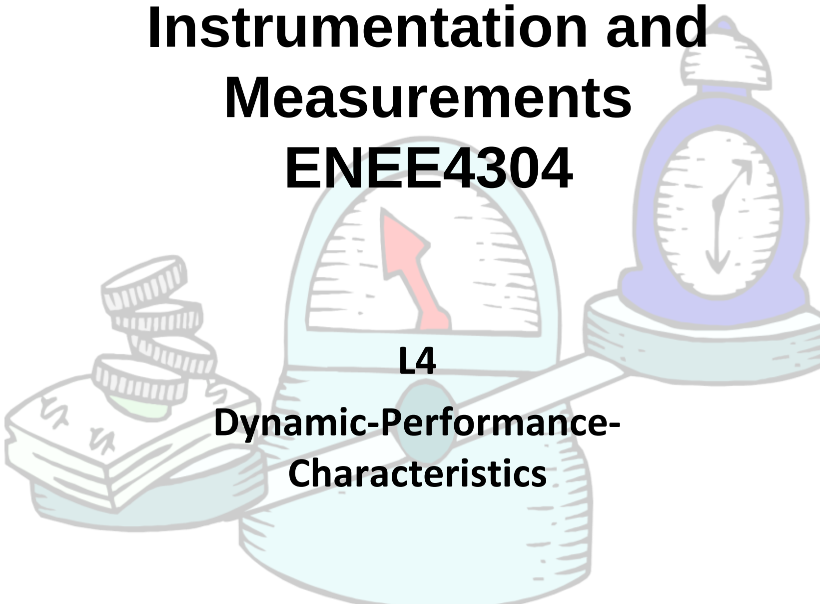
Instrumentation and Measurements ENEE4304

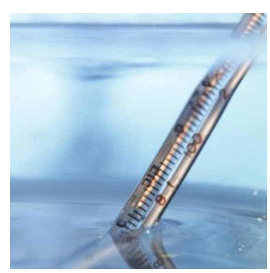


L4

Dynamic-Performance-Characteristics

Dynamic Performance

- The dynamic characteristics of a measuring instrument describe its behavior between the time a measured quantity changes value and the time when the instrument output attains a steady value in response.
- Because dynamic signals vary with time, the measurement system must be able to respond fast enough to keep up with the input signal.
- Further, we need to understand how the input signal is applied to the sensor because that plays a role in system response.



- Zero Order Systems
- Non-Zero Order Systems
  - 1st order
  - 2nd order
  - ….
  - Nth order

- In any linear, time-invariant measuring system, the following general relation can be written between input and output for time t > 0:

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + \cdots + a_1 \frac{dq_0}{dt} + a_0 q_0$$
$$= b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \cdots + b_1 \frac{dq_i}{dt} + b_0 q_i$$

- where qi is the measured quantity, q0 is the output reading and a0 . . . an, b0 . . . bm are constants.
- only certain special, simplified cases of it are applicable in normal measurement situations.
- The major point of importance is to have a practical appreciation of the manner in which various types of instruments respond when the measurand applied to them varies.

- If we limit consideration to that of step changes in the measured quantity only, then equation reduces to:

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + \cdots + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

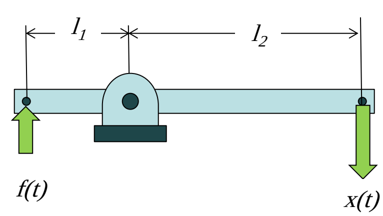
- Further simplification can be made by taking certain special cases the equation, which collectively apply to nearly all measurement systems.

Mechanical Zero-Order Systems

- The simplest model of a measurement system is the zero-order system model.
- This is represented by the zero-order differential equation:

$$a_o x = f(t)$$

$$x = \frac{1}{a_o} f(t) = Kf(t)$$



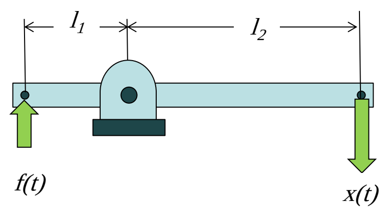
$$x(t)/l_2 = f(t)/l_1$$

$$K = l_2/l_1$$

- *K* is the static sensitivity or steady gain of the system.
- It is a measure of the amount of change in the output in response to the change in the input.

