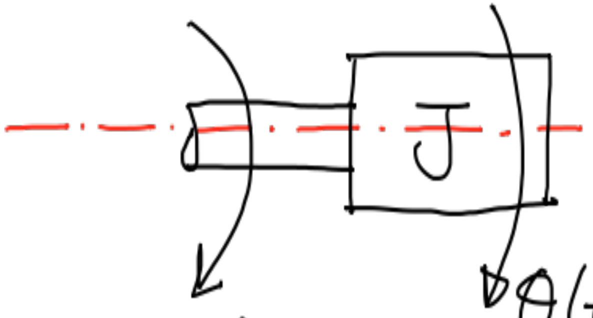
Rotational Mechanical Elements

$$2\pi \text{ rad} = 360^\circ$$



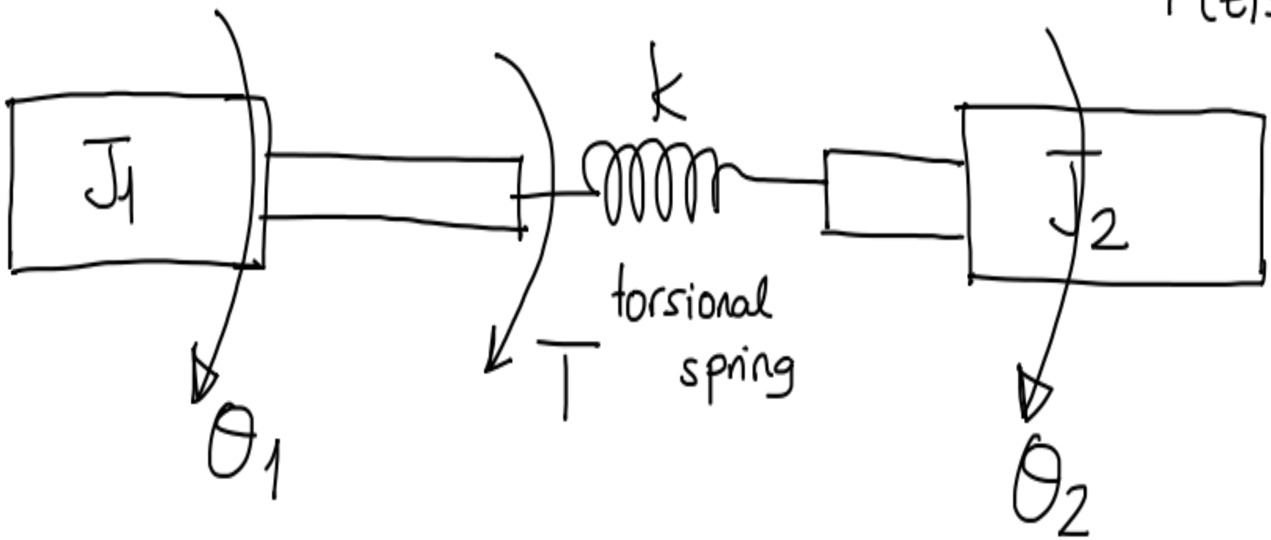
$T(t)$: torque $\qquad \theta(t)$ [rad]

$$T(t) = J\ddot{\theta}$$

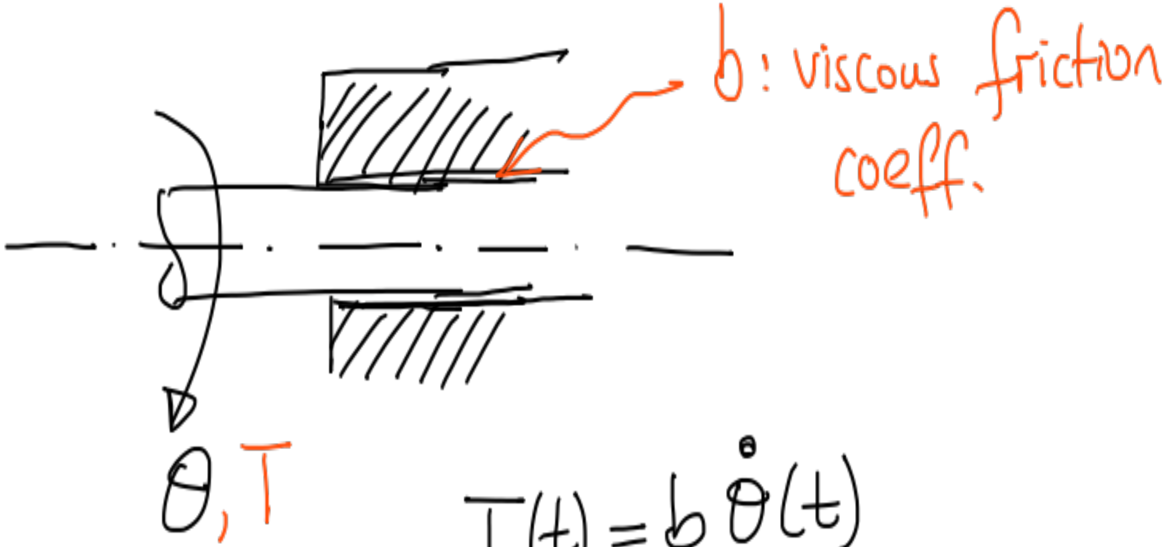
$$\dot{\theta} = \omega \ [\text{rad/sec}]$$

$$\ddot{\theta} = \alpha \ [\text{rad/sec}^2]$$

$$T(t) = k\,\underbrace{(\theta_1 - \theta_2)}_{\text{relative displacement}}$$



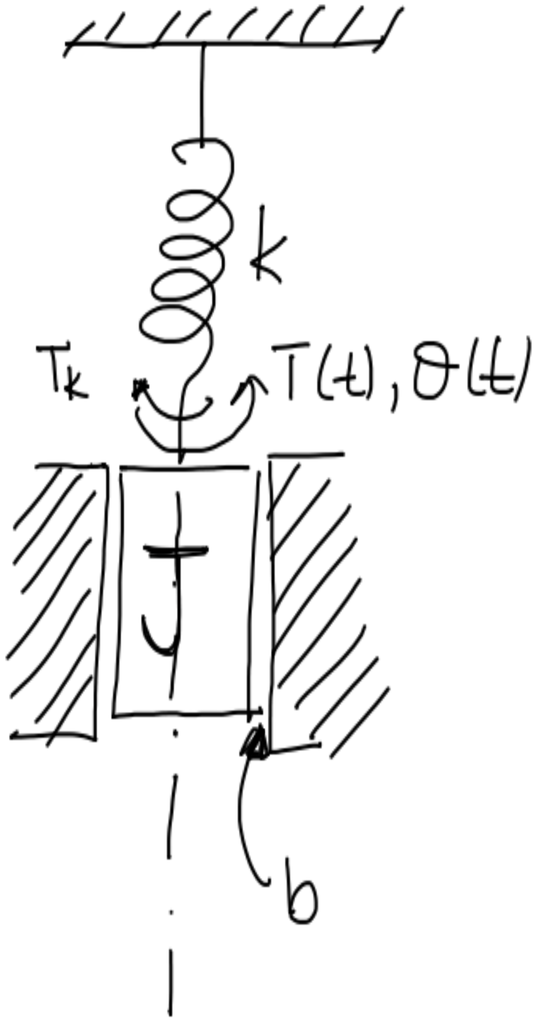
Friction



$$T(t) = b\,\dot{\theta}(t)$$

$$T(s) = b\,s\,\Theta(s)$$

EX: (Torsional pendulum system) J: inertia



Eq'n of Motion

$$J\ddot{\theta} = T(t) - T_k(t) - T_b(t)$$

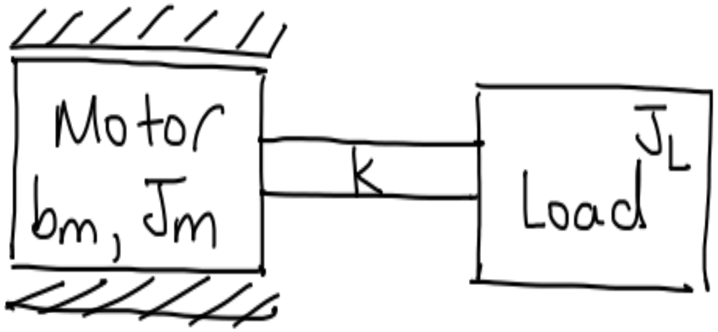
$$J\ddot{\theta} = T(t) - k\theta - b\dot{\theta}$$

