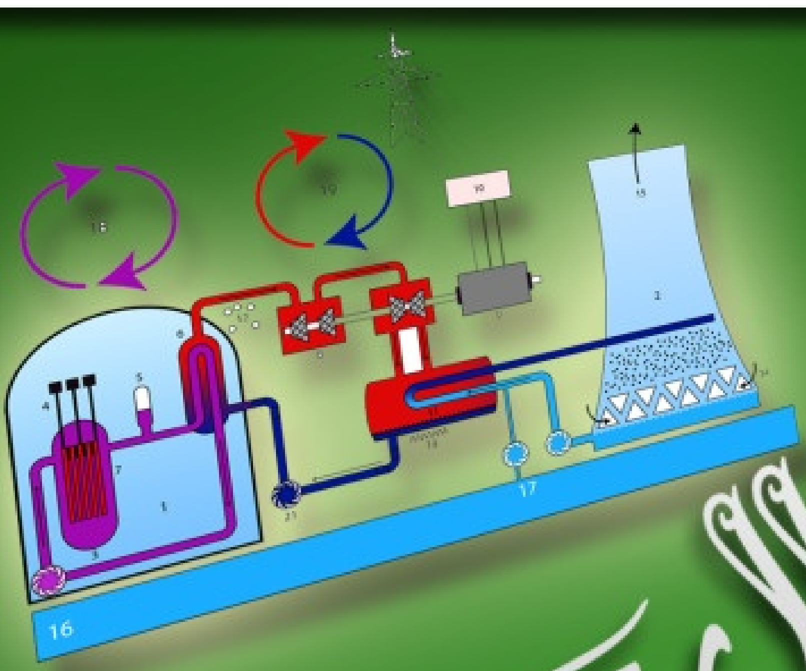

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# Power System Analysis

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1 2 3 4 5 6 7 8 9 0 DOC DOC 9 4 3 2 1 0 9

P/N 012235-0
Set ISBN 0-07-561634-3

Publisher: Kevin Kane
Executive editor: Betsy Jones
Sponsoring editor: Lynn Cox
Marketing manager: John Wannemacher
Project manager: Eve Strock
Production supervisor: Rich DeVitto
Compositor: York Graphic Services, Inc.
Printer: R. R. Donnelley & Sons, Inc.

**Library of Congress Cataloging-in-Publication Data**

Saadat, Hadi.
Power system analysis / Hadi Saadat
p. cm.
Includes bibliographical references and index.
ISBN 0-07-012235-0
1. Electric power systems. 2. System analysis. I. Title.
TK1011.S23 1999
621.31--dc21

**http://www.mhhc.com**



# CONTEN

**APPENDIXES**

# PREFACE

This book is intended for upper-division electrical engineering students studying power system analysis and design or as a reference for practicing engineers. As a reference, the book is written with self-study in mind. The text has grown out of many years of teaching the subject material to students in electrical engineering at various universities, including Michigan Technological University and Milwaukee School of Engineering.

Prerequisites for students using this text are physics and mathematics through differential equations and a circuit course. A background in electric machines is desirable, but not essential. Other required background materials, including *MATLAB* and an introduction to control systems, are provided in the appendixes.

In recent years, the analysis and design of power systems have been affected dramatically by the widespread use of personal computers. Personal computers have become so powerful and advanced that they can be used easily to perform steady-state and transient analysis of large interconnected power systems. Modern personal computers' ability to provide information, ask questions, and react to responses have enabled engineering educators to integrate computers into the curriculum. One of the difficulties of teaching power system analysis courses is not having a real system with which to experiment in the laboratory. Therefore, this book is written to supplement the teaching of power system analysis with a computer-simulated system. I developed many programs for power system analysis, giving students a valuable tool that allows them to spend more time on analysis and design of practical systems and less on programming, thereby enhancing the learning process. The book also provides a basis for further exploration of more advanced topics in power system analysis.

*MATLAB* is a matrix-based software package, which makes it ideal for power system analysis. *MATLAB*, with its extensive numerical resources, can be used to obtain numerical solutions that involve various types of vector-matrix operations.

In addition, *SIMULINK* provides a highly interactive environment for simulation of both linear and nonlinear dynamic systems. Both programs are integrated into discussions and problems. I developed a power system toolbox containing a set of M-files to help in typical power system analysis. In fact, all the examples and figures in this book have been generated by *MATLAB* functions and the use of this toolbox. The power system toolbox allows the student to analyze and design power systems without having to do detailed programming. Some of the programs, such as power flow, optimization, short-circuit, and stability analysis, were originally developed for a mainframe computer when I worked for power system consulting firms many years ago. These programs have been refined and modularized for interactive use with *MATLAB* for many problems related to the operation and analysis of power systems. These software modules are versatile, allowing some of the typical problems to be solved by several methods, thus enabling students to investigate alternative solution techniques. Furthermore, the software modules are structured in such a way that the user may mix them for other power system analyses.

This book has more than 140 illustrative examples that use *MATLAB* to assist in the analysis of power systems. Each example illustrates a specific concept and usually contains a script of the *MATLAB* commands used for the model creation and computation. Some examples are quite elaborate, in order to bring the practical world closer. The *MATLAB* M-files on the accompanying diskette can be copied to the user's computer and used to solve all the examples. The scripts can also be utilized with modifications as the foundation for solving the end-of-chapter problems.

The book is organized into 12 chapters and 3 appendixes. Each chapter begins with a introduction describing the topics students will encounter. **Chapter 1** is a brief overview of the development of power systems and a description of the major components in the power system. Included is a discussion of generating stations and transmission and subtransmission networks that convey the energy from the primary source to the load areas. **Chapter 2** reviews power concepts and three-phase systems. Typical students already will have studied much of this material. However, this specialized topic of networks may not be included in circuit theory courses, and the review here will reinforce these concepts. Before going into system analysis, we have to model all components of electrical power systems. **Chapter 3** addresses the steady-state presentation and modeling of synchronous machines and transformers. Also, the per unit system is presented, followed by the one-line diagram representation of the network.

**Chapter 4** discusses the parameters of a multicircuit transmission line. These parameters are computed for the balanced system on a per phase basis. **Chapter 5** thoroughly covers transmission line modeling and the performance and compensation of the transmission lines. This chapter provides the concepts and tools necessary for the preliminary transmission line design. **Chapter 6** presents a comprehensive coverage of the power flow solution of an interconnected power system during normal operation. First, the commonly used iterative techniques for the solution of nonlinear algebraic equation are discussed. Then several approaches to the solution of power flow are described. These techniques are applied to the solution of practical systems using the developed software modules.

**Chapter 7** covers some essential classical optimization of continuous functions and their application to optimal dispatch of generation. The programs developed here are designed to work in synergy with the power flow programs. **Chapter 8** deals with synchronous machine transient analysis. The voltage equations of the synchronous machine are first developed. These nonlinear equations are transformed into linear differential equations using Park's transformation. Analytical solution of the transformed equations can be obtained by the Laplace transform technique. However, *MATLAB* is used with ease to simulate the nonlinear differential equations of the synchronous machine directly in time-domain in matrix form for all modes of operation. Thus students can observe the dynamic response of the synchronous machine during short circuits and appreciate the significance and consequence of the change of machine parameters. The ultimate objective of this chapter is to develop simple network models of the synchronous generator for power system fault analysis and transient stability studies.

**Chapter 9** covers balanced fault analysis. The bus impedance matrix by the *building algorithms* is formulated and employed for the systematic computation of bus voltages and line currents during faults. **Chapter 10** discusses methods of symmetrical components that resolve the problem of an unbalanced circuit into a solution of a number of balanced circuits. Included are graphical displays of the symmetrical components transformation and some applications. The method is applied to the unbalanced fault, which once again allows the treatment of the problem on simple per phase basis. Algorithms have been developed to simulate different types of unbalanced faults. The software modules developed for unbalanced faults include single line-to-ground fault, line-to-line fault, and double line-to-ground fault.

**Chapter 11** covers power system stability problems. First, the dynamic behavior of a one-machine system due to a small disturbance is investigated, and the analytical solution of this linearized model is obtained. *MATLAB* and *SIMULINK* are used conveniently to simulate the system, and the model is extended to multimachine systems. Next, the transient stability using equal area criteria is discussed, and the result is represented graphically, providing physical insight into the dynamic behavior of the machine. An introduction to nonlinear differential equations and their numerical solutions is given. *MATLAB* is used to obtain the numerical solution of the swing equation of a one-machine system. Simulation is also obtained using the *SIMULINK* toolbox. A program compatible with the power flow programs is developed for the transient stability analysis of the multimachine systems.

**Chapter 12** is concerned with power system control and develops some of the control schemes required to operate the power system in the steady state. Simple models of the essential components used in control systems are presented. The automatic voltage regulator (AVR) and the load frequency control (LFC) are discussed. The automatic generation control (AGC) in single-area and multiarea systems, including tie-line power control, are analyzed. For each case, the responses to the real power demand are obtained. The generator responses with the AVR and various compensators, such as rate feedback and *Proportional Integral Derivative* (PID) controllers, are obtained. Both AGC and AVR systems are illustrated by several examples, and the responses are obtained using *MATLAB*. These analyses are supplemented by constructing the *SIMULINK* block diagram, which provides a highly interactive environment for simulation. Some basic materials of modern control theory are discussed, including the pole-placement state feedback design and the optimal controller designs using the linear quadratic regulator based on the *Riccati* equation. These modern techniques are then applied for simulation of the LFC systems.

**Appendix A** is a self-study *MATLAB* and *SIMULINK* tutorial focused on power and control systems and coordinated with the text. **Appendix B** includes a brief introduction to the fundamentals of control systems and is suitable for students without a background in control systems. **Appendix C** lists all functions, script files, and chapter examples. Answers to problems are given at the end of the book. The instructor's manual for this text contains the worked-out solutions for all of the book's problem.

The material in the text is designed to be fully covered in a two-semester undergraduate course sequence. The organization is flexible, allowing instructors to select the material that best suits the requirements of a one-quarter or a one-semester course. In a one-semester course, the first six chapters, which form the basis for power system analysis, should be covered. The material in Chapter 2 contains power concepts and three-phase systems, which are usually covered in circuit courses. This chapter can be excluded if the students are well prepared, or it can be used for review. Also, for students with electrical machinery background, Chapter 3 might be omitted. After the above coverage, additional material from the remaining chapters may then be appropriate, depending on the syllabus requirements and the individual preferences. One choice is to cover Chapter 7 (optimal dispatch of generation); another choice is Chapter 9 (balanced fault). The generator reactances required in Chapter 9 may be covered briefly from Section 8.7 without covering Chapter 8 in its entirety.

After reading the book, students should have a good perspective of power system analysis and an active knowledge of various numerical techniques that can be applied to the solution of large interconnected power systems. Students should





find *MATLAB* helpful in learning the material in the text, particularly in solving the problems at the end of each chapter.

I would like to express my appreciation and thanks to the following reviewers for their many helpful comments and suggestions: Professor Max D. Anderson, University of Missouri-Rolla; Professor Miroslav Begovic, Georgia Institute of Technology; Professor Karen L. Butler, Texas A&M University; Professor Kevin A. Clements, Worcester Polytechnic Institute; Professor Mariesa L. Crow, University of Missouri-Rolla; Professor Malik Elbuluk, University of Akron; Professor A. A. El-Keib, University of Alabama; Professor F. P. Emad, University of Maryland; Professor L. L. Grigsby, Auburn University; Professor Kwang Y. Lee, Pennsylvania State University; Professor M. A. Pai, University of Illinois-Urbana; Professor E. K. Stanek, University of Missouri-Rolla.

My sincere thanks to Lynn Kallas, who proofread the early version of the manuscript. Special thanks goes to the staff of McGraw-Hill: Lynn Cox, the editor, for her constant encouragement, Nina Kreiden, editorial coordinator, for her support, and Eve Strock, senior project manager, for her attention to detail during all phases of editing and production.

I wish to express my thanks to the Electrical Engineering and Computer Science Department of Milwaukee School of Engineering and to Professor Ray Palmer, chairman of the department, for giving me the opportunity to prepare this material.

Last, but not least, I thank my wife Jila and my children, Dana, Fred, and Cameron, who were a constant and active source of support throughout the endeavor.

# CHAPTER 1

# THE POWER SYSTEM: AN OVERVIEW

## 1.1 INTRODUCTION

Electric energy is the most popular form of energy, because it can be transported easily at high efficiency and reasonable cost.

The first electric network in the United States was established in 1882 at the Pearl Street Station in New York City by Thomas Edison. The station supplied dc power for lighting the lower Manhattan area. The power was generated by dc generators and distributed by underground cables. In the same year the first water-wheel driven generator was installed in Appleton, Wisconsin. Within a few years many companies were established producing energy for lighting – all operated under Edison's patents. Because of the excessive power loss, $RI^2$ at low voltage, Edison's companies could deliver energy only a short distance from their stations.

With the invention of the transformer (William Stanley, 1885) to raise the level of ac voltage for transmission and distribution and the invention of the induction motor (Nikola Tesla, 1888) to replace the dc motors, the advantages of the ac system became apparent, and made the ac system prevalent. Another advantage of the ac system is that due to lack of commutators in the ac generators, more power can be produced conveniently at higher voltages.

The first single-phase ac system in the United States was at Oregon City where power was generated by two 300 hp waterwheel turbines and transmitted at 4 kV to Portland. Southern California Edison Company installed the first three-phase system at 2.3 kV in 1893. Many electric companies were developed throughout the country. In the beginning, individual companies were operating at different frequencies anywhere from 25 Hz to 133 Hz. But, as the need for interconnection and parallel operation became evident, a standard frequency of 60 Hz was adopted throughout the U.S. and Canada. Most European countries selected the 50-Hz system. Transmission voltages have since risen steadily, and the extra high voltage (EHV) in commercial use is 765 kV, first put into operation in the United States in 1969.

For transmitting power over very long distances it may be more economical to convert the EHV ac to EHV dc, transmit the power over two lines, and invert it back to ac at the other end. Studies show that it is advantageous to consider dc lines when the transmission distance is 500 km or more. DC lines have no reactance and are capable of transferring more power for the same conductor size than ac lines. DC transmission is especially advantageous when two remotely located large systems are to be connected. The dc transmission tie line acts as an asynchronous link between the two rigid systems eliminating the instability problem inherent in the ac links. The main disadvantage of the dc link is the production of harmonics which requires filtering, and a large amount of reactive power compensation required at both ends of the line. The first $\pm 400$-kV dc line in the United States was the Pacific Intertie, 850 miles long between Oregon and California built in 1970.

The entire continental United States is interconnected in an overall network called the *power grid*. A small part of the network is federally and municipally owned, but the bulk is privately owned. The system is divided into several geographical regions called *power pools*. In an interconnected system, fewer generators are required as a reserve for peak load and spinning reserve. Also, interconnection makes the energy generation and transmission more economical and reliable, since power can readily be transferred from one area to others. At times, it may be cheaper for a company to buy bulk power from neighboring utilities than to produce it in one of its older plants.

## 1.2 ELECTRIC INDUSTRY STRUCTURE

The bulk generation of electricity in the United States is produced by integrated investor-owned utilities (IOU). A small portion of power generation is federally owned, such as the Tennessee Valley Authority and Bonneville Power Administration. Two separate levels of regulation currently regulate the United States electric system. One is the Federal Energy Regulatory Commission (FERC), which regulates the price of wholesale electricity, service terms, and conditions. The other is the Securities and Exchange Commission (SEC), which regulates the business structure of electric utilities.





The transmission system of electric utilities in the Unites States and Canada is interconnected into a large power grid known as the North American Power Systems Interconnection. The power grid is divided into several pools. The pools consist of several neighboring utilities which operate jointly to schedule generation in a cost-effective manner. A privately regulated organization called the North American Electric Reliability Council (NERC) is responsible for maintaining system standards and reliability. NERC works cooperatively with every provider and distributor of power to ensure reliability. NERC coordinates its efforts with FERC as well as other organizations such as the Edison Electric Institute (EEI). NERC currently has four distinct electrically separated areas. These areas are the Electric Reliability Council of Texas (ERCOT); the Western States Coordination Council (WSCC); the Eastern Interconnect, which includes all the states and provinces of Canada east of the Rocky Mountains (excluding Texas), and Hydro-Quebec, which has dc interconnects with the northeast. These electrically separate areas import and export power to each other but are not synchronized electrically.

The electric power industry in the United States is undergoing fundamental changes since the deregulation of the telecommunication, gas, and other industries. The generation business is rapidly becoming market-driven. This is a major change for an industry which, until the last decade, was characterized by large, vertically integrated monopolies. The implementation of open transmission access has resulted in wholesale and retail markets. In the future, utilities may possibly be divided into power generation, transmission, and retail segments. Generating utilities would sell directly to customers instead of to local distributors. This would eliminate the monopoly that distributors currently have. The distributors would sell their services as electricity distributors instead of being a retailer of electricity itself. The retail structure of power distribution would resemble the current structure of the telephone communication industry. The consumer would have a choice as to from which generator they purchase power. If the entire electric power industry were to be deregulated, final consumers could choose from generators across the country. Power brokers and power marketeers will assume a major role in this new competitive power industry. Currently, the ability to market electricity to retail end users exists, but only in a limited number of states in pilot programs.

Extensive efforts are being made to create a more competitive environment for electricity markets in order to promote greater efficiency. Thus, the power industry faces many new problems, with one of the highest priority issues being reliability, that is, bringing a steady, uninterruptable power supply to all electricity consumers. The restructuring and deregulation of electric utilities, together with recent progress in technology, introduce unprecedented challenges and opportunities for power systems research and open up new opportunities to young power engineers.

## 1.3 MODERN POWER SYSTEM

The power system of today is a complex interconnected network as shown in Figure 1.1 (page 7). A power system can be subdivided into four major parts:

- Generation
- Transmission and Subtransmission
- Distribution
- Loads

### 1.3.1 GENERATION

**Generators** — One of the essential components of power systems is the three-phase ac generator known as synchronous generator or alternator. Synchronous generators have two synchronously rotating fields: One field is produced by the rotor driven at synchronous speed and excited by dc current. The other field is produced in the stator windings by the three-phase armature currents. The dc current for the rotor windings is provided by excitation systems. In the older units, the exciters are dc generators mounted on the same shaft, providing excitation through slip rings. Today's systems use ac generators with rotating rectifiers, known as *brushless* excitation systems. The generator excitation system maintains generator voltage and controls the reactive power flow. Because they lack the commutator, ac generators can generate high power at high voltage, typically 30 kV. In a power plant, the size of generators can vary from 50 MW to 1500 MW.

The source of the mechanical power, commonly known as the *prime mover*, may be hydraulic turbines at waterfalls, steam turbines whose energy comes from the burning of coal, gas and nuclear fuel, gas turbines, or occasionally internal combustion engines burning oil. The estimated installed generation capacity in 1998 for the United States is presented in Table 1.1.

Steam turbines operate at relatively high speeds of 3600 or 1800 rpm. The generators to which they are coupled are cylindrical rotor, two-pole for 3600 rpm or four-pole for 1800 rpm operation. Hydraulic turbines, particularly those operating with a low pressure, operate at low speed. Their generators are usually a salient type rotor with many poles. In a power station several generators are operated in parallel in the power grid to provide the total power needed. They are connected at a common point called a *bus*.

Today the total installed electric generating capacity is about 760,000 MW. Assuming the United States population to be 270 million,

$$\text{Installed capacity per capita} = \frac{760 \times 10^9}{270 \times 10^6} = 2815 \text{ W}$$

To realize the significance of this figure, consider the average power of a person to be approximately 50 W. Therefore, the power of 2815 W is equivalent to

$$\frac{2815 \text{ W}}{50 \text{ W}} = 56 \text{ (power slave)}$$

The annual kWh consumption in the United States is about $3,550 \times 10^9$ kWh. The asset of the investment for investor-owned companies is about 200 billion dollars and they employ close to a half million people.

With today's emphasis on environmental consideration and conservation of fossil fuels, many alternate sources are considered for employing the untapped energy sources of the sun and the earth for generation of power. Some of these alternate sources which are being used to some extent are solar power, geothermal power, wind power, tidal power, and biomass. The aspiration for bulk generation of power in the future is the nuclear *fusion*. If nuclear fusion is harnessed economically, it would provide clean energy from an abundant source of fuel, namely water.

**Table 1.1** Installed Generation Capacity

| Type | Capacity, MW | Percent | Fuel |
|---|---|---|---|
| Steam Plant | 478,800 | 63 | Coal, gas, petroleum |
| Nuclear | 106,400 | 14 | Uranium |
| Hydro and pumped storage | 91,200 | 12 | Water |
| Gas Turbine | 60,800 | 8 | Gas, petroleum |
| Combined cycle | 15,200 | 2 | Gas, petroleum |
| Internal Combustion | 4,940 | 0.65 | Gas, petroleum |
| Others | 2,660 | 0:35 | Geothermal, solar, wind |
| Total | 760,000 | 100.00 | |

**Transformers** — Another major component of a power system is the transformer. It transfers power with very high efficiency from one level of voltage to another level. The power transferred to the secondary is almost the same as the primary, except for losses in the transformer, and the product $VI$ on the secondary side is approximately the same as the primary side. Therefore, using a step-up transformer of turns ratio $a$ will reduce the secondary current by a ratio of $1/a$. This will reduce losses in the line, which makes the transmission of power over long distances possible.

The insulation requirements and other practical design problems limit the generated voltage to low values, usually 30 kV. Thus, step-up transformers are used for transmission of power. At the receiving end of the transmission lines step-down transformers are used to reduce the voltage to suitable values for distribution or utilization. In a modern utility system, the power may undergo four or five transformations between generator and ultimate user.

## 1.3.2 TRANSMISSION AND SUBTRANSMISSION

The purpose of an overhead transmission network is to transfer electric energy from generating units at various locations to the distribution system which ultimately supplies the load. Transmission lines also interconnect neighboring utilities which permits not only economic dispatch of power within regions during normal conditions, but also the transfer of power between regions during emergencies.

Standard transmission voltages are established in the United States by the American National Standards Institute (ANSI). Transmission voltage lines operating at more than 60 kV are standardized at 69 kV, 115 kV, 138 kV, 161 kV, 230 kV, 345 kV, 500 kV, and 765 kV line-to-line. Transmission voltages above 230 kV are usually referred to as extra-high voltage (EHV).

Figure 1.1 shows an elementary diagram of a transmission and distribution system. High voltage transmission lines are terminated in substations, which are called *high-voltage substations, receiving substations*, or *primary substations*. The function of some substations is switching circuits in and out of service; they are referred to as *switching stations*. At the primary substations, the voltage is stepped down to a value more suitable for the next part of the journey toward the load. Very large industrial customers may be served from the transmission system.

The portion of the transmission system that connects the high-voltage substations through step-down transformers to the distribution substations are called the *subtransmission* network. There is no clear delineation between transmission and subtransmission voltage levels. Typically, the subtransmission voltage level ranges from 69 to 138 kV. Some large industrial customers may be served from the subtransmission system. Capacitor banks and reactor banks are usually installed in the substations for maintaining the transmission line voltage.

## 1.3.3 DISTRIBUTION

The distribution system is that part which connects the distribution substations to the consumers' service-entrance equipment. The primary distribution lines are usually in the range of 4 to 34.5 kV and supply the load in a well-defined geographical area. Some small industrial customers are served directly by the primary feeders.

The secondary distribution network reduces the voltage for utilization by commercial and residential consumers. Lines and cables not exceeding a few hun-





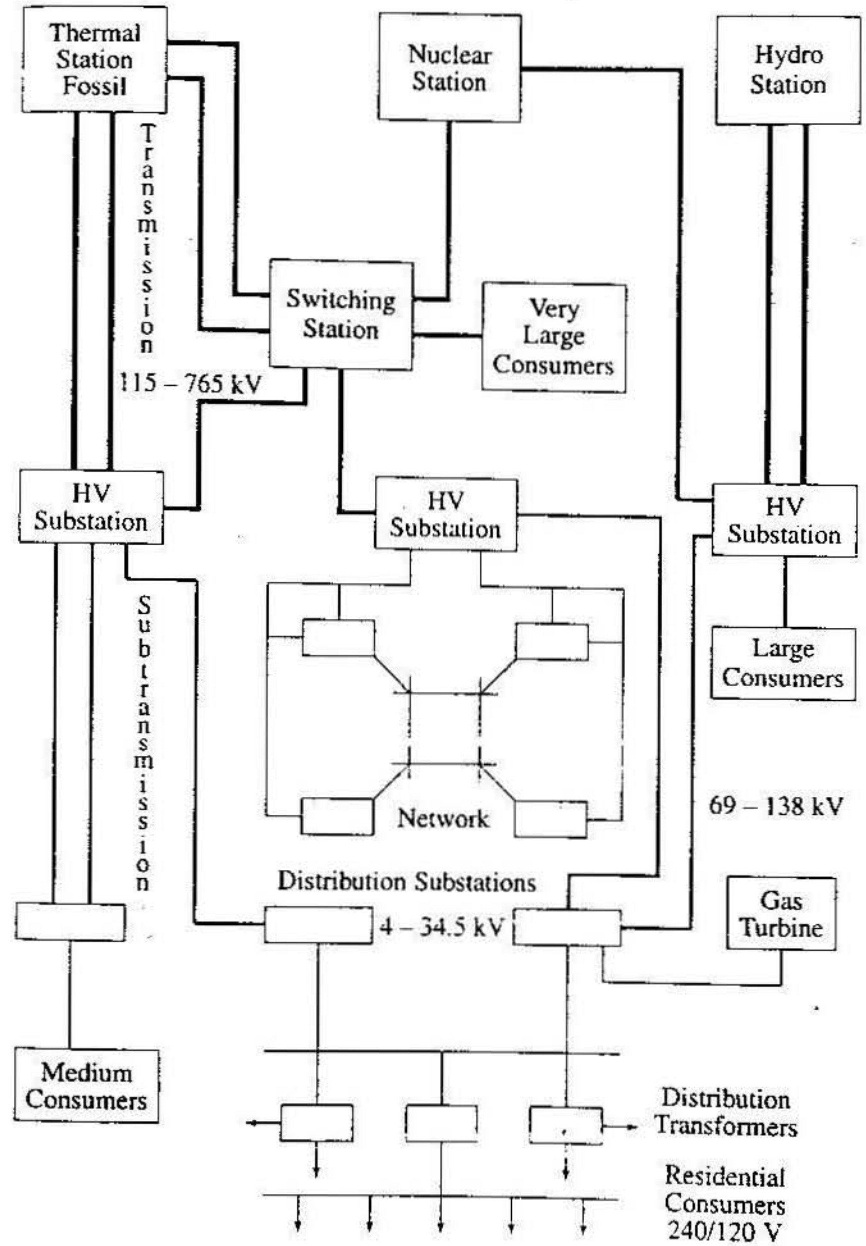



**FIGURE 1.1**
Basic components of a power system.

dred feet in length then deliver power to the individual consumers. The secondary distribution serves most of the customers at levels of 240/120 V, single-phase, three-wire; 208Y/120 V, three-phase, four-wire; or 480Y/277 V, three-phase, four-wire. The power for a typical home is derived from a transformer that reduces the primary feeder voltage to 240/120 V using a three-wire line.

Distribution systems are both *overhead* and *underground*. The growth of underground distribution has been extremely rapid and as much as 70 percent of new residential construction is served underground.

### 1.3.4 LOADS

Loads of power systems are divided into industrial, commercial, and residential. Very large industrial loads may be served from the transmission system. Large industrial loads are served directly from the subtransmission network, and small industrial loads are served from the primary distribution network. The industrial loads are composite loads, and induction motors form a high proportion of these load. These composite loads are functions of voltage and frequency and form a major part of the system load. Commercial and residential loads consist largely of lighting, heating, and cooling. These loads are independent of frequency and consume negligibly small reactive power.

The real power of loads are expressed in terms of kilowatts or megawatts. The magnitude of load varies throughout the day, and power must be available to consumers on demand.

The daily-load curve of a utility is a composite of demands made by various classes of users. The greatest value of load during a 24-hr period is called the *peak* or *maximum demand*. Smaller peaking generators may be commissioned to meet the peak load that occurs for only a few hours. In order to assess the usefulness of the generating plant the *load factor* is defined. The load factor is the ratio of average load over a designated period of time to the peak load occurring in that period. Load factors may be given for a day, a month, or a year. The yearly, or annual load factor is the most useful since a year represents a full cycle of time. The daily load factor is

$$\text{Daily L.F.} = \frac{\text{average load}}{\text{peak load}} \tag{1.1}$$

Multiplying the numerator and denominator of (1.1) by a time period of 24 hr, we have

$$\text{Daily L.F.} = \frac{\text{average load} \times 24 \text{ hr}}{\text{peak load} \times 24 \text{ hr}} = \frac{\text{energy consumed during 24 hr}}{\text{peak load} \times 24 \text{ hr}} \tag{1.2}$$

The annual load factor is

$$\text{Annual L.F.} = \frac{\text{total annual energy}}{\text{peak load} \times 8760 \text{ hr}} \tag{1.3}$$





Generally there is diversity in the peak load between different classes of loads, which improves the overall system load factor. In order for a power plant to operate economically, it must have a high system load factor. Today's typical system load factors are in the range of 55 to 70 percent.

There are a few other factors used by utilities. *Utilization factor* is the ratio of maximum demand to the installed capacity, and *plant factor* is the ratio of annual energy generation to the plant capacity × 8760 hr. These factors indicate how well the system capacity is utilized and operated.

A *MATLAB* function **barcycle(data)** is developed which obtains a plot of the load cycle for a given interval. The demand interval and the load must be defined by the variable **data** in a three-column matrix. The first two columns are the demand interval and the third column is the load value. The demand interval may be minutes, hours, or months, in ascending order. Hourly intervals must be expressed in military time.

#### Example 1.1

The daily load on a power system varies as shown in Table 1.2. Use the **barcycle** function to obtain a plot of the daily load curve. Using the given data compute the average load and the daily load factor (Figure 1.2).

**Table 1.2** Daily System Load

| Interval, hr | | | Load, MW |
|---|---|---|---|
| 12 A.M. | – | 2 A.M. | 6 |
| 2 | – | 6 | 5 |
| 6 | – | 9 | 10 |
| 9 | – | 12 | 15 |
| 12 P.M. | – | 2 P.M. | 12 |
| 2 | – | 4 | 14 |
| 4 | – | 6 | 16 |
| 6 | – | 8 | 18 |
| 8 | – | 10 | 16 |
| 10 | – | 11 | 12 |
| 11 | – | 12 A.M. | 6 |

The following commands

```
data = [ 0   2   6
         2   6   5
         6   9  10
         9  12  15
        12  14  12
```

```
          14  16   14
          16  18   16
          18  20   18
          20  22   16
          22  23   12
          23  24    6];
P = data(:,3);                                  % Column array of load
Dt = data(:, 2) - data(:,1); % Column array of demand interval
W = P'*Dt;                    % Total energy, area under the curve
Pavg = W/sum(Dt)                                       % Average load
Peak = max(P)                                             % Peak load
LF = Pavg/Peak*100                              % Percent load factor
barcycle(data)                                  % Plots the load cycle
xlabel('Time, hr'), ylabel('P,  MW')
```

result in

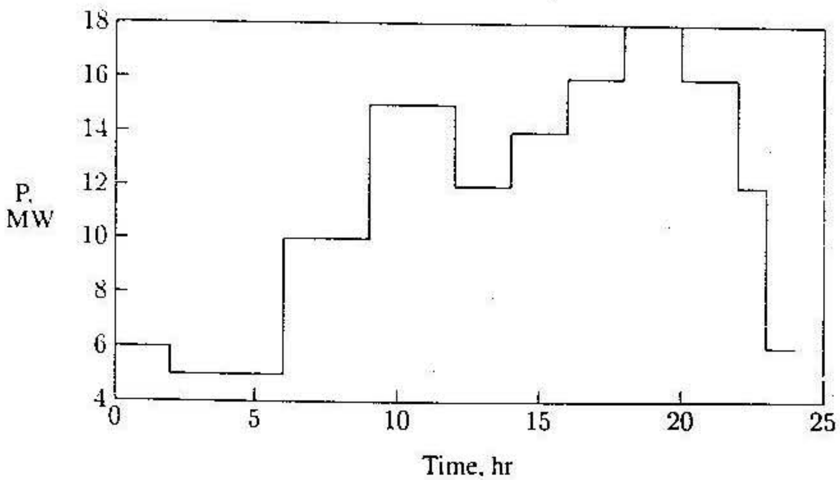



**FIGURE 1.2**
Daily load cycle for Example 1.1.

```
Pavg = 11.5417
Peak = 18
LF   = 64.12
```





## 1.4 SYSTEM PROTECTION

In addition to generators, transformers, and transmission lines, other devices are required for the satisfactory operation and protection of a power system. Some of the protective devices directly connected to the circuits are called *switchgear*. They include instrument transformers, circuit breakers, disconnect switches, fuses and lightning arresters. These devices are necessary to deenergize either for normal operation or on the occurrence of faults. The associated control equipment and protective relays are placed on *switchboard* in *control houses*.

## 1.5 ENERGY CONTROL CENTER

For reliable and economical operation of the power system it is necessary to monitor the entire system in a control center. The modern control center of today is called the *energy control center* (ECC). Energy control centers are equipped with on-line computers performing all signal processing through the remote acquisition system. Computers work in a hierarchical structure to properly coordinate different functional requirements in normal as well as emergency conditions. Every energy control center contains a control console which consists of a visual display unit (VDU), keyboard, and light pen. Computers may give alarms as advance warnings to the operators (dispatchers) when deviation from the normal state occurs. The dispatcher makes judgments and decisions and executes them with the aid of a computer. Simulation tools and software packages written in high-level language are implemented for efficient operation and reliable control of the system. This is referred to as SCADA, an acronym for "supervisory control and data acquisition."

## 1.6 COMPUTER ANALYSIS

For a power system to be practical it must be safe, reliable, and economical. Thus many analyses must be performed to design and operate an electrical system. However, before going into system analysis we have to model all components of electrical power systems. Therefore, in this text, after reviewing the concepts of power and three-phase circuits, we will calculate the parameters of a multi-circuit transmission line. Then, we will model the transmission line and look at the performance of the transmission line. Since transformers and generators are a part of the system, we will model these devices. Design of a power system, its operation and expansion requires much analysis. This text presents methods of power system analysis with the aid of a personal computer and the use of *MATLAB*. The *MATLAB* environment permits a nearly direct transition from mathematical expression to simulation. Some of the basic analysis covered in this text are:

- Evaluation of transmission line parameters
- Transmission line performance and compensation
- Power flow analysis
- Economic scheduling of generation
- Synchronous machine transient analysis
- Balanced fault
- Symmetrical components and unbalanced fault
- Stability studies
- Power system control

Many *MATLAB* functions are developed for the above studies thus allowing the student to concentrate on analysis and design of practical systems and spend less time on programming.

## PROBLEMS

**1.1.** The demand estimation is the starting point for planning the future electric power supply. The consistency of demand growth over the years has led to numerous attempts to fit mathematical curves to this trend. One of the simplest curves is

$$P = P_0 e^{a(t-t_0)}$$

where $a$ is the average per unit growth rate, $P$ is the demand in year $t$, and $P_0$ is the given demand at year $t_0$.

Assume the peak power demand in the United States in 1984 is 480 GW with an average growth rate of 3.4 percent. Using *MATLAB*, plot the predicated peak demand in GW from 1984 to 1999. Estimate the peak power demand for the year 1999.

**1.2.** In a certain country, the energy consumption is expected to double in 10 years. Assuming a simple exponential growth given by

$$P = P_0 e^{at}$$

calculate the growth rate $a$.





**1.3.** The annual load of a substation is given in the following table. During each month, the power is assumed constant at an average value. Using *MATLAB* and the **barcycle** function, obtain a plot of the annual load curve. Write the necessary statements to find the average load and the annual load factor.

| **Annual System Load** | |
|---|---|
| Interval, month | Load, MW |
| January | 8 |
| February | 6 |
| March | 4 |
| April | 2 |
| May | 6 |
| June | 12 |
| July | 16 |
| August | 14 |
| September | 10 |
| October | 4 |
| November | 6 |
| December | 8 |

# CHAPTER 2

# BASIC PRINCIPLES

## 2.1 INTRODUCTION

The concept of power is of central importance in electrical power systems and is the main topic of this chapter. The typical student will already have studied much of this material, and the review here will serve to reinforce the power concepts encountered in the electric circuit theory.

In this chapter, the flow of energy in an ac circuit is investigated. By using various trigonometric identities, the instantaneous power $p(t)$ is resolved into two components. A plot of these components is obtained using *MATLAB* to observe that ac networks not only consume energy at an average rate, but also borrow and return energy to its sources. This leads to the basic definitions of average power $P$ and reactive power $Q$. The volt-ampere $S$, which is a mathematical formulation based on the phasor forms of voltage and current, is introduced. Then the complex power balance is demonstrated, and the transmission inefficiencies caused by loads with low power factors are discussed and demonstrated by means of several examples.

Next, the transmission of complex power between two voltage sources is considered, and the dependency of real power on the voltage phase angle and the dependency of reactive power on voltage magnitude is established. *MATLAB* is used conveniently to demonstrate this idea graphically.

Finally, the balanced three-phase circuit is examined. An important property of a balanced three-phase system is that it delivers constant power. That is, the







power delivered does not fluctuate with time as in a single-phase system. For the purpose of analysis and modeling, the per-phase equivalent circuit is developed for the three-phase system under balanced condition.

## 2.2 POWER IN SINGLE-PHASE AC CIRCUITS

Figure 2.1 shows a single-phase sinusoidal voltage supplying a load.

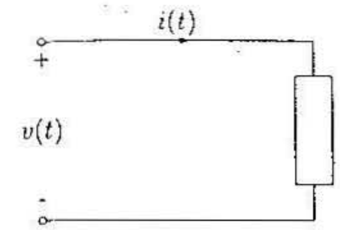



**FIGURE 2.1**
Sinusoidal source supplying a load.

Let the instantaneous voltage be

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{2.1}$$

and the instantaneous current be given by

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{2.2}$$

The instantaneous power $p(t)$ delivered to the load is the product of voltage $v(t)$ and current $i(t)$ given by

$$p(t) = v(t)\, i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{2.3}$$

In Example 2.1, *MATLAB* is used to plot the instantaneous power $p(t)$, and the result is shown in Figure 2.2. In studying Figure 2.2, we note that the frequency of the instantaneous power is twice the source frequency. Also, note that it is possible for the instantaneous power to be negative for a portion of each cycle. In a passive network, negative power implies that energy that has been stored in inductors or capacitors is now being extracted.

It is informative to write (2.3) in another form using the trigonometric identity

$$\cos A \cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B) \tag{2.4}$$

which results in

$$\begin{aligned} p(t) &= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)] \\ &= \frac{1}{2} V_m I_m \{\cos(\theta_v - \theta_i) + \cos[2(\omega t + \theta_v) - (\theta_v - \theta_i)]\} \\ &= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + \cos 2(\omega t + \theta_v)\cos(\theta_v - \theta_i) \\ &\quad + \sin 2(\omega t + \theta_v)\sin(\theta_v - \theta_i)] \end{aligned}$$

The *root-mean-square* (rms) value of $v(t)$ is $|V| = V_m/\sqrt{2}$ and the rms value of $i(t)$ is $|I| = I_m/\sqrt{2}$. Let $\theta = (\theta_v - \theta_i)$. The above equation, in terms of the rms values, is reduced to

$$p(t) = \underbrace{|V||I|\cos\theta[1 + \cos 2(\omega t + \theta_v)]}_{\substack{p_R(t) \\ \text{Energy flow into} \\ \text{the circuit}}} + \underbrace{|V||I|\sin\theta\sin 2(\omega t + \theta_v)}_{\substack{p_X(t) \\ \text{Energy borrowed and} \\ \text{returned by the circuit}}} \tag{2.5}$$

where $\theta$ is the angle between voltage and current, or the impedance angle. $\theta$ is positive if the load is inductive, (i.e., current is lagging the voltage) and $\theta$ is negative if the load is capacitive (i.e., current is leading the voltage).

The instantaneous power has been decomposed into two components. The first component of (2.5) is

$$p_R(t) = |V||I|\cos\theta + |V||I|\cos\theta\cos 2(\omega t + \theta_v)] \tag{2.6}$$

The second term in (2.6), which has a frequency twice that of the source, accounts for the sinusoidal variation in the absorption of power by the resistive portion of the load. Since the average value of this sinusoidal function is zero, the average power delivered to the load is given by

$$P = |V||I|\cos\theta \tag{2.7}$$

This is the power absorbed by the resistive component of the load and is also referred to as the *active power* or *real power*. The product of the rms voltage value and the rms current value $|V||I|$ is called the *apparent power* and is measured in units of volt ampere. The product of the apparent power and the cosine of the angle between voltage and current yields the real power. Because $\cos\theta$ plays a key role in the determination of the average power, it is called *power factor*. When the current lags the voltage, the power factor is considered lagging. When the current leads the voltage, the power factor is considered leading.

The second component of (2.5)

$$p_X(t) = |V||I|\sin\theta\sin 2(\omega t + \theta_v) \tag{2.8}$$





pulsates with twice the frequency and has an average value of zero. This component accounts for power oscillating into and out of the load because of its reactive element (inductive or capacitive). The amplitude of this pulsating power is called *reactive power* and is designated by $Q$.

$$Q = |V||I|\sin\theta \tag{2.9}$$

Both $P$ and $Q$ have the same dimension. However, in order to distinguish between the real and the reactive power, the term "var" is used for the reactive power (var is an acronym for the phrase "volt-ampere reactive"). For an inductive load, current is lagging the voltage, $\theta = (\theta_v - \theta_i) > 0$ and $Q$ is positive; whereas, for a capacitive load, current is leading the voltage, $\theta = (\theta_v - \theta_i) < 0$ and $Q$ is negative.

A careful study of Equations (2.6) and (2.8) reveals the following characteristics of the instantaneous power.

- For a pure resistor, the impedance angle is zero and the power factor is unity (UPF), so that the apparent and real power are equal. The electric energy is transformed into thermal energy.
- If the circuit is purely inductive, the current lags the voltage by 90° and the average power is zero. Therefore, in a purely inductive circuit, there is no transformation of energy from electrical to nonelectrical form. The instantaneous power at the terminal of a purely inductive circuit oscillates between the circuit and the source. When $p(t)$ is positive, energy is being stored in the magnetic field associated with the inductive elements, and when $p(t)$ is negative, energy is being extracted from the magnetic fields of the inductive elements.
- If the load is purely capacitive, the current leads the voltage by 90°, and the average power is zero, so there is no transformation of energy from electrical to nonelectrical form. In a purely capacitive circuit, the power oscillates between the source and the electric field associated with the capacitive elements.

**Example 2.1**

The supply voltage in Figure 2.1 is given by $v(t) = 100\cos\omega t$ and the load is inductive with impedance $Z = 1.25\angle 60°\ \Omega$. Determine the expression for the instantaneous current $i(t)$ and the instantaneous power $p(t)$. Use *MATLAB* to plot $i(t)$, $v(t)$, $p(t)$, $p_R(t)$, and $p_X(t)$ over an interval of 0 to $2\pi$.

$$I_{max} = \frac{100\angle 0°}{1.25\angle 60°} = 80\angle -60°\ \text{A}$$

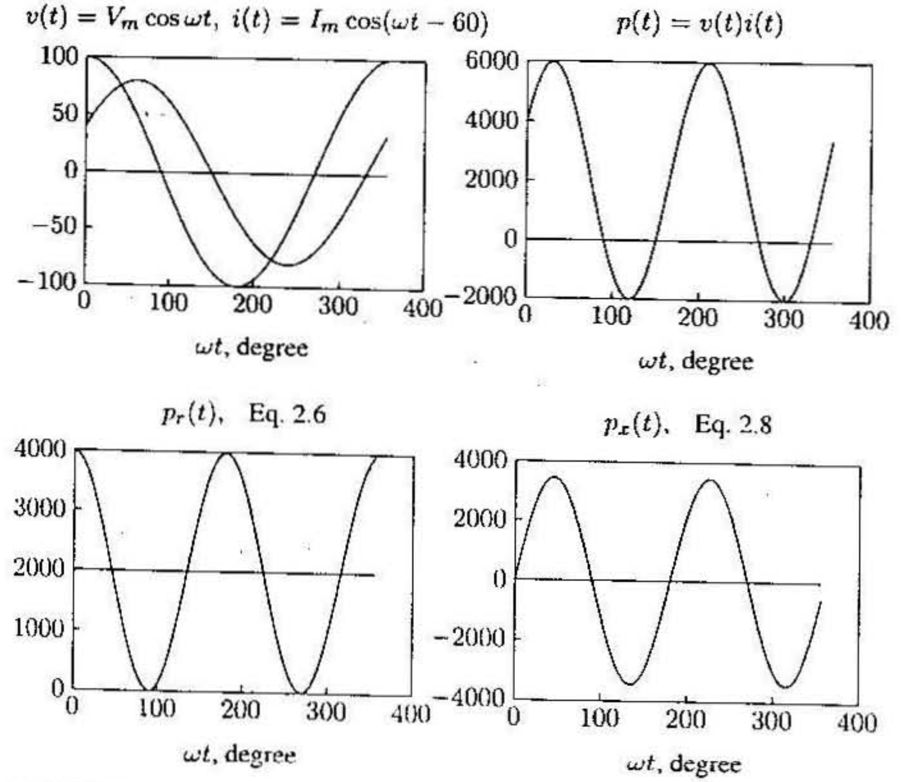



**FIGURE 2.2**
Instantaneous current, voltage, power, Eqs. 2.6 and 2.8.

therefore

$$i(t) = 80\cos(\omega t - 60°)\ \text{A}$$
$$p(t) = v(t)\,i(t) = 8000\cos\omega t\cos(\omega t - 60°)\ \text{W}$$

The following statements are used to plot the above instantaneous quantities and the instantaneous terms given by (2.6) and (2.8).

```
Vm = 100; thetav = 0;        % Voltage amplitude and phase angle
Z = 1.25;  gama = 60;        % Impedance magnitude and phase angle
thetai = thetav - gama;           % Current phase angle in degree
theta = (thetav - thetai)*pi/180;            % Degree to radian
Im = Vm/Z;                                  % Current amplitude
wt = 0:.05:2*pi;                              % wt from 0 to 2*pi
v = Vm*cos(wt);                          % Instantaneous voltage
```



```
i = Im*cos(wt + thetai*pi/180);        % Instantaneous current
p = v.*i;                                % Instantaneous power
V = Vm/sqrt(2); I=Im/sqrt(2);        % rms voltage and current
P = V*I*cos(theta);                          % Average power
Q = V*I*sin(theta);                         % Reactive power
S = P + j*Q                                  % Complex power
pr = P*(1 + cos(2*(wt + thetav)));               % Eq. (2.6)
px = Q*sin(2*(wt + thetav));                     % Eq. (2.8)
PP = P*ones(1, length(wt));%Average power of length w for plot
xline = zeros(1, length(wt));          %generates a zero vector
wt=180/pi*wt;                      % converting radian to degree
subplot(2,2,1), plot(wt, v, wt,i,wt, xline), grid
title(['v(t)=Vm coswt, i(t)=Im cos(wt+',num2str(thetai), ')']) 
xlabel('wt, degree')
subplot(2,2,2), plot(wt, p, wt, xline), grid
title('p(t)=v(t) i(t)'),xlabel('wt, degree')
subplot(2,2,3), plot(wt, pr, wt, PP,wt,xline), grid
title('pr(t)   Eq. 2.6'), xlabel('wt, degree')
subplot(2,2,4), plot(wt, px, wt, xline), grid
title('px(t)  Eq. 2.8'), xlabel('wt, degree'), subplot(111)
```

## 2.3 COMPLEX POWER

The rms voltage phasor of (2.1) and the rms current phasor of (2.2) shown in Figure 2.3 are

$$V = |V|\angle\theta_v \text{ and } I = |I|\angle\theta_i$$

The term $VI^*$ results in

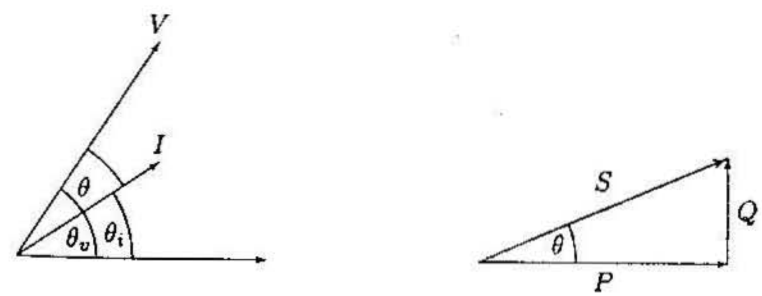



**FIGURE 2.3**
Phasor diagram and power triangle for an inductive load (lagging PF).

$$VI^* = |V||I|\angle\theta_v - \theta_i = |V||I|\angle\theta$$

$$= |V||I|\cos\theta + j|V||I|\sin\theta$$

The above equation defines a complex quantity where its real part is the average (real) power $P$ and its imaginary part is the reactive power $Q$. Thus, the complex power designated by $S$ is given by

$$S = VI^* = P + jQ \tag{2.10}$$

The magnitude of $S$, $|S| = \sqrt{P^2 + Q^2}$, is the apparent power; its unit is volt-amperes and the larger units are kVA or MVA. Apparent power gives a direct indication of heating and is used as a rating unit of power equipment. Apparent power has practical significance for an electric utility company since a utility company must supply both average and apparent power to consumers.

The reactive power $Q$ is positive when the phase angle $\theta$ between voltage and current (impedance angle) is positive (i.e., when the load impedance is inductive, and $I$ lags $V$). $Q$ is negative when $\theta$ is negative (i.e., when the load impedance is capacitive and $I$ leads $V$) as shown in Figure 2.4.

In working with Equation (2.10 ) it is convenient to think of $P$, $Q$, and $S$ as forming the sides of a right triangle as shown in Figures 2.3 and 2.4.


**FIGURE 2.4**
Phasor diagram and power triangle for a capacitive load (leading PF).

If the load impedance is $Z$ then

$$V = ZI \tag{2.11}$$

substituting for $V$ into (2.10) yields

$$S = VI^* = ZII^* = R|I|^2 + jX|I|^2 \tag{2.12}$$

From (2.12) it is evident that complex power $S$ and impedance $Z$ have the same angle. Because the power triangle and the impedance triangle are similar triangles, the impedance angle is sometimes called the *power angle*.

Similarly, substituting for $I$ from (2.11) into (2.10) yields

$$S = VI^* = \frac{VV^*}{Z^*} = \frac{|V|^2}{Z^*} \tag{2.13}$$





From (2.13), the impedance of the complex power $S$ is given by

$$Z = \frac{|V|^2}{S^*} \tag{2.14}$$

## 2.4 THE COMPLEX POWER BALANCE

From the conservation of energy, it is clear that real power supplied by the source is equal to the sum of real powers absorbed by the load. At the same time, a balance between the reactive power must be maintained. Thus the total complex power delivered to the loads in parallel is the sum of the complex powers delivered to each. Proof of this is as follows:

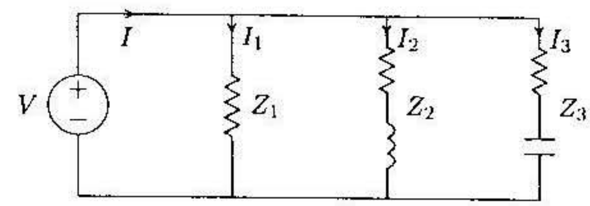



**FIGURE 2.5**
Three loads in parallel.

For the three loads shown in Figure 2.5, the total complex power is given by

$$S = VI^* = V[I_1 + I_2 + I_3]^* = VI_1^* + VI_2^* + VI_3^* \tag{2.15}$$

### Example 2.2

In the above circuit $V = 1200\angle 0°$ V, $Z_1 = 60 + j0\ \Omega$ , $Z_2 = 6 + j12\ \Omega$ and $Z_3 = 30 - j30\ \Omega$. Find the power absorbed by each load and the total complex power.

$$I_1 = \frac{1200\angle 0°}{60\angle 0} = 20 + j0 \text{ A}$$
$$I_2 = \frac{1200\angle 0°}{6 + j12} = 40 - j80 \text{ A}$$
$$I_3 = \frac{1200\angle 0°}{30 - j30} = 20 + j20 \text{ A}$$

$$S_1 = VI_1^* = 1200\angle 0°(20 - j0) = 24,000 \text{ W} + j0 \text{ var}$$
$$S_2 = VI_2^* = 1200\angle 0°(40 + j80) = 48,000 \text{ W} + j96,000 \text{ var}$$
$$S_3 = VI_3^* = 1200\angle 0°(20 - j20) = 24,000 \text{ W} - j24,000 \text{ var}$$

The total load complex power adds up to

$$S = S_1 + S_2 + S_3 = 96,000 \text{ W} + j72,000 \text{ var}$$

Alternatively, the sum of complex power delivered to the load can be obtained by first finding the total current.

$$I = I_1 + I_2 + I_3 = (20 + j0) + (40 - j80) + (20 + j20)$$
$$= 80 - j60 = 100\angle -36.87° \text{ A}$$

and

$$S = VI^* = (1200\angle 0°)(100\angle 36.87°) = 120,000\angle 36.87° \text{ VA}$$
$$= 96,000 \text{ W} + j72,000 \text{ var}$$

A final insight is contained in Figure 2.6, which shows the current phasor diagram and the complex power vector representation.

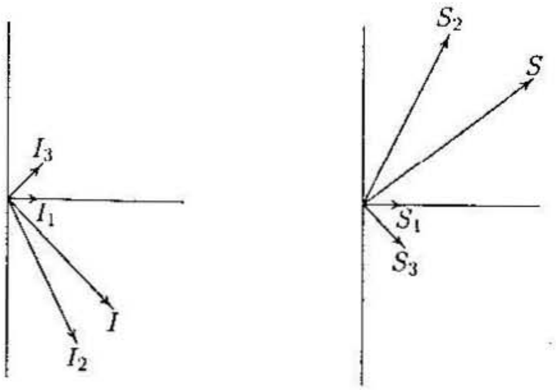



**FIGURE 2.6**
Current phasor diagram and power plane diagram.

The complex powers may also be obtained directly from (2.14)

$$S_1 = \frac{|V|^2}{Z_1^*} = \frac{(1200)^2}{60} = 24,000 \text{ W} + j\ 0$$
$$S_2 = \frac{|V|^2}{Z_2^*} = \frac{(1200)^2}{6 - j12} = 48,000 \text{ W} + j96,000 \text{ var}$$
$$S_3 = \frac{|V|^2}{Z_3^*} = \frac{(1200)^2}{30 + j30} = 24,000 \text{ W} - j24,000 \text{ var}$$





## 2.5 POWER FACTOR CORRECTION

It can be seen from (2.7) that the apparent power will be larger than $P$ if the power factor is less than 1. Thus the current $I$ that must be supplied will be larger for $PF < 1$ than it would be for $PF = 1$, even though the average power $P$ supplied is the same in either case. A larger current cannot be supplied without additional cost to the utility company. Thus, it is in the power company's (and its customer's) best interest that major loads on the system have power factors as close to 1 as possible. In order to maintain the power factor close to unity, power companies install banks of capacitors throughout the network as needed. They also impose an additional charge to industrial consumers who operate at low power factors. Since industrial loads are inductive and have low lagging power factors, it is beneficial to install capacitors to improve the power factor. This consideration is not important for residential and small commercial customers because their power factors are close to unity.

### Example 2.3

Two loads $Z_1 = 100 + j0\ \Omega$ and $Z_2 = 10 + j20\ \Omega$ are connected across a 200-V rms, 60-Hz source as shown in Figure 2.7.

(a) Find the total real and reactive power, the power factor at the source, and the total current.

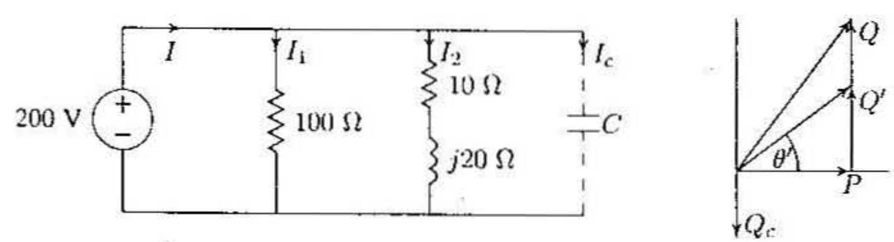



**FIGURE 2.7**
Circuit for Example 2.3 and the power triangle.

$$I_1 = \frac{200\angle 0°}{100} = 2\angle 0° \text{ A}$$
$$I_2 = \frac{200\angle 0°}{10 + j20} = 4 - j8 \text{ A}$$
$$S_1 = VI_1^* = 200\angle 0°(2 - j0) = 400 \text{ W} + j0 \text{ var}$$
$$S_2 = VI_2^* = 200\angle 0°(4 + j8) = 800 \text{ W} + j1600 \text{ var}$$

Total apparent power and current are

$$S = P + jQ = 1200 + j1600 = 2000\angle 53.13^\circ \text{ VA}$$
$$I = \frac{S^*}{V^*} = \frac{2000\angle -53.13^\circ}{200\angle 0^\circ} = 10\angle -53.13^\circ \text{ A}$$

Power factor at the source is

$$PF = \cos(53.13) = 0.6 \text{ lagging}$$

(b) Find the capacitance of the capacitor connected across the loads to improve the overall power factor to 0.8 lagging.

Total real power $P = 1200$ W at the new power factor 0.8 lagging. Therefore

$$\theta' = \cos^{-1}(0.8) = 36.87^\circ$$
$$Q' = P\tan\theta' = 1200\tan(36.87^\circ) = 900 \text{ var}$$
$$Q_c = 1600 - 900 = 700 \text{ var}$$
$$Z_c = \frac{|V|^2}{S_c^*} = \frac{(200)^2}{j700} = -j57.14\ \Omega$$
$$C = \frac{10^6}{2\pi(60)(57.14)} = 46.42\ \mu\text{F}$$

The total power and the new current are

$$S' = 1200 + j900 = 1500\angle 36.87^\circ$$
$$I' = \frac{S'^*}{V^*} = \frac{1500\angle -36.87^\circ}{200\angle 0^\circ} = 7.5\angle -36.87^\circ$$

Note the reduction in the supply current from 10 A to 7.5 A.

### Example 2.4

Three loads are connected in parallel across a 1400-V rms, 60-Hz single-phase supply as shown in Figure 2.8.

Load 1: Inductive load, 125 kVA at 0.28 power factor.

Load 2: Capacitive load, 10 kW and 40 kvar.

Load 3: Resistive load of 15 kW.

(a) Find the total kW, kvar, kVA, and the supply power factor.





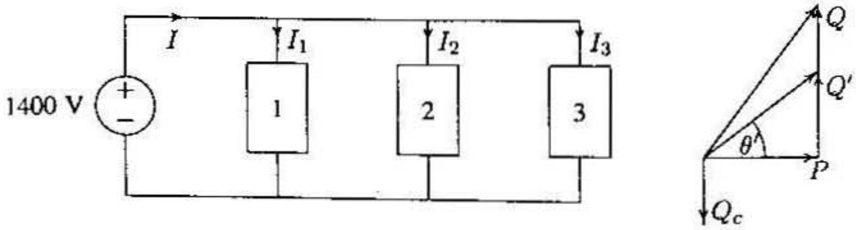



**FIGURE 2.8**
Circuit for Example 2.4.

An inductive load has a lagging power factor, the capacitive load has a leading power factor, and the resistive load has a unity power factor.

For Load 1:

$$\theta_1 = \cos^{-1}(0.28) = 73.74^\circ \text{ lagging}$$

The load complex powers are

$$S_1 = 125\angle 73.74 \text{ kVA} = 35 \text{ kW} + j120 \text{ kvar}$$
$$S_2 = 10 \text{ kW} - j40 \text{ kvar}$$
$$S_3 = 15 \text{ kW} + j0 \text{ kvar}$$

The total apparent power is

$$S = P + jQ = S_1 + S_2 + S_3$$
$$= (35 + j120) + (10 - j40) + (15 + j0)$$
$$= 60 \text{ kW} + j80 \text{ kvar} = 100\angle 53.13 \text{ kVA}$$

The total current is

$$I = \frac{S^*}{V^*} = \frac{100{,}000\angle -53.13^\circ}{1400\angle 0^\circ} = 71.43\angle -53.13^\circ \text{ A}$$

The supply power factor is

$$PF = \cos(53.13) = 0.6 \text{ lagging}$$

(b) A capacitor of negligible resistance is connected in parallel with the above loads to improve the power factor to 0.8 lagging. Determine the kvar rating of this capacitor and the capacitance in $\mu$F.

Total real power $P = 60$ kW at the new power factor of 0.8 lagging results in the new reactive power $Q'$.

$$\theta' = \cos^{-1}(0.8) = 36.87^\circ$$
$$Q' = 60\tan(36.87^\circ) = 45 \text{ kvar}$$

Therefore, the required capacitor kvar is

$$Q_c = 80 - 45 = 35 \text{ kvar}$$

and

$$X_c = \frac{|V|^2}{S_c^*} = \frac{1400^2}{j35,000} = -j56\ \Omega$$
$$C = \frac{10^6}{2\pi(60)(56)} = 47.37\ \mu\text{F}$$

and the new current is

$$I' = \frac{S'^*}{V^*} = \frac{60,000 - j45,000}{1400\angle 0^\circ} = 53.57\angle{-36.87^\circ}\ \text{A}$$

Note the reduction in the supply current from 71.43 A to 53.57 A.

## 2.6 COMPLEX POWER FLOW

Consider two ideal voltage sources connected by a line of impedance $Z = R + jX\ \Omega$ as shown in Figure 2.9.

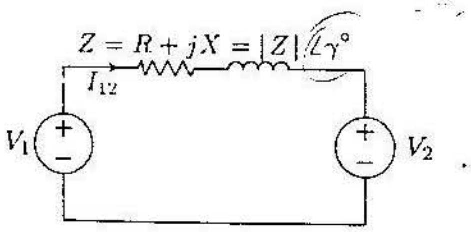



**FIGURE 2.9**
Two interconnected voltage sources.

Let the phasor voltage be $V_1 = |V_1|\angle\delta_1$ and $V_2 = |V_2|\angle\delta_2$. For the assumed direction of current

$$I_{12} = \frac{|V_1|\angle\delta_1 - |V_2|\angle\delta_2}{|Z|\angle\gamma} = \frac{|V_1|}{|Z|}\angle\delta_1 - \gamma - \frac{|V_2|}{|Z|}\angle\delta_2 - \gamma$$





The complex power $S_{12}$ is given by

$$S_{12} = V_1 I_{12}^* = |V_1|\angle\delta_1 \left[\frac{|V_1|}{|Z|}\angle\gamma - \delta_1 - \frac{|V_2|}{|Z|}\angle\gamma - \delta_2\right]$$
$$= \frac{|V_1|^2}{|Z|}\angle\gamma - \frac{|V_1||V_2|}{|Z|}\angle\gamma + \delta_1 - \delta_2$$

Thus, the real and reactive power at the sending end are

$$P_{12} = \frac{|V_1|^2}{|Z|}\cos\gamma - \frac{|V_1||V_2|}{|Z|}\cos(\gamma + \delta_1 - \delta_2) \quad (2.16)$$

$$Q_{12} = \frac{|V_1|^2}{|Z|}\sin\gamma - \frac{|V_1||V_2|}{|Z|}\sin(\gamma + \delta_1 - \delta_2) \quad (2.17)$$

Power system transmission lines have small resistance compared to the reactance. Assuming $R = 0$ (i.e., $Z = X\angle 90^\circ$ ), the above equations become

$$P_{12} = \frac{|V_1||V_2|}{X}\sin(\delta_1 - \delta_2) \quad (2.18)$$

$$Q_{12} = \frac{|V_1|}{X}[\,|V_1| - |V_2|\cos(\delta_1 - \delta_2)\,] \quad (2.19)$$

Since $R = 0$, there are no transmission line losses and the real power sent equals the real power received.

From the above results, for a typical power system with small $R/X$ ratio, the following important observations are made :

1. Equation (2.18) shows that small changes in $\delta_1$ or $\delta_2$ will have a significant effect on the real power flow, while small changes in voltage magnitudes will not have appreciable effect on the real power flow. Therefore, the flow of real power on a transmission line is governed mainly by the angle difference of the terminal voltages (i.e., $P_{12} \propto \sin\delta$ ), where $\delta = \delta_1 - \delta_2$. If $V_1$ leads $V_2$, $\delta$ is positive and the real power flows from node 1 to node 2. If $V_1$ lags $V_2$, $\delta$ is negative and power flows from node 2 to node 1.

2. Assuming $R = 0$, the theoretical maximum power (static transmission capacity) occurs when $\delta = 90^\circ$ and the maximum power transfer is given by

$$P_{max} = \frac{|V_1||V_2|}{X} \quad (2.20)$$

In Chapter 3 we learn that increasing $\delta$ beyond the static transmission capacity will result in loss of synchronism between the two machines.

3. For maintaining transient stability, the power system is usually operated with small load angle $\delta$. Also, from (2.19) the reactive power flow is determined by the magnitude difference of terminal voltages, (i.e., $Q \propto |V_1| - |V_2|$).

### Example 2.5

Two voltage sources $V_1 = 120\angle{-5}$ V and $V_2 = 100\angle 0$ V are connected by a short line of impedance $Z = 1 + j7\,\Omega$ as shown in Figure 2.9. Determine the real and reactive power supplied or received by each source and the power loss in the line.

$$I_{12} = \frac{120\angle{-5^\circ} - 100\angle 0^\circ}{1 + j7} = 3.135\angle{-110.02^\circ}\ \text{A}$$

$$I_{21} = \frac{100\angle 0^\circ - 120\angle - 5^\circ}{1 + j7} = 3.135\angle 69.98^\circ\ \text{A}$$

$$S_{12} = V_1 I_{12}^* = 376.2\angle 105.02^\circ = -97.5\ \text{W} + j363.3\ \text{var}$$

$$S_{21} = V_2 I_{21}^* = 313.5\angle{-69.98^\circ} = 107.3\ \text{W} - j294.5\ \text{var}$$

Line loss is given by

$$S_L = S_1 + S_2 = 9.8\ \ \text{W} + j68.8\ \text{var}$$

From the above results, since $P_1$ is negative and $P_2$ is positive, source 1 receives 97.5 W, and source 2 generates 107.3 W and the real power loss in the line is 9.8 W. The real power loss in the line can be checked by

$$P_L = R|I_{12}|^2 = (1)(3.135)^2 = 9.8\ \text{W}$$

Also, since $Q_1$ is positive and $Q_2$ is negative, source 1 delivers 363.3 var and source 2 receives 294.5 var, and the reactive power loss in the line is 68.6 var. The reactive power loss in the line can be checked by

$$Q_L = X|I_{12}|^2 = (7)(3.135)^2 = 68.8\ \text{var}$$

### Example 2.6

This example concerns the direction of power flow between two voltage sources. Write a *MATLAB* program for the system of Example 2.5 such that the phase angle of source 1 is changed from its initial value by $\pm 30^\circ$ in steps of $5^\circ$. Voltage magnitudes of the two sources and the voltage phase angle of source 2 is to be kept constant. Compute the complex power for each source and the line loss. Tabulate the real power and plot $P_1$, $P_2$, and $P_L$ versus voltage phase angle $\delta$. The following commands



```
E1 = input('Source # 1 Voltage Mag. = ');
a1 = input('Source# 1 Phase Angle  = ');
E2 = input('Source # 2 Voltage Mag. = ');
a2 = input('Source # 2 Phase Angle  = ');
R  = input('Line Resistance = ');
X  = input('Line Reactance = ');
Z  =  R + j*X;                              % Line impedance
a1 = (-30+a1:5:30+a1)';          % Change a1 by +/- 30, col. array
a1r = a1*pi/180;                        % Convert degree to radian
k = length(a1);
a2 = ones(k,1)*a2;   % Create col. array of same length for a2
a2r = a2*pi/180;                        % Convert degree to radian
V1 = E1.*cos(a1r) + j*E1.*sin(a1r);
V2 = E2.*cos(a2r) + j*E2.*sin(a2r);
I12 = (V1 - V2)./Z;  I21=-I12;
S1 = V1.*conj(I12);  P1 = real(S1);  Q1 = imag(S1);
S2 = V2.*conj(I21);  P2 = real(S2);  Q2 = imag(S2);
SL = S1+S2;          PL = real(SL);  QL = imag(SL);
Result1 = [a1, P1,  P2, PL];
disp('   Delta 1    P-1        P-2        P-L ')
disp(Result1)
plot(a1, P1,  a1, P2,  a1,PL)
xlabel('Source #1 Voltage Phase Angle')
ylabel(' P, Watts'),
text(-26, -550, 'P1'), text(-26, 600,'P2'),
text(-26, 100, 'PL')
```

result in

```
Source # 1 Voltage Mag.  = 120
Source # 1 Phase Angle   = -5
Source # 2 Voltage Mag.  = 100
Source # 2 Phase Angle   = 0
Line Resistance = 1
Line Reactance =  7
```

| Delta 1 | P-1 | P-2 | P-L |
|---|---|---|---|
| -35.0000 | -872.2049 | 967.0119 | 94.8070 |
| -30.0000 | -759.8461 | 832.1539 | 72.3078 |
| -25.0000 | -639.5125 | 692.4848 | 52.9723 |
| -20.0000 | -512.1201 | 549.0676 | 36.9475 |
| -15.0000 | -378.6382 | 402.9938 | 24.3556 |
| -10.0000 | -240.0828 | 255.3751 | 15.2923 |
| -5.0000 | -97.5084 | 107.3349 | 9.8265 |
| 0 | 48.0000 | -40.0000 | 8.0000 |

| | | | |
|---|---|---|---|
| 5.0000 | 195.3349 | -185.5084 | 9.8265 |
| 10.0000 | 343.3751 | -328.0828 | 15.2923 |
| 15.0000 | 490.9938 | -466.6382 | 24.3556 |
| 20.0000 | 637.0676 | -600.1201 | 36.9475 |
| 25.0000 | 780.4848 | -727.5125 | 52.9723 |

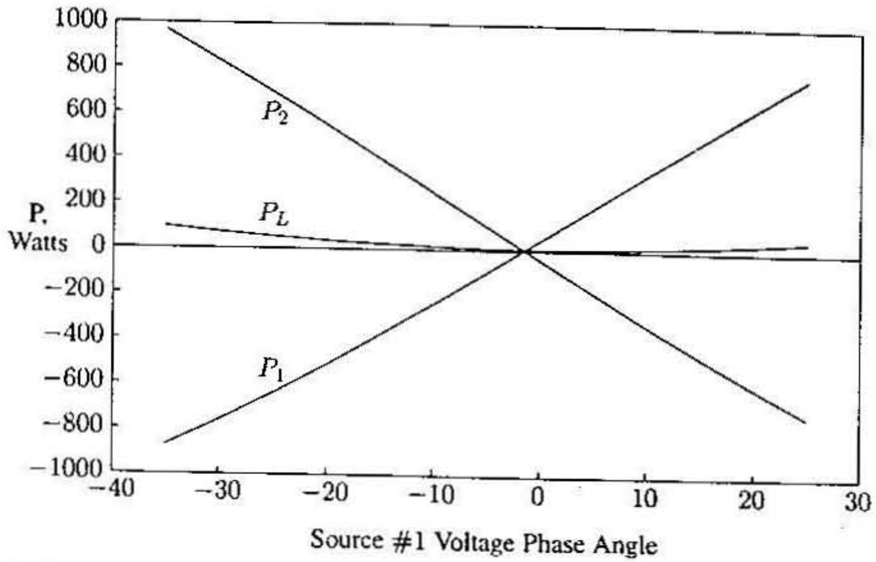



**FIGURE 2.10**
Real power versus voltage phase angle $\delta$.

Examination of Figure 2.10 shows that the flow of real power along the interconnection is determined by the angle difference of the terminal voltages. Problem 2.9 requires the development of a similar program for demonstrating the dependency of reactive power on the magnitude difference of terminal voltages.

## 2.7 BALANCED THREE-PHASE CIRCUITS

The generation, transmission and distribution of electric power is accomplished by means of three-phase circuits. At the generating station, three sinusoidal voltages are generated having the same amplitude but displaced in phase by 120°. This is called a *balanced source*. If the generated voltages reach their peak values in the sequential order ABC, the generator is said to have a *positive phase sequence*, shown in Figure 2.11(a). If the phase order is ACB, the generator is said to have a *negative phase sequence*, as shown in Figure 2.11(b).





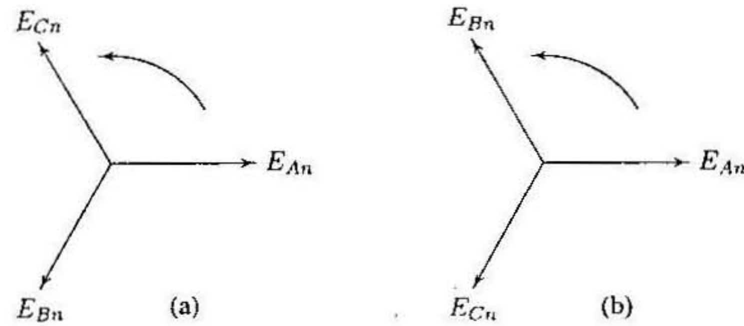



**FIGURE 2.11**
(a) Positive, or ABC, phase sequence. (b) Negative, or ACB, phase sequence.

In a three-phase system, the instantaneous power delivered to the external loads is constant rather than pulsating as it is in a single-phase circuit. Also, three-phase motors, having constant torque, start and run much better than single-phase motors. This feature of three-phase power, coupled with the inherent efficiency of its transmission compared to single-phase (less wire for the same delivered power), accounts for its universal use.

A power system has Y-connected generators and usually includes both Δ- and Y-connected loads. Generators are rarely Δ-connected, because if the voltages are not perfectly balanced, there will be a net voltage, and consequently a circulating current, around the Δ. Also, the phase voltages are lower in the Y-connected generator, and thus less insulation is required. Figure 2.12 shows a Y-connected generator supplying balanced Y-connected loads through a three-phase line. Assuming a positive phase sequence (phase order ABC) the generated voltages are:

$$\begin{aligned} E_{An} &= |E_p|\angle 0^\circ \\ E_{Bn} &= |E_p|\angle -120^\circ \\ E_{Cn} &= |E_p|\angle -240^\circ \end{aligned} \tag{2.21}$$

In power systems, great care is taken to ensure that the loads of transmission lines are balanced. For balanced loads, the terminal voltages of the generator $V_{An}$, $V_{Bn}$ and $V_{Cn}$ and the phase voltages $V_{an}$, $V_{bn}$ and $V_{cn}$ at the load terminals are balanced. For "phase A," these are given by

$$V_{An} = E_{An} - Z_G I_a \tag{2.22}$$

$$V_{an} = V_{An} - Z_L I_a \tag{2.23}$$

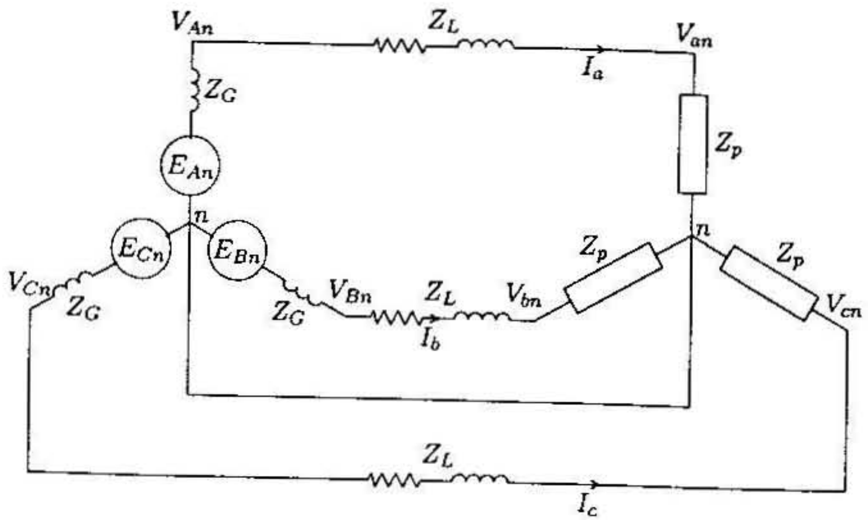



**FIGURE 2.12**
A Y-connected generator supplying a Y-connected load.

## 2.8 Y-CONNECTED LOADS

To find the relationship between the line voltages (line-to-line voltages) and the phase voltages (line-to-neutral voltages), we assume a positive, or ABC, sequence. We arbitrarily choose the line-to-neutral voltage of the a-phase as the reference, thus

$$\begin{aligned} V_{an} &= |V_p|\angle 0^\circ \\ V_{bn} &= |V_p|\angle -120^\circ \\ V_{cn} &= |V_p|\angle -240^\circ \end{aligned} \tag{2.24}$$

where $|V_p|$ represents the magnitude of the phase voltage (line-to-neutral voltage).

The line voltages at the load terminals in terms of the phase voltages are found by the application of Kirchhoff's voltage law

$$\begin{aligned} V_{ab} &= V_{an} - V_{bn} = |V_p|(1\angle 0^\circ - 1\angle -120^\circ) = \sqrt{3}|V_p|\angle 30^\circ \\ V_{bc} &= V_{bn} - V_{cn} = |V_p|(1\angle -120^\circ - 1\angle -240^\circ) = \sqrt{3}|V_p|\angle -90^\circ \\ V_{ca} &= V_{cn} - V_{an} = |V_p|(1\angle -240^\circ - 1\angle 0^\circ) = \sqrt{3}|V_p|\angle 150^\circ \end{aligned} \tag{2.25}$$

The voltage phasor diagram of the Y-connected loads of Figure 2.12 is shown in Figure 2.13. The relationship between the line voltages and phase voltages is demonstrated graphically.





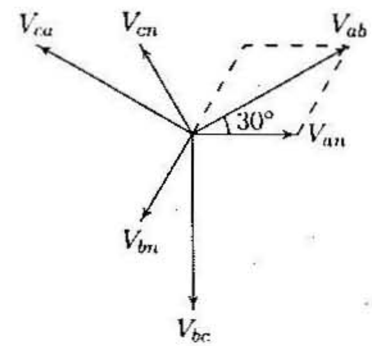



**FIGURE 2.13**
Phasor diagram showing phase and line voltages.

If the rms value of any of the line voltages is denoted by $V_L$, then one of the important characteristics of the Y-connected three-phase load may be expressed as

$$V_L = \sqrt{3}\,|V_p|\angle 30^\circ \tag{2.26}$$

Thus in the case of Y-connected loads, the magnitude of the line voltage is $\sqrt{3}$ times the magnitude of the phase voltage, and for a positive phase sequence, the set of line voltages leads the set of phase voltages by 30°.

The three-phase currents in Figure 2.12 also possess three-phase symmetry and are given by

$$\begin{aligned} I_a &= \frac{V_{an}}{Z_p} = |I_p|\angle -\theta \\ I_b &= \frac{V_{bn}}{Z_p} = |I_p|\angle -120^\circ - \theta \\ I_c &= \frac{V_{cn}}{Z_p} = |I_p|\angle -240^\circ - \theta \end{aligned} \tag{2.27}$$

where $\theta$ is the impedance phase angle.

The currents in lines are also the phase currents (the current carried by the phase impedances). Thus

$$I_L = I_p \tag{2.28}$$

## 2.9 Δ-CONNECTED LOADS

A balanced Δ-connected load (with equal phase impedances) is shown in Figure 2.14.

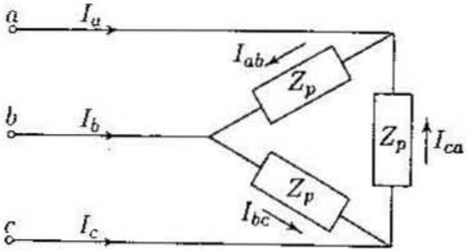



**FIGURE 2.14**
A Δ-connected load.

It is clear from the inspection of the circuit that the line voltages are the same as phase voltages.

$$V_L = V_p \tag{2.29}$$

Consider the phasor diagram shown in Figure 2.15, where the phase current $I_{ab}$ is arbitrarily chosen as reference, we have

$$\begin{aligned} I_{ab} &= |I_p|\angle 0^\circ \\ I_{bc} &= |I_p|\angle -120^\circ \\ I_{ca} &= |I_p|\angle -240^\circ \end{aligned} \tag{2.30}$$

where $|I_p|$ represents the magnitude of the phase current.

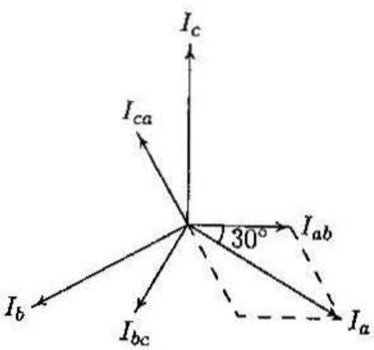



**FIGURE 2.15**
Phasor diagram showing phase and line currents.





The relationship between phase and line currents can be obtained by applying Kirchhoff's current law at the corners of Δ.

$$\begin{aligned} I_a &= I_{ab} - I_{ca} = |I_p|(1\angle 0^\circ - 1\angle -240^\circ) = \sqrt{3}|I_p|\angle -30^\circ \\ I_b &= I_{bc} - I_{ab} = |I_p|(1\angle -120^\circ - 1\angle 0^\circ) = \sqrt{3}|I_p|\angle -150^\circ \\ I_c &= I_{ca} - I_{bc} = |I_p|(1\angle -240^\circ - 1\angle -120^\circ) = \sqrt{3}|I_p|\angle 90^\circ \end{aligned} \tag{2.31}$$

The relationship between the line currents and phase currents is demonstrated graphically in Figure 2.15.

