Underdamped case $(0<\xi<1)$

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

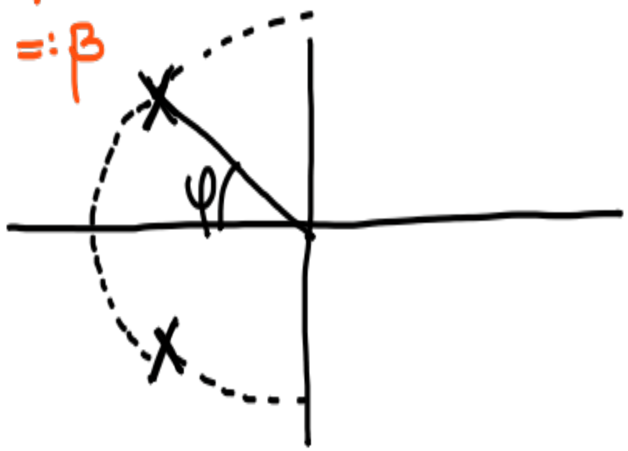
2^{nd} order prototype system.

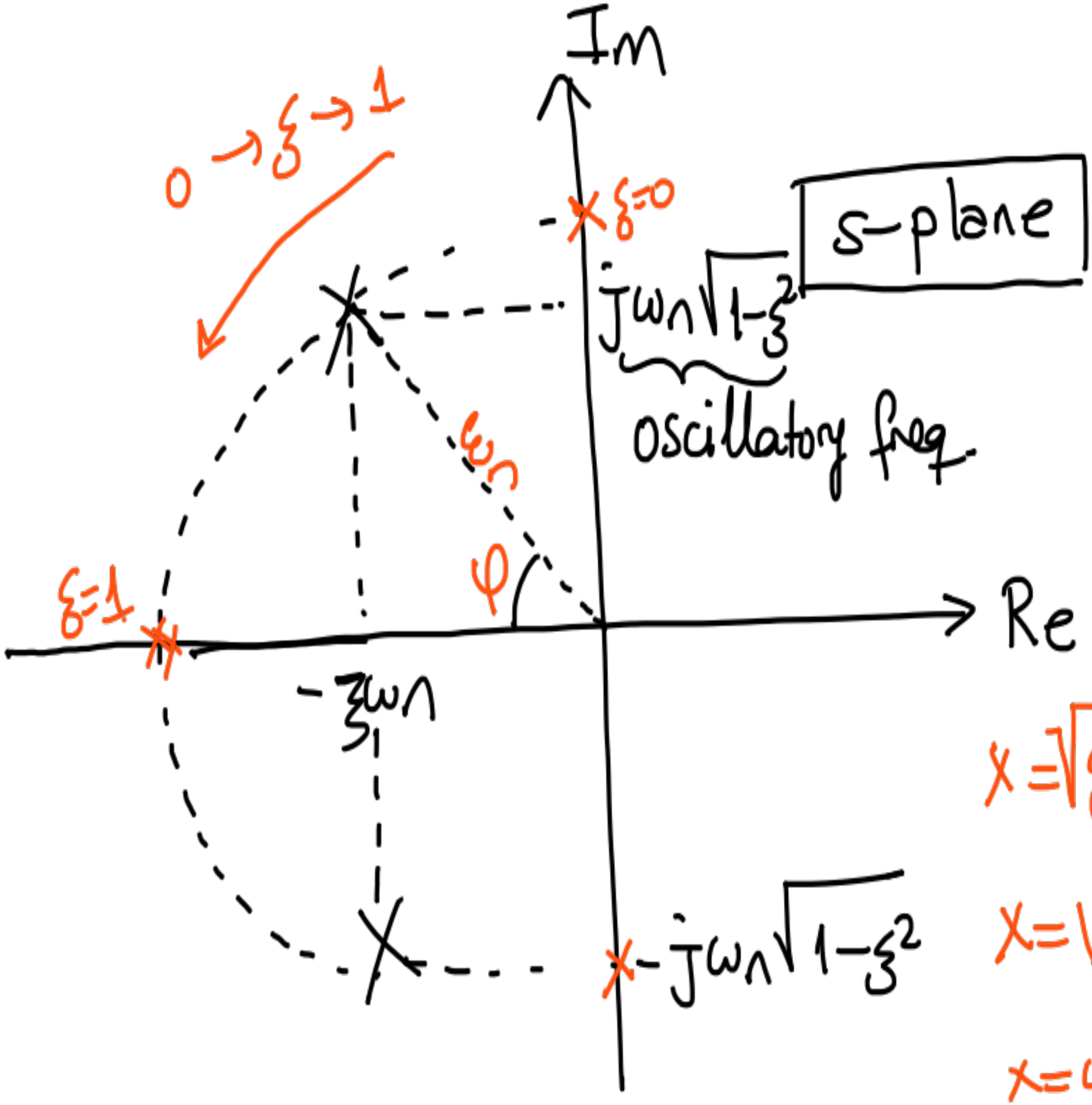
ξ: damping coeff.

ω_n: undamped natural freq. (rad/s)

poles of this poly.

$$s_{1,2} = \underbrace{-\xi\omega_n}_{Re} \pm \overbrace{j\omega_n\underbrace{\sqrt{1-\xi^2}}_{=:\beta}}^{Im} = -\xi\omega_n \pm j\omega_n\beta \in \mathbb{C}$$





$$\cos\varphi = \frac{-\xi\omega_n}{X}$$

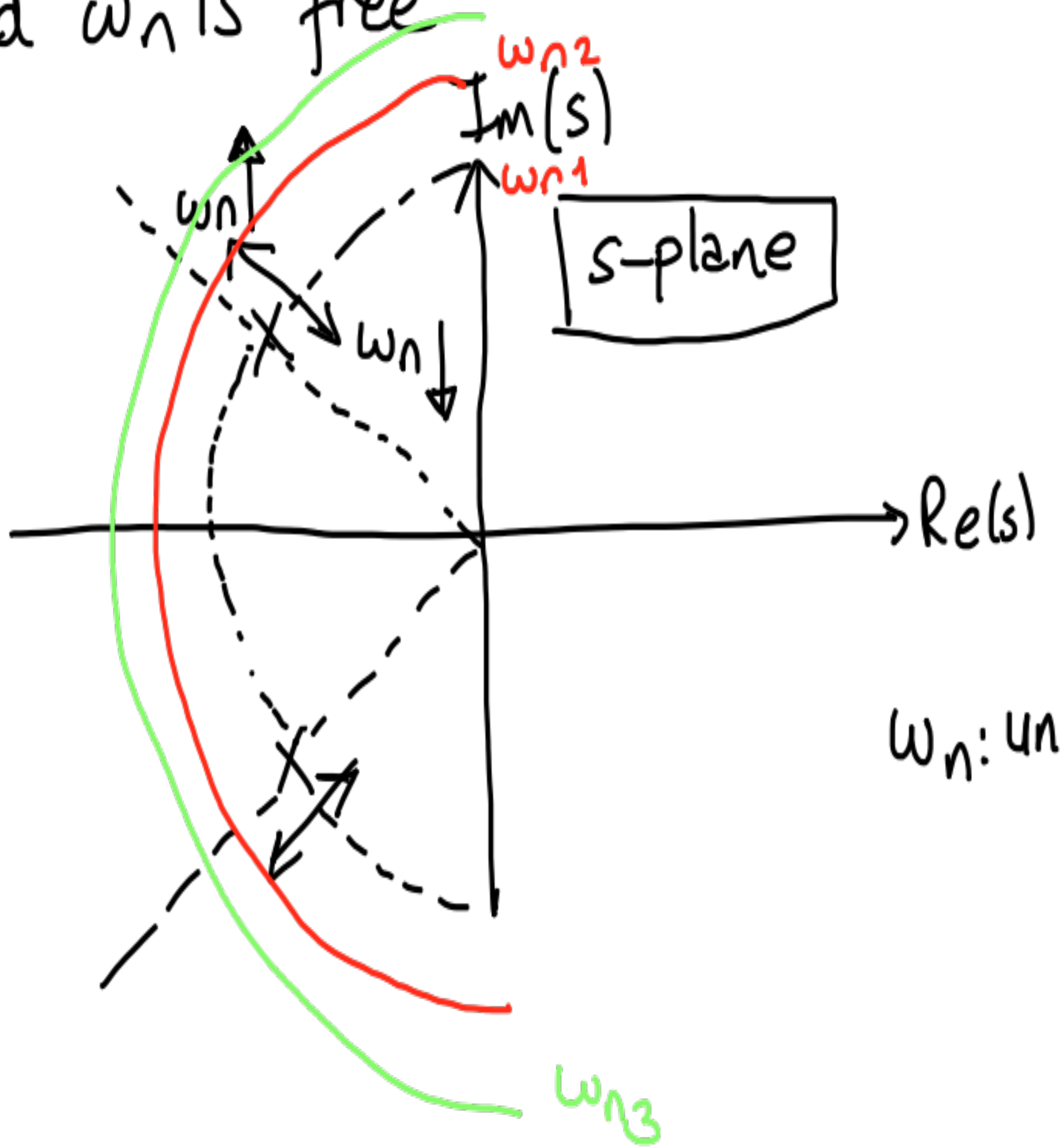
$$\Rightarrow \cos\varphi = \frac{-\xi\omega_n}{\omega_n} = -\xi$$