$$\Big\downarrow \mathcal{L} \quad \theta(0) = \dot{\theta}(0) = 0$$

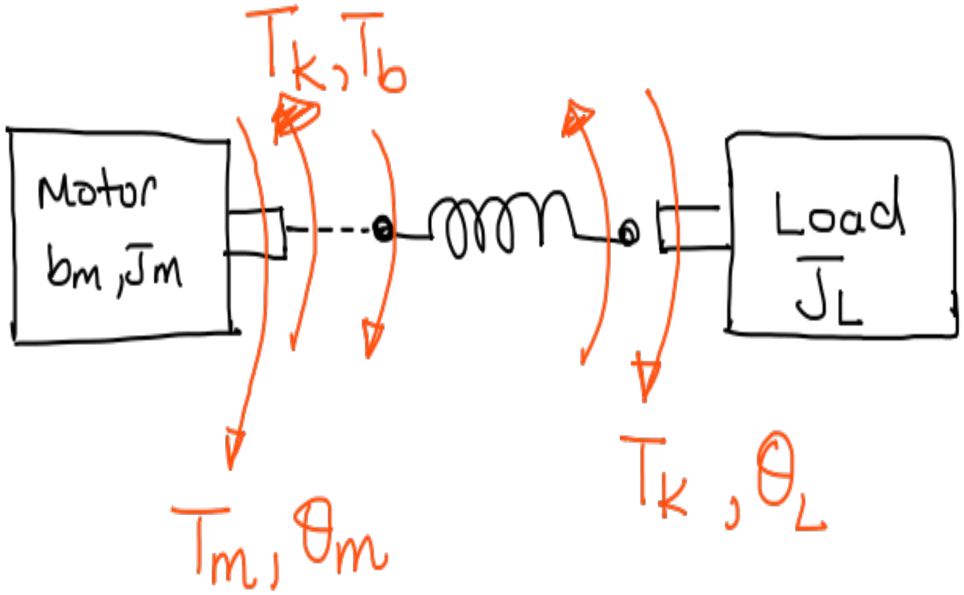
$$Js^2\Theta(s) = T(s) - k\Theta(s) - bs\Theta(s)$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + bs + k}$$ 2^{nd} order t.f.

Ex:



Free-body diagram



By Newton's Law:

$$①\quad J_m \ddot{\theta}_m = T_m(t) - b_m \dot{\theta}_m(t) - k\left(\theta_m(t) - \theta_L(t)\right)$$

$$②\quad J_L \ddot{\theta}_L = k\left(\theta_m(t) - \theta_L(t)\right)$$

$$①'\quad J_m s^2 \theta_m = T_m - b_m s \theta_m - k(\theta_m - \theta_L)$$

$$②'\quad J_L s^2 \theta_L = k(\theta_m - \theta_L)$$

SS representation of the system:

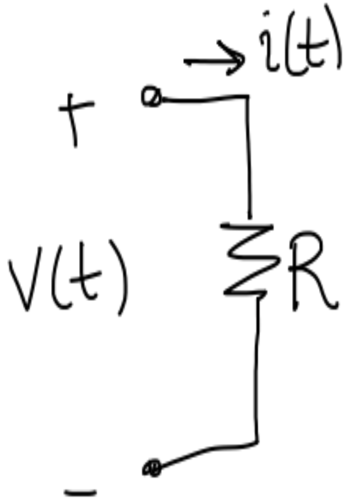
$\triangleq \Leftrightarrow :=$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangleq \begin{bmatrix} \theta_m \\ \theta_L \\ \omega_m \\ \omega_L \end{bmatrix}, \quad \vec{u} \triangleq T_m, \quad y \triangleq \theta_L$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/J_m & k/J_m & -\frac{b_m}{J_m} & 0 \\ k/J_L & -k/J_L & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/J_m \\ 0 \end{bmatrix}}_{B} u$$

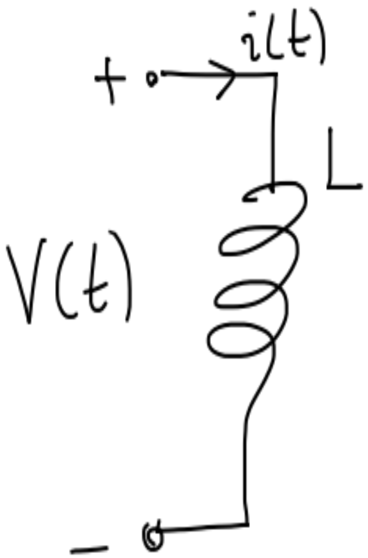
$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{0}_{D} \cdot u$$

Models of Electrical Circuits

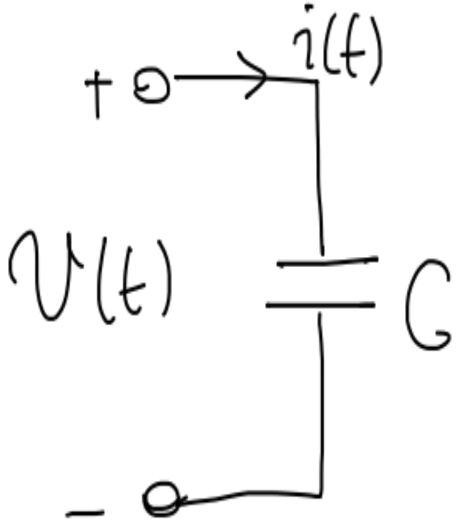


$$V(t) = R\,i(t)$$

Ohm's law



$$V(t) = L \frac{d}{dt} i(t)$$



$$i(t) = C \frac{d}{dt} v(t)$$

Assume $i(t)$ is the input, $v(t)$ is the output:

Resistance:

$$I(s) \longrightarrow \boxed{G(s)} \longrightarrow V(s)$$

$$G(s) = R$$

$v(t) = R\, i(t)$

$V(s) = R\, I(s)$

Inductance: (takes the derivative of the input)

$v(t) = L \frac{d}{dt} i(t) \implies G(s) = Ls$

$V(s) = L s I(s)$

Capacitence: (takes the integral of the input)

$i(t) = C \frac{d}{dt} V(t) \implies G(s) = \frac{1}{Cs}$ $\qquad \frac{1}{C}\int_0^t i(\tau)\, d\tau = v(t)$

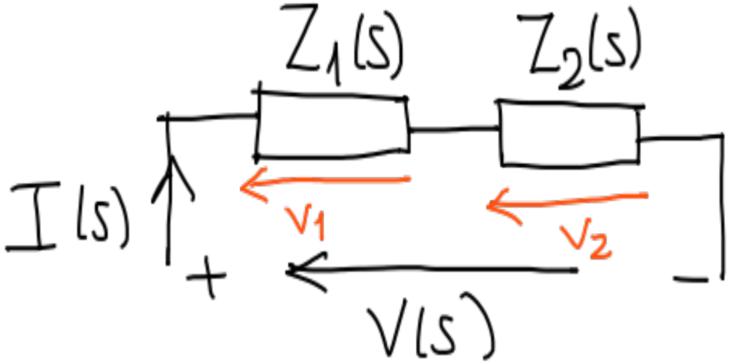
$I(s) = C s V(s)$

In sinusoidal steady state, Resistance, Inductance, Capacitance are generalized impedance to a sinusoidal alternating current