If the rms of any of the line currents is denoted by $I_L$, then one of the important characteristics of the Δ-connected three-phase load may be expressed as

$$I_L = \sqrt{3}|I_p| \angle -30^\circ \tag{2.32}$$

Thus in the case of Δ-connected loads, the magnitude of the line current is $\sqrt{3}$ times the magnitude of the phase current, and with positive phase sequence, the set of line currents lags the set of phase currents by 30°.

## 2.10 Δ-Y TRANSFORMATION

For analyzing network problems, it is convenient to replace the Δ-connected circuit with an equivalent Y-connected circuit. Consider the fictitious Y-connected circuit of $Z_Y$ Ω/phase which is equivalent to a balanced Δ-connected circuit of $Z_\Delta$ Ω/phase, as shown in Figure 2.16.

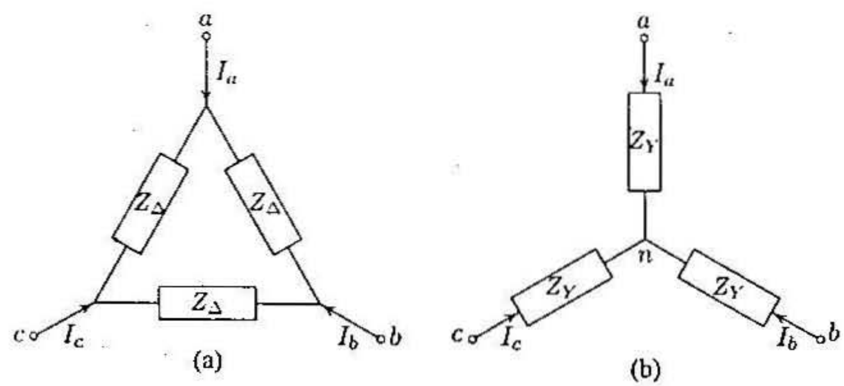



**FIGURE 2.16**
(a) Δ to (b) Y-connection.

For the Δ-connected circuit, the phase current $I_a$ is given by

$$I_a = \frac{V_{ab}}{Z_\Delta} + \frac{V_{ac}}{Z_\Delta} = \frac{V_{ab} + V_{ac}}{Z_\Delta} \tag{2.33}$$

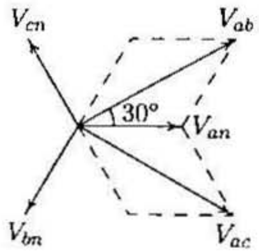



**FIGURE 2.17**
Phasor diagram showing phase and line voltages.

The phasor diagram in Figure 2.17 shows the relationship between balanced phase and line-to-line voltages. From this phasor diagram, we find

$$V_{ab} + V_{ac} = \sqrt{3}\,|V_{an}|\angle 30^\circ + \sqrt{3}\,|V_{an}|\angle -30^\circ \tag{2.34}$$
$$= 3V_{an} \tag{2.35}$$

Substituting in (2.33), we get

$$I_a = \frac{3V_{an}}{Z_\Delta}$$

or

$$V_{an} = \frac{Z_\Delta}{3} I_a \tag{2.36}$$

Now, for the Y-connected circuit, we have

$$V_{an} = Z_Y I_a \tag{2.37}$$

Thus, from (2.36) and (2.37), we find that

$$Z_Y = \frac{Z_\Delta}{3} \tag{2.38}$$

## 2.11 PER-PHASE ANALYSIS

The current in the neutral of the balanced Y-connected loads shown in Figure 2.12 is given by

$$I_n = I_a + I_b + I_c = 0 \tag{2.39}$$



Since the neutral carries no current, a neutral wire of any impedance may be replaced by any other impedance, including a short circuit and an open circuit. The return line may not actually exist, but regardless, a line of zero impedance is included between the two neutral points. The balanced power system problems are then solved on a "per-phase" basis. It is understood that the other two phases carry identical currents except for the phase shift.

We may then look at only one phase, say "phase A," consisting of the source $V_{An}$ in series with $Z_L$ and $Z_p$, as shown in Figure 2.18. The neutral is taken as datum and usually a single-subscript notation is used for phase voltages.

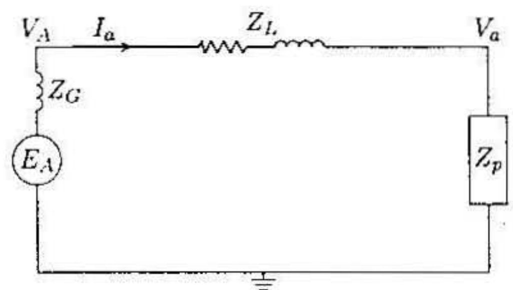



**FIGURE 2.18**
Single-phase circuit for per-phase analysis.

If the load in a three-phase circuit is connected in a $\Delta$, it can be transformed into a Y by using the $\Delta$-to-Y transformation. When the load is balanced, the impedance of each leg of the Y is one-third the impedance of each leg of the $\Delta$, as given by (2.38), and the circuit is modeled by the single-phase equivalent circuit.

## 2.12 BALANCED THREE-PHASE POWER

Consider a balanced three-phase source supplying a balanced Y- or $\Delta$- connected load with the following instantaneous voltages

$$\begin{aligned} v_{an} &= \sqrt{2}|V_p|\cos(\omega t + \theta_v) \\ v_{bn} &= \sqrt{2}|V_p|\cos(\omega t + \theta_v - 120^\circ) \\ v_{cn} &= \sqrt{2}|V_p|\cos(\omega t + \theta_v - 240^\circ) \end{aligned} \tag{2.40}$$

For a balanced load the phase currents are

$$\begin{aligned} i_a &= \sqrt{2}|I_p|\cos(\omega t + \theta_i) \\ i_b &= \sqrt{2}|I_p|\cos(\omega t + \theta_i - 120^\circ) \\ i_c &= \sqrt{2}|I_p|\cos(\omega t + \theta_i - 240^\circ) \end{aligned} \tag{2.41}$$

where $|V_p|$ and $|I_P|$ are the magnitudes of the rms phase voltage and current, respectively. The total instantaneous power is the sum of the instantaneous power of each phase, given by

$$p_{3\phi} = v_{an}i_a + v_{bn}i_b + v_{cn}i_c \tag{2.42}$$

Substituting for the instantaneous voltages and currents from (2.40) and (2.41) into (2.42)

$$\begin{aligned} p_{3\phi} &= 2|V_p||I_p|\cos(\omega t+\theta_v)\cos(\omega t+\theta_i)\\ &+2|V_p||I_p|\cos(\omega t+\theta_v-120°)\cos(\omega t+\theta_i-120°)\\ &+2|V_p||I_p|\cos(\omega t+\theta_v-240°)\cos(\omega t+\theta_i-240°)\end{aligned}$$

Using the trigonometric identity (2.4)

$$\begin{aligned} p_{3\phi} &= |V_p||I_p|[\cos(\theta_v-\theta_i)+\cos(2\omega t+\theta_v+\theta_i)]\\ &+|V_p||I_p|[\cos(\theta_v-\theta_i)+\cos(2\omega t+\theta_v+\theta_i-240°)]\\ &+|V_p||I_p|[\cos(\theta_v-\theta_i)+\cos(2\omega t+\theta_v+\theta_i-480°)]\end{aligned} \tag{2.43}$$

The three double frequency cosine terms in (2.43) are out of phase with each other by 120° and add up to zero, and the three-phase instantaneous power is

$$P_{3\phi} = 3|V_p||I_p|\cos\theta \tag{2.44}$$

$\theta = \theta_v - \theta_i$ is the angle between phase voltage and phase current or the impedance angle.

Note that although the power in each phase is pulsating, the total instantaneous power is constant and equal to three times the real power in each phase. Indeed, this constant power is the main advantage of the three-phase system over the single-phase system. Since the power in each phase is pulsating, the power, then, is made up of the real power and the reactive power. In order to obtain formula symmetry between real and reactive powers, the concept of complex or apparent power ($S$) is extended to three-phase systems by defining the three-phase reactive power as

$$Q_{3\phi} = 3|V_p||I_p|\sin\theta \tag{2.45}$$

Thus, the complex three-phase power is

$$S_{3\phi} = P_{3\phi} + jQ_3\phi \tag{2.46}$$

or

$$S_{3\phi} = 3V_pI_p^* \tag{2.47}$$

Equations (2.44) and (2.45) are sometimes expressed in terms of the rms magnitude of the line voltage and the rms magnitude of the line current. In a Y-connected load the phase voltage $|V_p| = |V_L|/\sqrt{3}$ and the phase current $I_p = I_L$.





In the Δ-connection $V_p = V_L$ and $|I_p| = |I_L|/\sqrt{3}$. Substituting for the phase voltage and phase currents in (2.44) and (2.45), the real and reactive powers for either connection are given by

$$P_{3\phi} = \sqrt{3}|V_L||I_L|\cos\theta \tag{2.48}$$

and

$$Q_{3\phi} = \sqrt{3}|V_L||I_L|\sin\theta \tag{2.49}$$

A comparison of the last two expressions with (2.44) and (2.45) shows that the equation for the power in a three-phase system is the same for either a Y or a Δ connection when the power is expressed in terms of line quantities.

When using (2.48) and (2.49) to calculate the total real and reactive power, remember that $\theta$ is the phase angle between the phase voltage and the phase current. As in the case of single-phase systems for the computation of power, it is best to use the complex power expression in terms of phase quantities given by (2.47). The rated power is customarily given for the three-phase and rated voltage is the line-to-line voltage. Thus, in using the per-phase equivalent circuit, care must be taken to use per-phase voltage by dividing the rated voltage by $\sqrt{3}$.

### Example 2.7

A three-phase line has an impedance of $2 + j4\ \Omega$ as shown in Figure 2.19.

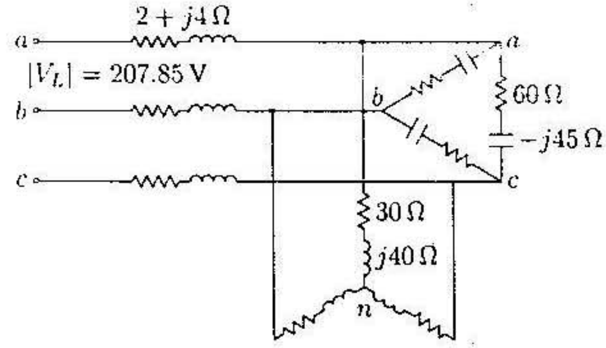



**FIGURE 2.19**
Three-phase circuit diagram for Example 2.7.

The line feeds two balanced three-phase loads that are connected in parallel. The first load is Y-connected and has an impedance of $30+j40\ \Omega$ per phase. The second load is Δ-connected and has an impedance of $60 - j45\ \Omega$. The line is energized at the sending end from a three-phase balanced supply of line voltage 207.85 V. Taking the phase voltage $V_a$ as reference, determine:

(a) The current, real power, and reactive power drawn from the supply.

(b) The line voltage at the combined loads.
(c) The current per phase in each load.
(d) The total real and reactive powers in each load and the line.

(a) The Δ-connected load is transformed into an equivalent Y. The impedance per phase of the equivalent Y is

$$Z_2 = \frac{60 - j45}{3} = 20 - j15 \ \Omega$$

The phase voltage is

$$V_1 = \frac{207.85}{\sqrt{3}} = 120 \text{ V}$$

The single-phase equivalent circuit is shown in Figure 2.20.

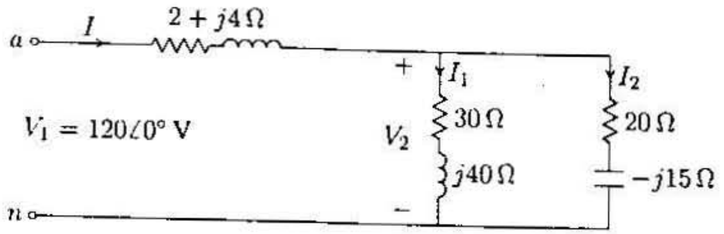



**FIGURE 2.20**
Single-phase equivalent circuit for Example 2.7.

The total impedance is

$$Z = 2 + j4 + \frac{(30 + j40)(20 - j15)}{(30 + j40) + (20 - j15)}$$
$$= 2 + j4 + 22 - j4 = 24 \ \Omega$$

With the phase voltage $V_{an}$ as reference, the current in phase $a$ is

$$I = \frac{V_1}{Z} = \frac{120\angle 0°}{24} = 5 \text{ A}$$

The three-phase power supplied is

$$S = 3V_1 I^* = 3(120\angle 0°)(5\angle 0°) = 1800 \text{ W}$$

(b) The phase voltage at the load terminal is

$$V_2 = 120\angle 0° - (2 + j4)(5\angle 0°) = 110 - j20$$
$$= 111.8\angle -10.3° \text{ V}$$





The line voltage at the load terminal is

$$V_{2ab} = \sqrt{3}\angle 30° \ V_2 = \sqrt{3}\,(111.8)\angle 19.7° = 193.64\angle 19.7° \text{ V}$$

(c) The current per phase in the Y-connected load and in the equivalent Y of the Δ load is

$$I_1 = \frac{V_2}{Z_1} = \frac{110 - j20}{30 + j40} = 1 - j2 = 2.236\angle -63.4° \text{ A}$$
$$I_2 = \frac{V_2}{Z_2} = \frac{110 - j20}{20 - j15} = 4 + j2 = 4.472\angle 26.56° \text{ A}$$

The phase current in the original Δ-connected load, i.e., $I_{ab}$ is given by

$$I_{ab} = \frac{I_2}{\sqrt{3}\angle -30°} = \frac{4.472\angle 26.56°}{\sqrt{3}\angle -30°} = 2.582\angle 56.56° \text{ A}$$

(d) The three-phase power absorbed by each load is

$$S_1 = 3V_2 I_1^* = 3(111.8\angle -10.3°)(2.236\angle 63.4°) = 450 \text{ W} + j600 \text{ var}$$
$$S_2 = 3V_2 I_2^* = 3(111.8\angle -10.3°)(4.472\angle -26.56°) = 1200 \text{ W} - j900 \text{ var}$$

The three-phase power absorbed by the line is

$$S_L = 3(R_L + jX_L)|I|^2 = 3(2 + j4)(5)^2 = 150 \text{ W} + j300 \text{ var}$$

It is clear that the sum of load powers and line losses is equal to the power delivered from the supply, i.e.,

$$S_1 + S_2 + S_L = (450 + j600) + (1200 - j900) + (150 + j300)$$
$$= 1800 \text{ W} + j0 \text{ var}$$

### Example 2.8

A three-phase line has an impedance of $0.4 + j2.7 \ \Omega$ per phase. The line feeds two balanced three-phase loads that are connected in parallel. The first load is absorbing 560.1 kVA at 0.707 power factor lagging. The second load absorbs 132 kW at unity power factor. The line-to-line voltage at the load end of the line is 3810.5 V. Determine:

(a) The magnitude of the line voltage at the source end of the line.
(b) Total real and reactive power loss in the line.
(c) Real power and reactive power supplied at the sending end of the line.

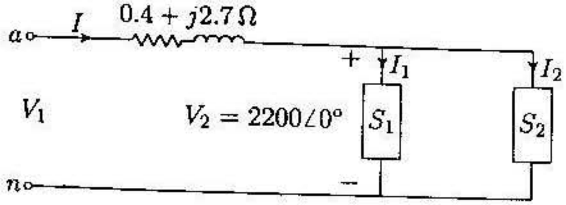



**FIGURE 2.21**
Single-phase equivalent diagram for Example 2.8.

(a) The phase voltage at the load terminals is

$$V_2 = \frac{3810.5}{\sqrt{3}} = 2200 \text{ V}$$

The single-phase equivalent circuit is shown in Figure 2.21.
The total complex power is

$$S_{R(3\phi)} = 560.1(0.707 + j0.707) + 132 = 528 + j396$$
$$= 660\angle 36.87^\circ \text{ kVA}$$

With the phase voltage $V_2$ as reference, the current in the line is

$$I = \frac{S^*_{R(3\phi)}}{3V_2^*} = \frac{660,000\angle -36.87^\circ}{3(2200\angle 0^\circ)} = 100\angle -36.87^\circ \text{ A}$$

The phase voltage at the sending end is

$$V_1 = 2200\angle 0^\circ + (0.4 + j2.7)100\angle -36.87^\circ = 2401.7\angle 4.58^\circ \text{ V}$$

The magnitude of the line voltage at the sending end of the line is

$$|V_{1L}| = \sqrt{3}|V_1| = \sqrt{3}(2401.7) = 4160 \text{ V}$$

(b) The three-phase power loss in the line is

$$S_{L(3\phi)} = 3R|I|^2 + j3X|I|^2 = 3(0.4)(100)^2 + j3(2.7)(100)^2$$
$$= 12 \text{ kW} + j81 \text{ kvar}$$

(c) The three-phase sending power is

$$S_{S(3\phi)} = 3V_1I^* = 3(2401.7\angle 4.58^\circ)(100\angle 36.87^\circ) = 540 \text{ kW} + j477 \text{ kvar}$$

It is clear that the sum of load powers and the line losses is equal to the power delivered from the supply, i.e.,

$$S_{S(3\phi)} = S_{R(3\phi)} + S_{L(3\phi)} = (528 + j396) + (12 + j81) = 540 \text{ kW} + j477 \text{ kvar}$$





## PROBLEMS

**2.1.** Modify the program in Example 2.1 such that the following quantities can be entered by the user:
The peak amplitude $V_m$, and the phase angle $\theta_v$ of the sinusoidal supply $v(t) = V_m \cos(\omega t + \theta_v)$. The impedance magnitude $Z$, and the phase angle $\gamma$ of the load.
The program should produce plots for $i(t)$, $v(t)$, $p(t)$, $p_r(t)$ and $p_x(t)$, similar to Example 2.1. Run the program for $V_m = 100$ V, $\theta_v = 0$ and the following loads:

An inductive load, $Z = 1.25\angle 60^\circ\Omega$
A capacitive load, $Z = 2.0\angle -30^\circ\Omega$
A resistive load, $Z = 2.5\angle 0^\circ\Omega$

(a) From $p_r(t)$ and $p_x(t)$ plots, estimate the real and reactive power for each load. Draw a conclusion regarding the sign of reactive power for inductive and capacitive loads.
(b) Using phasor values of current and voltage, calculate the real and reactive power for each load and compare with the results obtained from the curves.
(c) If the above loads are all connected across the same power supply, determine the total real and reactive power taken from the supply.

**2.2.** A single-phase load is supplied with a sinusoidal voltage

$$v(t) = 200\cos(377t)$$

The resulting instantaneous power is

$$p(t) = 800 + 1000\cos(754t - 36.87^\circ)$$

(a) Find the complex power supplied to the load.
(b) Find the instantaneous current $i(t)$ and the rms value of the current supplied to the load.
(c) Find the load impedance.
(d) Use *MATLAB* to plot $v(t)$, $p(t)$, and $i(t) = p(t)/v(t)$ over a range of 0 to 16.67 ms in steps of 0.1 ms. From the current plot, estimate the peak amplitude, phase angle and the angular frequency of the current, and verify the results obtained in part (b). Note in *MATLAB* the command for array or element-by-element division is ./.

**2.3.** An inductive load consisting of $R$ and $X$ in series feeding from a 2400-V rms supply absorbs 288 kW at a lagging power factor of 0.8. Determine $R$ and $X$.

**2.4.** An inductive load consisting of $R$ and $X$ in parallel feeding from a 2400-V rms supply absorbs 288 kW at a lagging power factor of 0.8. Determine $R$ and $X$.

**2.5.** Two loads connected in parallel are supplied from a single-phase 240-V rms source. The two loads draw a total real power of 400 kW at a power factor of 0.8 lagging. One of the loads draws 120 kW at a power factor of 0.96 leading. Find the complex power of the other load.

**2.6.** The load shown in Figure 2.22 consists of a resistance $R$ in parallel with a capacitor of reactance $X$. The load is fed from a single-phase supply through a line of impedance $8.4 + j11.2\ \Omega$. The rms voltage at the load terminal is $1200\angle 0°$ V rms, and the load is taking 30 kVA at 0.8 power factor leading.
(a) Find the values of $R$ and $X$.
(b) Determine the supply voltage $V$.

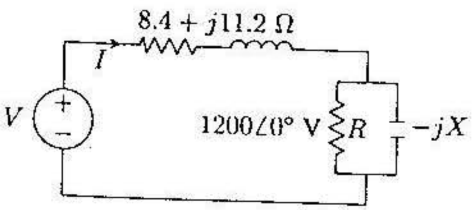



**FIGURE 2.22**
Circuit for Problem 2.6.

**2.7.** Two impedances, $Z_1 = 0.8 + j5.6\ \Omega$ and $Z_2 = 8 - j16\ \Omega$, and a single-phase motor are connected in parallel across a 200-V rms, 60-Hz supply as shown in Figure 2.23. The motor draws 5 kVA at 0.8 power factor lagging.

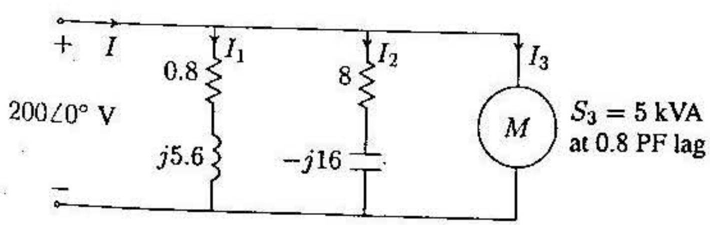



**FIGURE 2.23**
Circuit for Problem 2.7.



(a) Find the complex powers $S_1$, $S_2$ for the two impedances, and $S_3$ for the motor.
(b) Determine the total power taken from the supply, the supply current, and the overall power factor.
(c) A capacitor is connected in parallel with the loads. Find the kvar and the capacitance in $\mu$F to improve the overall power factor to unity. What is the new line current?

**2.8.** Two single-phase ideal voltage sources are connected by a line of impedance of $0.7 + j2.4\ \Omega$ as shown in Figure 2.24. $V_1 = 500\angle 16.26°$ V and $V_2 = 585\angle 0°$ V. Find the complex power for each machine and determine whether they are delivering or receiving real and reactive power. Also, find the real and the reactive power loss in the line.

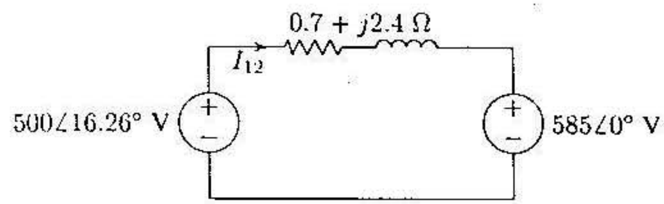



**FIGURE 2.24**
Circuit for Problem 2.8.

**2.9.** Write a *MATLAB* program for the system of Example 2.6 such that the voltage magnitude of source 1 is changed from 75 percent to 100 percent of the given value in steps of 1 V. The voltage magnitude of source 2 and the phase angles of the two sources is to be kept constant. Compute the complex power for each source and the line loss. Tabulate the reactive powers and plot $Q_1$, $Q_2$, and $Q_L$ versus voltage magnitude $|V_1|$. From the results, show that the flow of reactive power along the interconnection is determined by the magnitude difference of the terminal voltages.

**2.10.** A balanced three-phase source with the following instantaneous phase voltages

$$v_{an} = 2500\cos(\omega t)$$
$$v_{bn} = 2500\cos(\omega t - 120°)$$
$$v_{cn} = 2500\cos(\omega t - 240°)$$

supplies a balanced Y-connected load of impedance $Z = 250\angle 36.87°\ \Omega$ per phase.

(a) Using *MATLAB*, plot the instantaneous powers $p_a$, $p_b$, $p_c$ and their sum versus $\omega t$ over a range of 0:0.05: $2\pi$ on the same graph. Comment on the nature of the instantaneous power in each phase and the total three-phase real power.

(b) Use (2.44) to verify the total power obtained in part (a).

**2.11.** A 4157-V rms, three-phase supply is applied to a balanced Y-connected three-phase load consisting of three identical impedances of $48\angle 36.87°\,\Omega$. Taking the phase to neutral voltage $V_{an}$ as reference, calculate

(a) The phasor currents in each line.

(b) The total active and reactive power supplied to the load.

**2.12.** Repeat Problem 2.11 with the same three-phase impedances arranged in a $\Delta$ connection. Take $V_{ab}$ as reference.

**2.13.** A balanced delta connected load of $15 + j18\ \Omega$ per phase is connected at the end of a three-phase line as shown in Figure 2.25. The line impedance is $1 + j2\ \Omega$ per phase. The line is supplied from a three-phase source with a line-to-line voltage of 207.85 V rms. Taking $V_{an}$ as reference, determine the following:

(a) Current in phase $a$.

(b) Total complex power supplied from the source.

(c) Magnitude of the line-to-line voltage at the load terminal.

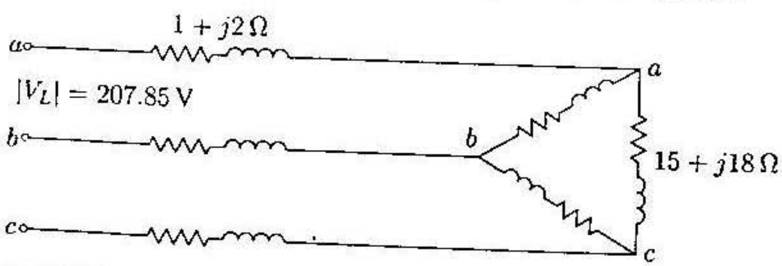



**FIGURE 2.25**
Circuit for Problem 2.13.

**2.14.** Three parallel three-phase loads are supplied from a 207.85-V rms, 60-Hz three-phase supply. The loads are as follows:

Load 1: A 15 hp motor operating at full-load, 93.25 percent efficiency, and 0.6 lagging power factor.



Load 2: A balanced resistive load that draws a total of 6 kW.
Load 3: A Y-connected capacitor bank with a total rating of 16 kvar.

(a) What is the total system kW, kvar, power factor, and the supply current per phase?

(b) What is the system power factor and the supply current per phase when the resistive load and induction motor are operating but the capacitor bank is switched off?

**2.15.** Three loads are connected in parallel across a 12.47 kV three-phase supply.

Load 1: Inductive load, 60 kW and 660 kvar.
Load 2: Capacitive load, 240 kW at 0.8 power factor.
Load 3: Resistive load of 60 kW.

(a) Find the total complex power, power factor, and the supply current.

(b) A Y-connected capacitor bank is connected in parallel with the loads. Find the total kvar and the capacitance per phase in $\mu$F to improve the overall power factor to 0.8 lagging. What is the new line current?

**2.16.** A balanced $\Delta$-connected load consisting of a pure resistances of 18 $\Omega$ per phase is in parallel with a purely resistive balanced Y-connected load of 12 $\Omega$ per phase as shown in Figure 2.26. The combination is connected to a three-phase balanced supply of 346.41-V rms (line-to-line) via a three-phase line having an inductive reactance of $j3\ \Omega$ per phase. Taking the phase voltage $V_{an}$ as reference, determine

(a) The current, real power, and reactive power drawn from the supply.

(b) The line-to-neutral and the line-to-line voltage of phase $a$ at the combined load terminals.

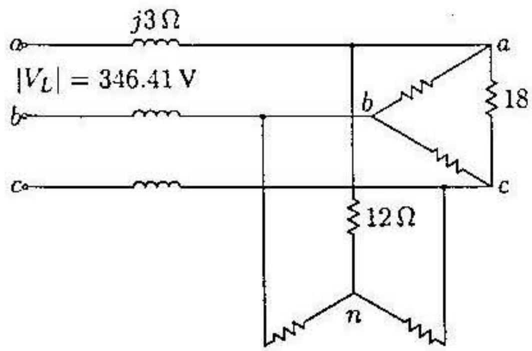



**FIGURE 2.26**
Circuit for Problem 2.16.

# CHAPTER 3

# GENERATOR AND TRANSFORMER MODELS; THE PER-UNIT SYSTEM

## 3.1 INTRODUCTION

Before the power systems network can be solved, it must first be modeled. The three-phase balanced system is represented on a per-phase basis, which was described in Section 2.10. The single-phase representation is also used for unbalanced systems by means of symmetrical components which is treated in a later chapter. In this chapter we deal with the balanced system, where transmission lines are represented by the $\pi$ model as described in Chapter 4. Other essential components of a power system are generators and transformers; their theory and construction are discussed in standard electric machine textbooks. In this chapter, we represent simple models of generators and transformers for steady-state balanced operation.

Next we review the one-line diagram of a power system showing generators, transformers, transmission lines, capacitors, reactors, and loads. The diagram is usually limited to major transmission systems. As a rule, distribution circuits and small loads are not shown in detail but are taken into account merely as lumped loads on substation busses.







In the analysis of power systems, it is frequently convenient to use the per-unit system. The advantage of this method is the elimination of transformers by simple impedances. The per-unit system is presented, followed by the impedance diagram of the network, expressed to a common MVA base.

## 3.2 SYNCHRONOUS GENERATORS

Large-scale power is generated by three-phase synchronous generators, known as *alternators*, driven either by steam turbines, hydroturbines, or gas turbines. The armature windings are placed on the stationary part called *stator*. The armature windings are designed for generation of balanced three-phase voltages and are arranged to develop the same number of magnetic poles as the field winding that is on the rotor. The field which requires a relatively small power (0.2–3 percent of the machine rating) for its excitation is placed on the rotor. The rotor is also equipped with one or more short-circuited windings known as *damper windings*. The rotor is driven by a prime mover at constant speed and its field circuit is excited by direct current. The excitation may be provided through slip rings and brushes by means of dc generators (referred to as *exciters*) mounted on the same shaft as the rotor of the synchronous machine. However, modern excitation systems usually use ac generators with rotating rectifiers, and are known as *brushless excitation*. The generator excitation system maintains generator voltage and controls the reactive power flow.

The rotor of the synchronous machine may be of cylindrical or salient construction. The cylindrical type of rotor, also called *round rotor*, has one distributed winding and a uniform air gap. These generators are driven by steam turbines and are designed for high speed 3600 or 1800 rpm (two- and four-pole machines, respectively) operation. The rotor of these generators has a relatively large axial length and small diameter to limit the centrifugal forces. Roughly 70 percent of large synchronous generators are cylindrical rotor type ranging from about 150 to 1500 MVA. The salient type of rotor has concentrated windings on the poles and nonuniform air gaps. It has a relatively large number of poles, short axial length, and large diameter. The generators in hydroelectric power stations are driven by hydraulic turbines, and they have salient-pole rotor construction.

### 3.2.1 GENERATOR MODEL

An elementary two-pole three-phase generator is illustrated in Figure 3.1. The stator contains three coils, $aa'$, $bb'$, and $cc'$, displaced from each other by 120 electrical degrees. The concentrated full-pitch coils shown here may be considered to represent distributed windings producing sinusoidal mmf waves concentrated on the magnetic axes of the respective phases. When the rotor is excited to produce

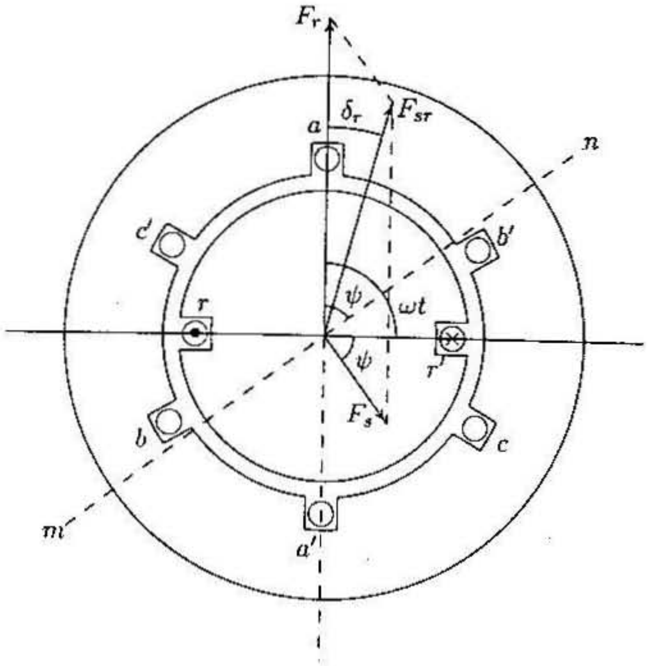



**FIGURE 3.1**
Elementary two-pole three-phase synchronous generator.

an air gap flux of $\phi$ per pole and is revolving at constant angular velocity $\omega$, the flux linkage of the coil varies with the position of the rotor mmf axis $\omega t$, where $\omega t$ is measured in electrical radians from coil $aa'$ magnetic axis. The flux linkage for an N-turn concentrated coil $aa'$ will be maximum ($N\phi$) at $\omega t = 0$ and zero at $\omega t = \pi/2$. Assuming distributed winding, the flux linkage $\lambda_a$ will vary as the cosine of the angle $\omega t$. Thus, the flux linkage with coil $a$ is

$$\lambda_a = N\phi \cos \omega t \tag{3.1}$$

The voltage induced in coil $aa'$ is obtained from Faraday's law as

$$\begin{aligned} e_a &= -\frac{d\lambda}{dt} = \omega N\phi \sin \omega t \\ &= E_{max} \sin \omega t \\ &= E_{max} \cos(\omega t - \frac{\pi}{2}) \end{aligned} \tag{3.2}$$

where

$$E_{max} = \omega N\phi = 2\pi f N\phi$$





Therefore, the rms value of the generated voltage is

$$E = 4.44 f N\phi \tag{3.3}$$

where $f$ is the frequency in hertz. In actual ac machine windings, the armature coil of each phase is distributed in a number of slots. Since the emfs induced in different slots are not in phase, their phasor sum is less than their numerical sum. Thus, a reduction factor $K_w$, called the *winding factor*, must be applied. For most three-phase windings $K_w$ is about 0.85 to 0.95. Therefore, for a distributed phase winding, the rms value of the generated voltage is

$$E = 4.44 K_w f N\phi \tag{3.4}$$

The magnetic field of the rotor revolving at constant speed induces three-phase sinusoidal voltages in the armature, displaced by $2\pi/3$ radians. The frequency of the induced armature voltages depends on the speed at which the rotor runs and on the number of poles for which the machine is wound. The frequency of the armature voltage is given by

$$f = \frac{P}{2}\frac{n}{60} \tag{3.5}$$

where $n$ is the rotor speed in rpm, referred to as *synchronous speed*. During normal conditions, the generator operates synchronously with the power grid. This results in three-phase balanced currents in the armature. Assuming current in phase $a$ is lagging the generated emf $e_a$ by an angle $\psi$, which is indicated by line $mn$ in Figure 3.1, the instantaneous armature currents are

$$\begin{aligned} i_a &= I_{max} \sin(\omega t - \psi) \\ i_b &= I_{max} \sin(\omega t - \psi - \frac{2\pi}{3}) \\ i_c &= I_{max} \sin(\omega t - \psi - \frac{4\pi}{3}) \end{aligned} \tag{3.6}$$

According to (3.2) the generated emf $e_a$ is maximum when the rotor magnetic axis is under phase $a$. Since $i_a$ is lagging $e_a$ by an angle $\psi$, when line $mn$ reaches the axis of coil $aa'$, current in phase $a$ reaches its maximum value. At any instant of time, each phase winding produces a sinusoidally distributed mmf wave with its peak along the axis of the phase winding. These sinusoidally distributed fields can be represented by vectors referred to as *space phasors*. The amplitude of the sinusoidally distributed mmf $f_a(\theta)$ is represented by the vector $F_a$ along the axis of phase $a$. Similarly, the amplitude of the mmfs $f_b(\theta)$ and $f_c(\theta)$ are shown by vectors $F_b$ and $F_c$ along their respective axis. The mmf amplitudes are proportional to the instantaneous value of the phase current, i.e.,

$$F_a = Ki_a = KI_{max}\sin(\omega t - \psi) = F_m \sin(\omega t - \psi)$$
$$F_b = Ki_b = KI_{max}\sin(\omega t - \psi - \frac{2\pi}{3}) = F_m \sin(\omega t - \psi - \frac{2\pi}{3}) \quad (3.7)$$
$$F_c = Ki_c = KI_{max}\sin(\omega t - \psi - \frac{4\pi}{3}) = F_m \sin(\omega t - \psi - \frac{4\pi}{3})$$

where $K$ is proportional to the number of armature turns per phase and is a function of the winding type. The resultant armature mmf is the vector sum of the above mmfs. A suitable method for finding the resultant mmf is to project these mmfs on line $mn$ and obtain the resultant in-phase and quadrature-phase components. The resultant in-phase components are

$$F_1 = F_m \sin(\omega t - \psi)\cos(\omega t - \psi) + F_m \sin(\omega t - \psi - \frac{2\pi}{3})$$
$$\cos(\omega t - \psi - \frac{2\pi}{3}) + F_m \sin(\omega t - \psi - \frac{4\pi}{3})\cos(\omega t - \psi - \frac{4\pi}{3})$$

Using the trigonometric identity $\sin\alpha\cos\alpha = (1/2)\sin 2\alpha$, the above expression becomes

$$F_1 = \frac{F_m}{2}[\sin 2(\omega t - \psi) + \sin 2(\omega t - \psi - \frac{2\pi}{3})$$
$$+ \sin 2(\omega t - \psi - \frac{4\pi}{3})]$$

The above expression is the sum of three sinusoidal functions displaced from each other by $2\pi/3$ radians, which adds up to zero, i.e., $F_1 = 0$.

The sum of quadrature components results in

$$F_2 = F_m \sin(\omega t - \psi)\sin(\omega t - \psi) + F_m \sin(\omega t - \psi - \frac{2\pi}{3})\sin(\omega t - \psi - \frac{2\pi}{3})$$
$$+ F_m \sin(\omega t - \psi - \frac{4\pi}{3})\sin(\omega t - \psi - \frac{4\pi}{3})$$

Using the trigonometric identity $\sin^2\alpha = (1/2)(1 - \cos 2\alpha)$, the above expression becomes

$$F_2 = \frac{F_m}{2}[3 - \cos 2(\omega t - \psi) + \cos 2(\omega t - \psi - \frac{2\pi}{3})$$
$$+ \cos 2(\omega t - \psi - \frac{4\pi}{3})]$$

The sinusoidal terms of the above expression are displaced from each other by $2\pi/3$ radians and add up to zero, with $F_2 = 3/2F_m$. Thus, the amplitude of the resultant armature mmf or stator mmf becomes

$$F_s = \frac{3}{2}F_m \quad (3.8)$$





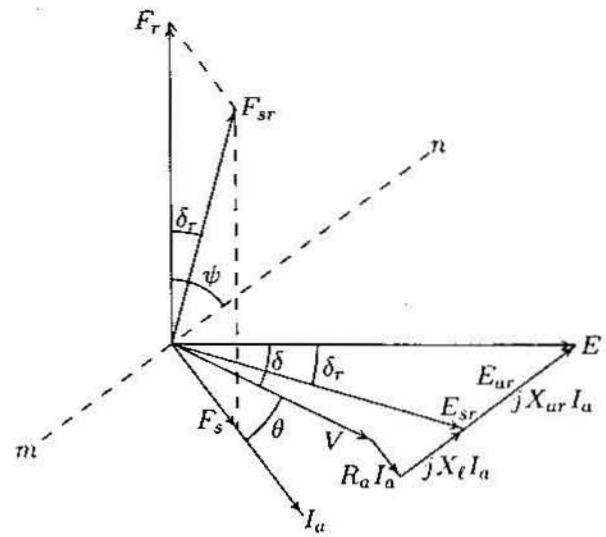



**FIGURE 3.2**
Combined phasor/vector diagram for one phase of a cylindrical rotor generator.

We thus conclude that the resultant armature mmf has a constant amplitude perpendicular to line $mn$, and rotates at a constant speed and in synchronism with the field mmf $F_r$. To see a demonstration of the rotating magnetic field, type **rotfield** at the *MATLAB* prompt.

A typical synchronous machine field alignment for operation as a generator is shown in Figure 3.2, using space vectors to represent the various fields. When the rotor is revolving at synchronous speed and the armature current is zero, the field mmf $F_r$ produces the no-load generated emf $E$ in each phase. The no-load generated voltage which is proportional to the field current is known as the *excitation voltage*. The phasor voltage for phase $a$, which is lagging $F_r$ by 90°, is combined with the mmf vector diagram as shown in Figure 3.2. This combined phasor/vector diagram leads to a circuit model for the synchronous machine. It must be emphasized that in Figure 3.2 mmfs are space vectors, whereas the emfs are time phasors. When the armature is carrying balanced three-phase currents, $F_s$ is produced perpendicular to line $mn$. The interaction of armature mmf and the field mmf, known as *armature reaction*, gives rise to the resultant air gap mmf $F_{sr}$. The resultant mmf $F_{sr}$ is the vector sum of the field mmf $F_r$ and the armature mmf $F_s$. The resultant mmf is responsible for the resultant air gap flux $\phi_{sr}$ that induces the generated emf on-load, shown by $E_{sr}$. The armature mmf $F_s$ induces the emf $E_{ar}$, known as the *armature reaction voltage*, which is perpendicular to $F_s$. The voltage $E_{ar}$ leads

$I_a$ by 90° and thus can be represented by a voltage drop across a reactance $X_{ar}$ due to the current $I_a$. $X_{ar}$ is known as the *reactance of the armature reaction*. The phasor sum of $E$ and $E_{ar}$ is shown by $E_{sr}$ perpendicular to $F_{sr}$, which represents the on-load generated emf.

$$E = E_{sr} + jX_{ar}I_a \tag{3.9}$$

The terminal voltage $V$ is less than $E_{sr}$ by the amount of resistive voltage drop $R_aI_a$ and leakage reactance voltage drop $X_\ell I_a$. Thus

$$E = V + [R_a + j(X_\ell + X_{ar})]I_a \tag{3.10}$$

or

$$E = V + [R_a + jX_s]I_a \tag{3.11}$$

where $X_s = (X_\ell + X_{ar})$ is known as the *synchronous reactance*. The cosine of the angle between $I$ and $V$, i.e., $\cos\theta$ represents the power factor at the generator terminals. The angle between $E$ and $E_{sr}$ is equal to the angle between the rotor mmf $F_r$ and the air gap mmf $F_{sr}$, shown by $\delta_r$. The power developed by the machine is proportional to the product of $F_r$, $F_{sr}$ and $\sin\delta_r$. The relative positions of these mmfs dictates the action of the synchronous machine. When $F_r$ is ahead of $F_{sr}$ by an angle $\delta_r$, the machine is operating as a generator and when $F_r$ falls behind $F_{sr}$, the machine will act as a motor. Since $E$ and $E_{sr}$ are proportional to $F_r$ and $F_{sr}$, respectively, the power developed by the machine is proportional to the products of $E$, $E_{sr}$, and $\sin\delta_r$. The angle $\delta_r$ is thus known as the *power angle*. This is a very important result because it relates the time angle between the phasor emfs with the space angle between the magnetic fields in the machine. Usually the developed power is expressed in terms of the excitation voltage $E$, the terminal voltage $V$, and $\sin\delta$. The angle $\delta$ is approximately equal to $\delta_r$ because the leakage impedance is very small compared to the magnetization reactance.

Due to the nonlinearity of the machine magnetization curve, the synchronous reactance is not constant. The unsaturated synchronous reactance can be found from the open- and short-circuit data. For operation at or near rated terminal voltage, it is usually assumed that the machine is equivalent to an unsaturated one whose magnetization curve is a straight line through the origin and the rated voltage point on the open-circuit characteristic. For steady-state analysis, a constant value known as the *saturated value of the synchronous reactance* corresponding to the rated voltage is used. A simple per-phase model for a cylindrical rotor generator based on (3.11) is obtained as shown in Figure 3.3. The armature resistance is generally much smaller than the synchronous reactance and is often neglected. The equivalent circuit connected to an infinite bus becomes that shown in Figure 3.4, and (3.11) reduces to

$$E = V + jX_sI_a \tag{3.12}$$





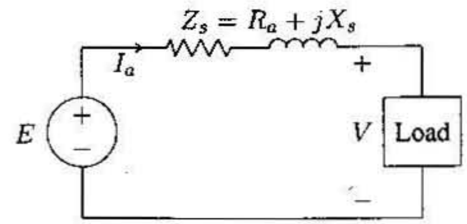



**FIGURE 3.3**
Synchronous machine equivalent circuit.

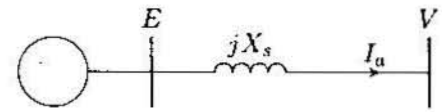
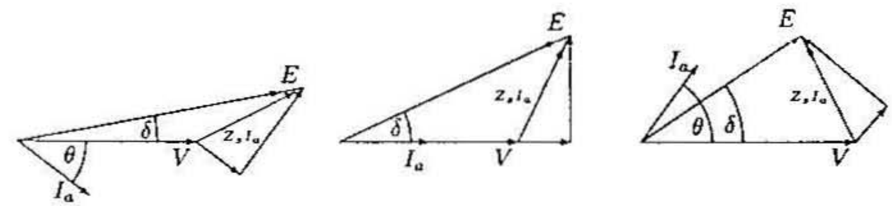



**FIGURE 3.4**
Synchronous machine connected to an infinite bus.

Figure 3.5 shows the phasor diagram of the generator with terminal voltage as reference for excitations corresponding to lagging, unity, and leading power factors. The voltage regulation of an alternator is a figure of merit used for comparison with other machines. It is defined as the percentage change in terminal voltage from no-load to rated load. This gives an indication of the change in field current required to maintain system voltage when going from no-load to rated load at some specific power factor.




(a) Lagging pf load (b) Upf load (c) Leading pf load

**FIGURE 3.5**
Synchronous generator phasor diagram.

$$\text{VR} = \frac{|V_{nl}| - |V_{rated}|}{|V_{rated}|} \times 100 = \frac{|E| - |V_{rated}|}{|V_{rated}|} \times 100 \tag{3.13}$$

The no-load voltage for a specific power factor may be determined by operating the machine at rated load conditions and then removing the load and observing the no-load voltage. Since this is not a practical method for very large machines, an accurate analytical method recommended by IEEE as given in reference [43] may be used. An approximate method that provides reasonable results is to consider a hypothetical linearized magnetization curve drawn to intersect the actual magnetization curve at rated voltage. The value of $E$ calculated from (3.12) is then used to find the field current from the linearized curve. Finally, the no-load voltage corresponding to this field current is found from the actual magnetization curve.

## 3.3 STEADY-STATE CHARACTERISTICS—CYLINDRICAL ROTOR

### 3.3.1 POWER FACTOR CONTROL

Most synchronous machines are connected to large interconnected electric power networks. These networks have the important characteristic that the system voltage at the point of connection is constant in magnitude, phase angle, and frequency. Such a point in a power system is referred to as an *infinite bus*. That is, the voltage at the generator bus will not be altered by changes in the generator's operating condition.

The ability to vary the rotor excitation is an important feature of the synchronous machine, and we now consider the effect of such a variation when the machine operates as a generator with constant mechanical input power. The per-phase equivalent circuit of a synchronous generator connected to an infinite bus is shown in Figure 3.4. Neglecting the armature resistance, the output power is equal to the power developed, which is assumed to remain constant given by

$$P_{3\phi} = \Re[3VI_a^*] = 3|V||I_a|\cos\theta \tag{3.14}$$

where $V$ is the phase-to-neutral terminal voltage assumed to remain constant. From (3.14) we see that for constant developed power at a fixed terminal voltage $V$, $I_a\cos\theta$ must be constant. Thus, the tip of the armature current phasor must fall on a vertical line as the power factor is varied by varying the field current as shown in Figure 3.6. From this diagram we have

$$cd = E_1\sin\delta_1 = X_sI_{a1}\cos\theta_1 \tag{3.15}$$

Thus $E_1\sin\delta_1$ is a constant, and the locus of $E_1$ is on the line $ef$. In Figure 3.6, phasor diagrams are drawn for three armature currents. Application of (3.12) for a lagging power factor armature current $I_{a1}$ results in $E_1$. If $\theta$ is zero, the generator operates at unity power factor and armature current has a minimum value, shown by $I_{a2}$, which results in $E_2$. Similarly, $E_3$ is obtained corresponding to $I_{a3}$ at a leading power factor. Figure 3.6 shows that the generation of reactive power can





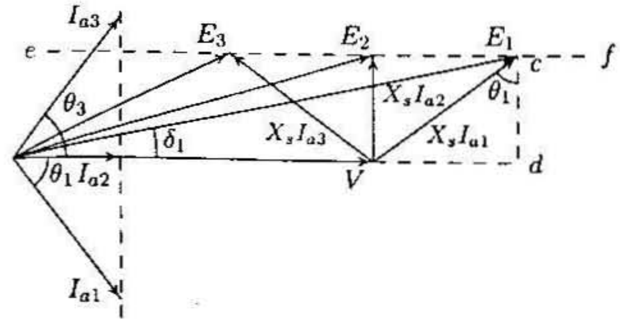



**FIGURE 3.6**
Variation of field current at constant power.

be controlled by means of the rotor excitation while maintaining a constant real power output. The variation in the magnitude of armature current as the excitation voltage is varied is best shown by a curve. Usually the field current is used as the abscissa instead of excitation voltage because the field current is readily measured. The curve of the armature current as the function of the field current resembles the letter V and is often referred to as the *V curve* of synchronous machines. These curves constitute one of the generator's most important characteristics. There is, of course, a limit beyond which the excitation cannot be reduced. This limit is reached when $\delta = 90°$. Any reduction in excitation below the stability limit for a particular load will cause the rotor to pull out of synchronism. The V curve is illustrated in Figure 3.7 (page 62) for the machine in Example 3.3.

### 3.3.2 POWER ANGLE CHARACTERISTICS

Consider the per-phase equivalent circuit shown in Figure 3.4. The three-phase complex power at the generator terminal is

$$S_{3\phi} = 3VI_a^* \tag{3.16}$$

Expressing the phasor voltages in polar form, the armature current is

$$I_a = \frac{|E|\angle\delta \; - |V|\angle 0}{|Z_s|\angle\gamma} \tag{3.17}$$

Substituting for $I_a^*$ in (3.16) results in

$$S_{3\phi} = 3\frac{|E||V|}{|Z_s|}\angle\gamma - \delta \; - 3\frac{|V|^2}{|Z_s|}\angle\gamma \tag{3.18}$$

Thus, the real power $P_{3\phi}$ and reactive power $Q_{3\phi}$ are

$$P_{3\phi} = 3\frac{|E||V|}{|Z_s|}\cos(\gamma - \delta) - 3\frac{|V|^2}{|Z_s|}\cos\gamma \tag{3.19}$$

$$Q_{3\phi} = 3\frac{|E||V|}{|Z_s|}\sin(\gamma - \delta) - 3\frac{|V|^2}{|Z_s|}\sin\gamma \tag{3.20}$$

If $R_a$ is neglected, then $Z_s = jX_s$ and $\gamma = 90°$. Equations (3.19) and (3.20) reduce to

$$P_{3\phi} = 3\frac{|E||V|}{X_s}\sin\delta \tag{3.21}$$

$$Q_{3\phi} = 3\frac{|V|}{X_s}(|E|\cos\delta - |V|) \tag{3.22}$$

Equation (3.21) shows that if $|E|$ and $|V|$ are held fixed and the power angle $\delta$ is changed by varying the mechanical driving torque, the power transfer varies sinusoidally with the angle $\delta$. From (3.21), the theoretical maximum power occurs when $\delta = 90°$

$$P_{max(3\phi)} = 3\frac{|E||V|}{X_s} \tag{3.23}$$

The behavior of the synchronous machine can be described as follows. If we start with $\delta = 0°$ and increase the driving torque, the machine accelerates, and the rotor mmf $F_r$ advances with respect to the resultant mmf $F_{sr}$. This results in an increase in $\delta$, causing the machine to deliver electric power. At some value of $\delta$ the machine reaches equilibrium where the electric power output balances the increased mechanical power owing to the increased driving torque. It is clear that if an attempt were made to advance $\delta$ further than 90° by increasing the driving torque, the electric power output would decrease from the $P_{max}$ point. Therefore, the excess driving torque continues to accelerate the machine, and the mmfs will no longer be magnetically coupled. The machine loses synchronism and automatic equipment disconnects it from the system. The value $P_{max}$ is called the *steady-state stability limit* or *static stability limit*. In general, stability considerations dictate that a synchronous machine achieve steady-state operation for a power angle at considerably less than 90°. The control of real power flow is maintained by the generator governor through the frequency-power control channel.

Equation (3.22) shows that for small $\delta$, $\cos\delta$ is nearly unity and the reactive power can be approximated to

$$Q_{3\phi} \simeq 3\frac{|V|}{X_s}(|E| - |V|) \tag{3.24}$$





From (3.24) we see that when $|E| > |V|$ the generator delivers reactive power to the bus, and the generator is said to be overexcited. If $|E| < |V|$, the reactive power delivered to the bus is negative; that is, the bus is supplying positive reactive power to the generator. Generators are normally operated in the overexcited mode since the generators are the main source of reactive power for inductive load throughout the system. Therefore, we conclude that the flow of reactive power is governed mainly by the difference in the excitation voltage $|E|$ and the bus bar voltage $|V|$. The adjustment in the excitation voltage for the control of reactive power is achieved by the generator excitation system.

**Example 3.1**

A 50-MVA, 30-kV, three-phase, 60-Hz synchronous generator has a synchronous reactance of 9 Ω per phase and a negligible resistance. The generator is delivering rated power at a 0.8 power factor lagging at the rated terminal voltage to an infinite bus.

(a) Determine the excitation voltage per phase $E$ and the power angle $\delta$.

(b) With the excitation held constant at the value found in (a), the driving torque is reduced until the generator is delivering 25 MW. Determine the armature current and the power factor.

(c) If the generator is operating at the excitation voltage of part (a), what is the steady-state maximum power the machine can deliver before losing synchronism? Also, find the armature current corresponding to this maximum power.

(a) The three-phase apparent power is

$$S_{3\phi} = 50\angle\cos^{-1}0.8 = 50\angle 36.87° \text{ MVA}$$
$$= 40 \text{ MW} + j30 \text{ Mvar}$$

The rated voltage per phase is

$$V = \frac{30}{\sqrt{3}} = 17.32\angle 0° \text{ kV}$$

The rated current is

$$I_a = \frac{S^*_{3\phi}}{3V^*} = \frac{(50\angle -36.87)10^3}{3(17.32\angle 0°)} = 962.25\angle -36.87° \text{ A}$$

The excitation voltage per phase from (3.12) is

$$E = 17320.5 + (j9)(962.25\angle -36.87) = 23558\angle 17.1° \text{ V}$$

The excitation voltage per phase (line to neutral) is 23.56 kV and the power angle is 17.1°.

(b) When the generator is delivering 25 MW from (3.21) the power angle is

$$\delta = \sin^{-1}\left[\frac{(25)(9)}{(3)(23.56)(17.32)}\right] = 10.591^\circ$$

The armature current is

$$I_a = \frac{(23,558\angle 10.591^\circ - 17,320\angle 0^\circ)}{j9} = 807.485\angle -53.43^\circ \quad \text{A}$$

The power factor is given by cos(53.43) = 0.596 lagging.

(c) The maximum power occurs at $\delta = 90^\circ$

$$P_{max(3\varphi)} = 3\frac{|E||V|}{X_s} = 3\frac{(23.56)(17.32)}{9} = 136 \quad \text{MW}$$

The armature current is

$$I_a = \frac{(23,558\angle 90^\circ - 17,320\angle 0^\circ)}{j9} = 3248.85\angle 36.32^\circ \quad \text{A}$$

The power factor is given by cos(36.32) = 0.8057 leading.

**Example 3.2**

The generator of Example 3.1 is delivering 40 MW at a terminal voltage of 30 kV. Compute the power angle, armature current, and power factor when the field current is adjusted for the following excitations.
(a) The excitation voltage is decreased to 79.2 percent of the value found in Example 3.1.
(b) The excitation voltage is decreased to 59.27 percent of the value found in Example 3.1.
(c) Find the minimum excitation below which the generator will lose synchronism.

(a) The new excitation voltage is

$$E = 0.792 \times 23,558 = 18,657 \quad \text{V}$$

From (3.21) the power angle is

$$\delta = \sin^{-1}\left[\frac{(40)(9)}{(3)(18.657)(17.32)}\right] = 21.8^\circ$$





The armature current is

$$I_a = \frac{(18657\angle 21.8^\circ - 17320\angle 0^\circ)}{j9} = 769.8\angle 0^\circ \quad \text{A}$$

The power factor is given by cos(0) = 1.

(b) The new excitation voltage is

$$E = 0.5927 \times 23,558 = 13,963 \quad \text{V}$$

From (3.21) the power angle is

$$\delta = \sin^{-1}\left[\frac{(40)(9)}{(3)(13.963)(17.32)}\right] = 29.748^\circ$$

The armature current is

$$I_a = \frac{(13,963\angle 29.748^\circ - 17,320\angle 0^\circ)}{j9} = 962.3\angle 36.87^\circ \quad \text{A}$$

From current phase angle, the power factor is cos 36.87 = 0.8 leading. The generator is underexcited and is actually receiving reactive power.

(c) From (3.23), the minimum excitation corresponding to $\delta = 90^\circ$ is

$$E = \frac{(40)(9)}{(3)(17.32)(1)} = 6.928 \quad \text{kV}$$

The armature current is

$$I_a = \frac{(6,928\angle 90^\circ - 17,320\angle 0^\circ)}{j9} = 2073\angle 68.2^\circ \quad \text{A}$$

The current phase angle shows that the power factor is cos 68.2 = 0.37 leading. The generator is underexcited and is receiving reactive power.

**Example 3.3**

For the generator of Example 3.1, construct the V curve for the rated power of 40 MW with varying field excitation from 0.4 power factor leading to 0.4 power factor lagging. Assume the open-circuit characteristic in the operating region is given by $E = 2000I_f$ V.

The following *MATLAB* command results in the V curve shown in Figure 3.7.

```
P = 40;                                        % real power, MW
V = 30/sqrt(3)+ j*0;                           % phase voltage, kV
Zs = j*9;                                      % synchronous impedance
ang = acos(0.4);
theta=ang:-0.01:-ang;%Angle 0.4 leading to 0.4 lagging pf
P = P*ones(1,length(theta));%generates array of same size
Iam = P./(3*abs(V)*cos(theta));     % current magnitude kA
Ia = Iam.*(cos(theta) + j*sin(theta));   % current phasor
E = V + Zs.*Ia;                      % excitation voltage phasor
Em = abs(E);                  % excitation voltage magnitude, kV
If = Em*1000/2000;                             % field current, A
plot(If, Iam), grid, xlabel('If - A')
ylabel('Ia - kA'), text(3.4, 1, 'Leading pf')
text(13, 1, 'Lagging pf'), text(9, .71, 'Upf')
```

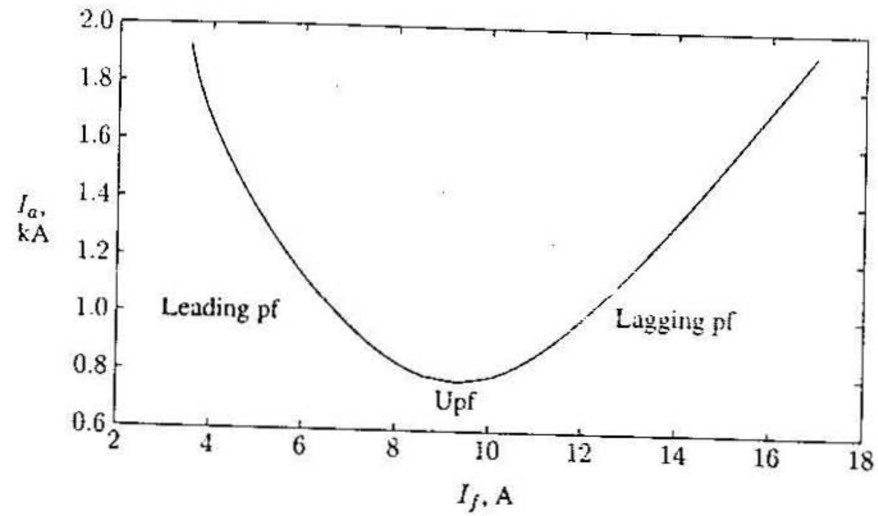



**FIGURE 3.7**
V curve for generator of Example 3.3.

## 3.4 SALIENT-POLE SYNCHRONOUS GENERATORS

The model developed in Section 3.2 is only valid for cylindrical rotor generators with uniform air gaps. The salient-pole rotor results in nonuniformity of the magnetic reluctance of the air gap. The reluctance along the polar axis, commonly referred to as the rotor *direct axis*, is appreciably less than that along the interpolar



axis, commonly referred to as the *quadrature axis*. Therefore, the reactance has a high value $X_d$ along the direct axis, and a low value $X_q$ along the quadrature axis. These reactances produce voltage drop in the armature and can be taken into account by resolving the armature current $I_a$ into two components $I_q$, in phase, and $I_d$ in time quadrature, with the excitation voltage. The phasor diagram with the armature resistance neglected is shown in Figure 3.8. It is no longer possible to rep-

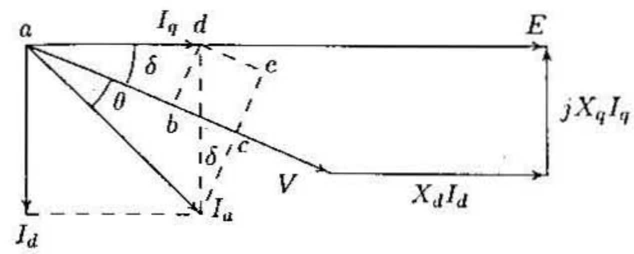



**FIGURE 3.8**
Phasor diagram for a salient-pole generator.

resent the machine by a simple equivalent circuit. The excitation voltage magnitude is

$$|E| = |V| \cos\delta + X_d I_d \tag{3.25}$$

The three-phase real power at the generator terminal is

$$P = 3|V||I_a| \cos\theta \tag{3.26}$$

The power component of the armature current can be expressed in terms of $I_d$ and $I_q$ as follows.

$$\begin{aligned} |I_a| \cos\theta &= ab + de \\ &= I_q \cos\delta + I_d \sin\delta \end{aligned} \tag{3.27}$$

Substituting from (3.27) into (3.26), we have

$$P = 3|V|(I_q \cos\delta + I_d \sin\delta) \tag{3.28}$$

Now from the phasor diagram given in Figure 3.8,

$$|V| \sin\delta = X_q I_q \tag{3.29}$$

or

$$I_q = \frac{|V| \sin\delta}{X_q} \tag{3.30}$$

Also, from (3.25), $I_d$ is given by

$$I_d = \frac{|E| - |V|\cos\delta}{X_d} \tag{3.31}$$

Substituting for $I_d$ and $I_q$ from (3.31) and (3.30) into (3.28), the real power with armature current neglected becomes

$$P_{3\phi} = 3\frac{|E||V|}{X_d}\sin\delta + 3|V|^2\frac{X_d - X_q}{2X_dX_q}\sin 2\delta \tag{3.32}$$

The power equation contains an additional term known as the *reluctance power*. Equations (3.25) and (3.32) can be utilized for steady-state analysis. For short-circuit analysis, assuming a high $X/R$ ratio, the power factor approaches zero and the quadrature component of current can often be neglected. In such a case, $X_d$ merely replaces the $X_s$ used for the cylindrical rotor machine. Generators are thus modeled by their direct axis reactance in series with a constant-voltage power source. Later in the text it will be shown that $X_d$ takes on different values, depending upon the transient time following the short circuit. These reactances are usually expressed in per-unit and are available from the manufacturer's data.

## 3.5 POWER TRANSFORMER

Transformers are essential elements in any power system. They allow the relatively low voltages from generators to be raised to a very high level for efficient power transmission. At the user end of the system, transformers reduce the voltage to values most suitable for utilization. In modern utility systems, the energy may undergo four or five transformations between generator and ultimate user. As a result, a given system is likely to have about five times more kVA of installed capacity of transformers than of generators.

## 3.6 EQUIVALENT CIRCUIT OF A TRANSFORMER

The equivalent circuit model of a single-phase transformer is shown in Figure 3.9. The equivalent circuit consists of an ideal transformer of ratio $N_1{:}N_2$ together with elements which represent the imperfections of the real transformer. An ideal transformer would have windings with zero resistance and a lossless, infinite permeability core. The voltage $E_1$ across the primary of the ideal transformer represents the rms voltage induced in the primary winding by the mutual flux $\phi$. This is the portion of the core flux which links both primary and secondary coils. Assuming





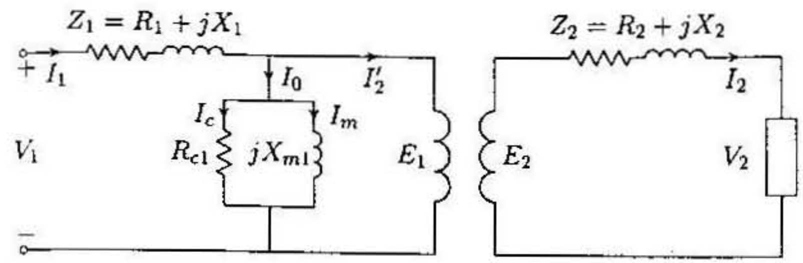



**FIGURE 3.9**
Equivalent circuit of a transformer.

sinusoidal flux $\phi = \Phi_{max}\cos\omega t$, the instantaneous voltage $e_1$ is

$$\begin{aligned} e_1 &= N_1\frac{d\phi}{dt} \\ &= -\omega N_1\Phi_{max}\sin\omega t \\ &= E_{1max}\cos(\omega t + 90^\circ) \end{aligned} \tag{3.33}$$

where

$$E_{1max} = 2\pi f N_1\Phi_{max} \tag{3.34}$$

or the rms voltage magnitude $E_1$ is

$$E_1 = 4.44 f N_1\Phi_{max} \tag{3.35}$$

It is important to note that the phasor flux is lagging the induced voltage $E_1$ by 90°. Similarly the rms voltage $E_2$ across the secondary of the ideal transformer represents the voltage induced in the secondary winding by the mutual flux $\phi$, given by

$$E_2 = 4.44 f N_2\Phi_{max} \tag{3.36}$$

In the ideal transformer, the core is assumed to have a zero reluctance and there is an exact mmf balanced between the primary and secondary. If $I_2'$ represents the component of current to neutralize the secondary mmf, then

$$I_2'N_1 = I_2N_2 \tag{3.37}$$

Therefore, for an ideal transformer, from (3.35) through (3.37) we have

$$\frac{E_1}{E_2} = \frac{I_2}{I_2'} = \frac{N_1}{N_2} \tag{3.38}$$

In a real transformer, the reluctance of the core is finite, and when the secondary current $I_2$ is zero, the primary current has a finite value. Since at no-load, induced voltage $E_1$ is almost equal to the supply voltage $V_1$, the induced voltage and the flux are sinusoidal. However, because of the nonlinear characteristics of the ferromagnetic core, the no-load current is not sinusoidal and contains odd harmonics. The third harmonic is particularly troublesome in certain three-phase connections of transformers. For the purpose of modeling, we assume a sinusoidal no-load current with the rms value of $I_0$, known as the *no-load current*. This current has a component $I_m$, in phase with flux, known as the *magnetizing current*, to set up the core flux. Since flux is lagging the induced voltage $E_1$ by 90°, $I_m$ is also lagging the induced voltage $E_1$ by 90°. Thus, this component can be represented in the circuit by the magnetizing reactance $jX_{m1}$. The other component of $I_0$ is $I_c$, which supplies the eddy-current and hysteresis losses in the core. Since this is a power component, it is in phase with $E_1$ and is represented by the resistance $R_{c1}$ as shown in Figure 3.9.

In a real transformer with finite reluctance, all of the flux is not common to both primary and secondary windings. The flux has three components: mutual flux, primary leakage flux, and secondary leakage flux. The leakage flux associated with one winding does not link the other, and the voltage drops caused by the leakage flux are expressed in terms of leakage reactances $X_1$ and $X_2$. Finally, $R_1$ and $R_2$ are included to represent the primary and secondary winding resistances.

To obtain the performance characteristics of a transformer, it is convenient to use an equivalent circuit model referred to one side of the transformer. From Kirchhoff's voltage law (KVL), the voltage equation of the secondary side is

$$E_2 = V_2 + Z_2 I_2 \tag{3.39}$$

From the relationship (3.38) developed for the ideal transformer, the secondary induced voltage and current are $E_2 = (N_2/N_1)E_1$ and $I_2 = (N_1/N_2)I_2'$, respectively. Upon substitution, (3.39) reduces to

$$\begin{aligned} E_1 &= \frac{N_1}{N_2}V_2 + \left(\frac{N_1}{N_2}\right)^2 Z_2 I_2' \\ &= V_2' + Z_2' I_2' \end{aligned} \tag{3.40}$$

where

$$Z_2' = R_2' + jX_2' = \left(\frac{N_1}{N_2}\right)^2 R_2 + j\left(\frac{N_1}{N_2}\right)^2 X_2$$

Relation (3.40) is the KVL equation of the secondary side referred to the primary, and the equivalent circuit of Figure 3.9 can be redrawn as shown in Figure 3.10, so the same effects are produced in the primary as would be produced in the secondary.





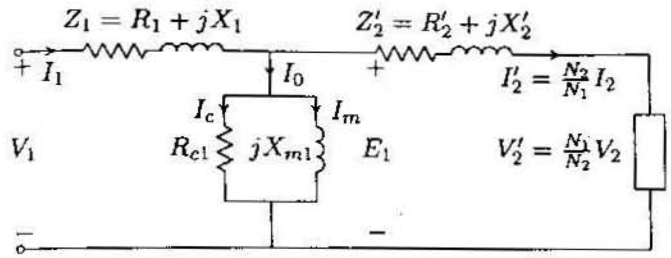



**FIGURE 3.10**
Exact equivalent circuit referred to the primary side.

On no-load, the primary voltage drop is very small, and $V_1$ can be used in place of $E_1$ for computing the no-load current $I_0$. Thus, the shunt branch can be moved to the left of the primary series impedance with very little loss of accuracy. In this manner, the primary quantities $R_1$ and $X_1$ can be combined with the referred secondary quantities $R_2'$ and $X_2'$ to obtain the equivalent primary quantities $R_{e1}$ and $X_{e1}$. The equivalent circuit is shown in Figure 3.11 where we have dispensed with the coils of the ideal transformer. From Figure 3.11

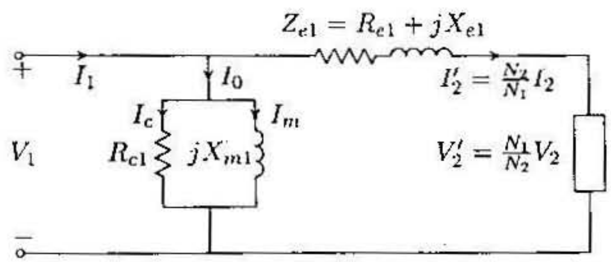



**FIGURE 3.11**
Approximate equivalent circuit referred to the primary.

$$V_1 = V_2' + (R_{e1} + jX_{e1})I_2' \tag{3.41}$$

where

$$R_{e1} = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2 \quad X_{e1} = X_1 + \left(\frac{N_1}{N_2}\right)^2 X_2 \quad \text{and} \quad I_2' = \frac{S_L^*}{3V_2'^*}$$

The equivalent circuit referred to the secondary is also shown in Figure 3.12. From Figure 3.12 the referred primary voltage $V_1'$ is given by

$$V_1' = V_2 + (R_{e2} + jX_{e2})I_2 \tag{3.42}$$

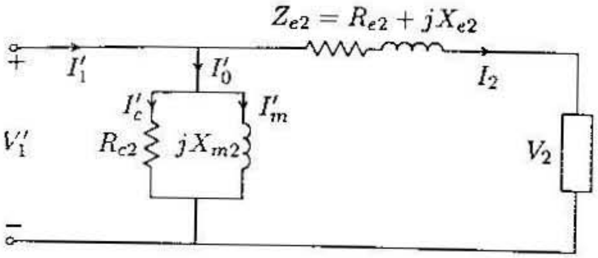



**FIGURE 3.12**
Approximate equivalent circuit referred to the secondary.

Power transformers are generally designed with very high permeability core and very small core loss. Consequently, a further approximation of the equivalent circuit can be made by omitting the shunt branch, as shown in Figure 3.13. The equivalent circuit referred to the secondary is also shown in Figure 3.13.

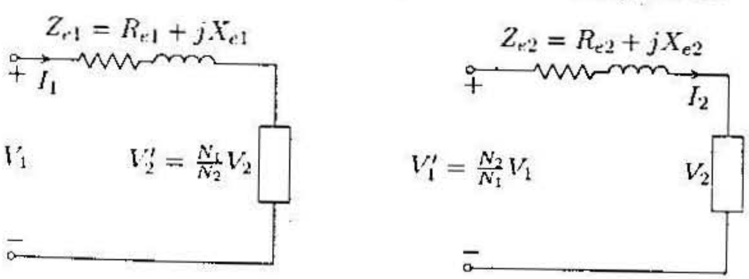



**FIGURE 3.13**
Simplified circuits referred to one side.

## 3.7 DETERMINATION OF EQUIVALENT CIRCUIT PARAMETERS

The parameters of the approximate equivalent circuit are readily obtained from open-circuit and short-circuit tests. In the open-circuit test, rated voltage is applied at the terminals of one winding while the other winding terminals are open-circuited. Instruments are connected to measure the input voltage $V_1$, the no-load input current $I_0$, and the input power $P_0$. If the secondary is open-circuited, the referred secondary current $I_2'$ will be zero, and only a small no-load current will be drawn from the supply. Also, the primary voltage drop $(R_1 + jX_1)I_0$ can be neglected, and the equivalent circuit reduces to the form shown in Figure 3.14.

Since the secondary winding copper loss (resistive power loss) is zero and the





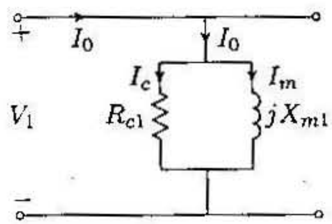



**FIGURE 3.14**
Equivalent circuit for the open-circuit test.

primary copper loss $R_1 I_0{}^2$ is negligible, the no-load input power $P_0$ represents the transformer core loss commonly referred to as *iron loss*. The shunt elements $R_c$ and $X_m$ may then be determined from the relations

$$R_{c1} = \frac{V_1{}^2}{P_0} \tag{3.43}$$

The two components of the no-load current are

$$I_c = \frac{V_1}{R_{c1}} \tag{3.44}$$

and

$$I_m = \sqrt{I_0{}^2 - I_c{}^2} \tag{3.45}$$

Therefore, the magnetizing reactance is

$$X_{m1} = \frac{V_1}{I_m} \tag{3.46}$$

In the short-circuit test, a reduced voltage $V_{sc}$ is applied at the terminals of one winding while the other winding terminals are short-circuited. Instruments are connected to measure the input voltage $V_{sc}$, the input current $I_{sc}$, and the input power $P_{sc}$. The applied voltage is adjusted until rated currents are flowing in the windings. The primary voltage required to produce rated current is only a few percent of the rated voltage. At the correspondingly low value of core flux, the exciting current and core losses are entirely negligible, and the shunt branch can be omitted. Thus, the power input can be taken to represent the winding copper loss. The transformer appears as a short when viewed from the primary with the equivalent leakage impedance $Z_{e1}$ consisting of the primary leakage impedance and the referred secondary leakage impedance as shown in Figure 3.15. The series elements

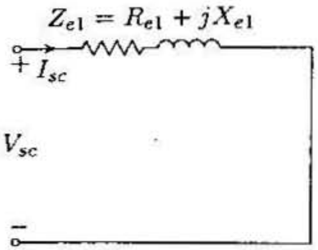



**FIGURE 3.15**
Equivalent circuit for the short-circuit test.

$R_{e1}$ and $X_{e1}$ may then be determined from the relations

$$Z_{e1} = \frac{V_{sc}}{I_{sc}}$$

and

$$R_{e1} = \frac{P_{sc}}{(I_{sc})^2} \tag{3.47}$$

Therefore, the equivalent leakage reactance is

$$X_{e1} = \sqrt{{Z_{e1}}^2 - {R_{e1}}^2} \tag{3.48}$$

## 3.8 TRANSFORMER PERFORMANCE

The equivalent circuit can now be used to predict the performance characteristics of the transformer. An important aspect is the transformer efficiency. Power transformer efficiencies very from 95 percent to 99 percent, the higher efficiencies being obtained from transformers with the greater ratings. The actual efficiency of a transformer in percent is given by

$$\eta = \frac{\text{output power}}{\text{input power}} \tag{3.49}$$

and the conventional efficiency of a transformer at $n$ fraction of the full-load power is given by

$$\eta = \frac{n \times S \times PF}{(n \times S \times PF) + n^2 \times P_{cu} + P_c} \tag{3.50}$$





where $S$ is the full-load rated volt-ampere, $P_{cu}$ is the full-load copper loss, and for a three-phase transformer, they are given by

$$\begin{aligned} S &= 3|V_2||I_2| \\ P_{cu} &= 3R_{e2}|I_2|^2 \end{aligned}$$

and $P_c$ is the iron loss at rated voltage. For varying $I_2$ at constant power factor, maximum efficiency occurs when

$$\frac{d\eta}{d|I_2|} = 0$$

For the above condition, it can be easily shown that maximum efficiency occurs when copper loss equals core loss at $n$ per-unit loading given by

$$n = \sqrt{\frac{P_c}{P_{cu}}} \tag{3.51}$$

Another important performance characteristic of a transformer is change in the secondary voltage from no-load to full-load. A figure of merit used to compare the relative performance of different transformers is the voltage regulation. Voltage regulation is defined as the change in the magnitude of the secondary terminal voltage from no-load to full-load expressed as a percentage of the full-load value.

$$\text{Regulation} = \frac{|V_{2nl}| - |V_2|}{|V_2|} \times 100 \tag{3.52}$$

where $V_2$ is the full-load rated voltage. $V_{2nl}$ in (3.52) can be calculated by using equivalent circuits referred to either primary or secondary. When the equivalent circuit is referred to the primary side, the primary no-load voltage is found from (3.41), and the voltage regulation becomes

$$\text{Regulation} = \frac{|V_1| - |V_2'|}{|V_2'|} \times 100 \tag{3.53}$$

When the equivalent circuit is referred to the secondary side, the secondary no-load voltage is found from (3.42), and the voltage regulation becomes

$$\text{Regulation} = \frac{|V_1'| - |V_2|}{|V_2|} \times 100 \tag{3.54}$$

An interesting feature arises with a capacitive load. Because partial resonance is set up between the capacitance and the reactance, the secondary voltage may actually tend to rise as the capacitive load value increases.

A program called **trans** is developed for obtaining the transformer performance characteristics. The command **trans** displays a menu with three options:

Option 1 calls upon the function **[Rc, Xm] = troct(Vo, Io, Po)** which prompts the user to enter the no-load test data and returns the shunt branch parameters. Then **Ze = trsct(Vsc, Isc, Psc)** is loaded which prompts the user to enter the short-circuit test data and returns the equivalent leakage impedance.

Option 2 calls upon the function **[Zelv, Zehv] = wz2eqz(Elv, Ehv, Zlv, Zhv)** which prompts the user to enter the individual winding impedances and the shunt branch. This function returns the referred equivalent circuit for both sides.

Option 3 prompts the user to enter the parameters of the equivalent circuit.

The above functions can be used independently when the arguments of the functions are defined in the *MATLAB* environment. If the above functions are typed without the parenthesis and the arguments, the user will be prompted to enter the required data.

After the selection of any of the above options, the program prompts the user to enter the load specifications and proceeds to obtain the transformer performance characteristics including an efficiency curve from 25 to 125 percent of full-load.