Mechanical Zero-Order Systems

- In a zero-order system, the output responds to the input signal instantaneously.
- If an input signal of magnitude *f(t) = A* were applied, the instrument would indicate *KA*.
- The scale of the measuring device would normally be calibrated to indicate A directly.



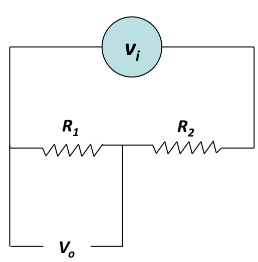
$$x(t)/l_2 = f(t)/l_1$$

$$K = l_2/l_1$$

$$a_o x = f(t)$$

$$x = \frac{1}{a_o} f(t) = Kf(t)$$

Electrical Zero-Order Systems

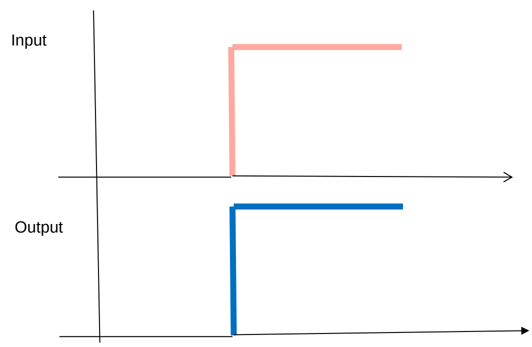


$$v_o = iR_1 = \frac{v_i}{R_1 + R_2} R_1$$

$$v_o = \frac{R_1}{R_1 + R_2} v_i$$

Zero Order Systems

- In general, systems without a storage or dissipative capability may be modeled as zero order system

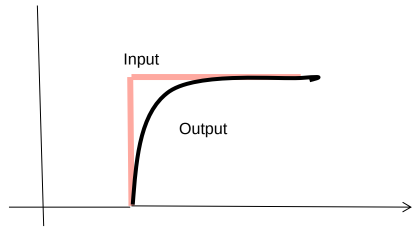


Non-zero Order Systems

- Measurement systems that contain storage or dissipative elements do not respond instantaneously to changes in input.
- In the bulb thermometer, when the ambient temperature changes, the liquid inside the bulb will need to store a certain amount of energy in order for it to reach the temperature of the environment.
- The temperature of the bulb sensor changes with time until this equilibrium is reached, which accounts physically for its lag in response.

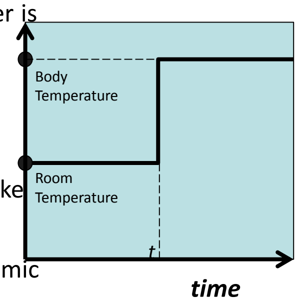
Non-zero Order Systems

- In general, systems with a storage or dissipative capability but negligible inertial forces may be modeled using a first-order differential equation.

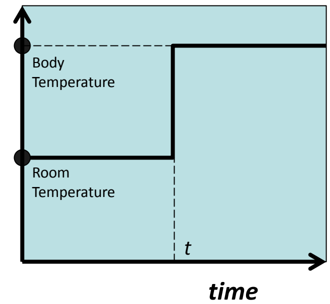


First Order Systems

- Consider the time response of a bulb thermometer for measuring body temperature.
- The thermometer, initially at room temperature, is placed under the tongue.
- Body temperature itself is constant (static) during the measurement, but the input signal to the thermometer is suddenly changed from room temperature to body temperature. This, is a step change in the measured signal.
- The thermometer must gain energy from its new environment to reach thermal equilibrium, and this take a finite amount of time.
- The ability of any measurement system to follow dynamic signals is a characteristic of the measuring system components.



