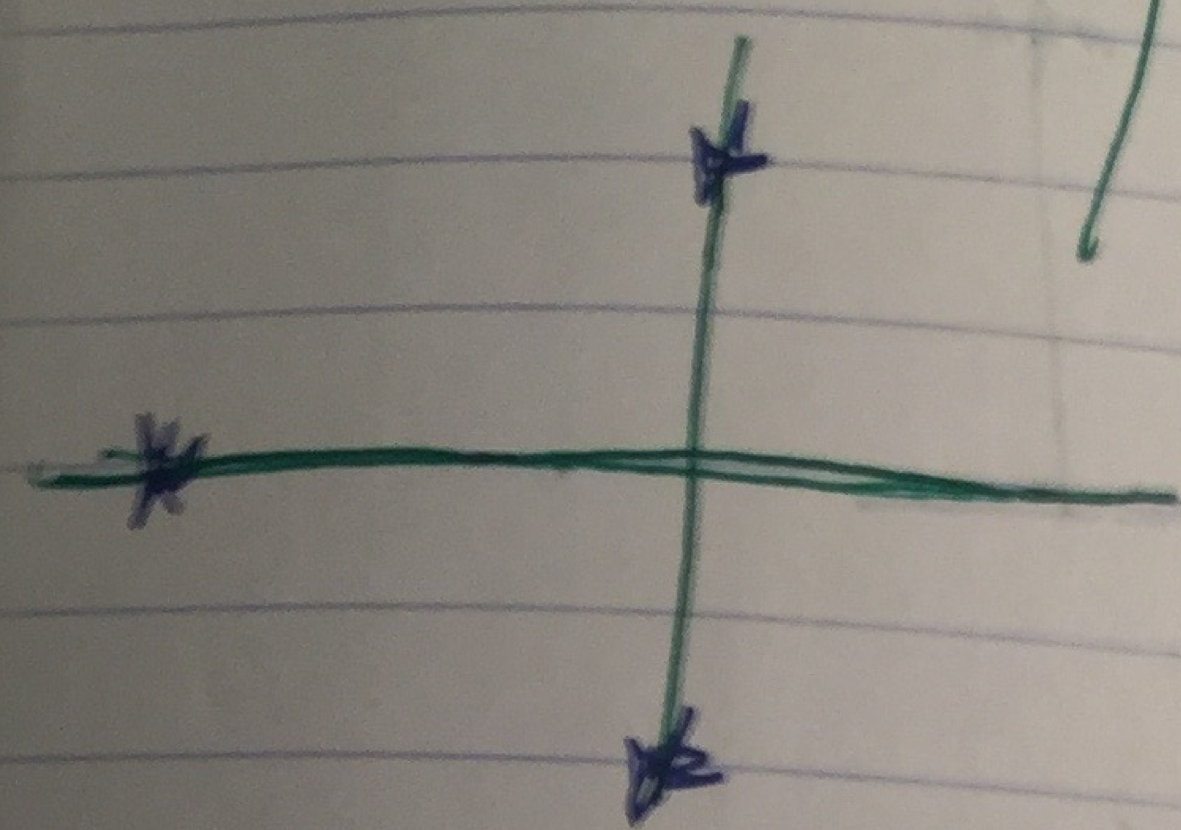
- For $K < 20$ { all the branches in the semi Left plan so the closed Loop system is assyptotically stable



- For $K > 20$ { the root locus have two poles at the right half plan, so it is have 2 sections from the branch at the ~~xxx~~ semi right plan, the closed loop system is unstable.