### Example 3.4

Data obtained from short-circuit and open-circuit tests of a 240-kVA, 4800/240-V, 60-Hz transformer are:

| Open-circuit test, low-side data | Short-circuit test, high-side data |
|---|---|
| $V_1 = 240$ V | $V_{sc} = 187.5$ V |
| $I_0 = 10$ A | $I_{sc} = 50$ A |
| $P_0 = 1440$ W | $P_{sc} = 2625$ W |

Determine the parameters of the equivalent circuit

The commands

```
trans
```

display the following menu





```
Type of parameters for input                          Select
To obtain equivalent circuit from tests                  1
To input individual winding impedances                   2
To input transformer equivalent impedance                3
To quit                                                  0

Select number of menu → 1
Enter Transformer rated power in kVA, S = 240
Enter rated low voltage in volts = 240
Enter rated high voltage in volts = 4800

Open circuit test data
Enter 'lv' within quotes for data ref. to low side or
enter 'hv' within quotes for data ref. to high side → 'lv'
Enter input voltage, in volts, Vo = 240
Enter no-load current in Amp, Io = 10
Enter no-load input power in Watt, Po = 1440

Short circuit test data
Enter 'lv' within quotes for data ref. to low side or
enter 'hv' within quotes for data ref. to high side → 'hv'
Enter reduced input voltage in volts, Vsc = 187.5
Enter input current in Amp, Isc = 50
Enter input power in Watt, Psc = 2625

Shunt branch ref. to LV side       Shunt branch ref. to HV side
Rc = 40.000 ohm                    Rc = 16000.000 ohm
Xm = 30.000 ohm                    Xm = 12000.000 ohm

Series branch ref. to LV side      Series branch ref. to HV side
Ze = 0.002625 + j 0.0090 ohm       Ze = 1.0500 + j 3.6000 ohm

Hit return to continue
```

At this point the user is prompted to enter the load apparent power, power factor, and voltage. The program then obtains the performance characteristics of the transformer including the efficiency curve from 25 to 125 percent of full load as shown in Figure 3.16.

```
Enter load kVA, S2 = 240
Enter load power factor, pf = 0.8
Enter 'lg' within quotes for lagging pf
or 'ld' within quotes for leading pf -> 'lg'
Enter load terminal voltage in volt, V2 = 240
```

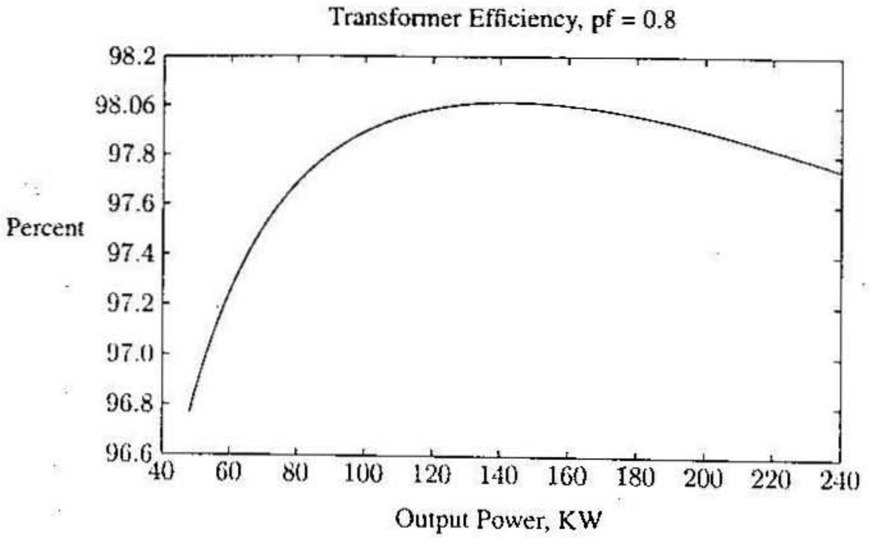



**FIGURE 3.16**
Efficiency curve of Example 3.4.

```
Secondary load voltage    =    240.000 V
Secondary load current    =   1000.000 A   at   -36.87 degrees
Current ref. to primary   =     50.000 A   at   -36.87 degrees
Primary no-load current   =      0.516 A   at   -53.13 degrees
Primary input current     =     50.495 A   at   -37.03 degrees
Primary input voltage     =   4951.278 V   at     1.30 degrees
Voltage regulation        =      3.152 %
Transformer efficiency    =     97.927 %

Maximum efficiency is 98.015 percent, occurs at 177.757 kVA
with 0.80 pf.
```

At the end of this analysis the program menu is displayed.

## 3.9 THREE-PHASE TRANSFORMER CONNECTIONS

Three-phase power is transformed by use of three-phase units. However, in large extra high voltage (EHV) units, the insulation clearances and shipping limitations may require a bank of three single-phase transformers connected in three-phase arrangements.





The primary and secondary windings can be connected in either wye (Y) or delta ($\Delta$) configurations. This results in four possible combinations of connections: Y–Y, $\Delta$–$\Delta$, Y–$\Delta$ and $\Delta$–Y shown by the simple schematic in Figure 3.17. In this diagram, transformer windings are indicated by heavy lines. The windings shown in parallel are located on the same core and their voltages are in phase. The Y–Y

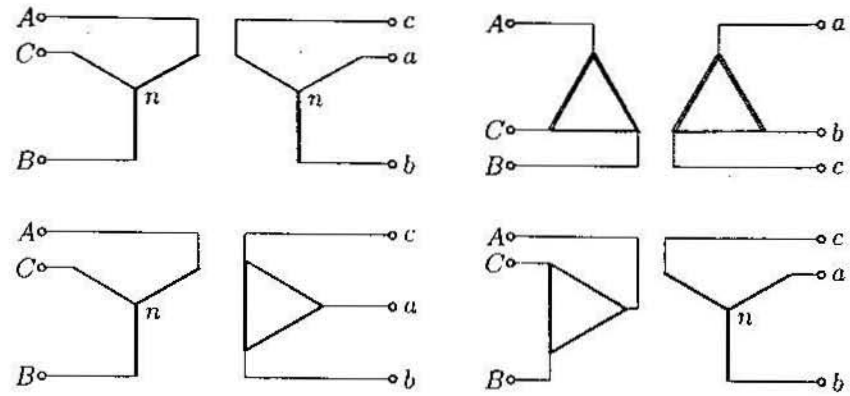



**FIGURE 3.17**
Three-phase transformer connections.

connection offers advantages of decreased insulation costs and the availability of the neutral for grounding purposes. However, because of problems associated with third harmonics and unbalanced operation, this connection is rarely used. To eliminate the harmonics, a third set of windings, called a *tertiary* winding, connected in $\Delta$ is normally fitted on the core to provide a path for the third harmonic currents. This is known as the *three-winding* transformer. The tertiary winding can be loaded with switched reactors or capacitors for reactive power compensation. The $\Delta$–$\Delta$ provides no neutral connection and each transformer must withstand full line-to-line voltage. The $\Delta$ connection does, however, provide a path for third harmonic currents to flow. This connection has the advantage that one transformer can be removed for repair and the remaining two can continue to deliver three-phase power at a reduced rating of 58 percent of the original bank. This is known as the V connection. The most common connection is the Y–$\Delta$ or $\Delta$–Y. This connection is more stable with respect to unbalanced loads, and if the Y connection is used on the high voltage side, insulation costs are reduced. The Y–$\Delta$ connection is commonly used to step down a high voltage to a lower voltage. The neutral point on the high voltage side can be grounded. This is desirable in most cases. The $\Delta$–Y connection is commonly used for stepping up to a high voltage.

### 3.9.1 THE PER-PHASE MODEL OF A THREE-PHASE TRANSFORMER

In Y–Y and Δ–Δ connections, the ratio of the line voltages on HV and LV sides are the same as the ratio of the phase voltages on the HV and LV sides. Furthermore, there is no phase shift between the corresponding line voltages on the HV and LV sides. However, the Y–Δ and the Δ–Y connections will result in a phase shift of 30° between the primary and secondary line-to-line voltages. The windings are arranged in accordance to the ASA (American Standards Association) such that the line voltage on the HV side leads the corresponding line voltage on the LV side by 30° regardless of which side is Y or Δ. Consider the Y–Δ schematic diagram shown in Figure 3.17. The positive phase sequence voltage phasor diagram for this connection is shown in Figure 3.18, where $V_{An}$ is taken as reference. Let the Y

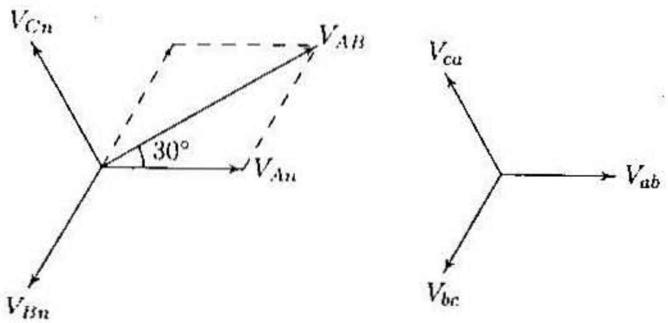



**FIGURE 3.18**
30° phase shift in line-to-line voltages of Y–Δ connection.

connection be the high voltage side shown by letter $H$ and the Δ connection the low voltage side shown by $X$. We consider phase $a$ only and use subscript $L$ for line and $P$ for phase quantities. If $N_H$ is the number of turns on one phase of the high voltage winding and $N_X$ is the number of turns on one phase of the low voltage winding, the transformer turns ratio is $a = N_H/N_X = V_{HP}/V_{XP}$. The relationship between the line voltage and phase voltage magnitudes is

$$V_{HL} = \sqrt{3}\, V_{HP}$$
$$V_{XL} = V_{XP}$$

Therefore, the ratio of the line voltage magnitudes for Y–Δ transformer is

$$\frac{V_{HL}}{V_{XL}} = \sqrt{3}\, a \qquad (3.55)$$

Because the core losses and magnetization current for power transformers are on the order of 1 percent of the maximum ratings, the shunt impedance is neglected





and only the winding resistance and leakage reactance are used to model the transformer. In dealing with Y–Δ or Δ–Y banks, it is convenient to replace the Δ connection by an equivalent Y connection and then work with only one phase. Since for balanced operations, the Y neutral and the neutral of the equivalent Y of the Δ connection are at the same potential, they can be connected together and represented by a neutral conductor. When the equivalent series impedance of one transformer is referred to the delta side, the Δ connected impedances of the transformer are replaced by equivalent Y-connected impedances, given by $Z_Y = Z_\Delta/3$. The per phase equivalent model with the shunt branch neglected is shown in Figure 3.19. $Z_{e1}$ and $Z_{e2}$ are the equivalent impedances based on the line-to-neutral connections, and the voltages are the line-to-neutral values.

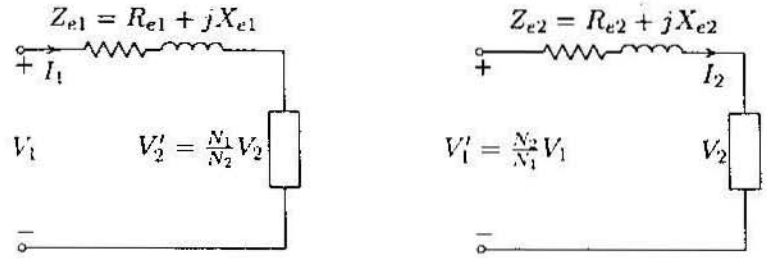



**FIGURE 3.19**
The per phase equivalent circuit.

## 3.10 AUTOTRANSFORMERS

Transformers can be constructed so that the primary and secondary coils are electrically connected. This type of transformer is called an autotransformer. A conventional two-winding transformer can be changed into an autotransformer by connecting the primary and secondary windings in series. Consider the two-winding transformer shown in Figure 3.20(a). The two-winding transformer is converted to an autotransformer arrangement as shown in Figure 3.20(b) by connecting the two windings electrically in series so that the polarities are additive. The winding from $X_1$ to $X_2$ is called the series winding, and the winding from $H_1$ to $H_2$ is called the common winding. From an inspection of this figure it follows that an autotransformer can operate as a step-up as well as a step-down transformer. In both cases, winding part $H_1H_2$ is common to the primary as well as the secondary side of the transformer. The performance of an autotransformer is governed by the fundamental considerations already discussed for transformers having two separate windings. For determining the power rating as an autotransformer, the ideal transformer relations are ordinarily used, which provides an adequate approximation to the actual transformer values.

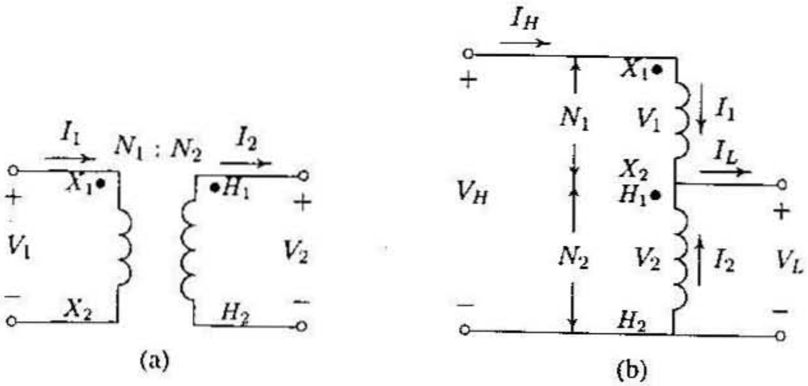



**FIGURE 3.20**
(a) Two-winding transformer, (b) reconnected as an autotransformer.

From Figure 3.20(a), the two-winding voltages and currents are related by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = a \tag{3.56}$$

and

$$\frac{I_2}{I_1} = \frac{N_1}{N_2} = a \tag{3.57}$$

where $a$ is the turns ratio of the two-winding transformer. From Figure 3.20(b), we have

$$V_H = V_2 + V_1 \tag{3.58}$$

Substituting for $V_1$ from (3.56) into (3.58) yields

$$V_H = V_2 + \frac{N_1}{N_2} V_2 \tag{3.59}$$

Since $V_2 = V_L$, the voltage relationship between the two sides of an autotransformer becomes

$$\begin{aligned} V_H &= V_L + \frac{N_1}{N_2} V_L \\ &= (1 + a) V_L \end{aligned} \tag{3.60}$$

or

$$\frac{V_H}{V_L} = 1 + a \tag{3.61}$$





Since the transformer is ideal, the mmf due to $I_1$ must be equal and opposite to the mmf produced by $I_2$. As a result, we have

$$N_2 I_2 = N_1 I_1 \tag{3.62}$$

From Kirchhoff's law, $I_2 = I_L - I_1$, and the above equation becomes

$$N_2(I_L - I_1) = N_1 I_1 \tag{3.63}$$

or

$$I_L = \frac{N_1 + N_2}{N_2} I_1 \tag{3.64}$$

Since $I_1 = I_H$, the current relationship between the two sides of an autotransformer becomes

$$\frac{I_L}{I_H} = 1 + a \tag{3.65}$$

The ratio of the apparent power rating of an autotransformer to a two-winding transformer, known as the *power rating advantage*, is found from

$$\frac{S_{auto}}{S_{2-w}} = \frac{(V_1 + V_2) I_1}{V_1 I_1} = 1 + \frac{N_2}{N_1} = 1 + \frac{1}{a} \tag{3.66}$$

From (3.66), we can see that a higher rating is obtained as an autotransformer with a higher number of turns of the common winding ($N_2$). The higher rating as an autotransformer is a consequence of the fact that only $S_{2-w}$ is transformed by the electromagnetic induction. The rest passes from the primary to secondary without being coupled through the transformer's windings. This is known as the *conducted power*. Compared with a two-winding transformer of the same rating, autotransformers are smaller, more efficient, and have lower internal impedance. Three-phase autotransformers are used extensively in power systems where the voltages of the two systems coupled by the transformers do not differ by a factor greater than about three.

**Example 3.5**

A two-winding transformer is rated at 60 kVA, 240/1200 V, 60 Hz. When operated as a conventional two-winding transformer at rated load, 0.8 power factor, its efficiency is 0.96. This transformer is to be used as a 1440/1200-V step-down autotransformer in a power distribution system.

(a) Assuming ideal transformer, find the transformer kVA rating when used as an autotransformer.

(b) Find the efficiency with the kVA loading of part (a) and 0.8 power factor.

The two-winding transformer rated currents are:

$$I_1 = \frac{60,000}{240} = 250 \ \text{A}$$
$$I_2 = \frac{60,000}{1200} = 50 \ \text{A}$$

The autotransformer connection is as shown in Figure 3.21.

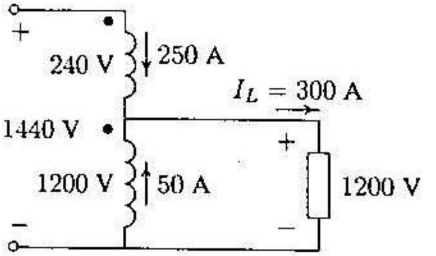



**FIGURE 3.21**
Auto transformer connection for Example 3.5.

(a) The autotransformer secondary current is

$$I_L = 250 + 50 = 300 \ \text{A}$$

With windings carrying rated currents, the autotransformer rating is

$$S = (1200)(300)(10^{-3}) = 360 \ \text{kVA}$$

Therefore, the power advantage of the autotransformer is

$$\frac{S_{auto}}{S_{2-w}} = \frac{360}{60} = 6$$

(b) When operated as a two-winding transformer at full-load, 0.8 power factor, the losses are found from the efficiency formula

$$\frac{(60)(0.8)}{(60)(0.8) + P_{loss}} = 0.96$$

Solving the above equation, the total transformer loss is

$$P_{loss} = \frac{48(1 - 0.96)}{0.96} = 2.0 \ \text{kW}$$





Since the windings are subjected to the same rated voltages and currents as the two-winding transformer, the autotransformer copper loss and the core loss at the rated values are the same as the two-winding transformer. Therefore, the autotransformer efficiency at rated load, 0.8 power factor, is

$$\eta = \frac{(360)(0.8)}{(360)(0.8) + 2} \times 100 = 99.31 \ \text{percent}$$

### 3.10.1 AUTOTRANSFORMER MODEL

When a two-winding transformer is connected as an autotransformer, its equivalent impedance expressed in per-unit is much smaller compared to the equivalent value of the two-winding connection. It can be shown that the effective per-unit impedance of an autotransformer is smaller by a factor equal to the reciprocal of the power advantage of the autotransformer connection. It is common practice to consider an autotransformer as a two-winding transformer with its two winding connected in series as shown in Figure 3.22, where the equivalent impedance is referred to the $N_1$-turn side.

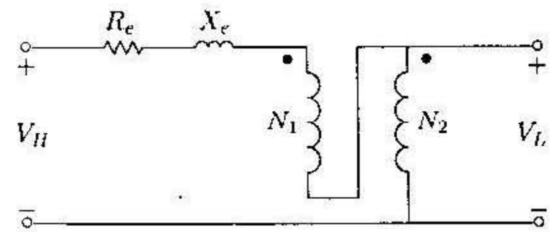



**FIGURE 3.22**
Autotransformer equivalent circuit.

## 3.11 THREE-WINDING TRANSFORMERS

Transformers having three windings are often used to interconnect three circuits which may have different voltages. These windings are called primary, secondary, and tertiary windings. Typical applications of three-winding transformers in power systems are for the supply of two independent loads at different voltages from the same source and interconnection of two transmission systems of different voltages. Usually the tertiary windings are used to provide voltage for auxiliary power purposes in the substation or to supply a local distribution system. In addition, the switched reactor or capacitors are connected to the tertiary bus for the purpose of reactive power compensation. Sometimes three-phase Y-Y transformers and Y–

connected autotransformers are provided with Δ–connected tertiary windings for harmonic suppression.

### 3.11.1 THREE-WINDING TRANSFORMER MODEL

If the exciting current of a three-winding transformer is neglected, it is possible to draw a simple single-phase equivalent T-circuit as shown in Figure 3.23.

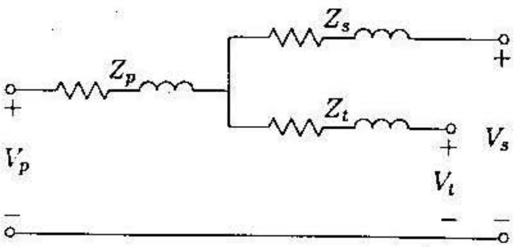



**FIGURE 3.23**
Equivalent circuit of three-winding transformer.

Three short-circuit tests are carried out on a three-winding transformer with $N_p$, $N_s$, and $N_t$ turns per phase on the three windings, respectively. The three tests are similar in that in each case one winding is open, one shorted, and reduced voltage is applied to the remaining winding. The following impedances are measured on the side to which the voltage is applied.

$Z_{ps}$ = impedance measured in the primary circuit with the secondary short-circuited and the tertiary open.
$Z_{pt}$ = impedance measured in the primary circuit with the tertiary short-circuited and the secondary open.
$Z'_{st}$ = impedance measured in the secondary circuit with the tertiary short-circuited and the primary open.

Referring $Z'_{st}$ to the primary side, we obtain

$$Z_{st} = \left(\frac{N_p}{N_s}\right)^2 Z'_{st} \tag{3.67}$$

If $Z_p$, $Z_s$, and $Z_t$ are the impedances of the three separate windings referred to the primary side, then

$$\begin{aligned} Z_{ps} &= Z_p + Z_s \\ Z_{pt} &= Z_p + Z_t \\ Z_{st} &= Z_s + Z_t \end{aligned} \tag{3.68}$$





Solving the above equations, we have

$$\begin{aligned} Z_p &= \frac{1}{2}(Z_{ps} + Z_{pt} - Z_{st}) \\ Z_s &= \frac{1}{2}(Z_{ps} + Z_{st} - Z_{pt}) \\ Z_t &= \frac{1}{2}(Z_{pt} + Z_{st} - Z_{ps}) \end{aligned} \tag{3.69}$$

## 3.12 VOLTAGE CONTROL OF TRANSFORMERS

Voltage control in transformers are required to compensate for varying voltage drops in the system and to control reactive power flow over transmission lines. Transformers may also be used to control phase angle and, therefore, active power flow. The two commonly used methods are tap changing transformers and regulating transformers.

### 3.12.1 TAP CHANGING TRANSFORMERS

Practically all power transformers and many distribution transformers have taps in one or more windings for changing the turns ratio. This method is the most popular since it can be used for controlling voltages at all levels. Tap changing, by altering the voltage magnitude, affects the distribution of vars and may therefore be used to control the flow of reactive power. There are two types of tap changing transformers

(i) Off-load tap changing transformers.
(ii) Tap changing under load (TCUL) transformers.

The off-load tap changing transformer requires the disconnection of the transformer when the tap setting is to be changed. Off-load tap changers are used when it is expected that the ratio will need to be changed only infrequently, because of load growth or some seasonal change. A typical transformer might have four taps in addition to the nominal setting, with spacing of 2.5 percent of full-load voltage between them. Such an arrangement provides for adjustments of up to 5 percent above or below the nominal voltage rating of the transformer.

Tap changing under load (TCUL) is used when changes in ratio may be frequent or when it is undesirable to de-energize the transformer to change a tap. A large number of units are now being built with load tap changing equipment. It is used on transformers and autotransformers for transmission tie, for bulk distribution units, and at other points of load service. Basically, a TCUL transformer is a transformer with the ability to change taps while power is connected. A TCUL transformer may have built-in voltage sensing circuitry that automatically changes taps to keep the system voltage constant. Such special transformers are very common in modern power systems. Special tap changing gear are required for TCUL transformers, and the position of taps depends on a number of factors and requires special consideration to arrive at an optimum location for the TCUL equipment. Step-down units usually have TCUL in the low voltage winding and de-energized taps in the high voltage winding. For example, the high voltage winding might be equipped with a nominal voltage turns ratio plus four 2.5 percent fixed tap settings to yield $\pm 5$ percent buck or boost voltage. In addition to this, there could be provision, on the low voltage windings, for 32 incremental steps of $\frac{5}{8}$ each, giving an automatic range of $\pm 10$ percent.

Tapping on both ends of a radial transmission line can be adjusted to compensate for the voltage drop in the line. Consider one phase of a three-phase transmission line with a step-up transformer at the sending end and a step-down transformer at the receiving end of the line. A single-line representation is shown in Figure 3.24, where $t_S$ and $t_R$ are the tap setting in per-unit. In this diagram, $V_1'$ is the supply phase voltage referred to the high voltage side, and $V_2'$ is the load phase voltage, also referred to the high voltage side. The impedance shown includes the

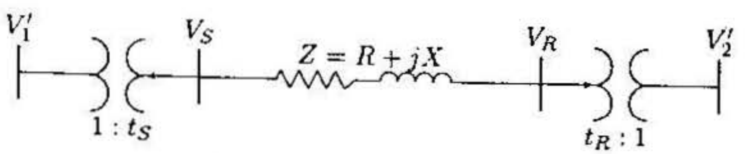



**FIGURE 3.24**
A radial line with tap changing transformers at both ends.

line impedance plus the referred impedances of the sending end and the receiving end transformers to the high voltage side. If $V_S$ and $V_R$ are the phase voltages at both ends of the line, we have

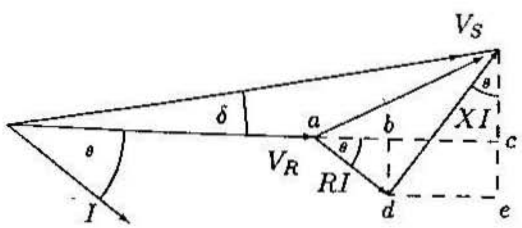



**FIGURE 3.25**
Voltage phasor diagram.





$$V_R = V_S + (R + jX)I \tag{3.70}$$

The phasor diagram for the above equation is shown in Figure 3.25.

The phase shift $\delta$ between the two ends of the line is usually small, and we can neglect the vertical component of $V_S$. Approximating $V_S$ by its horizontal component results in

$$\begin{aligned}|V_S| &= |V_R| + ab + de \\ &= |V_R| + |I|R\cos\theta + |I|X\sin\theta\end{aligned} \tag{3.71}$$

Substituting for $|I|$ from $P_\phi = |V_R||I|\cos\theta$ and $Q_\phi = |V_R||I|\sin\theta$ will result in

$$|V_S| = |V_R| + \frac{RP_\phi + XQ_\phi}{|V_R|} \tag{3.72}$$

Since $V_S = t_S V_1'$ and $V_R = t_R V_2'$, the above relation in terms of $V_1'$, and $V_2'$ becomes

$$t_S|V_1'| = t_R|V_2'| + \frac{RP_\phi + XQ_\phi}{t_R|V_2'|} \tag{3.73}$$

or

$$t_S = \frac{1}{|V_1'|}\left(t_R|V_2'| + \frac{RP_\phi + XQ_\phi}{t_R|V_2'|}\right) \tag{3.74}$$

Assuming the product of $t_S$ and $t_R$ is unity, i.e., $t_S t_R = 1$, and substituting for $t_R$ in (3.74), the following expression is found for $t_S$.

$$t_S = \sqrt{\frac{\frac{|V_2'|}{|V_1'|}}{1 - \frac{RP_\phi + XQ_\phi}{|V_1'||V_2'|}}} \tag{3.75}$$

### Example 3.6

A three-phase transmission line is feeding from a 23/230-kV transformer at its sending end. The line is supplying a 150-MVA, 0.8 power factor load through a step-down transformer of 230/23 kV. The impedance of the line and transformers at 230 kV is $18 + j60\ \Omega$. The sending end transformer is energized from a 23-kV supply. Determine the tap setting for each transformer to maintain the voltage at the load at 23 kV.

The load real and reactive power per phase are

$$P_\phi = \frac{1}{3}(150)(0.8) = 40 \text{ MW}$$
$$Q_\phi = \frac{1}{3}(150)(0.6) = 30 \text{ Mvar}$$

The source and the load phase voltages referred to the high voltage side are

$$|V_1'| = |V_2'| = \left(\frac{230}{23}\right)\left(\frac{23}{\sqrt{3}}\right) = \frac{230}{\sqrt{3}}$$

From (3.75), we have

$$t_S = \sqrt{\frac{1}{1 - \frac{(18)(40)+(60)(30)}{(230/\sqrt{3})^2}}} = 1.08 \text{ pu}$$

and

$$t_R = \frac{1}{1.08} = 0.926 \text{ pu}$$

### 3.12.2 REGULATING TRANSFORMERS OR BOOSTERS

Regulating transformers, also known as *boosters*, are used to change the voltage magnitude and phase angle at a certain point in the system by a small amount. A booster consists of an exciting transformer and a series transformer.

#### VOLTAGE MAGNITUDE CONTROL

Figure 3.26 shows the connection of a regulating transformer for phase $a$ of a three-phase system for voltage magnitude control. Other phases have identical arrangement. The secondary of the exciting transformer is tapped, and the voltage obtained from it is applied to the primary of the series transformer. The corresponding voltage on the secondary of the series transformer is added to the input voltage. Thus, the output voltage is

$$V'_{an} = V_{an} + \Delta V_{an} \tag{3.76}$$

Since the voltages are in phase, a booster of this type is called an *in-phase booster*. The output voltage can be adjusted by changing the excitation transformer taps. By changing the switch from position 1 to 2, the polarity of the voltage across the series transformer is reversed, so that the output voltage is now less than the input voltage.





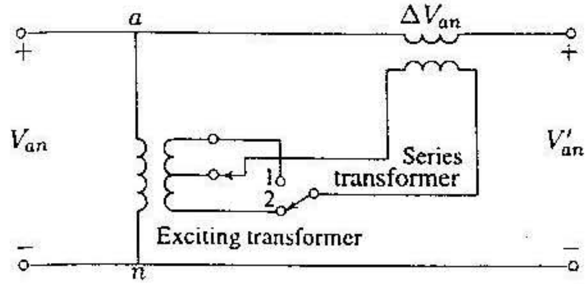



**FIGURE 3.26**
Regulating transformer for voltage magnitude control.

#### PHASE ANGLE CONTROL

Regulating transformers are also used to control the voltage phase angle. If the injected voltage is out of phase with the input voltage, the resultant voltage will have a phase shift with respect to the input voltage. Phase shifting is used to control active power flow at major intertie buses. A typical arrangement for phase $a$ of a three-phase system is shown in Figure 3.27.

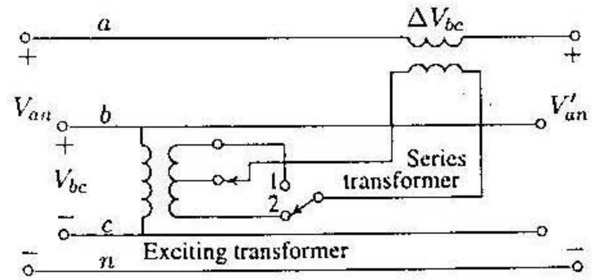



**FIGURE 3.27**
Regulating transformer for voltage phase angle control.

The series transformer of phase $a$ is supplied from the secondary of the exciting transformer $bc$. The injected voltage $\Delta V_{bc}$ is in quadrature with the voltage $V_{an}$, thus the resultant voltage $V'_{an}$ goes through a phase shift $\alpha$, as shown in Figure 3.28. The output voltage is

$$V'_{an} = V_{an} + \Delta V_{bc} \tag{3.77}$$

Similar connections are made for the remaining phases, resulting in a balanced three phase output voltage. The amount of phase shift can be adjusted by changing the excitation transformer taps. By changing the switch from position 1 to 2, the

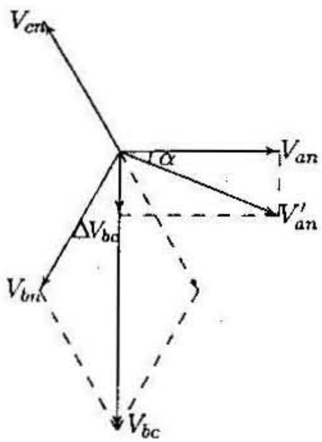



**FIGURE 3.28**
Voltage phasor diagram showing phase shifting effect for phase $a$.

output voltage can be made to lag or lead the input voltage. The advantages of the regulating transformers are

1. The main transformers are free from tappings.
2. The regulating transformers can be used at any intermediate point in the system.
3. The regulating transformers and the tap changing gears can be taken out of service for maintenance without affecting the system.

## 3.13 THE PER-UNIT SYSTEM

The solution of an interconnected power system having several different voltage levels requires the cumbersome transformation of all impedances to a single voltage level. However, power system engineers have devised the *per-unit system* such that the various physical quantities such as power, voltage, current and impedance are expressed as a decimal fraction or multiples of base quantities. In this system, the different voltage levels disappear, and a power network involving generators, transformers, and lines (of different voltage levels) reduces to a system of simple impedances. The per-unit value of any quantity is defined as

$$\text{Quantity in per-unit} = \frac{\text{actual quantity}}{\text{base value of quantity}} \tag{3.78}$$





For example,

$$S_{pu} = \frac{S}{S_B} \quad V_{pu} = \frac{V}{V_B} \quad I_{pu} = \frac{I}{I_B} \quad \text{and} \quad Z_{pu} = \frac{Z}{Z_B}$$

where the numerators (actual values) are phasor quantities or complex values and the denominators (base values) are always real numbers. A minimum of four base quantities are required to completely define a per-unit system: volt-ampere, voltage, current, and impedance. Usually, the three-phase base volt-ampere $S_B$ or $MVA_B$ and the line-to-line base voltage $V_B$ or $kV_B$ are selected. Base current and base impedance are then dependent on $S_B$ and $V_B$ and must obey the circuit laws. These are given by

$$I_B = \frac{S_B}{\sqrt{3}\,V_B} \tag{3.79}$$

and

$$Z_B = \frac{V_B/\sqrt{3}}{I_B} \tag{3.80}$$

Substituting for $I_B$ from (3.79), the base impedance becomes

$$Z_B = \frac{(V_B)^2}{S_B}.$$
$$Z_B = \frac{(kV_B)^2}{MVA_B} \tag{3.81}$$

The phase and line quantities expressed in per-unit are the same, and the circuit laws are valid, i.e.,

$$S_{pu} = V_{pu} I^*_{pu} \tag{3.82}$$

and

$$V_{pu} = Z_{pu} I_{pu} \tag{3.83}$$

The load power at its rated voltage can also be expressed by a per-unit impedance. If $S_{L(3\phi)}$ is the complex load power, the load current per phase at the phase voltage $V_P$ is given by

$$S_{L(3\phi)} = 3 V_P I^*_P \tag{3.84}$$

The phase current in terms of the ohmic load impedance is

$$I_P = \frac{V_P}{Z_P} \tag{3.85}$$

Substituting for $I_P$ from (3.85) into (3.84) results in the ohmic value of the load impedance

$$Z_P = \frac{3|V_P|^2}{S^*_{L(3\phi)}} = \frac{|V_{L-L}|^2}{S^*_{L(3\phi)}} \tag{3.86}$$

From (3.81) the load impedance in per-unit is

$$Z_{pu} = \frac{Z_P}{Z_B} = \left|\frac{V_{L-L}}{V_B}\right|^2 \frac{S_B}{S^*_{L(3\phi)}} \tag{3.87}$$

or

$$Z_{pu} = \frac{|V_{pu}|^2}{S^*_{L(pu)}} \tag{3.88}$$

## 3.14 CHANGE OF BASE

The impedance of individual generators and transformers, as supplied by the manufacturer, are generally in terms of percent or per-unit quantities based on their own ratings. The impedance of transmission lines are usually expressed by their ohmic values. For power system analysis, all impedances must be expressed in per unit on a common system base. To accomplish this, an arbitrary base for apparent power is selected; for example, 100 MVA. Then, the voltage bases must be selected. Once a voltage base has been selected for a point in a system, the remaining voltage bases are no longer independent; they are determined by the various transformer turns ratios. For example, if on a low-voltage side of a 34.5/115-kV transformer the base voltage of 36 kV is selected, the base voltage on the high-voltage side must be $36(115/34.5) = 120$ kV. Normally, we try to select the voltage bases that are the same as the nominal values.

Let $Z_{pu}^{old}$ be the per-unit impedance on the power base $S_B^{old}$ and the voltage base $V_B^{old}$, which is expressed by

$$Z_{pu}^{old} = \frac{Z_\Omega}{Z_B^{old}} = Z_\Omega \frac{S_B^{old}}{(V_B^{old})^2} \tag{3.89}$$

Expressing $Z_\Omega$ to a new power base and a new voltage base, results in the new per-unit impedance

$$Z_{pu}^{new} = \frac{Z_\Omega}{Z_B^{new}} = Z_\Omega \frac{S_B^{new}}{(V_B^{new})^2} \tag{3.90}$$





From (3.89) and (3.90), the relationship between the old and the new per-unit values is

$$Z_{pu}^{new} = Z_{pu}^{old} \frac{S_B^{new}}{S_B^{old}} \left(\frac{V_B^{old}}{V_B^{new}}\right)^2 \tag{3.91}$$

If the voltage bases are the same, (3.91) reduces to

$$Z_{pu}^{new} = Z_{pu}^{old} \frac{S_B^{new}}{S_B^{old}} \tag{3.92}$$

The advantages of the per-unit system for analysis are described below.

- The per-unit system gives us a clear idea of relative magnitudes of various quantities, such as voltage, current, power and impedance.
- The per-unit impedance of equipment of the same general type based on their own ratings fall in a narrow range regardless of the rating of the equipment. Whereas their impedance in ohms vary greatly with the rating.
- The per-unit values of impedance, voltage and current of a transformer are the same regardless of whether they are referred to the primary or the secondary side. This is a great advantage since the different voltage levels disappear and the entire system reduces to a system of simple impedance.
- The per-unit systems are ideal for the computerized analysis and simulation of complex power system problems.
- The circuit laws are valid in per-unit systems, and the power and voltage equations as given by (3.82) and (3.83) are simplified since the factors of $\sqrt{3}$ and 3 are eliminated in the per-unit system.

Example 3.7 demonstrates how a per-unit impedance diagram is obtained for a simple power system network.

**Example 3.7**

The one-line diagram of a three-phase power system is shown in Figure 3.29. Select a common base of 100 MVA and 22 kV on the generator side. Draw an impedance diagram with all impedances including the load impedance marked in per-unit. The manufacturer's data for each device is given as follow:

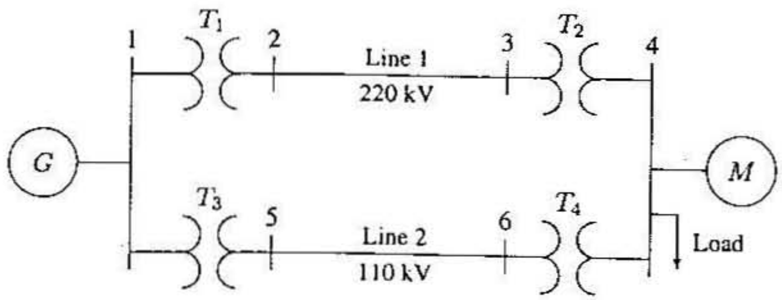



**FIGURE 3.29**
One-line diagram for Example 3.7.

| | | | |
|---|---|---|---|
| $G$: | 90 MVA | 22 kV | $X = 18$ % |
| $T_1$: | 50 MVA | 22/220 kV | $X = 10$ % |
| $T_2$: | 40 MVA | 220/11 kV | $X = 6.0$ % |
| $T_3$: | 40 MVA | 22/110 kV | $X = 6.4$ % |
| $T_4$: | 40 MVA | 110/11 kV | $X = 8.0$ % |
| $M$: | 66.5 MVA | 10.45 kV | $X = 18.5$ % |

The three-phase load at bus 4 absorbs 57 MVA, 0.6 power factor lagging at 10.45 kV. Line 1 and line 2 have reactances of 48.4 and 65.43 Ω, respectively.

First, the voltage bases must be determined for all sections of the network. The generator rated voltage is given as the base voltage at bus 1. This fixes the voltage bases for the remaining buses in accordance to the transformer turns ratios. The base voltage $V_{B1}$ on the LV side of $T_1$ is 22 kV. Hence the base on its HV side is

$$V_{B2} = 22\left(\frac{220}{22}\right) = 220 \quad \text{kV}$$

This fixes the base on the HV side of $T_2$ at $V_{B3} = 220$ kV, and on its LV side at

$$V_{B4} = 220\left(\frac{11}{220}\right) = 11 \quad \text{kV}$$

Similarly, the voltage base at buses 5 and 6 are

$$V_{B5} = V_{B6} = 22\left(\frac{110}{22}\right) = 110 \quad \text{kV}$$





Since generator and transformer voltage bases are the same as their rated values, their per-unit reactances on a 100 MVA base, from (3.92) are

$$G:\ X = 0.18\left(\frac{100}{90}\right) = 0.20 \quad \text{pu}$$
$$T_1:\ X = 0.10\left(\frac{100}{50}\right) = 0.20 \quad \text{pu}$$
$$T_2:\ X = 0.06\left(\frac{100}{40}\right) = 0.15 \quad \text{pu}$$
$$T_3:\ X = 0.064\left(\frac{100}{40}\right) = 0.16 \quad \text{pu}$$
$$T_4:\ X = 0.08\left(\frac{100}{40}\right) = 0.2 \quad \text{pu}$$

The motor reactance is expressed on its nameplate rating of 66.5 MVA and 10.45 kV. However, the base voltage at bus 4 for the motor is 11 kV. From (3.91) the motor reactance on a 100 MVA, 11-kV base is

$$M:\ X = 0.185\left(\frac{100}{66.5}\right)\left(\frac{10.45}{11}\right)^2 = 0.25 \quad \text{pu}$$

Impedance bases for lines 1 and 2, from (3.81) are

$$Z_{B2} = \frac{(220)^2}{100} = 484 \quad \Omega$$
$$Z_{B5} = \frac{(110)^2}{100} = 121 \quad \Omega$$

Line 1 and 2 per-unit reactances are

$$\text{Line 1:}\ X = \left(\frac{48.4}{484}\right) = 0.10 \quad \text{pu}$$
$$\text{Line 2:}\ X = \left(\frac{65.43}{121}\right) = 0.54 \quad \text{pu}$$

The load apparent power at 0.6 power factor lagging is given by

$$S_{L(3\phi)} = 57\angle 53.13^\circ \quad \text{MVA}$$

Hence, the load impedance in ohms is

$$Z_L = \frac{(V_{L-L})^2}{S^*_{L(3\phi)}} = \frac{(10.45)^2}{57\angle -53.13^\circ} = 1.1495 + j1.53267 \quad \Omega$$

The base impedance for the load is

$$Z_{B4} = \frac{(11)^2}{100} = 1.21 \quad \Omega$$

Therefore, the load impedance in per-unit is

$$Z_{L(pu)} = \frac{1.1495 + j1.53267}{1.21} = 0.95 + j1.2667 \quad \text{pu}$$

The per-unit equivalent circuit is shown in Figure 3.30.

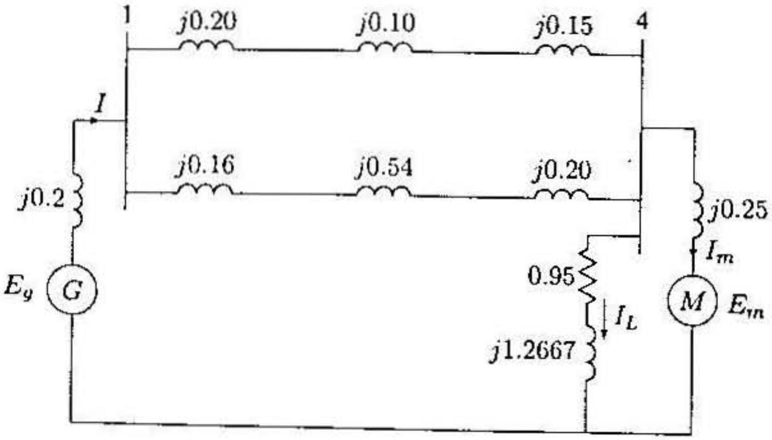



**FIGURE 3.30**
Per-unit impedance diagram for Example 3.7.

### Example 3.8

The motor of Example 3.7 operates at full-load 0.8 power factor leading at a terminal voltage of 10.45 kV.

(a) Determine the voltage at the generator bus bar (bus 1).
(b) Determine the generator and the motor internal emfs.

(a) The per-unit voltage at bus 4, taken as reference is

$$V_4 = \frac{10.45}{11} = 0.95\angle 0^\circ \quad \text{pu}$$

The motor apparent power at 0.8 power factor leading is given by

$$S_m = \frac{66.5}{100}\angle -36.87^\circ \quad \text{pu}$$





Therefore, current drawn by the motor is

$$I_m = \frac{S_m^*}{V_4^*} = \frac{0.665\angle 36.87}{0.95\angle 0^\circ} = 0.56 + j0.42 \quad \text{pu}$$

and current drawn by the load is

$$I_L = \frac{V_4}{Z_L} = \frac{0.95\angle 0^\circ}{0.95 + j1.2667} = 0.36 - j0.48 \quad \text{pu}$$

Total current drawn from bus 4 is

$$I = I_m + I_L = (0.56 + j0.42) + (0.36 - j0.48) = 0.92 - j0.06 \quad \text{pu}$$

The equivalent reactance of the parallel branches is

$$X_{\|} = \frac{0.45 \times 0.9}{0.45 + 0.9} = 0.3 \quad \text{pu}$$

The generator terminal voltage is

$$\begin{aligned} V_1 = V_4 + Z_{\|} I = 0.95\angle 0^\circ + j0.3(0.92 - j0.06) &= 0.968 + j0.276 \\ &= 1.0\angle 15.91^\circ \quad \text{pu} \\ &= 22\angle 15.91^\circ \quad \text{kV} \end{aligned}$$

(b) The generator internal emf is

$$\begin{aligned} E_g = V_1 + Z_g I = 0.968 + j0.276 + j0.20(0.92 - j0.06) &= 1.0826\angle 25.14^\circ \text{ pu} \\ &= 23.82\angle 25.14^\circ \quad \text{kV} \end{aligned}$$

and the motor internal emf is

$$\begin{aligned} E_m = V_4 - Z_m I_m = 0.95 + j0 - j0.25(0.56 + j0.42) &= 1.064\angle -7.56^\circ \quad \text{pu} \\ &= 11.71\angle -7.56^\circ \quad \text{kV} \end{aligned}$$

## PROBLEMS

**3.1.** A three-phase, 318.75-kVA, 2300-V alternator has an armature resistance of 0.35 Ω/phase and a synchronous reactance of 1.2 Ω/phase. Determine the no-load line-to-line generated voltage and the voltage regulation at

(a) Full-load kVA, 0.8 power factor lagging, and rated voltage.

(b) Full-load kVA, 0.6 power factor leading, and rated voltage.

**3.2.** A 60-MVA, 69.3-kV, three-phase synchronous generator has a synchronous reactance of 15 Ω/phase and negligible armature resistance.

(a) The generator is delivering rated power at 0.8 power factor lagging at the rated terminal voltage to an infinite bus bar. Determine the magnitude of the generated emf per phase and the power angle $\delta$.

(b) If the generated emf is 36 kV per phase, what is the maximum three-phase power that the generator can deliver before losing its synchronism?

(c) The generator is delivering 48 MW to the bus bar at the rated voltage with its field current adjusted for a generated emf of 46 kV per phase. Determine the armature current and the power factor. State whether power factor is lagging or leading?

**3.3.** A 24,000-kVA, 17.32-kV, 60-Hz, three-phase synchronous generator has a synchronous reactance of 5 Ω/phase and negligible armature resistance.

(a) At a certain excitation, the generator delivers rated load, 0.8 power factor lagging to an infinite bus bar at a line-to-line voltage of 17.32 kV. Determine the excitation voltage per phase.

(b) The excitation voltage is maintained at 13.4 kV/phase and the terminal voltage at 10 kV/phase. What is the maximum three-phase real power that the generator can develop before pulling out of synchronism?

(c) Determine the armature current for the condition of part (b).

**3.4.** A 34.64-kV, 60-MVA, three-phase salient-pole synchronous generator has a direct axis reactance of 13.5 Ω and a quadrature-axis reactance of 9.333 Ω. The armature resistance is negligible.

(a) Referring to the phasor diagram of a salient-pole generator shown in Figure 3.8, show that the power angle $\delta$ is given by

$$\delta = \tan^{-1}\left(\frac{X_q|I_a|\cos\theta}{V + X_q|I_a|\sin\theta}\right)$$

(b) Compute the load angle $\delta$ and the per phase excitation voltage $E$ when the generator delivers rated MVA, 0.8 power factor lagging to an infinite bus bar of 34.64-kV line-to-line voltage.

(c) The generator excitation voltage is kept constant at the value found in part (b). Use *MATLAB* to obtain a plot of the power angle curve, i.e., equation (3.32) over a range of $\delta$ = 0:0.05:180°. Use the command **Pmax, k = max(P); dmax = d(k)**, to obtain the steady-state maximum power **Pmax** and the corresponding power angle dmax.





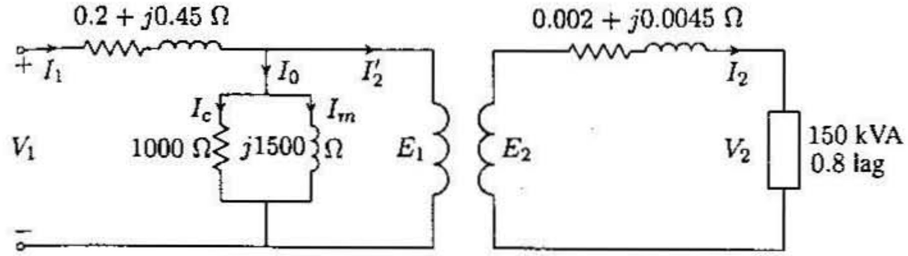



**FIGURE 3.31**
Transformer circuit for Problem 3.5

**3.5.** A 150-kVA, 2400/240-V single-phase transformer has the parameters as shown in Figure 3.31.

(a) Determine the equivalent circuit referred to the high-voltage side.

(b) Find the primary voltage and voltage regulation when transformer is operating at full load 0.8 power factor lagging and 240 V.

(c) Find the primary voltage and voltage regulation when the transformer is operating at full-load 0.8 power factor leading.

(d) Verify your answers by running the **trans** program in *MATLAB* and obtain the transformer efficiency curve.

**3.6.** A 60-kVA, 4800/2400-V single-phase transformer gave the following test results:

1. Rated voltage is applied to the low voltage winding and the high voltage winding is open-circuited. Under this condition, the current into the low voltage winding is 2.4 A and the power taken from the 2400 V source is 3456 W.

2. A reduced voltage of 1250 V is applied to the high voltage winding and the low voltage winding is short-circuited. Under this condition, the current flowing into the high voltage winding is 12.5 A and the power taken from the 1250 V source is 4375 W.

(a) Determine parameters of the equivalent circuit referred to the high voltage side.

(b) Determine voltage regulation and efficiency when transformer is operating at full-load, 0.8 power factor lagging, and a terminal voltage of 2400 V.

(c) What is the load kVA for maximum efficiency and the maximum efficiency at 0.8 power factor?

(d) Determine the efficiency when transformer is operating at 3/4 full-load, 0.8 power factor lagging, and a terminal voltage of 2400 V.

(e) Verify your answers by running the **trans** program in *MATLAB* and obtain the transformer efficiency curve.

**3.7.** A two-winding transformer rated at 9-kVA, 120/90-V, 60-HZ has a core loss of 200 W and a full-load copper loss of 500 W.

(a) The above transformer is to be connected as an auto transformer to supply a load at 120 V from a 210-V source. What kVA load can be supplied without exceeding the current rating of the windings? (For this part assume an ideal transformer.)

(b) Find the efficiency with the kVA loading of part (a) and 0.8 power factor.

**3.8.** Three identical 9-MVA, 7.2-kV/4.16-kV, single-phase transformers are connected in wye on the high-voltage side and delta on the low voltage side. The equivalent series impedance of each transformer referred to the high-voltage side is $0.12 + j0.82\ \Omega$ per phase. The transformer supplies a balanced three-phase load of 18 MVA, 0.8 power factor lagging at 4.16 kV. Determine the line-to-line voltage at the high-voltage terminals of the transformer.

**3.9.** A 400-MVA, 240-kV/24-kV, three-phase Y-Δ transformer has an equivalent series impedance of $1.2 + j6\ \Omega$ per phase referred to the high-voltage side. The transformer is supplying a three-phase load of 400-MVA, 0.8 power factor lagging at a terminal voltage of 24 kV (line to line) on its low-voltage side. The primary is supplied from a feeder with an impedance of $0.6 + j1.2\ \Omega$ per phase. Determine the line-to-line voltage at the high-voltage terminals of the transformer and the sending-end of the feeder.

**3.10.** In Problem 3.9, with transformer rated values as base quantities, express all impedances in per-unit. Working with per-unit values, determine the line-to-line voltage at the high-voltage terminals of the transformer and the sending-end of the feeder.

**3.11.** A three-phase, Y-connected, 75-MVA, 27-kV synchronous generator has a synchronous reactance of $9.0\ \Omega$ per phase. Using rated MVA and voltage as base values, determine the per-unit reactance. Then refer this per-unit value to a 100-MVA, 30-kV base.

**3.12.** A 40-MVA, 20-kV/400-kV, single-phase transformer has the following series impedances:

$$Z_1 = 0.9 + j1.8\ \Omega \text{ and } Z_2 = 128 + j288\ \Omega$$

Using the transformer rating as base, determine the per-unit impedance of the transformer from the ohmic value referred to the low-voltage side. Compute the per-unit impedance using the ohmic value referred to the high-voltage side.





**3.13.** Draw an impedance diagram for the electric power system shown in Figure 3.32 showing all impedances in per unit on a 100-MVA base. Choose 20-kV as the voltage base for generator. The three-phase power and line-line ratings are given below.

| | | | |
|---|---|---|---|
| $G_1$ : | 90 MVA | 20 kV | $X = 9\%$ |
| $T_1$ : | 80 MVA | 20/200 kV | $X = 16\%$ |
| $T_2$ : | 80 MVA | 200/20 kV | $X = 20\%$ |
| $G_2$ : | 90 MVA | 18 kV | $X = 9\%$ |
| Line: | | 200 kV | $X = 120\ \Omega$ |
| Load: | | 200 kV | $S = 48$ MW $+j64$ Mvar |

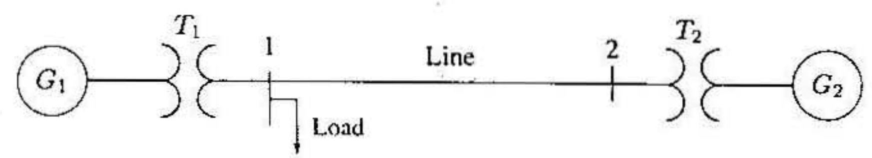



**FIGURE 3.32**
One-line diagram for Problem 3.13

**3.14.** The one-line diagram of a power system is shown in Figure 3.33.

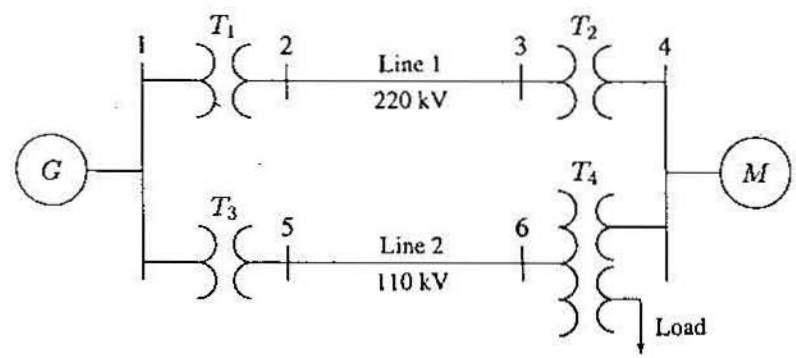



**FIGURE 3.33**
One-line diagram for Problem 3.14

The three-phase power and line-line ratings are given below.

| | | | |
|---|---|---|---|
| $G$: | 80 MVA | 22 kV | $X = 24\%$ |
| $T_1$: | 50 MVA | 22/220 kV | $X = 10\%$ |
| $T_2$: | 40 MVA | 220/22 kV | $X = 6.0\%$ |
| $T_3$: | 40 MVA | 22/110 kV | $X = 6.4\%$ |
| Line 1: | | 220 kV | $X = 121\ \Omega$ |
| Line 2: | | 110 kV | $X = 42.35\ \Omega$ |
| $M$: | 68.85 MVA | 20 kV | $X = 22.5\%$ |
| Load: | 10 Mvar | 4 kV | $\Delta$-connected capacitors |

The three-phase ratings of the three-phase transformer are

| | | |
|---|---|---|
| Primary: | Y-connected | 40MVA, 110 kV |
| Secondary: | Y-connected | 40 MVA, 22 kV |
| Tertiary: | $\Delta$-connected | 15 MVA, 4 kV |

The per phase measured reactances at the terminal of a winding with the second one short-circuited and the third open-circuited are

| | |
|---|---|
| $Z_{ps} = 9.6\%$ | 40 MVA, 110 kV/22 kV |
| $Z_{pt} = 7.2\%$ | 40 MVA, 110 kV/4 kV |
| $Z_{st} = 12\%$ | 40 MVA, 22kV/4 kV |

Obtain the T-circuit equivalent impedances of the three-winding transformer to the common 100-MVA base. Draw an impedance diagram showing all impedances in per-unit on a 100-MVA base. Choose 22 kV as the voltage base for generator.

**3.15.** The three-phase power and line-line ratings of the electric power system shown in Figure 3.34 are given below.

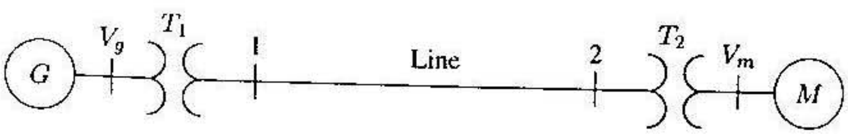



**FIGURE 3.34**
One-line diagram for Problem 3.15

| | | | |
|---|---|---|---|
| $G_1$: | 60 MVA | 20 kV | $X = 9\%$ |
| $T_1$: | 50 MVA | 20/200 kV | $X = 10\%$ |
| $T_2$: | 50 MVA | 200/20 kV | $X = 10\%$ |
| $M$: | 43.2 MVA | 18 kV | $X = 8\%$ |
| Line: | | 200 kV | $Z = 120 + j200\ \Omega$ |

(a) Draw an impedance diagram showing all impedances in per-unit on a 100-MVA base. Choose 20 kV as the voltage base for generator.





(b) The motor is drawing 45 MVA, 0.80 power factor lagging at a line-to-line terminal voltage of 18 kV. Determine the terminal voltage and the internal emf of the generator in per-unit and in kV.

**3.16.** The one-line diagram of a three-phase power system is as shown in Figure 3.35. Impedances are marked in per-unit on a 100-MVA, 400-kV base. The load at bus 2 is $S_2 = 15.93$ MW $-j33.4$ Mvar, and at bus 3 is $S_3 = 77$ MW $+j14$ Mvar. It is required to hold the voltage at bus 3 at $400\angle 0°$ kV. Working in per-unit, determine the voltage at buses 2 and 1.

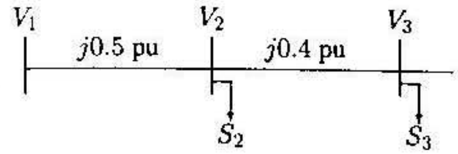



**FIGURE 3.35**
One-line diagram for Problem 3.16

**3.17.** The one-line diagram of a three-phase power system is as shown in Figure 3.36. The transformer reactance is 20 percent on a base of 100 MVA, 23/115 kV and the line impedance is $Z = j66.125\Omega$. The load at bus 2 is $S_2 = 184.8$ MW $+j6.6$ Mvar, and at bus 3 is $S_3 = 0$ MW $+j20$ Mvar. It is required to hold the voltage at bus 3 at $115\angle 0°$ kV. Working in per-unit, determine the voltage at buses 2 and 1.

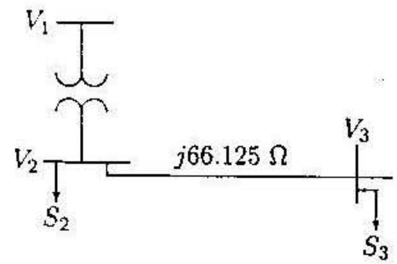



**FIGURE 3.36**
One-line diagram for Problem 3.17

# CHAPTER 4

# TRANSMISSION LINE PARAMETERS

## 4.1 INTRODUCTION

The purpose of a transmission network is to transfer electric energy from generating units at various locations to the distribution system which ultimately supplies the load. Transmission lines also interconnect neighboring utilities which permits not only economic dispatch of power within regions during normal conditions, but also transfer of power between regions during emergencies.

All transmission lines in a power system exhibit the electrical properties of resistance, inductance, capacitance, and conductance. The inductance and capacitance are due to the effects of magnetic and electric fields around the conductor. These parameters are essential for the development of the transmission line models used in power system analysis. The shunt conductance accounts for leakage currents flowing across insulators and ionized pathways in the air. The leakage currents are negligible compared to the current flowing in the transmission lines and may be neglected.

The first part of this chapter deals with the determination of inductance and capacitance of overhead lines. The concept of *geometric mean radius, GMR* and *geometric mean distance GMD* are discussed, and the function **[GMD, GMRL,**







**GMRC] = gmd** is developed for the evaluation of $GMR$ and $GMD$. This function is very useful for computing the inductance and capacitance of single-circuit or double-circuit transmission lines with bundled conductors. Alternatively, the function **[L, C] = gmd2LC** returns the line inductance in mH per km and the shunt capacitance in $\mu$F per km. Finally the effects of electromagnetic and electrostatic induction are discussed.

## 4.2 OVERHEAD TRANSMISSION LINES

A transmission circuit consists of conductors, insulators, and usually shield wires, as shown in Figure 4.1. Transmission lines are hung overhead from a tower usually made of steel, wood or reinforced concrete with its own right-of-way. Steel towers may be single-circuit or double-circuit designs. Multicircuit steel towers have been built, where the tower supports three to ten 69-kV lines over a given width of right-of-way. Less than 1 percent of the nation's total transmission lines are placed underground. Although underground ac transmission would present a solution to some of the environmental and aesthetic problems involved with overhead transmission lines, there are technical and economic reasons that make the use of underground ac transmission prohibitive.

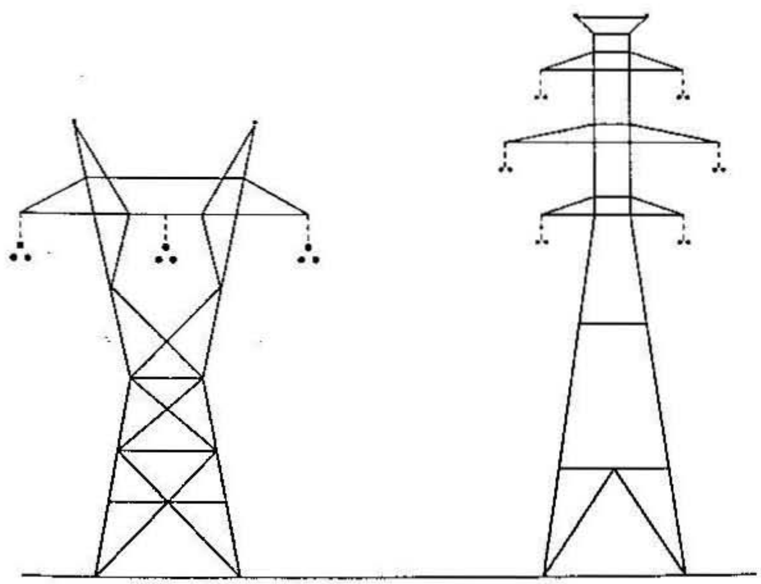


**FIGURE 4.1**
Typical lattice-type structure for 345-kV transmission line.

The selection of an economical voltage level for the transmission line is based on the amount of power and the distance of transmission. The voltage choice together with the selection of conductor size is mainly a process of weighing $RI^2$ losses, audible noise, and radio interference level against fixed charges on the investment. Standard transmission voltages are established in the United States by the American National Standards Institute (ANSI). Transmission voltage lines operating at more than 60 kV are standardized at 69 kV, 115 kV, 138 kV, 161 kV, 230 kV, 345 kV, 500 kV, 765 kV line-to-line. Transmission voltages above 230 kV are usually referred to as *extra-high voltage* (EHV) and those at 765 kV and above are referred to as *ultra-high voltage* (UHV). The most commonly used conductor materials for high voltage transmission lines are ACSR (aluminum conductor steel-reinforced), AAC (all-aluminum conductor), AAAC (all-aluminum alloy conductor), and ACAR (aluminum conductor alloy-reinforced). The reason for their popularity is their low relative cost and high strength-to-weight ratio as compared to copper conductors. Also, aluminum is in abundant supply, while copper is limited in quantity. A table of the most commonly used ACSR conductors is stored in file **acsr.m** Characteristics of other conductors can be found in conductor handbooks or manufacturer's literature. The conductors are stranded to have flexibility. The ACSR conductor consists of a center core of steel strands surrounded by layers of aluminum as shown in Figure 4.2. Each layer of strands is spiraled in the opposite direction of its adjacent layer. This spiraling holds the strands in place.

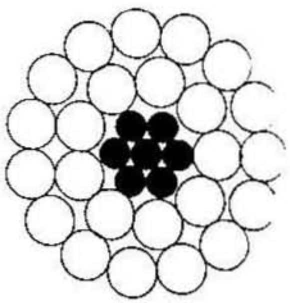


**FIGURE 4.2**
Cross-sectional view of a 24/7 ACSR conductor.

Conductor manufacturers provide the characteristics of the standard conductors with conductor sizes expressed in *circular mils* (cmil). One mil equals 0.001 inch, and for a solid round conductor the area in circular mils is defined as the square of diameter in mils. As an example, 1,000,000 cmil represents an area of a solid round conductor 1 inch in diameter. In addition, code words (bird names) have been assigned to each conductor for easy reference.

At voltages above 230 kV, it is preferable to use more than one conductor per phase, which is known as *bundling* of conductors. The bundle consists of two, three, or four conductors. Bundling increases the effective radius of the line's conductor and reduces the electric field strength near the conductors, which reduces corona power loss, audible noise, and radio interference. Another important advantage of bundling is reduced line reactance.





## 4.3 LINE RESISTANCE

The resistance of the conductor is very important in transmission efficiency evaluation and economic studies. The dc resistance of a solid round conductor at a specified temperature is given by

$$R_{dc} = \frac{\rho l}{A} \tag{4.1}$$

where $\rho$ = conductor resistivity
$l$ = conductor length
$A$ = conductor cross-sectional area

The conductor resistance is affected by three factors: frequency, spiraling, and temperature.

When ac flows in a conductor, the current distribution is not uniform over the conductor cross-sectional area and the current density is greatest at the surface of the conductor. This causes the ac resistance to be somewhat higher than the dc resistance. This behavior is known as *skin effect*. At 60 Hz, the ac resistance is about 2 percent higher than the dc resistance.

Since a stranded conductor is spiraled, each strand is longer than the finished conductor. This results in a slightly higher resistance than the value calculated from 4.1.

The conductor resistance increases as temperature increases. This change can be considered linear over the range of temperature normally encountered and may be calculated from

$$R_2 = R_1 \frac{T + t_2}{T + t_1} \tag{4.2}$$

where $R_2$ and $R_1$ are conductor resistances at $t_2$ and $t_1$-C°, respectively. $T$ is a temperature constant that depends on the conductor material. For aluminum $T \simeq 228$.

Because of the above effects, the conductor resistance is best determined from manufacturers' data.

## 4.4 INDUCTANCE OF A SINGLE CONDUCTOR

A current-carrying conductor produces a magnetic field around the conductor. The magnetic flux lines are concentric closed circles with direction given by the right-hand rule. With the thumb pointing in the direction of the current, the fingers of the right hand encircled the wire point in the direction of the magnetic field. When the current changes, the flux changes and a voltage is induced in the circuit. By definition, for nonmagnetic material, the inductance $L$ is the ratio of its total magnetic flux linkage to the current $I$, given by

$$L = \frac{\lambda}{I} \tag{4.3}$$

where $\lambda$ = flux linkages, in Weber turns.

Consider a long round conductor with radius $r$, carrying a current $I$ as shown in Figure 4.3.

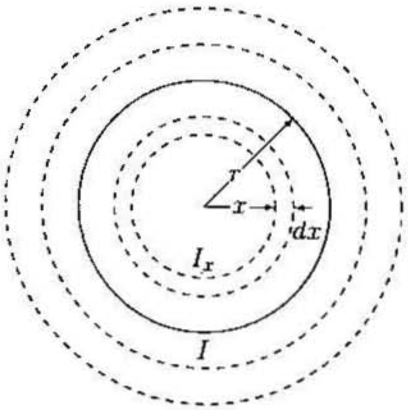



**FIGURE 4.3**
Flux linkage of a long round conductor.

The magnetic field intensity $H_x$, around a circle of radius $x$, is constant and tangent to the circle. The Ampere's law relating $H_x$ to the current $I_x$ is given by

$$\int_0^{2\pi x} H_x \cdot dl = I_x \tag{4.4}$$

or

$$H_x = \frac{I_x}{2\pi x} \tag{4.5}$$

where $I_x$ is the current enclosed at radius $x$. As shown in Figure 4.3, Equation (4.5) is all that is required for evaluating the flux linkage $\lambda$ of a conductor. The





inductance of the conductor can be defined as the sum of contributions from flux linkages internal and external to the conductor.

### 4.4.1 INTERNAL INDUCTANCE

A simple expression can be obtained for the internal flux linkage by neglecting the skin effect and assuming uniform current density throughout the conductor cross section, i.e.,

$$\frac{I}{\pi r^2} = \frac{I_x}{\pi x^2} \tag{4.6}$$

Substituting for $I_x$ in (4.5) yields

$$H_x = \frac{I}{2\pi r^2}x \tag{4.7}$$

For a nonmagnetic conductor with constant permeability $\mu_0$, the magnetic flux density is given by $B_x = \mu_0 H_x$, or

$$B_x = \frac{\mu_0 I}{2\pi r^2}x \tag{4.8}$$

where $\mu_0$ is the permeability of free space (or air) and is equal to $4\pi \times 10^{-7}$H/m. The differential flux $d\phi$ for a small region of thickness $dx$ and one meter length of the conductor is

$$d\phi_x = B_x dx \cdot 1 = \frac{\mu_0 I}{2\pi r^2}x dx \tag{4.9}$$

The flux $d\phi_x$ links only the fraction of the conductor from the center to radius $x$. Thus, on the assumption of uniform current density, only the fraction $\pi x^2/\pi r^2$ of the total current is linked by the flux, i.e.,

$$d\lambda_x = (\frac{x^2}{r^2})d\phi_x = \frac{\mu_0 I}{2\pi r^4}x^3 dx \tag{4.10}$$

The total flux linkage is found by integrating $d\lambda_x$ from 0 to $r$.

$$\lambda_{int} = \frac{\mu_0 I}{2\pi r^4}\int_0^r x^3 dx$$
$$= \frac{\mu_0 I}{8\pi} \text{ Wb/m} \tag{4.11}$$

From (4.3), the inductance due to the internal flux linkage is

$$L_{int} = \frac{\mu_0}{8\pi} = \frac{1}{2} \times 10^{-7} \text{ H/m} \tag{4.12}$$

Note that $L_{int}$ is independent of the conductor radius $r$.

### 4.4.2 INDUCTANCE DUE TO EXTERNAL FLUX LINKAGE

Consider $H_x$ external to the conductor at radius $x > r$ as shown in Figure 4.4. Since the circle at radius $x$ encloses the entire current, $I_x = I$ and in (4.5) $I_x$ is replaced by $I$ and the flux density at radius $x$ becomes

$$B_x = \mu_0 H_x = \frac{\mu_0 I}{2\pi x} \tag{4.13}$$

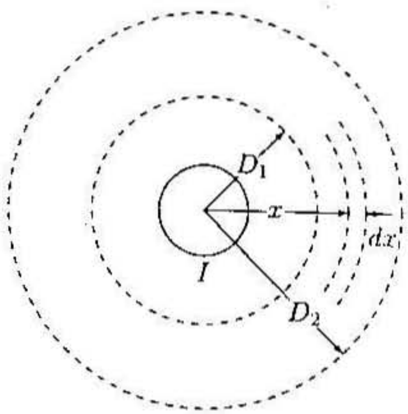



**FIGURE 4.4**
Flux linkage between $D_1$ and $D_2$.

Since the entire current $I$ is linked by the flux outside the conductor, the flux linkage $d\lambda_x$ is numerically equal to the flux $d\phi_x$. The differential flux $d\phi_x$ for a small region of thickness $dx$ and one meter length of the conductor is then given by

$$d\lambda_x = d\phi_x = B_x dx \cdot 1 = \frac{\mu_0 I}{2\pi x} dx \tag{4.14}$$

The external flux linkage between two points $D_1$ and $D_2$ is found by integrating $d\lambda_x$ from $D_1$ to $D_2$.

$$\begin{aligned} \lambda_{ext} &= \frac{\mu_0 I}{2\pi} \int_{D_1}^{D_2} \frac{1}{x} dx \\ &= 2 \times 10^{-7} I \ln \frac{D_2}{D_1} \quad \text{Wb/m} \end{aligned} \tag{4.15}$$

The inductance between two points external to a conductor is then

$$L_{ext} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \quad \text{H/m} \tag{4.16}$$





## 4.5 INDUCTANCE OF SINGLE-PHASE LINES

Consider one meter length of a single-phase line consisting of two solid round conductors of radius $r_1$ and $r_2$ as shown in Figure 4.5. The two conductors are separated by a distance $D$. Conductor 1 carries the phasor current $I_1$ referenced into the page and conductor 2 carries return current $I_2 = -I_1$. These currents set up magnetic field lines that links between the conductors as shown.

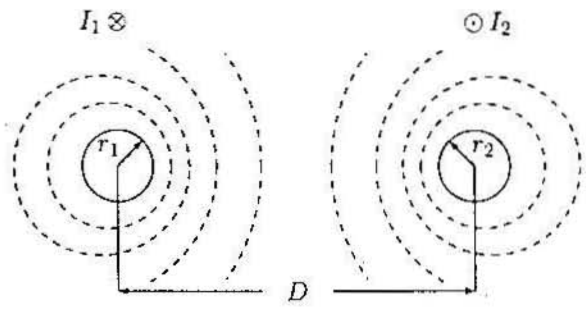



**FIGURE 4.5**
Single-phase two-wire line.

Inductance of conductor 1 due to internal flux is given by (4.12). The flux beyond $D$ links a net current of zero and does not contribute to the net magnetic flux linkages in the circuit. Thus, to obtain the inductance of conductor 1 due to the net external flux linkage, it is necessary to evaluate (4.16) from $D_1 = r_1$ to $D_2 = D$.

$$L_{1(ext)} = 2 \times 10^{-7} \ln \frac{D}{r_1} \quad \text{H/m} \tag{4.17}$$

The total inductance of conductor 1 is then

$$L_1 = \frac{1}{2} \times 10^{-7} + 2 \times 10^{-7} \ln \frac{D}{r_1} \quad \text{H/m} \tag{4.18}$$

Equation (4.18) is often rearranged as follows:

$$\begin{aligned} L_1 &= 2 \times 10^{-7} \left( \frac{1}{4} + \ln \frac{D}{r_1} \right) \\ &= 2 \times 10^{-7} \left( \ln e^{\frac{1}{4}} + \ln \frac{1}{r_1} + \ln \frac{D}{1} \right) \\ &= 2 \times 10^{-7} \left( \ln \frac{1}{r_1 e^{-1/4}} + \ln \frac{D}{1} \right) \end{aligned} \tag{4.19}$$

Let $r_1' = r_1 e^{-\frac{1}{4}}$, the inductance of conductor 1 becomes

$$L_1 = 2 \times 10^{-7} \ln \frac{1}{r_1'} + 2 \times 10^{-7} \ln \frac{D}{1} \quad \text{H/m} \tag{4.20}$$

Similarly, the inductance of conductor 2 is

$$L_2 = 2 \times 10^{-7} \ln \frac{1}{r_2'} + 2 \times 10^{-7} \ln \frac{D}{1} \quad \text{H/m} \tag{4.21}$$

If the two conductors are identical, $r_1 = r_2 = r$ and $L_1 = L_2 = L$, and the inductance per phase per meter length of the line is given by

$$L = 2 \times 10^{-7} \ln \frac{1}{r'} + 2 \times 10^{-7} \ln \frac{D}{1} \quad \text{H/m} \tag{4.22}$$

Examination of (4.22) reveals that the first term is only a function of the conductor radius. This term is considered as the inductance due to both the internal flux and that external to conductor 1 to a radius of 1 m. The second term of (4.22) is dependent only upon conductor spacing. This term is known as the *inductance spacing factor*. The above terms are usually expressed as inductive reactances at 60 Hz and are available in the manufacturers table in English units.

The term $r' = re^{-\frac{1}{4}}$ is known mathematically as the *self-geometric mean distance* of a circle with radius $r$ and is abbreviated by $GMR$. $r'$ can be considered as the radius of a fictitious conductor assumed to have no internal flux but with the same inductance as the actual conductor with radius $r$. $GMR$ is commonly referred to as *geometric mean radius* and will be designated by $D_s$. Thus, the inductance per phase in millihenries per kilometer becomes

$$L = 0.2 \ln \frac{D}{D_s} \quad \text{mH/km} \tag{4.23}$$

## 4.6 FLUX LINKAGE IN TERMS OF SELF- AND MUTUAL INDUCTANCES

The series inductance per phase for the above single-phase two-wire line can be expressed in terms of self-inductance of each conductor and their mutual inductance. Consider one meter length of the single-phase circuit represented by two coils characterized by the self-inductances $L_{11}$ and $L_{22}$ and the mutual inductance $L_{12}$. The magnetic polarity is indicated by dot symbols as shown in Figure 4.6.

The flux linkages $\lambda_1$ and $\lambda_2$ are given by

$$\begin{aligned} \lambda_1 &= L_{11} I_1 + L_{12} I_2 \\ \lambda_2 &= L_{21} I_1 + L_{22} I_2 \end{aligned} \tag{4.24}$$





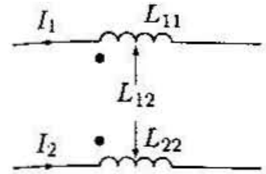



**FIGURE 4.6**
The single-phase line viewed as two magnetically coupled coils.

Since $I_2 = -I_1$, we have

$$\begin{aligned} \lambda_1 &= (L_{11} - L_{12}) I_1 \\ \lambda_2 &= (-L_{21} + L_{22}) I_2 \end{aligned} \tag{4.25}$$

Comparing (4.25) with (4.20) and (4.21), we conclude the following equivalent expressions for the self- and mutual inductances:

$$\begin{aligned} L_{11} &= 2 \times 10^{-7} \ln \frac{1}{r_1'} \\ L_{22} &= 2 \times 10^{-7} \ln \frac{1}{r_2'} \\ L_{12} &= L_{21} = 2 \times 10^{-7} \ln \frac{1}{D} \end{aligned} \tag{4.26}$$

The concept of self- and mutual inductance can be extended to a group of $n$ conductors. Consider $n$ conductors carrying phasor currents $I_1, I_2, \ldots, I_n$, such that

$$I_1 + I_2 + \cdots + I_i + \cdots + I_n = 0 \tag{4.27}$$

Generalizing (4.24), the flux linkages of conductor $i$ are

$$\lambda_i = L_{ii} I_i + \sum_{j=1}^{n} L_{ij} I_j \quad j \neq i \tag{4.28}$$

or

$$\lambda_i = 2 \times 10^{-7} \left( I_i \ln \frac{1}{r_i'} + \sum_{j=1}^{n} I_j \ln \frac{1}{D_{ij}} \right) \quad j \neq i \tag{4.29}$$

## 4.7 INDUCTANCE OF THREE-PHASE TRANSMISSION LINES

### 4.7.1 SYMMETRICAL SPACING

Consider one meter length of a three-phase line with three conductors, each with radius $r$, symmetrically spaced in a triangular configuration as shown in Figure 4.7.

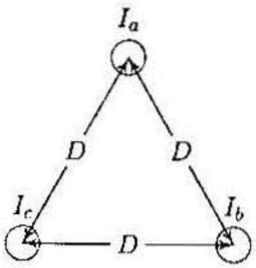



**FIGURE 4.7**
Three-phase line with symmetrical spacing.

Assuming balanced three-phase currents, we have

$$I_a + I_b + I_c = 0 \tag{4.30}$$

From (4.29) the total flux linkage of phase $a$ conductor is

$$\lambda_a = 2 \times 10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D} + I_c \ln\frac{1}{D}\right) \tag{4.31}$$

Substituting for $I_b + I_c = -I_a$

$$\begin{aligned}\lambda_a &= 2 \times 10^{-7}\left(I_a \ln\frac{1}{r'} - I_a \ln\frac{1}{D}\right)\\ &= 2 \times 10^{-7} I_a \ln\frac{D}{r'}\end{aligned} \tag{4.32}$$

Because of symmetry, $\lambda_b = \lambda_c = \lambda_a$, and the three inductances are identical. Therefore, the inductance per phase per kilometer length is

$$L = 0.2 \ln\frac{D}{D_s} \ \text{mH/km} \tag{4.33}$$

where $r'$ is the geometric mean radius, $GMR$, and is shown by $D_s$. For a solid round conductor, $D_s = re^{-\frac{1}{4}}$ for stranded conductor $D_s$ can be evaluated from (4.50). Comparison of (4.33) with (4.23) shows that inductance per phase for a three-phase circuit with equilateral spacing is the same as for one conductor of a single-phase circuit.





### 4.7.2 ASYMMETRICAL SPACING

Practical transmission lines cannot maintain symmetrical spacing of conductors because of construction considerations. With asymmetrical spacing, even with balanced currents, the voltage drop due to line inductance will be unbalanced. Consider one meter length of a three-phase line with three conductors, each with radius $r$. The Conductors are asymmetrically spaced with distances shown in Figure 4.8.

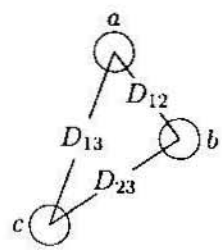



**FIGURE 4.8**
Three-phase line with asymmetrical spacing.

The application of (4.29) will result in the following flux linkages.

$$\begin{aligned}\lambda_a &= 2 \times 10^{-7}\left(I_a \ln\frac{1}{r'} + I_b \ln\frac{1}{D_{12}} + I_c \ln\frac{1}{D_{13}}\right)\\ \lambda_b &= 2 \times 10^{-7}\left(I_a \ln\frac{1}{D_{12}} + I_b \ln\frac{1}{r'} + I_c \ln\frac{1}{D_{23}}\right)\\ \lambda_c &= 2 \times 10^{-7}\left(I_a \ln\frac{1}{D_{13}} + I_b \ln\frac{1}{D_{23}} + I_c \ln\frac{1}{r'}\right)\end{aligned} \tag{4.34}$$

or in matrix form

$$\lambda = LI \tag{4.35}$$

where the symmetrical inductance matrix $L$ is given by

$$L = 2 \times 10^{-7}\begin{bmatrix} \ln\frac{1}{r'} & \ln\frac{1}{D_{12}} & \ln\frac{1}{D_{13}} \\ \ln\frac{1}{D_{12}} & \ln\frac{1}{r'} & \ln\frac{1}{D_{23}} \\ \ln\frac{1}{D_{13}} & \ln\frac{1}{D_{23}} & \ln\frac{1}{r'} \end{bmatrix} \tag{4.36}$$

For balanced three-phase currents with $I_a$ as reference, we have

$$\begin{aligned}I_b &= I_a\angle 240^\circ = a^2 I_a\\ I_c &= I_a\angle 120^\circ = aI_a\end{aligned} \tag{4.37}$$

where the operator $a = 1\angle 120°$ and $a^2 = 1\angle 240°$. Substituting in (4.34) results in

$$L_a = \frac{\lambda_a}{I_a} = 2\times 10^{-7}\left(\ln\frac{1}{r'} + a^2\ln\frac{1}{D_{12}} + a\ln\frac{1}{D_{13}}\right)$$
$$L_b = \frac{\lambda_b}{I_b} = 2\times 10^{-7}\left(a\ln\frac{1}{D_{12}} + \ln\frac{1}{r'} + a^2\ln\frac{1}{D_{23}}\right)$$
$$L_c = \frac{\lambda_c}{I_c} = 2\times 10^{-7}\left(a^2\ln\frac{1}{D_{13}} + a\ln\frac{1}{D_{23}} + \ln\frac{1}{r'}\right) \quad (4.38)$$

Examination of (4.38) shows that the phase inductances are not equal and they contain an imaginary term due to the mutual inductance.