$$X = \sqrt{\xi^2\omega_n^2 + \omega_n^2(1-\xi^2)}$$

$$X = \sqrt{\cancel{\xi^2\omega_n^2} + \omega_n^2 - \cancel{\xi^2\omega_n^2}}$$

$$X = \omega_n$$

Variation of poles when ξ is constant and ω_n is free

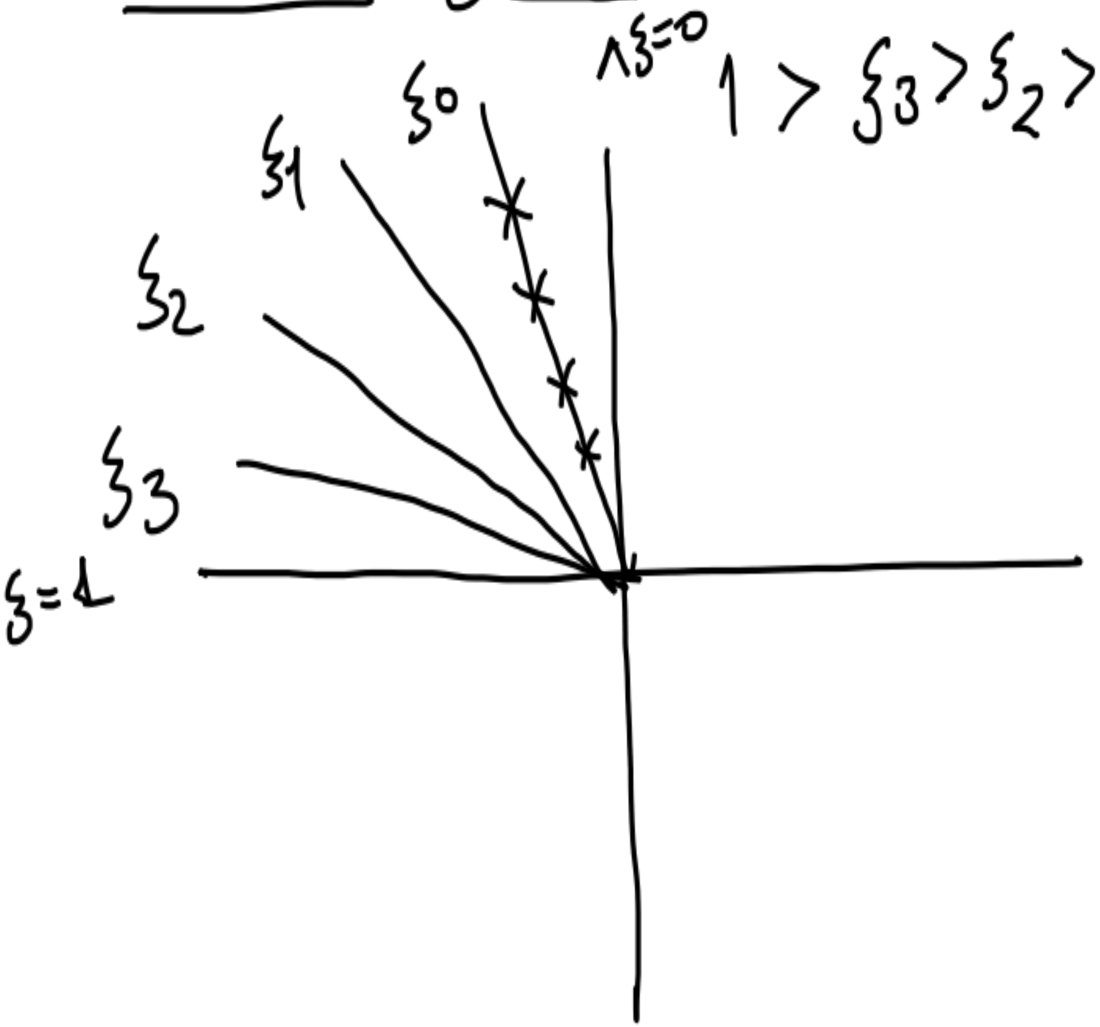


$\omega_{n3} > \omega_{n2} > \omega_{n1}$

ω_n: undamped natural freq

constant ξ regions

$$1 > \xi_3 > \xi_2 > \xi_1 > \xi_0 > 0$$

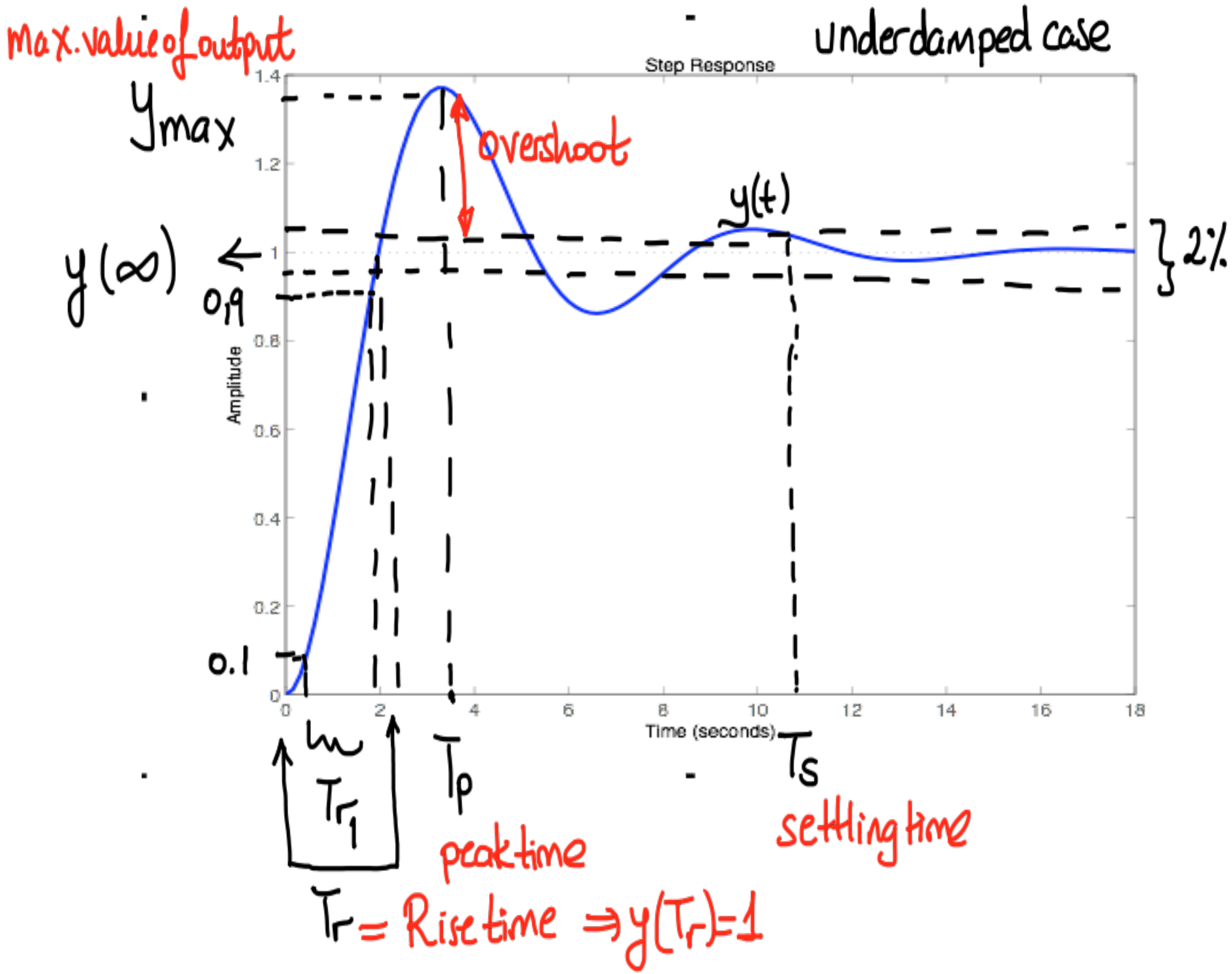


Definitions

ξ: damping ratio

ω_n: undamped natural freq. [rad/sec]

$\omega_d = \omega_n \sqrt{1-\xi^2}$ = damped freq.


underdamped case
Step Response
max. value of output
y_{max}
overshoot
$y(t)$
$y(\infty)$
2%
0,9
0.1
Amplitude
Time (seconds)
T_{r_1}
T_p
peak time
T_s
settling time
T_r = Rise time $\Rightarrow y(T_r) = 1$

T_{r_1} = (10 to 90% rise time) = Time required for the response to rise to 90% of its steady-state value (in this case = 1) from 10% of its steady-state value

$$T_{r_1} \cong \frac{2.16\xi + 0.6}{\omega_n} \qquad (0{,}3 \leq \xi \leq 0{,}8)$$

$T_{r_1}(\xi, \omega_n)$

T_p : peak-time $\Rightarrow y(T_p) = y_{max}$

$$\dot{y}(t) = \frac{\omega_n}{\beta} e^{-\xi\omega_n t} \sin(\omega_n \beta t) \overset{!}{=} 0$$

$$\Rightarrow t = T_p = \frac{\pi}{\omega_n \beta} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

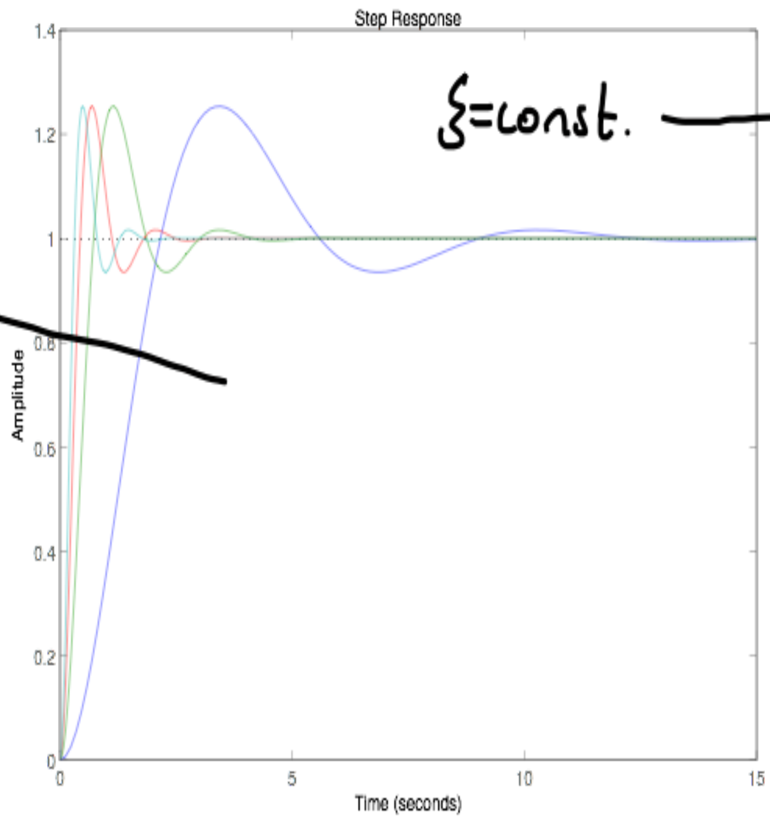
ξ varies in a small interval $\therefore$ T_p is mostly controlled by ω_n

as $\omega_n \uparrow$, $T_p \downarrow$

$$\boxed{y_{max} = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}$$

NOTE:

y_{max} depends only on ξ

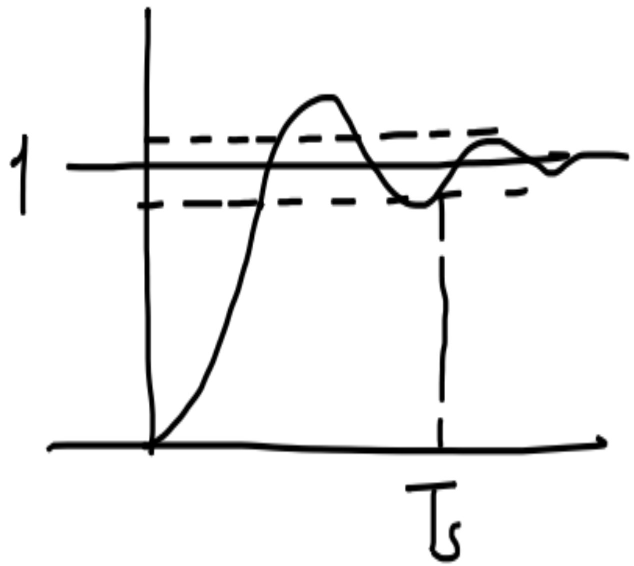


ξ=const. → overshoot is same

$\omega_n \uparrow$

T_s : settling time : time after which response remains within 2% of its steady-state value [sec]

$$y(t) = 1 - \frac{e^{-\xi \omega_n t}}{\beta} \sin(\,----\,)$$



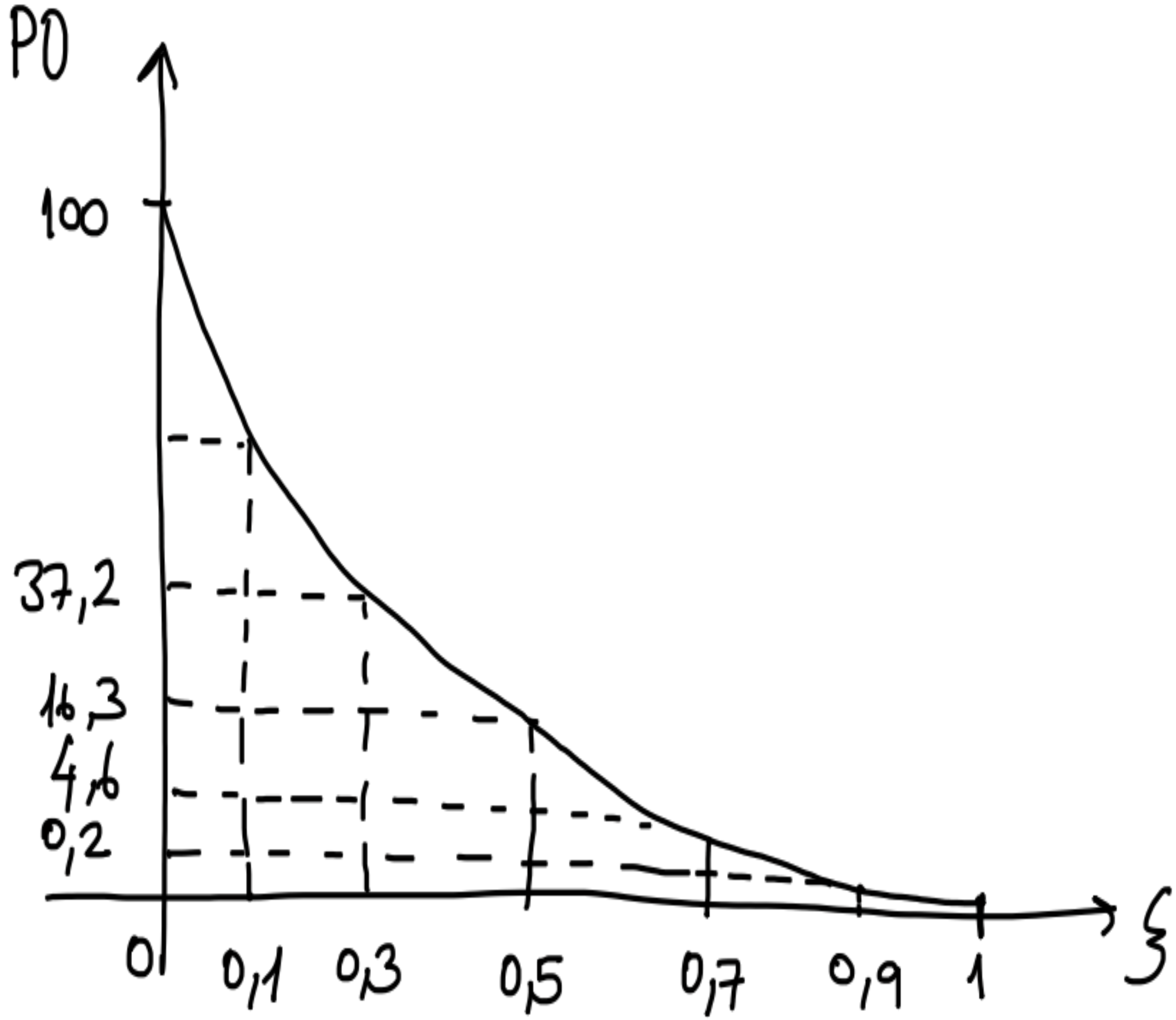
$$\Rightarrow 0.02 \overset{!}{=} \frac{e^{-\xi \omega_n T_s}}{\beta} \Rightarrow \boxed{T_s \cong \frac{4}{\xi \omega_n}}$$

P.O : Percent overshoot

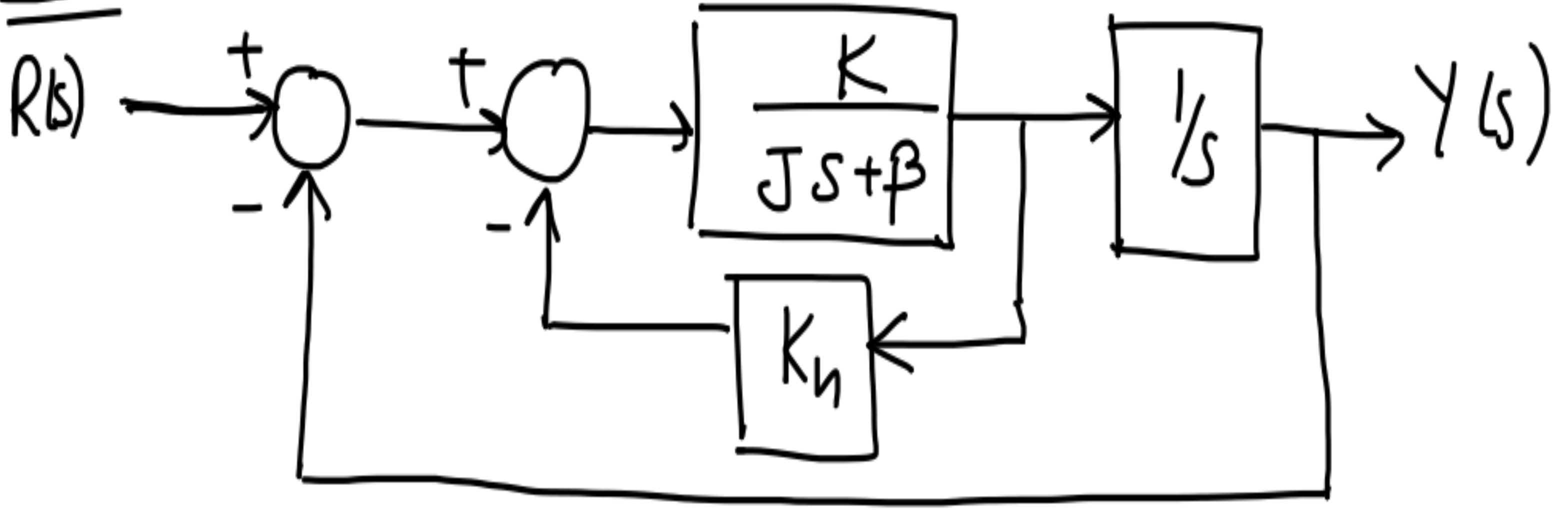
$$P.O = \frac{y_{max} - y(\infty)}{y(\infty)} \times 100$$

since $y(\infty) = 1 \Rightarrow y_{max} = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$

$$\Rightarrow \boxed{P.O = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100} \Rightarrow \boxed{\xi = \frac{-\ln \% PO}{\sqrt{\pi^2 + \ln^2 \% PO}}}$$


PO
100
37,2
16,3
4,6
0,2
0
0,1
0,3
0,5
0,7
0,9
1
ξ

HW



Determine K, K_h so that maximum overshoot for the unit step response is 0,2 and peak time is 1 sec. With the values of K, K_h, obtain rise-time and T_s assuming that $J = 1\,kg\,m^2$, $\beta = 1\,Nm\,rad/sec$

Performance Requirements

$$R_s \longrightarrow \boxed{\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}} \longrightarrow Y(s)$$

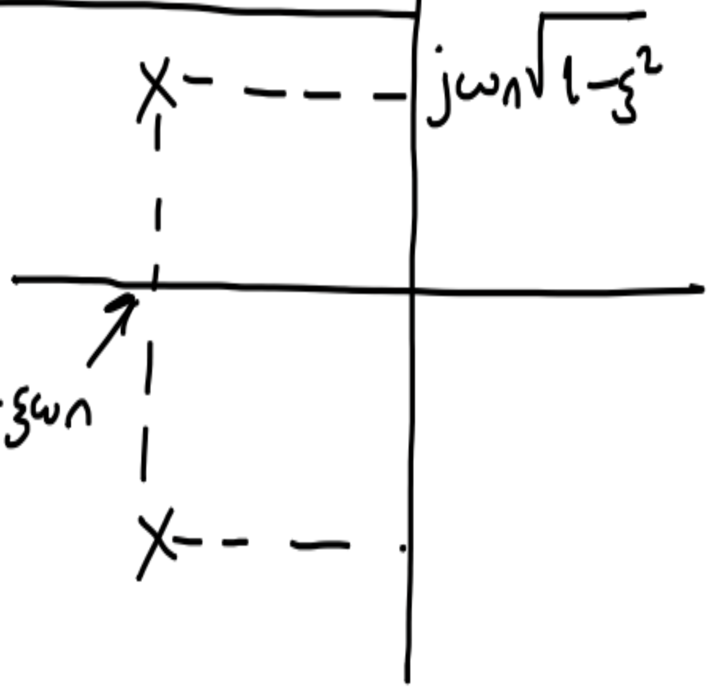
$$T_{r_1} \cong \frac{2,16\,\xi + 0,6}{\omega_n} \qquad (0,3 \leq \xi \leq 0,8)$$

$$\boxed{PO = 100\, e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

$$\boxed{y_{max} = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

$$\boxed{T_s \cong \frac{4}{\xi\omega_n}} = \frac{4}{Re(\text{dominant pole})}$$



Requirements:

✓ 1. Fast response: small T_r, T_p

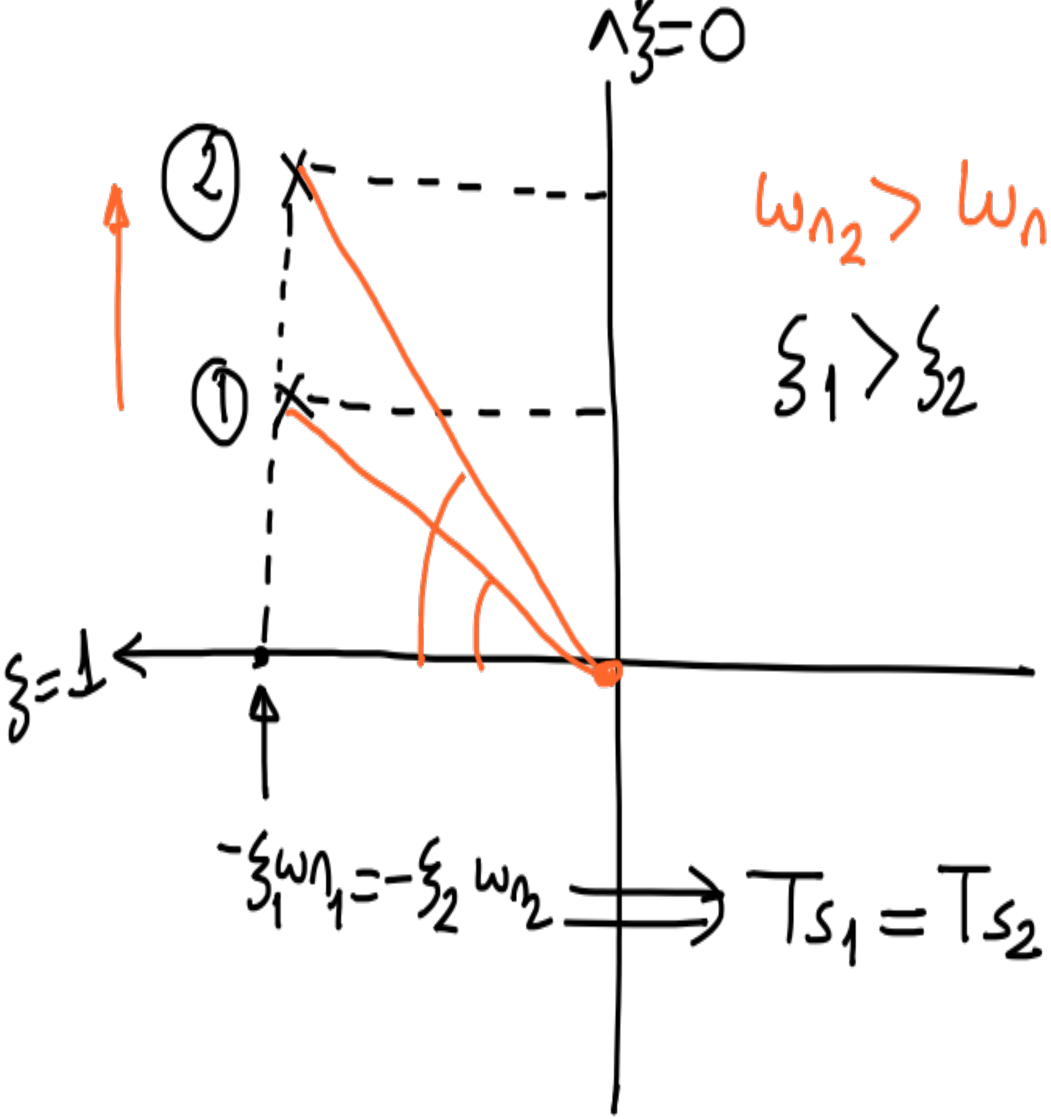
✓ 2. as close to unit-step as possible: $\rightarrow$ small O.S ($\zeta \uparrow$)
$\rightarrow$ small T_s

Observations:

1. $\zeta \uparrow \Rightarrow$ P.O $\downarrow$, $T_p \uparrow$ (fixed ω_n)

2. $\zeta \uparrow \Rightarrow T_s \downarrow$ (fixed ω_n)

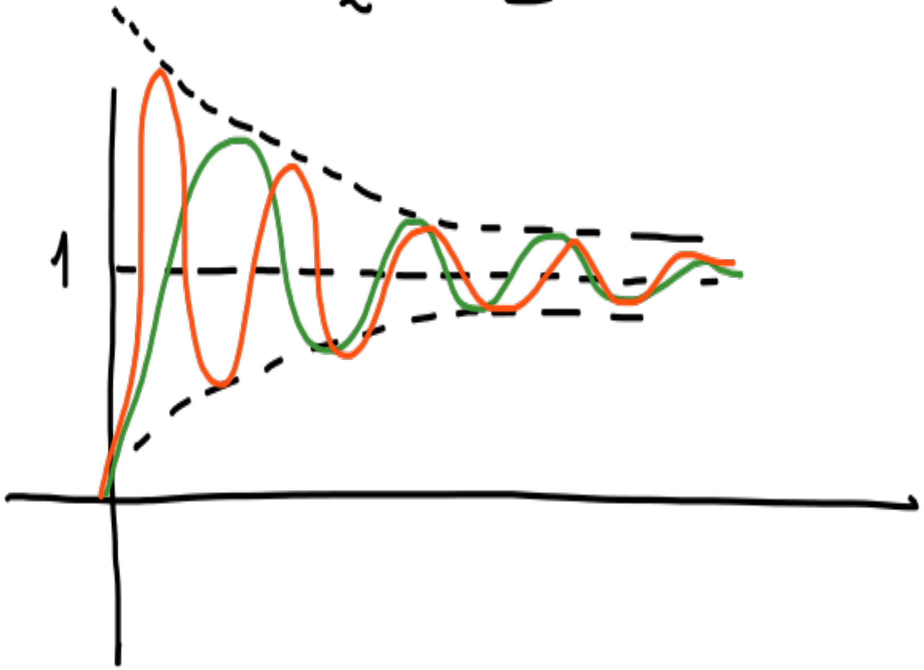
Fast response requirements:



$\omega_{n_2} > \omega_{n_1}$

$\xi_1 > \xi_2 \longrightarrow PO_2 > PO_1$

$-\xi_1 \omega_{n_1} = -\xi_2 \omega_{n_2} \Longrightarrow T_{s_1} = T_{s_2}$

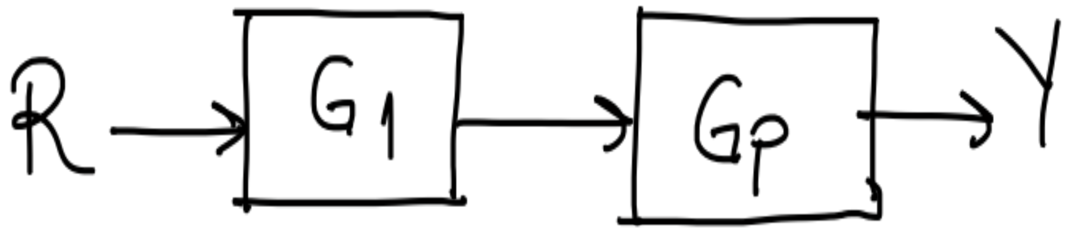


Effects of Feedback

① FB reduces sensitivity to parameter-variations

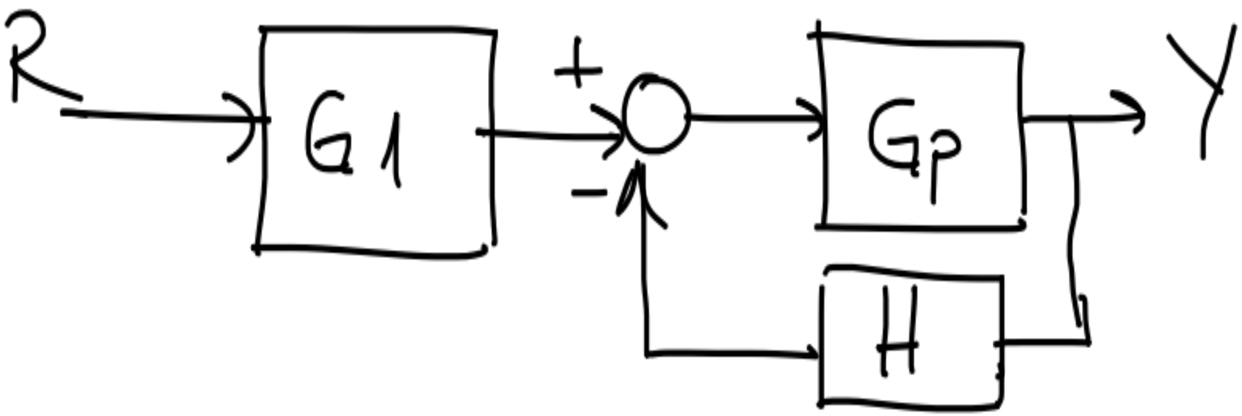
Consider the following configurations:

Open-loop case



$$\frac{Y}{R} = G_p G_1 = M_0$$

closed-loop case



$$\frac{Y}{R} = \frac{G_p G_1}{1 + G_p H} = M_1$$

Let's define sensitivity

$$S_{\alpha}^{M(\alpha)} = \frac{\%\ \text{change in } M}{\%\ \text{change in } \alpha} = \frac{\frac{dM}{M}}{\frac{d\alpha}{\alpha}} = \frac{\alpha}{M} \cdot \frac{dM}{d\alpha}$$

sensitivity

$$\boxed{S_{\alpha}^{M(\alpha)} = \frac{\alpha}{M} \cdot \frac{dM}{d\alpha}}$$

M: gain of the system

α: parameter

open-loop case:

$M_0 = G_p G_1$

$$S_{G_p}^{M_0} = \frac{G_p}{M_0} \cdot \frac{dM_0}{dG_p}$$

$$S_{G_p}^{M_0} = \frac{\cancel{G_p}}{\cancel{G_p} G_1} \cdot G_1 = 1$$

closed-loop case:

$$M_1 = \frac{G_p G_1}{1 + G_p H}$$

$$S_{G_p}^{M_1} = \frac{\cancel{G_p}(1+\cancel{G_p H})}{\cancel{G_p}\cancel{G_1}} \cdot \frac{\cancel{G_1}(1+G_p H) - H G_p \cancel{G_1}}{(1+G_p H)^{\cancel{2}}}$$

$$S_{G_p}^{M_1} = \frac{1}{1+G_p H}$$

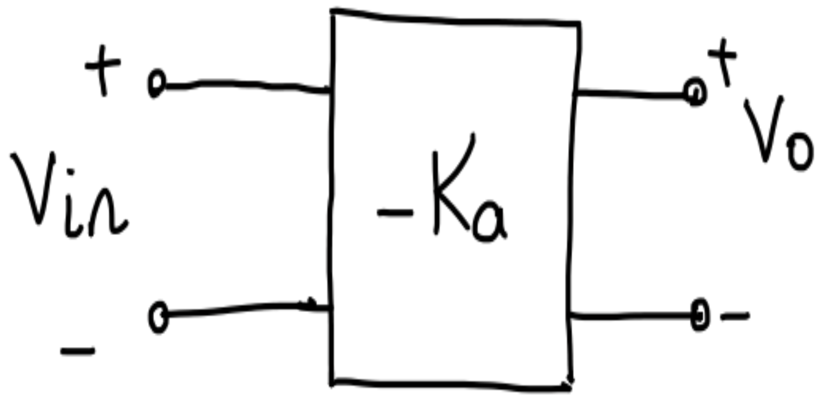
generally:

$G_p H \gg 1$

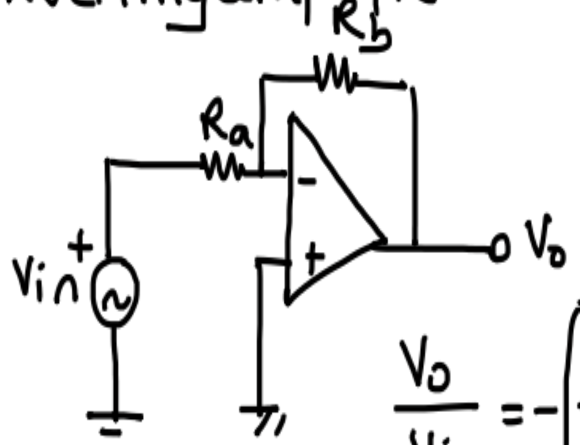
$\Rightarrow S_{G_p}^{M_1} \ll S_{G_p}^{M_0}$

FB and the sensitivity

EX: Consider a fb amplifier



inverting amplifier

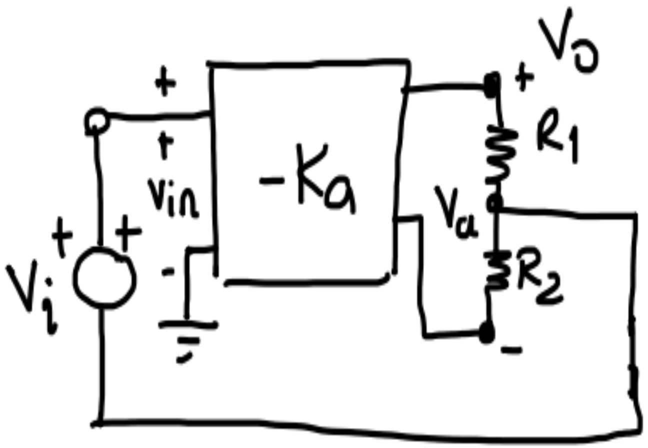


$$\frac{V_o}{V_{in}} = -\overbrace{\frac{R_b}{R_a}}^{K_a} = -K_a$$

$$V_o = -K_a V_{in}$$

$$\Rightarrow \frac{V_o}{V_{in}} = -K_a$$

$$\boxed{S^{-K_a}_{K_a} = 1}$$



$$V_o = -K_a V_{in}$$

$$V_o = -K_a (V_i + V_a)$$

$$V_a = \underbrace{\frac{R_2}{R_1+R_2}}_{\beta} V_o$$

$$V_o + \frac{K_a R_2}{R_1+R_2} V_o = -K_a V_i$$

$$T = \frac{V_o}{V_i} = \frac{-K_a}{1 + \frac{K_a R_2}{R_1+R_2}} = \frac{-K_a}{1+K_a\beta}$$

$$S_{K_a}^{T} = \frac{K_a}{T} \cdot \frac{dT}{dK_a} = \frac{K_{\cancel{a}}}{\frac{-\cancel{K}a}{\cancel{(1+\beta K_a)}}} \cdot \frac{-1-\beta \cancel{K_a}+\beta \cancel{K}a}{(1+\beta K_a)^{\cancel{2}}} = \frac{1}{1+\beta K_a}$$

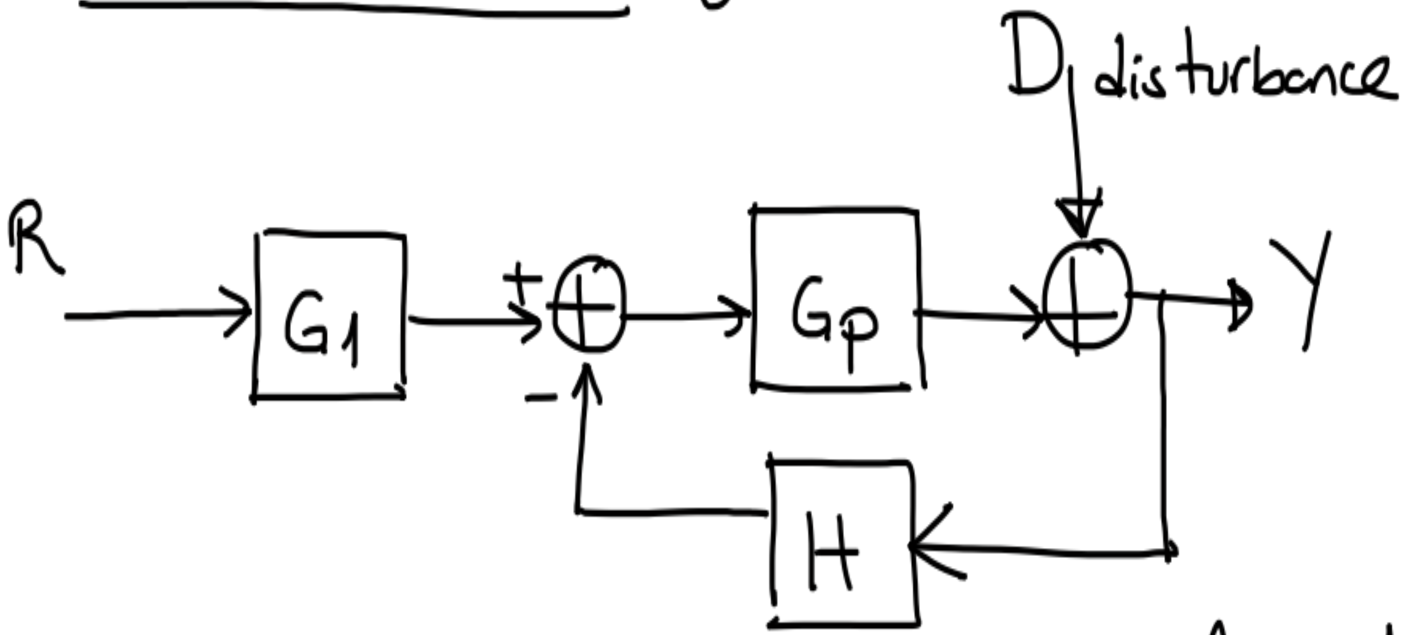
$$T = -\frac{K_a}{1+\beta K_a}$$

$$|1+\beta K_a| \gg 1$$

$$\Longrightarrow S_{K_a}^{-K_a} \gg S_{K_a}^{T}$$

if K_a is large $\Rightarrow$ sensitivity is reduced.

② Reduces Sensitivity to output disturbance

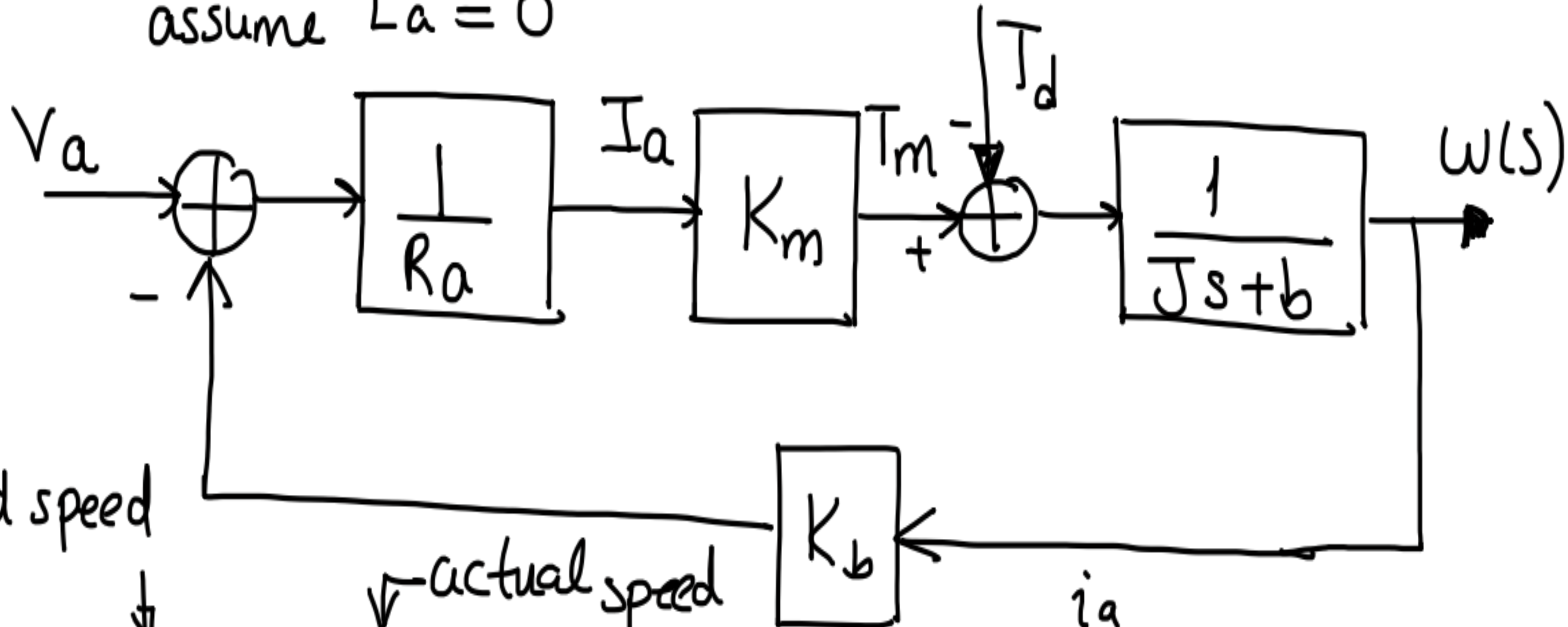


$$\left.\frac{Y}{D}\right|_{R=0} = (1+G_pH)^{-1} = \frac{1}{1+G_pH}$$

As $1+G_pH$ gets larger, disturbance effect on the output gets smaller

Ex:

assume $L_a = 0$

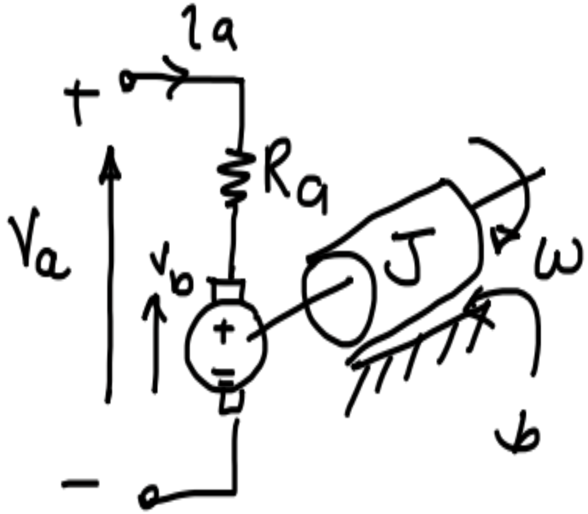


desired speed ↓ ↙ actual speed

$$E(s) \triangleq \omega_d(s) - \omega(s)$$

Assume $\omega_d = 0$. We want to minimize $E(s)$ so that $\omega_d(s) \equiv \omega(s)$ under the disturbace T_d

$$E_o(s)\Big|_{\omega_d=0} = \frac{\frac{1}{Js+b}}{1+\frac{1}{Js+b}\cdot K_b\cdot\frac{1}{R_a}\cdot K_m}\cdot T_d(s)$$

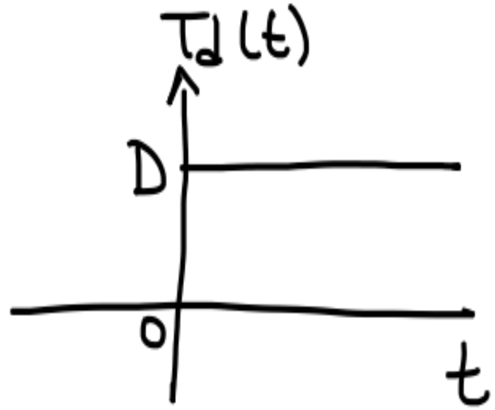


$V_a = R_a i_a + V_b \qquad T_m = (Js+b)\omega$

$V_b = K_b\dot{\theta} = K_b\omega$

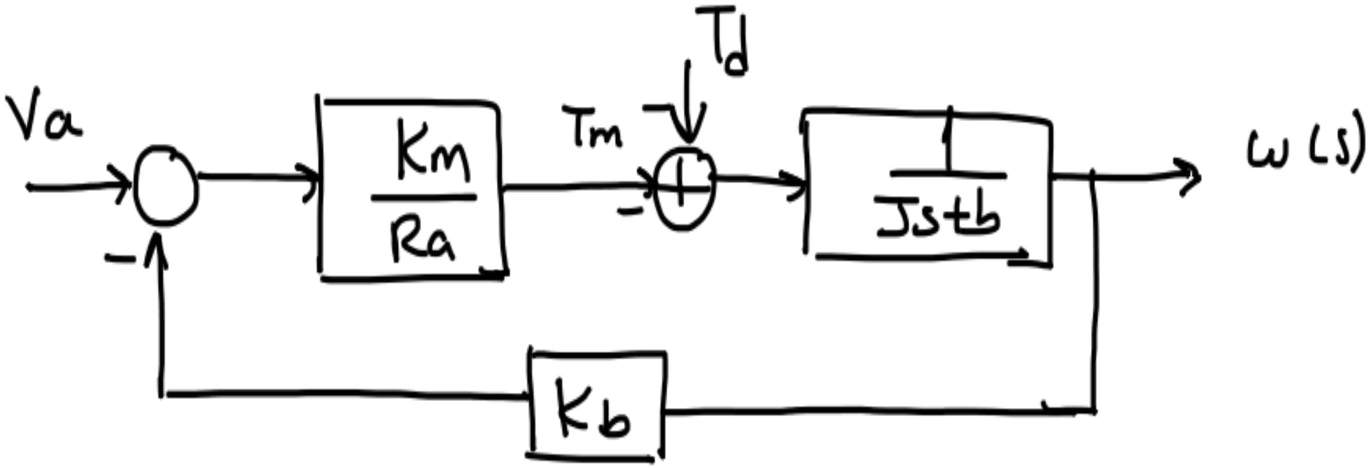
$T_m = i_a \cdot K_m$

$$E_o(s)\Big|_{\omega_d=0} = \frac{1}{Js + b + \frac{K_b K_m}{R_a}} \cdot T_d(s)$$

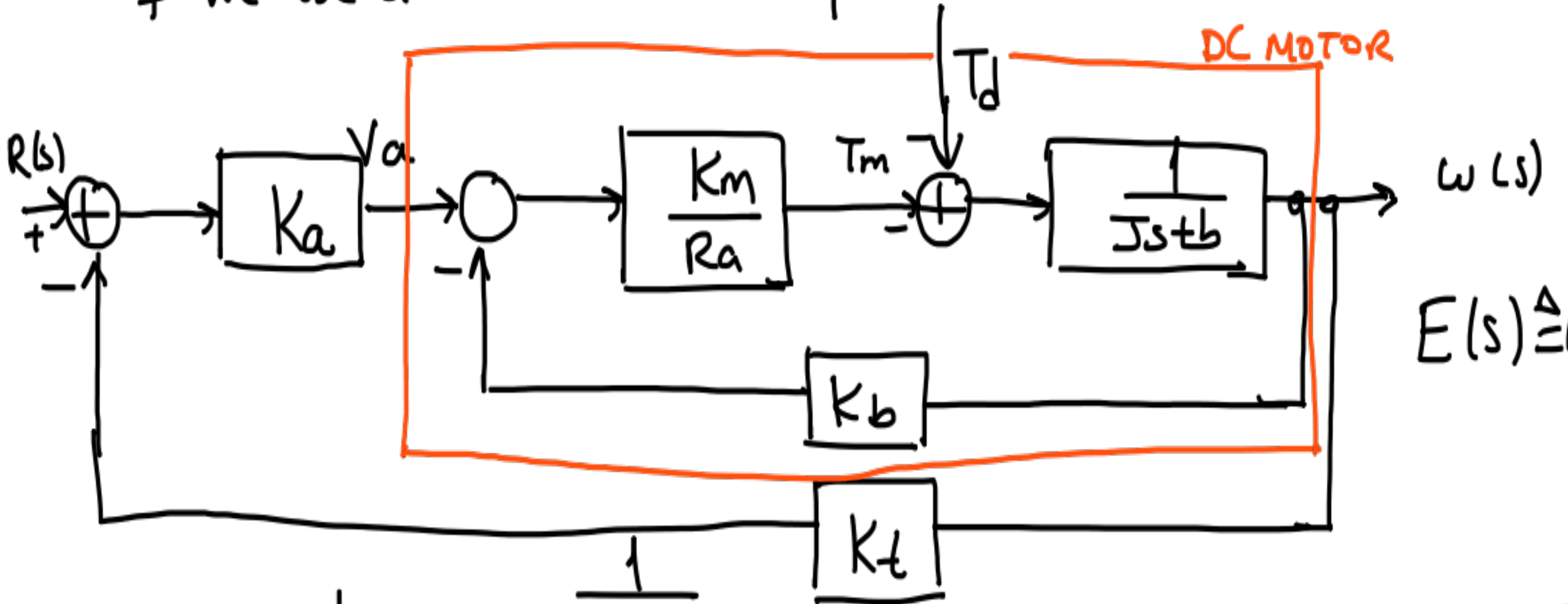


assume our disturbance effect is constant $\left(T_d(s) = \frac{D}{s}\right)$

$$\lim_{t \to \infty} e_o(t) = \lim_{s \to 0} s\,E_o(s) = \frac{D}{b + \frac{K_b K_m}{R_a}} = e_o(\infty)$$



if we use a tachometer speed control system then we have



$$E(s) \triangleq \omega_d(s) - \omega(s)$$

$$E_{cl}(s)\Big|_{R(s)=0} = \frac{\frac{1}{Js+b}}{1 + \frac{1}{Js+b} K_a \frac{K_m}{R_a}\left(K_t + \frac{K_b}{K_a}\right)} T_d(s) \simeq \frac{\frac{1}{Js+b}}{\frac{1}{Js+b} K_a \frac{K_m}{R_a}\left(K_t + \frac{K_b}{R_a}\right)} T_d$$

$$e_{cl}(\infty) = \lim_{s \to 0} s E_{cl}(s) = \frac{D}{\frac{K_a K_m}{R_a}\left(K_t + \frac{K_b}{R_a}\right)}$$

$$e_{ol}(\infty) = \frac{D}{b + \frac{K_b K_m}{R_a}}$$

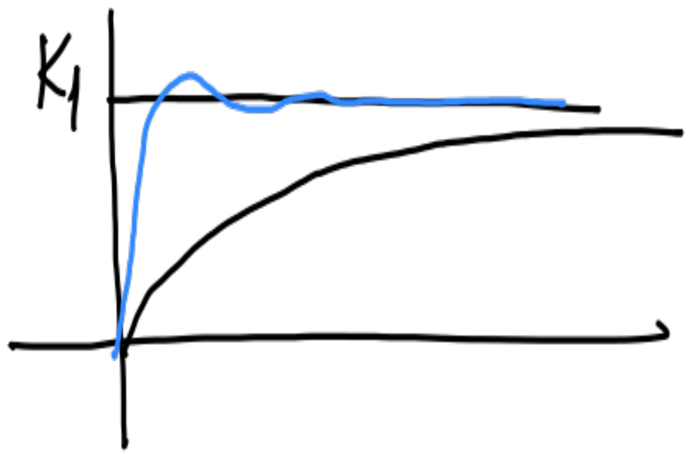
This can be made less than $e_o(\infty)$ for large amplifier gain K_a