- For $K=20$

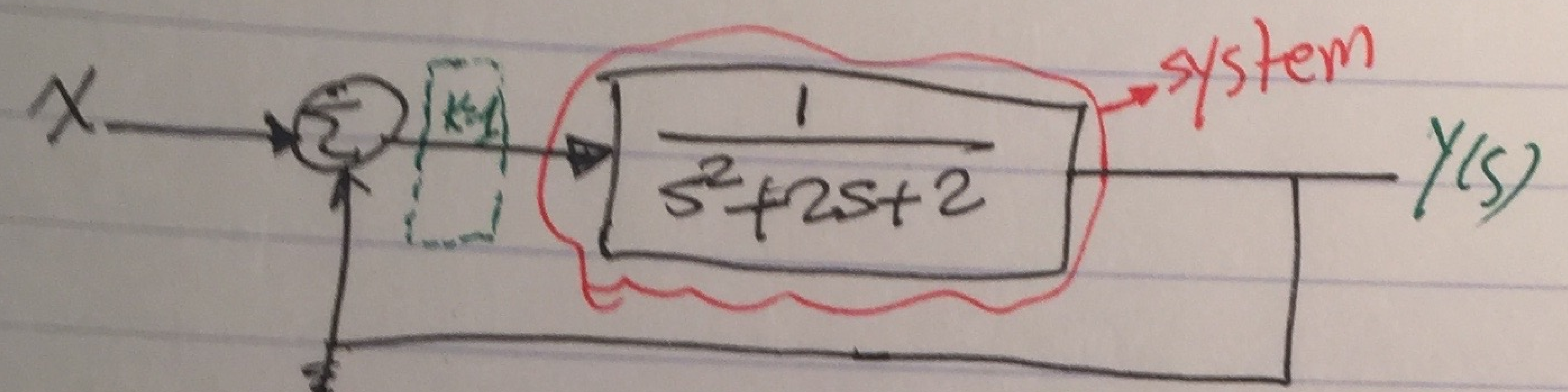


the root Locause have tow ymigynary pole ~~and~~ on the ymagynary axe and one pole in the ~~~~ semi left plane so the closed Loop system is simply stable.

EX

For this system Find.

1. The step response of the system.
2. plot the root Locause.
3. Discuss the possiblities for improving the SSE and the transfer of the system.
4. Improve the ~~xxx~~ error by at least 50%.



1. $H(s) = \dfrac{\frac{1}{s^2+2s+2}}{1+\frac{1}{s^2+2s+2}} = \dfrac{1}{s^2+2s+3} = \dfrac{Y(s)}{X(s)}$

$Y(s) = \dfrac{1}{s^2+2s+3} * X(s)$, the step response $X(s) = \frac{1}{s}$.

so we make Laplace inverse

$Y(s) = \dfrac{1}{s^2+2s+3} * \dfrac{1}{s} = \dfrac{1}{s(s^2+2s+3)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2s+3}$

-Note: the step response it is in time representation.