### 4.7.3 TRANSPOSE LINE

A per-phase model of the transmission line is required in most power system analysis. One way to regain symmetry in good measure and obtain a per-phase model is to consider transposition. This consists of interchanging the phase configuration every one-third the length so that each conductor is moved to occupy the next physical position in a regular sequence. Such a transposition arrangement is shown in Figure 4.9.

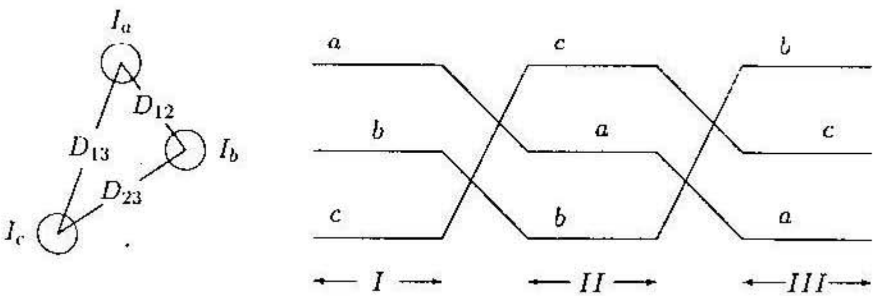



**FIGURE 4.9**
A transposed three-phase line.

Since in a transposed line each phase takes all three positions, the inductance per phase can be obtained by finding the average value of (4.38).

$$L = \frac{L_a + L_b + L_c}{3} \quad (4.39)$$

Noting $a + a^2 = 1\angle 120° + 1\angle 240° = -1$, the average of (4.38) becomes

$$L = \frac{2\times 10^{-7}}{3}\left(3\ln\frac{1}{r'} - \ln\frac{1}{D_{12}} - \ln\frac{1}{D_{23}} - \ln\frac{1}{D_{13}}\right)$$





or

$$L = 2\times 10^{-7}\left(\ln\frac{1}{r'} - \ln\frac{1}{(D_{12}D_{23}D_{13})^{\frac{1}{3}}}\right)$$
$$= 2\times 10^{-7}\ln\frac{(D_{12}D_{23}D_{13})^{\frac{1}{3}}}{r'} \quad (4.40)$$

or the inductance per phase per kilometer length is

$$L = 0.2\ln\frac{GMD}{D_S}\ \text{mH/km} \quad (4.41)$$

where

$$GMD = \sqrt[3]{D_{12}D_{23}D_{13}} \quad (4.42)$$

This again is of the same form as the expression for the inductance of one phase of a single-phase line. $GMD$ (geometric mean distance) is the equivalent conductor spacing. For the above three-phase line this is the cube root of the product of the three-phase spacings. $D_s$ is the geometric mean radius, $GMR$. For stranded conductor $D_s$ is obtained from the manufacturer's data. For solid conductor, $D_s = r' = re^{-\frac{1}{4}}$.

In modern transmission lines, transposition is not generally used. However, for the purpose of modeling, it is most practical to treat the circuit as transposed. The error introduced as a result of this assumption is very small.

## 4.8 INDUCTANCE OF COMPOSITE CONDUCTORS

In the evaluation of inductance, solid round conductors were considered. However, in practical transmission lines, stranded conductors are used. Also, for reasons of economy, most EHV lines are constructed with bundled conductors. In this section an expression is found for the inductance of composite conductors. The result can be used for evaluating the $GMR$ of stranded or bundled conductors. It is also useful in finding the equivalent $GMR$ and $GMD$ of parallel circuits. Consider a single-phase line consisting of two composite conductors $x$ and $y$ as shown in Figure 4.10. The current in $x$ is $I$ referenced into the page, and the return current in $y$ is $-I$. Conductor $x$ consists of $n$ identical strands or subconductors, each with radius $r_x$. Conductor $y$ consists of $m$ identical strands or subconductors, each with radius $r_y$. The current is assumed to be equally divided among the subconductors. The current per strand is $I/n$ in $x$ and $I/m$ in $y$. The application of (4.29) will result in the following expression for the total flux linkage of conductor $a$

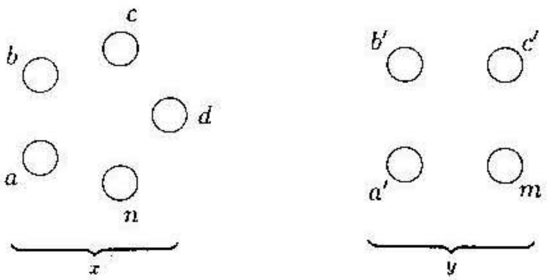



**FIGURE 4.10**
Single-phase line with two composite conductors.

$$\lambda_a = 2\times 10^{-7}\frac{I}{n}\left(\ln\frac{1}{r'_x} + \ln\frac{1}{D_{ab}} + \ln\frac{1}{D_{ac}} + \cdots + \ln\frac{1}{D_{an}}\right)$$
$$-2\times 10^{-7}\frac{I}{m}\left(\ln\frac{1}{D_{aa'}} + \ln\frac{1}{D_{ab'}} + \ln\frac{1}{D_{ac'}} + \cdots + \ln\frac{1}{D_{am}}\right)$$

or

$$\lambda_a = 2\times 10^{-7} I \ln\frac{\sqrt[m]{D_{aa'}D_{ab'}D_{ac'}\cdots D_{am}}}{\sqrt[n]{r'_x D_{ab}D_{ac}\cdots D_{an}}} \tag{4.43}$$

The inductance of subconductor $a$ is

$$L_a = \frac{\lambda_a}{I/n} = 2n\times 10^{-7}\ln\frac{\sqrt[m]{D_{aa'}D_{ab'}D_{ac'}\cdots D_{am}}}{\sqrt[n]{r'_x D_{ab}D_{ac}\cdots D_{an}}} \tag{4.44}$$

Using (4.29), the inductance of other subconductors in $x$ are similarly obtained. For example, the inductance of the subconductor $n$ is

$$L_n = \frac{\lambda_n}{I/n} = 2n\times 10^{-7}\ln\frac{\sqrt[m]{D_{na'}D_{nb'}D_{nc'}\cdots D_{nm}}}{\sqrt[n]{r'_x D_{na}D_{nb}\cdots D_{nc}}} \tag{4.45}$$

The average inductance of any one subconductor in group $x$ is

$$L_{av} = \frac{L_a + L_b + L_c + \cdots + L_n}{n} \tag{4.46}$$

Since all the subconductors of conductor $x$ are electrically parallel, the inductance of $x$ will be

$$L_x = \frac{L_{av}}{n} = \frac{L_a + L_b + L_c + \cdots + L_n}{n^2} \tag{4.47}$$

substituting the values of $L_a$, $L_b$, $L_c, \cdots, L_n$ in (4.47) results in

$$L_x = 2\times 10^{-7}\ln\frac{GMD}{GMR_x}\ \text{H/meter} \tag{4.48}$$





where

$$GMD = \sqrt[mn]{(D_{aa'}D_{ab'}\cdots D_{am})\cdots(D_{na'}D_{nb'}\cdots D_{nm})} \tag{4.49}$$

and

$$GMR_x = \sqrt[n^2]{(D_{aa}D_{ab}\cdots D_{an})\cdots(D_{na}D_{nb}\cdots D_{nn})} \tag{4.50}$$

where $D_{aa} = D_{bb}\cdots = D_{nn} = r'_x$

$GMD$ is the $mn$th root of the product of the $mn$th distances between $n$ strands of conductor $x$ and $m$ strands of conductor $y$. $GMR_x$ is the $n^2$ root of the product of $n^2$ terms consisting of $r'$ of every strand times the distance from each strand to all other strands within group $x$.

The inductance of conductor $y$ can also be similarly obtained. The geometric mean radius $GMR_y$ will be different. The geometric mean distance $GMD$, however, is the same.

### Example 4.1

A stranded conductor consists of seven identical strands each having a radius $r$ as shown in Figure 4.11. Determine the $GMR$ of the conductor in terms of $r$.

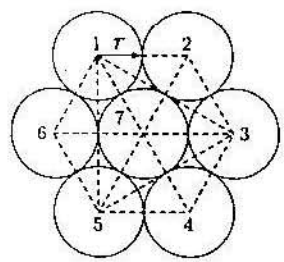



**FIGURE 4.11**
Cross section of a stranded conductor.

From Figure 4.11, the distance from strand 1 to all other strands is:

$$D_{12} = D_{16} = D_{17} = 2r$$
$$D_{14} = 4r$$
$$D_{13} = D_{15} = \sqrt{D_{14}^2 - D_{45}^2} = 2\sqrt{3}\,r$$

From (4.50) the $GMR$ of the above conductor is

$$GMR = \sqrt[49]{(r'\cdot 2r\cdot 2\sqrt{3}\,r\cdot 4r\cdot 2\sqrt{3}\,r\cdot 2r\cdot 2r)^6\cdot r'(2r)^6}$$

$$= r\sqrt[7]{(e)^{-\frac{1}{4}}(2)^6(3)^{\frac{6}{7}}(2)^{\frac{6}{7}}}$$
$$= 2.1767r$$

With a large number of strands the calculation of $GMR$ can become very tedious. Usually these are available in the manufacturer's data.

### 4.8.1 *GMR* OF BUNDLED CONDUCTORS

Extra-high voltage transmission lines are usually constructed with bundled conductors. Bundling reduces the line reactance, which improves the line performance and increases the power capability of the line. Bundling also reduces the voltage surface gradient, which in turn reduces corona loss, radio interference, and surge impedance. Typically, bundled conductors consist of two, three, or four subconductors symmetrically arranged in configuration as shown in Figure 4.12. The subconductors within a bundle are separated at frequent intervals by spacer-dampers. Spacer-dampers prevent clashing, provide damping, and connect the subconductors in parallel.

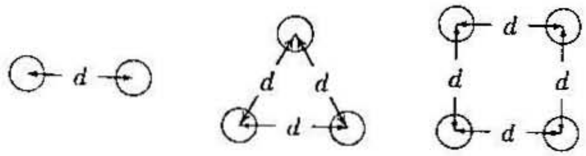



**FIGURE 4.12**
Examples of bundled arrangements.

The $GMR$ of the equivalent single conductor is obtained by using (4.50). If $D_s$ is the $GMR$ of each subconductor and $d$ is the bundle spacing, we have

for the two-subconductor bundle

$$D_s^b = \sqrt[4]{(D_s \times d)^2} = \sqrt{D_s \times d} \tag{4.51}$$

for the three-subconductor bundle

$$D_s^b = \sqrt[9]{(D_s \times d \times d)^3} = \sqrt[3]{D_s \times d^2} \tag{4.52}$$

for the four-subconductor bundle

$$D_s^b = \sqrt[16]{(D_s \times d \times d \times d \times 2^{\frac{1}{2}})^4} = 1.09\sqrt[4]{D_s \times d^3} \tag{4.53}$$





## 4.9 INDUCTANCE OF THREE-PHASE DOUBLE-CIRCUIT LINES

A three-phase double-circuit line consists of two identical three-phase circuits. The circuits are operated with $a_1$–$a_2$, $b_1$–$b_2$, and $c_1$–$c_2$ in parallel. Because of geometrical differences between conductors, voltage drop due to line inductance will be unbalanced. To achieve balance, each phase conductor must be transposed within its group and with respect to the parallel three-phase line. Consider a three-phase double-circuit line with relative phase positions $a_1b_1c_1$–$c_2b_2a_2$, as shown in Figure 4.13.

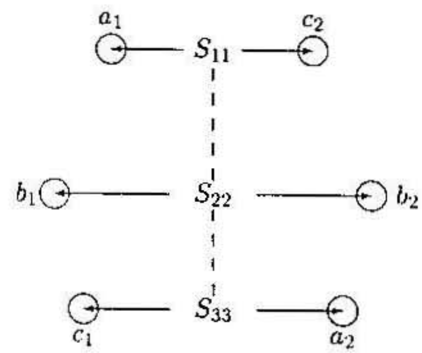



**FIGURE 4.13**
Transposed double-circuit line.

The method of $GMD$ can be used to find the inductance per phase. To do this, we group identical phases together and use (4.49) to find the $GMD$ between each phase group

$$D_{AB} = \sqrt[4]{D_{a_1b_1}D_{a_1b_2}D_{a_2b_1}D_{a_2b_2}}$$
$$D_{BC} = \sqrt[4]{D_{b_1c_1}D_{b_1c_2}D_{b_2c_1}D_{b_2c_2}}$$
$$D_{AC} = \sqrt[4]{D_{a_1c_1}D_{a_1c_2}D_{a_2c_1}D_{a_2c_2}} \tag{4.54}$$

The equivalent $GMD$ per phase is then

$$GMD = \sqrt[3]{D_{AB}D_{BC}D_{AC}} \tag{4.55}$$

Similarly, from (4.50), the $GMR$ of each phase group is

$$D_{SA} = \sqrt[4]{(D_s^b D_{a_1a_2})^2} = \sqrt{D_s^b D_{a_1a_2}}$$
$$D_{SB} = \sqrt[4]{(D_s^b D_{b_1b_2})^2} = \sqrt{D_s^b D_{b_1b_2}}$$
$$D_{SC} = \sqrt[4]{(D_s^b D_{c_1c_2})^2} = \sqrt{D_s^b D_{c_1c_2}} \tag{4.56}$$

where $D_s^b$ is the geometric mean radius of the bundled conductors given by (4.51)–(4.53). The equivalent geometric mean radius for calculating the per-phase inductance to neutral is

$$GMR_L = \sqrt[3]{D_{SA}D_{SB}D_{SC}} \tag{4.57}$$

The inductance per phase in millihenries per kilometer is

$$L = 0.2 \ln \frac{GMD}{GMR_L} \quad \text{mH/km} \tag{4.58}$$

## 4.10 LINE CAPACITANCE

Transmission line conductors exhibit capacitance with respect to each other due to the potential difference between them. The amount of capacitance between conductors is a function of conductor size, spacing, and height above ground. By definition, the capacitance $C$ is the ratio of charge $q$ to the voltage $V$, given by

$$C = \frac{q}{V} \tag{4.59}$$

Consider a long round conductor with radius $r$, carrying a charge of $q$ coulombs per meter length as shown in Figure 4.14.

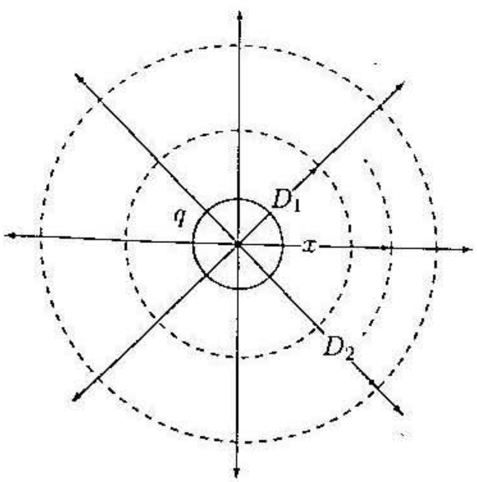



**FIGURE 4.14**
Electric field around a long round conductor.

The charge on the conductor gives rise to an electric field with radial flux lines. The total electric flux is numerically equal to the value of charge on the conductor. The intensity of the field at any point is defined as the force per unit charge and is termed *electric field intensity* designated as $E$. Concentric cylinders surrounding the conductor are equipotential surfaces and have the same electric flux density. From Gauss's law, for one meter length of the conductor, the electric flux density at a cylinder of radius $x$ is given by





$$D = \frac{q}{A} = \frac{q}{2\pi x(1)} \tag{4.60}$$

The electric field intensity E may be found from the relation

$$E = \frac{D}{\varepsilon_0} \tag{4.61}$$

where $\varepsilon_0$ is the permittivity of free space and is equal to $8.85 \times 10^{-12}$ F/m. Substituting (4.60) in (4.61) results in

$$E = \frac{q}{2\pi\varepsilon_0 x} \tag{4.62}$$

The potential difference between cylinders from position $D_1$ to $D_2$ is defined as the work done in moving a unit charge of one coulomb from $D_2$ to $D_1$ through the electric field produced by the charge on the conductor. This is given by

$$V_{12} = \int_{D_1}^{D_2} E\,dx = \int_{D_1}^{D_2} \frac{q}{2\pi\varepsilon_0 x} dx = \frac{q}{2\pi\varepsilon_0} \ln \frac{D_2}{D_1} \tag{4.63}$$

The notation $V_{12}$ implies the voltage drop from 1 relative to 2, that is, 1 is understood to be positive relative to 2. The charge $q$ carries its own sign.

## 4.11 CAPACITANCE OF SINGLE-PHASE LINES

Consider one meter length of a single-phase line consisting of two long solid round conductors each having a radius $r$ as shown in Figure 4.15. The two conductors are separated by a distance $D$. Conductor 1 carries a charge of $q_1$ coulombs/meter and conductor 2 carries a charge of $q_2$ coulombs/meter. The presence of the second conductor and ground disturbs the field of the first conductor. The distance of separation of the wires $D$ is great with respect to $r$ and the height of conductors is much larger compared with $D$. Therefore, the distortion effect is small and the charge is assumed to be uniformly distributed on the surface of the conductors.

Assuming conductor 1 alone to have a charge of $q_1$, the voltage between conductor 1 and 2 is

$$V_{12(q_1)} = \frac{q_1}{2\pi\varepsilon_0} \ln \frac{D}{r} \tag{4.64}$$

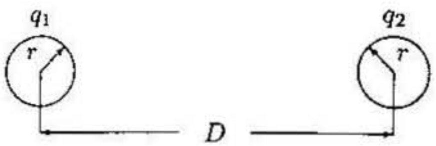



**FIGURE 4.15**
Single-phase two-wire line.

Now assuming only conductor 2, having a charge of $q_2$, the voltage between conductors 2 and 1 is

$$V_{21(q_2)} = \frac{q_2}{2\pi\varepsilon_0} \ln \frac{D}{r}$$

Since $V_{12(q_2)} = -V_{21(q_2)}$, we have

$$V_{12(q_2)} = \frac{q_2}{2\pi\varepsilon_0} \ln \frac{r}{D} \tag{4.65}$$

From the principle of superposition, the potential difference due to presence of both charges is

$$V_{12} = V_{12(q_1)} + V_{12(q_2)} = \frac{q_1}{2\pi\varepsilon_0} \ln \frac{D}{r} + \frac{q_2}{2\pi\varepsilon_0} \ln \frac{r}{D} \tag{4.66}$$

For a single-phase line $q_2 = -q_1 = -q$, and (4.66) reduces to

$$V_{12} = \frac{q}{\pi\varepsilon_0} \ln \frac{D}{r} \quad \text{F/m} \tag{4.67}$$

From (4.59), the capacitance between conductors is

$$C_{12} = \frac{\pi\varepsilon_0}{\ln \frac{D}{r}} \quad \text{F/m} \tag{4.68}$$

Equation (4.68) gives the line-to-line capacitance between the conductors. For the purpose of transmission line modeling, we find it convenient to define a capacitance $C$ between each conductor and a neutral as illustrated in Figure 4.16. Since the voltage to neutral is half of $V_{12}$, the capacitance to neutral $C = 2C_{12}$, or

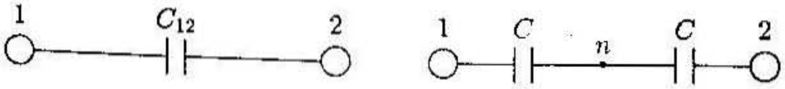



**FIGURE 4.16**
Illustration of capacitance to neutral.





$$C = \frac{2\pi\varepsilon_0}{\ln \frac{D}{r}} \quad \text{F/m} \tag{4.69}$$

Recalling $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m and converting to $\mu$F per kilometer, we have

$$C = \frac{0.0556}{\ln \frac{D}{r}} \quad \mu\text{F/km} \tag{4.70}$$

The capacitance per phase contains terms analogous to those derived for inductance per phase. However, unlike inductance where the conductor geometric mean radius ($GMR$) is used, in capacitance formula the actual conductor radius $r$ is used.

## 4.12 POTENTIAL DIFFERENCE IN A MULTICONDUCTOR CONFIGURATION

Consider $n$ parallel long conductors with charges $q_1, q_2, \ldots, q_n$ coulombs/meter as shown in Figure 4.17.

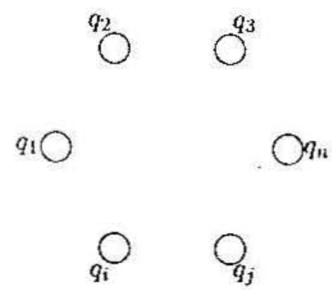



**FIGURE 4.17**
Multiconductor configuration.

Assume that the distortion effect is negligible and the charge is uniformly distributed around the conductor, with the following constraint

$$q_1 + q_2 + \cdots + q_n = 0 \tag{4.71}$$

Using superposition and (4.63), potential difference between conductors $i$ and $j$ due to the presence of all charges is

$$V_{ij} = \frac{1}{2\pi\varepsilon_0} \sum_{k=1}^{n} q_k \ln \frac{D_{kj}}{D_{ki}} \tag{4.72}$$

When $k = i$, $D_{ii}$ is the distance between the surface of the conductor and its center, namely its radius $r$.

## 4.13 CAPACITANCE OF THREE-PHASE LINES

Consider one meter length of a three-phase line with three long conductors, each with radius $r$, with conductor spacing as shown Figure 4.18.

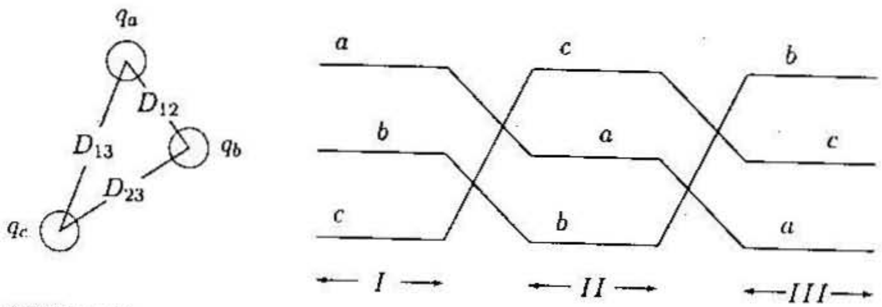



**FIGURE 4.18**
Three-phase transmission line.

Since we have a balanced three-phase system

$$q_a + q_b + q_c = 0 \tag{4.73}$$

We shall neglect the effect of ground and the shield wires. Assume that the line is transposed. We proceed with the calculation of the potential difference between $a$ and $b$ for each section of transposition. Applying (4.72) to the first section of the transposition, $V_{ab}$ is

$$V_{ab(I)} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{D_{12}}{r} + q_b \ln\frac{r}{D_{12}} + q_c \ln\frac{D_{23}}{D_{13}}\right) \tag{4.74}$$

Similarly, for the second section of the transposition, we have

$$V_{ab(II)} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{D_{23}}{r} + q_b \ln\frac{r}{D_{23}} + q_c \ln\frac{D_{13}}{D_{12}}\right) \tag{4.75}$$

and for the last section

$$V_{ab(III)} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{D_{13}}{r} + q_b \ln\frac{r}{D_{13}} + q_c \ln\frac{D_{12}}{D_{23}}\right) \tag{4.76}$$

The average value of $V_{ab}$ is

$$V_{ab} = \frac{1}{(3)2\pi\varepsilon_0}\left(q_a \ln\frac{D_{12}D_{23}D_{13}}{r^3} + q_b \ln\frac{r^3}{D_{12}D_{23}D_{13}} + q_c \ln\frac{D_{12}D_{23}D_{13}}{D_{12}D_{23}D_{13}}\right) \tag{4.77}$$





or

$$V_{ab} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{(D_{12}D_{23}D_{13})^{\frac{1}{3}}}{r} + q_b \ln\frac{r}{(D_{12}D_{23}D_{13})^{\frac{1}{3}}}\right) \tag{4.78}$$

Note that the $GMD$ of the conductor appears in the logarithm arguments and is given by

$$GMD = \sqrt[3]{D_{12}D_{23}D_{13}} \tag{4.79}$$

Therefore, $V_{ab}$ is

$$V_{ab} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{GMD}{r} + q_b \ln\frac{r}{GMD}\right) \tag{4.80}$$

Similarly, we find the average voltage $V_{ac}$ as

$$V_{ac} = \frac{1}{2\pi\varepsilon_0}\left(q_a \ln\frac{GMD}{r} + q_c \ln\frac{r}{GMD}\right) \tag{4.81}$$

Adding (4.80) and (4.81) and substituting for $q_b + q_c = -q_a$, we have

$$V_{ab} + V_{ac} = \frac{1}{2\pi\varepsilon}\left(2q_a \ln\frac{GMD}{r} - q_a \ln\frac{r}{GMD}\right) = \frac{3q_a}{2\pi\varepsilon_0}\ln\frac{GMD}{r} \tag{4.82}$$

For balanced three-phase voltages,

$$\begin{aligned} V_{ab} &= V_{an}\angle 0^\circ - V_{an}\angle -120^\circ \\ V_{ac} &= V_{an}\angle 0^\circ - V_{an}\angle -240^\circ \end{aligned} \tag{4.83}$$

Therefore,

$$V_{ab} + V_{ac} = 3V_{an} \tag{4.84}$$

Substituting in (4.82) the capacitance per phase to neutral is

$$C = \frac{q_a}{V_{an}} = \frac{2\pi\varepsilon_0}{\ln\frac{GMD}{r}} \quad \text{F/m} \tag{4.85}$$

or capacitance to neutral in $\mu$F per kilometer is

$$C = \frac{0.0556}{\ln\frac{GMD}{r}} \quad \mu\text{F/km} \tag{4.86}$$

This is of the same form as the expression for the capacitance of one phase of a single-phase line. $GMD$ (geometric mean distance) is the equivalent conductor spacing. For the above three-phase line this is the cube root of the product of the three-phase spacings.

## 4.14 EFFECT OF BUNDLING

The procedure for finding the capacitance per phase for a three-phase transposed line with bundle conductors follows the same steps as the procedure in Section 3.13. The capacitance per phase is found to be

$$C = \frac{2\pi\varepsilon_0}{\ln \frac{GMD}{r^b}} \text{ F/m} \tag{4.87}$$

The effect of bundling is to introduce an equivalent radius $r^b$. The equivalent radius $r^b$ is similar to the $GMR$ (geometric mean radius) calculated earlier for the inductance with the exception that radius $r$ of each subconductor is used instead of $D_s$. If $d$ is the bundle spacing, we obtain for the two-subconductor bundle

$$r^b = \sqrt{r \times d} \tag{4.88}$$

for the three-subconductor bundle

$$r^b = \sqrt[3]{r \times d^2} \tag{4.89}$$

for the four-subconductor bundle

$$r^b = 1.09\sqrt[4]{r \times d^3} \tag{4.90}$$

## 4.15 CAPACITANCE OF THREE-PHASE DOUBLE-CIRCUIT LINES

Consider a three-phase double-circuit line with relative phase positions $a_1b_1c_1$ - $c_2b_2a_2$, as shown in Figure 4.13. Each phase conductor is transposed within its group and with respect to the parallel three-phase line. The effect of shield wires and the ground are considered to be negligible for this balanced condition. Following the procedure of section 4.13, the average voltages $V_{ab}$, $V_{ac}$ and $V_{an}$ are calculated and the per-phase equivalent capacitance to neutral is obtained to be

$$C = \frac{2\pi\varepsilon_0}{\ln \frac{GMD}{GMR_c}} \text{ F/m} \tag{4.91}$$

or capacitance to neutral in $\mu$F per kilometer is

$$C = \frac{0.0556}{\ln \frac{GMD}{GMR_c}} \ \mu\text{F/km} \tag{4.92}$$

The expression for $GMD$ is the same as was found for inductance calculation and is given by (4.55). The $GMR_c$ of each phase group is similar to the $GMR_L$, with





the exception that in (4.56) $r^b$ is used instead of $D_s^b$. This will result in the following equations

$$r_A = \sqrt{r^b D_{a_1a_2}}$$
$$r_B = \sqrt{r^b D_{b_1b_2}}$$
$$r_C = \sqrt{r^b D_{c_1c_2}} \tag{4.93}$$

where $r^b$ is the geometric mean radius of the bundled conductors given by (4.88) – (4.90). The equivalent geometric mean radius for calculating the per-phase capacitance to neutral is

$$GMR_C = \sqrt[3]{r_A r_B r_C} \tag{4.94}$$

## 4.16 EFFECT OF EARTH ON THE CAPACITANCE

For an isolated charged conductor the electric flux lines are radial and are orthogonal to the cylindrical equipotential surfaces. The presence of earth will alter the distribution of electric flux lines and equipotential surfaces, which will change the effective capacitance of the line.

The earth level is an equipotential surface, therefore the flux lines are forced to cut the surface of the earth orthogonally. The effect of the presence of earth can be accounted for by the method of *image charges* introduced by Kelvin. To illustrate this method, consider a conductor with a charge $q$ coulombs/meter at a height $H$ above ground. Also, imagine a charge $-q$ placed at a depth $H$ below the surface of earth. This configuration without the presence of the earth surface will produce the same field distribution as a single charge and the earth surface. Thus, the earth can be replaced for the calculation of electric field potential by a fictitious charged conductor with charge equal and opposite to the charge on the actual conductor and at a depth below the surface of the earth the same as the height of the actual conductor above earth. This imaginary conductor is called the image of the actual conductor. The procedure of Section 4.13 can now be used for the computation of the capacitance.

The effect of the earth is to increase the capacitance. But normally the height of the conductor is large as compared to the distance between the conductors, and the earth effect is negligible. Therefore, for all line models used for balanced steady-state analysis, the effect of earth on the capacitance can be neglected. However, for unbalanced analysis such as unbalanced faults, the earth's effect as well as the shield wires should be considered.

### Example 4.2

A 500-kV three-phase transposed line is composed of one $ACSR$ $1,272,000$-cmil, $45/7$ Bittern conductor per phase with horizontal conductor configuration as shown in Figure 4.19. The conductors have a diameter of 1.345 in and a $GMR$ of 0.5328 in. Find the inductance and capacitance per phase per kilometer of the line.

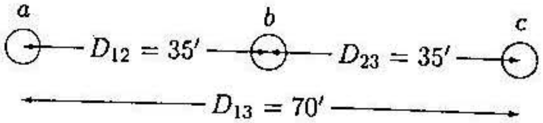



**FIGURE 4.19**
Conductor layout for Example 4.2.

Conductor radius is $r = \frac{1.345}{2\times 12} = 0.056$ ft, and $GMR_L = 0.5328/12 = 0.0444$ ft. $GMD$ is obtained using (4.42)

$$GMD = \sqrt[3]{35 \times 35 \times 70} = 44.097 \quad \text{ft}$$

From (4.58) the inductance per phase is

$$L = 0.2 \ln \frac{44.097}{0.0444} = 1.38 \quad \text{mH/km}$$

and from (4.92) the capacitance per phase is

$$C = \frac{0.0556}{\ln \frac{44.097}{0.056}} = 0.0083 \quad \mu\text{F/km}$$

### Example 4.3

The line in Example 4.2 is replaced by two $ACSR$ $636,000$-cmil, $24/7$ Rook conductors which have the same total cross-sectional area of aluminum as one Bittern conductor. The line spacing as measured from the center of the bundle is the same as before and is shown in Figure 4.20.

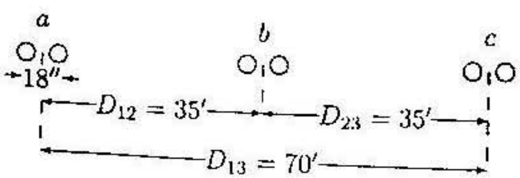



**FIGURE 4.20**
Conductor layout for Example 4.3.





The conductors have a diameter of 0.977 in and a $GMR$ of 0.3924 in. Bundle spacing is 18 in. Find the inductance and capacitance per phase per kilometer of the line and compare it with that of Example 4.2.

Conductor radius is $r = \frac{0.977}{2} = 0.4885$ in, and from Example 4.2 $GMD = 44.097$ ft. The equivalent geometric mean radius with two conductors per bundle, for calculating inductance and capacitance, are given by (4.51) and (4.88)

$$GMR_L = \frac{\sqrt{d \times D_s}}{12} = \frac{\sqrt{18 \times 0.3924}}{12} = 0.22147 \quad \text{ft}$$

and

$$GMR_c = \frac{\sqrt{d \times r}}{12} = \frac{\sqrt{18 \times 0.4885}}{12} = 0.2471 \quad \text{ft}$$

From (4.58) the inductance per phase is

$$L = 0.2 \ln \frac{44.097}{0.22147} = 1.0588 \quad \text{mH/km}$$

and from (4.92) the capacitance per phase is

$$C = \frac{0.0556}{\ln \frac{44.097}{0.2471}} = 0.0107 \quad \mu\text{F/km}$$

Comparing with the results of Example 4.2, there is a 23.3 percent reduction in the inductance and a 28.9 percent increase in the capacitance.

The function **[GMD, GMRL, GMRC] = gmd** is developed for the computation of $GMD$, $GMR_L$, and $GMR_C$ for single-circuit, double-circuit vertical, and horizontal transposed lines with up to four bundled conductors. A menu is displayed for the selection of any of the above three circuits. The user is prompted to input the phase spacing, number of bundled conductors and their spacing, conductor diameter, and the $GMR$ of the individual conductor. The specifications for some common $ACSR$ conductors are contained in a file named **acsr.m**. The command **acsr** will display the characteristics of $ACSR$ conductors. Also, the function **[L, C] = gmd2lc** in addition to the geometric mean values returns the inductance in mH per km and the capacitance in $\mu$F per km.

### Example 4.4

A 735-kV three-phase transposed line is composed of four $ACSR$, $954,000$-cmil, $45/7$ Rail conductors per phase with horizontal conductor configuration as shown in Figure 4.21. Bundle spacing is 46 cm. Use **acsr** in *MATLAB* to obtain the conductor size and the electrical characteristics for the Rail conductor. Find the inductance and capacitance per phase per kilometer of the line.

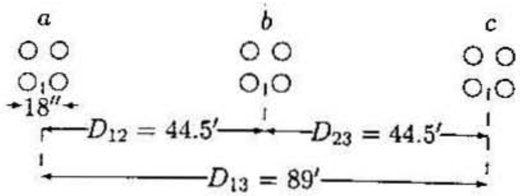



**FIGURE 4.21**
Conductor layout for Example 4.4.

The command **acsr** displays the conductor code name and the area in cmils for the ACSR conductors. The user is then prompted to enter the conductor code name within single quotes.

```
Enter ACSR code name within single quotes -> 'rail'
Al Area Strand  Diameter GMR  Resistance Ohm/km  Ampacity
 cmil   Al/St     cm      cm   60Hz 25C  60Hz 50C  Ampere
954000  45/7    2.959   1.173  0.0624    0.0683    1000
```

The following commands

```
[GMD, GMRL, GMRC] = gmd;
L=0.2*log(GMD/GMRL)          % mH/km        Eq. (4.58)
C = 0.0556/log(GMD/GMRC)     % micro F/km   Eq. (4.92)
```

result in

```
Number of three-phase circuits                  Enter
Single-circuit                                    1
Double-circuit vertical configuration             2
Double-circuit horizontal configuration           3
To quit                                           0

Select number of menu → 1
Enter spacing unit within quotes 'm' or 'ft' → 'ft'
Enter row vector [D12, D23, D13] = [44.5  44.5  89]
Cond. size, bundle spacing unit: 'cm' or 'in' → 'cm'
Conductor diameter in cm = 2.959
Geometric Mean Radius in cm = 1.173
No. of bundled cond. (enter 1 for single cond.) = 4
Bundle spacing in cm = 46
GMD = 56.06649 ft
GMRL = 0.65767 ft      GMRC = 0.69696 ft
L = 0.8891
C = 0.0127
```





### Example 4.5

A 345-kV double-circuit three-phase transposed line is composed of two $ACSR$, 1,431,000-cmil, 45/7 Bobolink conductors per phase with vertical conductor configuration as shown in Figure 4.22. The conductors have a diameter of 1.427 in and a $GMR$ of 0.564 in. The bundle spacing in 18 in. Find the inductance and capacitance per phase per kilometer of the line. The following commands

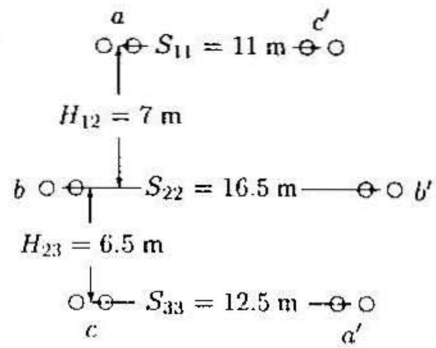



**FIGURE 4.22**
Conductor layout for Example 4.5.

```
[GMD, GMRL, GMRC] = gmd;
L=0.2*log(GMD/GMRL)          % mH/km        Eq. (4.58)
C = 0.0556/log(GMD/GMRC)     % micro F/km   Eq. (4.92)
```

result in

```
Number of three-phase circuits                  Enter
Single-circuit                                    1
Double-circuit vertical configuration             2
Double-circuit horizontal configuration           3
To quit                                           0

Select number of menu → 2

Circuit Arrangements
(1) abc-c'b'a'
(2) abc-a'b'c'

Enter (1 or 2) → 1
Enter spacing unit within quotes 'm' or 'ft' → 'm'
Enter row vector [S11, S22, S33] = [11  16.5  12.5]
Enter row vector [H12, H23] = [7  6.5]
Cond. size, bundle spacing unit: 'cm' or 'in' → 'in'
```

```
Conductor diameter in inch = 1.427
Geometric Mean Radius in inch = 0.564
No. of bundled cond. (enter 1 for single cond.) = 2
Bundle spacing in inch = 18
GMD = 11.21352 m
GMRL = 1.18731 m        GMRC = 1.25920 m
L = 0.4491
C = 0.0254
```

**Example 4.6**

A 345-kV double-circuit three-phase transposed line is composed of one $ACSR$, 556, 500-cmil, 26/7 Dove conductor per phase with horizontal conductor configuration as shown in Figure 4.23. The conductors have a diameter of 0.927 in and a $GMR$ of 0.3768 in. Bundle spacing is 18 in. Find the inductance and capacitance per phase per kilometer of the line. The following commands

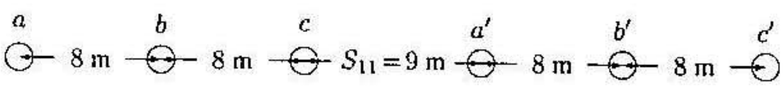



**FIGURE 4.23**
Conductor layout for Example 4.6.

```
[GMD, GMRL, GMRC] = gmd;
L=0.2*log(GMD/GMRL)           % mH/km        Eq. (4.58)
C = 0.0556/log(GMD/GMRC)      % micro F/km   Eq. (4.92)
```

result in

```
  Number of three-phase circuits              Enter
  Single-circuit                                1
  Double-circuit vertical configuration         2
  Double-circuit horizontal configuration       3
  To quit                                       0
Select number of menu → 3

Circuit Arrangements
(1) abc-a'b'c'
(2) abc-c'b'a'

Enter (1 or 2) → 1
Enter spacing unit within quotes 'm' or 'ft' → 'm'
Enter row vector [D12, D23, S13] = [8  8  16]
Enter distance between two circuits, S11 = 9
```





```
Cond. size, bundle spacing unit: 'cm' or 'in' → 'in'
Conductor diameter in inch = 0.927
Geometric Mean Radius in inch = 0.3768
No. of bundled cond. (enter 1 for single cond.) = 1
GMD = 14.92093 m
GMRL = 0.48915 m        GMRC = 0.54251 m
L = 0.6836
C = 0.0168
```

## 4.17 MAGNETIC FIELD INDUCTION

Transmission line magnetic fields affect objects in the proximity of the line. The magnetic fields, related to the currents in the line, induces voltage in objects that have a considerable length parallel to the line, such as fences, pipelines, and telephone wires.

The magnetic field is affected by the presence of earth return currents. Carson [14] presents an equation for computation of mutual resistance and inductance which are functions of the earth's resistivity. For balanced three-phase systems the total earth return current is zero. Under normal operating conditions, the magnetic field in proximity to balanced three-phase lines may be calculated considering the currents in the conductors and neglecting earth currents.

Magnetic fields have been reported to affect blood composition, growth, behavior, immune systems, and neural functions. There are general concerns regarding the biological effects of electromagnetic and electrostatic fields on people. Long-term effects are the subject of several worldwide research efforts.

**Example 4.7**

A three-phase untransposed transmission line and a telephone line are supported on the same towers as shown in Figure 4.24. The power line carries a 60-Hz balanced current of 200 A per phase. The telephone line is located directly below phase $b$. Assuming balanced three-phase currents in the power line, find the voltage per kilometer induced in the telephone line.

From (4.15) the flux linkage between conductors 1 and 2 due to current $I_a$ is

$$\lambda_{12(I_a)} = 0.2 I_a \ln \frac{D_{a2}}{D_{a1}} \quad \text{mWb/km}$$

Since $D_{b1} = D_{b2}$, $\lambda_{12}$ due to $I_b$ is zero. The flux linkage between conductors 1 and 2 due to current $I_c$ is

$$\lambda_{12(I_c)} = 0.2 I_c \ln \frac{D_{c2}}{D_{c1}} \quad \text{mWb/km}$$

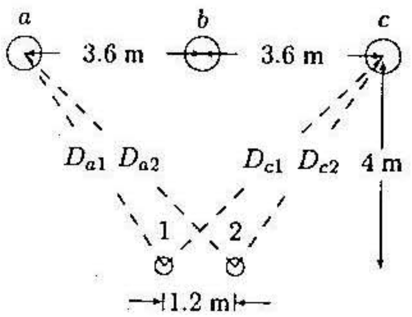



**FIGURE 4.24**
Conductor layout for Example 4.6.

Total flux linkage between conductors 1 and 2 due to all currents is

$$\lambda_{12} = 0.2 I_a \ln \frac{D_{a2}}{D_{a1}} + 0.2 I_c \ln \frac{D_{c2}}{D_{c1}} \quad \text{mWb/km}$$

For positive phase sequence, with $I_a$ as reference, $I_c = I_a \angle -240°$ and we have

$$\lambda_{12} = 0.2 I_a \left( \ln \frac{D_{a2}}{D_{a1}} + 1\angle -240° \ln \frac{D_{c2}}{D_{c1}} \right) \quad \text{mH/km}$$

With $I_a$ as reference, the instantaneous flux linkage is

$$\lambda_{12}(t) = \sqrt{2}\,|\lambda_{12}| \cos(\omega t + \alpha)$$

Thus, the induced voltage in the telephone line per kilometer length is

$$v = \frac{d\lambda_{12}(t)}{dt} = \sqrt{2}\,\omega |\lambda_{12}| \cos(\omega t + \alpha + 90°)$$

The rms voltage induced in the telephone line per kilometer is

$$V = \omega |\lambda_{12}| \angle \alpha + 90° = j\omega\lambda_{12}$$

From the circuits geometry

$$D_{a1} = D_{c2} = (3^2 + 4^2)^{\frac{1}{2}} = 5 \text{ m}$$
$$D_{a2} = D_{c1} = (4.2^2 + 4^2)^{\frac{1}{2}} = 5.8 \text{ m}$$

The total flux linkage is

$$\begin{aligned} \lambda_{12} &= 0.2 \times 200\angle 0° \ln \frac{5.8}{5} + 0.2 \times 200\angle -240° \ln \frac{5}{5.8} \\ &= 10.283\angle -30° \quad \text{mWb/km} \end{aligned}$$

The voltage induced in the telephone line per kilometer is

$$V = j\omega\lambda_{12} = j2\pi 60 (10.283\angle -30°)(10^{-3}) = 3.88\angle 60° \quad \text{V/km}$$





## 4.18 ELECTROSTATIC INDUCTION

Transmission line electric fields affect objects in the proximity of the line. The electric field produced by high voltage lines induces current in objects which are in the area of the electric fields. The effects of electric fields becomes of increasing concern at higher voltages. Electric fields, related to the voltage of the line, are the primary cause of induction to vehicles, buildings, and objects of comparable size. The human body is affected with exposure to electric discharges from charged objects in the field of the line. These may be steady current or spark discharges. The current densities in humans induced by electric fields of transmission lines are known to be much higher than those induced by magnetic fields.

The resultant electric field in proximity to a transmission line can be obtained by representing the earth effect by image charges located below the conductors at a depth equal to the conductor height.

## 4.19 CORONA

When the surface potential gradient of a conductor exceeds the dielectric strength of the surrounding air, ionization occurs in the area close to the conductor surface. This partial ionization is known as *corona*. The dielectric strength of air during fair weather and at NTP (25°C and 76 cm of Hg) is about 30 kV/cm.

Corona produces power loss, audible hissing sound in the vicinity of the line, ozone and radio and television interference. The audible noise is an environmental concern and occurs in foul weather. Radio interference occurs in the AM band. Rain and snow may produce moderate TVI in a low signal area. Corona is a function of conductor diameter, line configuration, type of conductor, and condition of its surface. Atmospheric conditions such as air density, humidity, and wind influence the generation of corona. Corona losses in rain or snow are many times the losses during fair weather. On a conductor surface, an irregularity such as a contaminating particle causes a voltage gradient that may become the point source of a discharge. Also, insulators are contaminated by dust or chemical deposits which will lower the disruptive voltage and increase the corona loss. The insulators are cleaned periodically to reduce the extent of the problem. Corona can be reduced by increasing the conductor size and the use of conductor bundling.

The power loss associated with corona can be represented by shunt conductance. However, under normal operating conditions $g$, which represents the resistive leakage between a phase and ground, has negligible effect on performance and is customarily neglected. (i.e., $g = 0$).

## PROBLEMS

**4.1.** A solid cylindrical aluminum conductor 25 km long has an area of 336,400 circular mils. Obtain the conductor resistance at (a) 20°C and (b) 50°C. The resistivity of aluminum at 20°C is $2.8 \times 10^{-8}$ Ω-m.

**4.2.** A transmission-line cable consists of 12 identical strands of aluminum, each 3 mm in diameter. The resistivity of aluminum strand at 20°C is $2.8 \times 10^{-8}$ Ω-m. Find the 50°C ac resistance per km of the cable. Assume a skin-effect correction factor of 1.02 at 60 Hz.

**4.3.** A three-phase transmission line is designed to deliver 190.5 MVA at 220 kV over a distance of 63 km. The total transmission line loss is not to exceed 2.5 percent of the rated line MVA. If the resistivity of the conductor material is $2.84 \times 10^{-8}$ Ω-m, determine the required conductor diameter and the conductor size in circular mils.

**4.4.** A single-phase transmission line 35 km long consists of two solid round conductors, each having a diameter of 0.9 cm. The conductor spacing is 2.5 m. Calculate the equivalent diameter of a fictitious hollow, thin-walled conductor having the same equivalent inductance as the original line. What is the value of the inductance per conductor?

**4.5.** Find the geometric mean radius of a conductor in terms of the radius $r$ of an individual strand for

(a) Three equal strands as shown in Figure 4.25(a)
(b) Four equal strands as shown in Figure 4.25(b)

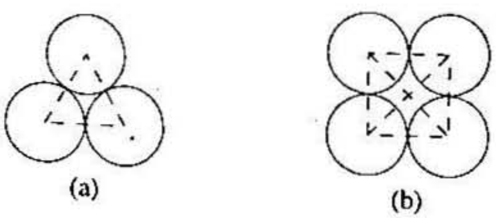



**FIGURE 4.25**
Cross section of the stranded conductor for Problem 4.5.

**4.6.** One circuit of a single-phase transmission line is composed of three solid 0.5-cm radius wires. The return circuit is composed of two solid 2.5-cm radius wires. The arrangement of conductors is as shown in Figure 4.26. Applying the concept of the $GMD$ and $GMR$, find the inductance of the complete line in millihenry per kilometer.





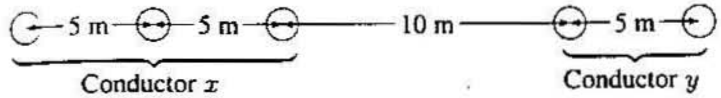



**FIGURE 4.26**
Conductor layout for Problem 4.6.

**4.7.** A three-phase, 60-Hz transposed transmission line has a flat horizontal configuration as shown in Figure 4.27. The line reactance is 0.486 Ω per kilometer. The conductor geometric mean radius is 2.0 cm. Determine the phase spacing $D$ in meters.

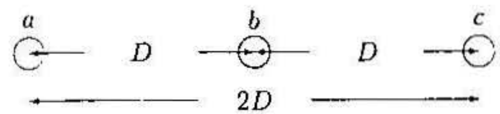



**FIGURE 4.27**
Conductor layout for Problem 4.7.

**4.8.** A three-phase transposed line is composed of one ACSR 159,000-cmil, 54/19 Lapwing conductor per phase with flat horizontal spacing of 8 m as shown in Figure 4.28. The $GMR$ of each conductor is 1.515 cm.
(a) Determine the inductance per phase per kilometer of the line.
(b) This line is to be replaced by a two-conductor bundle with 8 m spacing measured from the center of the bundles as shown in Figure 4.29. The spacing between the conductors in the bundle is 40 cm. If the line inductance per phase is to be 77 percent of the inductance in part (a), what would be the $GMR$ of each new conductor in the bundle?

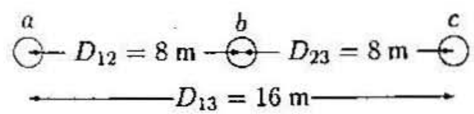



**FIGURE 4.28**
Conductor layout for Problem 4.8 (a).

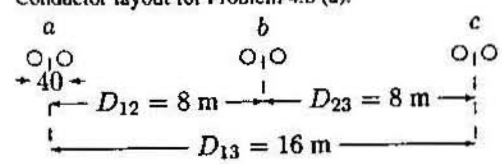



**FIGURE 4.29**
Conductor layout for Problem 4.8 (b).

**4.9.** A three-phase transposed line is composed of one ACSR, 1,431,000-cmil, 47/7 Bobolink conductor per phase with flat horizontal spacing of 11 m as shown in Figure 4.30. The conductors have a diameter of 3.625 cm and a $GMR$ of 1.439 cm. The line is to be replaced by a three-conductor bundle of ACSR, 477,000-cmil, 26/7 Hawk conductors having the same cross-sectional area of aluminum as the single-conductor line. The conductors have a diameter of 2.1793 cm and a $GMR$ of 0.8839 cm. The new line will also have a flat horizontal configuration, but it is to be operated at a higher voltage and therefore the phase spacing is increased to 14 m as measured from the center of the bundles as shown in Figure 4.31. The spacing between the conductors in the bundle is 45 cm. Determine

(a) The percentage change in the inductance.
(b) The percentage change in the capacitance.

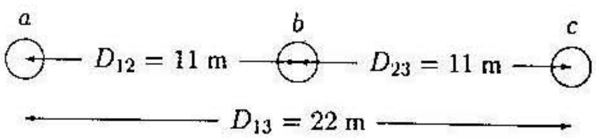



**FIGURE 4.30**
Conductor layout for Problem 4.9 (a).

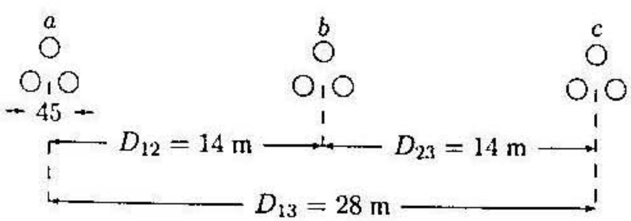



**FIGURE 4.31**
Conductor layout for Problem 4.9 (b).

**4.10.** A single-circuit three-phase transposed transmission line is composed of four ACSR, 1,272,000-cmil conductor per phase with horizontal configuration as shown in Figure 4.32. The bundle spacing is 45 cm. The conductor code name is *pheasant*. In *MATLAB*, use command **acsr** to find the conductor diameter and its $GMR$. Determine the inductance and capacitance per phase per kilometer of the line. Use function **[GMD, GMRL, GMRC] =gmd**, (4.58) and (4.92) in *MATLAB* to verify your results.





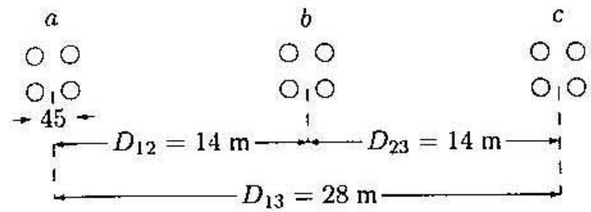



**FIGURE 4.32**
Conductor layout for Problem 4.10.

**4.11.** A double circuit three-phase transposed line is composed of two ACSR, 2,16,7000-cmil, 72/7 Kiwi conductor per phase with vertical configuration as shown in Figure 4.33. The conductors have a diameter of 4.4069 cm and a $GMR$ of 1.7374 cm. The bundle spacing is 45 cm. The circuit arrangement is $a_1b_1c_1$, $c_2b_2a_2$. Find the inductance and capacitance per phase per kilometer of the line. Find these values when the circuit arrangement is $a_1b_1c_1$, $a_2b_2c_2$. Use function **[GMD, GMRL, GMRC] =gmd**, (4.58) and (4.92) in *MATLAB* to verify your results.

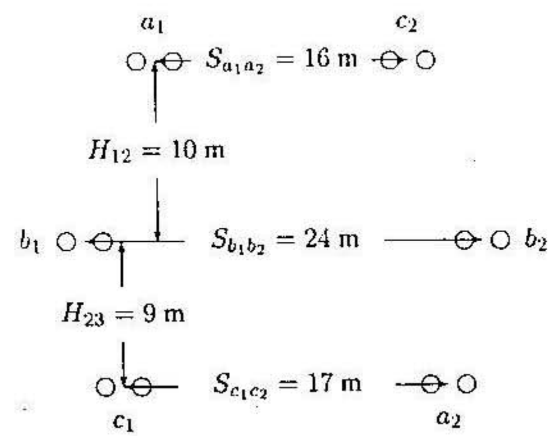



**FIGURE 4.33**
Conductor layout for Problem 4.11.

**4.12.** The conductors of a double-circuit three-phase transmission line are placed on the corner of a hexagon as shown in Figure 4.34. The two circuits are in parallel and are sharing the balanced load equally. The conductors of the circuits are identical, each having a radius $r$. Assume that the line is symmetrically transposed. Using the method of $GMD$, determine an expression for the capacitance per phase per meter of the line.

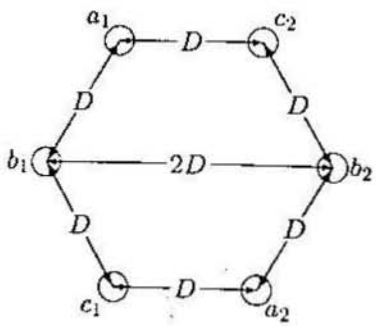



**FIGURE 4.34**
Conductor layout for Problem 4.12.

**4.13.** A 60-Hz, single-phase power line and a telephone line are parallel to each other as shown in Figure 4.35. The telephone line is symmetrically positioned directly below phase $b$. The power line carries an rms current of 226 A. Assume zero current flows in the ungrounded telephone wires. Find the magnitude of the voltage per km induced in the telephone line.

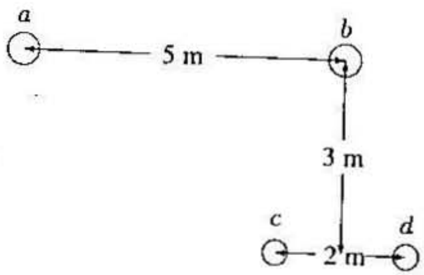



**FIGURE 4.35**
Conductor layout for Problem 4.13.

**4.14.** A three-phase, 60-Hz untransposed transmission line runs in parallel with a telephone line for 20 km. The power line carries a balanced three-phase rms current of $I_a = 320\angle 0°$ A, $I_b = 320\angle -120°$ A, and $I_c = 320\angle -240°$ A. The line configuration is as shown in Figure 4.36. Assume zero current flows in the ungrounded telephone wires. Find the magnitude of the voltage induced in the telephone line.

**4.15.** Since earth is an equipotential plane, the electric flux lines are forced to cut the surface of the earth orthogonally. The earth effect can be represented by placing an oppositely charged conductor a depth $H$ below the surface of the earth as shown in Figure 4.37(a). This configuration without the presence





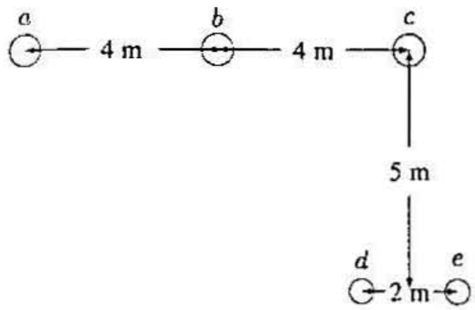
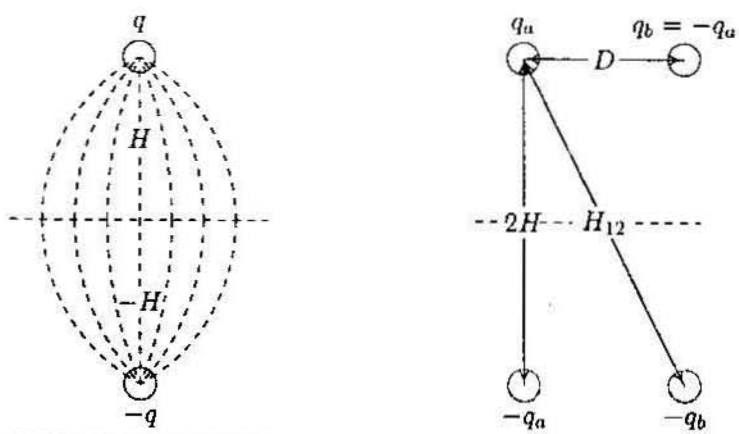



**FIGURE 4.36**
Conductor layout for Problem 4.14.

of the earth will produce the same field as a single charge and the earth surface. This imaginary conductor is called the image conductor. Figure 4.37(b) shows a single-phase line with its image conductors. Find the potential difference $V_{ab}$ and show that the equivalent capacitance to neutral is given by

$$C_{an} = C_{bn} = \frac{2\pi\varepsilon}{\ln\left(\frac{D}{r}\frac{2H}{H_{12}}\right)}$$




(a) Earth plane replaced by image conductor

(b) Single-phase line and its image

**FIGURE 4.37**
Conductor layout for Problem 4.15.

CHAPTER

5

# LINE MODEL AND PERFORMANCE

## 5.1 INTRODUCTION

In Chapter 4 the per-phase parameters of transmission lines were obtained. This chapter deals with the representation and performance of transmission lines under normal operating conditions. Transmission lines are represented by an equivalent model with appropriate circuit parameters on a "per-phase" basis. The terminal voltages are expressed from one line to neutral, the current for one phase and, thus, the three-phase system is reduced to an equivalent single-phase system.

The model used to calculate voltages, currents, and power flows depends on the length of the line. In this chapter the circuit parameters and voltage and current relations are first developed for "short" and "medium" lines. Problems relating to the regulation and losses of lines and their operation under conditions of fixed terminal voltages are then considered.

Next, long line theory is presented and expressions for voltage and current along the distributed line model are obtained. Propagation constant and characteristic impedance are defined, and it is demonstrated that the electrical power is being transmitted over the lines at approximately the speed of light. Since the terminal conditions at the two ends of the line are of primary importance, an equivalent







$\pi$ model is developed for the long lines. Several *MATLAB* functions are developed for calculation of line parameters and performance. Finally, line compensations are discussed for improving the line performance for unloaded and loaded transmission lines.

## 5.2 SHORT LINE MODEL

Capacitance may often be ignored without much error if the lines are less than about 80 km (50 miles) long, or if the voltage is not over 69 kV. The short line model is obtained by multiplying the series impedance per unit length by the line length.

$$
\begin{aligned}
Z &= (r + j\omega L)\ell \\
&= R + jX
\end{aligned}
\tag{5.1}
$$

where $r$ and $L$ are the per-phase resistance and inductance per unit length, respectively, and $\ell$ is the line length. The short line model on a per-phase basis is shown in Figure 5.1. $V_S$ and $I_S$ are the phase voltage and current at the sending end of the line, and $V_R$ and $I_R$ are the phase voltage and current at the receiving end of the line.

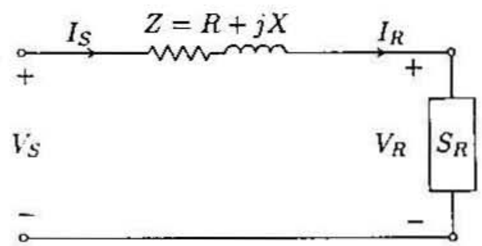



**FIGURE 5.1**
Short line model.

If a three-phase load with apparent power $S_{R(3\phi)}$ is connected at the end of the transmission line, the receiving end current is obtained by

$$I_R = \frac{S^*_{R(3\phi)}}{3V^*_R} \tag{5.2}$$

The phase voltage at the sending end is

$$V_S = V_R + ZI_R \tag{5.3}$$

and since the shunt capacitance is neglected, the sending end and the receiving end current are equal, i.e.,

$$I_S = I_R \tag{5.4}$$

The transmission line may be represented by a two-port network as shown in Figure 5.2, and the above equations can be written in terms of the generalized circuit constants commonly known as the $ABCD$ constants

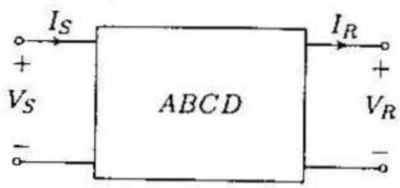



**FIGURE 5.2**
Two-port representation of a transmission line.

$$V_S = AV_R + BI_R \tag{5.5}$$

$$I_S = CV_R + DI_R \tag{5.6}$$

or in matrix form

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix} \tag{5.7}$$

According to (5.3) and (5.4), for short line model

$$A = 1 \qquad B = Z \qquad C = 0 \qquad D = 1 \tag{5.8}$$

Voltage regulation of the line may be defined as the percentage change in voltage at the receiving end of the line (expressed as percent of full-load voltage) in going from no-load to full-load.

$$\text{Percent } VR = \frac{|V_{R(NL)}| - |V_{R(FL)}|}{|V_{R(FL)}|} \times 100 \tag{5.9}$$

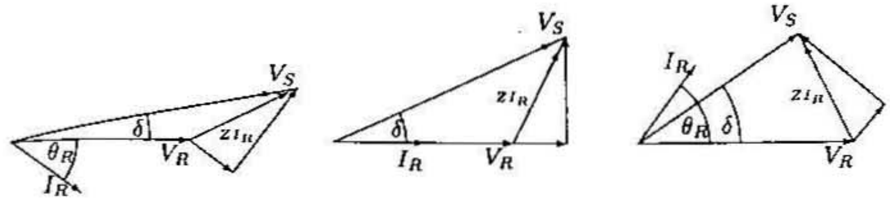
At no-load $I_R = 0$ and from (5.5)

$$V_{R(NL)} = \frac{V_S}{A} \tag{5.10}$$

For a short line, $A = 1$ and $V_{R(NL)} = V_S$. Voltage regulation is a measure of line voltage drop and depends on the load power factor. Voltage regulation will be








(a) Lagging pf load (b) Upf load (c) Leading pf load

**FIGURE 5.3**
Phasor diagram for short line.

poorer at low lagging power factor loads. With capacitive loads, i.e., leading power factor loads, regulation may become negative. This is demonstrated by the phasor diagram of Figure 5.3.

Once the sending end voltage is calculated the sending-end power is obtained by

$$S_{S(3\phi)} = 3V_S I_S^* \tag{5.11}$$

The total line loss is then given by

$$S_{L(3\phi)} = S_{S(3\phi)} - S_{R(3\phi)} \tag{5.12}$$

and the transmission line efficiency is given by

$$\eta = \frac{P_{R(3\phi)}}{P_{S(3\phi)}} \tag{5.13}$$

where $P_{R(3\phi)}$ and $P_{S(3\phi)}$ are the total real power at the receiving end and sending end of the line, respectively.

### Example 5.1

A 220-kV, three-phase transmission line is 40 km long. The resistance per phase is 0.15 Ω per km and the inductance per phase is 1.3263 mH per km. The shunt capacitance is negligible. Use the short line model to find the voltage and power at the sending end and the voltage regulation and efficiency when the line is supplying a three-phase load of

(a) 381 MVA at 0.8 power factor lagging at 220 kV.
(b) 381 MVA at 0.8 power factor leading at 220 kV.

(a) The series impedance per phase is

$$Z = (r + j\omega L)\ell = (0.15 + j2\pi \times 60 \times 1.3263 \times 10^{-3})40 = 6 + j20 \ \Omega$$

The receiving end voltage per phase is

$$V_R = \frac{220\angle 0^\circ}{\sqrt{3}} = 127\angle 0^\circ \text{ kV}$$

The apparent power is

$$S_{R(3\phi)} = 381\angle\cos^{-1} 0.8 = 381\angle 36.87^\circ = 304.8 + j228.6 \text{ MVA}$$

The current per phase is given by

$$I_R = \frac{S^*_{R(3\phi)}}{3V^*_R} = \frac{381\angle -36.87^\circ \times 10^3}{3 \times 127\angle 0^\circ} = 1000\angle -36.87^\circ \text{ A}$$

From (5.3) the sending end voltage is

$$\begin{aligned} V_S = V_R + ZI_R &= 127\angle 0^\circ + (6 + j20)(1000\angle -36.87^\circ)(10^{-3}) \\ &= 144.33\angle 4.93^\circ \text{ kV} \end{aligned}$$

The sending end line-to-line voltage magnitude is

$$|V_{S(L-L)}| = \sqrt{3}|V_S| = 250 \text{ kV}$$

The sending end power is

$$\begin{aligned} S_{S(3\phi)} = 3V_S I^*_S &= 3 \times 144.33\angle 4.93 \times 1000\angle 36.87^\circ \times 10^{-3} \\ &= 322.8 \text{ MW} + j288.6 \text{ Mvar} \\ &= 433\angle 41.8^\circ \text{ MVA} \end{aligned}$$

Voltage regulation is

$$\text{Percent } VR = \frac{250 - 220}{220} \times 100 = 13.6\%$$

Transmission line efficiency is

$$\eta = \frac{P_{R(3\phi)}}{P_{S(3\phi)}} = \frac{304.8}{322.8} \times 100 = 94.4\%$$

(b) The current for 381 MVA with 0.8 leading power factor is

$$I_R = \frac{S^*_{R(3\phi)}}{3V^*_R} = \frac{381\angle 36.87^\circ \times 10^3}{3 \times 127\angle 0^\circ} = 1000\angle 36.87^\circ \text{ A}$$





The sending end voltage is

$$\begin{aligned} V_S = V_R + ZI_R &= 127\angle 0^\circ + (6 + j20)(1000\angle 36.87^\circ)(10^{-3}) \\ &= 121.39\angle 9.29^\circ \text{ kV} \end{aligned}$$

The sending end line-to-line voltage magnitude is

$$|V_{S(L-L)}| = \sqrt{3}\, V_S = 210.26 \text{ kV}$$

The sending end power is

$$\begin{aligned} S_{S(3\phi)} = 3V_S I^*_S &= 3 \times 121.39\angle 9.29 \times 1000\angle -36.87^\circ \times 10^{-3} \\ &= 322.8 \text{ MW} - j168.6 \text{ Mvar} \\ &= 364.18\angle -27.58^\circ \text{ MVA} \end{aligned}$$

Voltage regulation is

$$\text{Percent } VR = \frac{210.26 - 220}{220} \times 100 = -4.43\%$$

Transmission line efficiency is

$$\eta = \frac{P_{R(3\phi)}}{P_{S(3\phi)}} = \frac{304.8}{322.8} \times 100 = 94.4\%$$

## 5.3 MEDIUM LINE MODEL

As the length of line increases, the line charging current becomes appreciable and the shunt capacitance must be considered. Lines above 80 km (50 miles) and below 250 km (150 miles) in length are termed as *medium length lines*. For medium length lines, half of the shunt capacitance may be considered to be lumped at each end of the line. This is referred to as the *nominal* $\pi$ model and is shown in Figure 5.4. $Z$ is the total series impedance of the line given by (5.1), and $Y$ is the total shunt admittance of the line given by

$$Y = (g + j\omega C)\ell \tag{5.14}$$

Under normal conditions, the shunt conductance per unit length, which represents the leakage current over the insulators and due to corona, is negligible and $g$ is assumed to be zero. $C$ is the line to neutral capacitance per km, and $\ell$ is the line length. The sending end voltage and current for the nominal $\pi$ model are obtained as follows:

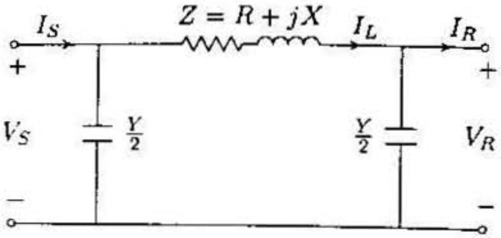



**FIGURE 5.4**
Nominal $\pi$ model for medium length line.

From KCL the current in the series impedance designated by $I_L$ is

$$I_L = I_R + \frac{Y}{2}V_R \tag{5.15}$$

From KVL the sending end voltage is

$$V_S = V_R + ZI_L \tag{5.16}$$

Substituting for $I_L$ from (5.15), we obtain

$$V_S = \left(1 + \frac{ZY}{2}\right) V_R + ZI_R \tag{5.17}$$

The sending end current is

$$I_S = I_L + \frac{Y}{2}V_S \tag{5.18}$$

Substituting for $I_L$ and $V_S$

$$I_S = Y\left(1 + \frac{ZY}{4}\right) V_R + \left(1 + \frac{ZY}{2}\right) I_R \tag{5.19}$$

Comparing (5.17) and (5.19) with (5.5) and (5.6), the $ABCD$ constants for the nominal $\pi$ model are given by

$$A = \left(1 + \frac{ZY}{2}\right) \qquad B = Z \tag{5.20}$$

$$C = Y\left(1 + \frac{ZY}{4}\right) \qquad D = \left(1 + \frac{ZY}{2}\right) \tag{5.21}$$

In general, the $ABCD$ constants are complex and since the $\pi$ model is a symmetrical two-port network, $A = D$. Furthermore, since we are dealing with a linear





passive, bilateral two-port network, the determinant of the transmission matrix in (5.7) is unity, i.e.,

$$AD - BC = 1 \tag{5.22}$$

Solving (5.7), the receiving end quantities can be expressed in terms of the sending end quantities by

$$\begin{bmatrix} V_R \\ I_R \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} V_S \\ I_S \end{bmatrix} \tag{5.23}$$

Two *MATLAB* functions are written for computation of the transmission matrix. Function **[ Z, Y, ABCD ] = rlc2abcd(r, L, C, g, f, Length)** is used when resistance in ohm, inductance in mH and capacitance in $\mu$F per unit length are specified, and function **[Z, Y, ABCD ] = zy2abcd(z, y, Length)** is used when series impedance in ohm and shunt admittance in siemens per unit length are specified. The above functions provide options for the nominal $\pi$ model and the equivalent $\pi$ model discussed in Section 5.4.

### Example 5.2

A 345-kV, three-phase transmission line is 130 km long. The resistance per phase is 0.036 $\Omega$ per km and the inductance per phase is 0.8 mH per km. The shunt capacitance is 0.0112 $\mu$F per km. The receiving end load is 270 MVA with 0.8 power factor lagging at 325 kV. Use the medium line model to find the voltage and power at the sending end and the voltage regulation.

The function **[Z, Y, ABCD] = rlc2abcd(r, L, C, g, f, Length)** is used to obtain the transmission matrix of the line. The following commands

```
r = .036; g = 0; f = 60;
L = 0.8;           % milli-Henry
C = 0.0112;        % micro-Farad
Length = 130;  VR3ph = 325;
VR = VR3ph/sqrt(3) + j*0;    % kV (receiving end phase voltage)
[Z, Y, ABCD] = rlc2abcd(r, L, C, g, f, Length);
AR = acos(0.8);
SR = 270*(cos(AR) + j*sin(AR));  %   MVA (receiving end power)
IR = conj(SR)/(3*conj(VR));      %  kA (receiving end current)
VsIs = ABCD* [VR; IR];           %      column vector [Vs; Is]
Vs = VsIs(1);
Vs3ph = sqrt(3)*abs(Vs);         % kV(sending end L-L voltage)
Is = VsIs(2); Ism = 1000*abs(Is);%     A (sending end current)
pfs= cos(angle(Vs)- angle(Is));  %  (sending end power factor)
Ss = 3*Vs*conj(Is);              %      MVA (sending end power)
REG = (Vs3ph/abs(ABCD(1,1)) - VR3ph)/VR3ph *100;
```

```
fprintf(' Is = %g A', Ism), fprintf('  pf = %g', pfs)
fprintf(' Vs = %g L-L kV', Vs3ph)
fprintf(' Ps = %g MW', real(Ss)),
fprintf('  Qs = %g Mvar', imag(Ss))
fprintf(' Percent voltage Reg. = %g', REG)
```

result in

```
Enter 1 for Medium line or 2 for long line → 1
Nominal π model
Z = 4.68 + j 39.2071 ohms
Y = 0 + j 0.000548899 siemens
```

$$ABCD = \begin{bmatrix} 0.98924 + j\,0.0012844 & 4.68 + j\,39.207 \\ -3.5251e{-}07 + j\,0.00054595 & 0.98924 + j\,0.0012844 \end{bmatrix}$$

```
Is = 421.132 A        pf = 0.869657
Vs = 345.002 L-L kV
Ps = 218.851 MW     Qs = 124.23 Mvar
Percent voltage Reg. = 7.30913
```

**Example 5.3**

A 345-kV, three-phase transmission line is 130 km long. The series impedance is $z = 0.036 + j0.3$ Ω per phase per km, and the shunt admittance is $y = j4.22 \times 10^{-6}$ siemens per phase per km. The sending end voltage is 345 kV, and the sending end current is 400 A at 0.95 power factor lagging. Use the medium line model to find the voltage, current and power at the receiving end and the voltage regulation.

The function **[Z, Y, ABCD] = zy2abcd(z, y, Length)** is used to obtain the transmission matrix of the line. The following commands

```
z = .036 + j* 0.3;  y = j*4.22/1000000;  Length = 130;
Vs3ph = 345;  Ism = 0.4;  %kA;
As = -acos(0.95);
Vs = Vs3ph/sqrt(3) + j*0;      % kV (sending end phase voltage)
Is = Ism*(cos(As) + j*sin(As));
[Z,Y, ABCD] = zy2abcd(z, y, Length);
VrIr = inv(ABCD)* [Vs; Is];    %        column vector [Vr; Ir]
Vr = VrIr(1);
Vr3ph = sqrt(3)*abs(Vr);        % kV(receiving end L-L voltage)
Ir = VrIr(2); Irm = 1000*abs(Ir);  % A (receiving end current)
pfr= cos(angle(Vr)- angle(Ir));  %(receiving end power factor)
Sr = 3*Vr*conj(Ir);                   % MVA (receiving end power)
```





```
REG = (Vs3ph/abs(ABCD(1,1)) - Vr3ph)/Vr3ph *100;
fprintf(' Ir = %g A', Irm), fprintf('  pf = %g', pfr)
fprintf(' Vr = %g L-L kV', Vr3ph)
fprintf(' Pr = %g MW', real(Sr))
fprintf('  Qr = %g Mvar', imag(Sr))
fprintf(' Percent voltage Reg. = %g', REG)
```

result in

```
Enter 1 for Medium line or 2 for long line → 1
Nominal π model
Z = 4.68 + j 39 ohms
Y = 0 + j 0.0005486 siemens
```

$$ABCD = \begin{bmatrix} 0.9893 + j\,0.0012837 & 4.68 + j\,39 \\ -3.5213e{-}07 + j\,0.00054565 & 0.9893 + j\,0.0012837 \end{bmatrix}$$

```
Ir = 441.832 A        pf = 0.88750
Vr = 330.68 L-L kV
Pr = 224.592 MW     Qr = 116.612 Mvar
Percent voltage Reg. = 5.45863
```

## 5.4 LONG LINE MODEL

For the short and medium length lines reasonably accurate models were obtained by assuming the line parameters to be lumped. For lines 250 km (150 miles) and longer and for a more accurate solution the exact effect of the distributed parameters must be considered. In this section expressions for voltage and current at any point on the line are derived. Then, based on these equations an equivalent $\pi$ model is obtained for the long line. Figure 5.5 shows one phase of a distributed line of length $\ell$ km.

The series impedance per unit length is shown by the lowercase letter $z$, and the shunt admittance per phase is shown by the lowercase letter $y$, where $z = r + j\omega L$ and $y = g + j\omega C$. Consider a small segment of line $\Delta x$ at a distance $x$ from the receiving end of the line. The phasor voltages and currents on both sides of this segment are shown as a function of distance. From Kirchhoff's voltage law

$$V(x + \Delta x) = V(x) + z\,\Delta x\, I(x) \tag{5.24}$$

or

$$\frac{V(x + \Delta x) - V(x)}{\Delta x} = z\, I(x) \tag{5.25}$$

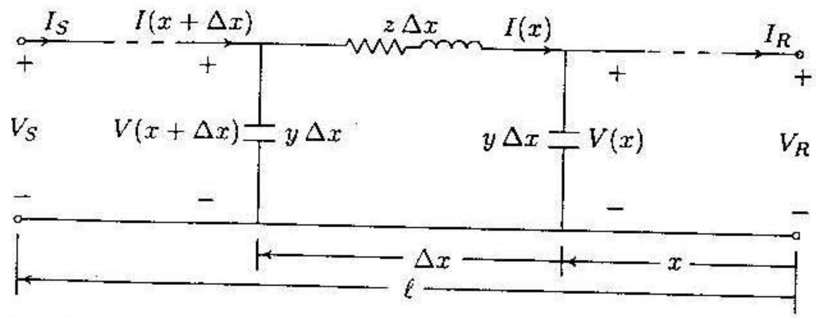



**FIGURE 5.5**
Long line with distributed parameters.

Taking the limit as $\Delta x \to 0$, we have

$$\frac{dV(x)}{dx} = z\,I(x) \tag{5.26}$$

Also, from Kirchhoff's current law

$$I(x + \Delta x) = I(x) + y\,\Delta x\,V(x + \Delta x) \tag{5.27}$$

or

$$\frac{I(x + \Delta x) - I(x)}{\Delta x} = y\,V(x + \Delta x) \tag{5.28}$$

Taking the limit as $\Delta x \to 0$, we have

$$\frac{dI(x)}{dx} = y\,V(x) \tag{5.29}$$

Differentiating (5.26) and substituting from (5.29), we get

$$\begin{aligned} \frac{d^2V(x)}{dx^2} &= z\,\frac{dI(x)}{dx} \\ &= zy\,V(x) \end{aligned} \tag{5.30}$$

Let

$$\gamma^2 = zy \tag{5.31}$$

The following second-order differential equation will result.

$$\frac{d^2V(x)}{dx^2} - \gamma^2\,V(x) = 0 \tag{5.32}$$



The solution of the above equation is

$$V(x) = A_1 e^{\gamma x} + A_2 e^{-\gamma x} \tag{5.33}$$

where $\gamma$, known as the *propagation constant*, is a complex expression given by (5.31) or

$$\gamma = \alpha + j\beta = \sqrt{zy} = \sqrt{(r + j\omega L)(g + j\omega C)} \tag{5.34}$$

The real part $\alpha$ is known as the *attenuation constant*, and the imaginary component $\beta$ is known as the *phase constant*. $\beta$ is measured in radian per unit length.