$$\boxed{G(s) = Z(s)}$$

Impedance Calculation

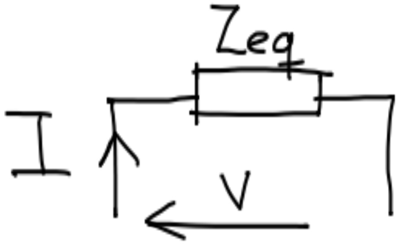
1. Series Connection



$$V_1(s) = I(s) Z_1(s)$$

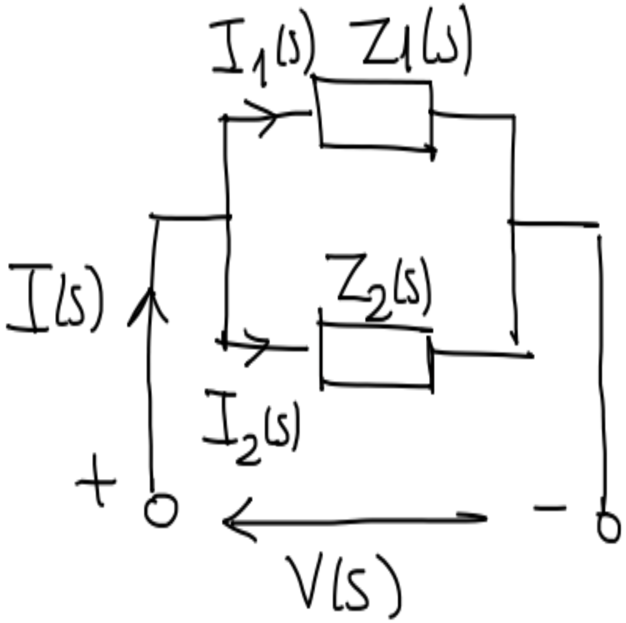
$$V_2(s) = I(s) Z_2(s)$$

$$V(s) = V_1(s) + V_2(s) = I(s)\left[Z_1(s) + Z_2(s)\right]$$



$$Z_{eq} = Z_1(s) + Z_2(s)$$

2. Parallel Connection:



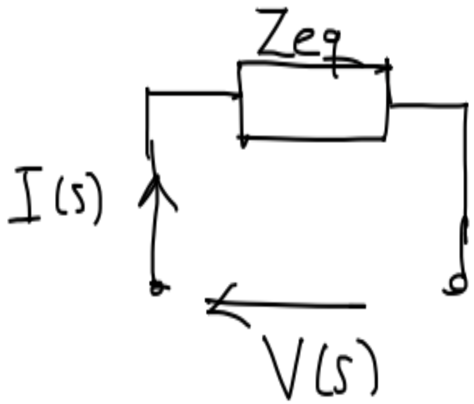
$$V(s) = I_1(s)\, Z_1(s)$$

$$V(s) = I_2(s)\, Z_2(s)$$

$$I(s) = I_1(s) + I_2(s)$$

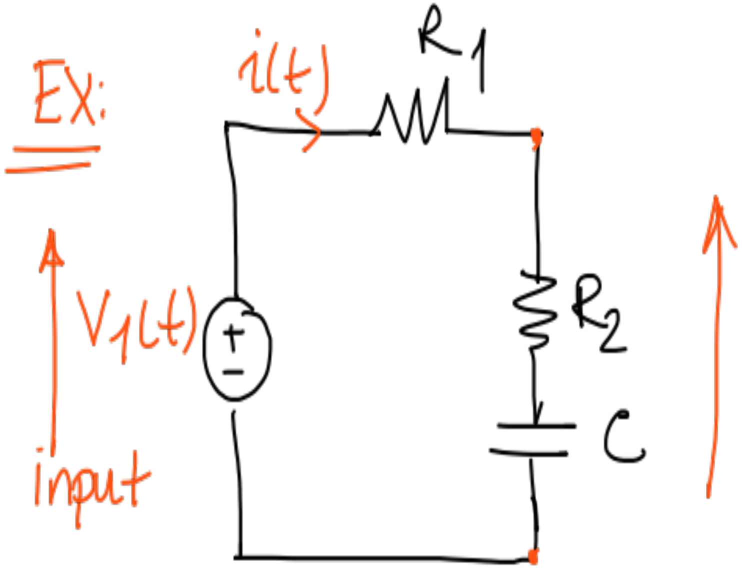
$$I(s) = \frac{V(s)}{Z_1(s)} + \frac{V(s)}{Z_2(s)} = V(s)\left[\frac{1}{Z_1} + \frac{1}{Z_2}\right]$$

$$V(s) = \overbrace{\boxed{\frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}}}^{Zeq} \cdot I(s)$$

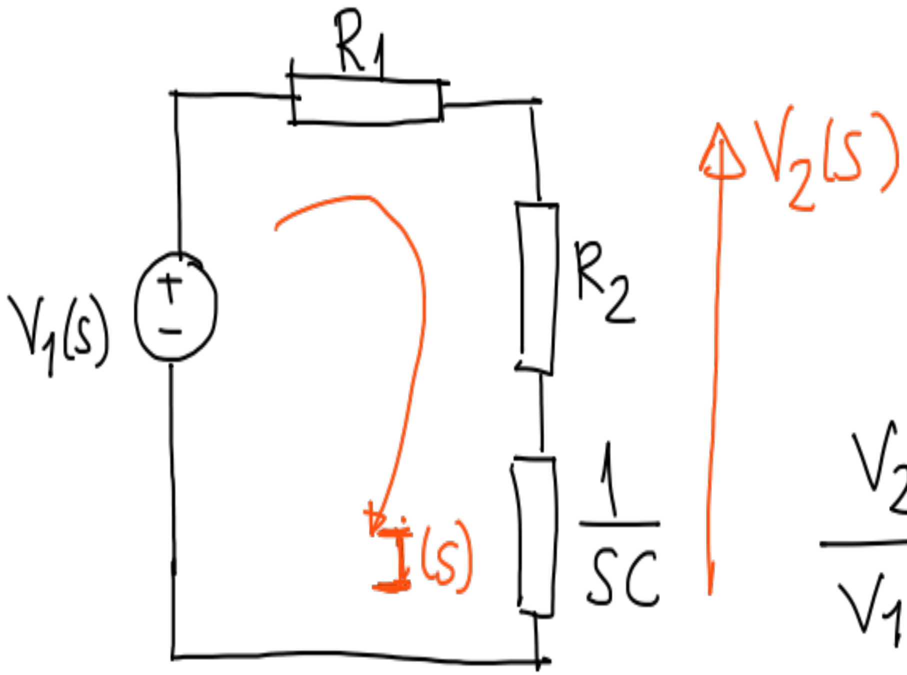


$$Zeq = \frac{Z_1(s)\, Z_2(s)}{Z_1 + Z_2}$$

Ex:



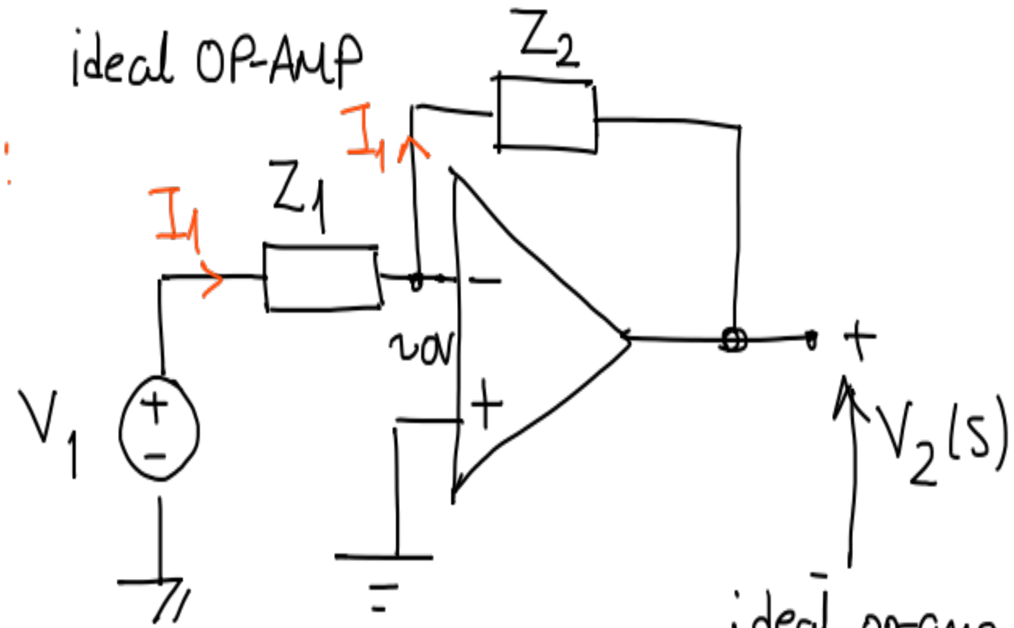
$$G(s) = \frac{V_2(s)}{V_1(s)} = ?$$



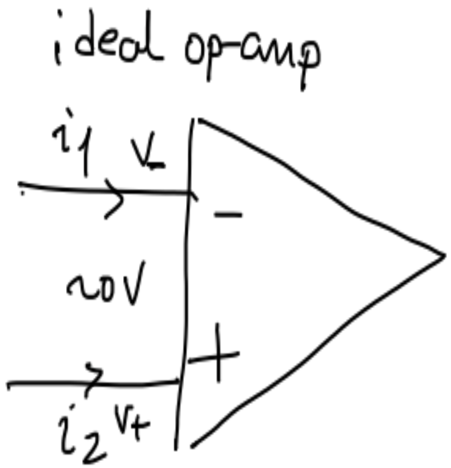
$$V_2(s) = \frac{V_1(s)}{R_1 + R_2 + \frac{1}{sC}} \times \left(R_2 + \frac{1}{sC}\right)$$

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{R_1 + R_2 + \frac{1}{sC}} = \frac{sR_2C + 1}{(R_1 + R_2)sC + 1}$$