$A = \dfrac{1}{3}$

$1 = A(s^2+2s+3) + (Bs+C)s$

$1 = As^2+2As+3A+Bs^2+Cs$, $0 = A+B \Rightarrow B = -\frac{1}{3}$

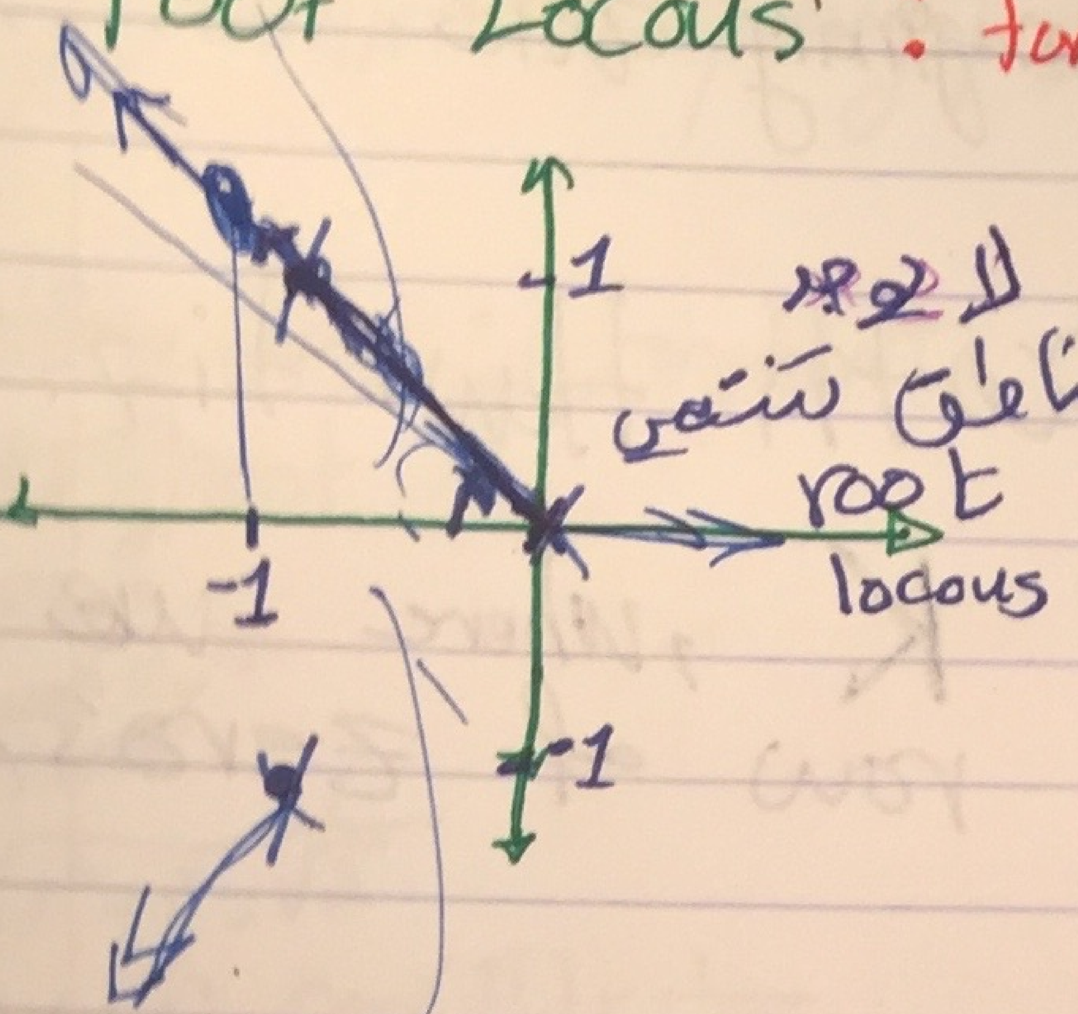
$0 = 2A+C \Rightarrow C = \frac{-2}{3}$

$= \dfrac{\frac{1}{3}}{s} + \dfrac{-\frac{1}{3}s}{s^2+2s+3} + \dfrac{-\frac{2}{3}}{s^2+2s+3}$

for [redacted]

open loop $= A*B = G(s)*H(s) = G(s)*1 = G(s) =$

[2] root Locous : for the system : the open Loop poles.



open Lopp system.

$$\frac{\boxed{1}^{\,k=1}}{(s^2+2s+2)} = \frac{}{(s+j+1)(s-j+1)}$$

$$s = \frac{-2 \mp \sqrt{4-4(2)}}{2}$$

$$= -1 \mp j\frac{2}{2} = -1 \mp j$$

$$s_1 = -1-j$$

$$s_2 = -1+j$$

[1] Number of branches = Number of poles = [redacted] 2.

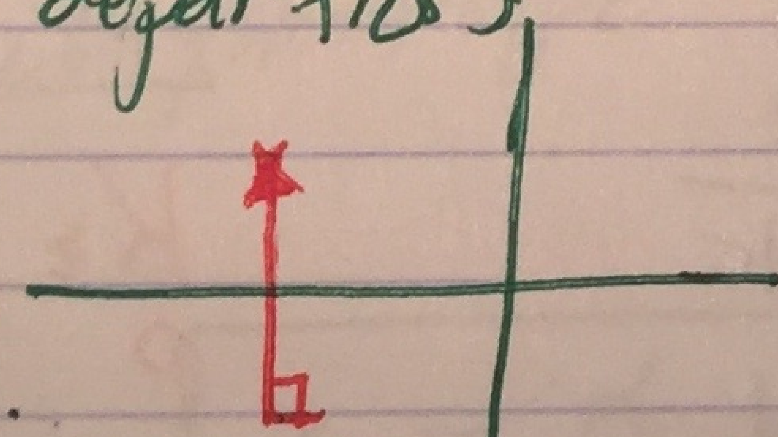
[2] The root Locous is symmetric with respect to the real axis.