From (5.26), the current is

$$\begin{aligned} I(x) &= \frac{1}{z}\frac{dV(x)}{dx} = \frac{\gamma}{z}(A_1 e^{\gamma x} - A_2 e^{-\gamma x}) \\ &= \sqrt{\frac{y}{z}}(A_1 e^{\gamma x} - A_2 e^{-\gamma x}) \end{aligned} \tag{5.35}$$

or

$$I(x) = \frac{1}{Z_c}(A_1 e^{\gamma x} - A_2 e^{-\gamma x}) \tag{5.36}$$

where $Z_c$ is known as the *characteristic impedance*, given by

$$Z_c = \sqrt{\frac{z}{y}} \tag{5.37}$$

To find the constants $A_1$ and $A_2$ we note that when $x = 0$, $V(x) = V_R$, and $I(x) = I_R$. From (5.33) and (5.36) these constants are found to be

$$\begin{aligned} A_1 &= \frac{V_R + Z_c I_R}{2} \\ A_2 &= \frac{V_R - Z_c I_R}{2} \end{aligned} \tag{5.38}$$

Upon substitution in (5.33) and (5.36), the general expressions for voltage and current along a long transmission line become

$$V(x) = \frac{V_R + Z_c I_R}{2} e^{\gamma x} + \frac{V_R - Z_c I_R}{2} e^{-\gamma x} \tag{5.39}$$

$$I(x) = \frac{\frac{V_R}{Z_c} + I_R}{2} e^{\gamma x} - \frac{\frac{V_R}{Z_c} - I_R}{2} e^{-\gamma x} \tag{5.40}$$

The equations for voltage and currents can be rearranged as follows:

$$V(x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2} V_R + Z_c \frac{e^{\gamma x} - e^{-\gamma x}}{2} I_R \tag{5.41}$$

$$I(x) = \frac{1}{Z_c}\frac{e^{\gamma x} - e^{-\gamma x}}{2} V_R + \frac{e^{\gamma x} + e^{-\gamma x}}{2} I_R \tag{5.42}$$

Recognizing the hyperbolic functions sinh, and cosh, the above equations are written as follows:

$$V(x) = \cosh \gamma x \, V_R + Z_c \sinh \gamma x \, I_R \tag{5.43}$$

$$I(x) = \frac{1}{Z_c} \sinh \gamma x \, V_R + \cosh \gamma x \, I_R \tag{5.44}$$

We are particularly interested in the relation between the sending end and the receiving end of the line. Setting $x = \ell$, $V(\ell) = V_s$ and $I(\ell) = I_s$, the result is

$$V_s = \cosh \gamma\ell \, V_R + Z_c \sinh \gamma\ell \, I_R \tag{5.45}$$

$$I_s = \frac{1}{Z_c} \sinh \gamma\ell \, V_R + \cosh \gamma\ell \, I_R \tag{5.46}$$

Rewriting the above equations in terms of the $ABCD$ constants as before, we have

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix} \tag{5.47}$$

where

$$A = \cosh \gamma\ell \qquad B = Z_c \sinh \gamma\ell \tag{5.48}$$

$$C = \frac{1}{Z_c} \sinh \gamma\ell \qquad D = \cosh \gamma\ell \tag{5.49}$$

Note that, as before, $A = D$ and $AD - BC = 1$.

It is now possible to find an accurate equivalent $\pi$ model, shown in Figure 5.6, to replace the $ABCD$ constants of the two-port network. Similar to the expressions (5.17) and (5.19) obtained for the nominal $\pi$, for the equivalent $\pi$ model we have

$$V_S = \left(1 + \frac{Z'Y'}{2}\right) V_R + Z' I_R \tag{5.50}$$

$$I_S = Y' \left(1 + \frac{Z'Y'}{4}\right) V_R + \left(1 + \frac{Z'Y'}{2}\right) I_R \tag{5.51}$$

Comparing (5.50) and (5.51) with (5.45) and (5.46), respectively, and making use of the identity

$$\tanh \frac{\gamma\ell}{2} = \frac{\cosh \gamma\ell - 1}{\sinh \gamma\ell} \tag{5.52}$$





the parameters of the equivalent $\pi$ model are obtained.

$$Z' = Z_c \sinh \gamma\ell = Z \frac{\sinh \gamma\ell}{\gamma\ell} \tag{5.53}$$

$$\frac{Y'}{2} = \frac{1}{Z_c} \tanh \frac{\gamma\ell}{2} = \frac{Y}{2} \frac{\tanh \gamma\ell/2}{\gamma\ell/2} \tag{5.54}$$

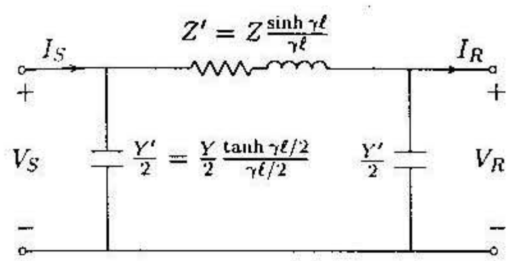



**FIGURE 5.6**
Equivalent $\pi$ model for long length line.

The functions **[Z, Y, ABCD ] = rlc2abcd(r, L, C, g, f, Length)** and **[Z, Y, ABCD ] = zy2abcd(z, y, Length)** with option 2 can be used for the evaluation of the transmission matrix and the equivalent $\pi$ parameters. However, Example 5.4 shows how these hyperbolic functions can be evaluated easily with simple *MATLAB* commands.

### Example 5.4

A 500-kV, three-phase transmission line is 250 km long. The series impedance is $z = 0.045 + j0.4\ \Omega$ per phase per km and the shunt admittance is $y = j4 \times 10^{-6}$ siemens per phase per km. Evaluate the equivalent $\pi$ model and the transmission matrix

The following commands

```
z = 0.045 + j*.4;     y = j*4.0/1000000; Length = 250;
gamma = sqrt(z*y);  Zc = sqrt(z/y);
A = cosh(gamma*Length);  B = Zc*sinh(gamma*Length);
C = 1/Zc * sinh(gamma*Length);  D = A;
ABCD = [A  B; C  D]
Z = B; Y = 2/Zc * tanh(gamma*Length/2)
```

result in

```
ABCD =
        0.9504 + 0.0055i  10.8778 +98.3624i
       -0.0000 + 0.0010i   0.9504 + 0.0055i
Z =
       10.8778 +98.3624i
Y =
        0.0000 + 0.0010i
```

## 5.5 VOLTAGE AND CURRENT WAVES

The rms expression for the phasor value of voltage at any point along the line is given by (5.33). Substituting $\alpha + j\beta$ for $\gamma$, the phasor voltage is

$$V(x) = A_1 e^{\alpha x} e^{j\beta x} + A_2 e^{-\alpha x} e^{-j\beta x}$$

Transforming from phasor domain to time domain, the instantaneous voltage as a function of $t$ and $x$ becomes

$$v(t,x) = \sqrt{2}\,\Re\, A_1 e^{\alpha x} e^{j(\omega t + \beta x)} + \sqrt{2}\,\Re\, A_2 e^{-\alpha x} e^{j(\omega t - \beta x)} \tag{5.55}$$

As $x$ increases (moving away from the receiving end), the first term becomes larger because of $e^{\alpha x}$ and is called the *incident wave*. The second term becomes smaller because of $e^{-\alpha x}$ and is called the *reflected wave*. At any point along the line, voltage is the sum of these two components.

$$v(t,x) = v_1(t,x) + v_2(t,x) \tag{5.56}$$

where

$$v_1(t,x) = \sqrt{2}\, A_1 e^{\alpha x} \cos(\omega t + \beta x) \tag{5.57}$$

$$v_2(t,x) = \sqrt{2}\, A_2 e^{-\alpha x} \cos(\omega t - \beta x) \tag{5.58}$$

As the current expression is similar to the voltage, the current can also be considered as the sum of incident and reflected current waves.

Equations (5.57 ) and (5.58 ) behave like traveling waves as we move along the line. This is similar to the disturbance in the water at some sending point. To see this, consider the reflected wave $v_2(t,x)$ and imagine that we ride along with the wave. To observe the instantaneous value, for example the peak amplitude requires that

$$\omega t - \beta x = 2K\pi \quad \text{or} \quad x = \frac{\omega}{\beta}t - \frac{2K\pi}{\beta}$$





Thus, to keep up with the wave and observe the peak amplitude we must travel with the speed

$$\frac{dx}{dt} = \frac{\omega}{\beta} \tag{5.59}$$

Thus, the velocity of propagation is given by

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} \tag{5.60}$$

The wavelength $\lambda$ or distance $x$ on the wave which results in a phase shift of $2\pi$ radian is

$$\beta\lambda = 2\pi$$

or

$$\lambda = \frac{2\pi}{\beta} \tag{5.61}$$

When line losses are neglected, i.e., when $g = 0$ and $r = 0$, the real part of the propagation constant $\alpha = 0$, and from (5.34) the phase constant becomes

$$\beta = \omega\sqrt{LC} \tag{5.62}$$

Also, the characteristic impedance is purely resistive and (5.37) becomes

$$Z_c = \sqrt{\frac{L}{C}} \tag{5.63}$$

which is commonly referred to as the *surge impedance*. Substituting for $\beta$ in (5.60) and (5.61), for a lossless line the velocity of propagation and the wavelength become

$$v = \frac{1}{\sqrt{LC}} \tag{5.64}$$

$$\lambda = \frac{1}{f\sqrt{LC}} \tag{5.65}$$

The expressions for the inductance per unit length $L$ and capacitance per unit length $C$ of a transmission line were derived in Chapter 4, given by (4.58) and (4.91). When the internal flux linkage of a conductor is neglected $GMR_L = GMR_C$, and upon substitution (5.64) and (5.65) become

$$v \simeq \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{5.66}$$

$$\lambda \simeq \frac{1}{f\sqrt{\mu_0\varepsilon_0}} \tag{5.67}$$

Substituting for $\mu_0 = 4\pi \times 10^{-7}$ and $\varepsilon_0 = 8.85 \times 10^{-12}$, the velocity of the wave is obtained to be approximately $3 \times 10^8$ m/sec, i.e., the velocity of light. At 60 Hz, the wavelength is 5000 km. Similarly, substituting for $L$ and $C$ in (5.63), we have

$$Z_c \simeq \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \ln \frac{GMD}{GMR_c}$$

$$\simeq 60 \ln \frac{GMD}{GMR_c} \tag{5.68}$$

For typical transmission lines the surge impedance varies from approximately 400Ω for 69-kV lines down to around 250 Ω for double-circuit 765-kV transmission lines.

For a lossless line $\gamma = j\beta$ and the hyperbolic functions $\cosh \gamma x = \cosh j\beta x = \cos \beta x$ and $\sinh \gamma x = \sinh j\beta x = j \sin \beta x$, the equations for the rms voltage and current along the line, given by (5.43) and (5.44), become

$$V(x) = \cos \beta x\, V_R + jZ_c \sin \beta x\, I_R \tag{5.69}$$

$$I(x) = j\frac{1}{Z_c} \sin \beta x\, V_R + \cos \beta x\, I_R \tag{5.70}$$

At the sending end $x = \ell$

$$V_S = \cos \beta\ell\, V_R + jZ_c \sin \beta\ell\, I_R \tag{5.71}$$

$$I_S = j\frac{1}{Z_c} \sin \beta\ell\, V_R + \cos \beta\ell\, I_R \tag{5.72}$$

For hand calculation it is easier to use (5.71) and (5.72), and for more accurate calculations (5.47) through (5.49) can be used in *MATLAB*. The terminal conditions are readily obtained from the above equations. For example, for the open-circuited line $I_R = 0$, and from (5.71) the no-load receiving end voltage is

$$V_{R(nl)} = \frac{V_S}{\cos \beta\ell} \tag{5.73}$$

At no-load, the line current is entirely due to the line charging capacitive current and the receiving end voltage is higher than the sending end voltage. This is evident from (5.73), which shows that as the line length increases $\beta\ell$ increases and $\cos \beta\ell$ decreases, resulting in a higher no-load receiving end voltage.

For a solid short circuit at the receiving end, $V_R = 0$ and (5.71) and (5.72) reduce to

$$V_S = jZ_c \sin \beta\ell\, I_R \tag{5.74}$$

$$I_S = \cos \beta\ell\, I_R \tag{5.75}$$

The above equations can be used to find the short circuit currents at both ends of the line.





## 5.6 SURGE IMPEDANCE LOADING

When the line is loaded by being terminated with an impedance equal to its characteristic impedance, the receiving end current is

$$I_R = \frac{V_R}{Z_c} \tag{5.76}$$

For a lossless line $Z_c$ is purely resistive. The load corresponding to the surge impedance at rated voltage is known as the ***surge impedance loading*** ($SIL$), given by

$$SIL = 3V_R I_R^* = \frac{3|V_R|^2}{Z_c} \tag{5.77}$$

Since $V_R = V_{Lrated}/\sqrt{3}$, $SIL$ in MW becomes

$$SIL = \frac{(kV_{Lrated})^2}{Z_c} \text{ MW} \tag{5.78}$$

Substituting for $I_R$ in (5.69) and $V_R$ in (5.70) will result in

$$V(x) = (\cos \beta x + j \sin \beta x)V_R \quad \text{or} \quad V(x) = V_R \angle \beta x \tag{5.79}$$

$$I(x) = (\cos \beta x + j \sin \beta x)I_R \quad \text{or} \quad I(x) = I_R \angle \beta x \tag{5.80}$$

Equations (5.79) and (5.80) show that in a lossless line under surge impedance loading the voltage and current at any point along the line are constant in magnitude and are equal to their sending end values. Since $Z_c$ has no reactive component, there is no reactive power in the line, $Q_S = Q_R = 0$. This indicates that for $SIL$, the reactive losses in the line inductance are exactly offset by reactive power supplied by the shunt capacitance or $\omega L|I_R|^2 = \omega C|V_R|^2$. From this relation, we find that $Z_c = V_R/I_R = \sqrt{L/C}$, which verifies the result in (5.63). $SIL$ for typical transmission lines varies from approximately 150 MW for 230-kV lines to about 2000 MW for 765-kV lines. $SIL$ is a useful measure of transmission line capacity as it indicates a loading where the line's reactive requirements are small. For loads significantly above $SIL$, shunt capacitors may be needed to minimize voltage drop along the line, while for light loads significantly below $SIL$, shunt inductors may be needed. Generally the transmission line full-load is much higher than $SIL$. The voltage profile for various loading conditions is illustrated in Figure 5.11 (page 182) in Example 5.9(h).

**Example 5.5**

A three-phase, 60-Hz, 500-kV transmission line is 300 km long. The line inductance is 0.97 mH/km per phase and its capacitance is 0.0115 $\mu$F/km per phase. Assume a lossless line.

(a) Determine the line phase constant $\beta$, the surge impedance $Z_C$, velocity of propagation $v$ and the line wavelength $\lambda$.
(b) The receiving end rated load is 800 MW, 0.8 power factor lagging at 500 kV. Determine the sending end quantities and the voltage regulation.

(a) For a lossless line, from (5.62) we have

$$\beta = \omega\sqrt{LC} = 2\pi \times 60\sqrt{0.97 \times 0.0115 \times 10^{-9}} = 0.001259 \text{ rad/km}$$

and from (5.63)

$$Z_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.97 \times 10^{-3}}{0.0115 \times 10^{-6}}} = 290.43 \;\Omega$$

Velocity of propagation is

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.97 \times 0.0115 \times 10^{-9}}} = 2.994 \times 10^5 \text{ km/s}$$

and the line wavelength is

$$\lambda = \frac{v}{f} = \frac{1}{60}(2.994 \times 10^5) = 4990 \text{ km}$$

(b) $\beta\ell = 0.001259 \times 300 = 0.3777 \text{ rad} = 21.641°$

The receiving end voltage per phase is .

$$V_R = \frac{500\angle 0°}{\sqrt{3}} = 288.675\angle 0° \text{ kV}$$

The receiving end apparent power is

$$S_{R(3\phi)} = \frac{800}{0.8}\angle\cos^{-1} 0.8 = 1000\angle 36.87° = 800 + j600 \text{ MVA}$$

The receiving end current per phase is given by

$$I_R = \frac{S^*_{R(3\phi)}}{3V^*_R} = \frac{1000\angle -36.87° \times 10^3}{3 \times 288.675\angle 0°} = 1154.7\angle -36.87° \text{ A}$$





From (5.71) the sending end voltage is

$$\begin{aligned} V_S &= \cos\beta\ell\, V_R + jZ_c \sin\beta\ell\, I_R \\ &= (0.9295)288.675\angle 0° + j(290.43)(0.3688)(1154.7\angle -36.87°)(10^{-3}) \\ &= 356.53\angle 16.1° \text{ kV} \end{aligned}$$

The sending end line-to-line voltage magnitude is

$$|V_{S(L-L)}| = \sqrt{3}\,|V_S| = 617.53 \text{ kV}$$

From (5.72) the sending end current is

$$\begin{aligned} I_S &= j\frac{1}{Z_c}\sin\beta\ell\, V_R + \cos\beta\ell\, I_R \\ &= j\frac{1}{290.43}(0.3688)(288.675\angle 0°)(10^3) + (0.9295)(1154.7\angle -36.87°) \\ &= 902.3\angle -17.9° \text{ A} \end{aligned}$$

The sending end power is

$$\begin{aligned} S_{S(3\phi)} &= 3V_S I^*_S = 3 \times 356.53\angle 16.1 \times 902.3\angle -17.9° \times 10^{-3} \\ &= 800 \text{ MW} + j539.672 \text{ Mvar} \\ &= 965.1\angle 34° \text{ MVA} \end{aligned}$$

Voltage regulation is

$$\text{Percent } VR = \frac{356.53/0.9295 - 288.675}{288.675} \times 100 = 32.87\%$$

The line performance of the above transmission line including the line resistance is obtained in Example 5.9 using the **lineperf** program. When a line is operating at the rated load, the exact solution results in $V_{S(L-L)} = 623.5\angle 15.57°$ kV, and $I_s = 903.1\angle -17.7°$ A. This shows that the lossless assumption yields acceptable results and is suitable for hand calculation.

## 5.7 COMPLEX POWER FLOW THROUGH TRANSMISSION LINES

Specific expressions for the complex power flow on a line may be obtained in terms of the sending end and receiving end voltage magnitudes and phase angles and the $ABCD$ constants. Consider Figure 5.2 where the terminal relations are given by (5.5) and (5.6). Expressing the $ABCD$ constants in polar form as $A = |A|\angle\theta_A$,

$B = |B|\angle\theta_B$, the sending end voltage as $V_S = |V_S|\angle\delta$, and the receiving end voltage as reference $V_R = |V_R|\angle 0$, from (5.5) $I_R$ can be written as

$$I_R = \frac{|V_S|\angle\delta - |A|\angle\theta_A|V_R|\angle 0}{|B|\angle\theta_B}$$
$$= \frac{|V_S|}{|B|}\angle\delta - \theta_B - \frac{|A||V_R|}{|B|}\angle\theta_A - \theta_B \tag{5.81}$$

The receiving end complex power is

$$S_{R(3\phi)} = P_{R(3\phi)} + jQ_{R(3\phi)} = 3V_R I_R^* \tag{5.82}$$

Substituting for $I_R$ from (5.81), we have

$$S_{R(3\phi)} = 3\frac{|V_S||V_R|}{|B|}\angle\theta_B - \delta - 3\frac{|A||V_R|^2}{|B|}\angle\theta_B - \theta_A \tag{5.83}$$

or in terms of the line-to-line voltages, we have

$$S_{R(3\phi)} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{|B|}\angle\theta_B - \delta - \frac{|A||V_{R(L-L)}|^2}{|B|}\angle\theta_B - \theta_A \tag{5.84}$$

The real and reactive power at the receiving end of the line are

$$P_{R(3\phi)} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{|B|}\cos(\theta_B - \delta) - \frac{|A||V_{R(L-L)}|^2}{|B|}\cos(\theta_B - \theta_A) \tag{5.85}$$
$$Q_{R(3\phi)} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{|B|}\sin(\theta_B - \delta) - \frac{|A||V_{R(L-L)}|^2}{|B|}\sin(\theta_B - \theta_A) \tag{5.86}$$

The sending end power is

$$S_{S(3\phi)} = P_{S(3\phi)} + jQ_{S(3\phi)} = 3V_S I_S^* \tag{5.87}$$

From (5.23), $I_S$ can be written as

$$I_S = \frac{|A|\angle\theta_A|V_S|\angle\delta - |V_R|\angle 0}{|B|\angle\theta_B} \tag{5.88}$$

Substituting for $I_S$ in (5.87) yields

$$P_{S(3\phi)} = \frac{|A||V_{S(L-L)}|^2}{|B|}\cos(\theta_B - \theta_A) - \frac{|V_{S(L-L)}||V_{R(L-L)}|}{|B|}\cos(\theta_B + \delta) \tag{5.89}$$
$$Q_{S(3\phi)} = \frac{|A||V_{S(L-L)}|^2}{|B|}\sin(\theta_B - \theta_A) - \frac{|V_{S(L-L)}||V_{R(L-L)}|}{|B|}\sin(\theta_B + \delta) \tag{5.90}$$





The real and reactive transmission line losses are

$$P_{L(3\phi)} = P_{S(3\phi)} - P_{R(3\phi)} \tag{5.91}$$
$$Q_{L(3\phi)} = Q_{S(3\phi)} - Q_{R(3\phi)} \tag{5.92}$$

The locus of all points obtained by plotting $Q_{R(3\phi)}$ versus $P_{R(3\phi)}$ for fixed line voltages and varying load angle $\delta$ is a circle known as the *receiving end power circle diagram*. A family of such circles with fixed receiving end voltage and varying sending end voltage is extremely useful in assessing the performance characteristics of the transmission line. A function called **pwrcirc(ABCD)** is developed for the construction of the receiving end power circle diagram, and its use is demonstrated in Example 5.9(g).

For a lossless line $B = jX'$, $\theta_A = 0$, $\theta_B = 90°$, and $A = \cos\beta\ell$, and the real power transferred over the line is given by

$$P_{3\phi} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{X'}\sin\delta \tag{5.93}$$

and the receiving end reactive power is

$$Q_{R3\phi} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{X'}\cos\delta - \frac{|V_{R(L-L)}|^2}{X'}\cos\beta\ell \tag{5.94}$$

For a given system operating at constant voltage, the power transferred is proportional to the sine of the power angle $\delta$. As the load increases, $\delta$ increases. For a lossless line, the maximum power that can be transmitted under stable steady-state condition occurs for an angle of 90°. However, a transmission system with its connected synchronous machines must also be able to withstand, without loss of stability, sudden changes in generation, load, and faults. To assure an adequate margin of stability, the practical operating load angle is usually limited to 35 to 45°.

## 5.8 POWER TRANSMISSION CAPABILITY

The power handling ability of a line is limited by the thermal loading limit and the stability limit. The increase in the conductor temperature, due to the real power loss, stretches the conductors. This will increase the sag between transmission towers. At higher temperatures this may result in irreversible stretching. The thermal limit is specified by the current-carrying capacity of the conductor and is available in the manufacturer's data. If the current-carrying capacity is denoted by $I_{thermal}$, the thermal loading limit of a line is

$$S_{thermal} = 3V_{\phi rated}I_{thermal} \tag{5.95}$$

The expression for real power transfer over the line for a lossless line is given by (5.93). The theoretical maximum power transfer is when $\delta = 90°$. The practical operating load angle for the line alone is limited to no more than 30 to 45°. This is because of the generator and transformer reactances which, when added to the line, will result in a larger $\delta$ for a given load. For planning and other purposes, it is very useful to express the power transfer formula in terms of $SIL$, and construct the line loadability curve. For a lossless line $X' = Z_c \sin \beta\ell$, and (5.93) may be written as

$$P_{3\phi} = \left(\frac{|V_{S(L-L)}|}{V_{rated}}\right)\left(\frac{|V_{R(L-L)}|}{V_{rated}}\right)\left(\frac{V_{rated}^2}{Z_c}\right)\frac{\sin\delta}{\sin\beta\ell} \tag{5.96}$$

The first two terms within parenthesis are the per-unit voltages denoted by $V_{Spu}$ and $V_{Rpu}$, and the third term is recognized as $SIL$. Equation (5.96) may be written as

$$P_{3\phi} = \frac{|V_{Spu}||V_{Rpu}|SIL}{\sin\beta\ell}\sin\delta$$
$$= \frac{|V_{Spu}||V_{Rpu}|SIL}{\sin(\frac{2\pi\ell}{\lambda})}\sin\delta \tag{5.97}$$

The function **loadabil(L, C, f)** obtains the loadability curve and thermal limit curve of the line. The loadability curve as obtained in Figure 5.12 (page 182) for Example 5.9(i) shows that for short and medium lines the thermal limit dictates the maximum power transfer. Whereas, for longer lines the limit is set by the practical line loadability curve. As we see in the next section, for longer lines it may be necessary to use series capacitors in order to increase the power transfer over the line.

**Example 5.6**

A three-phase power of 700-MW is to be transmitted to a substation located 315 km from the source of power. For a preliminary line design assume the following parameters:

$V_S = 1.0$ per unit, $V_R = 0.9$ per unit, $\lambda = 5000$ km, $Z_c = 320\ \Omega$, and $\delta = 36.87°$

(a) Based on the practical line loadability equation determine a nominal voltage level for the transmission line.

(b) For the transmission voltage level obtained in (a) calculate the theoretical maximum power that can be transferred by the transmission line.

(a) From (5.61), the line phase constant is

$$\beta\ell = \frac{2\pi}{\lambda}\ell \text{ rad}$$
$$= \frac{360}{\lambda}\ell = \frac{360}{5000}(315) = 22.68°$$





From the practical line loadability given by (5.97), we have

$$700 = \frac{(1.0)(0.9)(SIL)}{\sin(22.68°)}\sin(36.87°)$$

Thus

$$SIL = 499.83 \text{ MW}$$

From (5.78)

$$kV_L = \sqrt{(Z_c)(SIL)} = \sqrt{(320)(499.83)} = 400 \text{ kV}$$

(b) The equivalent line reactance for a lossless line is given by

$$X' = Z_c \sin\beta\ell = 320\sin(22.68) = 123.39\ \Omega$$

For a lossless line, the maximum power that can be transmitted under steady state condition occurs for a load angle of 90°. Thus, from (5.93), assuming $|V_S| = 1.0$ pu and $|V_R| = 0.9$ pu, the theoretical maximum power is

$$P_{3\phi(max)} = \frac{(400)(0.9)(400)}{123.39}(1) = 1167 \text{ MW}$$

## 5.9 LINE COMPENSATION

We have noted that a transmission line loaded to its surge impedance loading has no net reactive power flow into or out of the line and will have approximately a flat voltage profile along its length. On long transmission lines, light loads appreciably less than $SIL$ result in a rise of voltage at the receiving end, and heavy loads appreciably greater than $SIL$ will produce a large dip in voltage. The voltage profile of a long line for various loading conditions is shown in Figure 5.11 (page 182). Shunt reactors are widely used to reduce high voltages under light load or open line conditions. If the transmission system is heavily loaded, shunt capacitors, static var control, and synchronous condensers are used to improve voltage, increase power transfer, and improve the system stability.

### 5.9.1 SHUNT REACTORS

Shunt reactors are applied to compensate for the undesirable voltage effects associated with line capacitance. The amount of reactor compensation required on a transmission line to maintain the receiving end voltage at a specified value can be obtained as follows.

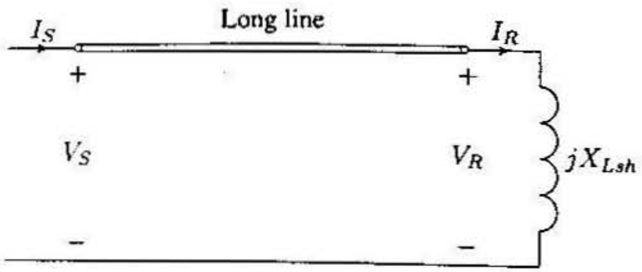



**FIGURE 5.7**
Shunt reactor compensation.

Consider a reactor of reactance $X_{Lsh}$, connected at the receiving end of a long transmission line as shown in Figure 5.7. The receiving end current is

$$I_R = \frac{V_R}{jX_{Lsh}} \tag{5.98}$$

Substituting $I_R$ into (5.71) results in

$$V_S = V_R(\cos\beta\ell + \frac{Z_c}{X_{Lsh}}\sin\beta\ell)$$

Note that $V_S$ and $V_R$ are in phase, which is consistent with the fact that no real power is being transmitted over the line. Solving for $X_{Lsh}$ yields

$$X_{Lsh} = \frac{\sin\beta\ell}{\frac{V_S}{V_R} - \cos\beta\ell} Z_c \tag{5.99}$$

For $V_S = V_R$, the required inductor reactance is

$$X_{Lsh} = \frac{\sin\beta\ell}{1 - \cos\beta\ell} Z_c \tag{5.100}$$

To find the relation between $I_S$ and $I_R$, we substitute for $V_R$ from (5.98) into (5.72)

$$I_S = \left(-\frac{1}{Z_c}\sin\beta\ell\, X_{Lsh} + \cos\beta\ell\right) I_R$$

Substituting for $X_{Lsh}$ from (5.100) for the case when $V_S = V_R$ results in

$$I_S = -I_R \tag{5.101}$$





With one reactor only at the receiving end, the voltage profile will not be uniform, and the maximum rise occurs at the midspan. It is left as an exercise to show that for $V_S = V_R$, the voltage at the midspan is given by

$$V_m = \frac{V_R}{\cos\frac{\beta\ell}{2}} \tag{5.102}$$

Also, the current at the midspan is zero. The function **openline(ABCD)** is used to find the receiving end voltage of an open line and to determine the Mvar of the reactor required to maintain the no-load receiving end voltage at a specified value. Example 5.9(d) illustrates the reactor compensation. Installing reactors at both ends of the line will improve the voltage profile and reduce the tension at midspan.

### Example 5.7

For the transmission line of Example 5.5:
(a) Calculate the receiving end voltage when line is terminated in an open circuit and is energized with 500 kV at the sending end.
(b) Determine the reactance and the Mvar of a three-phase shunt reactor to be installed at the receiving end to keep the no-load receiving end voltage at the rated value.

(a) The line is energized with 500 kV at the sending end. The sending end voltage per phase is

$$V_S = \frac{500\angle 0^\circ}{\sqrt{3}} = 288.675 \text{ kV}$$

From Example 5.5, $Z_c = 290.43$ and $\beta\ell = 21.641°$.

When the line is open $I_R = 0$ and from (5.71) the no-load receiving end voltage is given by

$$V_{R(nl)} = \frac{V_S}{\cos\beta\ell} = \frac{288.675}{0.9295} = 310.57 \text{ kV}$$

The no-load receiving end line-to-line voltage is

$$V_{R(L-L)(nl)} = \sqrt{3}\,V_{R(nl)} = 537.9 \text{ kV}$$

(b) For $V_S = V_R$, the required inductor reactance given by (5.100) is

$$X_{Lsh} = \frac{\sin(21.641°)}{1 - \cos(21.641°)}(290.43) = 1519.5 \ \Omega$$

The three-phase shunt reactor rating is

$$Q_{3\phi} = \frac{(kV_{Lrated})^2}{X_{Lsh}} = \frac{(500)^2}{1519.5} = 164.53 \text{ Mvar}$$

### 5.9.2 SHUNT CAPACITOR COMPENSATION

Shunt capacitors are used for lagging power factor circuits created by heavy loads. The effect is to supply the requisite reactive power to maintain the receiving end voltage at a satisfactory level. Capacitors are connected either directly to a bus bar or to the tertiary winding of a main transformer and are disposed along the route to minimize the losses and voltage drops. Given $V_S$ and $V_R$, (5.85) and (5.86) can be used conveniently to compute the required capacitor Mvar at the receiving end for a specified load. A function called **shntcomp(ABCD)** is developed for this purpose, and its use is demonstrated in Example 5.9(f).

### 5.9.3 SERIES CAPACITOR COMPENSATION

Series capacitors are connected in series with the line, usually located at the midpoint, and are used to reduce the series reactance between the load and the supply point. This results in improved transient and steady-state stability, more economical loading, and minimum voltage dip on load buses. Series capacitors have the good characteristics that their reactive power production varies concurrently with the line loading. Studies have shown that the addition of series capacitors on EHV transmission lines can more than double the transient stability load limit of long lines at a fraction of the cost of a new transmission line.

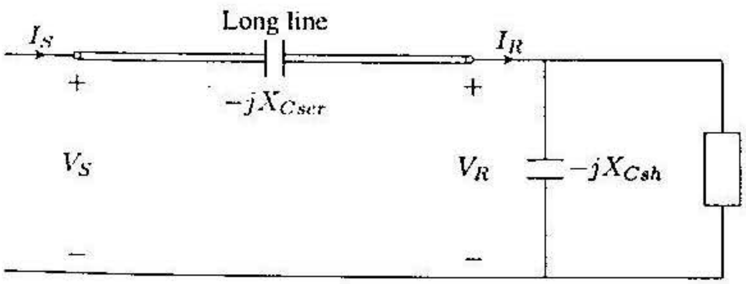



**FIGURE 5.8**
Shunt and series capacitor compensation.

With the series capacitor switched on as shown in Figure 5.8, from (5.93), the power transfer over the line for a lossless line becomes

$$P_{3\phi} = \frac{|V_{S(L-L)}||V_{R(L-L)}|}{X' - X_{Cser}} \sin\delta \tag{5.103}$$

Where $X_{Cser}$ is the series capacitor reactance. The ratio $X_{Cser}/X'$ expressed as a percentage is usually referred to as the *percentage compensation*. The percentage compensation is in the range of 25 to 70 percent.





One major drawback with series capacitor compensation is that special protective devices are required to protect the capacitors and bypass the high current produced when a short circuit occurs. Also, inclusion of series capacitors establishes a resonant circuit that can oscillate at a frequency below the normal synchronous frequency when stimulated by a disturbance. This phenomenon is referred to as *subsynchronous resonance* (SSR). If the synchronous frequency minus the electrical resonant frequency approaches the frequency of one of the turbine-generator natural torsional modes, considerable damage to the turbine-generator may result. If $L'$ is the lumped line inductance corrected for the effect of distribution and $C_{ser}$ is the capacitance of the series capacitor, the subsynchronous resonant frequency is

$$f_r = f_s\sqrt{\frac{1}{L'C_{ser}}} \tag{5.104}$$

where $f_s$ is the synchronous frequency. The function **sercomp(ABCD)** can be used to obtain the line performance for a specified percentage compensation. Finally, when line is compensated with both series and shunt capacitors, for the specified terminal voltages, the function **srshcomp(ABCD)** is used to obtain the line performance and the required shunt capacitor. These compensations are also demonstrated in Example 5.9(f).

#### Example 5.8

The transmission line in Example 5.5 supplies a load of 1000 MVA, 0.8 power factor lagging at 500 kV.
(a) Determine the Mvar and the capacitance of the shunt capacitors to be installed at the receiving end to keep the receiving end voltage at 500 kV when the line is energized with 500 kV at the sending end.
(b) Only series capacitors are installed at the midpoint of the line providing 40 percent compensation. Find the sending end voltage and voltage regulation.

(a) From Example 5.5, $Z_c = 290.43$ and $\beta\ell = 21.641°$. Thus, the equivalent line reactance for a lossless line is given by

$$X' = Z_c \sin\beta\ell = 290.43\sin(21.641°) = 107.11 \ \Omega$$

The receiving end power is

$$S_{R(3\phi)} = 1000\angle\cos^{-1}(0.8) = 800 + j600 \ \text{MVA}$$

For the above operating condition, the power angle $\delta$ is obtained from (5.93)

$$800 = \frac{(500)(500)}{107.11}\sin\delta$$

which results in $\delta = 20.044°$. Using the approximate relation given by (5.94), the net reactive power at the receiving end is

$$Q_{R(3\phi)} = \frac{(500)(500)}{107.11}\cos(20.044°) - \frac{(500)^2}{107.11}\cos(21.641°) = 23.15 \text{ Mvar}$$

Thus, the required capacitor Mvar is $S_C = j23.15 - j600 = -j576.85$

The capacitive reactance is given by

$$X_C = \frac{|V_L|^2}{S_C^*} = \frac{(500)^2}{j576.85} = -j433.38 \ \Omega$$

or

$$C = \frac{10^6}{2\pi(60)(433.38)} = 6.1 \ \mu\text{F}$$

The shunt compensation for the above transmission line including the line resistance is obtained in Example 5.9(f) using the **lineperf** program. The exact solution results in 613.8 Mvar for capacitor reactive power as compared to 576.85 Mvar obtained from the approximate formula for the lossless line. This represents approximately an error of 6 percent.

(b) For 40 percent compensation, the series capacitor reactance per phase is

$$X_{ser} = 0.4X' = 0.4(107.1) = 42.84 \ \Omega$$

The new equivalent $\pi$ circuit parameters are given by

$$Z' = j(X' - X_{ser}) = j(107.1 - 42.84) = j64.26 \ \Omega$$
$$Y' = = j\frac{2}{Z_c}\tan(\beta\ell/2) = j\frac{2}{290.43}\tan(21.641°/2) = j0.001316 \text{ siemens}$$

The new $B$ constant is $B = j64.26$ and the new $A$-constant is given by

$$A = 1 + \frac{Z'Y'}{2} = 1 + \frac{(j64.26)(j0.001316)}{2} = 0.9577$$

The receiving end voltage per phase is

$$V_R = \frac{500}{\sqrt{3}} = 288.675 \text{ kV}$$

and the receiving end current is

$$I_R = \frac{S_{R(3\phi)}^*}{3V_R^*} = \frac{1000\angle -36.87°}{3 \times 288.675\angle 0°} = 1.1547\angle -36.87° \text{ kA}$$





Thus, the sending end voltage is

$$V_S = AV_R + BI_R = 0.9577 \times 288.675 + j64.26 \times 1.1547\angle -36.87° = 326.4\angle 10.47° \text{ kV}$$

and the line-to-line voltage magnitude is $|V_{S(L-L)}| = \sqrt{3}\,V_S = 565.4$ kV. Voltage regulation is

$$\text{Percent } VR = \frac{565.4/0.958 - 500}{500} \times 100 = 18\%$$

The exact solution obtained in Example 5.9(f) results in $V_{S(L-L)} = 571.9$ kV. This represents an error of 1.0 percent.

## 5.10 LINE PERFORMANCE PROGRAM

A program called **lineperf** is developed for the complete analysis and compensation of a transmission line. The command **lineperf** displays a menu with five options for the computation of the parameters of the $\pi$ models and the transmission constants. Selection of these options will call upon the following functions.

**[Z, Y, ABCD] = rlc2abcd(r, L, C, g, f, Length)** computes and returns the $\pi$ model parameters and the transmission constants when $r$ in ohm, $L$ in mH, and $C$ in $\mu$F per unit length, frequency, and line length are specified.

**[Z, Y, ABCD] = zy2abcd(z, y, Length)** computes and returns the $\pi$ model parameters and the transmission constants when impedance and admittance per unit length are specified.

**[Z, Y, ABCD] = pi2abcd(Z, Y)** returns the ABCD constants when the $\pi$ model parameters are specified.

**[Z, Y, ABCD] = abcd2pi(A, B, C)** returns the $\pi$ model parameters when the transmission constants are specified.

**[L , C] = gmd2lc** computes and returns the inductance and capacitance per phase when the line configuration and conductor dimensions are specified.

**[r, L, C, f] = abcd2rlc(ABCD)** returns the line parameters per unit length and frequency when the transmission constants are specified.

Any of the above functions can be used independently when the arguments of the functions are defined in the *MATLAB* environment. If the above functions are typed without the parenthesis and the arguments, the user will be prompted to enter the required data. Next the **lineperf** loads the program **listmenu** which displays a list of eight options for transmission line analysis and compensation. Selection of these options will call upon the following functions.

**givensr(ABCD)** prompts the user to enter $V_R$, $P_R$ and $Q_R$. This function computes $V_S$, $P_S$, $Q_S$, line losses, voltage regulation, and transmission efficiency.

**givenss(ABCD)** prompts the user to enter $V_S$, $P_S$ and $Q_S$. This function computes $V_R$, $P_R$, $Q_R$, line losses, voltage regulation, and transmission efficiency.

**givenzl(ABCD)** prompts the user to enter $V_R$ and the load impedance. This function computes $V_S$, $P_S$, $Q_S$, line losses, voltage regulation, and transmission efficiency.

**openline(ABCD)** prompts the user to enter $V_S$. This function computes $V_R$ for the open-ended line. Also, the reactance and the Mvar of the necessary reactor to maintain the receiving end voltage at a specified value are obtained. In addition, the function plots the voltage profile of the line.

**shcktlin(ABCD)** prompts the user to enter $V_S$. This function computes the current at both ends of the line for a solid short circuit at the receiving end.

Option 6 is for capacitive compensation and calls upon **compmenu** which displays three options. Selection of these options will call upon the following functions.

**shntcomp(ABCD)** prompts the user to enter $V_S$, $P_R$, $Q_R$ and the desired $V_R$. This function computes the capacitance and the Mvar of the shunt capacitor bank to be installed at the receiving end in order to maintain the specified $V_R$. Then, $V_S$, $P_S$, $Q_S$, line losses, voltage regulation, and transmission efficiency are found.

**sercomp(ABCD)** prompts the user to enter $V_R$, $P_R$, $Q_R$, power, and the percentage compensation (i.e., $X_{Cser}/X_{line} \times 100$ ). This function computes the Mvar of the specified series capacitor and $V_S$, $P_S$, $Q_S$, line losses, voltage regulation, and transmission efficiency for the compensated line.

**srshcomp(ABCD)** prompts the user to enter $V_S$, $P_R$, $Q_R$, the desired $V_R$ and the percentage series capacitor compensation. This function computes the capacitance and the Mvar of a shunt capacitor to be installed at the receiving end in order to maintain the specified $V_R$. Also, $V_S$, $P_S$, $Q_S$, line losses, voltage regulation, and transmission efficiency are obtained for the compensated line.





Option 7 loads the **pwrcirc(ABCD)** which prompts for the receiving end voltage. This function constructs the receiving end power circle diagram for various values of $V_S$ from $V_R$ up to $1.3V_R$.

Option 8 calls upon **profmenu** which displays two options. Selection of these options will call upon the following functions:

**vprofile(r, L, C, f)** prompts the user to enter $V_S$, rated MVA, power factor, $V_R$, $P_R$, and $Q_R$. This function displays a graph consisting of voltage profiles for line length up to 1/8 of the line wavelength for the following cases: open-ended line, line terminated in $SIL$, short-circuited line, and full-load.

**loadabil(L, C, f)** prompts the user for $V_S$, $V_R$, rated line voltage, and current-carrying capacity of the line. This function displays a graph consisting of the practical line loadability curve for $\delta = 30°$, the theoretical stability limit curve, and the thermal limit. This function assumes a lossless line and the plots are obtained for a line length up to 1/4 of the line wavelength.

Any of the above functions can be used independently when the arguments of the functions are defined in the *MATLAB* environment. The $ABCD$ constant is entered as a matrix. If the above functions are typed without the parenthesis and the arguments, the user will be prompted to enter the required data.

### Example 5.9

A three-phase, 60-Hz, 550-kV transmission line is 300 km long. The line parameters per phase per unit length are found to be

$$r = 0.016\ \Omega/\text{km} \quad L = 0.97\ \text{mH/km} \quad C = 0.0115\ \mu\text{F/km}$$

(a) Determine the line performance when load at the receiving end is 800 MW, 0.8 power factor lagging at 500 kV.

The command:

```
lineperf
```

displays the following menu

| Type of parameters for input | Select |
|---|---|
| Parameters per unit length<br>r (Ω), g (siemens), L (mH), C (μF) | 1 |
| Complex z and y per unit length<br>r + j*x (Ω), g + j*b (siemens) | 2 |
| Nominal π or Eq. π model | 3 |
| A, B, C, D constants | 4 |
| Conductor configuration and dimension | 5 |
| To quit | 0 |

```
Select number of menu → 1
Enter line length = 300
Enter frequency in Hz = 60
Enter line resistance/phase in Ω/unit length, r = 0.016
Enter line inductance in mH per unit length, L = 0.97
Enter line capacitance in μF per unit length, C = .0115
Enter line conductance in siemens per unit length, g = 0
Enter 1 for medium line or 2 for long line → 2
```

Equivalent π model

```
Z' = 4.57414 + j 107.119 ohms
Y' = 6.9638e-07 + j 0.00131631 siemens
Zc = 290.496 + j -6.35214 ohms
αℓ = 0.00826172 neper βℓ = 0.377825 radian = 21.6478°
```

$$\text{ABCD} = \begin{bmatrix} 0.9295 + j0.0030478 & 4.5741 + j107.12 \\ -1.3341e-06 + j0.0012699 & 0.9295 + j0.0030478 \end{bmatrix}$$

At this point the program **listmenu** is automatically loaded and displays the following menu.

| Transmission line performance<br>Analysis | Select |
|---|---|
| To calculate sending end quantities<br>for specified receiving end MW, Mvar | 1 |
| To calculate receiving end quantities<br>for specified sending end MW, Mvar | 2 |
| To calculate sending end quantities<br>when load impedance is specified | 3 |
| Open-end line and reactive compensation | 4 |
| Short-circuited line | 5 |
| Capacitive compensation | 6 |
| Receiving end circle diagram | 7 |
| Loadability curve and voltage profile | 8 |
| To quit | 0 |





```
Select number of menu → 1

Enter receiving end line-line voltage kV = 500
Enter receiving end voltage phase angle° = 0
Enter receiving end 3-phase power MW = 800
Enter receiving end 3-phase reactive power
(+ for lagging and - for leading power factor) Mvar = 600
```

Line performance for specified receiving end quantities

```
Vr = 500 kV (L-L) at 0°
Pr = 800 MW Qr = 600 Mvar
Ir = 1154.7 A at -36.8699° PFr = 0.8 lagging
Vs = 623.511 kV (L-L) at 15.5762°
Is = 903.113 A at -17.6996°, PFs = 0.836039 lagging
Ps = 815.404 MW,     Qs = 535.129 Mvar
PL = 15.4040 MW,     QL = -64.871 Mvar
Percent Voltage Regulation = 34.1597
Transmission line efficiency = 98.1108
```

At the end of this analysis the **listmenu** (Analysis Menu) is displayed.

(b) Determine the receiving end quantities and the line performance when 600 MW and 400 Mvar are being transmitted at 525 kV from the sending end.

Selecting option 2 of the **listmenu** results in

```
Enter sending end line-line voltage kV = 525
Enter sending end voltage phase angle° = 0
Enter sending end 3-phase power MW = 600
Enter sending end 3-phase reactive power
(+ for lagging and - for leading power factor) Mvar = 400

Line performance for specified sending end quantities

Vs = 525 kV (L-L) at 0°
Ps = 600 MW,      Qs = 400 Mvar
Is = 793.016 A at -33.6901°, PFs = 0.83205 lagging
Vr = 417.954 kV (L-L) at -16.3044°
Ir = 1002.6 A at -52.16° PFr = 0.810496 lagging
Pr = 588.261 MW,     Qr = 425.136 Mvar
PL = 11.7390 MW,     QL = -25.136 Mvar
Percent Voltage Regulation = 35.1383
Transmission line efficiency = 98.0435
```

(c) Determine the sending end quantities and the line performance when the receiving end load impedance is 290 Ω at 500 kV.

Selecting option 3 of the listmenu results in

```
Enter receiving end line-line voltage kV = 500
Enter receiving end voltage phase angle° = 0
Enter sending end complex load impedance 290 + j * 0

Line performance for specified load impedance

Vr = 500 kV (L-L) at 0°
Ir = 995.431 A at 0° PFr = 1
Pr = 862.069 MW,     Qr = 0 Mvar
Vs = 507.996 kV (L-L) at 21.5037°
Is = 995.995 A at 21.7842°, PFs = 0.999988 leading
Ps = 876.341 MW Qs = -4.290 Mvar
PL = 14.272 MW QL = -4.290 Mvar
Percent Voltage Regulation = 9.30464
Transmission line efficiency = 98.3714
```

(d) Find the receiving end voltage when the line is terminated in an open circuit and is energized with 500 kV at the sending end. Also, determine the reactance and the Mvar of a three-phase shunt reactor to be installed at the receiving end in order to limit the no-load receiving end voltage to 500 kV.





Selecting option 4 of the **listmenu** results in

```
Enter sending end line-line voltage kV = 500
Enter sending end voltage phase angle° = 0

Open line and shunt reactor compensation

Vs = 500 kV (L-L) at 0°
Vr = 537.92 kV (L-L) at -0.00327893°
Is = 394.394 A at 89.8723°, PFs = 0.0022284 leading
Desired no load receiving end voltage = 500 kV
Shunt reactor reactance = 1519.4 Ω
Shunt reactor rating = 164.538 Mvar
```

The voltage profile for the uncompensated and the compensated line is also found as shown in Figure 5.9.

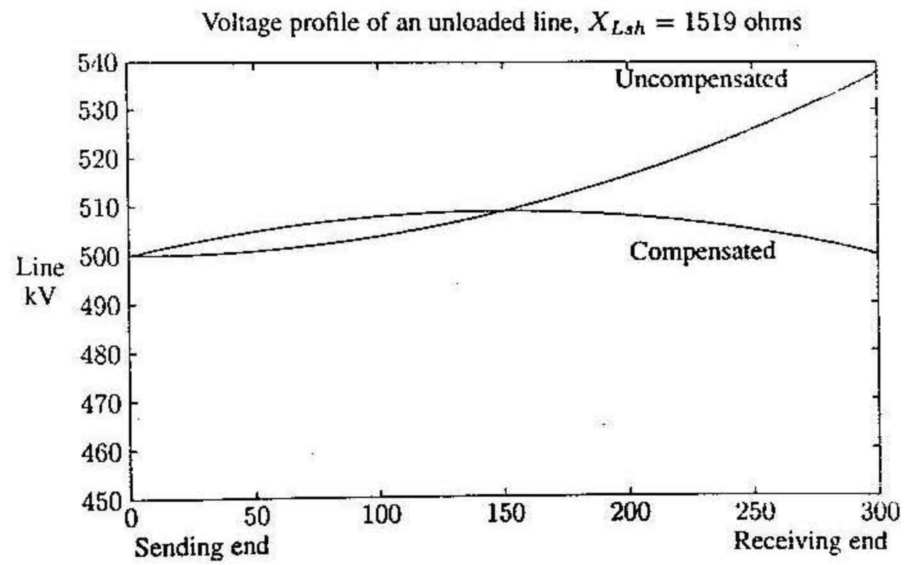



**FIGURE 5.9**
Compensated and uncompensated voltage profile of open-ended line.

(e) Find the receiving end and the sending end currents when the line is terminated in a short circuit.

Selecting option 5 of the **listmenu** results in

```
Enter sending end line-line voltage kV = 500
Enter sending end voltage phase angle° = 0
```

```
Line short-circuited at the receiving end

Vs = 500 kV (L-L) at 0°
Ir = 2692.45 A at -87.5549°
Is = 2502.65 A at -87.367°
```

(f) The line loading in part (a) resulted in a voltage regulation of 34.16 percent, which is unacceptably high. To improve the line performance, the line is compensated with series and shunt capacitors. For the loading condition in (a):

(1) Determine the Mvar and the capacitance of the shunt capacitors to be installed at the receiving end to keep the receiving end voltage at 500 kV when the line is energized with 500 kV at the sending end.

Selecting option 6 will display the **compmenu** as follows:

```
        Capacitive compensation
            Analysis                          Select

Shunt capacitive compensation                    1

Series capacitive compensation                   2

Series and shunt capacitive compensation         3

To quit                                          0
```

Selecting option 1 of the **compmenu** results in

```
Enter sending end line-line voltage kV = 500
Enter desired receiving end line-line voltage kV = 500
Enter receiving end voltage phase angle° = 0
Enter receiving end 3-phase power MW = 800
Enter receiving end 3-phase reactive power
(+ for lagging and - for leading power factor) Mvar = 600
```





```
Shunt capacitive compensation

Vs = 500 kV (L-L) at 20.2479°
Vr = 500 kV (L-L) at 0°
Pload = 800 MW,     Qload = 600 Mvar
Load current = 1154.7 A at -36.8699°, PFl = 0.8 lagging
Required shunt capacitor: 407.267 Ω, 6.51314 μF,613.849 Mvar
Shunt capacitor current = 708.811 A at 90°
Pr = 800.000 MW,     Qr = -13.849 Mvar
Ir = 923.899 A at 0.991732°, PFr = 0.99985 leading
Is = 940.306 A at 24.121° PFs = 0.997716 leading
Ps = 812.469 MW,     Qs = -55.006 Mvar
PL = 12.469 MW,     QL = -41.158 Mvar
Percent Voltage Regulation = 7.58405
Transmission line efficiency = 98.4653
```

(2) Determine the line performance when the line is compensated by series capacitors for 40 percent compensation with the load condition in (a) at 500 kV.

Selecting option 2 of the **compmenu** results in

```
Enter receiving end line-line voltage kV = 500
Enter receiving end voltage phase angle° = 0
Enter receiving end 3-phase power MW = 800
Enter receiving end 3-phase reactive power
(+ for lagging and - for leading power factor) Mvar = 600
Enter percent compensation for series capacitor
(Recommended range 25 to 75% of the line reactance) = 40
```

```
Series capacitor compensation

Vr = 500 kV (L-L) at 0°
Pr = 800 MW,     Qr = 600 Mvar
Required series capacitor: 42.8476 Ω, 61.9074 μF, 47.4047 Mvar
Subsynchronous resonant frequency = 37.9473 Hz
Ir = 1154.7 A at -36.8699°, PFr = 0.8 lagging
Vs = 571.904 kV (L-L) at 9.95438°
Is = 932.258 A at -18.044°, PFs = 0.882961 lagging
Ps = 815.383 MW,     Qs = 433.517 Mvar
PL = 15.383 MW,     QL = -166.483 Mvar
Percent Voltage Regulation = 19.4322
Transmission line efficiency = 98.1134
```

(3) The line has 40 percent series capacitor compensation and supplies the load in (a). Determine the Mvar and the capacitance of the shunt capacitors to be installed at the receiving end to keep the receiving end voltage at 500 kV when line is energized with 500 kV at the sending end.

Selecting option 3 of the **compmenu** results in

```
Enter sending end line-line voltage kV = 500
Enter desired receiving end line-line voltage kV = 500
Enter receiving end voltage phase angle° = 0
Enter receiving end 3-phase power MW = 800
Enter receiving end 3-phase reactive power
(+ for lagging and - for leading power factor) Mvar = 600
Enter percent compensation for series capacitor
(Recommended range 25 to 75% of the line reactance) = 40
```

```
Series and shunt capacitor compensation

Vs = 500 kV (L-L) at 12.0224°
Vr = 500 kV (L-L) at 0°
Pload = 800 MW,     Qload = 600 Mvar
Load current = 1154.7 A at -36.8699°, PFl = 0.8 lagging
Required shunt capacitor: 432.736 Ω, 6.1298 μF, 577.72 Mvar
Shunt capacitor current = 667.093 A at 90°
Required series capacitor: 42.8476 Ω, 61.9074 μF, 37.7274 Mvar
Subsynchronous resonant frequency = 37.9473 Hz
Pr = 800 MW,     Qr = 22.2804 Mvar
Ir = 924.119 A at -1.5953°, PFr = 0.999612 lagging
Is = 951.165 A at 21.5977°, PFs = 0.986068 leading
Ps = 812.257 MW,     Qs = -137.023 Mvar
PL = 12.257 MW,     QL = -159.304 Mvar
Percent Voltage Regulation = 4.41619
Transmission line efficiency = 98.491
```

(g) Construct the receiving end circle diagram.

Selecting option 7 of the **listmenu** results in

```
Enter receiving end line-line voltage kV = 500
```

A plot of the receiving end circle diagram is obtained as shown in Figure 5.10.





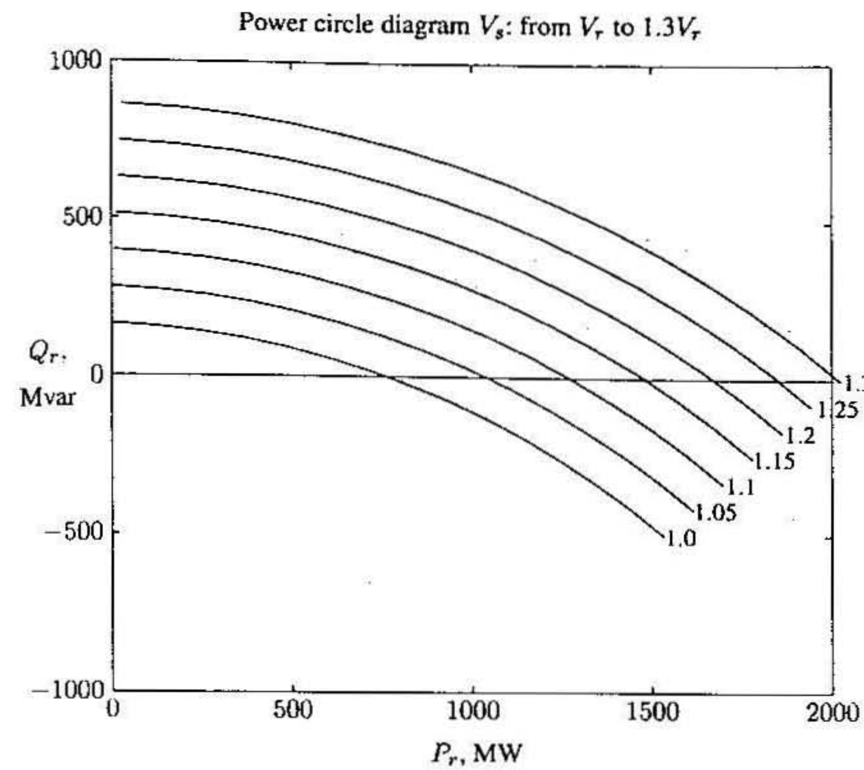



**FIGURE 5.10**
Receiving end circle diagram.

(h) Determine the line voltage profile for the following cases: no-load, rated load, line terminated in the $SIL$, and short-circuited line.

Selecting option 8 of the **listmenu** results in

```
        Voltage profile and line loadability
                  Analysis                        Select

Voltage profile curves                              1
Line loadability curve                              2
To quit                                             0
```

Selecting option 1 of the **profmenu** results in

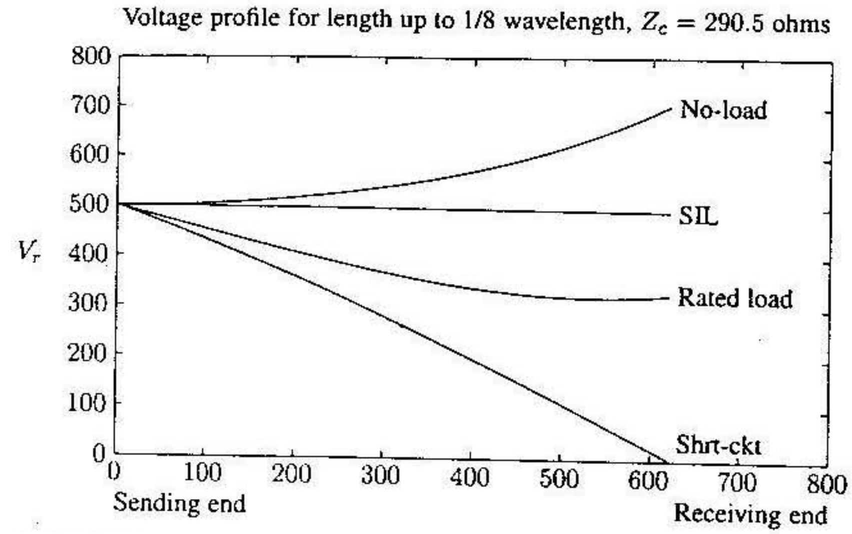



**FIGURE 5.11**
Voltage profile for length up to 1/8 wavelength.

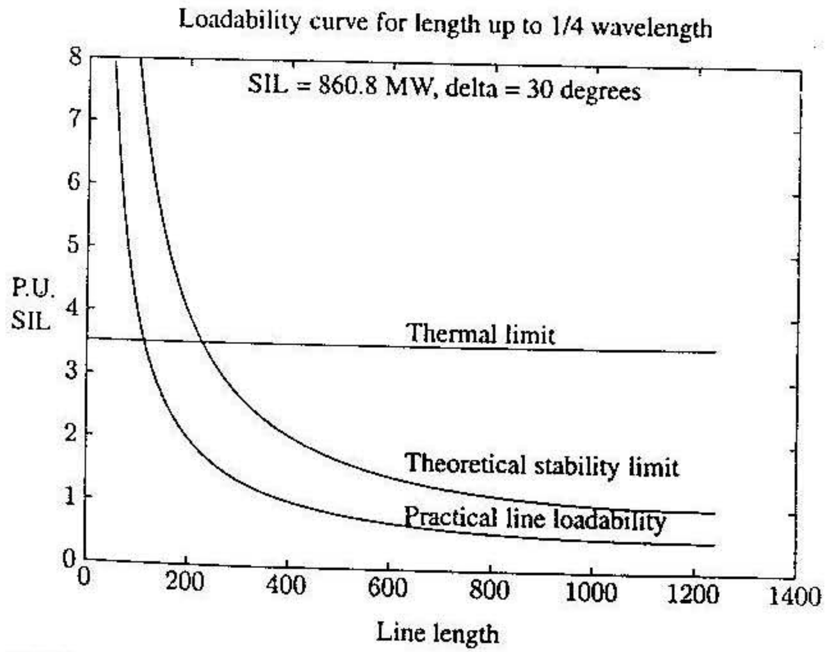



**FIGURE 5.12**
Line loadability curve for length up to 1/4 wavelength.





```
Enter sending end line-line voltage kV = 500
Enter rated sending end power, MVA = 1000
Enter power factor = 0.8
```

A plot of the voltage profile is obtained as shown in Figure 5.11 (page 182).

(i) Obtain the line loadability curves.
Selecting option 2 of the **profmenu** results in

```
Enter sending end line-line voltage kV = 500
Enter receiving end line-line voltage kV = 500
Enter rated line-line voltage kV = 500
Enter line current-carrying capacity, Amp/phase = 3500
```

The line loadability curve is obtained as shown in Figure 5.12 (page 182).

## PROBLEMS

**5.1.** A 69-kV, three-phase short transmission line is 16 km long. The line has a per phase series impedance of $0.125 + j0.4375\ \Omega$ per km. Determine the sending end voltage, voltage regulation, the sending end power, and the transmission efficiency when the line delivers

(a) 70 MVA, 0.8 lagging power factor at 64 kV.
(b) 120 MW, unity power factor at 64 kV.

Use **lineperf** program to verify your results.

**5.2.** Shunt capacitors are installed at the receiving end to improve the line performance of Problem 5.1. The line delivers 70 MVA, 0.8 lagging power factor at 64 kV. Determine the total Mvar and the capacitance per phase of the Y-connected capacitors when the sending end voltage is

(a) 69 kV.
(b) 64 kV.

*Hint*: Use (5.85) and (5.86) to compute the power angle $\delta$ and the receiving end reactive power.

(c) Use **lineperf** to obtain the compensated line performance.

**5.3.** A 230-kV, three-phase transmission line has a per phase series impedance of $z = 0.05 + j0.45\ \Omega$ per km and a per phase shunt admittance of $y = j3.4 \times 10^{-6}$ siemens per km. The line is 80 km long. Using the nominal $\pi$ model, determine

(a) The transmission line ABCD constants.

Find the sending end voltage and current, voltage regulation, the sending end power and the transmission efficiency when the line delivers

(b) 200 MVA, 0.8 lagging power factor at 220 kV.
(c) 306 MW, unity power factor at 220 kV.

Use **lineperf** program to verify your results.

**5.4.** Shunt capacitors are installed at the receiving end to improve the line performance of Problem 5.3. The line delivers 200 MVA, 0.8 lagging power factor at 220 kV.

(a) Determine the total Mvar and the capacitance per phase of the Y-connected capacitors when the sending end voltage is 220 kV. *Hint:* Use (5.85) and (5.86) to compute the power angle $\delta$ and the receiving end reactive power.
(b) Use **lineperf** to obtain the compensated line performance.

**5.5.** A three-phase, 345-kV, 60-Hz transposed line is composed of two ACSR, 1,113,000-cmil, 45/7 Bluejay conductors per phase with flat horizontal spacing of 11 m. The conductors have a diameter of 3.195 cm and a $GMR$ of 1.268 cm. The bundle spacing is 45 cm. The resistance of each conductor in the bundle is 0.0538 $\Omega$ per km and the line conductance is negligible. The line is 150 km long. Using the nominal $\pi$ model, determine the ABCD constant of the line. Use **lineperf** and option 5 to verify your results.

**5.6.** The ABCD constants of a three-phase, 345-kV transmission line are

$$A = D = 0.98182 + j0.0012447$$
$$B = 4.035 + j58.947$$
$$C = j0.00061137$$

The line delivers 400 MVA at 0.8 lagging power factor at 345 kV. Determine the sending end quantities, voltage regulation, and transmission efficiency.

**5.7.** Write a *MATLAB* function named **[ABCD] = abcdm(z, y, Lngt)** to evaluate and return the ABCD transmission matrix for a medium-length transmission line where **z** is the per phase series impedance per unit length, **y** is the shunt admittance per unit length, and **Lngt** is the line length. Then, write a program that uses the above function and computes the receiving end quantities, voltage regulation, and the line efficiency when sending end quantities are specified. The program should prompt for the following quantities:

The sending end line-to-line voltage magnitude in kV
The sending end voltage phase angle in degrees





The three-phase sending end real power in MW
The three-phase sending end reactive power in Mvar

Use your program to obtain the solution for the following case.

A three-phase transmission line has a per phase series impedance of $z = 0.03 + j0.4$ $\Omega$ per km and a per phase shunt admittance of $y = j4.0 \times 10^{-6}$ siemens per km. The line is 125 km long. Obtain the ABCD transmission matrix. Determine the receiving end quantities, voltage regulation, and the line efficiency when the line is sending 407 MW, 7.833 Mvar at 350 kV.

**5.8.** Obtain the solution for Problems 5.8 through 5.13 using the **lineperf** program. Then, solve each problem using hand calculations.

A three-phase, 765-kV, 60-Hz transposed line is composed of four ACSR, 1,431,000-cmil, 45/7 Bobolink conductors per phase with flat horizontal spacing of 14 m. The conductors have a diameter of 3.625 cm and a $GMR$ of 1.439 cm. The bundle spacing is 45 cm. The line is 400 km long, and for the purpose of this problem, a lossless line is assumed.

(a) Determine the transmission line surge impedance $Z_c$, phase constant $\beta$, wavelength $\lambda$, the surge impedance loading SIL, and the ABCD constant.
(b) The line delivers 2000 MVA at 0.8 lagging power factor at 735 kV. Determine the sending end quantities and voltage regulation.
(c) Determine the receiving end quantities when 1920 MW and 600 Mvar are being transmitted at 765 kV at the sending end.
(d) The line is terminated in a purely resistive load. Determine the sending end quantities and voltage regulation when the receiving end load resistance is 264.5 $\Omega$ at 735 kV.

**5.9.** The transmission line in Problem 5.8 is energized with 765 kV at the sending end when the load at the receiving end is removed.

(a) Find the receiving end voltage.
(b) Determine the reactance and the Mvar of a three-phase shunt reactor to be installed at the receiving end in order to limit the no-load receiving end voltage to 735 kV.

**5.10.** The transmission line in Problem 5.8 is energized with 765 kV at the sending end when a three-phase short-circuit occurs at the receiving end. Determine the receiving end current and the sending end current.

**5.11.** Shunt capacitors are installed at the receiving end to improve the line performance of Problem 5.8. The line delivers 2000 MVA, 0.8 lagging power factor. Determine the total Mvar and the capacitance per phase of the Y-connected capacitors to keep the receiving end voltage at 735 kV when the sending end voltage is 765 kV. *Hint:* Use (5.93) and (5.94) to compute the power angle $\delta$ and the receiving end reactive power. Find the sending end quantities and voltage regulation for the compensated line.

**5.12.** Series capacitors are installed at the midpoint of the line in Problem 5.8, providing 40 percent compensation. Determine the sending end quantities and the voltage regulation when the line delivers 2000 MVA at 0.8 lagging power factor at 735 kV.

**5.13.** Series capacitors are installed at the midpoint of the line in Problem 5.8, providing 40 percent compensation. In addition, shunt capacitors are installed at the receiving end. The line delivers 2000 MVA, 0.8 lagging power factor. Determine the total Mvar and the capacitance per phase of the series and shunt capacitors to keep the receiving end voltage at 735 kV when the sending end voltage is 765 kV. Find the sending end quantities and voltage regulation for the compensated line.

**5.14.** The transmission line in Problem 5.8 has a per phase resistance of 0.011 $\Omega$ per km. Using the **lineperf** program, perform the following analysis and present a summary of the calculation along with your conclusions and recommendations.

(a) Determine the sending end quantities for the specified receiving end quantities of $735\angle 0°$, 1600 MW, 1200 Mvar.
(b) Determine the receiving end quantities for the specified sending end quantities of $765\angle 0°$, 1920 MW, 600 Mvar.
(c) Determine the sending end quantities for a load impedance of $282.38 + j0\ \Omega$ at 735 kV.
(d) Find the receiving end voltage when the line is terminated in an open circuit and is energized with 765 kV at the sending end. Also, determine the reactance and the Mvar of a three-phase shunt reactor to be installed at the receiving end in order to limit the no-load receiving end voltage to 765 kV. Obtain the voltage profile for the uncompensated and the compensated line.
(e) Find the receiving end and the sending end current when the line is terminated in a three-phase short circuit.
(f) For the line loading of part (a), determine the Mvar and the capacitance of the shunt capacitors to be installed at the receiving end to keep the receiving end voltage at 735 kV when line is energized with 765 kV. Obtain the line performance of the compensated line.
(g) Determine the line performance when the line is compensated by series capacitor for 40 percent compensation with the load condition in part (a) at 735 kV.
(h) The line has 40 percent series capacitor compensation and supplies the load in part (a). Determine the Mvar and the capacitance of the shunt capacitors to be installed at the receiving end to keep the receiving end voltage at 735 kV when line is energized with 765 kV at the sending end.
(i) Obtain the receiving end circle diagram.
(j) Obtain the line voltage profile for a sending end voltage of 765 kV.
(k) Obtain the line loadability curves when the sending end voltage is 765 kV, and the receiving end voltage is 735 kV. The current-carrying capacity of the line is 5000 A per phase.





**5.15.** The ABCD constants of a lossless three-phase, 500-kV transmission line are

$$A = D = 0.86 + j0$$
$$B = 0 + j130.2$$
$$C = j0.002$$

(a) Obtain the sending end quantities and the voltage regulation when line delivers 1000 MVA at 0.8 lagging power factor at 500 kV.

To improve the line performance, series capacitors are installed at both ends in each phase of the transmission line. As a result of this, the compensated ABCD constants become

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}jX_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}jX_c \\ 0 & 1 \end{bmatrix}$$

where $X_c$ is the total reactance of the series capacitor. If $X_c = 100\ \Omega$

(b) Determine the compensated ABCD constants.
(c) Determine the sending end quantities and the voltage regulation when line delivers 1000 MVA at 0.8 lagging power factor at 500 kV.

**5.16.** A three-phase 420-kV, 60-HZ transmission line is 463 km long and may be assumed lossless. The line is energized with 420 kV at the sending end. When the load at the receiving end is removed, the voltage at the receiving end is 700 kV, and the per phase sending end current is $646.6\angle 90°$ A.

(a) Find the phase constant $\beta$ in radians per km and the surge impedance $Z_c$ in $\Omega$.
(b) Ideal reactors are to be installed at the receiving end to keep $|V_S| = |V_R| = 420$ kV when load is removed. Determine the reactance per phase and the required three-phase kvar.

**5.17.** A three-phase power of 3600 MW is to be transmitted via four identical 60-Hz transmission lines for a distance of 300 km. From a preliminary line design, the line phase constant and surge impedance are given by $\beta = 9.46 \times 10^{-4}$ radian/km and $Z_c = 343\ \Omega$, respectively.

Based on the practical line loadability criteria determine the suitable nominal voltage level in kV for each transmission line. Assume $V_S = 1.0$ per unit, $V_R = 0.9$ per unit, and the power angle $\delta = 36.87°$.

**5.18.** Power system studies on an existing system have indicated that 2400 MW are to be transmitted for a distance of 400 km. The voltage levels being considered include 345 kV, 500 kV, and 765 kV. For a preliminary design based on the practical line loadability, you may assume the following surge impedances

345 kV $Z_C = 320\ \Omega$
500 kV $Z_C = 290\ \Omega$
765 kV $Z_C = 265\ \Omega$

The line wavelength may be assumed to be 5000 km. The practical line loadability may be based on a load angle $\delta$ of 35°. Assume $|V_S| = 1.0$ pu and $|V_R| = 0.9$ pu. Determine the number of three-phase transmission circuits required for each voltage level. Each transmission tower may have up to two circuits. To limit the corona loss, all 500-kV lines must have at least two conductors per phase, and all 765-kV lines must have at least four conductors per phase. The bundle spacing is 45 cm. The conductor size should be such that the line would be capable of carrying current corresponding to at least 5000 MVA. Use **acsr** command in *MATLAB* to find a suitable conductor size. Following are the minimum recommended spacings between adjacent phase conductors at various voltage levels.

| Voltage level, kV | Spacing meter |
|---|---|
| 345 | 7.0 |
| 500 | 9.0 |
| 765 | 12.5 |

(a) Select a suitable voltage level, and conductor size, and tower structure. Use **lineperf** program and option 1 to obtain the voltage regulation and transmission efficiency based on a receiving end power of 3000 MVA at 0.8 power factor lagging at the selected rated voltage. Modify your design and select a conductor size for a line efficiency of at least 94 percent for the above specified load.

(b) Obtain the line performance including options 4–8 of the **lineperf** program for your final selection. Summarize the line characteristics and the required line compensation.



# CHAPTER 6

# POWER FLOW ANALYSIS

## 6.1 INTRODUCTION

In the previous chapters, modeling of the major components of an electric power system was discussed. This chapter deals with the steady-state analysis of an interconnected power system during normal operation. The system is assumed to be operating under balanced condition and is represented by a single-phase network. The network contains hundreds of nodes and branches with impedances specified in per unit on a common MVA base.

Network equations can be formulated systematically in a variety of forms. However, the node-voltage method, which is the most suitable form for many power system analyses, is commonly used. The formulation of the network equations in the nodal admittance form results in complex linear simultaneous algebraic equations in terms of node currents. When node currents are specified, the set of linear equations can be solved for the node voltages. However, in a power system, powers are known rather than currents. Thus, the resulting equations in terms of power, known as the *power flow equation*, become nonlinear and must be solved by iterative techniques. Power flow studies, commonly referred to as *load flow*, are the backbone of power system analysis and design. They are necessary for planning, operation, economic scheduling and exchange of power between utilities. In addition, power flow analysis is required for many other analyses such as transient stability and contingency studies.

In this chapter, the bus admittance matrix of the node-voltage equation is formulated, and a *MATLAB* function named **ybus** is developed for the systematic formation of the bus admittance matrix. Next, two commonly used iterative techniques, namely Gauss-Seidel and Newton-Raphson methods for the solution of nonlinear algebraic equations, are discussed. These techniques are employed in the solution of power flow problems. Three programs **lfgauss**, **lfnewton**, and **decouple** are developed for the solution of power flow problems by Gauss-Seidel, Newton-Raphson, and the fast decoupled power flow, respectively.

## 6.2 BUS ADMITTANCE MATRIX

In order to obtain the node-voltage equations, consider the simple power system shown in Figure 6.1 where impedances are expressed in per unit on a common MVA base and for simplicity resistances are neglected. Since the nodal solution is based upon Kirchhoff's current law, impedances are converted to admittance, i.e.,

$$y_{ij} = \frac{1}{z_{ij}} = \frac{1}{r_{ij} + jx_{ij}}$$

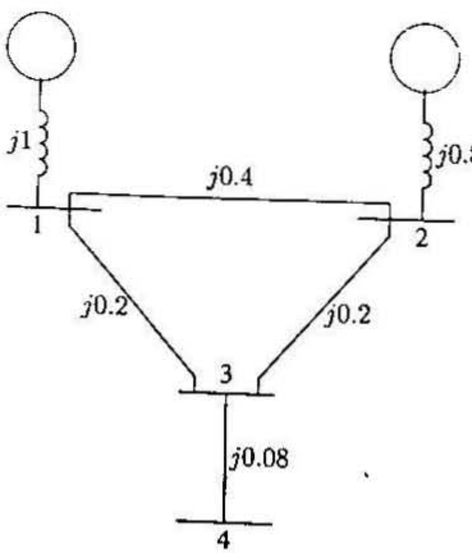



**FIGURE 6.1**
The impedance diagram of a simple system.





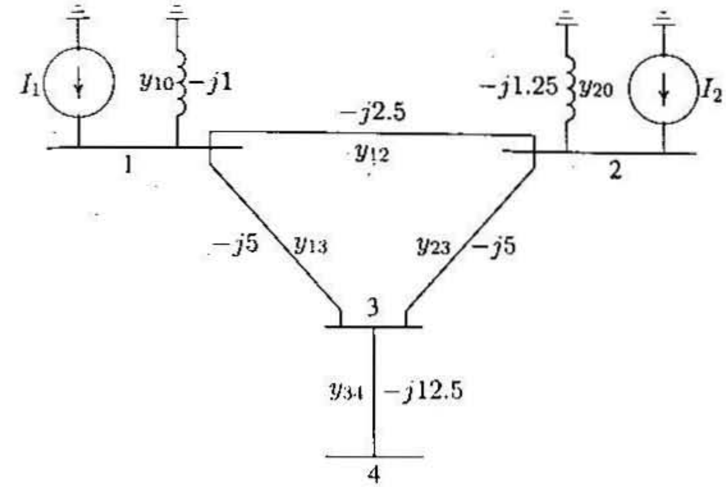



**FIGURE 6.2**
The admittance diagram for system of Figure 6.1.

The circuit has been redrawn in Figure 6.2 in terms of admittances and transformation to current sources. Node 0 (which is normally ground) is taken as reference. Applying KCL to the independent nodes 1 through 4 results in

$$\begin{aligned}
I_1 &= y_{10}V_1 + y_{12}(V_1 - V_2) + y_{13}(V_1 - V_3)\\
I_2 &= y_{20}V_2 + y_{12}(V_2 - V_1) + y_{23}(V_2 - V_3)\\
0 &= y_{23}(V_3 - V_2) + y_{13}(V_3 - V_1) + y_{34}(V_3 - V_4)\\
0 &= y_{34}(V_4 - V_3)
\end{aligned}$$

Rearranging these equations yields

$$\begin{aligned}
I_1 &= (y_{10} + y_{12} + y_{13})V_1 - y_{12}V_2 - y_{13}V_3\\
I_2 &= -y_{12}V_1 + (y_{20} + y_{12} + y_{23})V_2 - y_{23}V_3\\
0 &= -y_{13}V_1 - y_{23}V_2 + (y_{13} + y_{23} + y_{34})V_3 - y_{34}V_4\\
0 &= -y_{34}V_3 + y_{34}V_4
\end{aligned}$$

We introduce the following admittances

$$\begin{aligned}
Y_{11} &= y_{10} + y_{12} + y_{13}\\
Y_{22} &= y_{20} + y_{12} + y_{23}
\end{aligned}$$

$$Y_{33} = y_{13} + y_{23} + y_{34}$$
$$Y_{44} = y_{34}$$
$$Y_{12} = Y_{21} = -y_{12}$$
$$Y_{13} = Y_{31} = -y_{13}$$
$$Y_{23} = Y_{32} = -y_{23}$$
$$Y_{34} = Y_{43} = -y_{34}$$

The node equation reduces to

$$I_1 = Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 + Y_{14}V_4$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 + Y_{24}V_4$$
$$I_3 = Y_{31}V_1 + Y_{32}V_2 + Y_{33}V_3 + Y_{34}V_4$$
$$I_4 = Y_{41}V_1 + Y_{42}V_2 + Y_{43}V_3 + Y_{44}V_4$$

In the above network, since there is no connection between bus 1 and 4, $Y_{14} = Y_{41} = 0$; similarly $Y_{24} = Y_{42} = 0$.

Extending the above relation to an $n$ bus system, the node-voltage equation in matrix form is

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_i \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1i} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2i} & \cdots & Y_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ Y_{i1} & Y_{i2} & \cdots & Y_{ii} & \cdots & Y_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{ni} & \cdots & Y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_i \\ \vdots \\ V_n \end{bmatrix} \tag{6.1}$$

or

$$\mathbf{I}_{bus} = \mathbf{Y}_{bus}\,\mathbf{V}_{bus} \tag{6.2}$$

where $\mathbf{I}_{bus}$ is the vector of the injected bus currents (i.e., external current sources). The current is positive when flowing towards the bus, and it is negative if flowing away from the bus. $\mathbf{V}_{bus}$ is the vector of bus voltages measured from the reference node (i.e., node voltages). $\mathbf{Y}_{bus}$ is known as the *bus admittance matrix*. The diagonal element of each node is the sum of admittances connected to it. It is known as the *self-admittance* or *driving point admittance*, i.e.,

$$Y_{ii} = \sum_{j=0}^{n} y_{ij} \qquad j \neq i \tag{6.3}$$

The off-diagonal element is equal to the negative of the admittance between the nodes. It is known as the *mutual admittance* or *transfer admittance*, i.e.,

$$Y_{ij} = Y_{ji} = -y_{ij} \tag{6.4}$$





When the bus currents are known, (6.2) can be solved for the $n$ bus voltages.

$$\mathbf{V}_{bus} = \mathbf{Y}_{bus}^{-1}\,\mathbf{I}_{bus} \tag{6.5}$$

The inverse of the bus admittance matrix is known as the *bus impedance matrix* $Z_{bus}$. The admittance matrix obtained with one of the buses as reference is nonsingular. Otherwise the nodal matrix is singular.

Inspection of the bus admittance matrix reveals that the matrix is symmetric along the leading diagonal, and we need to store the upper triangular nodal admittance matrix only. In a typical power system network, each bus is connected to only a few nearby buses. Consequently, many off-diagonal elements are zero. Such a matrix is called *sparse*, and efficient numerical techniques can be applied to compute its inverse. By means of an appropriately ordered triangular decomposition, the inverse of a sparse matrix can be expressed as a product of sparse matrix factors, thereby giving an advantage in computational speed, storage and reduction of round-off errors. However, $Z_{bus}$, which is required for short-circuit analysis, can be obtained directly by the method of *building algorithm* without the need for matrix inversion. This technique is discussed in Chapter 9.