Ex:



$$G(s) = \frac{V_2(s)}{V_1(s)}$$



$$I_1 = \frac{V_1(s)}{Z_1(s)}$$

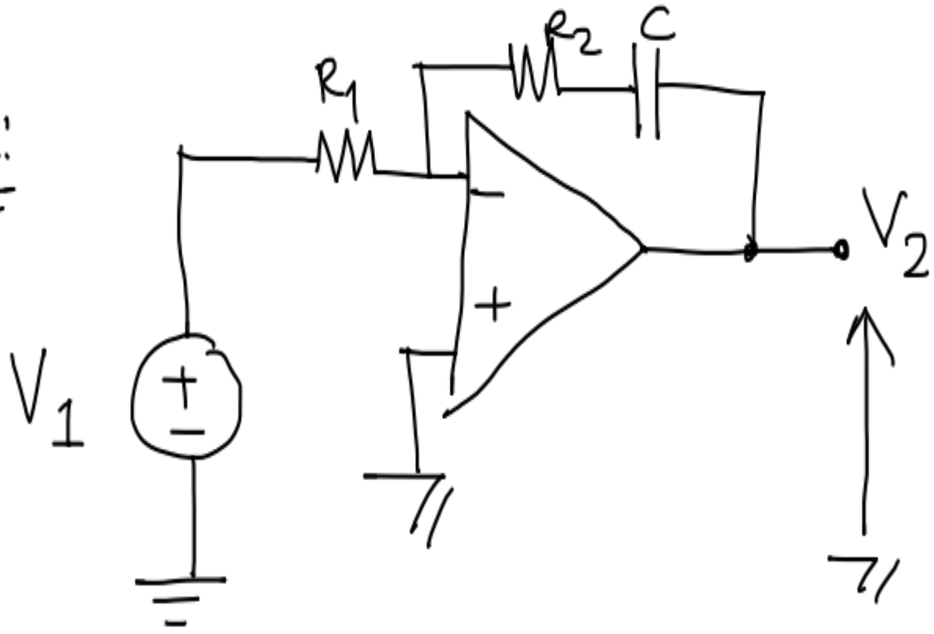
KCL

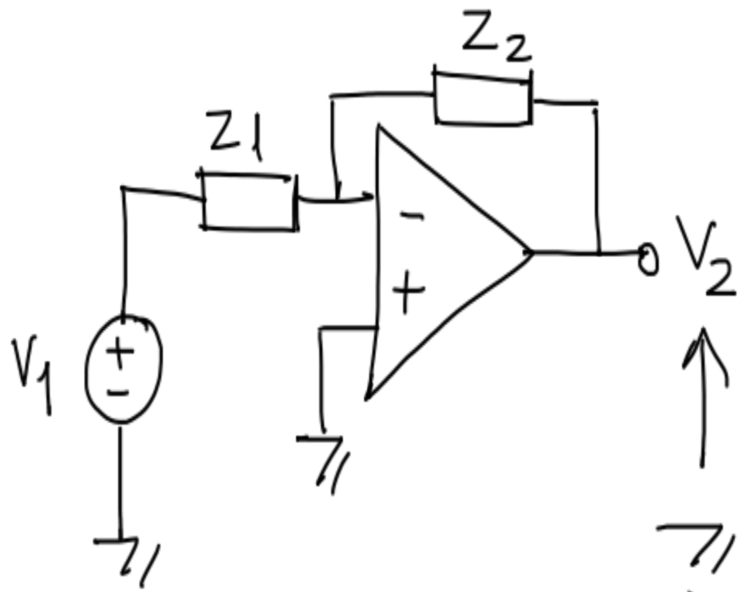
$$I_1 Z_2 + V_2 = 0$$

$$\boxed{\begin{array}{l} i_1 = i_2 = 0A \\ V_- = V_+ \end{array}}$$

$$V_2 = -I_1 Z_2 = -\frac{V_1}{Z_1} Z_2 \Rightarrow V_2 = -\frac{Z_2}{Z_1} V_1 \Rightarrow G(s) = \frac{V_2}{V_1} = -\frac{Z_2}{Z_1}$$

Ex:





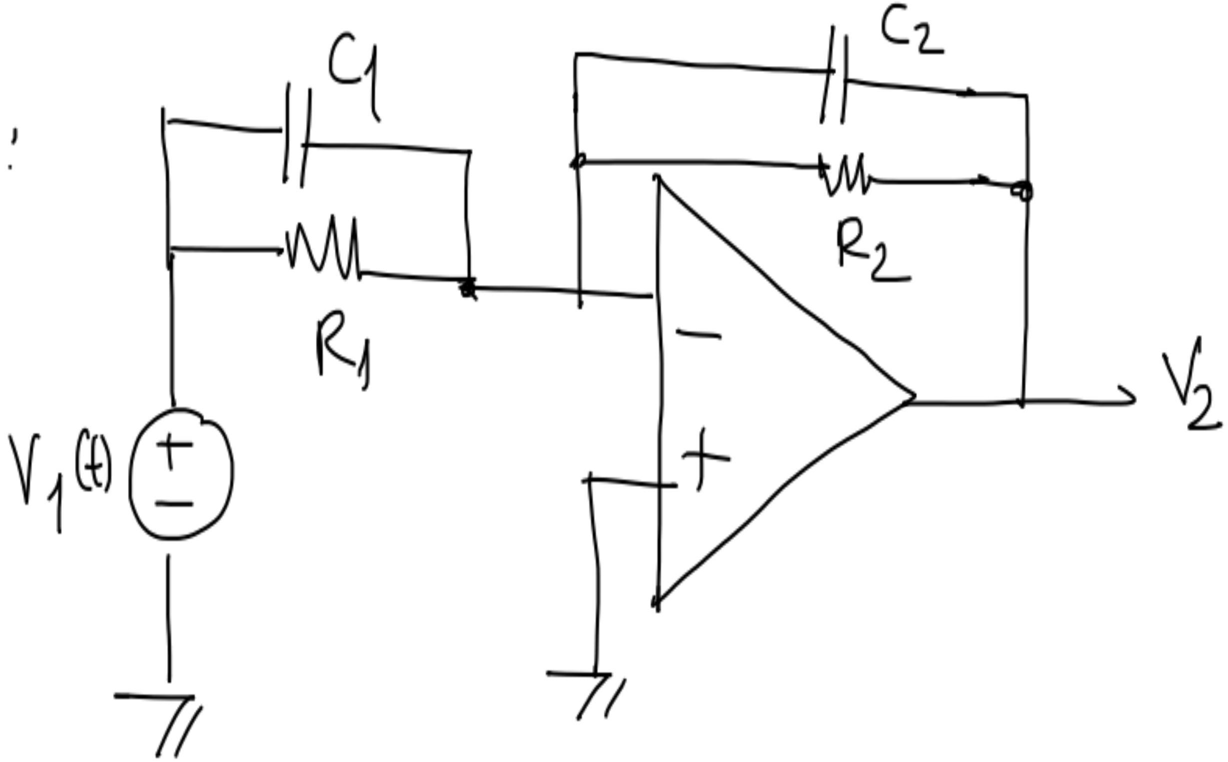
$$G(s) = -\frac{Z_2(s)}{Z_1(s)}$$

$$Z_1 = R_1$$

$$Z_2 = R_2 + \frac{1}{sC}$$

$$G(s) = -\frac{R_2 + \frac{1}{sC}}{R_1} = -\frac{sR_2C + 1}{sR_1C}$$

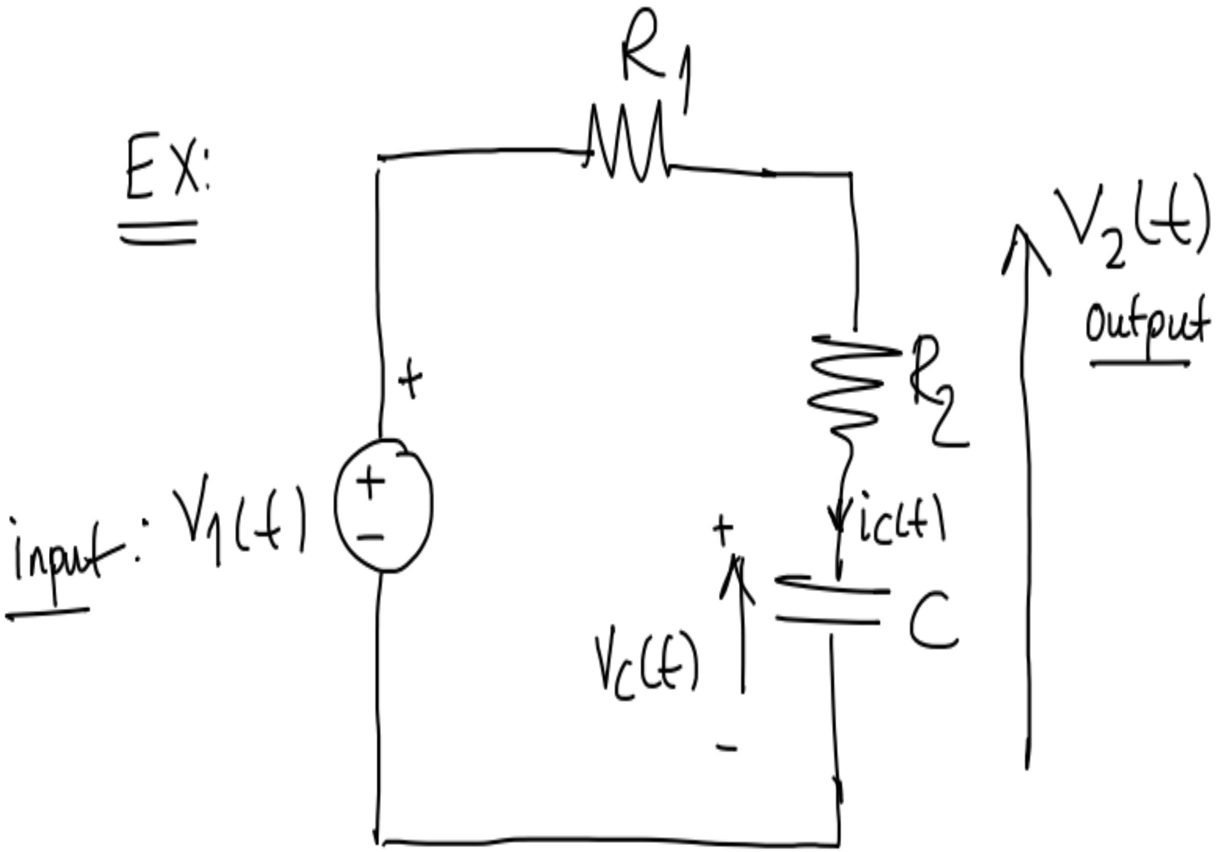
HW :