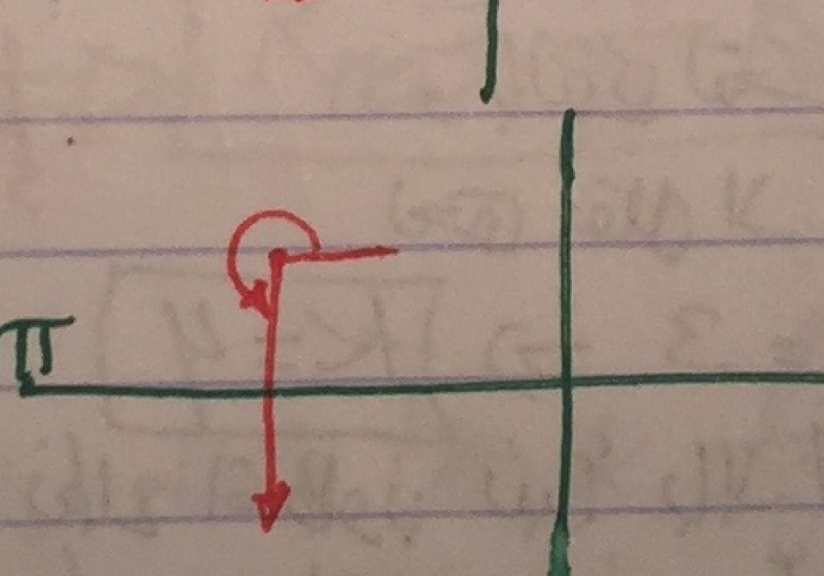
[3] points of the real axis that belong to the locous Φ.

- The angle of departion

$$\varphi_{-1+j} = 0 - \left(\frac{\pi}{2}\right) + (2(0)+1)\pi$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$





$$\varphi_{-1-j} = 0 - \frac{3\pi}{2} + (2(0)+1)\pi$$

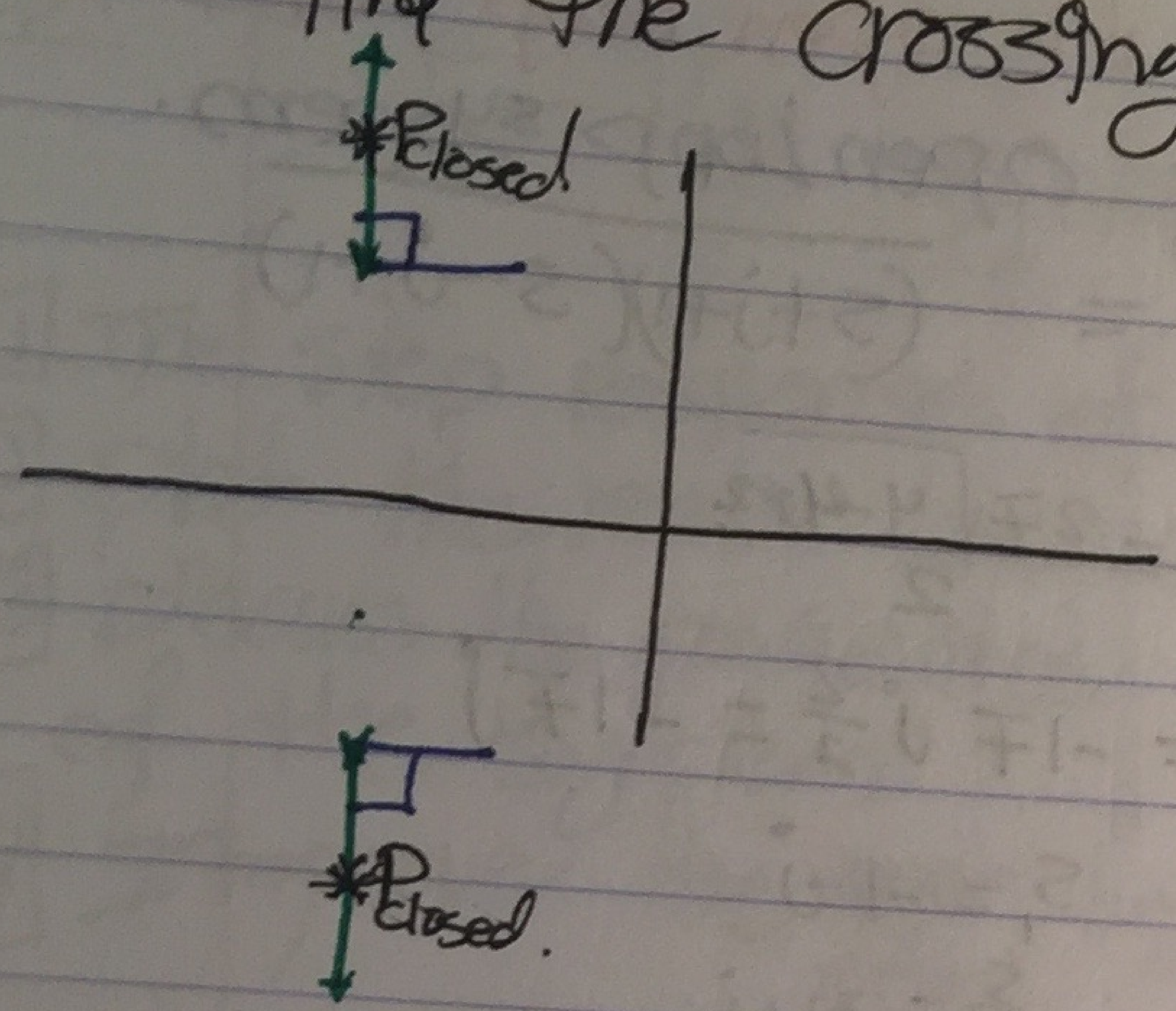
$$= \pi - \frac{3\pi}{2} = \frac{-\pi}{2}$$