First Order Systems



- Suppose a bulb thermometer originally at temperature T_o is suddenly exposed to a fluid temperature T_∞.
- Develop a model that simulates the thermometer output response.
- Rate of energy stored = Rate of energy in

$$\dot{E}_{stored} = Q_{in}$$

$$mC_p \frac{dT}{dt} = hA(T_\infty - T)$$

$$mC_p \frac{dT}{dt} + hAT = hAT_\infty$$

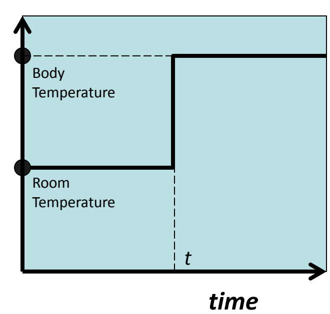
$$\frac{mC_p}{hA}\frac{dT}{dt} + T = T_\infty$$

- *m- Mass*
- *Cp- specific heat*
- *A- contact area ; h – heat transfer coeff.*

First Order Systems

$$\frac{mC_p}{hA}\frac{dT}{dt} + T = T_\infty$$

$$\tau \frac{dT}{dt} + T = T_\infty$$



- The ratio ***mCp/hA*** has a units of seconds and is called the time constant , *τ.*
- *m- sensor mass*
- *Cp- specific heat of sensor material*
- *A- contact area ; h – heat transfer coeff.*

1st Order Systems

- Examples:
  - Bulb Thermometer
  - RC Circuits
  - Terminal velocity

- Mathematical Model:

$$\tau \frac{dx}{dt} + x = f(t)$$

τ: Time constant

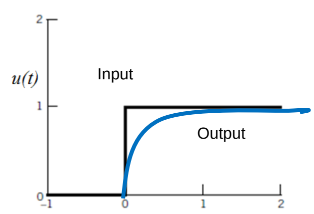
$f(t)$: Input (quantity to be measured)

x: Output (instrument response)

1st Order Systems with Step Input

$$f(t) = Ku(t)$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$\tau \frac{dx}{dt} + x = Ku(t)$$

$$x(0) = x_0$$

Transfer function *G*(*s*) for First Order System (additional example)

- A good example of a first-order element is provided by a temperature sensor with an electrical output signal, e.g. a thermocouple or thermistor.
- The bare element (not enclosed in a sheath) is placed inside a fluid
- Initially at time $t = 0-$, the sensor temperature is equal to the fluid temperature, i.e. $T(0-) = T_F(0-)$.
- If the fluid temperature is suddenly raised at $t = 0$, the sensor is no longer in a steady state, and its dynamic behavior is described by the **heat balance equation**:

rate of heat inflow − rate of heat outflow=

rate of change of sensor heat content

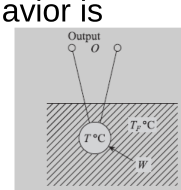


Figure 4.1 Temperature sensor in fluid.

- **rate of heat inflow − rate of heat outflow= rate of change of sensor heat content**
- Assuming that $T_F > T$, then the rate of heat outflow will be zero, and the rate of heat inflow W will be proportional to the temperature difference ($T_F - T$). we have:
- Rate of heat inflow : $W = UA(T_F - T)$ watts ;

Where U- [W m^−2 °C−1] is the overall heat transfer coefficient between fluid and sensor

and A is the effective heat transfer area [m^2].

- The increase of heat content of the sensor is
- $MC[T - T(0-)]$ joules, where M –is the sensor mass [kg] and C –is the specific heat of the sensor material [J kg−1 °C−1]

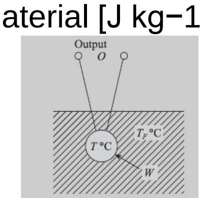


Figure 4.1 Temperature sensor in fluid.

$$\text{rate of increase of sensor heat content} = MC\frac{\mathrm{d}}{\mathrm{d}t}[T - T(0-)]$$

$$UA(\Delta T_F - \Delta T) = MC\frac{\mathrm{d}\Delta T}{\mathrm{d}t}$$

$$\frac{MC}{UA}\frac{\mathrm{d}\Delta T}{\mathrm{d}t} + \Delta T = \Delta T_F$$

$$\tau\frac{\mathrm{d}\Delta T}{\mathrm{d}t} + \Delta T = \Delta T_F$$

Laplace

$$\tau[s\Delta\bar{T}(s) - \Delta T(0-)] + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

$$\tau s\Delta\bar{T}(s) + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

- Thus, assuming M and C are constants:

$$\text{rate of increase of sensor heat content} = MC\frac{\mathrm{d}}{\mathrm{d}t}[T - T(0-)]$$