$$\frac{V_2(s)}{V_1(s)} = ?$$

State-Space Models of Electrical Circuits

Ex:



obtain SS equations in the form of

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} + D\vec{u}$$

$\vec{x}(t)$: state vector $\vec{x}(t) \in \mathbb{R}^n$

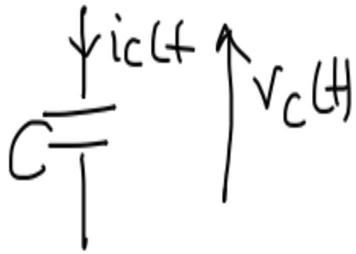
$\vec{u}(t)$: input vector $\vec{u}(t) \in \mathbb{R}^m$

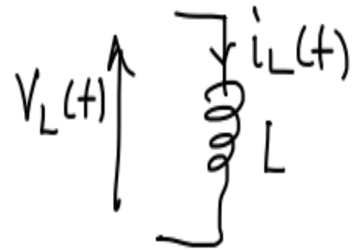
$\vec{y}(t)$: output vector $\vec{y}(t) \in \mathbb{R}^p$

For our ex: $n = 1$, $m = 1$, $p = 1$

$$i_C(t) = \frac{V_1(t) - V_C(t)}{R_1 + R_2}$$

$$C \frac{d}{dt} V_C(t) = i_C(t)$$





$$L \frac{d}{dt} i_L(t) = V_L(t)$$

Voltage accross the terminals of a capacitor is a state

current " " " " " inductance " " "

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

I need to write this term, in terms of states and inputs

$$C \frac{d}{dt} V_C(t) = i_C(t)$$

$x \triangleq V_C(t)$

$u \triangleq V_1(t)$

$y \triangleq V_2(t)$

$$y = \frac{u - x}{R_1 + R_2} \cdot R_2 + x$$

$$C \frac{d}{dt} V_C(t) = \frac{V_1(t) - V_C(t)}{R_1 + R_2}$$

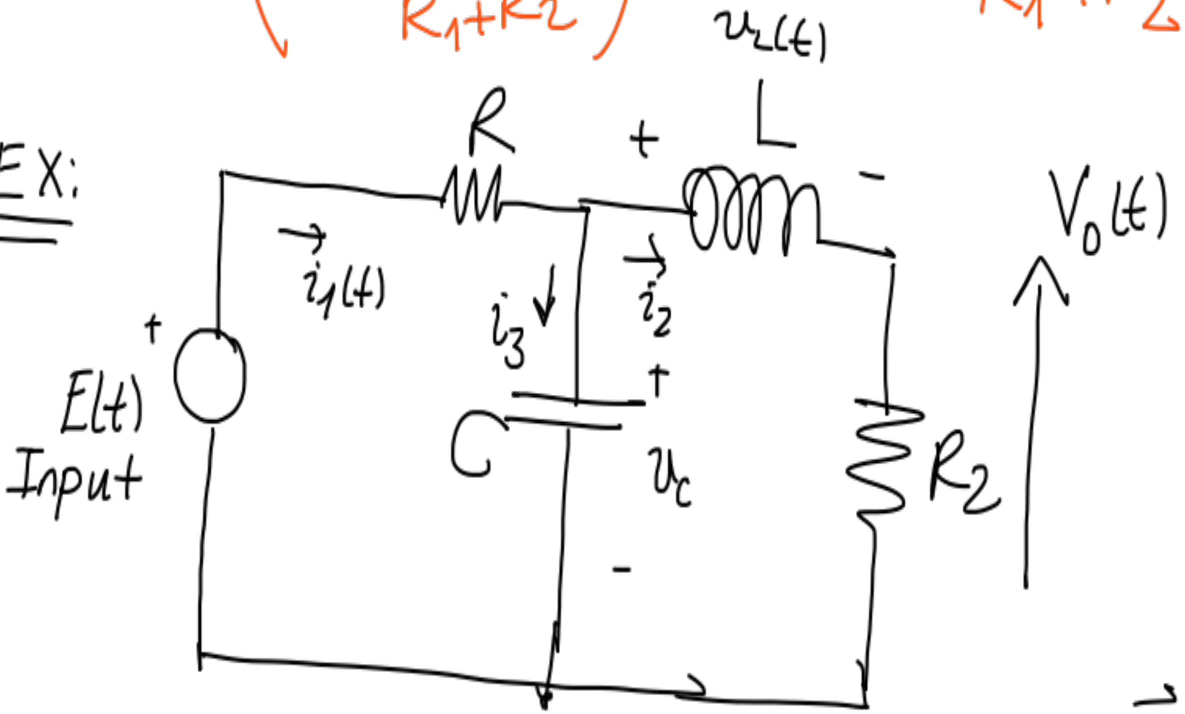
A B

$$\frac{d}{dt} x(t) = \frac{u(t) - x(t)}{(R_1 + R_2)C} = -\frac{1}{(R_1+R_2)C} x(t) + \frac{1}{(R_1+R_2)C} u(t)$$

$$y(t) = \frac{V_1 - V_C}{R_1 + R_2} \times R_2 + V_C$$

$$C = \left(1 - \frac{R_2}{R_1+R_2}\right) \qquad D = \frac{R_2}{R_1+R_2}$$

EX:



obtain SS equations for this system?

① $C\dfrac{d}{dt}v_C = i_3(t)$

② $L\dfrac{d}{dt}i_2 = v_L(t)$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} i_2(t) \\ V_C(t) \end{bmatrix}, \quad u(t) = E(t)$$

$$C\frac{d}{dt}V_C(t) = -i_2(t) + \frac{E(t)-V_C(t)}{R_1}$$

$$y = i_2 R_2$$

$$y = x_1 R_2$$

$$L\frac{d}{dt}i_2(t) = V_C(t) - R_2 i_2(t)$$

$$C\frac{d}{dt}x_2 = -x_1 + \frac{u - x_2}{R_1}$$

$$L\frac{d}{dt}x_1 = x_2 - R_2 x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{R_1 C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{R_1 C} \end{bmatrix} u$$

$$y = \begin{bmatrix} R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Assume a SS equation is given as follows:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$$

$$y = Cx + Du \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n}$$

$$D \in \mathbb{R}^{p\times m}$$

---

using Laplace TF:

$$s\underline{X}(s) = A\underline{X}(s) + B\underline{U}(s) \Longrightarrow \underbrace{[sI_{n\times n} - A]}\underline{X}(s) = B\underline{U}(s)$$

$$Y(s) = C\underline{X}(s) + D\underline{U}(s) \qquad \Rightarrow \underline{X}(s) = [sI - A]^{-1} B\underline{U}(s)$$

$$Y(s) = \left[C(sI - A)^{-1} B + D \right] U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

Matlab

```
>> A=[-----];
>> B=[-----];
>> C=[-----];
>> D=[-----];
>> system=ss(A,B,C,D)
>> tf(system)
```

In general n^{th} order differential equation can be written as

$$y^{(n)}(t)+q_{n-1}y^{(n-1)}(t)+\cdots+q_0 y(t)$$
$$=p_m u^{(m)}(t)+p_{m-1}u^{(m-1)}(t)+\cdots+p_0 u(t)$$

where y is the output, u is the input

- since this eq'n is an ODE $\Rightarrow$ q_i, p_i constant
- if $m \leq n$ $\Rightarrow$ then this eq'n has a unique sol'n. provided that $y(0), y'(0), y''(0), \ldots, y^{(n-1)}(0)$ are given.

Def'n: (n) is the order of the diff eq'n.