- repeated poles : From Graph No repeated poles

or by calculation $= 0$

$$\sum \frac{1}{s+P_i} - \sum \frac{1}{s+Z_i} = 0$$

- To find the crossing with the imaginary axis.



By Routh Hurwitz.

Find K, where we have row of zeros.

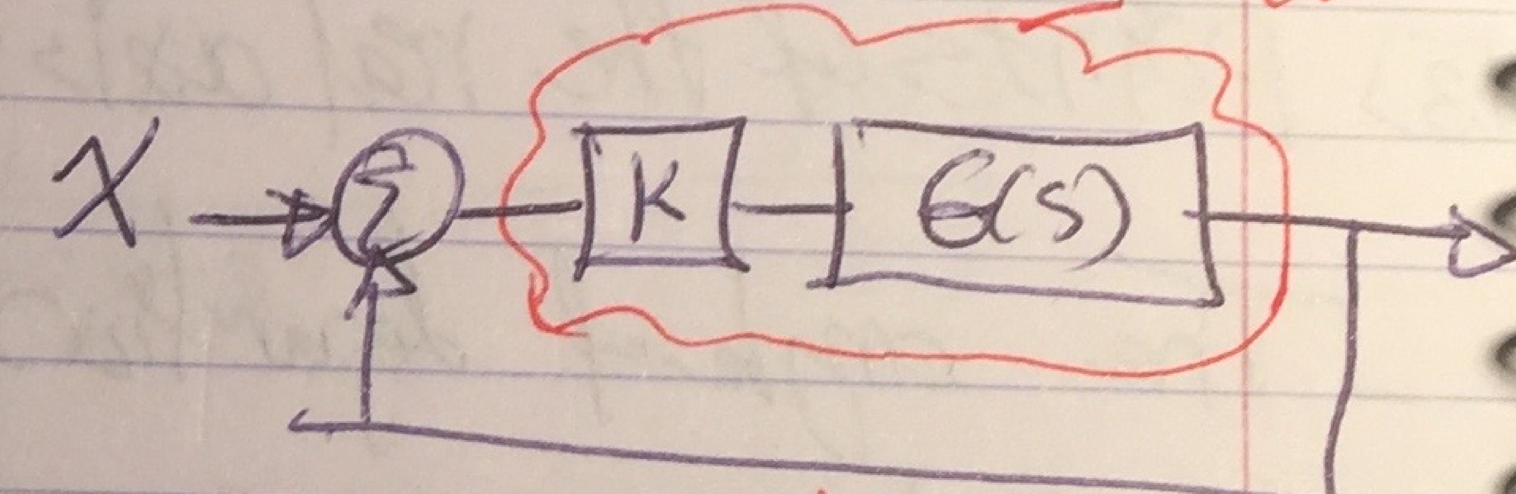
Because unity feedback the error function is the same for direct path function. No poles in the origin from G so zero order