Based on (6.3) and (6.4), the bus admittance matrix for the network in Figure 6.2 obtained by inspection is

$$Y_{bus} = \begin{bmatrix} -j8.50 & j2.50 & j5.00 & 0 \\ j2.50 & -j8.75 & j5.00 & 0 \\ j5.00 & j5.00 & -j22.50 & j12.50 \\ 0 & 0 & j12.50 & -j12.50 \end{bmatrix}$$

A function called **Y = ybus(zdata)** is written for the formation of the bus admittance matrix. **zdata** is the line data input and contains four columns. The first two columns are the line bus numbers and the remaining columns contain the line resistance and reactance in per unit. The function returns the bus admittance matrix. The algorithm for the bus admittance program is very simple and basic to power system programming. Therefore, it is presented here for the reader to study and understand the method of solution. In the program, the line impedances are first converted to admittances. $Y$ is then initialized to zero. In the first loop, the line data is searched, and the off-diagonal elements are entered. Finally, in a nested loop, line data is searched to find the elements connected to a bus, and the diagonal elements are thus formed.

The following is a program for building the bus admittance matrix:

```
function[Y] = ybus(zdata)
nl=zdata(:,1); nr=zdata(:,2); R=zdata(:,3); X=zdata(:,4);
nbr=length(zdata(:,1)); nbus = max(max(nl), max(nr));
Z = R + j*X;                                    %branch impedance
```

```
y= ones(nbr,1)./Z;                          %branch admittance
Y = zeros(nbus,nbus);                     % initialize Y to zero
for k = 1:nbr;   % formation of the off diagonal elements
    if nl(k) > 0 & nr(k) > 0
    Y(nl(k),nr(k)) = Y(nl(k),nr(k)) - y(k);
    Y(nr(k),nl(k)) = Y(nl(k),nr(k));
    end
end
for n = 1:nbus       % formation of the diagonal elements
    for k = 1:nbr
    if nl(k) == n | nr(k) == n
    Y(n,n) = Y(n,n) + y(k);
    else, end
    end
end
```

**Example 6.1**

The emfs shown in Figure 6.1 are $E_1 = 1.1\angle 0°$ and $E_2 = 1.0\angle 0°$. Use the function **Y = ybus(zdata)** to obtain the bus admittance matrix. Find the bus impedance matrix by inversion, and solve for the bus voltages.

With source transformation, the equivalent current sources are

$$I_1 = \frac{1.1}{j1.0} = -j1.1 \text{ pu}$$

$$I_2 = \frac{1.0}{j0.8} = -j1.25 \text{ pu}$$

The following commands

```
%     From  To   R    X
z = [ 0     1    0    1.0
      0     2    0    0.8
      1     2    0    0.4
      1     3    0    0.2
      2     3    0    0.2
      3     4    0    0.08];
Y = ybus(z)                                 % bus admittance matrix
Ibus = [-j*1.1; -j*1.25; 0; 0];  % vector of bus currents
Zbus = inv(Y)                              % bus impedance matrix
Vbus = Zbus*Ibus
```

result in





```
Y =
     0 - 8.50i    0 + 2.50i    0 +  5.00i    0 +  0.00i
     0 + 2.50i    0 - 8.75i    0 +  5.00i    0 +  0.00i
     0 + 5.00i    0 + 5.00i    0 - 22.50i    0 + 12.50i
     0 + 0.00i    0 + 0.00i    0 + 12.50i    0 - 12.50i
Zbus =
     0 + 0.50i    0 + 0.40i    0 + 0.450i    0 + 0.450i
     0 + 0.40i    0 + 0.48i    0 + 0.440i    0 + 0.440i
     0 + 0.45i    0 + 0.44i    0 + 0.545i    0 + 0.545i
     0 + 0.45i    0 + 0.44i    0 + 0.545i    0 + 0.625i
Vbus =
     1.0500
     1.0400
     1.0450
     1.0450
```

The solution of equation $\mathbf{I}_{bus} = \mathbf{Y}_{bus}\mathbf{V}_{bus}$ by inversion is very inefficient. It is not necessary to obtain the inverse of $\mathbf{Y}_{bus}$. Instead, direct solution is obtained by optimally ordered triangular factorization. In *MATLAB*, the solution of linear simultaneous equations $AX = B$ is obtained by using the matrix division operator \ (i.e., $X = A \setminus B$), which is based on the triangular factorization and Gaussian elimination. This technique is superior in both execution time and numerical accuracy. It is two to three times as fast and produces residuals on the order of machine accuracy.

In Example 6.1, obtain the direct solution by replacing the statements `Zbus = inv(Y)` and `Vbus = Zbus*Ibus` with `Vbus = Y\ Ibus`.

## 6.3 SOLUTION OF NONLINEAR ALGEBRAIC EQUATIONS

The most common techniques used for the iterative solution of nonlinear algebraic equations are Gauss-Seidel, Newton-Raphson, and Quasi-Newton methods. The Gauss-Seidel and Newton-Raphson methods are discussed for one-dimensional equation, and are then extended to $n$-dimensional equations.

### 6.3.1 GAUSS-SEIDEL METHOD

The Gauss-Seidel method is also known as the method of successive displacements. To illustrate the technique, consider the solution of the nonlinear equation given by

$$f(x) = 0 \tag{6.6}$$

The above function is rearranged and written as

$$x = g(x) \tag{6.7}$$

If $x^{(k)}$ is an initial estimate of the variable $x$, the following iterative sequence is formed.

$$x^{(k+1)} = g(x^{(k)}) \tag{6.8}$$

A solution is obtained when the difference between the absolute value of the successive iteration is less than a specified accuracy, i.e.,

$$|x^{(k+1)} - x^{(k)}| \leq \epsilon \tag{6.9}$$

where $\epsilon$ is the desired accuracy.

### Example 6.2

Use the Gauss-Seidel method to find a root of the following equation

$$f(x) = x^3 - 6x^2 + 9x - 4 = 0$$

Solving for $x$, the above expression is written as

$$\begin{aligned} x &= -\frac{1}{9}x^3 + \frac{6}{9}x^2 + \frac{4}{9} \\ &= g(x) \end{aligned}$$

The *MATLAB* **plot** command is used to plot $g(x)$ and $x$ over a range of 0 to 4.5, as shown in Figure 6.3. The intersections of $g(x)$ and $x$ results in the roots of $f(x)$. From Figure 6.3 two of the roots are found to be 1 and 4. Actually, there is a repeated root at $x = 1$. Apply the Gauss-Seidel algorithm, and use an initial estimate of

$$x^{(0)} = 2$$

From (6.8), the first iteration is

$$x^{(1)} = g(2) = -\frac{1}{9}(2)^3 + \frac{6}{9}(2)^2 + \frac{4}{9} = 2.2222$$

The second iteration is

$$x^{(2)} = g(2.2222) = -\frac{1}{9}(2.2222)^3 + \frac{6}{9}(2.2222)^2 + \frac{4}{9} = 2.5173$$

The subsequent iterations result in 2.8966, 3.3376, 3.7398, 3.9568, 3.9988 and 4.0000. The process is repeated until the change in variable is within the desired





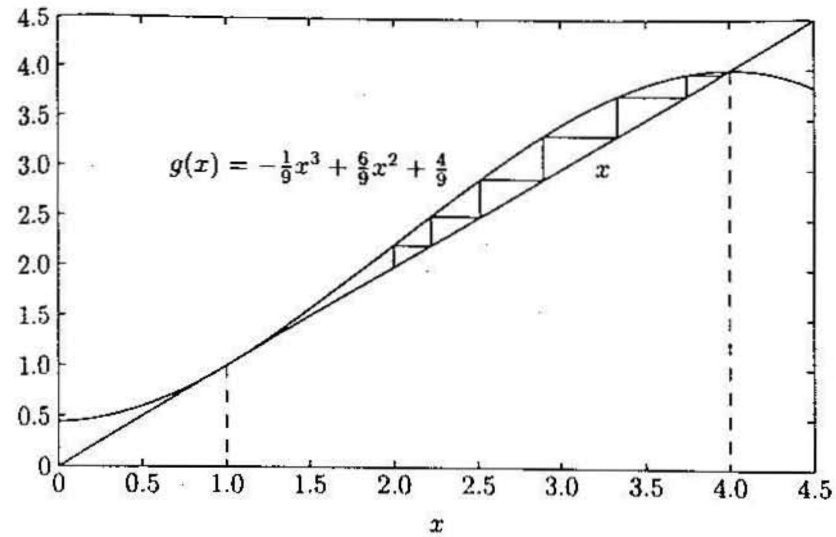



**FIGURE 6.3**
Graphical illustration of the Gauss-Seidel method.

accuracy. It can be seen that the Gauss-Seidel method needs many iterations to achieve the desired accuracy, and there is no guarantee for the convergence. In this example, since the initial estimate was within a "boxed in" region, the solution converged in a zigzag fashion to one of the roots. In fact, if the initial estimate was outside this region, say $x^{(0)} = 6$, the process would diverge. A test of convergence, especially for the $n$-dimensional case, is difficult, and no general methods are known.

The following commands show the procedure for the solution of the given equation starting with an initial estimate of $x^{(0)} = 2$.

```
dx=1;            % Change in variable is set to a high value
x=2;                                         % Initial estimate
iter = 0;                                    % Iteration counter
disp('Iter    g           dx          x')%Heading for results
while abs(dx) >= 0.001 & iter < 100 %Test for convergence
iter = iter + 1;                            % No. of iterations
g = -1/9*x^3+6/9*x^2+4/9 ;
dx = g-x;                                  % Change in variable
x = x + dx;                          % Successive approximation
fprintf('%g', iter), disp([g, dx, x])
end
```

The result is

```
Iter      g          dx         x
  1     2.2222     0.2222     2.2222
  2     2.5173     0.2951     2.5173
  3     2.8966     0.3793     2.8966
  4     3.3376     0.4410     3.3376
  5     3.7398     0.4022     3.7398
  6     3.9568     0.2170     3.9568
  7     3.9988     0.0420     3.9988
  8     4.0000     0.0012     4.0000
  9     4.0000     0.0000     4.0000
```

In some cases, an acceleration factor can be used to improve the rate of convergence. If $\alpha > 1$ is the acceleration factor, the Gauss-Seidel algorithm becomes

$$x^{(k+1)} = x^{(k)} + \alpha[g(x^{(k)}) - x^{(k)}] \tag{6.10}$$

### Example 6.3

Find a root of the equation in Example 6.2, using the Gauss-Seidel method with an acceleration factor of $\alpha = 1.25$:

Starting with an initial estimate of $x^{(0)} = 2$ and using (6.10), the first iteration is

$$\begin{aligned} g(2) &= -\frac{1}{9}(2)^3 + \frac{6}{9}(2)^2 + \frac{4}{9} = 2.2222 \\ x^{(1)} &= 2 + 1.25[2.2222 - 2] = 2.2778 \end{aligned}$$

The second iteration is

$$\begin{aligned} g(2.2778) &= -\frac{1}{9}(2.2778)^3 + \frac{6}{9}(2.2778)^2 + \frac{4}{9} = 2.5902 \\ x^{(2)} &= 2.2778 + 1.25[2.5902 - 2.2778] = 2.6683 \end{aligned}$$

The subsequent iterations result in 3.0801, 3.1831, 3.7238, 4.0084, 3.9978 and 4.0005. The effect of acceleration is shown graphically in Figure 6.4. Care must be taken not to use a very large acceleration factor since the larger step size may result in an overshoot. This can cause an increase in the number of iterations or even result in divergence. In the *MATLAB* command of Example 6.2, replace the command before the end statement by $x = x + 1.25 * dx$ to reflect the effect of the acceleration factor and run the program.





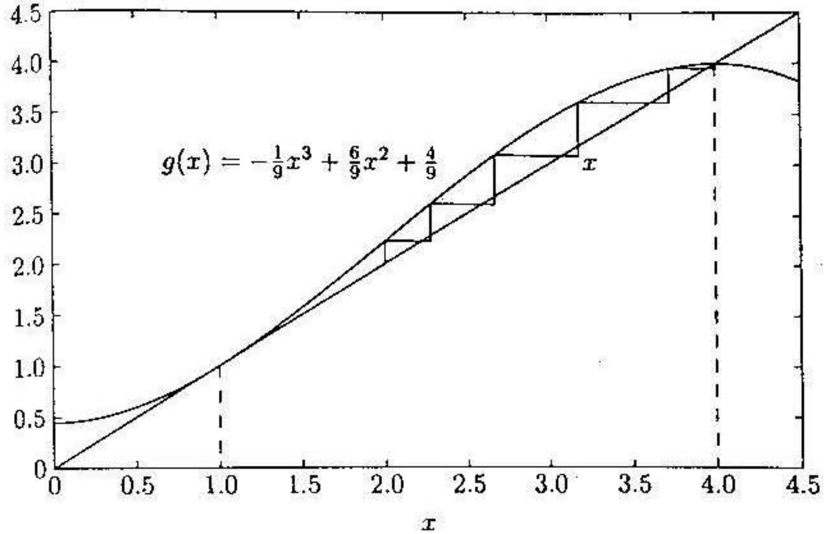



**FIGURE 6.4**
Graphical illustration of the Gauss-Seidel method using acceleration factor.

We now consider the system of $n$ equations in $n$ variables

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= c_1 \\ f_2(x_1, x_2, \cdots, x_n) &= c_2 \\ \cdots\cdots&\cdots\cdots \\ f_n(x_1, x_2, \cdots, x_n) &= c_n \end{aligned} \tag{6.11}$$

Solving for one variable from each equation, the above functions are rearranged and written as

$$\begin{aligned} x_1 &= c_1 + g_1(x_1, x_2, \cdots, x_n) \\ x_2 &= c_2 + g_2(x_1, x_2, \cdots, x_n) \\ \cdots\cdots&\cdots\cdots \\ x_n &= c_n + g_n(x_1, x_2, \cdots, x_n) \end{aligned} \tag{6.12}$$

The iteration procedure is initiated by assuming an approximate solution for each of the independent variables $(x_1^{(0)}, x_2^{(0)} \cdots, x_n^{(0)})$. Equation (6.12) results in a new approximate solution $(x_1^{(1)}, x_2^{(1)} \cdots, x_n^{(1)})$. In the Gauss-Seidel method, the updated values of the variables calculated in the preceding equations are immediately used in the solution of the subsequent equations. At the end of each iteration, the calculated values of all variables are tested against the previous values. If all changes in the variables are within the specified accuracy, a solution has converged, otherwise another iteration must be performed. The rate of convergence can often be increased by using a suitable acceleration factor $\alpha$, and the iterative sequence becomes

$$x_i^{(k+1)} = x_i^{(k)} + \alpha(x_{i\,cal}^{(k+1)} - x_i^{(k)}) \tag{6.13}$$

### 6.3.2 NEWTON-RAPHSON METHOD

The most widely used method for solving simultaneous nonlinear algebraic equations is the Newton-Raphson method. Newton's method is a successive approximation procedure based on an initial estimate of the unknown and the use of Taylor's series expansion. Consider the solution of the one-dimensional equation given by

$$f(x) = c \tag{6.14}$$

If $x^{(0)}$ is an initial estimate of the solution, and $\Delta x^{(0)}$ is a small deviation from the correct solution, we must have

$$f(x^{(0)} + \Delta x^{(0)}) = c$$

Expanding the left-hand side of the above equation in Taylor's series about $x^{(0)}$ yields

$$f(x^{(0)}) + \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)} + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\right)^{(0)} (\Delta x^{(0)})^2 + \cdots = c$$

Assuming the error $\Delta x^{(0)}$ is very small, the higher-order terms can be neglected, which results in

$$\Delta c^{(0)} \simeq \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)}$$

where

$$\Delta c^{(0)} = c - f(x^{(0)})$$

Adding $\Delta x^{(0)}$ to the initial estimate will result in the second approximation

$$x^{(1)} = x^{(0)} + \frac{\Delta c^{(0)}}{\left(\frac{df}{dx}\right)^{(0)}}$$





Successive use of this procedure yields the Newton-Raphson algorithm

$$\Delta c^{(k)} = c - f(x^{(k)}) \tag{6.15}$$

$$\Delta x^{(k)} = \frac{\Delta c^{(k)}}{\left(\frac{df}{dx}\right)^{(k)}} \tag{6.16}$$

$$x^{(k+1)} = x^{(k)} + \Delta x^{(k)} \tag{6.17}$$

(6.16) can be rearranged as

$$\Delta c^{(k)} = j^{(k)} \Delta x^{(k)} \tag{6.18}$$

where

$$j^{(k)} = \left(\frac{df}{dx}\right)^{(k)}$$

The relation in (6.18) demonstrates that the nonlinear equation $f(x) - c = 0$ is approximated by the tangent line on the curve at $x^{(k)}$. Therefore, a linear equation is obtained in terms of the small changes in the variable. The intersection of the tangent line with the $x$-axis results in $x^{(k+1)}$. This idea is demonstrated graphically in Example 6.4.

#### Example 6.4

Use the Newton-Raphson method to find a root of the equation given in Example 6.2. Assume an initial estimate of $x^{(0)} = 6$.

The *MATLAB* **plot** command is used to plot $f(x) = x^3 - 6x^2 + 9x - 4$ over a range of 0 to 6 as shown in Figure 6.5. The intersections of $f(x)$ with the $x$-axis results in the roots of $f(x)$. From Figure 6.5, two of the roots are found to be 1 and 4. Actually, there is a repeated root at $x = 1$.

Also, Figure 6.5 gives a graphical description of the Newton-Raphson method. Starting with an initial estimate of $x^{(0)} = 6$, we extrapolate along the tangent to its intersection with the $x$-axis and take that as the next approximation. This is continued until successive $x$-values are sufficiently close.

The analytical solution given by the Newton-Raphson algorithm is

$$\frac{df(x)}{dx} = 3x^2 - 12x + 9$$

$$\Delta c^{(0)} = c - f(x^{(0)}) = 0 - [(6)^3 - 6(6)^2 + 9(6) - 4] = -50$$

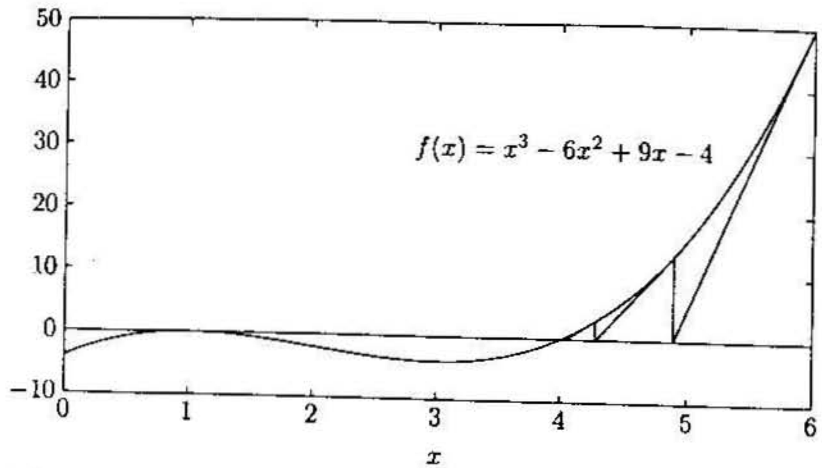



**FIGURE 6.5**
Graphical illustration of the Newton-Raphson algorithm.

$$\left(\frac{df}{dx}\right)^{(0)} = 3(6)^2 - 12(6) + 9 = 45$$

$$\Delta x^{(0)} = \frac{\Delta c^{(0)}}{\left(\frac{df}{dx}\right)^{(0)}} = \frac{-50}{45} = -1.1111$$

Therefore, the result at the end of the first iteration is

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = 6 - 1.1111 = 4.8889$$

The subsequent iterations result in

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 4.8889 - \frac{13.4431}{22.037} = 4.2789$$

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 4.2789 - \frac{2.9981}{12.5797} = 4.0405$$

$$x^{(4)} = x^{(3)} + \Delta x^{(3)} = 4.0405 - \frac{0.3748}{9.4914} = 4.0011$$

$$x^{(5)} = x^{(4)} + \Delta x^{(4)} = 4.0011 - \frac{0.0095}{9.0126} = 4.0000$$





We see that Newton's method converges considerably more rapidly than the Gauss-Seidel method. The method may converge to a root different from the expected one or diverge if the starting value is not close enough to the root.

The following commands show the procedure for the solution of the given equation by the Newton-Raphson method.

```
dx=1;          % Change in variable is set to a high value
x=input('Enter initial estimate -> '); % Initial estimate
iter = 0;                              % Iteration counter
disp('iter   Dc        J        dx        x') % Heading
while abs(dx) >= 0.001 & iter < 100  Test for convergence
iter = iter + 1;                           % No. of iterations
Dc = 0 - (x^3 - 6*x^2 + 9*x - 4);                % Residual
J = 3*x^2 - 12*x + 9;                          % Derivative
dx= Dc/J;                              %Change in variable
x=x + dx;                             % Successive solution
fprintf('%g', iter), disp([Dc, J, dx, x])
end
```

The result is

```
Enter the initial estimate -> 6
iter   Dc          J          dx          x
  1  -50.0000   45.0000   -1.1111    4.8889
  2  -13.4431   22.0370   -0.6100    4.2789
  3   -2.9981   12.5797   -0.2383    4.0405
  4   -0.3748    9.4914   -0.0395    4.0011
  5   -0.0095    9.0126   -0.0011    4.0000
  6   -0.0000    9.0000   -0.0000    4.0000
```

Now consider the $n$-dimensional equations given by (6.11). Expanding the left-hand side of the equations (6.11) in the Taylor's series about the initial estimates and neglecting all higher order terms, leads to the expression

$$\begin{aligned}
(f_1)^{(0)} + \left(\frac{\partial f_1}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_1}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \cdots + \left(\frac{\partial f_1}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)} &= c_1 \\
(f_2)^{(0)} + \left(\frac{\partial f_2}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_2}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \cdots + \left(\frac{\partial f_2}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)} &= c_2 \\
&\vdots \\
(f_n)^{(0)} + \left(\frac{\partial f_n}{\partial x_1}\right)^{(0)} \Delta x_1^{(0)} + \left(\frac{\partial f_n}{\partial x_2}\right)^{(0)} \Delta x_2^{(0)} + \cdots + \left(\frac{\partial f_n}{\partial x_n}\right)^{(0)} \Delta x_n^{(0)} &= c_n
\end{aligned}$$

or in matrix form

$$\begin{bmatrix} c_1 - (f_1)^{(0)} \\ c_2 - (f_2)^{(0)} \\ \vdots \\ c_n - (f_n)^{(0)} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^{(0)} & \left(\frac{\partial f_1}{\partial x_2}\right)^{(0)} & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)^{(0)} \\ \left(\frac{\partial f_2}{\partial x_1}\right)^{(0)} & \left(\frac{\partial f_2}{\partial x_2}\right)^{(0)} & \cdots & \left(\frac{\partial f_2}{\partial x_n}\right)^{(0)} \\ \vdots & \vdots & \ddots & \cdots \\ \left(\frac{\partial f_n}{\partial x_1}\right)^{(0)} & \left(\frac{\partial f_n}{\partial x_2}\right)^{(0)} & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)^{(0)} \end{bmatrix} \begin{bmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ \vdots \\ \Delta x_n^{(0)} \end{bmatrix}$$

In short form, it can be written as

$$\Delta C^{(k)} = J^{(k)} \Delta X^{(k)}$$

or

$$\Delta X^{(k)} = [J^{(k)}]^{-1} \Delta C^{(k)} \tag{6.19}$$

and the Newton-Raphson algorithm for the $n$-dimensional case becomes

$$X^{(k+1)} = X^{(k)} + \Delta X^{(k)} \tag{6.20}$$

where

$$\Delta X^{(k)} = \begin{bmatrix} \Delta x_1^{(k)} \\ \Delta x_2^{(k)} \\ \vdots \\ \Delta x_n^{(k)} \end{bmatrix} \quad \text{and} \quad \Delta C^{(k)} = \begin{bmatrix} c_1 - (f_1)^{(k)} \\ c_2 - (f_2)^{(k)} \\ \vdots \\ c_n - (f_n)^{(k)} \end{bmatrix} \tag{6.21}$$

$$J^{(k)} = \begin{bmatrix} \left(\frac{\partial f_1}{\partial x_1}\right)^{(k)} & \left(\frac{\partial f_1}{\partial x_2}\right)^{(k)} & \cdots & \left(\frac{\partial f_1}{\partial x_n}\right)^{(k)} \\ \left(\frac{\partial f_2}{\partial x_1}\right)^{(k)} & \left(\frac{\partial f_2}{\partial x_2}\right)^{(k)} & \cdots & \left(\frac{\partial f_2}{\partial x_n}\right)^{(k)} \\ \vdots & \vdots & \ddots & \cdots \\ \left(\frac{\partial f_n}{\partial x_1}\right)^{(k)} & \left(\frac{\partial f_n}{\partial x_2}\right)^{(k)} & \cdots & \left(\frac{\partial f_n}{\partial x_n}\right)^{(k)} \end{bmatrix} \tag{6.22}$$

$J^{(k)}$ is called the *Jacobian matrix*. Elements of this matrix are the partial derivatives evaluated at $X^{(k)}$. It is assumed that $J^{(k)}$ has an inverse during each iteration. Newton's method, as applied to a set of nonlinear equations, reduces the problem to solving a set of linear equations in order to determine the values that improve the accuracy of the estimates.

The solution of (6.19) by inversion is very inefficient. It is not necessary to obtain the inverse of $J^{(k)}$. Instead, a direct solution is obtained by optimally ordered triangular factorization. In *MATLAB*, the solution of linear simultaneous equations $\Delta C = J \Delta X$ is obtained by using the matrix division operator \ (i.e., $\Delta X = J \setminus \Delta C$) which is based on the triangular factorization and Gaussian elimination.



**Example 6.5**

Use the Newton-Raphson method to find the intersections of the curves

$$\begin{aligned} x_1^2 + x_2^2 &= 4 \\ e^{x_1} + x_2 &= 1 \end{aligned}$$

Graphically, the solution to this system is represented by the intersections of the circle $x_1^2 + x_2^2 = 4$ with the curve $e^{x_1} + x_2 = 1$. Figure 6.6 shows that these are near $(1, -1.7)$ and $(-1.8, 0.8)$.

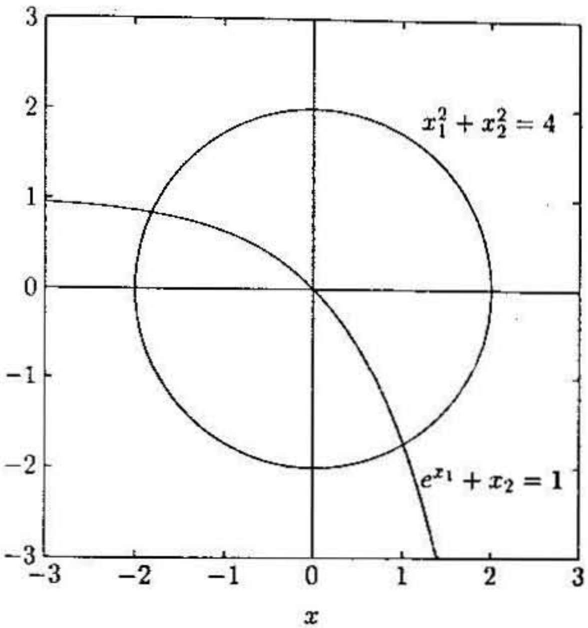



**FIGURE 6.6**
Graphs of Example 6.5.

Taking partial derivatives of the above functions results in the Jacobian matrix

$$J = \begin{bmatrix} 2x_1 & 2x_2 \\ e^{x_1} & 1 \end{bmatrix}$$

The Newton-Raphson algorithm for the above system is presented in the following statements.

```
iter = 0;                               % Iteration counter
```

```
x=input('Enter initial estimates, col. vector[x1;x2]->');
Dx = [1; 1];  % Change in variable is set to a high value
C=[4; 1];
disp('Iter   DC        Jacobian matrix        Dx          x');
                                        % Heading for results
while max(abs(Dx)) >= 0.0001 & iter <10 %Convergence test
iter=iter+1;                               % Iteration counter
f = [x(1)^2+x(2)^2;  exp(x(1))+x(2)];             % Functions
DC = C - f;                                       % Residuals
J = [2*x(1)       2*x(2)                    % Jacobian matrix
     exp(x(1))         1];
Dx=J\DC;                                % Change in variables
x=x+Dx;                                % Successive solutions
fprintf('%g', iter), disp([DC, J, Dx, x])         % Results
end
```

When the program is run, the user is prompted to enter the initial estimate. Let us try an initial estimate given by [0.5; -1].

Enter Initial estimates, col. vector $[x_1;\ x_2] \rightarrow$ [0.5; −1]

| Iter | $\Delta C$ | Jacobian matrix | | $\Delta x$ | $x$ |
|---|---|---|---|---|---|
| 1 | 2.7500 | 1.0000 | -2.0000 | 0.8034 | 1.3034 |
| | 0.3513 | 1.6487 | 1.0000 | -0.9733 | -1.9733 |
| 2 | -1.5928 | 2.6068 | -3.9466 | -0.2561 | 1.0473 |
| | -0.7085 | 3.6818 | 1.0000 | 0.2344 | -1.7389 |
| 3 | -0.1205 | 2.0946 | -3.4778 | -0.0422 | 1.0051 |
| | -0.1111 | 2.8499 | 1.0000 | 0.0092 | -1.7296 |
| 4 | -0.0019 | 2.0102 | -3.4593 | -0.0009 | 1.0042 |
| | -0.0025 | 2.7321 | 1.0000 | 0.0000 | -1.7296 |
| 5 | -0.0000 | 2.0083 | -3.4593 | -0.0000 | 1.0042 |
| | -0.0000 | 2.7296 | 1.0000 | -0.0000 | -1.7296 |

After five iterations, the solution converges to $x_1 = 1.0042$ and $x_2 = -1.7296$ accurate to four decimal places. Starting with an initial value of $[-0.5;\ 1]$, which is closer to the other intersection, results in $x_1 = -1.8163$ and $x_2 = 0.8374$.

**Example 6.6**

Starting with the initial values, $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$, solve the following system of equations by the Newton-Raphson method.

$$\begin{aligned} x_1^2 - x_2^2 + x_3^2 &= 11 \\ x_1x_2 + x_2^2 - 3x_3 &= 3 \\ x_1 - x_1x_3 + x_2x_3 &= 6 \end{aligned}$$





Taking partial derivatives of the above functions results in the Jacobian matrix

$$J = \begin{bmatrix} 2x_1 & -2x_2 & 2x_3 \\ x_2 & x_1 + 2x_2 & -3 \\ 1 - x_3 & x_3 & -x_1 + x_2 \end{bmatrix}$$

The following statements solve the given system of equations by the Newton-Raphson algorithm

```
Dx=[10;10;10]; %Change in variable is set to a high value
x=[1; 1; 1];                              % Initial estimate
C=[11; 3; 6];
iter = 0;                                % Iteration counter
while max(abs(Dx))>=.0001 & iter<10;%Test for convergence
iter = iter + 1                         % No. of iterations
F = [x(1)^2-x(2)^2+x(3)^2                         % Functions
     x(1)*x(2)+x(2)^2-3*x(3)
     x(1)-x(1)*x(3)+x(2)*x(3)];
DC =C - F                                         % Residuals
J = [2*x(1)   -2*x(2)        2*x(3)         % Jacobian matrix
     x(2)     x(1)+2*x(2)   -3
     1-x(3)   x(3)          -x(1)+x(2)]
Dx=J\DC                                  %Change in variable
x=x+Dx                                  % Successive solution
end
```

The program results for the first iteration are

```
DC =                  J =
        10                   2     -2      2
         4                   1      3     -3
         5                   0      1      0
Dx =                  x =
        4.750                5.750
        5.000                6.000
        5.250                6.250
```

After six iterations, the solution converges to $x_1 = 2.0000$, $x_2 = 3.0000$, and $x_3 = 4.0000$.

Newton's method has the advantage of converging quadratically when we are near a root. However, more functional evaluations are required during each iteration. A very important limitation is that it does not generally converge to a solution from an arbitrary starting point.

## 6.4 POWER FLOW SOLUTION

Power flow studies, commonly known as *load flow*, form an important part of power system analysis. They are necessary for planning, economic scheduling, and control of an existing system as well as planning its future expansion. The problem consists of determining the magnitudes and phase angle of voltages at each bus and active and reactive power flow in each line.

In solving a power flow problem, the system is assumed to be operating under balanced conditions and a single-phase model is used. Four quantities are associated with each bus. These are voltage magnitude $|V|$, phase angle $\delta$, real power $P$, and reactive power $Q$. The system buses are generally classified into three types.

**Slack bus** One bus, known as *slack* or *swing bus*, is taken as reference where the magnitude and phase angle of the voltage are specified. This bus makes up the difference between the scheduled loads and generated power that are caused by the losses in the network.

**Load buses** At these buses the active and reactive powers are specified. The magnitude and the phase angle of the bus voltages are unknown. These buses are called P-Q buses.

**Regulated buses** These buses are the *generator buses*. They are also known as *voltage-controlled buses*. At these buses, the real power and voltage magnitude are specified. The phase angles of the voltages and the reactive power are to be determined. The limits on the value of the reactive power are also specified. These buses are called P-V buses.

### 6.4.1 POWER FLOW EQUATION

Consider a typical bus of a power system network as shown in Figure 6.7. Transmission lines are represented by their equivalent $\pi$ models where impedances have been converted to per unit admittances on a common MVA base.

Application of KCL to this bus results in

$$\begin{aligned} I_i &= y_{i0}V_i + y_{i1}(V_i - V_1) + y_{i2}(V_i - V_2) + \cdots + y_{in}(V_i - V_n) \\ &= (y_{i0} + y_{i1} + y_{i2} + \cdots + y_{in})V_i - y_{i1}V_1 - y_{i2}V_2 - \cdots - y_{in}V_n \end{aligned} \tag{6.23}$$

or

$$I_i = V_i \sum_{j=0}^{n} y_{ij} - \sum_{j=1}^{n} y_{ij}V_j \qquad j \neq i \tag{6.24}$$





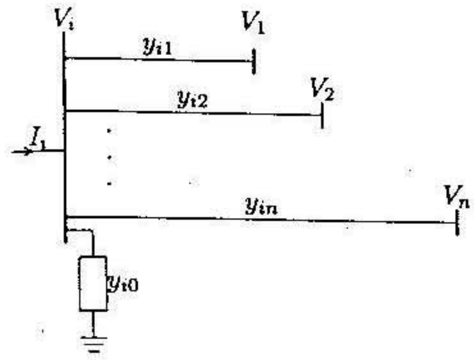



**FIGURE 6.7**
A typical bus of the power system.

The real and reactive power at bus $i$ is

$$P_i + jQ_i = V_i I_i^* \tag{6.25}$$

or

$$I_i = \frac{P_i - jQ_i}{V_i^*} \tag{6.26}$$

Substituting for $I_i$ in (6.24) yields

$$\frac{P_i - jQ_i}{V_i^*} = V_i \sum_{j=0}^{n} y_{ij} - \sum_{j=1}^{n} y_{ij}V_j \qquad j \neq i \tag{6.27}$$

From the above relation, the mathematical formulation of the power flow problem results in a system of algebraic nonlinear equations which must be solved by iterative techniques.

## 6.5 GAUSS-SEIDEL POWER FLOW SOLUTION

In the power flow study, it is necessary to solve the set of nonlinear equations represented by (6.27) for two unknown variables at each node. In the Gauss-Seidel method (6.27) is solved for $V_i$, and the iterative sequence becomes

$$V_i^{(k+1)} = \frac{\frac{P_i^{sch} - jQ_i^{sch}}{V_i^{*(k)}} + \sum y_{ij}V_j^{(k)}}{\sum y_{ij}} \qquad j \neq i \tag{6.28}$$

where $y_{ij}$ shown in lowercase letters is the actual admittance in per unit. $P_i^{sch}$ and $Q_i^{sch}$ are the net real and reactive powers expressed in per unit. In writing the KCL, current entering bus $i$ was assumed positive. Thus, for buses where real and reactive powers are injected into the bus, such as generator buses, $P_i^{sch}$ and $Q_i^{sch}$ have positive values. For load buses where real and reactive powers are flowing away from the bus, $P_i^{sch}$ and $Q_i^{sch}$ have negative values. If (6.27) is solved for $P_i$ and $Q_i$, we have

$$P_i^{(k+1)} = \Re\{V_i^{*(k)}[V_i^{(k)}\sum_{j=0}^{n} y_{ij} - \sum_{j=1}^{n} y_{ij}V_j^{(k)}]\} \qquad j \neq i \tag{6.29}$$

$$Q_i^{(k+1)} = -\Im\{V_i^{*(k)}[V_i^{(k)}\sum_{j=0}^{n} y_{ij} - \sum_{j=1}^{n} y_{ij}V_j^{(k)}]\} \qquad j \neq i \tag{6.30}$$

The power flow equation is usually expressed in terms of the elements of the bus admittance matrix. Since the off-diagonal elements of the bus admittance matrix $Y_{bus}$, shown by uppercase letters, are $Y_{ij} = -y_{ij}$, and the diagonal elements are $Y_{ii} = \sum y_{ij}$, (6.28) becomes

$$V_i^{(k+1)} = \frac{\frac{P_i^{sch} - jQ_i^{sch}}{V_i^{*(k)}} - \sum_{j \neq i} Y_{ij}V_j^{(k)}}{Y_{ii}} \tag{6.31}$$

and

$$P_i^{(k+1)} = \Re\{V_i^{*(k)}[V_i^{(k)}Y_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} Y_{ij}V_j^{(k)}]\} \quad j \neq i \tag{6.32}$$

$$Q_i^{(k+1)} = -\Im\{V_i^{*(k)}[V_i^{(k)}Y_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} Y_{ij}V_j^{(k)}]\} \quad j \neq i \tag{6.33}$$

$Y_{ii}$ includes the admittance to ground of line charging susceptance and any other fixed admittance to ground. In Section 6.7, a model is presented for transformers containing off-nominal ratio, which includes the effect of transformer tap setting.

Since both components of voltage are specified for the slack bus, there are $2(n-1)$ equations which must be solved by an iterative method. Under normal operating conditions, the voltage magnitude of buses are in the neighborhood of 1.0 per unit or close to the voltage magnitude of the slack bus. Voltage magnitude at load buses are somewhat lower than the slack bus value, depending on the reactive power demand, whereas the scheduled voltage at the generator buses are somewhat higher. Also, the phase angle of the load buses are below the reference angle in accordance to the real power demand, whereas the phase angle of the generator





buses may be above the reference value depending on the amount of real power flowing into the bus. Thus, for the Gauss-Seidel method, an initial voltage estimate of $1.0 + j0.0$ for unknown voltages is satisfactory, and the converged solution correlates with the actual operating states.

For P-Q buses, the real and reactive powers $P_i^{sch}$ and $Q_i^{sch}$ are known. Starting with an initial estimate, (6.31) is solved for the real and imaginary components of voltage. For the voltage-controlled buses (P-V buses) where $P_i^{sch}$ and $|V_i|$ are specified, first (6.33) is solved for $Q_i^{(k+1)}$, and then is used in (6.31) to solve for $V_i^{(k+1)}$. However, since $|V_i|$ is specified, only the imaginary part of $V_i^{(k+1)}$ is retained, and its real part is selected in order to satisfy

$$(e_i^{(k+1)})^2 + (f_i^{(k+1)})^2 = |V_i|^2 \tag{6.34}$$

or

$$e_i^{(k+1)} = \sqrt{|V_i|^2 - (f_i^{(k+1)})^2} \tag{6.35}$$

where $e_i^{(k+1)}$ and $f_i^{(k+1)}$ are the real and imaginary components of the voltage $V_i^{(k+1)}$ in the iterative sequence.

The rate of convergence is increased by applying an acceleration factor to the approximate solution obtained from each iteration.

$$V_i^{(k+1)} = V_i^{(k)} + \alpha(V_{i\,cal}^{(k)} - V_i^{(k)}) \tag{6.36}$$

where $\alpha$ is the acceleration factor. Its value depends upon the system. The range of 1.3 to 1.7 is found to be satisfactory for typical systems.

The updated voltages immediately replace the previous values in the solution of the subsequent equations. The process is continued until changes in the real and imaginary components of bus voltages between successive iterations are within a specified accuracy, i.e.,

$$|e_i^{(k+1)} - e_i^{(k)}| \leq \epsilon$$
$$|f_i^{(k+1)} - f_i^{(k)}| \leq \epsilon \tag{6.37}$$

For the power mismatch to be reasonably small and acceptable, a very tight tolerance must be specified on both components of the voltage. A voltage accuracy in the range of 0.00001 to 0.00005 pu is satisfactory. In practice, the method for determining the completion of a solution is based on an accuracy index set up on the power mismatch. The iteration continues until the magnitude of the largest element in the $\Delta P$ and $\Delta Q$ columns is less than the specified value. A typical power mismatch accuracy is 0.001 pu

Once a solution is converged, the net real and reactive powers at the slack bus are computed from (6.32) and (6.33).

## 6.6 LINE FLOWS AND LOSSES

After the iterative solution of bus voltages, the next step is the computation of line flows and line losses. Consider the line connecting the two buses $i$ and $j$ in Figure 6.8. The line current $I_{ij}$, measured at bus $i$ and defined positive in the direction

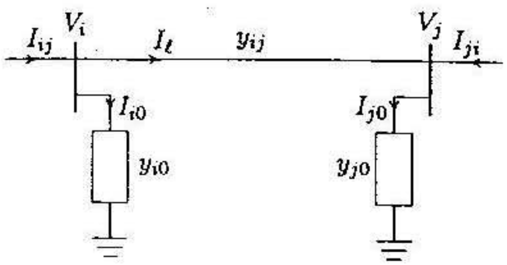



**FIGURE 6.8**
Transmission line model for calculating line flows.

$i \to j$ is given by

$$I_{ij} = I_\ell + I_{i0} = y_{ij}(V_i - V_j) + y_{i0}V_i \tag{6.38}$$

Similarly, the line current $I_{ji}$ measured at bus $j$ and defined positive in the direction $j \to i$ is given by

$$I_{ji} = -I_\ell + I_{j0} = y_{ij}(V_j - V_i) + y_{j0}V_j \tag{6.39}$$

The complex powers $S_{ij}$ from bus $i$ to $j$ and $S_{ji}$ from bus $j$ to $i$ are

$$S_{ij} = V_i I_{ij}^* \tag{6.40}$$

$$S_{ji} = V_j I_{ji}^* \tag{6.41}$$

The power loss in line $i - j$ is the algebraic sum of the power flows determined from (6.40) and (6.41), i.e.,

$$S_{L\,ij} = S_{ij} + S_{ji} \tag{6.42}$$

The power flow solution by the Gauss-Seidel method is demonstrated in the following two examples.

### Example 6.7

Figure 6.9 shows the one-line diagram of a simple three-bus power system with generation at bus 1. The magnitude of voltage at bus 1 is adjusted to 1.05 per





unit. The scheduled loads at buses 2 and 3 are as marked on the diagram. Line impedances are marked in per unit on a 100-MVA base and the line charging susceptances are neglected.

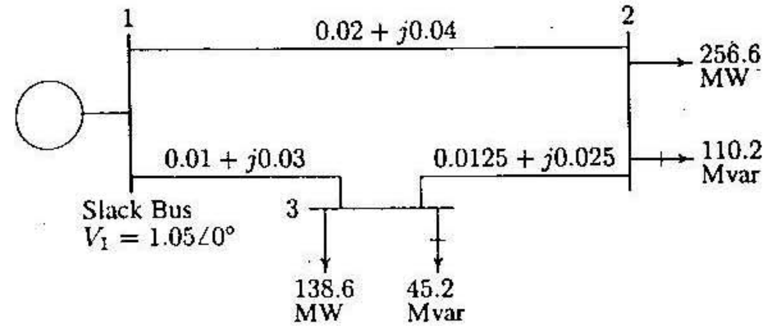



**FIGURE 6.9**
One-line diagram of Example 6.7 (impedances in pu on 100-MVA base).

(a) Using the Gauss-Seidel method, determine the phasor values of the voltage at the load buses 2 and 3 (P-Q buses) accurate to four decimal places.
(b) Find the slack bus real and reactive power.
(c) Determine the line flows and line losses. Construct a power flow diagram showing the direction of line flow.

(a) Line impedances are converted to admittances

$$y_{12} = \frac{1}{0.02 + j0.04} = 10 - j20$$

Similarly, $y_{13} = 10 - j30$ and $y_{23} = 16 - j32$. The admittances are marked on the network shown in Figure 6.10.
At the P-Q buses, the complex loads expressed in per units are

$$S_2^{sch} = -\frac{(256.6 + j110.2)}{100} = -2.566 - j1.102 \quad \text{pu}$$

$$S_3^{sch} = -\frac{(138.6 + j45.2)}{100} = -1.386 - j0.452 \quad \text{pu}$$

Since the actual admittances are readily available in Figure 6.10, for hand calculation, we use (6.28). Bus 1 is taken as reference bus (slack bus). Starting from an initial estimate of $V_2^{(0)} = 1.0 + j0.0$ and $V_3^{(0)} = 1.0 + j0.0$, $V_2$ and $V_3$ are computed from (6.28) as follows

$$V_2^{(1)} = \frac{\frac{P_2^{sch} - jQ_2^{sch}}{V_2^{*(0)}} + y_{12}V_1 + y_{23}V_3^{(0)}}{y_{12} + y_{23}}$$

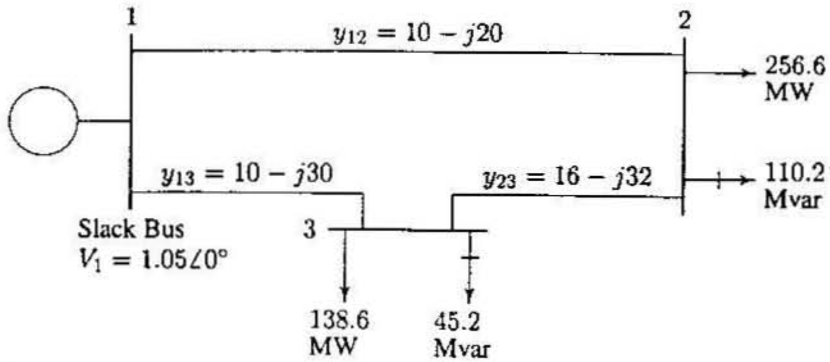



**FIGURE 6.10**
One-line diagram of Example 6.7 (admittances in pu on 100-MVA base).

$$= \frac{\frac{-2.566+j1.102}{1.0-j0} + (10 - j20)(1.05 + j0) + (16 - j32)(1.0 + j0)}{(26 - j52)}$$
$$= 0.9825 - j0.0310$$

and

$$V_3^{(1)} = \frac{\frac{P_3^{sch} - jQ_3^{sch}}{V_3^{*(0)}} + y_{13}V_1 + y_{23}V_2^{(1)}}{y_{13} + y_{23}}$$
$$= \frac{\frac{-1.386+j0.452}{1-j0} + (10 - j30)(1.05 + j0) + (16 - j32)(0.9825 - j0.0310)}{(26 - j62)}$$
$$= 1.0011 - j0.0353$$

For the second iteration we have

$$V_2^{(2)} = \frac{\frac{-2.566+j1.102}{0.9825+j0.0310} + (10 - j20)(1.05 + j0) + (16 - j32)(1.0011 - j0.0353)}{(26 - j52)}$$
$$= 0.9816 - j0.0520$$

and

$$V_3^{(2)} = \frac{\frac{-1.386+j0.452}{1.0011+j0.0353} + (10 - j30)(1.05 + j0) + (16 - j32)(0.9816 - j0.052)}{(26 - j62)}$$
$$= 1.0008 - j0.0459$$

The process is continued and a solution is converged with an accuracy of $5 \times 10^{-5}$ per unit in seven iterations as given below.

$$V_2^{(3)} = 0.9808 - j0.0578 \qquad V_3^{(3)} = 1.0004 - j0.0488$$





$$V_2^{(4)} = 0.9803 - j0.0594 \qquad V_3^{(4)} = 1.0002 - j0.0497$$
$$V_2^{(5)} = 0.9801 - j0.0598 \qquad V_3^{(5)} = 1.0001 - j0.0499$$
$$V_2^{(6)} = 0.9801 - j0.0599 \qquad V_3^{(6)} = 1.0000 - j0.0500$$
$$V_2^{(7)} = 0.9800 - j0.0600 \qquad V_3^{(7)} = 1.0000 - j0.0500$$

The final solution is

$$V_2 = 0.9800 - j0.0600 = 0.98183\angle{-3.5035°} \quad \text{pu}$$
$$V_3 = 1.0000 - j0.0500 = 1.00125\angle{-2.8624°} \quad \text{pu}$$

(b) With the knowledge of all bus voltages, the slack bus power is obtained from (6.27)

$$P_1 - jQ_1 = V_1^*[V_1(y_{12} + y_{13}) - (y_{12}V_2 + y_{13}V_3)]$$
$$= 1.05[1.05(20 - j50) - (10 - j20)(0.98 - j.06) - (10 - j30)(1.0 - j0.05)]$$
$$= 4.095 - j1.890$$

or the slack bus real and reactive powers are $P_1 = 4.095$ pu $= 409.5$ MW and $Q_1 = 1.890$ pu $= 189$ Mvar.

(c) To find the line flows, first the line currents are computed. With line charging capacitors neglected, the line currents are

$$I_{12} = y_{12}(V_1 - V_2) = (10 - j20)[(1.05 + j0) - (0.98 - j0.06)] = 1.9 - j0.8$$
$$I_{21} = -I_{12} = -1.9 + j0.8$$
$$I_{13} = y_{13}(V_1 - V_3) = (10 - j30)[(1.05 + j0) - (1.0 - j0.05)] = 2.0 - j1.0$$
$$I_{31} = -I_{13} = -2.0 + j1.0$$
$$I_{23} = y_{23}(V_2 - V_3) = (16 - j32)[(0.98 - j0.06) - (1 - j0.05)] = -.64 + j.48$$
$$I_{32} = -I_{23} = 0.64 - j0.48$$

The line flows are

$$S_{12} = V_1 I_{12}^* = (1.05 + j0.0)(1.9 + j0.8) = 1.995 + j0.84 \text{ pu}$$
$$= 199.5 \text{ MW} + j84.0 \text{ Mvar}$$
$$S_{21} = V_2 I_{21}^* = (0.98 - j0.06)(-1.9 - j0.8) = -1.91 - j0.67 \text{ pu}$$
$$= -191.0 \text{ MW} - j67.0 \text{ Mvar}$$
$$S_{13} = V_1 I_{13}^* = (1.05 + j0.0)(2.0 + j1.0) = 2.1 + j1.05 \text{ pu}$$
$$= 210.0 \text{ MW} + j105.0 \text{ Mvar}$$

$$S_{31} = V_3 I_{31}^* = (1.0 - j0.05)(-2.0 - j1.0) = -2.05 - j0.90 \text{ pu}$$
$$= -205.0 \text{ MW} - j90.0 \text{ Mvar}$$
$$S_{23} = V_2 I_{23}^* = (0.98 - j0.06)(-0.656 + j0.48) = -0.656 - j0.432 \text{ pu}$$
$$= -65.6 \text{ MW} - j43.2 \text{ Mvar}$$
$$S_{32} = V_3 I_{32}^* = (1.0 - j0.05)(0.64 + j0.48) = 0.664 + j0.448 \text{ pu}$$
$$= 66.4 \text{ MW} + j44.8 \text{ Mvar}$$

and the line losses are

$$S_{L\,12} = S_{12} + S_{21} = 8.5 \text{ MW} + j17.0 \text{ Mvar}$$
$$S_{L\,13} = S_{13} + S_{31} = 5.0 \text{ MW} + j15.0 \text{ Mvar}$$
$$S_{L\,23} = S_{23} + S_{32} = 0.8 \text{ MW} + j1.60 \text{ Mvar}$$

The power flow diagram is shown in Figure 6.11, where real power direction is indicated by → and the reactive power direction is indicated by ↦. The values within parentheses are the real and reactive losses in the line.

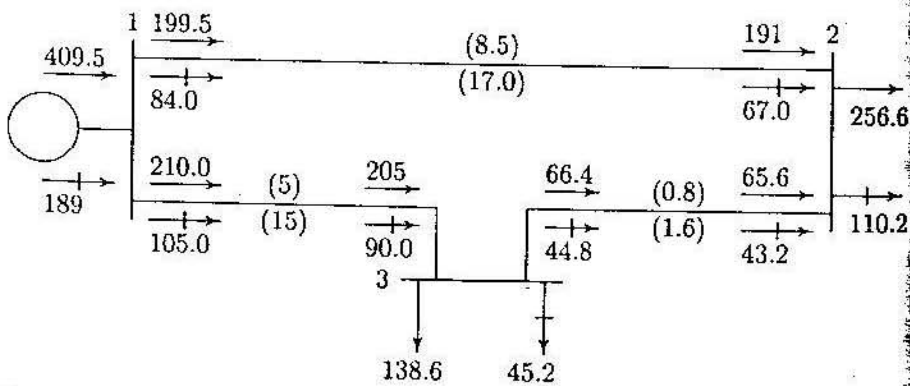



**FIGURE 6.11**
Power flow diagram of Example 6.7 (powers in MW and Mvar).

### Example 6.8

Figure 6.12 shows the one-line diagram of a simple three-bus power system with generators at buses 1 and 3. The magnitude of voltage at bus 1 is adjusted to 1.05 pu. Voltage magnitude at bus 3 is fixed at 1.04 pu with a real power generation of 200 MW. A load consisting of 400 MW and 250 Mvar is taken from bus 2. Line impedances are marked in per unit on a 100 MVA base, and the line charging susceptances are neglected. Obtain the power flow solution by the Gauss-Seidel method including line flows and line losses.





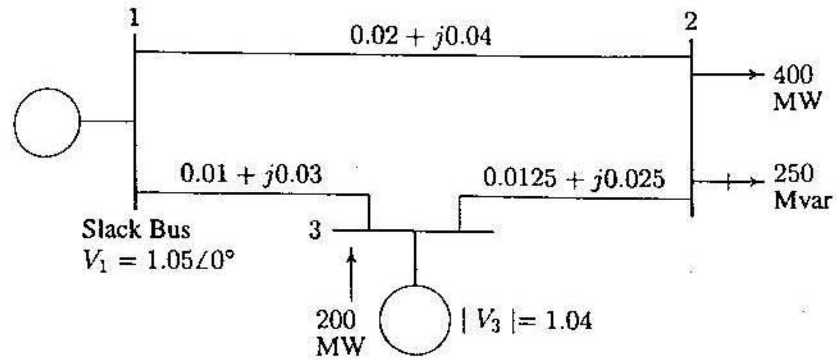



**FIGURE 6.12**
One-line diagram of Example 6.8 (impedances in pu on 100-MVA base).

Line impedances converted to admittances are $y_{12} = 10 - j20$, $y_{13} = 10 - j30$ and $y_{23} = 16 - j32$. The load and generation expressed in per units are

$$S_2^{sch} = -\frac{(400 + j250)}{100} = -4.0 - j2.5 \quad \text{pu}$$
$$P_3^{sch} = \frac{200}{100} = 2.0 \quad \text{pu}$$

Bus 1 is taken as the reference bus (slack bus). Starting from an initial estimate of $V_2^{(0)} = 1.0 + j0.0$ and $V_3^{(0)} = 1.04 + j0.0$, $V_2$ and $V_3$ are computed from (6.28).

$$V_2^{(1)} = \frac{\frac{P_2^{sch} - jQ_2^{sch}}{V_2^{*(0)}} + y_{12}V_1 + y_{23}V_3^{(0)}}{y_{12} + y_{23}}$$
$$= \frac{\frac{-4.0 + j2.5}{1.0 - j0} + (10 - j20)(1.05 + j0) + (16 - j32)(1.04 + j0)}{(26 - j52)}$$
$$= 0.97462 - j0.042307$$

Bus 3 is a regulated bus where voltage magnitude and real power are specified. For the voltage-controlled bus, first the reactive power is computed from (6.30)

$$Q_3^{(1)} = -\Im\{V_3^{*(0)}[V_3^{(0)}(y_{13} + y_{23}) - y_{13}V_1 - y_{23}V_2^{(1)}]\}$$
$$= -\Im\{(1.04 - j0)[(1.04 + j0)(26 - j62) - (10 - j30)(1.05 + j0) - (16 - j32)(0.97462 - j0.042307)]\}$$
$$= 1.16$$

The value of $Q_3^{(1)}$ is used as $Q_3^{sch}$ for the computation of voltage at bus 3. The complex voltage at bus 3, denoted by $V_{c3}^{(1)}$, is calculated

$$V_{c3}^{(1)} = \frac{\frac{P_3^{sch} - jQ_3^{sch}}{V_3^{*(0)}} + y_{13}V_1 + y_{23}V_2^{(1)}}{y_{13} + y_{23}}$$
$$= \frac{\frac{2.0 - j1.16}{1.04 - j0} + (10 - j30)(1.05 + j0) + (16 - j32)(0.97462 - j0.042307)}{(26 - j62)}$$
$$= 1.03783 - j0.005170$$

Since $|V_3|$ is held constant at 1.04 pu, only the imaginary part of $V_{c3}^{(1)}$ is retained, i.e, $f_3^{(1)} = -0.005170$, and its real part is obtained from

$$e_3^{(1)} = \sqrt{(1.04)^2 - (0.005170)^2} = 1.039987$$

Thus

$$V_3^{(1)} = 1.039987 - j0.005170$$

For the second iteration, we have

$$V_2^{(2)} = \frac{\frac{P_2^{sch} - jQ_2^{sch}}{V_2^{*(1)}} + y_{12}V_1 + y_{23}V_3^{(1)}}{y_{12} + y_{23}}$$
$$= \frac{\frac{-4.0 + j2.5}{.97462 + j.042307} + (10 - j20)(1.05) + (16 - j32)(1.039987 + j0.005170)}{(26 - j52)}$$
$$= 0.971057 - j0.043432$$

$$Q_3^{(2)} = -\Im\{V_3^{*(1)}[V_3^{(1)}(y_{13} + y_{23}) - y_{13}V_1 - y_{23}V_2^{(2)}]\}$$
$$= -\Im\{(1.039987 + j0.005170)[(1.039987 - j0.005170)(26 - j62) - (10 - j30)(1.05 + j0) - (16 - j32)(0.971057 - j0.043432)]\}$$
$$= 1.38796$$

$$V_{c3}^{(2)} = \frac{\frac{P_3^{sch} - jQ_3^{sch}}{V_3^{*(1)}} + y_{13}V_1 + y_{23}V_2^{(2)}}{y_{13} + y_{23}}$$
$$= \frac{\frac{2.0 - j1.38796}{1.039987 + j0.00517} + (10 - j30)(1.05) + (16 - j32)(.971057 - j.043432)}{(26 - j62)}$$
$$= 1.03908 - j0.00730$$





Since $|V_3|$ is held constant at 1.04 pu, only the imaginary part of $V_{c3}^{(2)}$ is retained, i.e, $f_3^{(2)} = -0.00730$, and its real part is obtained from

$$e_3^{(2)} = \sqrt{(1.04)^2 - (0.00730)^2} = 1.039974$$

or

$$V_3^{(2)} = 1.039974 - j0.00730$$

The process is continued and a solution is converged with an accuracy of $5 \times 10^{-5}$ pu in seven iterations as given below.

$$V_2^{(3)} = 0.97073 - j0.04479 \quad Q_3^{(3)} = 1.42904 \quad V_3^{(3)} = 1.03996 - j0.00833$$
$$V_2^{(4)} = 0.97065 - j0.04533 \quad Q_3^{(4)} = 1.44833 \quad V_3^{(4)} = 1.03996 - j0.00873$$
$$V_2^{(5)} = 0.97062 - j0.04555 \quad Q_3^{(5)} = 1.45621 \quad V_3^{(5)} = 1.03996 - j0.00893$$
$$V_2^{(6)} = 0.97061 - j0.04565 \quad Q_3^{(6)} = 1.45947 \quad V_3^{(6)} = 1.03996 - j0.00900$$
$$V_2^{(7)} = 0.97061 - j0.04569 \quad Q_3^{(7)} = 1.46082 \quad V_3^{(7)} = 1.03996 - j0.00903$$

The final solution is

$$V_2 = 0.97168\angle -2.6948^\circ \quad \text{pu}$$

$$S_3 = 2.0 + j1.4617 \quad \text{pu}$$
$$V_3 = 1.04\angle -.498^\circ \quad \text{pu}$$
$$S_1 = 2.1842 + j1.4085 \quad \text{pu}$$

Line flows and line losses are computed as in Example 6.7, and the results expressed in MW and Mvar are

$$S_{12} = 179.36 + j118.734 \quad S_{21} = -170.97 - j101.947 \quad S_{L12} = 8.39 + j16.79$$
$$S_{13} = 39.06 + j22.118 \quad S_{31} = -38.88 - j\,21.569 \quad S_{L13} = 0.18 + j0.548$$
$$S_{23} = -229.03 - j148.05 \quad S_{32} = 238.88 + j167.746 \quad S_{L23} = 9.85 + j19.69$$

The power flow diagram is shown in Figure 6.13, where real power direction is indicated by $\rightarrow$ and the reactive power direction is indicated by $\mapsto$. The values within parentheses are the real and reactive losses in the line.

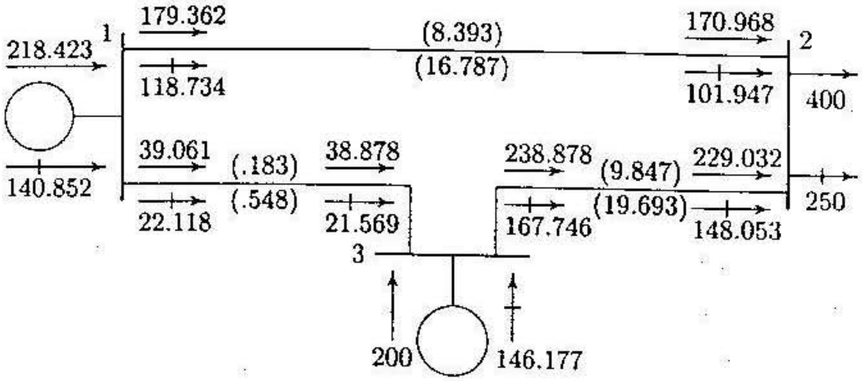



**FIGURE 6.13**
Power flow diagram of Example 6.8 (powers in MW and Mvar).

## 6.7 TAP CHANGING TRANSFORMERS

In Section 2.6 it was shown that the flow of real power along a transmission line is determined by the angle difference of the terminal voltages, and the flow of reactive power is determined mainly by the magnitude difference of terminal voltages. Real and reactive powers can be controlled by use of tap changing transformers and regulating transformers.

In a tap changing transformer, when the ratio is at the nominal value, the transformer is represented by a series admittance $y_t$ in per unit. With off-nominal ratio, the per unit admittance is different from both sides of the transformer, and the admittance must be modified to include the effect of the off-nominal ratio. Consider a transformer with admittance $y_t$ in series with an ideal transformer representing the off-nominal tap ratio 1:$a$ as shown in Figure 6.14. $y_t$ is the admittance in per unit based on the nominal turn ratio and $a$ is the per unit off-nominal tap position allowing for small adjustment in voltage of usually $\pm 10$ percent. In the case of phase shifting transformers, $a$ is a complex number. Consider a fictitious bus $x$ between the turn ratio and admittance of the transformer. Since the complex power on either side of the ideal transformer is the same, it follows that if the voltage goes through a positive phase angle shift, the current will go through a negative phase angle shift. Thus, for the assumed direction of currents, we have

$$V_x = \frac{1}{a} V_j \tag{6.43}$$

$$I_i = -a^* I_j \tag{6.44}$$

The current $I_i$ is given by

$$I_i = y_t(V_i - V_x)$$





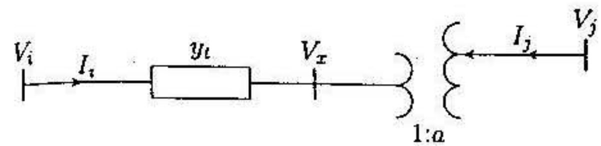



**FIGURE 6.14**
Transformer with tap setting ratio $a$:1

Substituting for $V_x$, we have

$$I_i = y_t V_i - \frac{y_t}{a} V_j \tag{6.45}$$

Also, from (6.44) we have

$$I_j = -\frac{1}{a^*} I_i$$

substituting for $I_i$ from (6.45) we have

$$I_j = -\frac{y_t}{a^*} V_i + \frac{y_t}{|a|^2} V_j \tag{6.46}$$

writing (6.45) and (6.46) in matrix form results in

$$\begin{bmatrix} I_i \\ I_j \end{bmatrix} = \begin{bmatrix} y_t & -\frac{y_t}{a} \\ -\frac{y_t}{a^*} & \frac{y_t}{|a|^2} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix} \tag{6.47}$$

For the case when $a$ is real, the $\pi$ model shown in Figure 6.15 represents the admittance matrix in (6.47). In the $\pi$ model, the left side corresponds to the non-tap side and the right side corresponds to the tap side of the transformer.

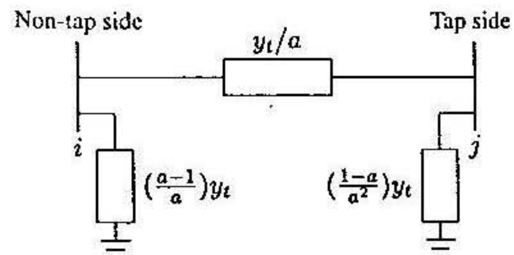



**FIGURE 6.15**
Equivalent circuit for a tap changing transformer.

## 6.8 POWER FLOW PROGRAMS

Several computer programs have been developed for the power flow solution of practical systems. Each method of solution consists of four programs. The program for the Gauss-Seidel method is **lfgauss**, which is preceded by **lfybus**, and is followed by **busout** and **lineflow**. Programs **lfybus**, **busout**, and **lineflow** are designed to be used with two more power flow programs. These are **lfnewton** for the Newton-Raphson method and **decouple** for the fast decoupled method. The following is a brief description of the programs used in the Gauss-Seidel method.

**lfybus** This program requires the line and transformer parameters and transformer tap settings specified in the input file named **linedata**. It converts impedances to admittances and obtains the bus admittance matrix. The program is designed to handle parallel lines.

**lfgauss** This program obtains the power flow solution by the Gauss-Seidel method and requires the files named **busdata** and **linedata**. It is designed for the direct use of load and generation in MW and Mvar, bus voltages in per unit, and angle in degrees. Loads and generation are converted to per unit quantities on the base MVA selected. A provision is made to maintain the generator reactive power of the voltage-controlled buses within their specified limits. The violation of reactive power limit may occur if the specified voltage is either too high or too low. After a few iterations ($10^{th}$ iteration in the Gauss method), the var calculated at the generator buses are examined. If a limit is reached, the voltage magnitude is adjusted in steps of 0.5 percent up to $\pm5$ percent to bring the var demand within the specified limits.

**busout** This program produces the bus output result in a tabulated form. The bus output result includes the voltage magnitude and angle, real and reactive power of generators and loads, and the shunt capacitor/reactor Mvar. Total generation and total load are also included as outlined in the sample case.

**lineflow** This program prepares the line output data. It is designed to display the active and reactive power flow entering the line terminals and line losses as well as the net power at each bus. Also included are the total real and reactive losses in the system. The output of this portion is also shown in the sample case.





## 6.9 DATA PREPARATION

In order to perform a power flow analysis by the Gauss-Seidel method in the *MATLAB* environment, the following variables must be defined: power system base MVA, power mismatch accuracy, acceleration factor, and maximum number of iterations. The name (in lowercase letters) reserved for these variables are **basemva**, **accuracy**, **accel**, and **maxiter**, respectively. Typical values are as follows:

```
basemva = 100;    accuracy = 0.001;
accel   = 1.6;    maxiter = 80;
```

The initial step in the preparation of input file is the numbering of each bus. Buses are numbered sequentially. Although the numbers are sequentially assigned, the buses need not be entered in sequence. In addition, the following data files are required.

**BUS DATA FILE – busdata** The format for the bus entry is chosen to facilitate the required data for each bus in a single row. The information required must be included in a matrix called **busdata**. Column 1 is the bus number. Column 2 contains the bus code. Columns 3 and 4 are voltage magnitude in per unit and phase angle in degrees. Columns 5 and 6 are load MW and Mvar. Column 7 through 10 are MW, Mvar, minimum Mvar and maximum Mvar of generation, in that order. The last column is the injected Mvar of shunt capacitors. The bus code entered in column 2 is used for identifying load, voltage-controlled, and slack buses as outlined below:

**1** This code is used for the slack bus. The only necessary information for this bus is the voltage magnitude and its phase angle.

**0** This code is used for load buses. The loads are entered positive in megawatts and megavars. For this bus, initial voltage estimate must be specified. This is usually 1 and 0 for voltage magnitude and phase angle, respectively. If voltage magnitude and phase angle for this type of bus are specified, they will be taken as the initial starting voltage for that bus instead of a flat start of 1 and 0.

**2** This code is used for the voltage-controlled buses. For this bus, voltage magnitude, real power generation in megawatts, and the minimum and maximum limits of the megavar demand must be specified.

**LINE DATA FILE – linedata** Lines are identified by the node-pair method. The information required must be included in a matrix called **linedata**. Columns 1 and 2 are the line bus numbers. Columns 3 through 5 contain the line resistance, reactance, and one-half of the total line charging susceptance in per unit on the specified

MVA base. The last column is for the transformer tap setting; for lines, 1 must be entered in this column. The lines may be entered in any sequence or order with the only restriction being that if the entry is a transformer, the left bus number is assumed to be the tap side of the transformer.

The IEEE 30 bus system is used to demonstrate the data preparation and the use of the power flow programs by the Gauss-Seidel method.

### Example 6.9

Figure 6.16 is part of the American Electric Power Service Corporation network which is being made available to the electric utility industry as a standard test case for evaluating various analytical methods and computer programs for the solution of power system problems. Use the **lfgauss** program to obtain the power solution by the Gauss-Seidel method. Bus 1 is taken as the slack bus with its voltage adjusted to $1.06\angle 0^\circ$ pu. The data for the voltage-controlled buses is

| Regulated Bus Data | | | |
|---|---|---|---|
| Bus No. | Voltage Magnitude | Min. Mvar Capacity | Max. Mvar Capacity |
| 2 | 1.043 | -40 | 50 |
| 5 | 1.010 | -40 | 40 |
| 8 | 1.010 | -10 | 40 |
| 11 | 1.082 | -6 | 24 |
| 13 | 1.071 | -6 | 24 |

Transformer tap setting are given in the table below. The left bus number is assumed to be the tap side of the transformer.

| Transformer Data | |
|---|---|
| Transformer Designation | Tap Setting pu |
| 4 – 12 | 0.932 |
| 6 – 9 | 0.978 |
| 6 – 10 | 0.969 |
| 28 – 27 | 0.968 |

The data for the injected $Q$ due to shunt capacitors is

| Injected Q due to Capacitors | |
|---|---|
| Bus No. | Mvar |
| 10 | 19 |
| 24 | 4.3 |





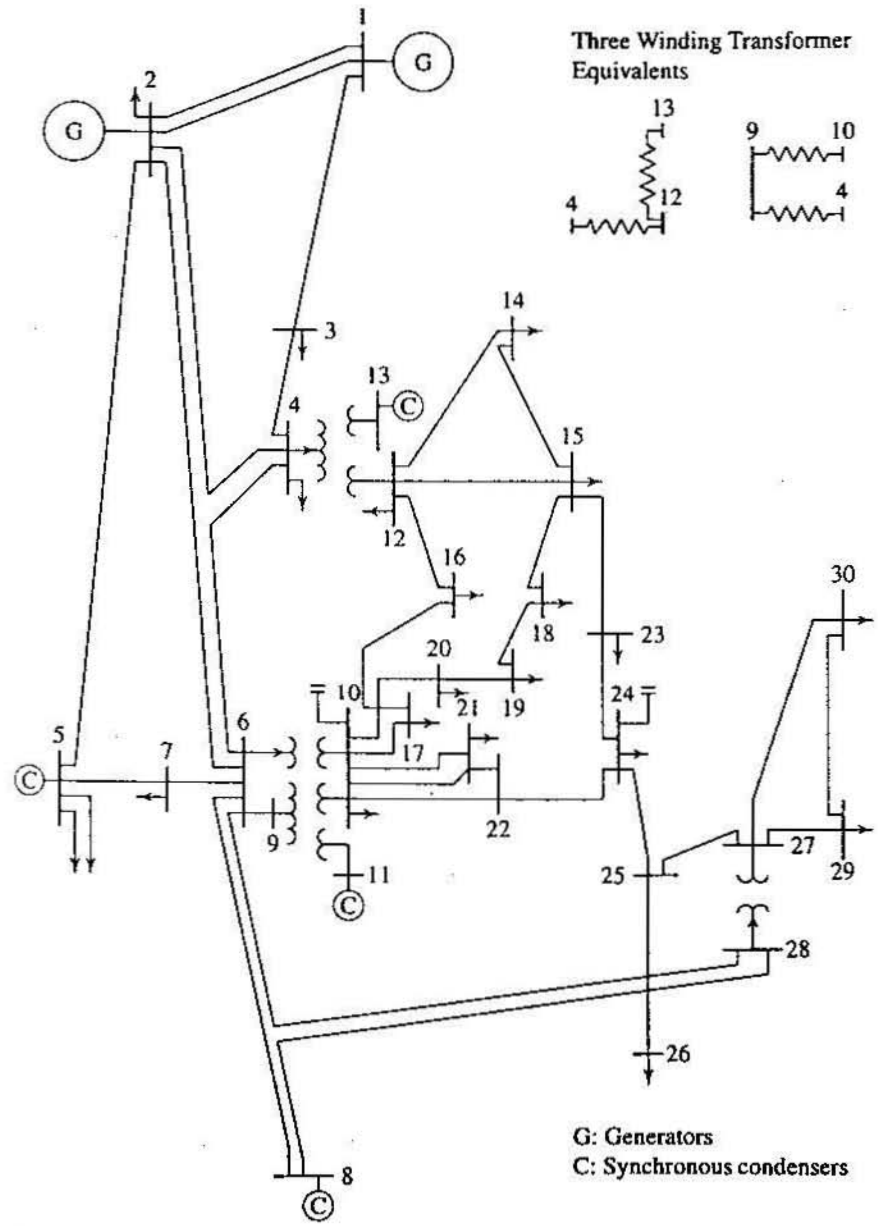



**FIGURE 6.16**
30-bus IEEE sample system.

Generation and loads are as given in the data prepared for use in the *MATLAB* environment in the matrix defined as **busdata**. Code 0, code 1, and code 2 are used for the load buses, the slack bus and the voltage-controlled buses, respectively. Values for **basemva**, **accuracy**, **accel** and **maxiter** must be specified. Line data are as given in the matrix called **linedata**. The last column of this data must contain 1 for lines, or the tap setting values for transformers with off-nominal turn ratio. The control commands required are **lfybus**, **lfgauss** and **lineflow**. A **diary** command may be used to save the output to the specified file name. The power flow data and the commands required are as follows.

```
clear          % clears all variables from workspace.
basemva = 100;  accuracy = 0.001; accel = 1.8; maxiter = 100;
%   IEEE 30-BUS TEST SYSTEM (American Electric Power)
%       Bus Bus Voltage Angle --Load-- ---Generator---Injected
%        No code Mag. Degree MW   Mvar   MW  Mvar Qmin Qmax Mvar
busdata=[1  1  1.06   0    0.0   0.0   0.0  0.0   0   0    0
         2  2  1.043  0   21.70 12.7  40.0  0.0 -40  50    0
         3  0  1.0    0    2.4   1.2   0.0  0.0   0   0    0
         4  0  1.06   0    7.6   1.6   0.0  0.0   0   0    0
         5  2  1.01   0   94.2  19.0   0.0  0.0 -40  40    0
         6  0  1.0    0    0.0   0.0   0.0  0.0   0   0    0
         7  0  1.0    0   22.8  10.9   0.0  0.0   0   0    0
         8  2  1.01   0   30.0  30.0   0.0  0.0 -10  40    0
         9  0  1.0    0    0.0   0.0   0.0  0.0   0   0    0
        10  0  1.0    0    5.8   2.0   0.0  0.0   0   0   19
        11  2  1.082  0    0.0   0.0   0.0  0.0  -6  24    0
        12  0  1.0    0   11.2   7.5   0    0     0   0    0
        13  2  1.071  0    0.0   0.0   0    0    -6  24    0
        14  0  1.0    0    6.2   1.6   0    0     0   0    0
        15  0  1.0    0    8.2   2.5   0    0     0   0    0
        16  0  1.0    0    3.5   1.8   0    0     0   0    0
        17  0  1.0    0    9.0   5.8   0    0     0   0    0
        18  0  1.0    0    3.2   0.9   0    0     0   0    0
        19  0  1.0    0    9.5   3.4   0    0     0   0    0
        20  0  1.0    0    2.2   0.7   0    0     0   0    0
        21  0  1.0    0   17.5  11.2   0    0     0   0    0
        22  0  1.0    0    0.0   0.0   0    0     0   0    0
        23  0  1.0    0    3.2   1.6   0    0     0   0    0
        24  0  1.0    0    8.7   6.7   0    0     0   0  4.3
        25  0  1.0    0    0.0   0.0   0    0     0   0    0
        26  0  1.0    0    3.5   2.3   0    0     0   0    0
        27  0  1.0    0    0.0   0.0   0    0     0   0    0
        28  0  1.0    0    0.0   0.0   0    0     0   0    0
        29  0  1.0    0    2.4   0.9   0    0     0   0    0
        30  0  1.0    0   10.6   1.9   0    0     0   0    0];
```





```
% Line Data
%
%          Bus   bus    R        X        1/2 B  1 for Line code or
%          nl    nr     pu       pu        pu    tap setting value
linedata=[1      2    0.0192   0.0575    0.02640     1
          1      3    0.0452   0.1852    0.02040     1
          2      4    0.0570   0.1737    0.01840     1
          3      4    0.0132   0.0379    0.00420     1
          2      5    0.0472   0.1983    0.02090     1
          2      6    0.0581   0.1763    0.01870     1
          4      6    0.0119   0.0414    0.00450     1
          5      7    0.0460   0.1160    0.01020     1
          6      7    0.0267   0.0820    0.00850     1
          6      8    0.0120   0.0420    0.00450     1
          6      9    0.0      0.2080    0.0       0.978
          6     10    0.0      0.5560    0.0       0.969
          9     11    0.0      0.2080    0.0         1
          9     10    0.0      0.1100    0.0         1
          4     12    0.0      0.2560    0.0       0.932
         12     13    0.0      0.1400    0.0         1
         12     14    0.1231   0.2559    0.0         1
         12     15    0.0662   0.1304    0.0         1
         12     16    0.0945   0.1987    0.0         1
         14     15    0.2210   0.1997    0.0         1
         16     17    0.0824   0.1923    0.0         1
         15     18    0.1073   0.2185    0.0         1
         18     19    0.0639   0.1292    0.0         1
         19     20    0.0340   0.0680    0.0         1
         10     20    0.0936   0.2090    0.0         1
         10     17    0.0324   0.0845    0.0         1
         10     21    0.0348   0.0749    0.0         1
         10     22    0.0727   0.1499    0.0         1
         21     22    0.0116   0.0236    0.0         1
         15     23    0.1000   0.2020    0.0         1
         22     24    0.1150   0.1790    0.0         1
         23     24    0.1320   0.2700    0.0         1
         24     25    0.1885   0.3292    0.0         1
         25     26    0.2544   0.3800    0.0         1
         25     27    0.1093   0.2087    0.0         1
         28     27    0.0000   0.3960    0.0       0.968
         27     29    0.2198   0.4153    0.0         1
         27     30    0.3202   0.6027    0.0         1
         29     30    0.2399   0.4533    0.0         1
          8     28    0.0636   0.2000    0.0214      1
          6     28    0.0169   0.0599    0.065       1];
```

```
%
   lfybus                     % Forms the bus admittance matrix
   lfgauss          % Power flow solution by Gauss-Seidel method
   busout          % Prints the power flow solution on the screen
   lineflow   % Computes and displays the line flow and losses
```

The **lfgauss**, **busout** and the **lineflow** produce the following tabulated results.