- Defining ΔT = T − T(0−) and ΔTF = TF − TF(0−) to be the deviations in temperatures from initial steady-state conditions, the differential equation describing the sensor temperature changes is
- i.e.

$$UA(\Delta T_F - \Delta T) = MC\frac{\mathrm{d}\Delta T}{\mathrm{d}t}$$

$$\frac{MC}{UA}\frac{\mathrm{d}\Delta T}{\mathrm{d}t} + \Delta T = \Delta T_F$$

- This is a **linear differential equation** in which dΔT/dt and ΔT are multiplied by constant coefficients; the equation is **first order.**
- The quantity MC/UA has the dimensions of time and is referred to as the **time constant** τ for the system

- The differential equation is now:

$$\tau \frac{d\Delta T}{dt} + \Delta T = \Delta T_F$$

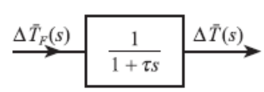
- While the above differential equation is a perfectly adequate description of the dynamics of the sensor, it is not the most useful representation.
- The transfer function based on the Laplace transform of the differential equation provides a convenient framework for studying the dynamics of multi-element systems.

$$\tau[s\Delta\bar{T}(s) - \Delta T(0-)] + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

- where ΔT(0−) is the temperature deviation at initial conditions prior to $t = 0$.
- By definition, $\Delta T(0-) = 0$, giving: $\tau s\Delta\bar{T}(s) + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$

- i.e. $(\tau s + 1)\Delta\bar{T}(s) = \Delta\bar{T}_F(s)$
- The transfer function $G(s)$:

$$G(s) = \frac{\Delta\bar{T}(s)}{\Delta\bar{T}_F(s)} = \frac{1}{1 + \tau s}$$

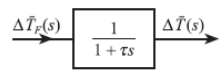


First-order function block

- The above transfer function only relates changes in sensor temperature to changes in fluid temperature.
- The overall relationship between changes in sensor output signal O and fluid temperature is:

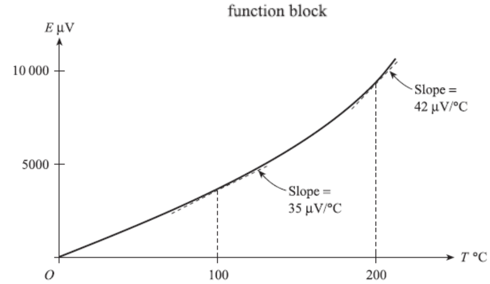
$$\frac{\Delta\bar{O}(s)}{\Delta\bar{T}_F(s)} = \frac{\Delta O}{\Delta T}\frac{\Delta\bar{T}(s)}{\Delta\bar{T}_F(s)}$$

- Thus for a copper–constantan thermocouple measuring small fluctuations in temperature around 100 °C, ΔE/ΔT is found by evaluating dE/dT at 100 °C to give ΔE/ΔT = 35 μV /°C.
- Thus if the time constant of the thermocouple is 10s the overall dynamic relationship between changes in e.m.f. and fluid temperature is:



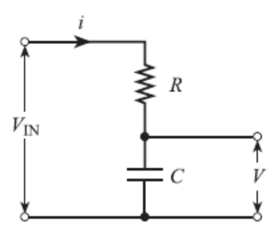
First-order function block

$$\frac{\Delta \bar{E}(s)}{\Delta \bar{T}_F(s)} = 35 \times \frac{1}{1 + 10s}$$



First Order System -Electrical

Electrical



$$\tau_E = RC = R_E C_E; \quad R_E = R, \quad C_E = C$$

$$V_{IN} - V = iR$$

Charge $q = CV$, current $i = \frac{dq}{dt} = \frac{CdV}{dt}$

$$RC\frac{dV}{dt} + V = V_{IN}$$

i.e.