3

$$e_p = \frac{1}{1+K_p} \quad , \quad K_p = \lim_{s \to 0} \frac{1}{s^2+2s+2} = \frac{1}{2}$$

$G(s)$ (transfer function ليس ال).

$$e_p = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$



عوضنا ضرب ال K

$$e_p(K) = \frac{1}{1+\frac{K}{2}} = \frac{2}{2+K}$$

$$K_p = \lim_{s \to 0} K * G(s) = \lim_{s \to 0} \frac{K}{s^2+2s+2} = \frac{K}{2}$$

وبالتالي لتحسين ال error يجب أن نزيد قيمة ال K حتى نقلل ال error، مثلاً إذا أردنا أن نجعله $\frac{1}{3}$

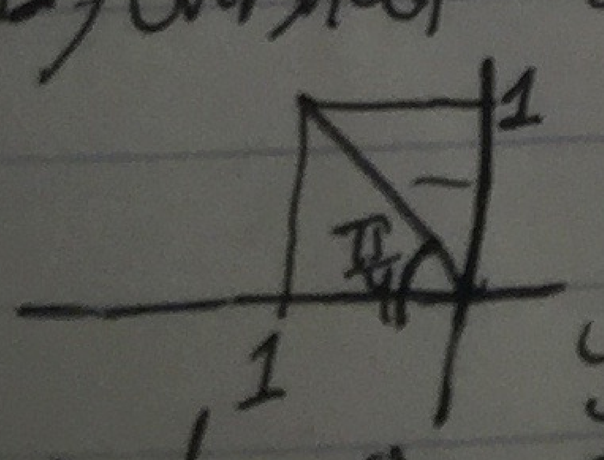
$$\frac{1}{1+\frac{K}{2}} = \frac{1}{3} \Rightarrow 1+\frac{K}{2} = 3 \Rightarrow \boxed{K=4}$$

* إذا كان بالإمكان أن نطلع المطلوب نبقى بال static controller
أما إذا ما قدر يطلع المطلوب يجب أن ندخل ال Dynamic

- هنا عشان $K \to \infty$ فإنه $e_p = 0$. لازم أضيف Dynamic controller.
- في ال transient نناقش ال overshoot, settling time, damping freq.

a) settling time: Because the real part constant, so we can't change it

b) overshoot: we can change it, but to limit, since



minimum over shoot.

(Natural frequency).

c) damping frequency: (imaginary part for pole): from $(1 \to \infty)$

when $K \uparrow \Rightarrow$ oscillation frequency $\uparrow \Rightarrow$ error $\downarrow$ ($\omega_{n\,min} = \sqrt{2} \to \infty_{max}$)

يجب أن نضيف ال Dynamic controller.

$K\uparrow$, osillation frequency $\uparrow$, Error $\downarrow$

Damping Controller:

Lag compinsator ← لا يصبح صفر ال error

PI controller ← ال error يصبح صفر

setling time : Constant

overshoot : ↑ يجب ان نعرف الـ minimum overshoot

$= \sqrt{1^2+1^2} = \frac{1}{\sqrt{2}}$.

$= \cos(\xi) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

- تحسين ال error باستخدام ال lag comp. يعني

$\frac{K_{new}}{K_{old}} = \frac{\alpha}{\beta}$.

$e_{new} < e_{old} \Rightarrow K_s > K_{s,old}$.

K_s: static gain

$G_{lag} = \frac{s+\alpha}{s+\beta}$, $|\alpha| > |\beta|$

$K_{s,old} = \lim_{s \to 0} s^i G(s)$, i: system type

$K_{old} = K \frac{\pi z_i}{\pi p_i}$

$K_{new} = K \frac{\pi z_i}{\pi p_i} * \frac{\alpha}{\beta} \Rightarrow \boxed{K_{new} = \frac{\alpha}{\beta} K_{old}}$ → $\alpha > \beta$ وبالتالي عززنا تأثيرنا على ال error.