Power Flow Solution by Gauss-Seidel Method
Maximum Power mismatch = 0.000951884
No. of iterations = 34

| Bus No. | Voltage Mag. | Angle Degree | Load MW | Load Mvar | Generation MW | Generation Mvar | Injected Mvar |
|---|---|---|---|---|---|---|---|
| 1 | 1.060 | 0.000 | 0.000 | 0.000 | 260.950 | -17.010 | 0.00 |
| 2 | 1.043 | -5.496 | 21.700 | 12.700 | 40.000 | 48.826 | 0.00 |
| 3 | 1.022 | -8.002 | 2.400 | 1.200 | 0.000 | 0.000 | 0.00 |
| 4 | 1.013 | -9.659 | 7.600 | 1.600 | 0.000 | 0.000 | 0.00 |
| 5 | 1.010 | -14.380 | 94.200 | 19.000 | 0.000 | 35.995 | 0.00 |
| 6 | 1.012 | -11.396 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 7 | 1.003 | -13.149 | 22.800 | 10.900 | 0.000 | 0.000 | 0.00 |
| 8 | 1.010 | -12.114 | 30.000 | 30.000 | 0.000 | 30.759 | 0.00 |
| 9 | 1.051 | -14.432 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 10 | 1.044 | -16.024 | 5.800 | 2.000 | 0.000 | 0.000 | 19.00 |
| 11 | 1.082 | -14.432 | 0.000 | 0.000 | 0.000 | 16.113 | 0.00 |
| 12 | 1.057 | -15.301 | 11.200 | 7.500 | 0.000 | 0.000 | 0.00 |
| 13 | 1.071 | -15.300 | 0.000 | 0.000 | 0.000 | 10.406 | 0.00 |
| 14 | 1.043 | -16.190 | 6.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 15 | 1.038 | -16.276 | 8.200 | 2.500 | 0.000 | 0.000 | 0.00 |
| 16 | 1.045 | -15.879 | 3.500 | 1.800 | 0.000 | 0.000 | 0.00 |
| 17 | 1.039 | -16.187 | 9.000 | 5.800 | 0.000 | 0.000 | 0.00 |
| 18 | 1.028 | -16.881 | 3.200 | 0.900 | 0.000 | 0.000 | 0.00 |
| 19 | 1.025 | -17.049 | 9.500 | 3.400 | 0.000 | 0.000 | 0.00 |
| 20 | 1.029 | -16.851 | 2.200 | 0.700 | 0.000 | 0.000 | 0.00 |
| 21 | 1.032 | -16.468 | 17.500 | 11.200 | 0.000 | 0.000 | 0.00 |
| 22 | 1.033 | -16.455 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 23 | 1.027 | -16.660 | 3.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 24 | 1.022 | -16.829 | 8.700 | 6.700 | 0.000 | 0.000 | 4.30 |
| 25 | 1.019 | -16.423 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 26 | 1.001 | -16.835 | 3.500 | 2.300 | 0.000 | 0.000 | 0.00 |
| 27 | 1.026 | -15.913 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 28 | 1.011 | -12.056 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 29 | 1.006 | -17.133 | 2.400 | 0.900 | 0.000 | 0.000 | 0.00 |
| 30 | 0.994 | -18.016 | 10.600 | 1.900 | 0.000 | 0.000 | 0.00 |
| Total | | | 283.400 | 126.200 | 300.950 | 125.089 | 23.30 |





Line Flow and Losses

| Line from | Line to | Power at bus & line flow MW | Mvar | MVA | Line loss MW | Line loss Mvar | Transformer tap |
|---|---|---|---|---|---|---|---|
| 1 | | 260.950 | -17.010 | 261.504 | | | |
| | 2 | 177.743 | -22.140 | 179.117 | 5.461 | 10.517 | |
| | 3 | 83.197 | 5.125 | 83.354 | 2.807 | 7.079 | |
| 2 | | 18.300 | 36.126 | 40.497 | | | |
| | 1 | -172.282 | 32.657 | 175.350 | 5.461 | 10.517 | |
| | 4 | 45.702 | 2.720 | 45.783 | 1.106 | -0.519 | |
| | 5 | 82.990 | 1.704 | 83.008 | 2.995 | 8.178 | |
| | 6 | 61.905 | -0.966 | 61.913 | 2.047 | 2.263 | |
| 3 | | -2.400 | -1.200 | 2.683 | | | |
| | 1 | -80.390 | 1.954 | 80.414 | 2.807 | 7.079 | |
| | 4 | 78.034 | -3.087 | 78.095 | 0.771 | 1.345 | |
| 4 | | -7.600 | -1.600 | 7.767 | | | |
| | 2 | -44.596 | -3.239 | 44.713 | 1.106 | -0.519 | |
| | 3 | -77.263 | 4.432 | 77.390 | 0.771 | 1.345 | |
| | 6 | 70.132 | -17.624 | 72.313 | 0.605 | 1.181 | |
| | 12 | 44.131 | 14.627 | 46.492 | 0.000 | 4.686 | 0.932 |
| 5 | | -94.200 | 16.995 | 95.721 | | | |
| | 2 | -79.995 | 6.474 | 80.256 | 2.995 | 8.178 | |
| | 7 | -14.210 | 10.467 | 17.649 | 0.151 | -1.687 | |
| 6 | | 0.000 | 0.000 | 0.000 | | | |
| | 2 | -59.858 | 3.229 | 59.945 | 2.047 | 2.263 | |
| | 4 | -69.527 | 18.805 | 72.026 | 0.605 | 1.181 | |
| | 7 | 37.537 | -1.915 | 37.586 | 0.368 | -0.598 | |
| | 8 | 29.534 | -3.712 | 29.766 | 0.103 | -0.558 | |
| | 9 | 27.687 | -7.318 | 28.638 | 0.000 | 1.593 | 0.978 |
| | 10 | 15.828 | 0.656 | 15.842 | 0.000 | 1.279 | 0.969 |
| | 28 | 18.840 | -9.575 | 21.134 | 0.060 | -13.085 | |
| 7 | | -22.800 | -10.900 | 25.272 | | | |
| | 5 | 14.361 | -12.154 | 18.814 | 0.151 | -1.687 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| | 6 | -37.170 | 1.317 | 37.193 | 0.368 | -0.598 |
| 8 | | -30.000 | 0.759 | 30.010 | | |
| | 6 | -29.431 | 3.154 | 29.599 | 0.103 | -0.558 |
| | 28 | -0.570 | -2.366 | 2.433 | 0.000 | -4.368 |
| 9 | | 0.000 | 0.000 | 0.000 | | |
| | 6 | -27.687 | 8.911 | 29.086 | 0.000 | 1.593 |
| | 11 | 0.003 | -15.653 | 15.653 | -0.000 | 0.461 |
| | 10 | 27.731 | 6.747 | 28.540 | 0.000 | 0.811 |
| 10 | | -5.800 | 17.000 | 17.962 | | |
| | 6 | -15.828 | 0.623 | 15.840 | 0.000 | 1.279 |
| | 9 | -27.731 | -5.936 | 28.359 | 0.000 | 0.811 |
| | 20 | 9.018 | 3.569 | 9.698 | 0.081 | 0.180 |
| | 17 | 5.347 | 4.393 | 6.920 | 0.014 | 0.037 |
| | 21 | 15.723 | 9.846 | 18.551 | 0.110 | 0.236 |
| | 22 | 7.582 | 4.487 | 8.811 | 0.052 | 0.107 |
| 11 | | 0.000 | 16.113 | 16.113 | | |
| | 9 | -0.003 | 16.114 | 16.114 | -0.000 | 0.461 |
| 12 | | -11.200 | -7.500 | 13.479 | | |
| | 4 | -44.131 | -9.941 | 45.237 | 0.000 | 4.686 |
| | 13 | -0.021 | -10.274 | 10.274 | 0.000 | 0.132 |
| | 14 | 7.852 | 2.428 | 8.219 | 0.074 | 0.155 |
| | 15 | 17.852 | 6.968 | 19.164 | 0.217 | 0.428 |
| | 16 | 7.206 | 3.370 | 7.955 | 0.053 | 0.112 |
| 13 | | 0.000 | 10.406 | 10.406 | | |
| | 12 | 0.021 | 10.406 | 10.406 | 0.000 | 0.132 |
| 14 | | -6.200 | -1.600 | 6.403 | | |
| | 12 | -7.778 | -2.273 | 8.103 | 0.074 | 0.155 |
| | 15 | 1.592 | 0.708 | 1.742 | 0.006 | 0.006 |
| 15 | | -8.200 | -2.500 | 8.573 | | |
| | 12 | -17.634 | -6.540 | 18.808 | 0.217 | 0.428 |
| | 14 | -1.586 | -0.702 | 1.734 | 0.006 | 0.006 |
| | 18 | 6.009 | 1.741 | 6.256 | 0.039 | 0.079 |
| | 23 | 5.004 | 2.963 | 5.815 | 0.031 | 0.063 |
| 16 | | -3.500 | -1.800 | 3.936 | | |
| | 12 | -7.152 | -3.257 | 7.859 | 0.053 | 0.112 |
| | 17 | 3.658 | 1.440 | 3.931 | 0.012 | 0.027 |





| | | | | | | |
|---|---|---|---|---|---|---|
| 17 | | -9.000 | -5.800 | 10.707 | | |
| | 16 | -3.646 | -1.413 | 3.910 | 0.012 | 0.027 |
| | 10 | -5.332 | -4.355 | 6.885 | 0.014 | 0.037 |
| 18 | | -3.200 | -0.900 | 3.324 | | |
| | 15 | -5.970 | -1.661 | 6.197 | 0.039 | 0.079 |
| | 19 | 2.779 | 0.787 | 2.888 | 0.005 | 0.010 |
| 19 | | -9.500 | -3.400 | 10.090 | | |
| | 18 | -2.774 | -0.777 | 2.881 | 0.005 | 0.010 |
| | 20 | -6.703 | -2.675 | 7.217 | 0.017 | 0.034 |
| 20 | | -2.200 | -0.700 | 2.309 | | |
| | 19 | 6.720 | 2.709 | 7.245 | 0.017 | 0.034 |
| | 10 | -8.937 | -3.389 | 9.558 | 0.081 | 0.180 |
| 21 | | -17.500 | -11.200 | 20.777 | | |
| | 10 | -15.613 | -9.609 | 18.333 | 0.110 | 0.236 |
| | 22 | -1.849 | -1.627 | 2.463 | 0.001 | 0.001 |
| 22 | | 0.000 | 0.000 | 0.000 | | |
| | 10 | -7.531 | -4.380 | 8.712 | 0.052 | 0.107 |
| | 21 | 1.850 | 1.628 | 2.464 | 0.001 | 0.001 |
| | 24 | 5.643 | 2.795 | 6.297 | 0.043 | 0.067 |
| 23 | | -3.200 | -1.600 | 3.578 | | |
| | 15 | -4.972 | -2.900 | 5.756 | 0.031 | 0.063 |
| | 24 | 1.771 | 1.282 | 2.186 | 0.006 | 0.012 |
| 24 | | -8.700 | -2.400 | 9.025 | | |
| | 22 | -5.601 | -2.728 | 6.230 | 0.043 | 0.067 |
| | 23 | -1.765 | -1.270 | 2.174 | 0.006 | 0.012 |
| | 25 | -1.322 | 1.604 | 2.079 | 0.008 | 0.014 |
| 25 | | 0.000 | 0.000 | 0.000 | | |
| | 24 | 1.330 | -1.590 | 2.073 | 0.008 | 0.014 |
| | 26 | 3.520 | 2.372 | 4.244 | 0.044 | 0.066 |
| | 27 | -4.866 | -0.786 | 4.929 | 0.026 | 0.049 |
| 26 | | -3.500 | -2.300 | 4.188 | | |
| | 25 | -3.476 | -2.306 | 4.171 | 0.044 | 0.066 |
| 27 | | 0.000 | 0.000 | 0.000 | | |
| | 25 | 4.892 | 0.835 | 4.963 | 0.026 | 0.049 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 28 | -18.192 | -4.152 | 18.660 | -0.000 | 1.310 | |
| | 29 | 6.178 | 1.675 | 6.401 | 0.086 | 0.162 | |
| | 30 | 7.093 | 1.663 | 7.286 | 0.162 | 0.304 | |
| 28 | | 0.000 | 0.000 | 0.000 | | | |
| | 27 | 18.192 | 5.463 | 18.994 | -0.000 | 1.310 | 0.968 |
| | 8 | 0.570 | -2.003 | 2.082 | 0.000 | -4.368 | |
| | 6 | -18.780 | -3.510 | 19.106 | 0.060 | -13.085 | |
| 29 | | -2.400 | -0.900 | 2.563 | | | |
| | 27 | -6.093 | -1.513 | 6.278 | 0.086 | 0.162 | |
| | 30 | 3.716 | 0.601 | 3.764 | 0.034 | 0.063 | |
| 30 | | -10.600 | -1.900 | 10.769 | | | |
| | 27 | -6.932 | -1.359 | 7.064 | 0.162 | 0.304 | |
| | 29 | -3.683 | -0.537 | 3.722 | 0.034 | 0.063 | |
| Total loss | | | | | 17.594 | 22.233 | |

## 6.10 NEWTON-RAPHSON POWER FLOW SOLUTION

Because of its quadratic convergence, Newton's method is mathematically superior to the Gauss-Seidel method and is less prone to divergence with ill-conditioned problems. For large power systems, the Newton-Raphson method is found to be more efficient and practical. The number of iterations required to obtain a solution is independent of the system size, but more functional evaluations are required at each iteration. Since in the power flow problem real power and voltage magnitude are specified for the voltage-controlled buses, the power flow equation is formulated in polar form. For the typical bus of the power system shown in Figure 6.7, the current entering bus $i$ is given by (6.24). This equation can be rewritten in terms of the bus admittance matrix as

$$I_i = \sum_{j=1}^{n} Y_{ij} V_j \tag{6.48}$$

In the above equation, $j$ includes bus $i$. Expressing this equation in polar form, we have

$$I_i = \sum_{j=1}^{n} |Y_{ij}||V_j| \angle \theta_{ij} + \delta_j \tag{6.49}$$

The complex power at bus $i$ is

$$P_i - jQ_i = V_i^* I_i \tag{6.50}$$



Substituting from (6.49) for $I_i$ in (6.50),

$$P_i - jQ_i = |V_i| \angle -\delta_i \sum_{j=1}^{n} |Y_{ij}||V_j| \angle \theta_{ij} + \delta_j \tag{6.51}$$

Separating the real and imaginary parts,

$$P_i = \sum_{j=1}^{n} |V_i||V_j||Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \tag{6.52}$$

$$Q_i = -\sum_{j=1}^{n} |V_i||V_j||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \tag{6.53}$$

Equations (6.52) and (6.53) constitute a set of nonlinear algebraic equations in terms of the independent variables, voltage magnitude in per unit, and phase angle in radians. We have two equations for each load bus, given by (6.52) and (6.53), and one equation for each voltage-controlled bus, given by (6.52). Expanding (6.52) and (6.53) in Taylor's series about the initial estimate and neglecting all higher order terms results in the following set of linear equations.

$$\left[\begin{array}{c} \Delta P_2^{(k)} \\ \vdots \\ \Delta P_n^{(k)} \\ \hline \Delta Q_2^{(k)} \\ \vdots \\ \Delta Q_n^{(k)} \end{array}\right] = \left[\begin{array}{ccc|ccc} \frac{\partial P_2}{\partial \delta_2}^{(k)} & \cdots & \frac{\partial P_2}{\partial \delta_n}^{(k)} & \frac{\partial P_2}{\partial |V_2|}^{(k)} & \cdots & \frac{\partial P_2}{\partial |V_n|}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_n}{\partial \delta_2}^{(k)} & \cdots & \frac{\partial P_n}{\partial \delta_n}^{(k)} & \frac{\partial P_n}{\partial |V_2|}^{(k)} & \cdots & \frac{\partial P_n}{\partial |V_n|}^{(k)} \\ \hline \frac{\partial Q_2}{\partial \delta_2}^{(k)} & \cdots & \frac{\partial Q_2}{\partial \delta_n}^{(k)} & \frac{\partial Q_2}{\partial |V_2|}^{(k)} & \cdots & \frac{\partial Q_2}{\partial |V_n|}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_n}{\partial \delta_2}^{(k)} & \cdots & \frac{\partial Q_n}{\partial \delta_n}^{(k)} & \frac{\partial Q_n}{\partial |V_2|}^{(k)} & \cdots & \frac{\partial Q_n}{\partial |V_n|}^{(k)} \end{array}\right] \left[\begin{array}{c} \Delta \delta_2^{(k)} \\ \vdots \\ \Delta \delta_n^{(k)} \\ \hline \Delta |V_2^{(k)}| \\ \vdots \\ \Delta |V_n^{(k)}| \end{array}\right]$$

In the above equation, bus 1 is assumed to be the slack bus. The Jacobian matrix gives the linearized relationship between small changes in voltage angle $\Delta \delta_i^{(k)}$ and voltage magnitude $\Delta |V_i^{(k)}|$ with the small changes in real and reactive power $\Delta P_i^{(k)}$ and $\Delta Q_i^{(k)}$. Elements of the Jacobian matrix are the partial derivatives of (6.52) and (6.53), evaluated at $\Delta \delta_i^{(k)}$ and $\Delta |V_i^{(k)}|$. In short form, it can be written as

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta |V| \end{bmatrix} \tag{6.54}$$

For voltage-controlled buses, the voltage magnitudes are known. Therefore, if m buses of the system are voltage-controlled, m equations involving $\Delta Q$ and $\Delta V$

and the corresponding columns of the Jacobian matrix are eliminated. Accordingly, there are $n-1$ real power constraints and $n-1-m$ reactive power constraints, and the Jacobian matrix is of order $(2n-2-m) \times (2n-2-m)$. $J_1$ is of the order $(n-1) \times (n-1)$, $J_2$ is of the order $(n-1) \times (n-1-m)$, $J_3$ is of the order $(n-1-m) \times (n-1)$, and $J_4$ is of the order $(n-1-m) \times (n-1-m)$.

The diagonal and the off-diagonal elements of $J_1$ are

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{j \neq i} |V_i||V_j||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \tag{6.55}$$

$$\frac{\partial P_i}{\partial \delta_j} = -|V_i||V_j||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \qquad j \neq i \tag{6.56}$$

The diagonal and the off-diagonal elements of $J_2$ are

$$\frac{\partial P_i}{\partial |V_i|} = 2|V_i||Y_{ii}| \cos\theta_{ii} + \sum_{j \neq i} |V_j||Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \tag{6.57}$$

$$\frac{\partial P_i}{\partial |V_j|} = |V_i||Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \qquad j \neq i \tag{6.58}$$

The diagonal and the off-diagonal elements of $J_3$ are

$$\frac{\partial Q_i}{\partial \delta_i} = \sum_{j \neq i} |V_i||V_j||Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \tag{6.59}$$

$$\frac{\partial Q_i}{\partial \delta_j} = -|V_i||V_j||Y_{ij}| \cos(\theta_{ij} - \delta_i + \delta_j) \qquad j \neq i \tag{6.60}$$

The diagonal and the off-diagonal elements of $J_4$ are

$$\frac{\partial Q_i}{\partial |V_i|} = -2|V_i||Y_{ii}| \sin\theta_{ii} - \sum_{j \neq i} |V_j||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \tag{6.61}$$

$$\frac{\partial Q_i}{\partial |V_j|} = -|V_i||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) \qquad j \neq i \tag{6.62}$$

The terms $\Delta P_i^{(k)}$ and $\Delta Q_i^{(k)}$ are the difference between the scheduled and calculated values, known as the *power residuals*, given by

$$\Delta P_i^{(k)} = P_i^{sch} - P_i^{(k)} \tag{6.63}$$

$$\Delta Q_i^{(k)} = Q_i^{sch} - Q_i^{(k)} \tag{6.64}$$





The new estimates for bus voltages are

$$\delta_i^{(k+1)} = \delta_i^{(k)} + \Delta\delta_i^{(k)} \tag{6.65}$$

$$|V_i^{(k+1)}| = |V_i^{(k)}| + \Delta|V_i^{(k)}| \tag{6.66}$$

The procedure for power flow solution by the Newton-Raphson method is as follows:

1. For load buses, where $P_i^{sch}$ and $Q_i^{sch}$ are specified, voltage magnitudes and phase angles are set equal to the slack bus values, or 1.0 and 0.0, i.e., $|V_i^{(0)}| = 1.0$ and $\delta_i^{(0)} = 0.0$. For voltage-regulated buses, where $|V_i|$ and $P_i^{sch}$ are specified, phase angles are set equal to the slack bus angle, or 0, i.e., $\delta_i^{(0)} = 0$.
2. For load buses, $P_i^{(k)}$ and $Q_i^{(k)}$ are calculated from (6.52) and (6.53) and $\Delta P_i^{(k)}$ and $\Delta Q_i^{(k)}$ are calculated from (6.63) and (6.64).
3. For voltage-controlled buses, $P_i^{(k)}$ and $\Delta P_i^{(k)}$ are calculated from (6.52) and (6.63), respectively.
4. The elements of the Jacobian matrix ($J_1$, $J_2$, $J_3$, and $J_4$) are calculated from (6.55) – (6.62).
5. The linear simultaneous equation (6.54) is solved directly by optimally ordered triangular factorization and Gaussian elimination.
6. The new voltage magnitudes and phase angles are computed from (6.65) and (6.66).
7. The process is continued until the residuals $\Delta P_i^{(k)}$ and $\Delta Q_i^{(k)}$ are less than the specified accuracy, i.e.,

$$|\Delta P_i^{(k)}| \leq \epsilon$$
$$|\Delta Q_i^{(k)}| \leq \epsilon \tag{6.67}$$

The power flow solution by the Newton-Raphson method is demonstrated in the following example.

### Example 6.10

Obtain the power flow solution by the Newton-Raphson method for the system of Example 6.8.

Line impedances converted to admittances are $y_{12} = 10 - j20$, $y_{13} = 10 - j30$, and $y_{23} = 16 - j32$. This results in the bus admittance matrix

$$Y_{bus} = \begin{bmatrix} 20 - j50 & -10 + j20 & -10 + j30 \\ -10 + j20 & 26 - j52 & -16 + j32 \\ -10 + j30 & -16 + j32 & 26 - j62 \end{bmatrix}$$

Converting the bus admittance matrix to polar form with angles in radian yields

$$Y_{bus} = \begin{bmatrix} 53.85165\angle -1.9029 & 22.36068\angle 2.0344 & 31.62278\angle 1.8925 \\ 22.36068\angle 2.0344 & 58.13777\angle -1.1071 & 35.77709\angle 2.0344 \\ 31.62278\angle 1.8925 & 35.77709\angle 2.0344 & 67.23095\angle -1.1737 \end{bmatrix}$$

From (6.52) and (6.53), the expressions for real power at bus 2 and 3 and the reactive power at bus 2 are

$$\begin{aligned} P_2 &= |V_2||V_1||Y_{21}|\cos(\theta_{21} - \delta_2 + \delta_1) + |V_2^2||Y_{22}|\cos\theta_{22} + \\ &\quad |V_2||V_3||Y_{23}|\cos(\theta_{23} - \delta_2 + \delta_3) \\ P_3 &= |V_3||V_1||Y_{31}|\cos(\theta_{31} - \delta_3 + \delta_1) + |V_3||V_2||Y_{32}|\cos(\theta_{32} - \\ &\quad \delta_3 + \delta_2) + |V_3^2||Y_{33}|\cos\theta_{33} \\ Q_2 &= -|V_2||V_1||Y_{21}|\sin(\theta_{21} - \delta_2 + \delta_1) - |V_2^2||Y_{22}|\sin\theta_{22} - \\ &\quad |V_2||V_3||Y_{23}|\sin(\theta_{23} - \delta_2 + \delta_3) \end{aligned}$$

Elements of the Jacobian matrix are obtained by taking partial derivatives of the above equations with respect to $\delta_2$, $\delta_3$ and $|V_2|$.

$$\begin{aligned} \frac{\partial P_2}{\partial \delta_2} &= |V_2||V_1||Y_{21}|\sin(\theta_{21} - \delta_2 + \delta_1) + |V_2||V_3||Y_{23}| \\ &\quad \sin(\theta_{23} - \delta_2 + \delta_3) \\ \frac{\partial P_2}{\partial \delta_3} &= -|V_2||V_3||Y_{23}|\sin(\theta_{23} - \delta_2 + \delta_3) \\ \frac{\partial P_2}{\partial |V_2|} &= |V_1||Y_{21}|\cos(\theta_{21} - \delta_2 + \delta_1) + 2|V_2||Y_{22}|\cos\theta_{22} + \\ &\quad |V_3||Y_{23}|\cos(\theta_{23} - \delta_2 + \delta_3) \\ \frac{\partial P_3}{\partial \delta_2} &= -|V_3||V_2||Y_{32}|\sin(\theta_{32} - \delta_3 + \delta_2) \\ \frac{\partial P_3}{\partial \delta_3} &= |V_3||V_1||Y_{31}|\sin(\theta_{31} - \delta_3 + \delta_1) + |V_3||V_2||Y_{32}| \\ &\quad \sin(\theta_{32} - \delta_3 + \delta_2) \\ \frac{\partial P_3}{\partial |V_2|} &= |V_3||Y_{32}|\cos(\theta_{32} - \delta_3 + \delta_2) \end{aligned}$$





$$\begin{aligned} \frac{\partial Q_2}{\partial \delta_2} &= |V_2||V_1||Y_{21}|\cos(\theta_{21} - \delta_2 + \delta_1) + |V_2||V_3||Y_{23}| \\ &\quad \cos(\theta_{23} - \delta_2 + \delta_3) \\ \frac{\partial Q_2}{\partial \delta_3} &= -|V_2||V_3||Y_{23}|\cos(\theta_{23} - \delta_2 + \delta_3) \\ \frac{\partial Q_2}{\partial |V_2|} &= -|V_1||Y_{21}|\sin(\theta_{21} - \delta_2 + \delta_1) - 2|V_2||Y_{22}|\sin\theta_{22} - \\ &\quad |V_3||Y_{23}|\sin(\theta_{23} - \delta_2 + \delta_3) \end{aligned}$$

The load and generation expressed in per units are

$$\begin{aligned} S_2^{sch} &= -\frac{(400 + j250)}{100} = -4.0 - j2.5 \quad \text{pu} \\ P_3^{sch} &= \frac{200}{100} = 2.0 \quad \text{pu} \end{aligned}$$

The slack bus voltage is $V_1 = 1.05\angle 0$ pu, and the bus 3 voltage magnitude is $|V_3| = 1.04$ pu. Starting with an initial estimate of $|V_2^{(0)}| = 1.0$, $\delta_2^{(0)} = 0.0$, and $\delta_3^{(0)} = 0.0$, the power residuals are computed from (6.63) and (6.64)

$$\begin{aligned} \Delta P_2^{(0)} &= P_2^{sch} - P_2^{(0)} = -4.0 - (-1.14) = -2.8600 \\ \Delta P_3^{(0)} &= P_3^{sch} - P_3^{(0)} = 2.0 - (0.5616) = 1.4384 \\ \Delta Q_2^{(0)} &= Q_2^{sch} - Q_2^{(0)} = -2.5 - (-2.28) = -0.2200 \end{aligned}$$

Evaluating the elements of the Jacobian matrix with the initial estimate, the set of linear equations in the first iteration becomes

$$\begin{bmatrix} -2.8600 \\ 1.4384 \\ -0.2200 \end{bmatrix} = \begin{bmatrix} 54.28000 & -33.28000 & 24.86000 \\ -33.28000 & 66.04000 & -16.64000 \\ -27.14000 & 16.64000 & 49.72000 \end{bmatrix} \begin{bmatrix} \Delta\delta_2^{(0)} \\ \Delta\delta_3^{(0)} \\ \Delta|V_2^{(0)}| \end{bmatrix}$$

Obtaining the solution of the above matrix equation, the new bus voltages in the first iteration are

$$\begin{aligned} \Delta\delta_2^{(0)} &= -0.045263 & \delta_2^{(1)} &= 0 + (-0.045263) = -0.045263 \\ \Delta\delta_3^{(0)} &= -0.007718 & \delta_3^{(1)} &= 0 + (-0.007718) = -0.007718 \\ \Delta|V_2^{(0)}| &= -0.026548 & |V_2^{(1)}| &= 1 + (-0.026548) = 0.97345 \end{aligned}$$

Voltage phase angles are in radians. For the second iteration, we have

$$\begin{bmatrix} -0.099218 \\ 0.021715 \\ -0.050914 \end{bmatrix} = \begin{bmatrix} 51.724675 & -31.765618 & 21.302567 \\ -32.981642 & 65.656383 & -15.379086 \\ -28.538577 & 17.402838 & 48.103589 \end{bmatrix} \begin{bmatrix} \Delta\delta_2^{(1)} \\ \Delta\delta_3^{(1)} \\ \Delta|V_2^{(1)}| \end{bmatrix}$$

and

$$\Delta\delta_2^{(1)} = -0.001795 \qquad \delta_2^{(2)} = -0.045263 + (-0.001795) = -0.04706$$
$$\Delta\delta_3^{(1)} = -0.000985 \qquad \delta_3^{(2)} = -0.007718 + (-0.000985) = -0.00870$$
$$\Delta|V_2^{(1)}| = -0.001767 \qquad |V_2^{(2)}| = 0.973451 + (-0.001767) = 0.971684$$

For the third iteration, we have

$$\begin{bmatrix} -0.000216 \\ 0.000038 \\ -0.000143 \end{bmatrix} = \begin{bmatrix} 51.596701 & -31.693866 & 21.147447 \\ -32.933865 & 65.597585 & -15.351628 \\ -28.548205 & 17.396932 & 47.954870 \end{bmatrix} \begin{bmatrix} \Delta\delta_2^{(2)} \\ \Delta\delta_3^{(2)} \\ \Delta|V_2^{(2)}| \end{bmatrix}$$

and

$$\Delta\delta_2^{(2)} = -0.000038 \qquad \delta_2^{(3)} = -0.047058 + (-0.0000038) = -0.04706$$
$$\Delta\delta_3^{(2)} = -0.0000024 \qquad \delta_3^{(3)} = -0.008703 + (-0.0000024) = 0.008705$$
$$\Delta|V_2^{(2)}| = -0.0000044 \qquad |V_2^{(3)}| = 0.971684 + (-0.0000044) = 0.97168$$

The solution converges in 3 iterations with a maximum power mismatch of $2.5 \times 10^{-4}$ with $V_2 = 0.97168\angle{-2.696^\circ}$ and $V_3 = 1.04\angle{-0.4988^\circ}$. From (6.52) and (6.53), the expressions for reactive power at bus 3 and the slack bus real and reactive powers are

$$Q_3 = -|V_3||V_1||Y_{31}|\sin(\theta_{31} - \delta_3 + \delta_1) - |V_3||V_2||Y_{32}|$$
$$\sin(\theta_{32} - \delta_3 + \delta_2) - |V_3|^2|Y_{33}|\sin\theta_{33}$$
$$P_1 = |V_1|^2|Y_{11}|\cos\theta_{11} + |V_1||V_2||Y_{12}|\cos(\theta_{12} - \delta_1 + \delta_2) + |V_1||V_3|$$
$$|Y_{13}|\cos(\theta_{13} - \delta_1 + \delta_3)$$
$$Q_1 = -|V_1|^2|Y_{11}|\sin\theta_{11} - |V_1||V_2||Y_{12}|\sin(\theta_{12} - \delta_1 + \delta_2) - |V_1||V_3|$$
$$|Y_{13}|\sin(\theta_{13} - \delta_1 + \delta_3)$$

Upon substitution, we have

$$Q_3 = 1.4617 \text{ pu}$$
$$P_1 = 2.1842 \text{ pu}$$
$$Q_1 = 1.4085 \text{ pu}$$

Finally, the line flows are calculated in the same manner as the line flow calculations in the Gauss-Seidel method described in Example 6.7, and the power flow diagram is as shown in Figure 6.13.

A program named **lfnewton** is developed for power flow solution by the Newton-Raphson method for practical power systems. This program must be preceded by the **lfybus** program. **busout** and **lineflow** programs can be used to print the load flow solution and the line flow results. The format is the same as the Gauss-Seidel. The following is a brief description of the **lfnewton** program.





**lfnewton** This program obtains the power flow solution by the Newton-Raphson method and requires the **busdata** and the **linedata** files described in Section 6.9. It is designed for the direct use of load and generation in MW and Mvar, bus voltages in per unit, and angle in degrees. Loads and generation are converted to per unit quantities on the base MVA selected. A provision is made to maintain the generator reactive power of the voltage-controlled buses within their specified limits. The violation of reactive power limit may occur if the specified voltage is either too high or too low. In the second iteration, the var calculated at the generator buses are examined. If a limit is reached, the voltage magnitude is adjusted in steps of 0.5 percent up to ±5 percent to bring the var demand within the specified limits.

### Example 6.11

Obtain the power flow solution for the IEEE-30 bus test system by the Newton-Raphson method.

The data required is the same as in Example 6.9 with the following commands

```
clear        % clears all variables from the workspace.
basemva = 100;  accuracy = 0.001;  maxiter = 12;

busdata =   [ same as in Example 6.9 ];
linedata = [ same as in Example 6.9 ];

     lfybus                   % Forms the bus admittance matrix
     lfnewton   % Power flow solution by Newton-Raphson method
     busout     % Prints the power flow solution on the screen
     lineflow % Computes and displays the line flow and losses
```

The output of **lfnewton** is

```
          Power Flow Solution by Newton-Raphson Method
             Maximum Power mismatch = 7.54898e-07
                    No. of iterations = 4

Bus Voltage  Angle    -----Load-----    --Generation-- Injected
No. Mag.     Degree    MW      Mvar      MW      Mvar      Mvar

1    1.060    0.000   0.000    0.000  260.998  -17.021     0.00
2    1.043   -5.497  21.700   12.700   40.000   48.822     0.00
3    1.022   -8.004   2.400    1.200    0.000    0.000     0.00
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 1.013 | -9.661 | 7.600 | 1.600 | 0.000 | 0.000 | 0.00 |
| 5 | 1.010 | -14.381 | 94.200 | 19.000 | 0.000 | 35.975 | 0.00 |
| 6 | 1.012 | -11.398 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 7 | 1.003 | -13.150 | 22.800 | 10.900 | 0.000 | 0.000 | 0.00 |
| 8 | 1.010 | -12.115 | 30.000 | 30.000 | 0.000 | 30.826 | 0.00 |
| 9 | 1.051 | -14.434 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 10 | 1.044 | -16.024 | 5.800 | 2.000 | 0.000 | 0.000 | 19.00 |
| 11 | 1.082 | -14.434 | 0.000 | 0.000 | 0.000 | 16.119 | 0.00 |
| 12 | 1.057 | -15.302 | 11.200 | 7.500 | 0.000 | 0.000 | 0.00 |
| 13 | 1.071 | -15.302 | 0.000 | 0.000 | 0.000 | 10.423 | 0.00 |
| 14 | 1.042 | -16.191 | 6.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 15 | 1.038 | -16.278 | 8.200 | 2.500 | 0.000 | 0.000 | 0.00 |
| 16 | 1.045 | -15.880 | 3.500 | 1.800 | 0.000 | 0.000 | 0.00 |
| 17 | 1.039 | -16.188 | 9.000 | 5.800 | 0.000 | 0.000 | 0.00 |
| 18 | 1.028 | -16.884 | 3.200 | 0.900 | 0.000 | 0.000 | 0.00 |
| 19 | 1.025 | -17.052 | 9.500 | 3.400 | 0.000 | 0.000 | 0.00 |
| 20 | 1.029 | -16.852 | 2.200 | 0.700 | 0.000 | 0.000 | 0.00 |
| 21 | 1.032 | -16.468 | 17.500 | 11.200 | 0.000 | 0.000 | 0.00 |
| 22 | 1.033 | -16.455 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 23 | 1.027 | -16.662 | 3.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 24 | 1.022 | -16.830 | 8.700 | 6.700 | 0.000 | 0.000 | 4.30 |
| 25 | 1.019 | -16.424 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 26 | 1.001 | -16.842 | 3.500 | 2.300 | 0.000 | 0.000 | 0.00 |
| 27 | 1.026 | -15.912 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 28 | 1.011 | -12.057 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 29 | 1.006 | -17.136 | 2.400 | 0.900 | 0.000 | 0.000 | 0.00 |
| 30 | 0.995 | -18.015 | 10.600 | 1.900 | 0.000 | 0.000 | 0.00 |
| Total | | | 283.400 | 126.200 | 300.998 | 125.144 | 23.30 |

The output of the **lineflow** is the same as the line flow output of Example 6.9 with the power mismatch as dictated by the Newton-Raphson method.

## 6.11 FAST DECOUPLED POWER FLOW SOLUTION

Power system transmission lines have a very high $X/R$ ratio. For such a system, real power changes $\Delta P$ are less sensitive to changes in the voltage magnitude and are most sensitive to changes in phase angle $\Delta\delta$. Similarly, reactive power is less sensitive to changes in angle and are mainly dependent on changes in voltage magnitude. Therefore, it is reasonable to set elements $J_2$ and $J_3$ of the Jacobian matrix to zero. Thus, (6.54) becomes

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_4 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|V| \end{bmatrix} \tag{6.68}$$





or

$$\Delta P = J_1 \Delta\delta = [\frac{\partial P}{\partial \delta}]\, \Delta\delta \tag{6.69}$$

$$\Delta Q = J_4 \Delta|V| = [\frac{\partial Q}{\partial |V|}]\, \Delta|V| \tag{6.70}$$

(6.69) and (6.70) show that the matrix equation is separated into two decoupled equations requiring considerably less time to solve compared to the time required for the solution of (6.54). Furthermore, considerable simplification can be made to eliminate the need for recomputing $J_1$ and $J_4$ during each iteration. This procedure results in the decoupled power flow equations developed by Stott and Alsac[75–76]. The diagonal elements of $J_1$ described by (6.55) may be written as

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{j=1}^{n} |V_i||V_j||Y_{ij}| \sin(\theta_{ij} - \delta_i + \delta_j) - |V_i|^2 |Y_{ii}| \sin\theta_{ii}$$

Replacing the first term of the above equation with $-Q_i$, as given by (6.53), results in

$$\frac{\partial P_i}{\partial \delta_i} = -Q_i - |V_i|^2 |Y_{ii}| \sin\theta_{ii}$$
$$= -Q_i - |V_i|^2 B_{ii}$$

Where $B_{ii} = |Y_{ii}| \sin\theta_{ii}$ is the imaginary part of the diagonal elements of the bus admittance matrix. $B_{ii}$ is the sum of susceptances of all the elements incident to bus $i$. In a typical power system, the self-susceptance $B_{ii} \gg Q_i$, and we may neglect $Q_i$. Further simplification is obtained by assuming $|V_i|^2 \approx |V_i|$, which yields

$$\frac{\partial P_i}{\partial \delta_i} = -|V_i| B_{ii} \tag{6.71}$$

Under normal operating conditions, $\delta_j - \delta_i$ is quite small. Thus, in (6.56) assuming $\theta_{ii} - \delta_i + \delta_j \approx \theta_{ii}$, the off-diagonal elements of $J_1$ becomes

$$\frac{\partial P_i}{\partial \delta_j} = -|V_i||V_j| B_{ij}$$

Further simplification is obtained by assuming $|V_j| \approx 1$

$$\frac{\partial P_i}{\partial \delta_j} = -|V_i| B_{ij} \tag{6.72}$$

Similarly, the diagonal elements of $J_4$ described by (6.61) may be written as

$$\frac{\partial Q_i}{\partial |V_i|} = -|V_i||Y_{ii}|\sin\theta_{ii} - \sum_{j=1}^{n} |V_i||V_j||Y_{ij}|\sin(\theta_{ij} - \delta_i + \delta_j)$$

replacing the second term of the above equation with $-Q_i$, as given by (6.53), results in

$$\frac{\partial Q_i}{\partial |V_i|} = -|V_i||Y_{ii}|\sin\theta_{ii} + Q_i$$

Again, since $B_{ii} = Y_{ii}\sin\theta_{ii} \gg Q_i$, $Q_i$ may be neglected and (6.61) reduces to

$$\frac{\partial Q_i}{\partial |V_i|} = -|V_i|B_{ii} \tag{6.73}$$

Likewise in (6.62), assuming $\theta_{ij} - \delta_i + \delta_j \approx \theta_{ij}$ yields

$$\frac{\partial Q_i}{\partial |V_j|} = -|V_i|B_{ij} \tag{6.74}$$

With these assumptions, equations (6.69) and (6.70) take the following form

$$\frac{\Delta P}{|V_i|} = -B'\ \Delta\delta \tag{6.75}$$

$$\frac{\Delta Q}{|V_i|} = -B''\ \Delta|V| \tag{6.76}$$

Here, $B'$ and $B''$ are the imaginary part of the bus admittance matrix $Y_{bus}$. Since the elements of this matrix are constant, they need to be triangularized and inverted only once at the beginning of the iteration. $B'$ is of order of $(n-1)$. For voltage-controlled buses where $|V_i|$ and $P_i$ are specified and $Q_i$ is not specified, the corresponding row and column of $Y_{bus}$ are eliminated. Thus, $B''$ is of order of $(n-1-m)$, where $m$ is the number of voltage-regulated buses. Therefore, in the fast decoupled power flow algorithm, the successive voltage magnitude and phase angle changes are

$$\Delta\delta = -[B']^{-1}\frac{\Delta P}{|V|} \tag{6.77}$$

$$\Delta|V| = -[B'']^{-1}\frac{\Delta Q}{|V|} \tag{6.78}$$

The fast decoupled power flow solution requires more iterations than the Newton-Raphson method, but requires considerably less time per iteration, and a power flow solution is obtained very rapidly. This technique is very useful in contingency analysis where numerous outages are to be simulated or a power flow solution is required for on-line control.





**Example 6.12**

Obtain the power flow solution by the fast decoupled method for the system of Example 6.8.

The bus admittance matrix of the system as obtained in Example 6.10 is

$$Y_{bus} = \begin{bmatrix} 20-j50 & -10+j20 & -10+j30 \\ -10+j20 & 26-j52 & -16+j32 \\ -10+j30 & -16+j32 & 26-j62 \end{bmatrix}$$

In this system, bus 1 is the slack bus and the corresponding bus susceptance matrix for evaluation of phase angles $\Delta\delta_2$ and $\Delta\delta_3$ is

$$B' = \begin{bmatrix} -52 & 32 \\ 32 & -62 \end{bmatrix}$$

The inverse of the above matrix is

$$[B']^{-1} = \begin{bmatrix} -0.028182 & -0.014545 \\ -0.014545 & -0.023636 \end{bmatrix}$$

From (6.52) and (6.53), the expressions for real power at bus 2 and 3 and the reactive power at bus 2 are

$$P_2 = |V_2||V_1||Y_{21}|\cos(\theta_{21} - \delta_2 + \delta_1) + |V_2^2||Y_{22}|\cos\theta_{22}$$
$$+|V_2||V_3||Y_{23}|\cos(\theta_{23} - \delta_2 + \delta_3)$$
$$P_3 = |V_3||V_1||Y_{31}|\cos(\theta_{31} - \delta_3 + \delta_1) + |V_3||V_2||Y_{32}|\cos(\theta_{32}$$
$$-\delta_3 + \delta_2) + |V_3^2||Y_{33}|\cos\theta_{33}$$
$$Q_2 = -|V_2||V_1||Y_{21}|\sin(\theta_{21} - \delta_2 + \delta_1) - |V_2^2||Y_{22}|\sin\theta_{22}$$
$$-|V_2||V_3||Y_{23}|\sin(\theta_{23} - \delta_2 + \delta_3)$$

The load and generation expressed in per units are

$$S_2^{sch} = -\frac{(400 + j250)}{100} = -4.0 - j2.5 \quad \text{pu}$$

$$P_3^{sch} = \frac{200}{100} = 2.0 \quad \text{pu}$$

The slack bus voltage is $V_1 = 1.05\angle 0$ pu, and the bus 3 voltage magnitude is $|V_3| = 1.04$ pu. Starting with an initial estimate of $|V_2^{(0)}| = 1.0$, $\delta_2^{(0)} = 0.0$, and $\delta_3^{(0)} = 0.0$, the power residuals are computed from (6.63) and (6.64)

$$\Delta P_2^{(0)} = P_2^{sch} - P_2^{(0)} = -4.0 - (-1.14) = -2.86$$
$$\Delta P_3^{(0)} = P_3^{sch} - P_3^{(0)} = 2.0 - (0.5616) = 1.4384$$
$$\Delta Q_2^{(0)} = Q_2^{sch} - Q_2^{(0)} = -2.5 - (-2.28) = -0.22$$

The fast decoupled power flow algorithm given by (6.77) becomes

$$\begin{bmatrix} \Delta\delta_2^{(0)} \\ \Delta\delta_3^{(0)} \end{bmatrix} = -\begin{bmatrix} -0.028182 & -0.014545 \\ -0.014545 & -0.023636 \end{bmatrix} \begin{bmatrix} \frac{-2.8600}{1.0} \\ \frac{1.4384}{1.04} \end{bmatrix} = \begin{bmatrix} -0.060483 \\ -0.008909 \end{bmatrix}$$

Since bus 3 is a regulated bus, the corresponding row and column of $B'$ are eliminated and we get

$$B'' = [-52]$$

From (6.78), we have

$$\Delta|V_2| = -\left[\frac{-1}{52}\right]\left[\frac{-.22}{1.0}\right] = -0.0042308$$

The new bus voltages in the first iteration are

$$\begin{aligned} \Delta\delta_2^{(0)} &= -0.060483 & \delta_2^{(1)} &= 0 + (-0.060483) = -0.060483 \\ \Delta\delta_3^{(0)} &= -0.008989 & \delta_3^{(1)} &= 0 + (-0.008989) = -0.008989 \\ \Delta|V_2^{(0)}| &= -0.0042308 & |V_2^{(1)}| &= 1 + (-0.0042308) = 0.995769 \end{aligned}$$

The voltage phase angles are in radians. The process is continued until power residuals are within a specified accuracy. The result is tabulated in the table below.

| Iter | $\delta_2$ | $\delta_3$ | $\|V_2\|$ | $\Delta P_2$ | $\Delta P_3$ | $\Delta Q_2$ |
|---|---|---|---|---|---|---|
| 1 | -0.060482 | -0.008909 | 0.995769 | -2.860000 | 1.438400 | -0.220000 |
| 2 | -0.056496 | -0.007952 | 0.965274 | 0.175895 | -0.070951 | -1.579042 |
| 3 | -0.044194 | -0.008690 | 0.965711 | 0.640309 | -0.457039 | 0.021948 |
| 4 | -0.044802 | -0.008986 | 0.972985 | -0.021395 | 0.001195 | 0.365249 |
| 5 | -0.047665 | -0.008713 | 0.973116 | -0.153368 | 0.112899 | 0.006657 |
| 6 | -0.047614 | -0.008645 | 0.971414 | 0.000520 | 0.002610 | -0.086136 |
| 7 | -0.046936 | -0.008702 | 0.971333 | 0.035980 | -0.026190 | -0.004067 |
| 8 | -0.046928 | -0.008720 | 0.971732 | 0.000948 | -0.001411 | 0.020119 |
| 9 | -0.047087 | -0.008707 | 0.971762 | -0.008442 | 0.006133 | 0.001558 |
| 10 | -0.047094 | -0.008702 | 0.971669 | -0.000470 | 0.000510 | -0.004688 |
| 11 | -0.047057 | -0.008705 | 0.971660 | 0.001971 | -0.001427 | -0.000500 |
| 12 | -0.047054 | -0.008706 | 0.971681 | 0.000170 | -0.000163 | 0.001087 |
| 13 | -0.047063 | -0.008706 | 0.971684 | -0.000458 | 0.000330 | 0.000151 |
| 14 | -0.047064 | -0.008706 | 0.971680 | -0.000053 | 0.000048 | -0.000250 |

Converting phase angles to degrees the final solution is $V_2 = 0.97168\angle{-2.696°}$ and $V_3 = 1.04\angle{-0.4988°}$. Using (6.52) and (6.53) as in Example 6.10, the reactive





power at bus 3 and the slack bus real and reactive powers are

$$\begin{aligned} Q_3 &= 1.4617 \text{ pu} \\ P_1 &= 2.1842 \text{ pu} \\ Q_1 &= 1.4085 \text{ pu} \end{aligned}$$

The fast decoupled power flow for this example has taken 14 iterations with the maximum power mismatch of $2.5 \times 10^{-4}$ pu compared to the Newton-Raphson method which took only three iterations. The highest $X/R$ ratio of the transmission lines in this example is 3. For systems with a higher $X/R$ ratio, the fast decoupled power flow method converges in relatively fewer iterations. However, the number of iterations is a function of system size.

Finally, the line flows are calculated in the same manner as the line flow calculations in the Gauss-Seidel method described in Example 6.7, and the power flow diagram is as shown in Figure 6.13.

A program named **decouple** is developed for power flow solution by the fast decoupled method for practical power systems. This program must be preceded by the **lfybus** program. **busout** and **lineflow** programs can be used to print the load flow solution and the line flow results. The format is the same as the Gauss-Seidel method. The following is a brief description of the **decouple** program:

**decouple** This program finds the power flow solution by the fast decouple method and requires the **busdata** and the **linedata** files described in Section 6.9. It is designed for the direct use of load and generation in MW and Mvar, bus voltages in per unit, and angle in degrees. Loads and generation are converted to per unit quantities on the base MVA selected. A provision is made to maintain the generator reactive power of the voltage-controlled buses within their specified limits. The violation of reactive power limit may occur if the specified voltage is either too high or too low. In the 10th iteration, the vars calculated at the generator buses are examined. If a limit is reached, the voltage magnitude is adjusted in steps of 0.5 percent up to ±5 percent to bring the var demand within the specified limits.

### Example 6.13

Obtain the power flow solution for the IEEE-30 bus test system by the fast decoupled method.

Data required is the same as in Example 6.9 with the following commands

```
clear      % clears all variables from the workspace.
basemva = 100;  accuracy = 0.001;  maxiter = 20;

busdata =  [ same as in Example 6.9 ];
linedata = [ same as in Example 6.9 ];

     lfybus                   % Forms the bus admittance matrix
     decouple   % Power flow solution by fast decoupled method
     busout     % Prints the power flow solution on the screen
     lineflow % Computes and displays the line flow and losses
```

The output of **decouple** is

```
        Power Flow Solution by Fast Decoupled Method
           Maximum Power mismatch = 0.000919582
                  No. of iterations = 15
```

| Bus No. | Voltage Mag. | Angle Degree | -----Load----- MW | Mvar | --Generation-- MW | Mvar | Injected Mvar |
|---|---|---|---|---|---|---|---|
| 1 | 1.060 | 0.000 | 0.000 | 0.000 | 260.998 | -17.021 | 0.00 |
| 2 | 1.043 | -5.497 | 21.700 | 12.700 | 40.000 | 48.822 | 0.00 |
| 3 | 1.022 | -8.004 | 2.400 | 1.200 | 0.000 | 0.000 | 0.00 |
| 4 | 1.013 | -9.662 | 7.600 | 1.600 | 0.000 | 0.000 | 0.00 |
| 5 | 1.010 | -14.381 | 94.200 | 19.000 | 0.000 | 35.975 | 0.00 |
| 6 | 1.012 | -11.398 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 7 | 1.003 | -13.149 | 22.800 | 10.900 | 0.000 | 0.000 | 0.00 |
| 8 | 1.010 | -12.115 | 30.000 | 30.000 | 0.000 | 30.828 | 0.00 |
| 9 | 1.051 | -14.434 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 10 | 1.044 | -16.024 | 5.800 | 2.000 | 0.000 | 0.000 | 19.00 |
| 11 | 1.082 | -14.434 | 0.000 | 0.000 | 0.000 | 16.120 | 0.00 |
| 12 | 1.057 | -15.303 | 11.200 | 7.500 | 0.000 | 0.000 | 0.00 |
| 13 | 1.071 | -15.303 | 0.000 | 0.000 | 0.000 | 10.421 | 0.00 |
| 14 | 1.042 | -16.198 | 6.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 15 | 1.038 | -16.276 | 8.200 | 2.500 | 0.000 | 0.000 | 0.00 |
| 16 | 1.045 | -15.881 | 3.500 | 1.800 | 0.000 | 0.000 | 0.00 |
| 17 | 1.039 | -16.188 | 9.000 | 5.800 | 0.000 | 0.000 | 0.00 |
| 18 | 1.028 | -16.882 | 3.200 | 0.900 | 0.000 | 0.000 | 0.00 |
| 19 | 1.025 | -17.051 | 9.500 | 3.400 | 0.000 | 0.000 | 0.00 |
| 20 | 1.029 | -16.852 | 2.200 | 0.700 | 0.000 | 0.000 | 0.00 |
| 21 | 1.032 | -16.468 | 17.500 | 11.200 | 0.000 | 0.000 | 0.00 |
| 22 | 1.033 | -16.454 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 23 | 1.027 | -16.661 | 3.200 | 1.600 | 0.000 | 0.000 | 0.00 |
| 24 | 1.022 | -16.829 | 8.700 | 6.700 | 0.000 | 0.000 | 4.30 |
| 25 | 1.019 | -16.423 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 26 | 1.001 | -16.840 | 3.500 | 2.300 | 0.000 | 0.000 | 0.00 |
| 27 | 1.026 | -15.912 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 28 | 1.011 | -12.057 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 29 | 1.006 | -17.136 | 2.400 | 0.900 | 0.000 | 0.000 | 0.00 |
| 30 | 0.995 | -18.014 | 10.600 | 1.900 | 0.000 | 0.000 | 0.00 |
| Total | | | 283.400 | 126.200 | 300.998 | 125.145 | 23.30 |





The output of the **lineflow** is the same as the line flow output of Example 6.9 with the power mismatch as dictated by the fast decoupled method.

## PROBLEMS

**6.1.** A power system network is shown in Figure 6.17. The generators at buses 1 and 2 are represented by their equivalent current sources with their reactances in per unit on a 100-MVA base. The lines are represented by $\pi$ model where series reactances and shunt reactances are also expressed in per unit on a 100 MVA base. The loads at buses 3 and 4 are expressed in MW and Mvar.

(a) Assuming a voltage magnitude of 1.0 per unit at buses 3 and 4, convert the loads to per unit impedances. Convert network impedances to admittances and obtain the bus admittance matrix by inspection.

(b) Use the function **Y = ybus(zdata)** to obtain the bus admittance matrix. The function argument **zdata** is a matrix containing the line bus numbers, resistance and reactance. (See Example 6.1.)

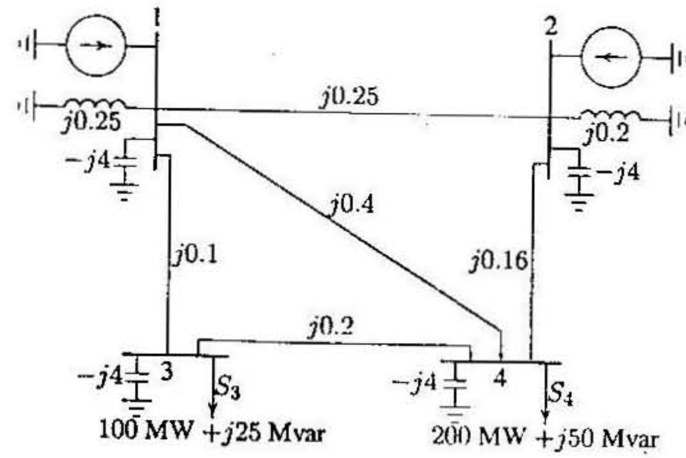



**FIGURE 6.17**
One-line diagram for Problem 6.1.

**6.2.** A power system network is shown in Figure 6.18. The values marked are impedances in per unit on a base of 100 MVA. The currents entering buses 1 and 2 are

$$I_1 = 1.38 - j2.72 \text{ pu}$$
$$I_2 = 0.69 - j1.36 \text{ pu}$$

(a) Determine the bus admittance matrix by inspection.
(b) Use the function **Y = ybus(zdata)** to obtain the bus admittance matrix. The function argument **zdata** is a matrix containing the line bus numbers, resistance and reactance. (See Example 6.1.) Write the necessary *MATLAB* commands to obtain the bus voltages.

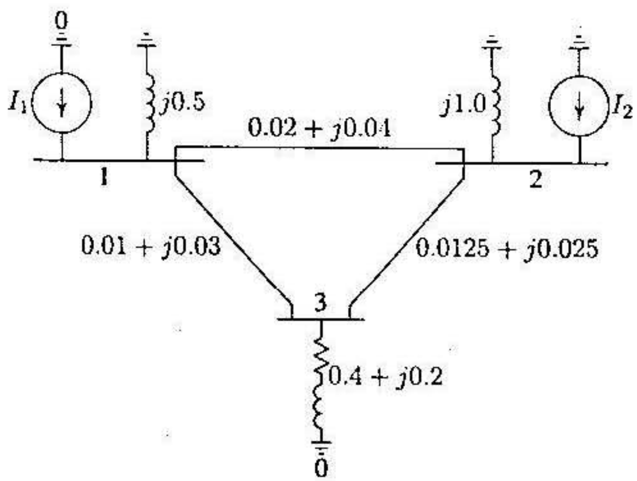



**FIGURE 6.18**
One-line diagram for Problem 6.2.

**6.3.** Use Gauss-Seidel method to find the solution of the following equations

$$x_1 + x_1 x_2 = 10$$
$$x_1 + x_2 = 6$$

with the following initial estimates
(a) $x_1^{(0)} = 1$ and $x_2^{(0)} = 1$
(b) $x_1^{(0)} = 1$ and $x_2^{(0)} = 2$
Continue the iterations until $|\Delta x_1^{(k)}|$ and $|\Delta x_2^{(k)}|$ are less than 0.001.





**6.4.** A fourth-order polynomial equation is given by

$$x^4 - 21x^3 + 147x^2 - 379x + 252 = 0$$

(a) Use Newton-Raphson method and hand calculations to find one of the roots of the polynomial equation. Start with the initial estimate of $x^{(0)} = 0$ and continue until $|\Delta x^{(k)}| < 0.001$.
(b) Write a *MATLAB* program to find the roots of the above polynomial by Newton-Raphson method. The program should prompt the user to input the initial estimate. Run using the initial estimates of 0, 3, 6, 10.
(c) Check your answers using the *MATLAB* function **r = roots(A)**, where **A** is a row vector containing the polynomial coefficients in descending powers.

**6.5.** Use Newton-Raphson method and hand calculation to find the solution of the following equations:

$$x_1^2 - 2x_1 - x_2 = 3$$
$$x_1^2 + x_2^2 = 41$$

(a) Start with the initial estimates of $x_1^{(0)} = 2$, $x_2^{(0)} = 3$. Perform three iterations.
(b) Write a *MATLAB* program to find one of the solutions of the above equations by Newton-Raphson method. The program should prompt the user to input the initial estimates. Run the program with the above initial estimates.

**6.6.** In the power system network shown in Figure 6.19, bus 1 is a slack bus with $V_1 = 1.0\angle 0°$ per unit and bus 2 is a load bus with $S_2 = 280$ MW + $j60$ Mvar. The line impedance on a base of 100 MVA is $Z = 0.02 + j0.04$ per unit.
(a) Using Gauss-Seidel method, determine $V_2$. Use an initial estimate of $V_2^{(0)} = 1.0 + j0.0$ and perform four iterations.
(b) If after several iterations voltage at bus 2 converges to $V_2 = 0.90 - j0.10$, determine $S_1$ and the real and reactive power loss in the line.

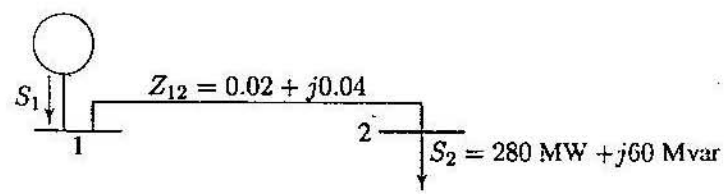



**FIGURE 6.19**
One-line diagram for Problem 6.6.

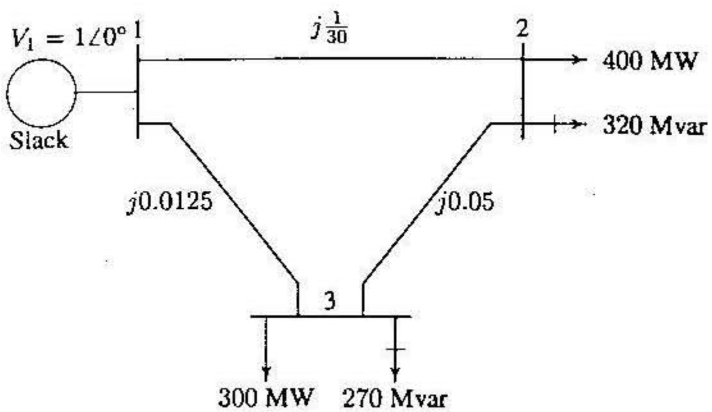



**FIGURE 6.20**
One-line diagram for Problem 6.7.

**6.7.** Figure 6.20 shows the one-line diagram of a simple three-bus power system with generation at bus 1. The voltage at bus 1 is $V_1 = 1.0\angle 0°$ per unit. The scheduled loads on buses 2 and 3 are marked on the diagram. Line impedances are marked in per unit on a 100-MVA base. For the purpose of hand calculations, line resistances and line charging susceptances are neglected.

(a) Using Gauss-Seidel method and initial estimates of $V_2^{(0)} = 1.0 + j0$ and $V_3^{(0)} = 1.0 + j0$, determine $V_2$ and $V_3$. Perform two iterations.

(b) If after several iterations the bus voltages converge to

$$V_2 = 0.90 - j0.10 \text{ pu}$$
$$V_3 = 0.95 - j0.05 \text{ pu}$$

determine the line flows and line losses and the slack bus real and reactive power. Construct a power flow diagram and show the direction of the line flows.

(c) Check the power flow solution using the **lfgauss** and other required programs. (Refer to Example 6.9.) Use a power accuracy of 0.00001 and an acceleration factor of 1.0.

**6.8.** Figure 6.21 shows the one-line diagram of a simple three-bus power system with generation at buses 1 and 3. The voltage at bus 1 is $V_1 = 1.025\angle 0°$ per unit. Voltage magnitude at bus 3 is fixed at 1.03 pu with a real power generation of 300 MW. A load consisting of 400 MW and 200 Mvar is taken from bus 2. Line impedances are marked in per unit on a 100-MVA base. For the



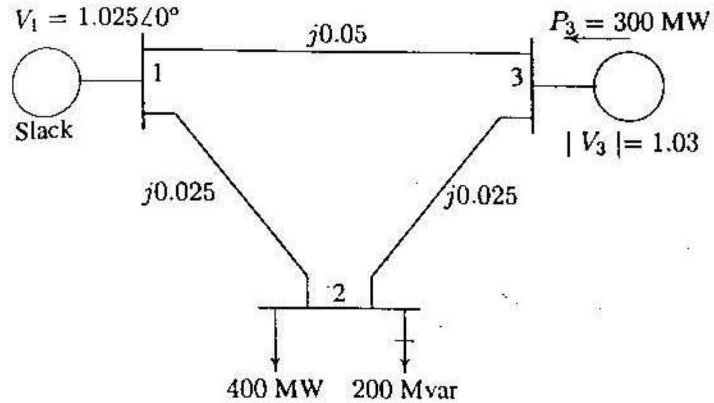



**FIGURE 6.21**
One-line diagram for Problem 6.8.

purpose of hand calculations, line resistances and line charging susceptances are neglected.

(a) Using Gauss-Seidel method and initial estimates of $V_2^{(0)} = 1.0 + j0$ and $V_3^{(0)} = 1.03 + j0$ and keeping $|V_3| = 1.03$ pu, determine the phasor values of $V_2$ and $V_3$. Perform two iterations.

(b) If after several iterations the bus voltages converge to

$$V_2 = 1.001243\angle -2.1° = 1.000571 - j0.0366898 \text{ pu}$$
$$V_3 = 1.03\angle 1.36851° = 1.029706 + j0.0246 \text{ pu}$$

determine the line flows and line losses and the slack bus real and reactive power. Construct a power flow diagram and show the direction of the line flows.

(c) Check the power flow solution using the **lfgauss** and other required programs. (Refer to Example 6.9.)

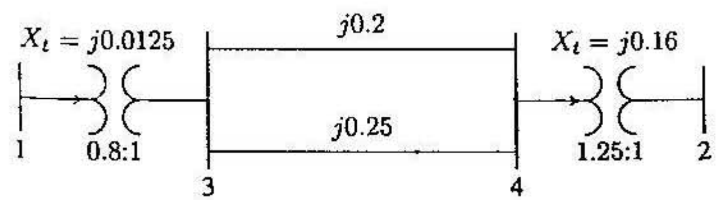



**FIGURE 6.22**
One-line diagram for Problem 6.9.

**6.9.** The one-line diagram of a four-bus power system is as shown in Figure 6.22. Reactances are given in per unit on a common MVA base. Transformers $T_1$ and $T_2$ have tap settings of 0.8:1, and 1.25:1 respectively. Obtain the bus admittance matrix.

**6.10.** In the two-bus system shown in Figure 6.23, bus 1 is a slack bus with $V_1 = 1.0\angle 0°$ pu. A load of 150 MW and 50 Mvar is taken from bus 2. The line admittance is $y_{12} = 10\angle -73.74°$ pu on a base of 100 MVA. The expression for real and reactive power at bus 2 is given by

$$P_2 = 10|V_2||V_1|\cos(106.26° - \delta_2 + \delta_1) + 10|V_2|^2\cos(-73.74°)$$
$$Q_2 = -10|V_2||V_1|\sin(106.26° - \delta_2 + \delta_1) - 10|V_2|^2\sin(-73.74°)$$

Using Newton-Raphson method, obtain the voltage magnitude and phase angle of bus 2. Start with an initial estimate of $|V_2|^{(0)} = 1.0$ pu and $\delta_2^{(0)} = 0°$. Perform two iterations.

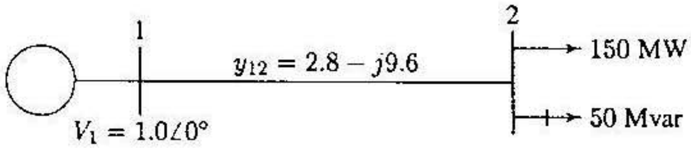



**FIGURE 6.23**
One-line diagram for Problem 6.10.

**6.11.** In the two-bus system shown in Figure 6.24, bus 1 is a slack bus with $V_1 = 1.0\angle 0°$ pu. A load of 100 MW and 50 Mvar is taken from bus 2. The line impedance is $z_{12} = 0.12 + j0.16$ pu on a base of 100 MVA. Using Newton-Raphson method, obtain the voltage magnitude and phase angle of bus 2. Start with an initial estimate of $|V_2|^{(0)} = 1.0$ pu and $\delta_2^{(0)} = 0°$. Perform two iterations.

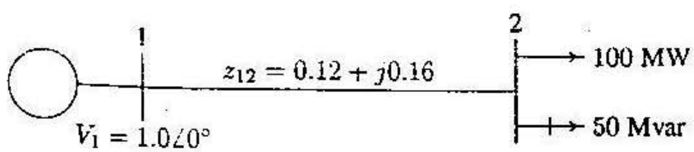



**FIGURE 6.24**
One-line diagram for Problem 6.11.





**6.12.** Figure 6.25 shows the one-line diagram of a simple three-bus power system with generation at buses 1 and 2. The voltage at bus 1 is $V = 1.0\angle 0°$ per unit. Voltage magnitude at bus 2 is fixed at 1.05 pu with a real power generation of 400 MW. A load consisting of 500 MW and 400 Mvar is taken from bus 3. Line admittances are marked in per unit on a 100 MVA base. For the purpose of hand calculations, line resistances and line charging susceptances are neglected.

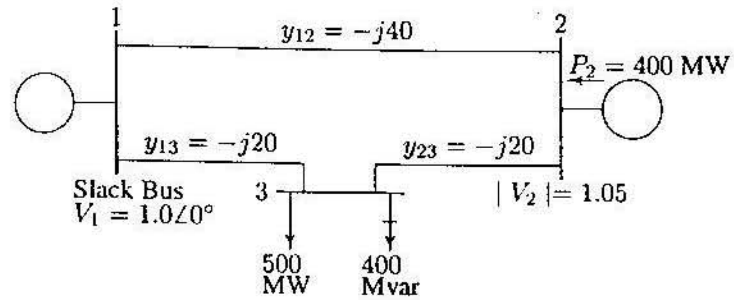



**FIGURE 6.25**
One-line diagram for Problem 6.12

(a) Show that the expression for the real power at bus 2 and real and reactive power at bus 3 are

$$P_2 = 40|V_2||V_1|\cos(90° - \delta_2 + \delta_1) + 20|V_2||V_3|\cos(90° - \delta_2 + \delta_3)$$
$$P_3 = 20|V_3||V_1|\cos(90° - \delta_3 + \delta_1) + 20|V_3||V_2|\cos(90° - \delta_3 + \delta_2)$$
$$Q_3 = -20|V_3||V_1|\sin(90° - \delta_3 + \delta_1) - 20|V_3||V_2|\sin(90° - \delta_3 + \delta_2) + 40|V_3|^2$$

(b) Using Newton-Raphson method, start with the initial estimates of $V_2^{(0)} = 1.0 + j0$ and $V_3^{(0)} = 1.0 + j0$, and keeping $|V_2| = 1.05$ pu, determine the phasor values of $V_2$ and $V_3$. Perform two iterations.
(c) Check the power flow solution for Problem 6.12 using **lfnewton** and other required programs. Assume the regulated bus (bus # 2) reactive power limits are between 0 and 600 Mvar.

**6.13.** For Problem 6.12:
(a) Obtain the power flow solution using the fast decoupled algorithm. Perform two iterations.
(b) Check the power flow solution for Problem 6.12 using **decouple** and other required programs. Assume the regulated bus (bus # 2) reactive power limits are between 0 and 600 Mvar.

**6.14.** The 26-bus power system network of an electric utility company is shown in Figure 6.26 (page 256). Obtain the power flow solution by the following

methods:
(a) Gauss-Seidel power flow (see Example 6.9).
(b) Newton-Raphson power flow (see Example 6.11).
(c) Fast decoupled power flow (see Example 6.13).

The load data is as follows.

| LOAD DATA | | | | | |
|---|---|---|---|---|---|
| Bus No. | Load MW | Load Mvar | Bus No. | Load MW | Load Mvar |
| 1 | 51.0 | 41.0 | 14 | 24.0 | 12.0 |
| 2 | 22.0 | 15.0 | 15 | 70.0 | 31.0 |
| 3 | 64.0 | 50.0 | 16 | 55.0 | 27.0 |
| 4 | 25.0 | 10.0 | 17 | 78.0 | 38.0 |
| 5 | 50.0 | 30.0 | 18 | 153.0 | 67.0 |
| 6 | 76.0 | 29.0 | 19 | 75.0 | 15.0 |
| 7 | 0.0 | 0.0 | 20 | 48.0 | 27.0 |
| 8 | 0.0 | 0.0 | 21 | 46.0 | 23.0 |
| 9 | 89.0 | 50.0 | 22 | 45.0 | 22.0 |
| 10 | 0.0 | 0.0 | 23 | 25.0 | 12.0 |
| 11 | 25.0 | 15.0 | 24 | 54.0 | 27.0 |
| 12 | 89.0 | 48.0 | 25 | 28.0 | 13.0 |
| 13 | 31.0 | 15.0 | 26 | 40.0 | 20.0 |

Voltage magnitude, generation schedule, and the reactive power limits for the regulated buses are tabulated below. Bus 1, whose voltage is specified as $V_1 = 1.025\angle 0°$, is taken as the slack bus.

| GENERATION DATA | | | | |
|---|---|---|---|---|
| Bus No. | Voltage Mag. | Generation MW | Mvar Limits Min. | Mvar Limits Max. |
| 1 | 1.025 | | | |
| 2 | 1.020 | 79.0 | 40.0 | 250.0 |
| 3 | 1.025 | 20.0 | 40.0 | 150.0 |
| 4 | 1.050 | 100.0 | 40.0 | 80.0 |
| 5 | 1.045 | 300.0 | 40.0 | 160.0 |
| 26 | 1.015 | 60.0 | 15.0 | 50.0 |

The Mvar of the shunt capacitors installed at substations and the transformer tap settings are given below.





| SHUNT CAPACITORS | |
|---|---|
| Bus No. | Mvar |
| 1 | 4.0 |
| 4 | 2.0 |
| 5 | 5.0 |
| 6 | 2.0 |
| 11 | 1.5 |
| 12 | 2.0 |
| 15 | 0.5 |
| 19 | 5.0 |

| TRANSFORMER TAP | |
|---|---|
| Designation | Tap Setting |
| 2 – 3 | 0.960 |
| 2 – 13 | 0.960 |
| 3 – 13 | 1.017 |
| 4 – 8 | 1.050 |
| 4 – 12 | 1.050 |
| 6 – 19 | 0.950 |
| 7 – 9 | 0.950 |

The line and transformer data containing the series resistance and reactance in per unit and one-half the total capacitance in per unit susceptance on a 100-MVA base are tabulated below.

| LINE AND TRANSFORMER DATA | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Bus No. | Bus No. | $R$, pu | $X$, pu | $\frac{1}{2}B$, pu | Bus No. | Bus No. | $R$, pu | $X$, pu | $\frac{1}{2}B$, pu |
| 1 | 2 | 0.0005 | 0.0048 | 0.0300 | 10 | 22 | 0.0069 | 0.0298 | 0.005 |
| 1 | 18 | 0.0013 | 0.0110 | 0.0600 | 11 | 25 | 0.0960 | 0.2700 | 0.010 |
| 2 | 3 | 0.0014 | 0.0513 | 0.0500 | 11 | 26 | 0.0165 | 0.0970 | 0.004 |
| 2 | 7 | 0.0103 | 0.0586 | 0.0180 | 12 | 14 | 0.0327 | 0.0802 | 0.000 |
| 2 | 8 | 0.0074 | 0.0321 | 0.0390 | 12 | 15 | 0.0180 | 0.0598 | 0.000 |
| 2 | 13 | 0.0035 | 0.0967 | 0.0250 | 13 | 14 | 0.0046 | 0.0271 | 0.001 |
| 2 | 26 | 0.0323 | 0.1967 | 0.0000 | 13 | 15 | 0.0116 | 0.0610 | 0.000 |
| 3 | 13 | 0.0007 | 0.0054 | 0.0005 | 13 | 16 | 0.0179 | 0.0888 | 0.001 |
| 4 | 8 | 0.0008 | 0.0240 | 0.0001 | 14 | 15 | 0.0069 | 0.0382 | 0.000 |
| 4 | 12 | 0.0016 | 0.0207 | 0.0150 | 15 | 16 | 0.0209 | 0.0512 | 0.000 |
| 5 | 6 | 0.0069 | 0.0300 | 0.0990 | 16 | 17 | 0.0990 | 0.0600 | 0.000 |
| 6 | 7 | 0.0053 | 0.0306 | 0.0010 | 16 | 20 | 0.0239 | 0.0585 | 0.000 |
| 6 | 11 | 0.0097 | 0.0570 | 0.0001 | 17 | 18 | 0.0032 | 0.0600 | 0.038 |
| 6 | 18 | 0.0037 | 0.0222 | 0.0012 | 17 | 21 | 0.2290 | 0.4450 | 0.000 |
| 6 | 19 | 0.0035 | 0.0660 | 0.0450 | 19 | 23 | 0.0300 | 0.1310 | 0.000 |
| 6 | 21 | 0.0050 | 0.0900 | 0.0226 | 19 | 24 | 0.0300 | 0.1250 | 0.002 |
| 7 | 8 | 0.0012 | 0.0069 | 0.0001 | 19 | 25 | 0.1190 | 0.2249 | 0.004 |
| 7 | 9 | 0.0009 | 0.0429 | 0.0250 | 20 | 21 | 0.0657 | 0.1570 | 0.000 |
| 8 | 12 | 0.0020 | 0.0180 | 0.0200 | 20 | 22 | 0.0150 | 0.0366 | 0.000 |
| 9 | 10 | 0.0010 | 0.0493 | 0.0010 | 21 | 24 | 0.0476 | 0.1510 | 0.000 |
| 10 | 12 | 0.0024 | 0.0132 | 0.0100 | 22 | 23 | 0.0290 | 0.0990 | 0.000 |
| 10 | 19 | 0.0547 | 0.2360 | 0.0000 | 22 | 24 | 0.0310 | 0.0880 | 0.000 |
| 10 | 20 | 0.0066 | 0.0160 | 0.0010 | 23 | 25 | 0.0987 | 0.1168 | 0.000 |

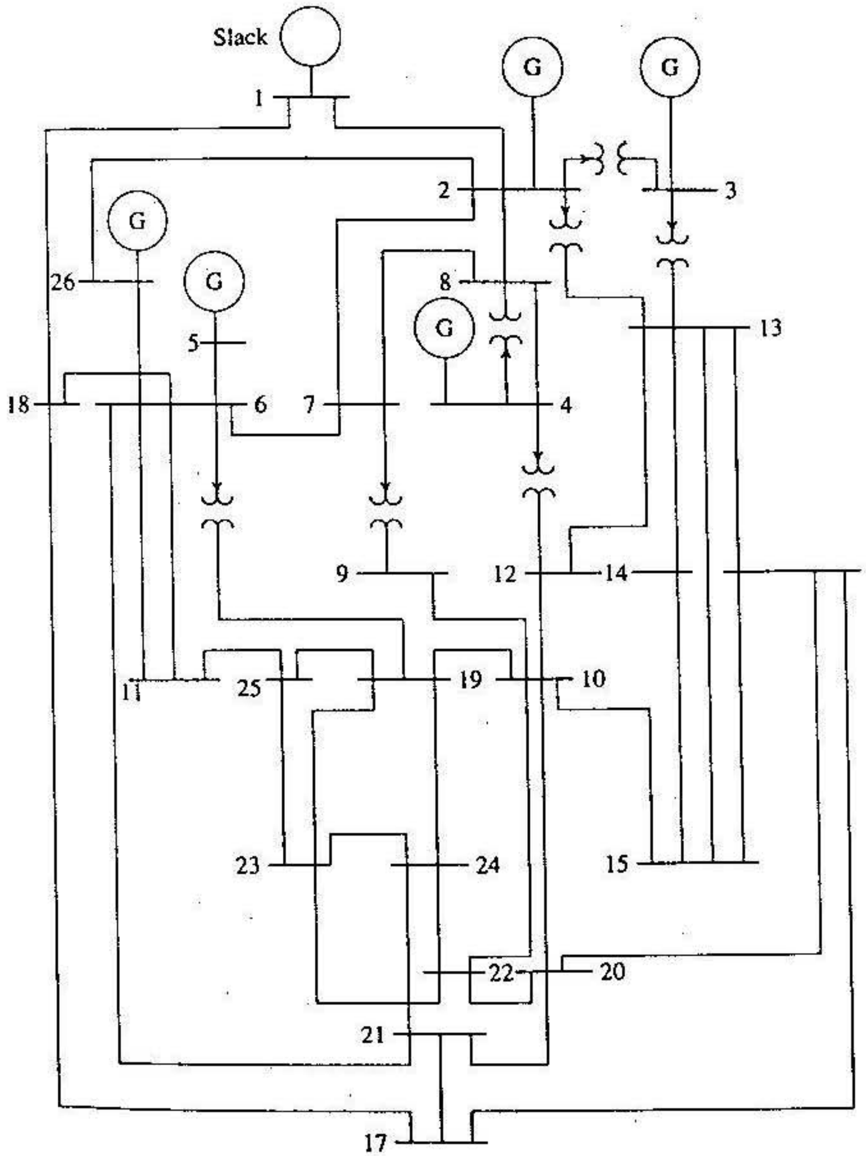



**FIGURE 6.26**
One-line diagram for Problem 6.14.



CHAPTER

# 7

# OPTIMAL DISPATCH OF GENERATION

## 7.1 INTRODUCTION

The formulation of power flow problem and its solutions were discussed in Chapter 6. One type of bus in the power flow was the voltage-controlled bus, where real power generation and voltage magnitude were specified. The power flow solution provided the voltage phase angle and the reactive power generation. In a practical power system, the power plants are not located at the same distance from the center of loads and their fuel costs are different. Also, under normal operating conditions, the generation capacity is more than the total load demand and losses. Thus, there are many options for scheduling generation. In an interconnected power system, the objective is to find the real and reactive power scheduling of each power plant in such a way as to minimize the operating cost. This means that the generator's real and reactive power are allowed to vary within certain limits so as to meet a particular load demand with minimum fuel cost. This is called the *optimal power flow* (OPF) problem. The OPF is used to optimize the power flow solution of large scale power system. This is done by minimizing selected objective functions while maintaining an acceptable system performance in terms of generator capability limits and the output of the compensating devices. The objective functions, also known as *cost functions*, may present economic costs, system security, or other objectives. Efficient reactive power planning enhances economic operation as well as system security. The OPF has been studied by many researchers and many algorithms using different objective functions and methods have been presented [11, 12, 22, 23, 40, 42, 54, 78].

In this chapter, we will limit our analysis to the economic dispatch of real power generation. The classical optimization of continuous functions is introduced. The application of constraints to optimization problems is presented. Following this, the incremental production cost of generation is introduced. The economic dispatch of generation for minimization of the total operating cost with transmission losses neglected is obtained. Next, the transmission loss formula is derived and the economic dispatch of generation based on the loss formula is obtained. A program named **bloss** is developed for the evaluation of the transmission loss **B** coefficients which can be used following any one of the power flow programs **lfgauss**, **lfnewton**, or **decouple** discussed in Chapter 6. Also, a general program called **dispatch** is developed for the optimal scheduling of real power generation and can be used in conjunction with the **bloss** program.

## 7.2 NONLINEAR FUNCTION OPTIMIZATION

### Unconstrained Parameter Optimization

Nonlinear function optimization is an important tool in computer-aided design and is part of a broader class of optimization called *nonlinear programming*. The underlying theory and the computational methods are discussed in many books. The basic goal is the minimization of some nonlinear objective cost function subject to nonlinear equality and inequality constraints.

The mathematical tools that are used to solve unconstrained parameter optimization problems come directly from multivariable calculus. The necessary condition to minimize the cost function

$$f(x_1, x_2, \ldots, x_n) \tag{7.1}$$

is obtained by setting derivative of $f$ with respect to the variables equal to zero, i.e.,

$$\frac{\partial f}{\partial x_i} = 0 \qquad i = 1, \ldots, n \tag{7.2}$$

or

$$\nabla f = 0 \tag{7.3}$$





where

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) \tag{7.4}$$

which is known as the *gradient vector*. The terms associated with second derivatives is given by

$$H = \frac{\partial^2 f}{\partial x_i \, \partial x_j} \tag{7.5}$$

The above equation results in a symmetric matrix called the *Hessian matrix* of the function.

Once the derivative of $f$ is vanished at local extrema $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, for $f$ to have a relative minimum, the Hessian matrix evaluated at $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ must be a positive definite matrix. This condition requires that all the eigenvalues of the Hessian matrix evaluated at $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ be positive.

In summary, the unconstrained minimum of a function is found by setting its partial derivatives (with respect to the parameters that may be varied) equal to zero and solving for the parameter values. Among the sets of parameter values obtained, those at which the matrix of second partial derivatives of the cost function is positive definite are local minima. If there is a single local minimum, it is also the global minimum; otherwise, the cost function must be evaluated at each of the local minima to determine which one is the global minimum.

### Example 7.1

Find the minimum of

$$f(x_1, x_2, \ldots, x_n) = x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + x_2x_3 - 8x_1 - 16x_2 - 32x_3 + 110$$

Equating the first derivatives to zero, results in

$$\frac{\partial f}{\partial x_1} = 2x_1 + x_2 - 8 = 0$$
$$\frac{\partial f}{\partial x_2} = x_1 + 4x_2 + x_3 - 16 = 0$$
$$\frac{\partial f}{\partial x_3} = x_2 + 6x_3 - 32 = 0$$

or

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix}$$

The solution of the above linear simultaneous equation is readily obtained (in *MATLAB* use $X = A\backslash B$) and is given by $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (3, 2, 5)$. The function evaluated at this point is $f(3, 2, 5) = 2$. To see if this point is a minimum, we evaluate the second derivatives and form the Hessian matrix

$$H(\hat{X}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

Using the *MATLAB* function **eig(H)**, the eigenvalues are found to be 1.55, 4.0 and 6.45, which are all positive. Thus, the Hessian matrix is a positive definite matrix and (3, 2, 5) is a minimum point.

### 7.2.1 CONSTRAINED PARAMETER OPTIMIZATION: EQUALITY CONSTRAINTS

This type of problem arises when there are functional dependencies among the parameters to be chosen. The problem is to minimize the cost function

$$f(x_1, x_2, \ldots, x_n) \tag{7.6}$$

subject to the equality constraints

$$g_i(x_1, x_2, \ldots, x_n) = 0 \qquad i = 1, 2, \ldots, k \tag{7.7}$$

Such problems may be solved by the *Lagrange multiplier* method. This provides an augmented cost function by introducing $k$-vector $\lambda$ of undetermined quantities. The unconstrained cost function becomes

$$\mathcal{L} = f + \sum_{i=1}^{k} \lambda_i g_i \tag{7.8}$$

The resulting necessary conditions for constrained local minima of $\mathcal{L}$ are the following:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{i=1}^{k} \lambda_i \frac{\partial g_i}{\partial x_i} = 0 \tag{7.9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = g_i = 0 \tag{7.10}$$

Note that Equation (7.10) is simply the original constraints.





**Example 7.2**

Use the Lagrange multiplier method for solving constrained parameter optimizations to determine the minimum distance from origin of the $xy$ plane to a circle described by

$$(x - 8)^2 + (y - 6)^2 = 25$$

The minimum distance is obtained by minimization of the distance square, given by

$$f(x, y) = x^2 + y^2$$

The *MATLAB* **plot** command is used to plot the circle as shown in Figure 7.1.

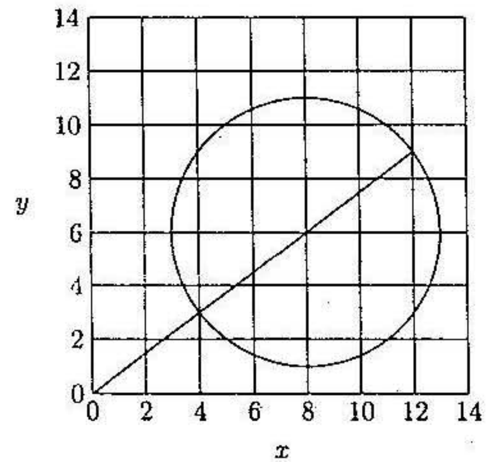



**FIGURE 7.1**
Constraint function of Example 7.2

From this graph, clearly the minimum distance is 5, located at point (4, 3).

Now let us use Lagrange multiplier to minimize $f(x, y)$ subject to the constraint described by the circle equation. Forming the Lagrange function, we obtain

$$\mathcal{L} = x^2 + y^2 + \lambda[(x - 8)^2 + (y - 6)^2 - 25]$$

The necessary conditions for extrema are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda(2x - 16) = 0 \quad \text{or} \quad 2x(\lambda + 1) = 16\lambda$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + \lambda(2y - 12) = 0 \quad \text{or} \quad 2y(\lambda + 1) = 12\lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (x - 8)^2 + (y - 6)^2 - 25 = 0$$

The solution of the above three equations will provide optimal points. In this problem, a direct solution can be obtained as follows:

Eliminating $\lambda$ from the first two equations results in

$$y = \frac{3}{4}x$$

Substituting for $y$ in the third equation yields

$$\frac{25}{16}x^2 - 25x + 75 = 0$$

The solutions of the above quadratic equations are $x = 4$ and $x = 12$. Thus, the corresponding extrema are at points (4, 3) with $\lambda = 1$, and (12, 9) with $\lambda = -3$. From Figure 7.1, it is clear that the minimum distance is at point (4, 3) and the maximum distance is at point (12, 9). To distinguish these points, the second derivatives are obtained and the Hessian matrices evaluated at these points are formed. The matrix with positive eigenvalues is a positive definite matrix and the parameters correspond to the minimum point.

In many problems, a direct solution is not possible and the above equations are solved iteratively. Many iterative schemes are available. The simplest search method is to assume a value for $\lambda$ and compute $\Delta f$. If $\Delta f$ is zero, the estimated $\lambda$ corresponds to the optimum solution. If not, depending on the sign of $\Delta f$, $\lambda$ is increased or decreased, and another solution is obtained. With two solutions, a better value of $\lambda$ is obtained by extrapolation and the process is continued until $\Delta f$ is within a specified accuracy. A significantly superior method applicable to continuous functions is the Newton-Raphson method. One way to apply the Newton-Raphson method to the problem at hand is as follows: From the first two equations, $x$ and $y$ are found. These are

$$x = \frac{8\lambda}{\lambda + 1}$$

$$y = \frac{6\lambda}{\lambda + 1}$$

Substituting into the third equation results in

$$f(\lambda) = \frac{100\lambda^2}{(\lambda + 1)^2} - \frac{200\lambda}{\lambda + 1} + 75 = 0$$





This is a nonlinear equation in terms of $\lambda$ and can be solved by the Newton-Raphson method. The Newton-Raphson method is a successive approximation procedure based on an initial estimate of the unknown and the use of Taylor's series expansion (see Chapter 6 for more details). For a one-dimensional case,

$$\Delta\lambda^{(k)} = \frac{-\Delta f(\lambda)^k}{(\frac{df}{d\lambda})^k} \tag{7.11}$$

and

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \tag{7.12}$$

Starting with an estimated value of $\lambda$, a new value is found in the direction of steepest decent (negative gradient). The process is repeated in the direction of negative gradient until $\Delta f(\lambda)$ is less than a specified accuracy. This algorithm is known as the *gradient method*. For the above function, the gradient is

$$\frac{df(\lambda)}{d\lambda} = \frac{200\lambda}{(\lambda+1)^3} - \frac{200}{(\lambda+1)^2} = \frac{-200}{(\lambda+1)^3}$$

The following commands show the procedure for the solution of the given equation by the Newton-Raphson method.