$$\Rightarrow e_{cl}(\infty) \ll e_o(\infty)$$

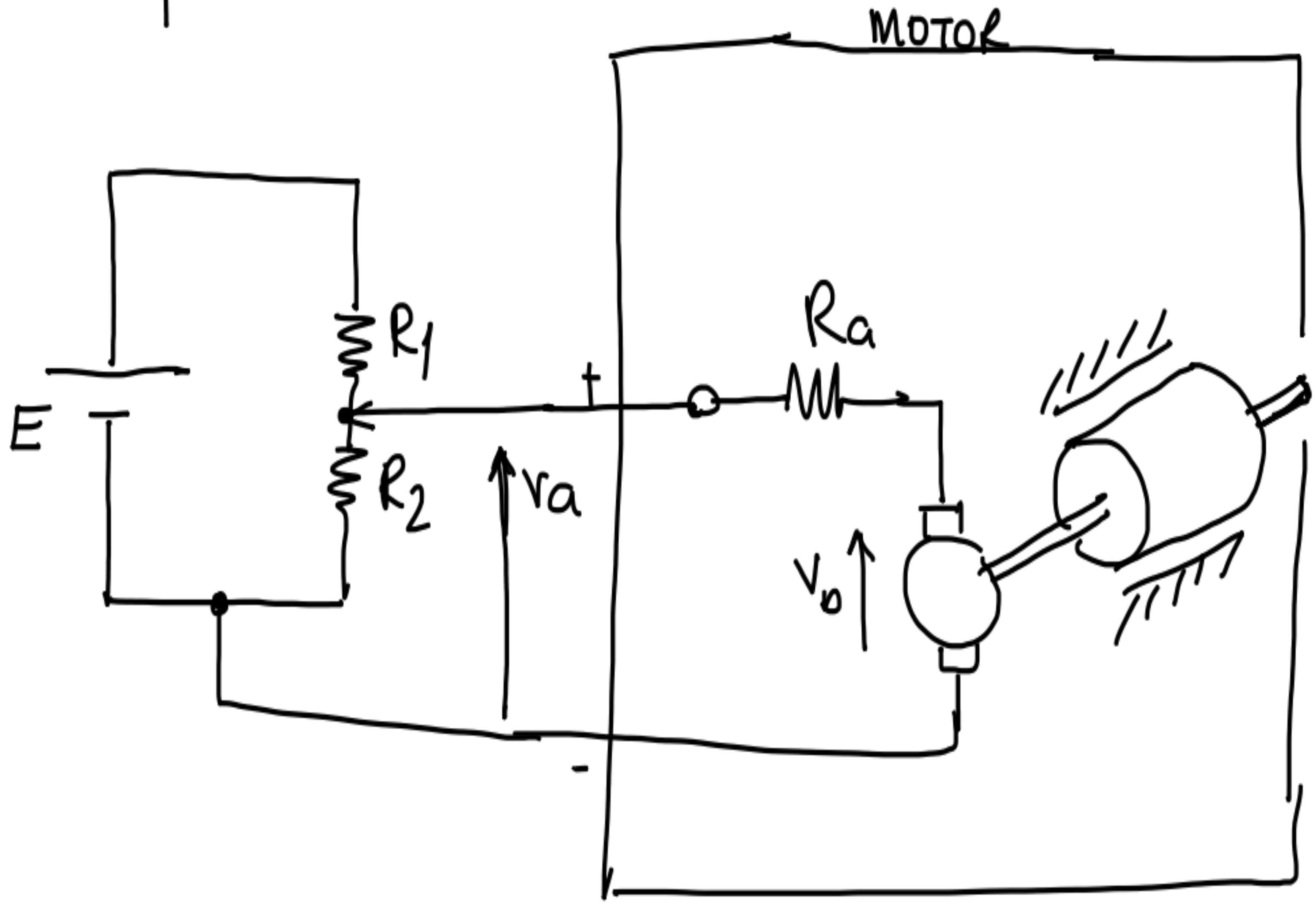
③ Control of Transient Response

$$\left.\frac{\omega(s)}{V_a(s)}\right|_{T_d=0} = \frac{K_m}{R_a \cdot b + K_b \cdot K_m + Js} = \frac{\overbrace{\dfrac{K_m}{R_a b + K_b K_m}}^{K_1}}{1 + \underbrace{\dfrac{J}{R_a b + K_b K_m}}_{\tau_1} s} = \frac{K_1}{1+\tau_1 s}$$

$$\left.\frac{\omega(s)}{V_a(s)}\right|_{T_d=0} = \frac{K_1}{1+\tau_1 s}$$



Let us assume that $V_a(s)$ is adjusted by using a potentiometer then

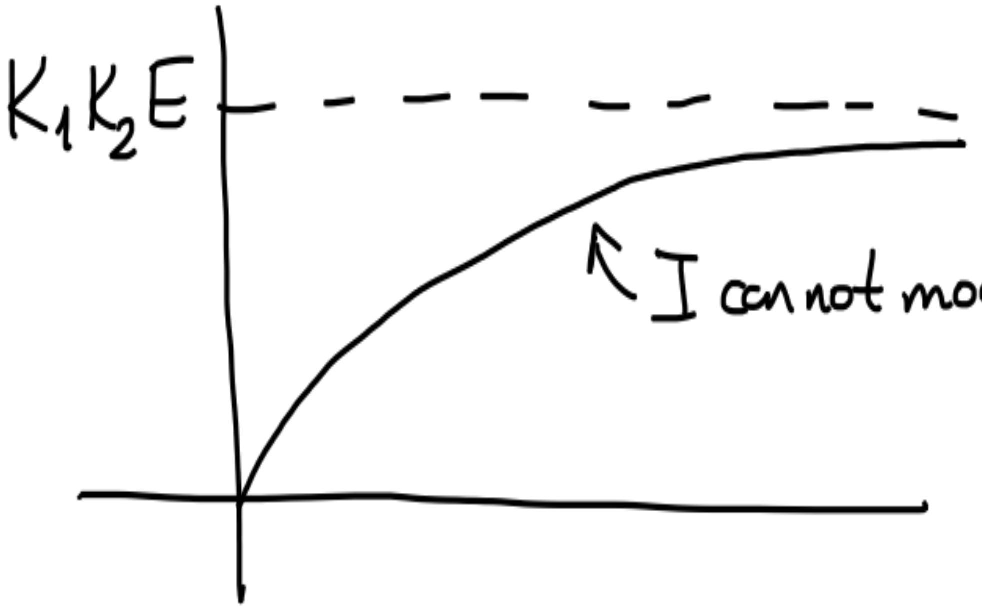


$$V_a = \underbrace{\frac{R_2}{R_1+R_2}}_{K_2} E$$

$$\omega(s) = \frac{K_1}{1+\tau_1 s} \cdot \frac{K_2 E}{s} = K_1 K_2 E \left(\frac{1}{s} - \frac{\tau_1}{1+\tau_1 s} \right)$$

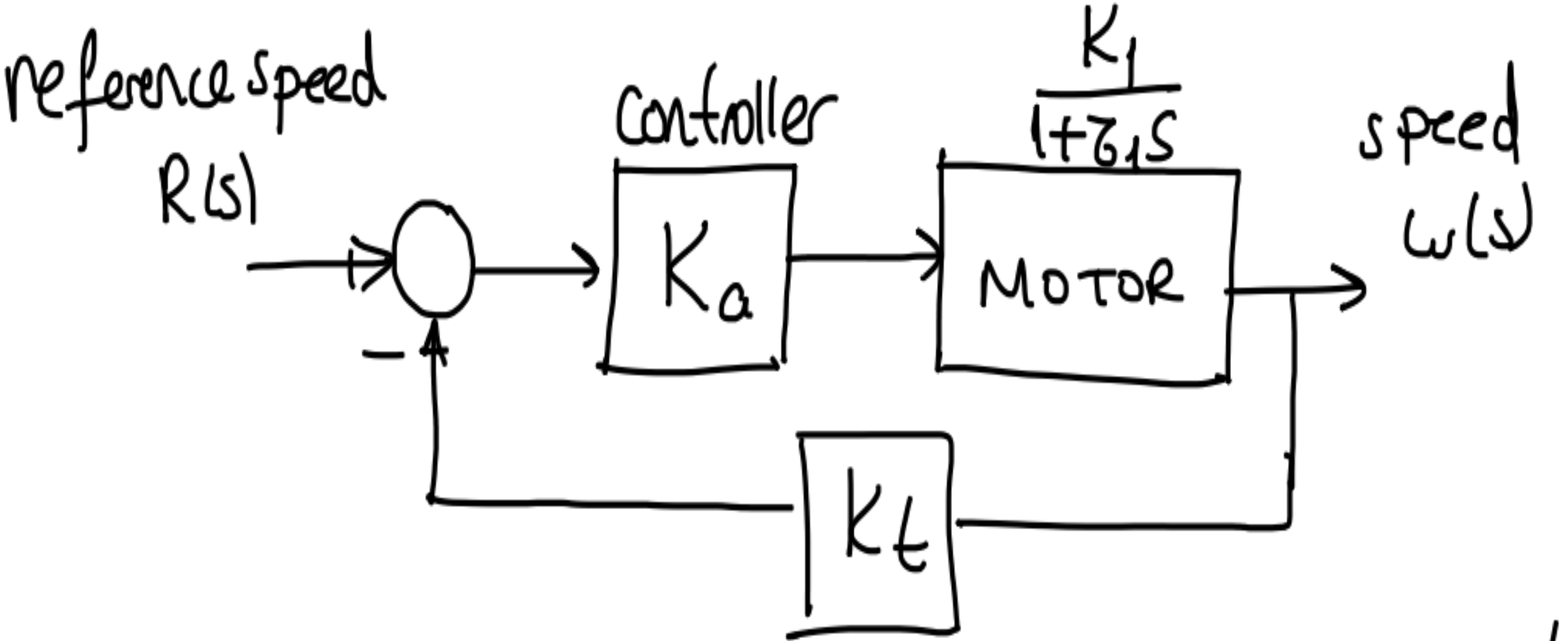
where $K_2 = \dfrac{R_2}{R_1 + R_2}$

$$\omega(t) = K_1 K_2 E \left(1 - e^{-\frac{t}{\tau_1}} \right)$$



I cannot modify this transient by adjusting K_2.