Assume that all initial cond'ns are identical to zero, i.e., $y(0)=y'(0)=y''(0)=\cdots=y^{(n-1)}(0)=0$

then, we can take Laplace transform of both sides

$$\left(s^n+q_{n-1}s^{n-1}+\cdots+q_1 s+q_0\right)Y(s)=\left(p_m s^m+p_{m-1}s^{m-1}+\cdots+p_0\right)U(s)$$

or

$$\frac{Y(s)}{U(s)}=\frac{p_m s^m+\cdots+p_0}{s^n+q_{n-1}s^{n-1}+\cdots+q_0}=G(s) \Leftarrow \text{Transfer function}$$

$\Rightarrow$

$$\frac{Y(s)}{U(s)} = p_m \frac{s^m + \cdots + \frac{p_0}{p_m}}{s^n + \cdots + q_0}$$

$(m \leq n)$

proper system.

$A, B, C, D \leftarrow m = n$ proper

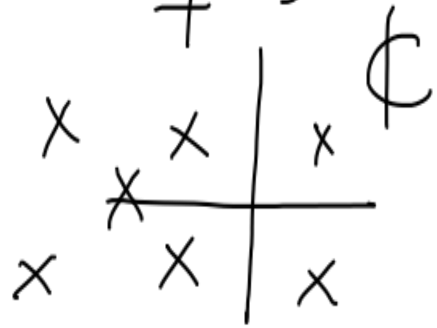
$A, B, C, D = 0 \leftarrow m < n$: strictly proper

$\cancel{A, B, C, D}$ $m > n$: nonproper

do not have a realization

we need prediction.

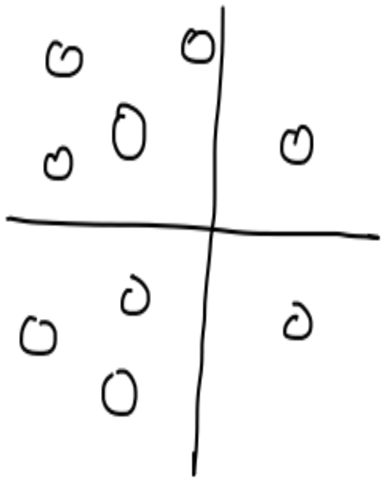
$$\frac{Y(s)}{U(s)} = p_m \frac{(s-b_1)(s-b_2)\cdots(s-b_m)}{(s-a_1)(s-a_2)\cdots(s-a_n)} := K\frac{p(s)}{q(s)}$$



$s=a_1,\ s=a_2,\ \ldots,\ s=a_n$

singular points $\Longrightarrow$ POLES

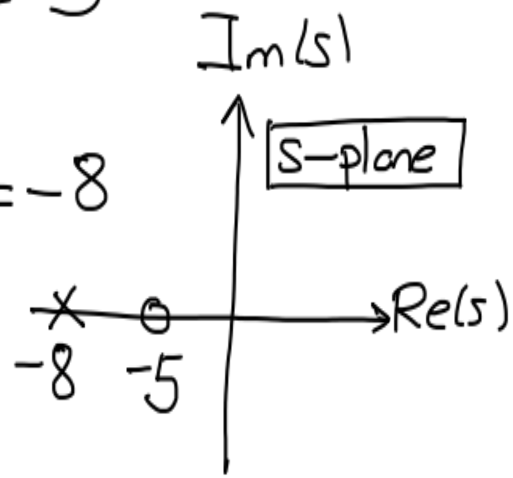
$s=b_1,\ s=b_2,\ \ldots,\ s=b_m \Longrightarrow$ ZEROS



$$\frac{Y(s)}{U(s)} = 4\frac{s+5}{s+8}$$

$\longrightarrow z_1=-5$

$\longrightarrow p_1=-8$



Given an input $U(s)$ we can obtain $y(t)$ as follows:

Given $\begin{matrix} G(s) \\ U(s) \end{matrix} \Rightarrow G(s) = \dfrac{Y(s)}{U(s)} \Rightarrow Y(s) = G(s)U(s)$

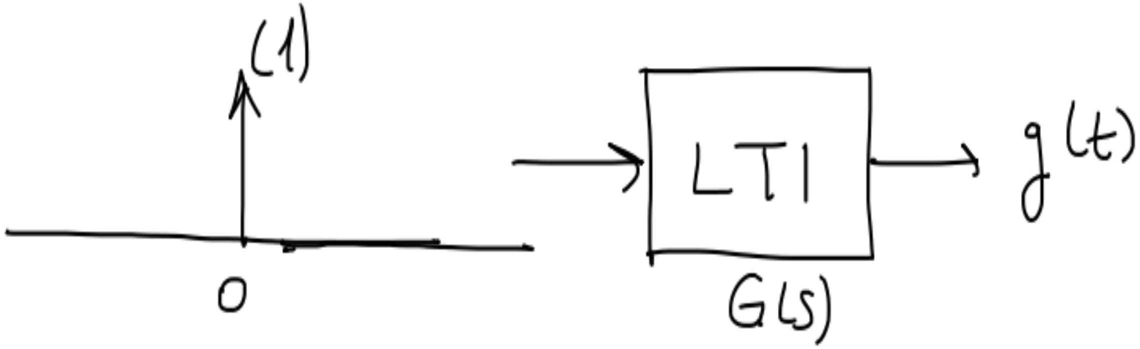
$$Y(s) = G(s)U(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$g(t) = \mathcal{L}^{-1}\{G(s)\}$ is called impulse response of the system

$$y(t) = \underbrace{\int_0^\infty g(\tau)u(t-\tau)d\tau}_{\text{convolution}} = \underbrace{\int_0^\infty g(t-\tau)u(\tau)d\tau}_{\text{convolution}}$$

if $u(t) = \delta(t)$



$$\downarrow$$

$$U(s) = 1$$

$$Y(s) = G(s)\,U(s) = G(s)\,1 = G(s)$$

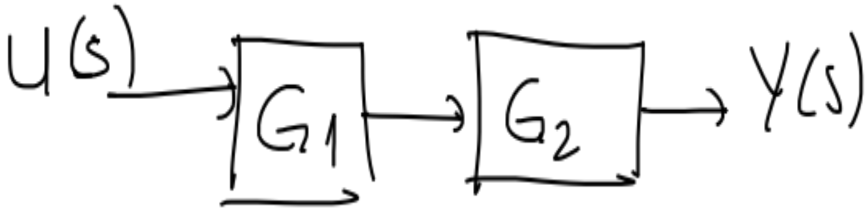
$$Y(s) = G(s)$$

$$\downarrow \mathcal{L}^{-1}$$

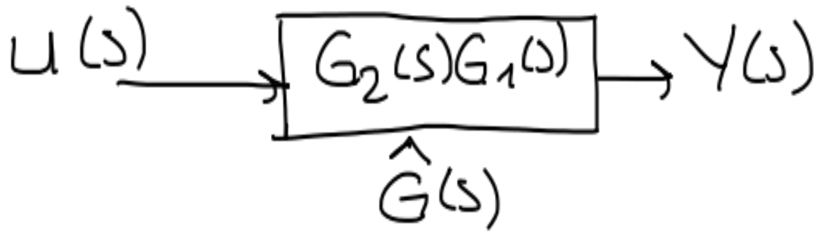
$$y(t) = g(t)$$

Block Diagrams

$$Y(s) = G(s)\,U(s)$$



$$\equiv \; Y(s) = \underbrace{G_2(s)\,G_1(s)}\,U(s)$$



series connection
tandem connection
cascade connection

in MATLAB

```
G1 = tf(num1, den1)
G2 = tf(num2, den2)
Ghat = G1 * G2    or    Ghat = series(G1, G2)
```