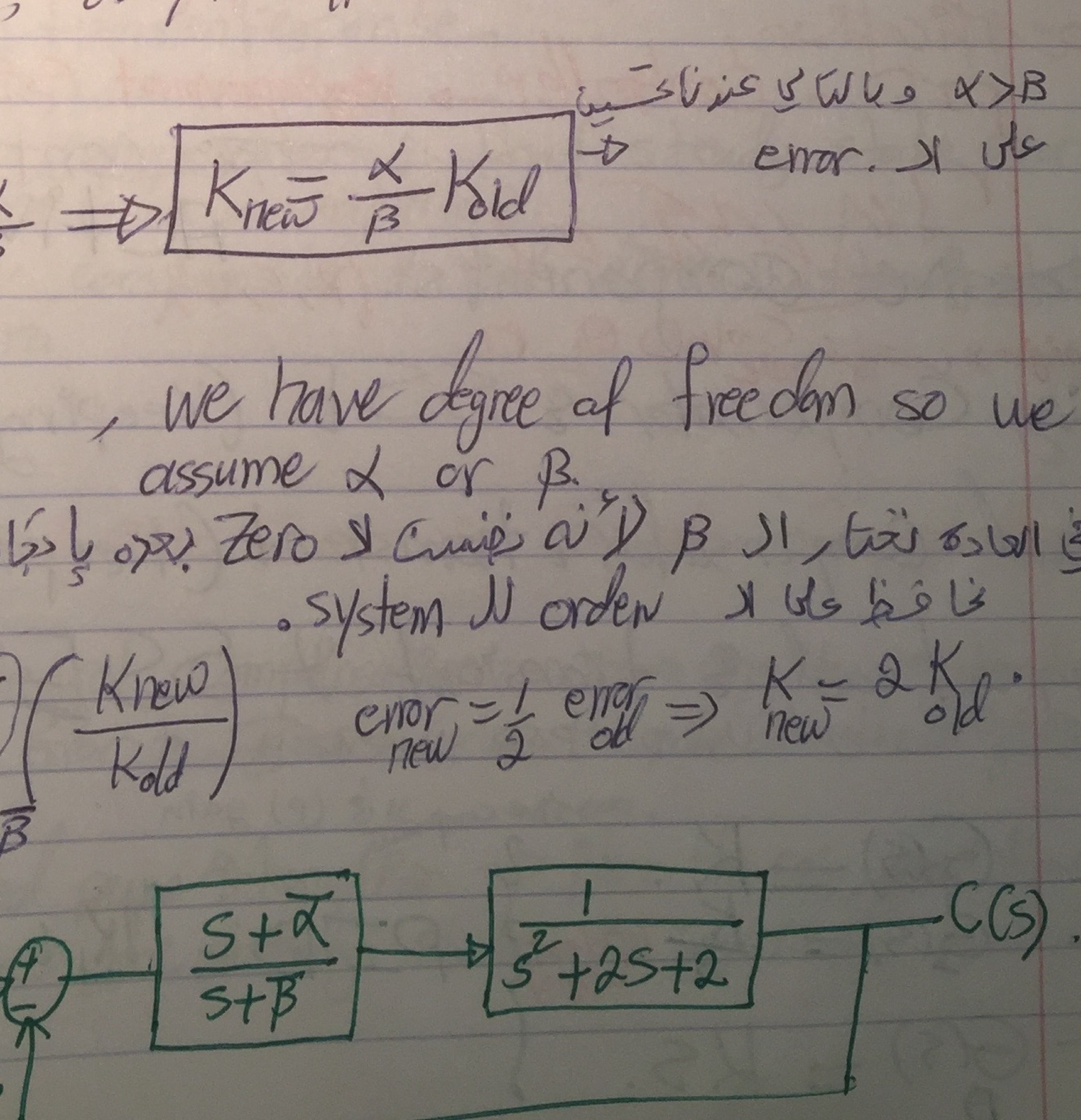
$\boxed{4}$ $\frac{K_{new}}{K_{old}} = \frac{\alpha}{\beta}$, we have degree of freedom so we assume α or β.

في العادة نختار ال β لأنه نفس ال Zero بعده بإتجاه اليسار، وذلك لنحافظ على ال order لل system.

so $\bar{\alpha} = \bar{\beta} \left(\frac{K_{new}}{K_{old}} \right)$ $\quad error_{new} = \frac{1}{2} error_{old} \Rightarrow K_{new} = 2K_{old}$.

نفترض قيمة $= 2\bar{\beta}$

So



استعدت ان استخدم فقط (R,L,C) passive components ووفرنا ال Amplifier.