Tachometer feedback: P PI, PD, PID, etc.



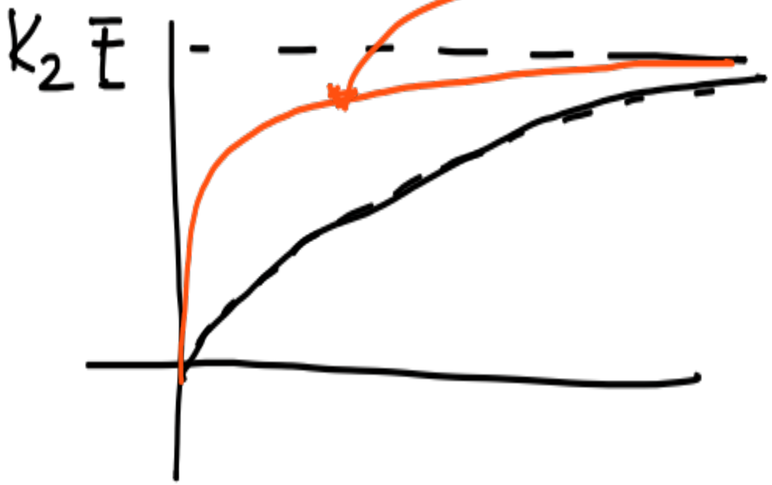
$$R(s) = \frac{K_2 E}{s}$$

$$\frac{\omega(s)}{R(s)} = \frac{\frac{K_a K_1}{1+\tau_1 s}}{1 + \frac{K_a K_1 K_t}{1+\tau_1 s}} =: \frac{K_a K_1 / \tau_1}{s + p}$$

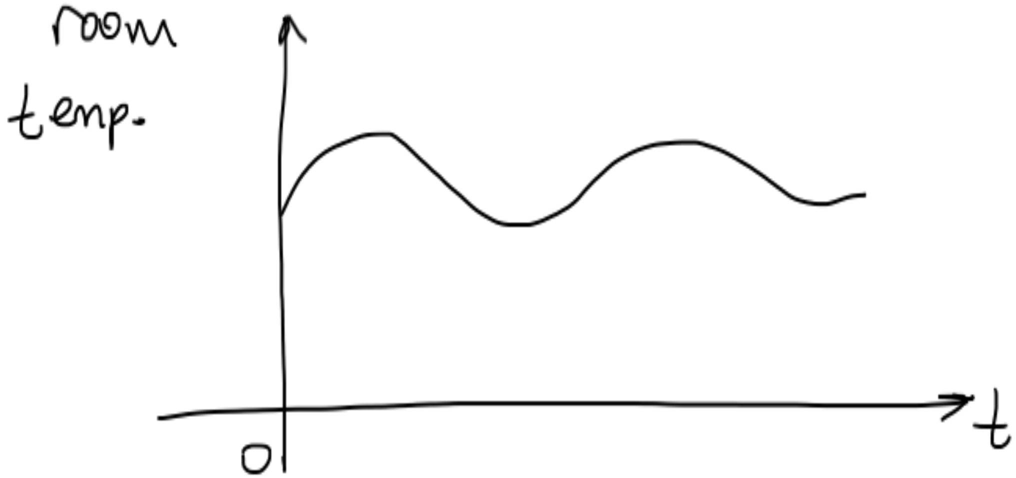
where $p \triangleq \dfrac{1 + K_1 K_t K_a}{\tau_1}$

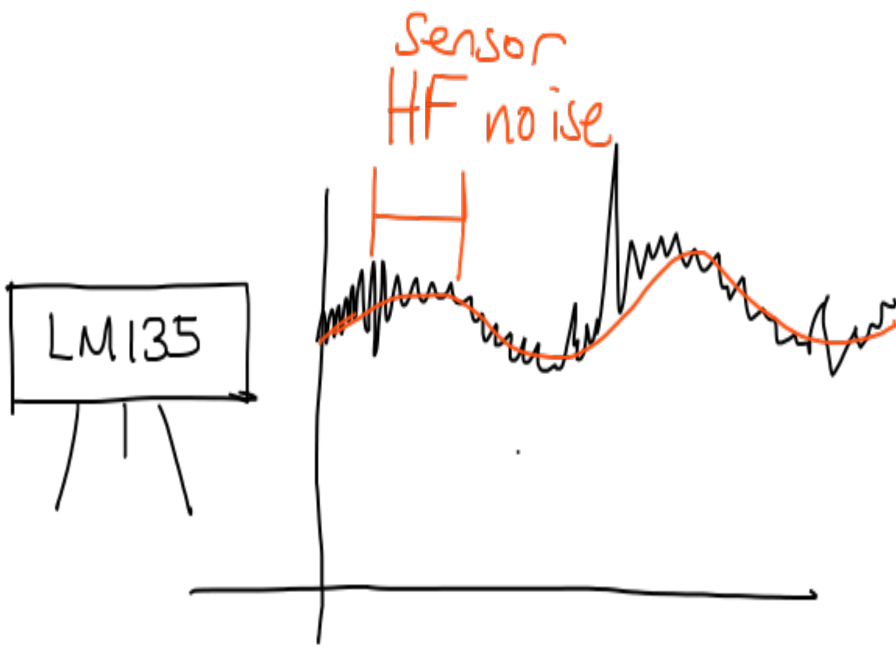
$$\Rightarrow \omega(t) = \frac{K_a K_1}{\tau_1 p}\left(1 - e^{-pt}\right) K_2 E$$

where $p \triangleq \dfrac{1 + K_1 K_t \cdot K_a}{\tau_1}$

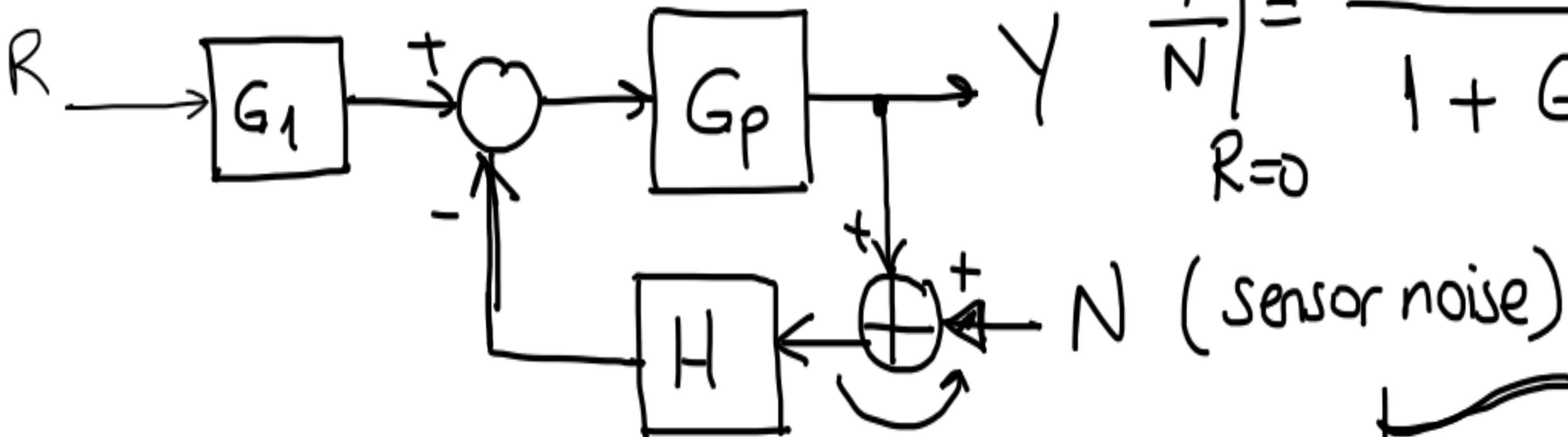


4 Reduction of sensor noise





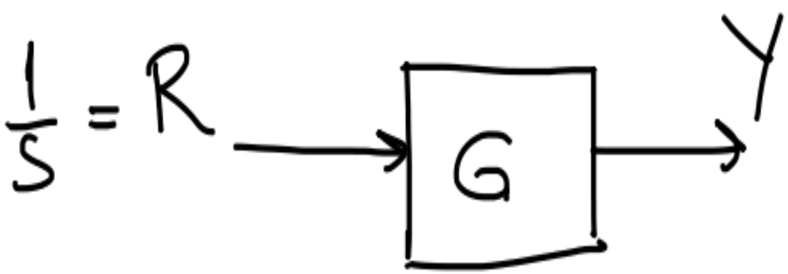
$$\left.\frac{Y}{N}\right|_{R=0} = \frac{-G_pH}{1+G_pH}$$



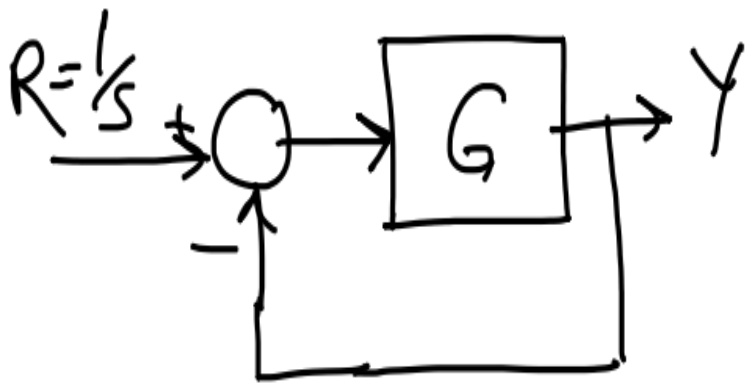
$$G_pH(s) = G_pH(j\omega)$$

To attenuate sensor noise $|G_p H|$ must be low at frequencies of the sensor noise. (mostly at high freq.)

(5) Reduction of Steady-State Error!



open-loop syst.



closed-loop syst.

$$E_o = R - Y = R - GR = R(1-G)$$

$$|e_o(\infty)| = \lim_{s \to 0} sE_o(s) = 1 - G(0) \cong G_o$$

$$E_{cl} = R - Y = R - \frac{G}{1+G} R = \frac{1}{1+G} R \Rightarrow e_{cl}(\infty) = \frac{1}{1+G(0)} \cong 1/G(0)$$