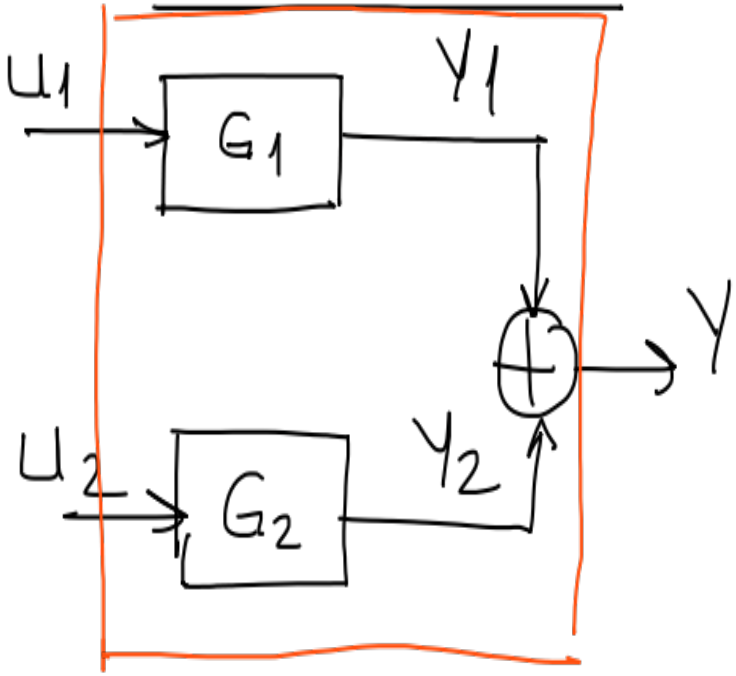
For example

$$G(s) = \frac{4s+5}{s^2+7s+8}$$

in MATLAB, you can define G as follows:

```
>> G=tf([4 5],[1 7 8]);
```

<u>Parallel Connection:</u>



$$Y_1 = G_1 U_1$$

$$Y_2 = G_2 U_2$$

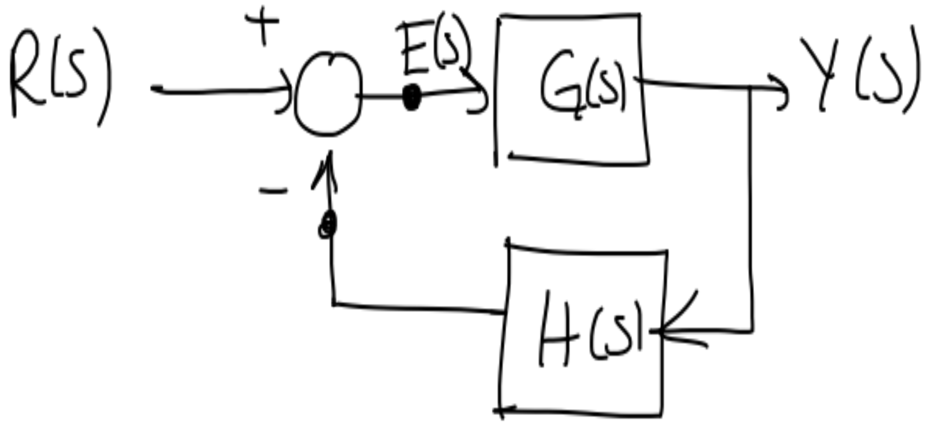
$$\Rightarrow Y = Y_1 + Y_2$$

$$Y = G_1 U_1 + G_2 U_2$$

$$Y = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

in MATLAB `>> parallel(G1,G2)`

Feedback Connection :



$$E(s) = R(s) - H(s)Y(s)$$

$$Y(s) = G(s)E(s)$$

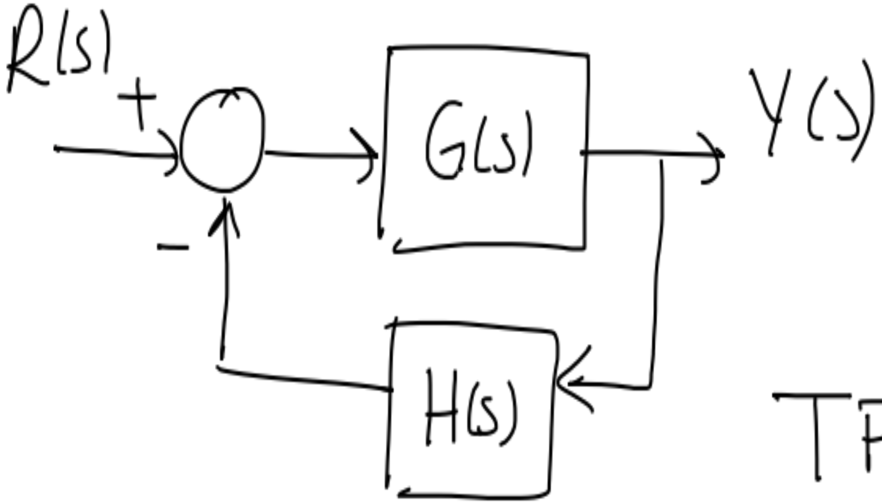
$$\Rightarrow Y(s) = G(s)\left[R(s) - H(s)Y(s)\right] = G(s)R(s) - G(s)H(s)Y(s)$$

$$\Rightarrow Y(s)\left[1 + G(s)H(s)\right] = G(s)R(s)$$

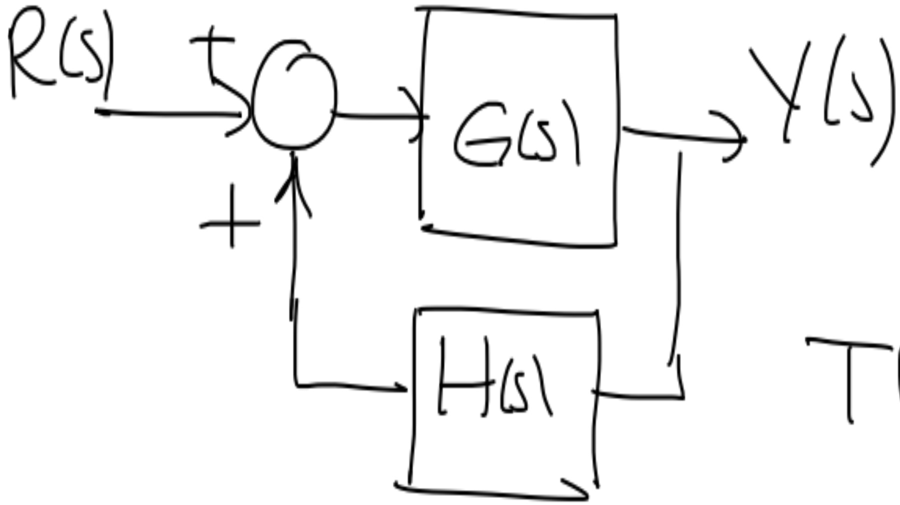
$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$G(s)$: forword t.f

$G(s)H(s)$: openloop t.f

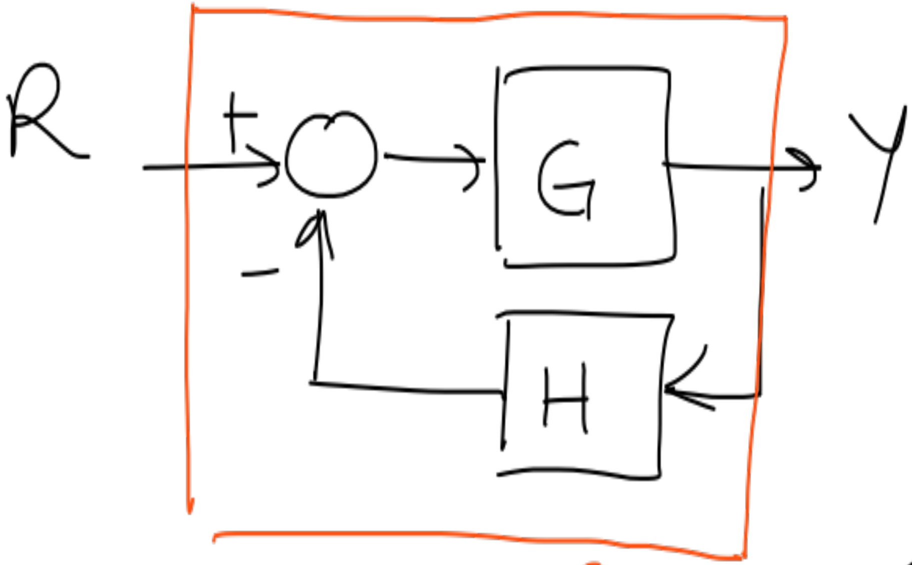


$$TF = \frac{\text{Forward TF}}{1 + \text{Open Loop TF}}$$



$$TF = \frac{G(s)}{1 - G(s)H(s)}$$

In MATLAB!



$$G_{cl} = \frac{G}{1+GH}$$

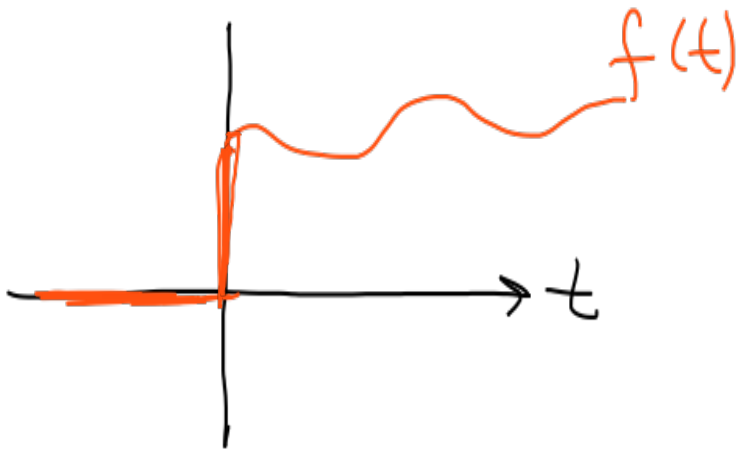
```
G = tf(num1, den1);
H = tf(num2, den2);
Gcl = feedback(G, H);
```