$$\tau_E \frac{dV}{dt} + V = V_{IN}, \ \tau_E = RC$$

First Order System -Mechanical

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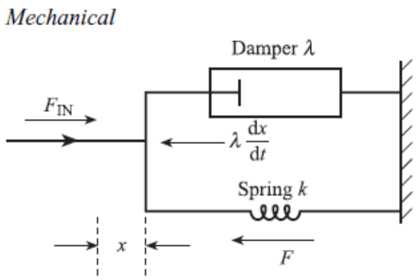


$$\tau_M = \frac{\lambda}{k} = R_M C_M; \quad R_M = \lambda, \quad C_M = \frac{1}{k}$$

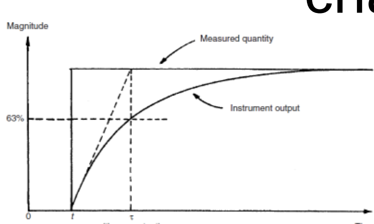
$$F_{IN} - F = \lambda \frac{dx}{dt}, \quad \lambda \text{ N s m}^{-1} = \text{damping constant}$$

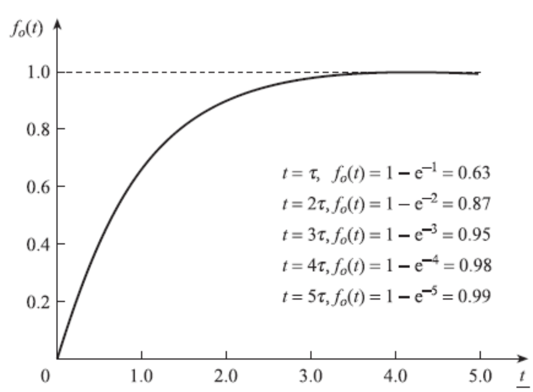
$$\text{Displacement } x = \frac{F}{k}, \quad k \text{ N m}^{-1} = \text{spring stiffness}$$

$$\frac{\lambda}{k}\frac{dF}{dt} + F = F_{IN}$$

$$\tau_M \frac{dF}{dt} + F = F_{IN}, \; \tau_M = \frac{\lambda}{k}$$

First order instrument characteristic





2^{nd} Order Systems

- Example:
  - Spring – mass damper
  - RLC Circuits
  - Accelerometers

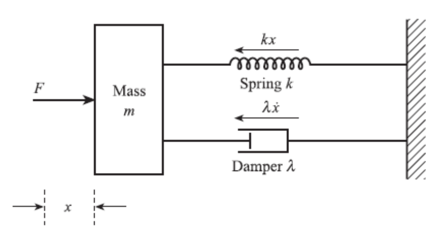
- Mathematical Model:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n{}^2x = f(t)$$

ζ	Damping ratio (dimensionless)
ω_n	Natural frequency (1/s)
$f(t)$:	Input (quantity to be measured)
x:	Output (instrument response)

Mass–spring–damper model of elastic force sensor (2^{nd} order)

- The elastic sensor which converts a force input F into a displacement output x, is a good example of a second-order element.
- The diagram is a conceptual model of the element, which incorporates***: a mass m [kg], a spring of stiffness k [Nm^−1], and a damper of constant λ [N s m^−1].***
- The system is initially at rest at time t = 0− so that the initial velocity x(0−) = 0 and the initial acceleration y(0−) = 0.