Some Mathematical Preliminaries

Laplace Transform:

Definition: (Laplace TF): Consider a piecewise cont's function $f(t)$ s.t

$$f(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

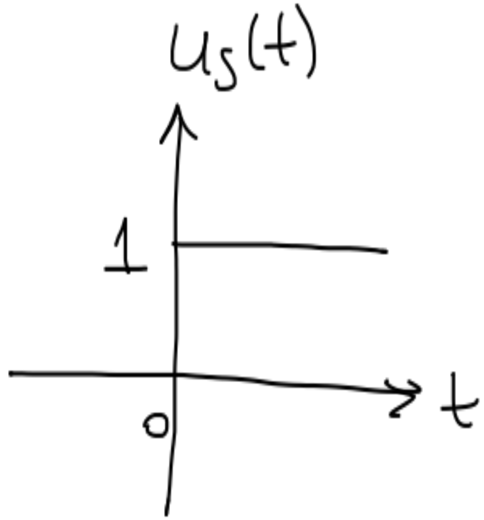


the Laplace Transform of $f(t)$ is defined as

$$F(s) := \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt \qquad Re(s) > 0 \quad s \in \mathbb{C}$$

EX: $f(t) = u_s(t)$: step function

$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



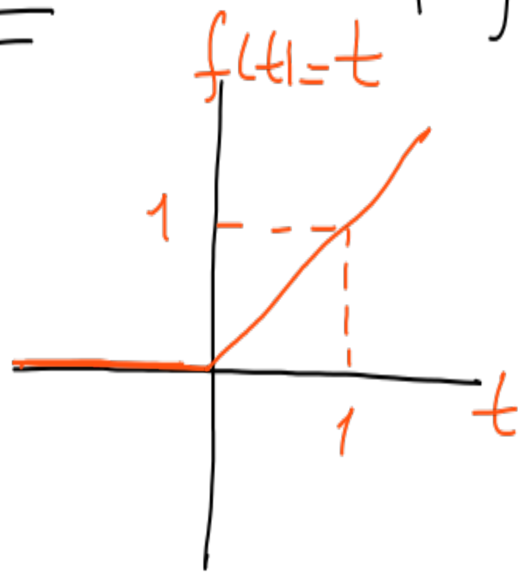
$$\Rightarrow F(s) := \int_0^\infty u_s(t) e^{-st}\, dt = \int_0^\infty 1 e^{-st} = -\frac{1}{s} e^{-st} \Big|_0^\infty = 0 - \left(-\frac{1}{s}\right)$$

$$= \frac{1}{s}$$

$$\boxed{u_s(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}}$$

unit-step function

EX: (unit-ramp function)



$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow F(s) = \int_0^\infty t e^{-st}\, dt = -t \frac{1}{s} e^{-st} \Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\, dt$$

$$= \frac{1}{s} \int_0^\infty e^{-st}\, dt = -\frac{1}{s^2} e^{-st} \Big|_0^\infty = 0 - \left(-\frac{1}{s^2}\right) = 1/s^2$$

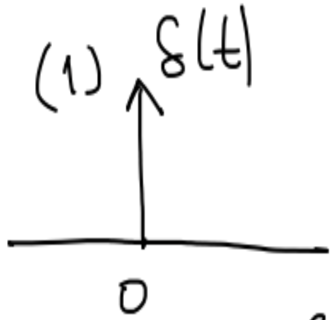
— NOTE —

Integration by parts:

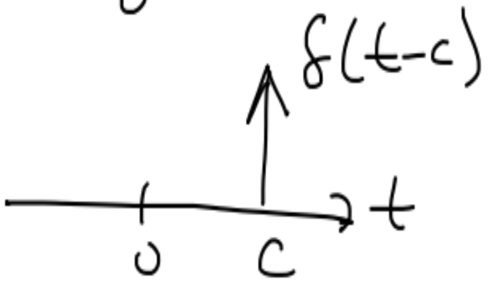
$$\int_0^\infty \begin{pmatrix}\text{derivated} \\ \text{part}\end{pmatrix} \begin{pmatrix}\text{integrated} \\ \text{part}\end{pmatrix} = \begin{pmatrix}\text{do not} \\ \text{take} \\ \text{deriv.}\end{pmatrix} \begin{pmatrix}\text{integrate}\end{pmatrix} \Bigg|_0^\infty - \int_0^\infty \begin{pmatrix}\text{both take} \\ \text{derivative and} \\ \text{integration}\end{pmatrix}$$

$$\boxed{f(t)=t \quad \overset{\mathcal{L}}{\Longleftrightarrow} \quad \frac{1}{s^2}}$$

Ex: (Laplace Transform of a unit-impulse function)



$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \quad \xRightarrow{\text{Shifting property}} \quad \int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = f(0)$$



$$\int_{-\infty}^{+\infty} \delta(t-c) = 1 \quad \Longrightarrow \quad \int_{-\infty}^{+\infty} f(t)\,\delta(t-c)\,dt = f(c)$$

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)\underbrace{e^{-st}}_{f(t)}\,dt = e^{0} = 1$$

$$\mathcal{L}\{\delta(t-c)\} = \int_{-\infty}^{+\infty} \delta(t-c)e^{-st}\,dt = e^{-sc}$$

Ex:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at} f(t) e^{-st}\, dt$$

$$= \int_0^\infty f(t) e^{-\underbrace{(s-a)}_{=s_1}t}\, dt = \int_0^\infty f(t) e^{-s_1 t}\, dt = F(s_1)$$

since $s_1 = s - a$

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

EX:

$$\mathcal{L}\{e^{at}\} = \mathcal{L}\{e^{at}\cdot 1\} = \frac{1}{s}\Bigg|_{s=s-a} = \frac{1}{s-a}$$

EX:

$$\mathcal{L}\{e^{(a+jb)t}\} = \frac{1}{s-a-jb} = \frac{1}{s-(a+jb)}$$

Ex: $\mathcal{L}\{\cos \omega t\} = ?$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Euler's identity

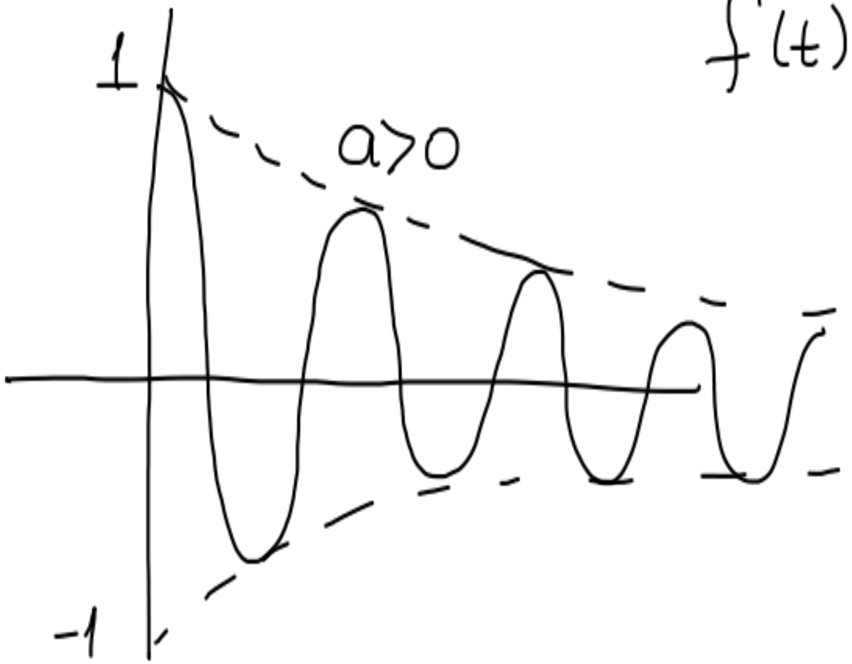
bel. me, $\mathcal{L}$ tf is a linear tf.

$$\mathcal{L}\{\cos \omega t\} = \mathcal{L}\left\{\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{j\omega t}\} + \frac{1}{2}\mathcal{L}\{e^{-j\omega t}\}$$

$$= \frac{1}{2} \cdot \frac{1}{s - j\omega} + \frac{1}{2} \cdot \frac{1}{s + j\omega} = \frac{s}{s^2 + \omega^2}$$

Similarly $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$

Ex: $\mathcal{L}\{e^{-at}\underbrace{\cos\omega t}_{f(t)}\} = \left.\dfrac{s}{s^2+\omega^2}\right|_{s=s+a} = \dfrac{s+a}{(s+a)^2+\omega^2}$



Ex: $\mathcal{L}\{e^{-at}\sin\omega t\} = \left.\dfrac{\omega}{s^2+\omega^2}\right|_{s\leftarrow s+a}$

$= \dfrac{\omega}{(s+a)^2+\omega^2}$