- This is a **second-order linear differential equation**

undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ rad/s

damping ratio $\xi = \frac{\lambda}{2\sqrt{km}}$

$$m/k = 1/\omega_n^2,\ \lambda/k = 2\xi/\omega_n.$$

$$\frac{1}{\omega_n^2}\frac{d^2\Delta x}{dt^2} + \frac{2\xi}{\omega_n}\frac{d\Delta x}{dt} + \Delta x = \frac{1}{k}\Delta F$$

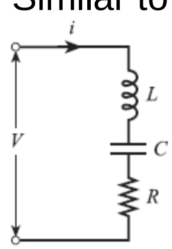
Linear second-order differential equation

Second Order System -Electrical

- Transfer function for a second-order element

$$G(s) = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\xi}{\omega_n}s + 1}$$

- Similar to series RLC circuit



$$V = iR + \frac{q}{C} + L\frac{di}{dt}$$

where $i = \frac{dq}{dt}$ (q = charge on the capacitance)

thus $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V$

$$\omega_n = 1/\sqrt{LC} \text{ and } \xi = (R/2)\sqrt{C/L}.$$

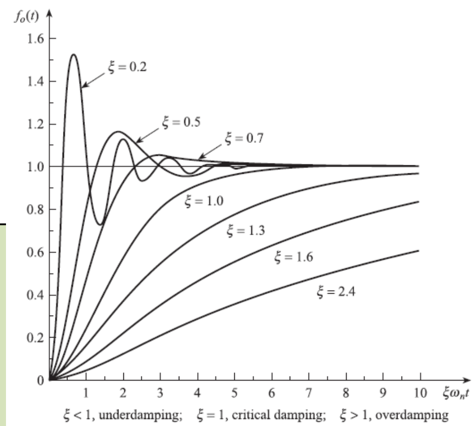
2^{nd} Order Systems with step input

$$f(t) = Ku(t)$$

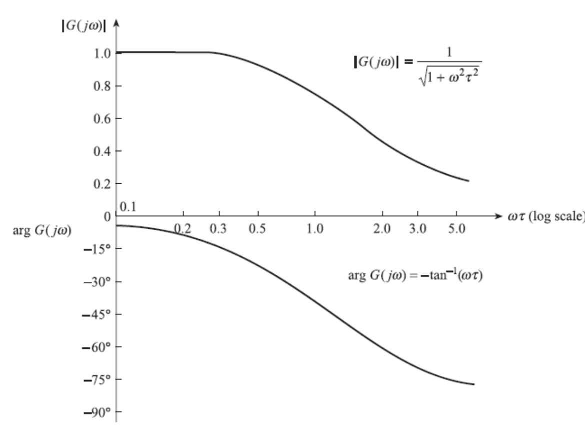
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + {\omega_n}^2 x = Af(t)$$

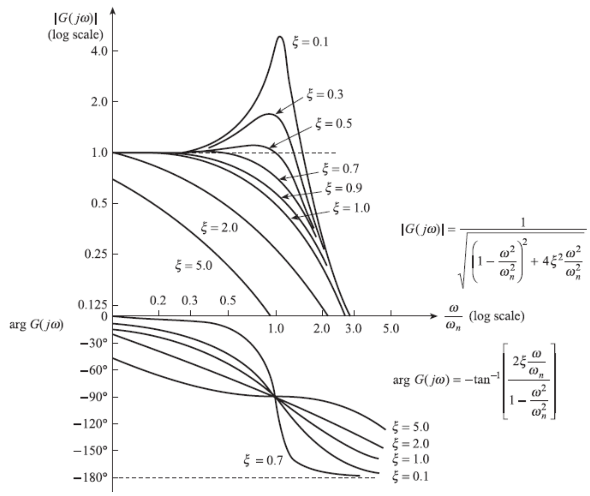
ζ	Damping ratio (dimensionless)
ω_n	Natural frequency (1/s)
$f(t)$:	Input (quantity to be measured)
x:	Output (instrument response)
A:	Arbitrary constant



Frequency Response (1^{st} order)



Frequency Response (2nd order)



$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}}}$$

$$\arg G(j\omega) = -\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right]$$

Example – Automobile Accelerometer

- Consider the accelerometer used in seismic and vibration engineering to determine the motion of large bodies to which the accelerometer is attached.
- The acceleration of the large body places the piezoelectric crystal into compression or tension, causing a surface charge to develop on the crystal.
- The charge is proportional to the motion. As the large body moves, the mass of the accelerometer will move with an inertial response.
- The stiffness of the spring, k, provides a restoring force to move the accelerometer mass back to equilibrium while internal frictional damping, c, opposes any displacement away from equilibrium.