```
iter = 0;                                   % Iteration counter
Df = 10;                  % Error in Df is set to a high value
Lambda = input('Enter estimated value of Lambda = ');
fprintf('\n ')
disp(['    Iter       Df          J      DLambda    Lambda' ...
'       x          y'])
while abs(Df)  >= 0.0001                   % Test for convergence
iter = iter + 1;                            % No. of iterations
x = 8*Lambda/(Lambda + 1);
y = 6*Lambda/(Lambda + 1);
Df = (x- 8)^2 + (y - 6)^2 - 25;                       % Residual
J = -200/(Lambda + 1)^3;
Delambda =-Df/J;                            % Change in variable
disp([iter, Df, J, Delambda, Lambda, x, y])
Lambda = Lambda + Delambda;                % Successive solution
end
```

When the program is run, the user is prompted to enter the initial estimate for $\lambda$. Using a value of $\lambda = 0.4$, the result is

```
Enter estimated value of Lambda = 0.4
```

| Iter | $\Delta f$ | $J$ | $\Delta\lambda$ | $\lambda$ | $x$ | $y$ |
|---|---|---|---|---|---|---|
| 1 | 26.0240 | -72.8863 | 0.3570 | 0.4000 | 2.2857 | 1.7134 |
| 2 | 7.3934 | -36.8735 | 0.2005 | 0.7570 | 3.4468 | 2.5851 |
| 3 | 1.0972 | -26.6637 | 0.0411 | 0.9575 | 3.9132 | 2.9349 |
| 4 | 0.0337 | -25.0505 | 0.0013 | 0.9987 | 3.9973 | 2.9980 |
| 5 | 0.0000 | -25.0001 | 0.0000 | 1.0000 | 4.0000 | 3.0000 |

After five iterations, the solution converges to $\lambda = 1.0$, $x = 4$, and $y = 3$, corresponding to the minimum length. If the program is run with an initial estimate of $-2$, the solution converges to $\lambda = -3$, $x = 12$, $y = 9$, which corresponds to the maximum length.

### 7.2.2 CONSTRAINT PARAMETER OPTIMIZATION: INEQUALITY CONSTRAINTS

Practical optimization problems contain inequality constraints as well as equality constraints. The problem is to minimize the cost function

$$f(x_1, x_2, \ldots, x_n) \tag{7.13}$$

subject to the equality constraints

$$g_i(x_1, x_2, \ldots, x_n) = 0 \qquad i = 1, 2, \ldots, k \tag{7.14}$$

and the inequality constraints

$$u_j(x_1, x_2, \ldots, x_n) \le 0 \qquad i = 1, 2, \ldots, m \tag{7.15}$$

The Lagrange multiplier is extended to include the inequality constraints by introducing $m$-vector $\mu$ of undetermined quantities. The unconstrained cost function becomes

$$\mathcal{L} = f + \sum_{i=1}^{k} \lambda_i g_i + \sum_{j=1}^{m} \mu_j u_j \tag{7.16}$$

The resulting necessary conditions for constrained local minima of $\mathcal{L}$ are the following:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad i = 1, \ldots, n \tag{7.17}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = g_i = 0 \quad i = 1, \ldots, k \tag{7.18}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_j} = u_j \le 0 \quad j = 1, \ldots, m \tag{7.19}$$

$$\mu_j u_j = 0 \;\; \& \;\; \mu_j > 0 \quad j = 1, \ldots, m \tag{7.20}$$





Note that Equation (7.18) is simply the original equality constraints. Suppose $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is a relative minimum. The inequality constraints in (7.19) is said to be inactive if strict inequality holds at $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ and $\mu_j = 0$. On the other hand, when strict equality holds, the constraint is active at this point, (i.e., if the constraint $\mu_j u_j(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) = 0$ and $\mu_j > 0$. This is known as the *Kuhn-Tucker* necessary condition.

**Example 7.3**

Solve Example 7.2 with an additional inequality constraint defined below. The problem is to find the minimum value of the function

$$f(x, y) = x^2 + y^2$$

subject to one equality constraint

$$g(x, y) = (x - 8)^2 + (y - 6)^2 - 25 = 0$$

and one inequality constraint,

$$u(x, y) = 2x + y \ge 12$$

The unconstrained cost function from (7.16) is

$$\mathcal{L} = x^2 + y^2 + \lambda[(x - 8)^2 + (y - 6)^2 - 25] + \mu(2x + y - 12)$$

The resulting necessary conditions for constrained local minima of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 2\lambda(x - 8) + 2\mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + 2\lambda(y - 6) + \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (x - 8)^2 + (y - 6)^2 - 25 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 2x + y - 12 = 0$$

Eliminating $\mu$ from the first two equations result in

$$(2x - 4y)(1 + \lambda) + 8\lambda = 0$$

From the fourth condition, we have

$$y = 12 - 2x$$

Substituting for $y$ in the above equation, yields

$$x = \frac{4\lambda + 4.8}{1 + \lambda}$$

Now substituting for $x$ in the previous equation, we get

$$y = \frac{4\lambda + 2.4}{1 + \lambda}$$

Substituting for $x$ and $y$ in the third condition (equality constraint) results in an equation in terms of $\lambda$

$$\left(\frac{4\lambda + 4.8}{1 + \lambda} - 8\right)^2 + \left(\frac{4\lambda + 2.4}{1 + \lambda} - 6\right)^2 - 25 = 0$$

from which we have the following equation

$$\lambda^2 + 2\lambda + 0.36 = 0$$

Roots of the above equation are $\lambda = -0.2$ and $\lambda = -1.8$. Substituting for these values of $\lambda$ in the expression for $x$ and $y$, the corresponding extrema are

$$(x, y) = (5, 2) \quad \text{for} \quad \lambda = -0.2, \quad \mu = -5.6$$
$$(x, y) = (3, 6) \quad \text{for} \quad \lambda = -1.8, \quad \mu = -12$$

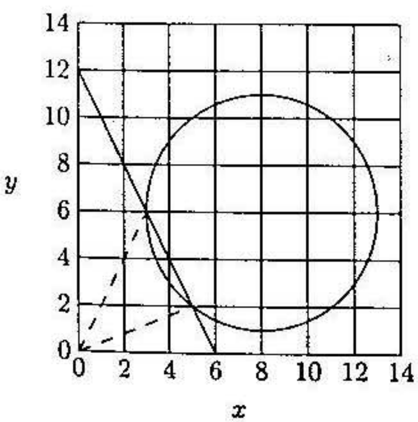



**FIGURE 7.2**
Constraint functions of Example 7.3.

The minimum distance from the cost function is 5.385, located at point $(5, 2)$, and the maximum distance is 6.71 located at point $(3, 6)$.

Adding the inequality constraint $2x + y \geq 12$ to the graphs in Figure 7.1, the solution is verified graphically as shown in Figure 7.2.





## 7.3 OPERATING COST OF A THERMAL PLANT

The factors influencing power generation at minimum cost are operating efficiencies of generators, fuel cost, and transmission losses. The most efficient generator in the system does not guarantee minimum cost as it may be located in an area where fuel cost is high. Also, if the plant is located far from the load center, transmission losses may be considerably higher and hence the plant may be overly uneconomical. Hence, the problem is to determine the generation of different plants such that the total operating cost is minimum. The operating cost plays an important role in the economic scheduling and are discussed here.

The input to the thermal plant is generally measured in Btu/h, and the output is measured in MW. A simplified input-output curve of a thermal unit known as *heat-rate* curve is given in Figure 7.3(a). Converting the ordinate of heat-rate

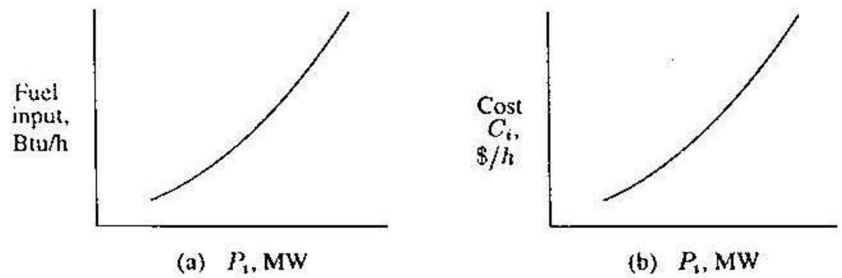



**FIGURE 7.3**
(a) Heat-rate curve. (b) Fuel-cost curve.

curve from Btu/h to \$/h results in the *fuel-cost* curve shown in Figure 7.3(b). In all practical cases, the fuel cost of generator $i$ can be represented as a quadratic function of real power generation

$$C_i = \alpha_i + \beta_i P_i + \gamma_i P_i^2 \tag{7.21}$$

An important characteristic is obtained by plotting the derivative of the fuel-cost curve versus the real power. This is known as the *incremental fuel-cost* curve shown in Figure 7.4.

$$\frac{dC_i}{dP_i} = 2\gamma_i P_i + \beta_i \tag{7.22}$$

The incremental fuel-cost curve is a measure of how costly it will be to produce the next increment of power. The total operating cost includes the fuel cost, and the cost of labor, supplies and maintenance. These costs are assumed to be a fixed percentage of the fuel cost and are generally included in the incremental fuel-cost curve.

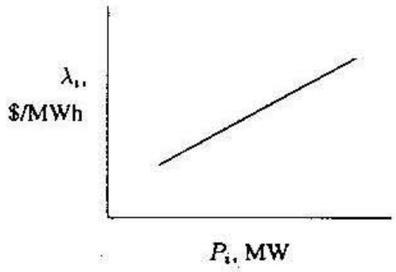



**FIGURE 7.4**
Typical incremental fuel-cost curve.

## 7.4 ECONOMIC DISPATCH NEGLECTING LOSSES AND NO GENERATOR LIMITS

The simplest economic dispatch problem is the case when transmission line losses are neglected. That is, the problem model does not consider system configuration and line impedances. In essence, the model assumes that the system is only one bus with all generation and loads connected to it as shown schematically in Figure 7.5.

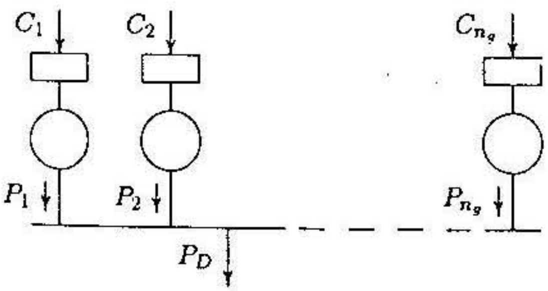



**FIGURE 7.5**
Plants connected to a common bus.

Since transmission losses are neglected, the total demand $P_D$ is the sum of all generation. A cost function $C_i$ is assumed to be known for each plant. The problem is to find the real power generation for each plant such that the objective function (i.e., total production cost) as defined by the equation

$$\begin{aligned} C_t &= \sum_{i=1}^{n_g} C_i \\ &= \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \end{aligned} \tag{7.23}$$





is minimum, subject to the constraint

$$\sum_{i=1}^{n_g} P_i = P_D \tag{7.24}$$

where $C_t$ is the total production cost, $C_i$ is the production cost of $i$th plant, $P_i$ is the generation of $i$th plant, $P_D$ is the total load demand, and $n_g$ is the total number of dispatchable generating plants.

A typical approach is to augment the constraints into objective function by using the Lagrange multipliers

$$\mathcal{L} = C_t + \lambda \left( P_D - \sum_{i=1}^{n_g} P_i \right) \tag{7.25}$$

The minimum of this unconstrained function is found at the point where the partials of the function to its variables are zero.

$$\frac{\partial \mathcal{L}}{\partial P_i} = 0 \tag{7.26}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{7.27}$$

First condition, given by (7.26), results in

$$\frac{\partial C_t}{\partial P_i} + \lambda(0 - 1) = 0$$

Since

$$C_t = C_1 + C_2 + \cdots + C_{n_g}$$

then

$$\frac{\partial C_t}{\partial P_i} = \frac{dC_i}{dP_i} = \lambda$$

and therefore the condition for optimum dispatch is

$$\frac{dC_i}{dP_i} = \lambda \qquad i = 1, \ldots, n_g \tag{7.28}$$

or

$$\beta_i + 2\gamma_i P_i = \lambda \tag{7.29}$$

Second condition, given by (7.27), results in

$$\sum_{i=1}^{n_g} P_i = P_D \tag{7.30}$$

Equation (7.30) is precisely the equality constraint that was to be imposed. In summary, when losses are neglected with no generator limits, for most economic operation, all plants must operate at equal incremental production cost while satisfying the equality constraint given by (7.30). In order to find the solution, (7.29) is solved for $P_i$

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i} \tag{7.31}$$

The relations given by (7.31) are known as the *coordination equations*. They are functions of $\lambda$. An analytical solution can be obtained for $\lambda$ by substituting for $P_i$ in (7.30), i.e.,

$$\sum_{i=1}^{n_g} \frac{\lambda - \beta_i}{2\gamma_i} = P_D \tag{7.32}$$

or

$$\lambda = \frac{P_D + \sum_{i=1}^{n_g} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_g} \frac{1}{2\gamma_i}} \tag{7.33}$$

The value of $\lambda$ found from (7.33) is substituted in (7.31) to obtain the optimal scheduling of generation.

The solution for economic dispatch neglecting losses was found analytically. However when losses are considered the resulting equations as seen in Section 7.6 are nonlinear and must be solved iteratively. Thus, an iterative procedure is introduced here and (7.31) is solved iteratively. In an iterative search technique, starting with two values of $\lambda$, a better value of $\lambda$ is obtained by extrapolation, and the process is continued until $\Delta P_i$ is within a specified accuracy. However, as mentioned earlier, a rapid solution is obtained by the use of the gradient method. To do this, (7.32) is written as

$$f(\lambda) = P_D \tag{7.34}$$

Expanding the left-hand side of the above equation in Taylor's series about an operating point $\lambda^{(k)}$, and neglecting the higher-order terms results in

$$f(\lambda)^{(k)} + \left(\frac{df(\lambda)}{d\lambda}\right)^{(k)} \Delta\lambda^{(k)} = P_D \tag{7.35}$$





or

$$\Delta\lambda^{(k)} = \frac{\Delta P^{(k)}}{\left(\frac{df(\lambda)}{d\lambda}\right)^{(k)}} = \frac{\Delta P^{(k)}}{\sum \left(\frac{dP_i}{d\lambda}\right)^{(k)}} \tag{7.36}$$

or

$$\Delta\lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum \frac{1}{2\gamma_i}} \tag{7.37}$$

and therefore,

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \tag{7.38}$$

where

$$\Delta P^{(k)} = P_D - \sum_{i=1}^{n_g} P_i^{(k)} \tag{7.39}$$

The process is continued until $\Delta P^{(k)}$ is less than a specified accuracy.

**Example 7.4**

The fuel-cost functions for three thermal plants in \$/h are given by

$$C_1 = 500 + 5.3P_1 + 0.004P_1^2$$
$$C_2 = 400 + 5.5P_2 + 0.006P_2^2$$
$$C_3 = 200 + 5.8P_3 + 0.009P_3^2$$

where $P_1$, $P_2$, and $P_3$ are in MW. The total load, $P_D$, is 800 MW. Neglecting line losses and generator limits, find the optimal dispatch and the total cost in \$/h

(a) by analytical method using (7.33)
(b) by graphical demonstration.
(c) by iterative technique using the gradient method.

(a) From (7.33), $\lambda$ is found to be

$$\lambda = \frac{800 + \frac{5.3}{0.008} + \frac{5.5}{0.012} + \frac{5.8}{0.018}}{\frac{1}{0.008} + \frac{1}{0.012} + \frac{1}{0.018}} = \frac{800 + 1443.0555}{263.8889} = 8.5 \quad \$/\text{MWh}$$

Substituting for $\lambda$ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{8.5 - 5.3}{2(0.004)} = 400.0000$$
$$P_2 = \frac{8.5 - 5.5}{2(0.006)} = 250.0000$$
$$P_3 = \frac{8.5 - 5.8}{2(0.009)} = 150.0000$$

(b) From (7.28), the necessary conditions for optimal dispatch are

$$\frac{dC_1}{dP_1} = 5.3 + 0.008P_1 = \lambda$$
$$\frac{dC_2}{dP_2} = 5.5 + 0.012P_2 = \lambda$$
$$\frac{dC_3}{dP_3} = 5.8 + 0.018P_3 = \lambda$$

subject to

$$P_1 + P_2 + P_3 = P_D$$

To demonstrate the concept of equal incremental cost for optimal dispatch, we can use *MATLAB* **plot** command to plot the incremental cost of each plant on the same graph as shown in Figure 7.6. To obtain a solution, various values of $\lambda$ could be tried until one is found which produces $\sum P_i = P_D$. For each $\lambda$, if $\sum P_i < P_D$, we increase $\lambda$ otherwise, if $\sum P_i > P_D$, we reduce $\lambda$. Therefore, the horizontal dashed-line shown in the graph is moved up or down until at the optimum point $\hat{\lambda}$, $\sum P_i = P_D$. For this example, with $P_D = 800$ MW, the optimal dispatch is $P_1 = 400$, $P_2 = 250$, and $P_3 = 150$ at $\hat{\lambda} = 8.5\,\$/\text{MWh}$.

(c) For the numerical solution using the gradient method, assume the initial value of $\lambda^{(1)} = 6.0$. From coordination equations, given by (7.31), $P_1$, $P_2$, and $P_3$ are

$$P_1^{(1)} = \frac{6.0 - 5.3}{2(0.004)} = 87.5000$$
$$P_2^{(1)} = \frac{6.0 - 5.5}{2(0.006)} = 41.6667$$
$$P_3^{(1)} = \frac{6.0 - 5.8}{2(0.009)} = 11.1111$$

Since $P_D = 800$ MW, the error $\Delta P$ from (7.39) is

$$\Delta P^{(1)} = 800 - (87.5 + 41.6667 + 11.1111) = 659.7222$$





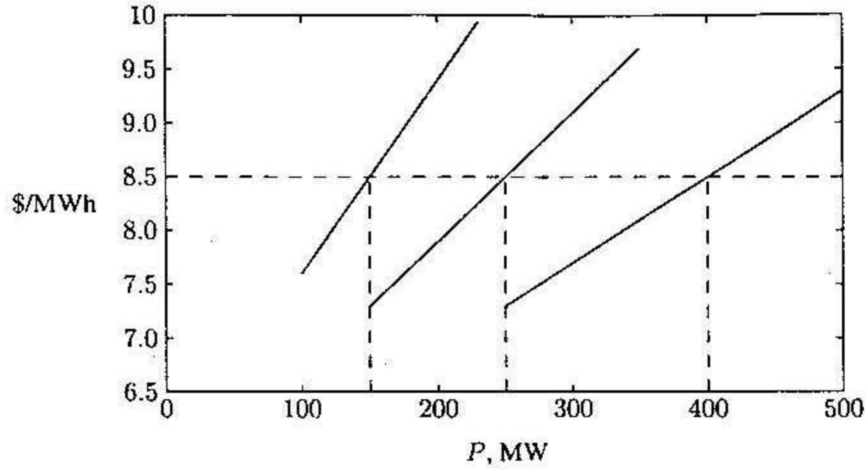



**FIGURE 7.6**
Illustrating the concept of equal incremental cost production cost.

From (7.37)

$$\Delta\lambda^{(1)} = \frac{659.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{659.7222}{263.8888} = 2.5$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(2)} = 6.0 + 2.5 = 8.5$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{8.5 - 5.3}{2(0.004)} = 400.0000$$
$$P_2^{(2)} = \frac{8.5 - 5.5}{2(0.006)} = 250.0000$$
$$P_3^{(2)} = \frac{8.5 - 5.8}{2(0.009)} = 150.0000$$

and

$$\Delta P^{(2)} = 800 - (400 + 250 + 150) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations. Therefore, the optimal dispatch are

$$P_1 = 400 \quad \text{MW}$$

$$P_2 = 250 \quad \text{MW}$$
$$P_3 = 150 \quad \text{MW}$$
$$\hat{\lambda} = 8.5 \quad \$/\text{MWh}$$

and the total fuel cost is

$$C_t = 500 + 5.3(400) + 0.004(400)^2 + 400 + 5.5(250) + 0.006(250)^2$$
$$+200 + 5.8(150) + 0.009(150)^2 = 6,682.5 \quad \$/\text{h}$$

To demonstrate the above method, the following simple program is written for Example 7.4.

```
alpha =[500; 400; 200];
beta = [5.3; 5.5; 5.8]; gamma=[.004; .006; .009];
PD=800;
DelP = 10;          % Error in DelP is set to a high value
lambda = input('Enter estimated value of Lambda = ');
fprintf(' ')
disp(['   Lambda      P1        P2        P3        DP'...
 '     grad       Delambda'])
iter = 0;                              % Iteration counter
while abs(DelP)  >= 0.001           % Test for convergence
iter = iter + 1;                       % No. of iterations
P = (lambda - beta)./(2*gamma);    % Coordination equation
DelP =PD - sum(P);                             % Residual
J = sum(ones(length(gamma), 1)./(2*gamma));% Gradient sum
Delambda = DelP/J;                    % Change in variable
disp([lambda, P(1), P(2), P(3), DelP, J, Delambda])
lambda = lambda + Delambda;          % Successive solution
end
totalcost = sum(alpha + beta.*P + gamma.*P.^2)
```

When the program is run, the result is

```
Enter estimated value of Lambda = 6

Lambda     P1      P2       P3        DP      grad    Delambda
6.0000  87.500  41.6667  11.1111  659.7222  263.8889  2.500
8.5000 400.000 250.0000 150.0000    0.0000  263.8889  0.000

 totalcost =

          6682.5
```





A general program called **dispatch** is developed for the optimal dispatch problem. The program returns the system $\lambda$, the optimal dispatch generation vector $P$, and the total cost. The following reserved variables are required by the **dispatch** program:

**Pdt** This reserved name must be used to specify the total load in MW. If **Pdt** is not specified the user is prompted to input the total load. If **dispatch** is used following any of the power flow programs, the total load is automatically passed by the power flow program.

**cost** This reserved name must be used to specify the cost function coefficients. The coefficients are arranged in the *MATLAB* matrix format. Each row contains the coefficients of the cost function in ascending powers of $P$.

**mwlimits** This name is reserved for the generator's real power limits and are discussed in Section 7.5. This entry is specified in matrix form with the first column representing the minimum value and the second column representing the maximum value. If **mwlimits** is not specified, the program obtains the optimal dispatch of generation with no limits.

**B B0 B00** These names are reserved for the loss formula coefficient matrices and are discussed in Section 7.6. If these variables are not specified, optimal dispatch of generation is obtained neglecting losses.

The total generation cost of a thermal power system can be obtained with the aid of the **gencost** command. This program can be used following any of the power flow programs or the **dispatch** program, provided cost function matrix is defined.

### Example 7.5

Neglecting generator limits and system losses, use **dispatch** program to obtain the optimal dispatch of generation for thermal plants specified in Example 7.4.

We use the following command:

```
cost = [500   5.3    0.004
        400   5.5    0.006
        200   5.8    0.009];
Pdt = 800;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power(system lambda) = 8.5$/MWh
Optimal Dispatch of Generation:

      400.0000
      250.0000
      150.0000
   Total generation cost = 6682.50 $/h
```

## 7.5 ECONOMIC DISPATCH NEGLECTING LOSSES AND INCLUDING GENERATOR LIMITS

The power output of any generator should not exceed its rating nor should it be below that necessary for stable boiler operation. Thus, the generations are restricted to lie within given minimum and maximum limits. The problem is to find the real power generation for each plant such that the objective function (i.e., total production cost) as defined by (7.23) is minimum, subject to the constraint given by (7.24) and the inequality constraints given by

$$P_{i(min)} \leq P_i \leq P_{i(max)} \qquad i = 1, \ldots, n_g \tag{7.40}$$

Where $P_{i(min)}$ and $P_{i(max)}$ are the minimum and maximum generating limits respectively for plant $i$.

The Kuhn-Tucker conditions complement the Lagrangian conditions to include the inequality constraints as additional terms. The necessary conditions for the optimal dispatch with losses neglected becomes

$$\begin{aligned} \frac{dC_i}{dP_i} &= \lambda \qquad \text{for} \quad P_{i(min)} < P_i < P_{i(max)} \\ \frac{dC_i}{dP_i} &\leq \lambda \qquad \text{for} \quad P_i = P_{i(max)} \\ \frac{dC_i}{dP_i} &\geq \lambda \qquad \text{for} \quad P_i = P_{i(min)} \end{aligned} \tag{7.41}$$

The numerical solution is the same as before. That is, for an estimated $\lambda$, $P_i$ are found from the coordination Equation (7.31) and iteration is continued until $\sum P_i = P_D$. As soon as any plant reaches a maximum or minimum, the plant becomes pegged at the limit. In effect, the plant output becomes a constant, and only the unviolated plants must operate at equal incremental cost.





**Example 7.6**

Find the optimal dispatch and the total cost in \$/h for the thermal plants of Example 7.4 when the total load is 975 MW with the following generator limits (in MW):

$$\begin{aligned} 200 &\leq P_1 \leq 450 \\ 150 &\leq P_2 \leq 350 \\ 100 &\leq P_3 \leq 225 \end{aligned}$$

Assume the initial value of $\lambda^{(1)} = 6.0$. From coordination equations given by (7.31), $P_1$, $P_2$, and $P_3$ are

$$\begin{aligned} P_1^{(1)} &= \frac{6.0 - 5.3}{2(0.004)} = 87.5000 \\ P_2^{(1)} &= \frac{6.0 - 5.5}{2(0.006)} = 41.6667 \\ P_3^{(1)} &= \frac{6.0 - 5.8}{2(0.009)} = 11.1111 \end{aligned}$$

Since $P_D = 975$ MW, the error $\Delta P$ from (7.39) is

$$\Delta P^{(1)} = 975 - (87.5 + 41.6667 + 11.1111) = 834.7222$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{834.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{834.7222}{263.8888} = 3.1632$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(2)} = 6.0 + 3.1632 = 9.1632$$

Continuing the process, for the second iteration, we have

$$\begin{aligned} P_1^{(2)} &= \frac{9.1632 - 5.3}{2(0.004)} = 482.8947 \\ P_2^{(2)} &= \frac{9.1632 - 5.5}{2(0.006)} = 305.2632 \\ P_3^{(2)} &= \frac{9.1632 - 5.8}{2(0.009)} = 186.8421 \end{aligned}$$

and

$$\Delta P^{(2)} = 975 - (482.8947 + 305.2632 + 186.8421) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations. However, $P_1$ exceeds its upper limit. Thus, this plant is pegged at its upper limit. Hence $P_1 = 450$ and is kept constant at this value. Thus, the new imbalance in power is

$$\Delta P^{(2)} = 975 - (450 + 305.2632 + 186.8421) = 32.8947$$

From (7.37)

$$\Delta\lambda^{(2)} = \frac{32.8947}{\frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{32.8947}{138.8889} = 0.2368$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(3)} = 9.1632 + 0.2368 = 9.4$$

For the third iteration, we have

$$P_1^{(3)} = 450$$
$$P_2^{(3)} = \frac{9.4 - 5.5}{2(0.006)} = 325$$
$$P_3^{(3)} = \frac{9.4 - 5.8}{2(0.009)} = 200$$

and

$$\Delta P^{(3)} = 975 - (450 + 325 + 200) = 0.0$$

$\Delta P^{(3)} = 0$, and the equality constraint is met and $P_2$ and $P_3$ are within their limits. Thus, the optimal dispatch is

$$P_1 = 450 \quad \text{MW}$$
$$P_2 = 325 \quad \text{MW}$$
$$P_3 = 200 \quad \text{MW}$$
$$\hat{\lambda} = 9.4 \quad \$/\text{MWh}$$

and the total fuel cost is

$$C_t = 500 + 5.3(450) + 0.004(450)^2 + 400 + 5.5(325) + 0.006(325)^2$$
$$+200 + 5.8(200) + 0.009(200)^2 = 8,236.25 \quad \$/\text{h}$$

The following commands can be used to obtain the optimal dispatch of generation including generator limits.





```
cost = [500    5.3    0.004
        400    5.5    0.006
        200    5.8    0.009];
mwlimits=[200  450
          150  350
          100  225];
Pdt = 975;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power(system lambda) = 9.4$/MWh
Optimal Dispatch of Generation:

     450
     325
     200

Total generation cost = 8236.25 $/h
```

## 7.6 ECONOMIC DISPATCH INCLUDING LOSSES

When transmission distances are very small and load density is very high, transmission losses may be neglected and the optimal dispatch of generation is achieved with all plants operating at equal incremental production cost. However, in a large interconnected network where power is transmitted over long distances with low load density areas, transmission losses are a major factor and affect the optimum dispatch of generation. One common practice for including the effect of transmission losses is to express the total transmission loss as a quadratic function of the generator power outputs. The simplest quadratic form is

$$P_L = \sum_{i=1}^{n_g}\sum_{j=1}^{n_g} P_i B_{ij} P_j \qquad (7.42)$$

A more general formula containing a linear term and a constant term, referred to as *Kron's loss formula*, is

$$P_L = \sum_{i=1}^{n_g}\sum_{j=1}^{n_g} P_i B_{ij} P_j + \sum_{i=1}^{n_g} B_{0i} P_i + B_{00} \qquad (7.43)$$

The coefficients $B_{ij}$ are called *loss coefficients* or *B-coefficients*. $B$-coefficients are assumed constant, and reasonable accuracy can be expected provided the actual operating conditions are close to the base case where the $B$-constants were computed. There are various ways of arriving at a loss equation. A method for obtaining these $B$-coefficients is presented in Section 7.7.

The economic dispatching problem is to minimize the overall generating cost $C_i$, which is the function of plant output

$$C_t = \sum_{i=1}^{n_g} C_i$$
$$= \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \tag{7.44}$$

subject to the constraint that generation should equal total demands plus losses, i.e.,

$$\sum_{i=1}^{n_g} P_i = P_D + P_L \tag{7.45}$$

satisfying the inequality constraints, expressed as follows:

$$P_{i(min)} \leq P_i \leq P_{i(max)} \qquad i = 1, \ldots, n_g \tag{7.46}$$

where $P_{i(min)}$ and $P_{i(max)}$ are the minimum and maximum generating limits, respectively, for plant $i$.

Using the Lagrange multiplier and adding additional terms to include the inequality constraints, we obtain

$$\mathcal{L} = C_t + \lambda(P_D + P_L - \sum_{i=1}^{n_g} P_i) + \sum_{i=1}^{n_g} \mu_{i(max)}(P_i - P_{i(max)}) +$$
$$\sum_{i=1}^{n_g} \mu_{i(min)}(P_i - P_{i(min)}) \tag{7.47}$$

The constraints should be understood to mean the $\mu_{i(max)} = 0$ when $P_i < P_{i(max)}$ and that $\mu_{i(min)} = 0$ when $P_i > P_{i(min)}$. In other words, if the constraint is not violated, its associated $\mu$ variable is zero and the corresponding term in (7.47) does not exist. The constraint only becomes active when violated. The minimum of this unconstrained function is found at the point where the partials of the function to its variables are zero.

$$\frac{\partial \mathcal{L}}{\partial P_i} = 0 \tag{7.48}$$





$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \tag{7.49}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_{i(max)}} = P_i - P_{i(max)} = 0 \tag{7.50}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_{i(min)}} = P_i - P_{i(min)} = 0 \tag{7.51}$$

Equations (7.50) and (7.51) imply that $P_i$ should not be allowed to go beyond its limit, and when $P_i$ is within its limits $\mu_{i(min)} = \mu_{i(max)} = 0$ and the Kuhn-Tucker function becomes the same as the Lagrangian one. First condition, given by (7.48), results in

$$\frac{\partial C_t}{\partial P_i} + \lambda(0 + \frac{\partial P_L}{\partial P_i} - 1) = 0$$

Since

$$C_t = C_1 + C_2 + \cdots + C_{n_g}$$

then

$$\frac{\partial C_t}{\partial P_i} = \frac{dC_i}{dP_i}$$

and therefore the condition for optimum dispatch is

$$\frac{dC_i}{dP_i} + \lambda \frac{\partial P_L}{\partial P_i} = \lambda \qquad i = 1, \ldots, n_g \tag{7.52}$$

The term $\frac{\partial P_L}{\partial P_i}$ is known as the incremental transmission loss. Second condition, given by (7.49), results in

$$\sum_{i=1}^{n_g} P_i = P_D + P_L \tag{7.53}$$

Equation (7.53) is precisely the equality constraint that was to be imposed.

Classically, Equation (7.52) is rearranged as

$$\left(\frac{1}{1 - \frac{\partial P_L}{\partial P_i}}\right) \frac{dC_i}{dP_i} = \lambda \qquad i = 1, \ldots, n_g \tag{7.54}$$

or

$$L_i \frac{dC_i}{dP_i} = \lambda \qquad i = 1, \ldots, n_g \tag{7.55}$$

where $L_i$ is known as the *penalty factor* of plant $i$ and is given by

$$L_i = \frac{1}{1 - \frac{\partial P_L}{\partial P_i}} \tag{7.56}$$

Hence, the effect of transmission loss is to introduce a penalty factor with a value that depends on the location of the plant. Equation (7.55) shows that the minimum cost is obtained when the incremental cost of each plant multiplied by its penalty factor is the same for all plants.

The incremental production cost is given by (7.22), and the incremental transmission loss is obtained from the loss formula (7.43) which yields

$$\frac{\partial P_L}{\partial P_i} = 2\sum_{j=1}^{n_g} B_{ij}P_j + B_{0i} \tag{7.57}$$

Substituting the expression for the incremental production cost and the incremental transmission loss in (7.52) results in

$$\beta_i + 2\gamma_i P_i + 2\lambda \sum_{j=1}^{n_g} B_{ij}P_j + B_{0i}\lambda = \lambda$$

or

$$\left(\frac{\gamma_i}{\lambda} + B_{ii}\right) P_i + \sum_{\substack{j=1\\ j\neq i}}^{n_g} B_{ij}P_j = \frac{1}{2}\left(1 - B_{0i} - \frac{\beta_i}{\lambda}\right) \tag{7.58}$$

Extending (7.58) to all plants results in the following linear equations in matrix form

$$\begin{bmatrix} \frac{\gamma_1}{\lambda}+B_{11} & B_{12} & \cdots & B_{1n_g} \\ B_{21} & \frac{\gamma_2}{\lambda}+B_{22} & \cdots & B_{2n_g} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n_g1} & B_{n_g2} & \cdots & \frac{\gamma_{n_g}}{\lambda}+B_{n_gn_g} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_g} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1-B_{01}-\frac{\beta_1}{\lambda} \\ 1-B_{02}-\frac{\beta_2}{\lambda} \\ \vdots \\ 1-B_{0n_g}-\frac{\beta_{n_g}}{\lambda} \end{bmatrix} \tag{7.59}$$

or in short form

$$\boldsymbol{EP} = \boldsymbol{D} \tag{7.60}$$

To find the optimal dispatch for an estimated value of $\lambda^{(1)}$, the simultaneous linear equation given by (7.60) is solved. In *MATLAB* use the command $\mathbf{P} = \mathbf{E}\backslash\mathbf{D}$.





Then the iterative process is continued using the gradient method. To do this, from (7.58), $P_i$ at the $k$th iteration is expressed as

$$P_i^{(k)} = \frac{\lambda^{(k)}(1 - B_{0i}) - \beta_i - 2\lambda^{(k)}\sum_{j\neq i} B_{ij}P_j^{(k)}}{2(\gamma_i + \lambda^{(k)}B_{ii})} \tag{7.61}$$

Substituting for $P_i$ from (7.61) in (7.53) results in

$$\sum_{i=1}^{n_g} \frac{\lambda^{(k)}(1 - B_{0i}) - \beta_i - 2\lambda^{(k)}\sum_{j\neq i} B_{ij}P_j^{(k)}}{2(\gamma_i + \lambda^{(k)}B_{ii})} = P_D + P_L^{(k)} \tag{7.62}$$

or

$$f(\lambda)^{(k)} = P_D + P_L^{(k)} \tag{7.63}$$

Expanding the left-hand side of the above equation in Taylor's series about an operating point $\lambda^{(k)}$, and neglecting the higher-order terms results in

$$f(\lambda)^{(k)} + \left(\frac{df(\lambda)}{d\lambda}\right)^{(k)} \Delta\lambda^{(k)} = P_D + P_L^{(k)} \tag{7.64}$$

or

$$\begin{aligned} \Delta\lambda^{(k)} &= \frac{\Delta P^{(k)}}{\left(\frac{df(\lambda)}{d\lambda}\right)^{(k)}} \\ &= \frac{\Delta P^{(k)}}{\sum\left(\frac{dP_i}{d\lambda}\right)^{(k)}} \end{aligned} \tag{7.65}$$

where

$$\sum_{i=1}^{n_g}\left(\frac{\partial P_i}{\partial \lambda}\right)^{(k)} = \sum_{i=1}^{n_g} \frac{\gamma_i(1 - B_{0i}) + B_{ii}\beta_i - 2\gamma_i\sum_{j\neq i} B_{ij}P_j^{(k)}}{2(\gamma_i + \lambda^{(k)}B_{ii})^2} \tag{7.66}$$

and therefore,

$$\lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \tag{7.67}$$

where

$$\Delta P^{(k)} = P_D + P_L^{(k)} - \sum_{i=1}^{n_g} P_i^{(k)} \tag{7.68}$$

The process is continued until $\Delta P^{(k)}$ is less than a specified accuracy.

If an approximate loss formula expressed by

$$P_L = \sum_{i=1}^{n_g} B_{ii} P_i^2 \tag{7.69}$$

is used, $B_{ij} = 0$, $B_{00} = 0$, and solution of the simultaneous equation given by (7.61) reduces to the following simple expression

$$P_i^{(k)} = \frac{\lambda^{(k)} - \beta_i}{2(\gamma_i + \lambda^{(k)} B_{ii})} \tag{7.70}$$

and (7.66) reduces to

$$\sum_{i=1}^{n_g} \left(\frac{\partial P_i}{\partial \lambda}\right)^{(k)} = \sum_{i=1}^{n_g} \frac{\gamma_i + B_{ii}\beta_i}{2(\gamma_i + \lambda^{(k)} B_{ii})^2} \tag{7.71}$$

**Example 7.7**

The fuel cost in \$/h of three thermal plants of a power system are

$$C_1 = 200 + 7.0P_1 + 0.008P_1^2 \quad \$/h$$
$$C_2 = 180 + 6.3P_2 + 0.009P_2^2 \quad \$/h$$
$$C_3 = 140 + 6.8P_3 + 0.007P_3^2 \quad \$/h$$

where $P_1$, $P_2$, and $P_3$ are in MW. Plant outputs are subject to the following limits

$$10 \text{ MW} \le 85 \text{ MW}$$
$$10 \text{ MW} \le 80 \text{ MW}$$
$$10 \text{ MW} \le 70 \text{ MW}$$

For this problem, assume the real power loss is given by the simplified expression

$$P_{L(pu)} = 0.0218P_{1(pu)}^2 + 0.0228P_{2(pu)}^2 + 0.0179P_{3(pu)}^2$$

where the loss coefficients are specified in per unit on a 100-MVA base. Determine the optimal dispatch of generation when the total system load is 150 MW.

In the cost function $P_i$ is expressed in MW. Therefore, the real power loss in terms of MW generation is

$$P_L = \left[0.0218\left(\frac{P_1}{100}\right)^2 + 0.0228\left(\frac{P_2}{100}\right)^2 + 0.0179\left(\frac{P_3}{100}\right)^2\right] \times 100 \quad \text{MW}$$
$$= 0.000218P_1^2 + 0.000228P_2^2 + 0.000179P_3^2 \quad \text{MW}$$





For the numerical solution using the gradient method, assume the initial value of $\lambda^{(1)} = 8.0$. From coordination equations, given by (7.70), $P_1^{(1)}$, $P_2^{(1)}$, and $P_3^{(1)}$ are

$$P_1^{(1)} = \frac{8.0 - 7.0}{2(0.008 + 8.0 \times 0.000218)} = 51.3136 \quad \text{MW}$$
$$P_2^{(1)} = \frac{8.0 - 6.3}{2(0.009 + 8.0 \times 0.000228)} = 78.5292 \quad \text{MW}$$
$$P_3^{(1)} = \frac{8.0 - 6.8}{2(0.007 + 8.0 \times 0.000179)} = 71.1575 \quad \text{MW}$$

The real power loss is

$$P_L^{(1)} = 0.000218(51.3136)^2 + 0.000228(78.5292)^2 + 0.000179(71.1575)^2 = 2.886$$

Since $P_D = 150$ MW, the error $\Delta P^{(1)}$ from (7.68) is

$$\Delta P^{(1)} = 150 + 2.8864 - (51.3136 + 78.5292 + 71.1575) = -48.1139$$

From (7.71)

$$\sum_{i=1}^{3}\left(\frac{\partial P_i}{\partial \lambda}\right)^{(1)} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 8.0 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 8.0 \times 0.000228)^2}$$
$$+ \frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 8.0 \times 0.000179)^2} = 152.4924$$

From (7.65)

$$\Delta\lambda^{(1)} = \frac{-48.1139}{152.4924} = -0.31552$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(2)} = 8.0 - 0.31552 = 7.6845$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{7.6845 - 7.0}{2(0.008 + 7.6845 \times 0.000218)} = 35.3728 \quad \text{MW}$$
$$P_2^{(2)} = \frac{7.6845 - 6.3}{2(0.009 + 7.6845 \times 0.000228)} = 64.3821 \quad \text{MW}$$
$$P_3^{(2)} = \frac{7.6845 - 6.8}{2(0.007 + 7.6845 \times 0.000179)} = 52.8015 \quad \text{MW}$$

The real power loss is

$$P_L^{(2)} = 0.000218(35.3728)^2 + 0.000228(64.3821)^2 + 0.000179(52.8015)^2 = 1.717$$

Since $P_D = 150$ MW, the error $\Delta P^{(2)}$ from (7.68) is

$$\Delta P^{(2)} = 150 + 1.7169 - (35.3728 + 64.3821 + 52.8015) = -0.8395$$

From (7.71)

$$\sum_{i=1}^{3}\left(\frac{\partial P_i}{\partial \lambda}\right)^{(2)} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 7.684 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 7.684 \times 0.000228)^2}$$
$$+ \frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 7.6845 \times 0.000179)^2} = 154.588$$

From (7.65)

$$\Delta \lambda^{(2)} = \frac{-0.8395}{154.588} = -0.005431$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(3)} = 7.6845 - 0.005431 = 7.679$$

For the third iteration, we have

$$P_1^{(3)} = \frac{7.679 - 7.0}{2(0.008 + 7.679 \times 0.000218)} = 35.0965 \quad \text{MW}$$
$$P_2^{(3)} = \frac{7.679 - 6.3}{2(0.009 + 7.679 \times 0.000228)} = 64.1369 \quad \text{MW}$$
$$P_3^{(3)} = \frac{7.679 - 6.8}{2(0.007 + 7.679 \times 0.000179)} = 52.4834 \quad \text{MW}$$

The real power loss is

$$P_L^{(3)} = 0.000218(35.0965)^2 + 0.000228(64.1369)^2 + 0.000179(52.4834)^2 = 1.699$$

Since $P_D = 150$ MW, the error $\Delta P^{(3)}$ from (7.68) is

$$\Delta P^{(3)} = 150 + 1.6995 - (35.0965 + 64.1369 + 52.4834) = -0.01742$$

From (7.71)

$$\sum_{i=1}^{3}\left(\frac{\partial P_i}{\partial \lambda}\right)^{(3)} = \frac{0.008 + 0.000218 \times 7.0}{2(0.008 + 7.679 \times 0.000218)^2} + \frac{0.009 + 0.000228 \times 6.3}{2(0.009 + 7.679 \times 0.000228)^2}$$
$$+ \frac{0.007 + 0.000179 \times 6.8}{2(0.007 + 7.679 \times 0.000179)^2} = 154.624$$





From (7.65)

$$\Delta \lambda^{(3)} = \frac{-0.01742}{154.624} = -0.0001127$$

Therefore, the new value of $\lambda$ is

$$\lambda^{(4)} = 7.679 - 0.0001127 = 7.6789$$

Since $\Delta\lambda^{(3)}$, is small the equality constraint is met in four iterations, and the optimal dispatch for $\lambda = 7.6789$ are

$$P_1^{(4)} = \frac{7.6789 - 7.0}{2(0.008 + 7.679 \times 0.000218)} = 35.0907 \quad \text{MW}$$
$$P_2^{(4)} = \frac{7.6789 - 6.3}{2(0.009 + 7.679 \times 0.000228)} = 64.1317 \quad \text{MW}$$
$$P_3^{(4)} = \frac{7.6789 - 6.8}{2(0.007 + 7.679 \times 0.000179)} = 52.4767 \quad \text{MW}$$

The real power loss is

$$P_L^{(4)} = 0.000218(35.0907)^2 + 0.000228(64.1317)^2 + 0.000179(52.4767)^2 = 1.699$$

and the total fuel cost is

$$C_t = 200 + 7.0(35.0907) + 0.008(35.0907)^2 + 180 + 6.3(64.1317) +$$
$$0.009(64.1317)^2 + 140 + 6.8(52.4767) + 0.007(52.4767)^2 = 1592.65 \text{ \$/h}$$

The **dispatch** program can be used to find the optimal dispatch of generation. The program is designed for the loss coefficients to be expressed in per unit. The loss coefficients are arranged in a matrix form with the variable name $B$. The base MVA must be specified by the variable name **basemva**. If base mva is not specified, it is set to 100 MVA.

We use the following commands

```
cost = [200  7.0    0.008
        180  6.3    0.009
        140  6.8    0.007];
mwlimits =[10  85
           10  80
           10  70];
Pdt = 150;
B = [0.0218        0         0
```

```
        0    0.0228        0
        0         0   0.0179];
basemva = 100;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power(system lambda) =
7.678935$/MWh
Optimal Dispatch of Generation:

   35.0907
   64.1317
   52.4767

Total system loss = 1.6991 MW
Total generation cost = 1592.65 $/h
```

**Example 7.8**

Figure 7.7 (page 295) shows the one-line diagram of a power system described in Example 7.9. The $B$ matrices of the loss formula for this system are found in Example 7.9. They are given in per unit on a 100 MVA base as follows

$$B = \begin{bmatrix} 0.0218 & 0.0093 & 0.0028 \\ 0.0093 & 0.0228 & 0.0017 \\ 0.0028 & 0.0017 & 0.0179 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.0003 & 0.0031 & 0.0015 \end{bmatrix}$$

$$B_{00} = 0.00030523$$

Cost functions, generator limits, and total loads are given in Example 7.7. Use **dispatch** program to obtain the optimal dispatch of generation.

We use the following commands.

```
cost = [200  7.0    0.008
        180  6.3    0.009
        140  6.8    0.007];
mwlimits =[10  85
           10  80
           10  70];
Pdt = 150;
```





```
B =  [0.0218   0.0093   0.0028
      0.0093   0.0228   0.0017
      0.0028   0.0017   0.0179];
B0 = [0.0003   0.0031   0.0015];
B00 = 0.00030523;
basemva = 100;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power (system lambda) =
7.767785 $/MWh
Optimal Dispatch of Generation:

   33.4701
   64.0974
   55.1011

Total generation cost =    1599.98 $/h
```

## 7.7 DERIVATION OF LOSS FORMULA

One of the major steps in the optimal dispatch of generation is to express the system losses in terms of the generator's real power outputs. There are several methods of obtaining the loss formula. One method developed by Kron and adopted by Kirchmayer is the *loss coefficient* or *B-coefficient* method.

The total injected complex power at bus $i$, denoted by $S_i$, is given by

$$S_i = P_i + jQ_i = V_i I_i^* \tag{7.72}$$

The summation of powers over all buses gives the total system losses

$$P_L + jQ_L = \sum_{i=1}^{n} V_i I_i^* = V_{bus}^T I_{bus}^* \tag{7.73}$$

where $P_L$ and $Q_L$ are the real and reactive power loss of the system. $V_{bus}$ is the column vector of the nodal bus voltages and $I_{bus}$ is the column vector of the injected bus currents. The expression for the bus currents in terms of bus voltage was derived in Chapter 6 and is given by (6.2) as

$$I_{bus} = Y_{bus} V_{bus} \tag{7.74}$$

where $Y_{bus}$ is the bus admittance matrix with ground as reference. Solving for $V_{bus}$, we have

$$\begin{aligned} V_{bus} &= Y_{bus}^{-1} I_{bus} \\ &= Z_{bus} I_{bus} \end{aligned} \tag{7.75}$$

The inverse of the bus admittance matrix is known as the *bus impedance matrix.* The bus admittance matrix is nonsingular if there are shunt elements (such as shunt capacitive susceptance) connected to the ground (bus number 0). As discussed in Chapter 6, the bus admittance matrix is sparse and its inverse can be expressed as a product of sparse matrix factors. Actually $Z_{bus}$, which is also required for short-circuit analysis, can be obtained directly by the method of *building algorithm* without the need for matrix inversion. This technique is discussed in Chapter 9.

Substituting for $V_{bus}$ from (7.75) into (7.73), results in

$$\begin{aligned} P_L + jQ_L &= [Z_{bus} I_{bus}]^T I_{bus}^* \\ &= I_{bus}^T Z_{bus}^T I_{bus}^* \end{aligned} \tag{7.76}$$

$Z_{bus}$ is a symmetrical matrix; therefore, $Z_{bus}^T = Z_{bus}$, and the total system loss becomes

$$P_L + jQ_L = I_{bus}^T Z_{bus} I_{bus}^* \tag{7.77}$$

The expression in (7.77) can also be expressed with the use of index notation as

$$P_L + jQ_L = \sum_{i=1}^{n} \sum_{j=1}^{n} I_i Z_{ij} I_j^* \tag{7.78}$$

Since the bus impedance matrix is symmetrical, i.e., $Z_{ij} = Z_{ji}$, the above equation may be rewritten as

$$P_L + jQ_L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} Z_{ij}(I_i I_j^* + I_j I_i^*) \tag{7.79}$$

The quantity inside the parentheses in (7.79) is real; thus the power loss can be broken into its real and imaginary components as

$$P_L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_{ij}(I_i I_j^* + I_j I_i^*) \tag{7.80}$$

$$Q_L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}(I_i I_j^* + I_j I_i^*) \tag{7.81}$$





where $R_{ij}$ and $X_{ij}$ are the real and imaginary elements of the bus impedance matrix, respectively. Again, since $R_{ij} = R_{ji}$, the real power loss equation can be converted back into

$$P_L = \sum_{i=1}^{n} \sum_{j=1}^{n} I_i R_{ij} I_j^* \tag{7.82}$$

Or in matrix form, the equation for the system real power loss becomes

$$P_L = I_{bus}^T R_{bus} I_{bus}^* \tag{7.83}$$

where $R_{bus}$ is the real part of the bus impedance matrix. In order to obtain the general formula for the system power loss in terms of generator powers, we define the total load current as the sum of all individual load currents, i.e.,

$$I_{L1} + I_{L2} + \cdots + I_{Ln_d} = I_D \tag{7.84}$$

where $n_d$ is the number of load buses and $I_D$ is the total load currents. Now the individual bus currents are assumed to vary as a constant complex fraction of the total load current, i.e.,

$$I_{Lk} = \ell_k I_D \qquad k = 1, 2, \ldots, n_d \tag{7.85}$$

or

$$\ell_k = \frac{I_{Lk}}{I_D} \tag{7.86}$$

Assuming bus 1 to be the reference bus (slack bus), expanding the first row in (7.75) results in

$$V_1 = Z_{11} I_1 + Z_{12} I_2 + \cdots + Z_{1n} In \tag{7.87}$$

If $n_g$ is the number of generator buses and $n_d$ is the number of load buses, the above equation can be written in terms of the load currents and generator currents as

$$V_1 = \sum_{i=1}^{n_g} Z_{1i} I_{gi} + \sum_{k=1}^{n_d} Z_{1k} I_{Lk} \tag{7.88}$$

Substituting for $I_{Lk}$ from (7.85) into (7.88), we have

$$\begin{aligned} V_1 &= \sum_{i=1}^{n_g} Z_{1i} I_{gi} + I_D \sum_{k=1}^{n_d} \ell_k Z_{1k} \\ &= \sum_{i=1}^{n_g} Z_{1i} I_{gi} + I_D T \end{aligned} \tag{7.89}$$

where

$$T = \sum_{k=1}^{n_d} \ell_k Z_{1k} \tag{7.90}$$

If $I_0$ is defined as the current flowing away from bus 1, with all other load currents set to zero, we have

$$V_1 = -Z_{11} I_0 \tag{7.91}$$

Substituting for $V_1$ in (7.89) and solving for $I_D$, we have

$$I_D = -\frac{1}{T}\sum_{i=1}^{n_g} Z_{1i} I_{gi} - \frac{1}{T} Z_{11} I_0 \tag{7.92}$$

Substituting for $I_D$ from (7.92) into (7.85), the load currents become

$$I_{Lk} = -\frac{\ell_k}{T}\sum_{i=1}^{n_g} Z_{1i} I_{gi} - \frac{\ell_k}{T} Z_{11} I_0 \tag{7.93}$$

Let

$$\rho = -\frac{\ell_k}{T} \tag{7.94}$$

Then

$$I_{Lk} = \rho_k \sum_{i=1}^{n_g} Z_{1i} I_{gi} + \rho_k Z_{11} I_0 \tag{7.95}$$

Augmenting the generator currents with the above relation in matrix form, we have

$$\begin{bmatrix} I_{g1} \\ I_{g2} \\ \vdots \\ I_{gn_g} \\ I_{L1} \\ I_{L2} \\ \vdots \\ I_{Ln_d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \rho_1 Z_{11} & \rho_1 Z_{12} & \cdots & \rho_1 Z_{1n_g} & \rho_1 Z_{11} \\ \rho_2 Z_{11} & \rho_2 Z_{12} & \cdots & \rho_2 Z_{1n_g} & \rho_2 Z_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_k Z_{11} & \rho_k Z_{12} & \cdots & \rho_k Z_{1n_g} & \rho_k Z_{11} \end{bmatrix} \begin{bmatrix} I_{g1} \\ I_{g2} \\ \vdots \\ I_{gn_g} \\ \\ \\ \\ I_0 \end{bmatrix} \tag{7.96}$$

Showing the above matrix by $C$, (7.96) becomes

$$I_{bus} = C I_{new} \tag{7.97}$$





Substituting for $I_{bus}$ in (7.83), we have

$$\begin{aligned} P_L &= [C I_{new}]^T R_{bus} C^* I_{new}^* \\ &= I_{new}^T C^T R_{bus} C^* I_{new}^* \end{aligned} \tag{7.98}$$

If $S_{gi}$ is the complex power at bus $i$, the generator current is

$$\begin{aligned} I_{gi} = \frac{S_{gi}^*}{V_i^*} &= \frac{P_{gi} - jQ_{gi}}{V_i^*} \\ &= \frac{1 - j\frac{Q_{gi}}{P_{gi}}}{V_i^*} P_{gi} \end{aligned} \tag{7.99}$$

or

$$I_{gi} = \psi_i P_{gi} \tag{7.100}$$

where

$$\psi_i = \frac{1 - j\frac{Q_{gi}}{P_{gi}}}{V_i^*} \tag{7.101}$$

Adding the current $I_0$ to the column vector current $I_{gi}$ in (7.100) results in

$$\begin{bmatrix} I_{g1} \\ I_{g2} \\ \vdots \\ I_{gn_g} \\ I_0 \end{bmatrix} = \begin{bmatrix} \psi_1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \psi_{n_g} & 0 \\ 0 & 0 & \cdots & 0 & I_0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \vdots \\ P_{gn_g} \\ 1 \end{bmatrix} \tag{7.102}$$

or in short form

$$I_{new} = \Psi P_{G1} \tag{7.103}$$

where

$$P_{G1} = \begin{bmatrix} P_{g1} \\ P_{g2} \\ \vdots \\ P_{gn_g} \\ 1 \end{bmatrix} \tag{7.104}$$

Substituting from (7.103) for $I_{new}$ in (7.98), the loss equation becomes

$$\begin{aligned} P_L &= [\Psi P_{G1}]^T C^T R_{bus} C^* \Psi^* P_{G1}^* \\ &= P_{G1}^T \Psi^T C^T R_{bus} C^* \Psi^* P_{G1}^* \end{aligned} \tag{7.105}$$

The resultant matrix in the above equation is complex and the real power loss is found from its real part, thus

$$P_L = P_{G1}^T \Re[H] P_{G1}^* \tag{7.106}$$

where

$$H = \Psi^T C^T R_{bus} C^* \Psi^* \tag{7.107}$$

Since elements of the matrix $H$ are complex, its real part must be used for computing the real power loss. It is found that $H$ is a *Hermitian matrix*. This means that $H$ is symmetrical and $H = H^*$. Thus, real part of $H$ is found from

$$\Re[H] = \frac{H + H^*}{2} \tag{7.108}$$

The above matrix is partitioned as follows

$$\Re[H] = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_g} & B_{01}/2 \\ B_{21} & B_{22} & \cdots & B_{2n_g} & B_{02}/2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n_g1} & B_{n_g2} & \cdots & B_{n_gn_g} & B_{0n_g}/2 \\ B_{01}/2 & B_{02}/2 & \cdots & B_{0n_g}/2 & B_{00} \end{bmatrix} \tag{7.109}$$

Substituting for $\Re[H]$ into (7.106), yields

$$P_L = [P_{g1}\, P_{g2} \cdots P_{gn_g}\; 1] \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_g} & B_{01}/2 \\ B_{21} & B_{22} & \cdots & B_{2n_g} & B_{02}/2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n_g1} & B_{n_g2} & \cdots & B_{n_gn_g} & B_{0n_g}/2 \\ B_{01}/2 & B_{02}/2 & \cdots & B_{0n_g}/2 & B_{00} \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \cdots \\ P_{gn_g} \\ 1 \end{bmatrix} \tag{7.110}$$

or

$$P_L = [\, P_{g1}\;\; P_{g2}\; \cdots\; P_{gn_g}\,] \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_g} \\ B_{21} & B_{22} & \cdots & B_{2n_g} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n_g1} & B_{n_g2} & \cdots & B_{n_gn_g} \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \cdots \\ P_{gn_g} \end{bmatrix} + [\, P_{g1}\;\; P_{g2}\; \cdots\; P_{gn_g}\,] \begin{bmatrix} B_{01}/2 \\ B_{02}/2 \\ \cdots \\ B_{0n_g}/2 \end{bmatrix} + B_{00} \tag{7.111}$$





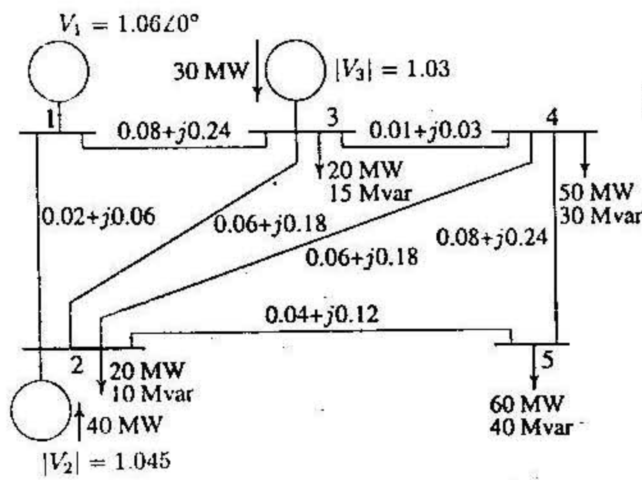
To find the loss coefficients, first a power flow solution is obtained for the initial operating state. This provides the voltage magnitude and phase angles at all buses. From these results, load currents $I_{Lk}$, the total load current $I_D$, and $\ell_k$ are obtained. Next the bus matrix $Z_{bus}$ is found. This can be obtained by converting the bus admittance matrix found from lfybus or directly from the *building algorithm* described in Chapter 9. Next the transformation matrices $C$ and $\Psi$ and $H$ are obtained. Finally the $B$-coefficients are evaluated from (7.109). It should be noted that the $B$-coefficients are functions of the system operating state. If a new scheduling of generation is not drastically different from the initial operating condition, the loss coefficients may be assumed constant. A program named **bloss** is developed for the computation of the $B$-coefficients. This program requires the power flow solution and can be used following any of the power flow programs such as **lfgauss**, **lfnewton**, or **decouple**. The $B$-coefficients obtained are based on the generation in per unit. When generation are expressed in MW, the loss coefficients are

$$B_{ij} = B_{ij\,pu}/S_B \quad B_{0i} = B_{0i\,pu} \;\text{ and }\; B_{00} = B_{00\,pu} \times S_B$$

where $S_B$ is the base MVA.

### Example 7.9

Figure 7.7 shows the one-line diagram of a simple 5-bus power system with generator at buses 1, 2, and 3. Bus 1, with its voltage set at $1.06\angle 0°$ pu, is taken as the slack bus. Voltage magnitude and real power generation at buses 2 and 3 are 1.045 pu, 40 MW, and 1.030 pu, 30 MW, respectively.




Capacitive susceptance

| Line | $\frac{1}{2}B$ |
|---|---|
| 1 – 2 | 0.030 |
| 1 – 3 | 0.025 |
| 2 – 3 | 0.020 |
| 2 – 4 | 0.020 |
| 2 – 5 | 0.015 |
| 3 – 4 | 0.010 |
| 4 – 5 | 0.025 |

**FIGURE 7.7**
One-line diagram of Example 7.9 (impedances in pu on 100-MVA base).

The load MW and Mvar values are shown on the diagram. Line impedances and one-half of the line capacitive susceptance are given in per unit on a 100-MVA base. Obtain the power flow solution and use the **bloss** program to obtain the loss coefficients in per unit.

We use the following commands

```
clear
basemva = 100;  accuracy = 0.0001;  maxiter = 10;

%       Bus Bus Voltage Angle -Load- ----Generator----Injected
%       No code Mag. Degree  MW   Mvar   MW Mvar   Qmin Qmax Mvar
busdata=[1   1  1.06   0.0    0     0     0    0    10   50   0
         2   2  1.045  0.0   20    10    40   30    10   50   0
         3   2  1.03   0.0   20    15    30   10    10   40   0
         4   0  1.00   0.0   50    30     0    0     0    0   0
         5   0  1.00   0.0   60    40     0    0     0    0  0];
%         Bus bus    R       X       1/2 B   1 for lines code or
%         nl  nr    pu      pu       pu    tap setting value
linedata=[1   2    0.02    0.06    0.030     1
          1   3    0.08    0.24    0.025     1
          2   3    0.06    0.18    0.020     1
          2   4    0.06    0.18    0.020     1
          2   5    0.04    0.12    0.015     1
          3   4    0.01    0.03    0.010     1
          4   5    0.08    0.24    0.025     1];
lfybus                          % form the bus admittance matrix
lfnewton              % Power flow solution by Newton-Raphson method
busout                % Prints the power flow solution on the screen
bloss                        % Obtains the loss formula coefficients
```

The result is

```
            Power Flow Solution by Newton-Raphson Method
               Maximum Power mismatch = 1.43025e-05
                      No. of iterations = 3

Bus Voltage  Angle    -----Load-----     --Generation-- Injected
No. Mag.     Degree     MW      Mvar      MW      Mvar     Mvar
1   1.060    0.000    0.000     0.00    83.051    7.271    0.00
2   1.045   -1.782   20.000    10.00    40.000   41.811    0.00
3   1.030   -2.664   20.000    15.00    30.000   24.148    0.00
4   1.019   -3.243   50.000    30.00     0.000    0.000    0.00
5   0.990   -4.405   60.000    40.00     0.000    0.000    0.00
Total               150.000    95.000  153.051   73.230    0.00
```





```
B =
      0.0218     0.0093     0.0028
      0.0093     0.0228     0.0017
      0.0028     0.0017     0.0179

B0 =
      0.0003     0.0031     0.0015
B00 =
      3.0523e-04
Total system loss = 3.05248 MW
```

As we have seen, any of the power flow programs, together with the **bloss** and **dispatch** programs can be used to obtain the optimal dispatch of generation. The **dispatch** program produces a variable named **dpslack**. This is the difference (absolute value) between the scheduled slack generation determined from the coordination equation, and the slack generation, obtained from the power flow solution. A power flow solution obtained with the new scheduling of generation results in a new loss coefficients, which can be used to solve the coordination equation again. This process can be continued until **dpslack** is within a specified tolerance. This procedure is demonstrated in the following example.

#### Example 7.10

The generation cost and the real power limits of the generators of the power system in Example 7.9 is given in Example 7.4 and Example 7.6. Obtain the optimal dispatch of generation. Continue the optimization process until the difference (absolute value) between the scheduled slack generation, determined from the coordination equation, and the slack generation, obtained from the power flow solution, is within 0.001 MW.

We use the following commands

```
clear
basemva = 100;  accuracy = 0.0001;  maxiter = 10;

%       Bus Bus  Voltage Angle --Load-- --Generator-- Injected
%       No  code Mag.  Degree MW  Mvar MW  Mvar Qmin Qmax Mvar

busdata=[1   1   1.06   0.0    0    0    0    0   10   50    0
         2   2   1.045  0.0   20   10   40   30   10   50    0
         3   2   1.03   0.0   20   15   30   10   10   40    0
         4   0   1.00   0.0   50   30    0    0    0    0    0
         5   0   1.00   0.0   60   40    0    0    0    0   0];
```

```
%         Bus bus    R      X       1/2 B  1 for lines code or
%         nl  nr     pu     pu       pu    tap setting value

linedata=[1   2     0.02    0.06    0.030     1
          1   3     0.08    0.24    0.025     1
          2   3     0.06    0.18    0.020     1
          2   4     0.06    0.18    0.020     1
          2   5     0.04    0.12    0.015     1
          3   4     0.01    0.03    0.010     1
          4   5     0.08    0.24    0.025     1];

     cost = [200  7.0     0.008
             180  6.3     0.009
             140  6.8     0.007];

     mwlimits =[10  85
                10  80
                10  70];

     lfybus                     % forms the bus admittance matrix
     lfnewton    % Power flow solution by Newton-Raphson method
     busout      % Prints the power flow solution on the screen
     bloss              % Obtains the loss formula coefficients
     gencost           % Computes the total generation cost $/h
     dispatch          % Obtains optimum dispatch of generation
          % dpslack is the difference (absolute value) between
          % the scheduled slack generation determined from the
             % coordination equation, and the slack generation
                      % obtained from the power flow solution.

     while dpslack > 0.001                % Test for convergence
     lfnewton                            % New power flow solution
     bloss                         % Loss coefficients are updated
     dispatch %Optimum dispatch of gen.with new B-coefficients
     end
     busout               % Prints the final power flow solution
     gencost % Generation cost with optimum scheduling of gen.
```

The result is

```
          Power Flow Solution by Newton-Raphson Method
             Maximum Power mismatch = 1.43025e-05
                    No. of iterations = 3
```





```
Bus Voltage  Angle     -----Load-----     --Generation-- Injected
No. Mag.     Degree     MW       Mvar      MW       Mvar     Mvar

1   1.060    0.000     0.000     0.00    83.051    7.271     0.00
2   1.045   -1.782    20.000    10.00    40.000   41.811     0.00
3   1.030   -2.664    20.000    15.00    30.000   24.148     0.00
4   1.019   -3.243    50.000    30.00     0.000    0.000     0.00
5   0.990   -4.405    60.000    40.00     0.000    0.000     0.00
Total                150.000    95.000  153.051   73.230     0.00

     B =
          0.0218    0.0093    0.0028
          0.0093    0.0228    0.0017
          0.0028    0.0017    0.0179

     B0 =
          0.0003    0.0031    0.0015
     B00 =
          3.0523e-04
     Total system loss = 3.05248 MW

     Total generation cost =     1633.24 $/h
     Incremental cost of delivered power (system lambda) =
     7.767608 $/MWh
     Optimal Dispatch of Generation:

        33.4558
        64.1101
        55.1005

     Absolute value of the slack bus real power mismatch,
     dpslack = 0.4960 pu
```

In this example the final optimal dispatch of generation was obtained in six iterations. The results for final loss coefficients and final optimal dispatch of generation is presented below

```
     B =
          0.0472    0.0130    0.0036
          0.0130    0.0130    0.0010
          0.0036    0.0010    0.0115

     B0 =
          0.0047    0.0012    0.0004
```

```
B00 =
     3.0516e-04

Total system loss = 2.15691 MW
Incremental cost of delivered power (system lambda) =
7.759051 $/MWh
Optimal Dispatch of Generation:

   23.5581
   69.5593
   59.0368

Absolute value of the slack bus real power mismatch,
dpslack = 0.0009 pu

      Power Flow Solution by Newton-Raphson Method
         Maximum Power mismatch = 1.90285e-08
                No. of iterations = 4
```

| Bus No. | Voltage Mag. | Angle Degree | -----Load----- MW | Mvar | --Generation-- MW | Mvar | Injected Mvar |
|---|---|---|---|---|---|---|---|
| 1 | 1.060 | 0.000 | 0.000 | 0.000 | 23.649 | 25.727 | 0.00 |
| 2 | 1.045 | -0.282 | 20.000 | 10.000 | 69.518 | 30.767 | 0.00 |
| 3 | 1.030 | -0.495 | 20.000 | 15.000 | 58.990 | 14.052 | 0.00 |
| 4 | 1.019 | -1.208 | 50.000 | 30.000 | 0.000 | 0.000 | 0.00 |
| 5 | 0.990 | -2.729 | 60.000 | 40.000 | 0.000 | 0.000 | 0.00 |
| Total | | | 150.000 | 95.000 | 152.154 | 70.545 | 0.00 |

```
Total generation cost =     1596.96 $/h
```

The total generation cost for the initial operating condition is 1,633.24 \$/h and the total generation cost with optimal dispatch of generation is 1,596.96 \$/h. This results in a savings of 36.27 \$/h.

### Example 7.11

Figure 7.8 is the 26-bus power system network of Problem 6.14. Bus 1 is taken as the slack bus with its voltage adjusted to $1.025\angle 0°$ pu. The data for the voltage-controlled buses is





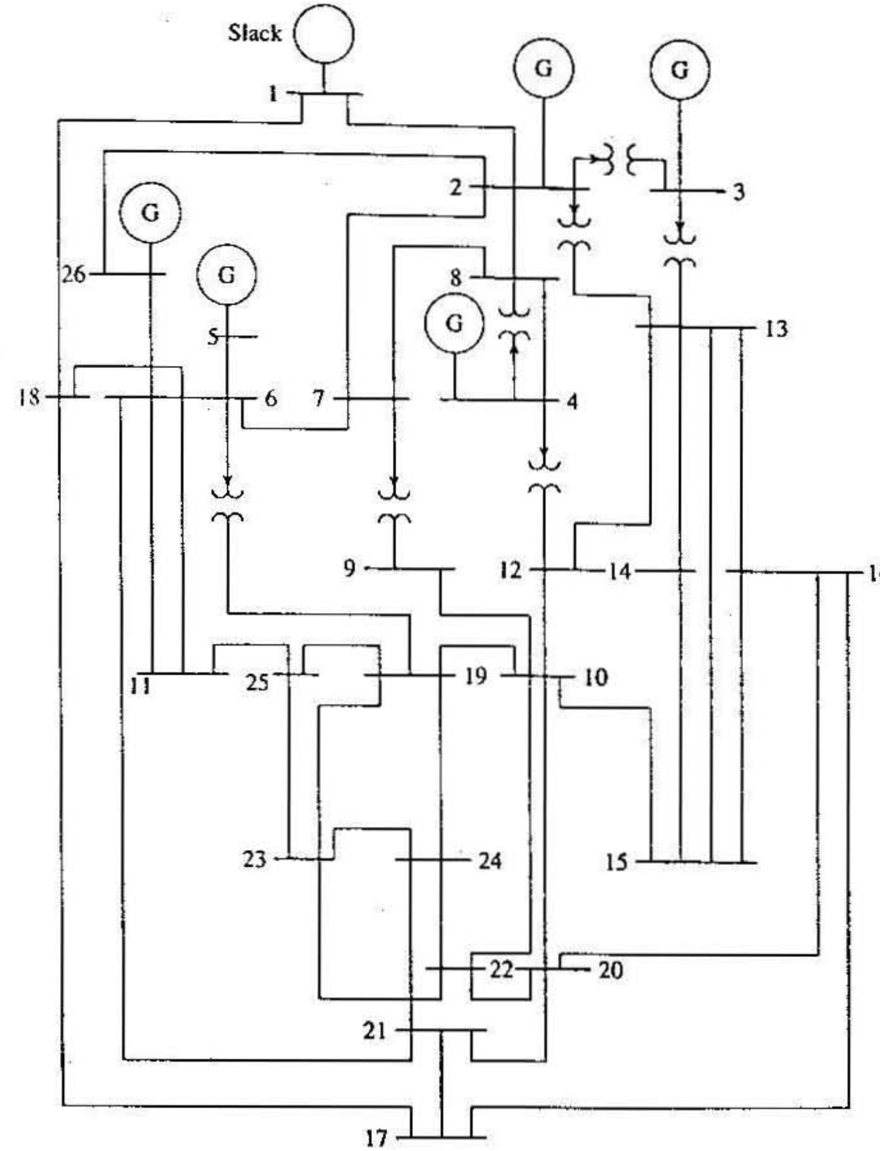



**FIGURE 7.8**
One-line diagram of Example 7.11.

| REGULATED BUS DATA | | | |
|---|---|---|---|
| Bus No. | Voltage Magnitude | Min. Mvar Capacity | Max. Mvar Capacity |
| 2 | 1.020 | 40 | 250 |
| 3 | 1.025 | 40 | 150 |
| 4 | 1.050 | 40 | 80 |
| 5 | 1.045 | 40 | 160 |
| 26 | 1.015 | 15 | 50 |

Transformer tap settings are given in the table below. The left bus number is assumed to be the tap side of the transformer.

| TRANSFORMER DATA | |
|---|---|
| Transformer Designation | Tap Setting Per Unit |
| 2 – 3 | 0.960 |
| 2 – 13 | 0.960 |
| 3 – 13 | 1.017 |
| 4 – 8 | 1.050 |
| 4 – 12 | 1.050 |
| 6 – 19 | 0.950 |
| 7 – 9 | 0.950 |

The shunt capacitive data is

| SHUNT CAPACITOR DATA | |
|---|---|
| Bus No. | Mvar |
| 1 | 4.0 |
| 4 | 2.0 |
| 5 | 5.0 |
| 6 | 2.0 |
| 9 | 3.0 |
| 11 | 1.5 |
| 12 | 2.0 |
| 15 | 0.5 |
| 19 | 5.0 |

Generation and loads are as given in the data prepared for use in the *MATLAB* environment in the matrix defined as **busdata**. Code 0, code 1, and code 2 are used for the load buses, the slack bus, and the voltage-controlled buses, respectively. Values for **basemva**, **accuracy**, and **maxiter** must be specified. Line data are as given in the matrix called **linedata**. The last column of this data must contain 1 for lines, or the tap setting values for transformers with off-nominal turn ratio. The





generator's operating costs in \$/h, with $P_i$ in MW are as follow:

$$C_1 = 240 + 7.0P_1 + 0.0070P_1^2$$
$$C_2 = 200 + 10.0P_2 + 0.0095P_2^2$$
$$C_3 = 220 + 8.5P_3 + 0.0090P_3^2$$
$$C_4 = 200 + 11.0P_4 + 0.0090P_4^2$$
$$C_5 = 220 + 10.5P_5 + 0.0080P_5^2$$
$$C_{26} = 190 + 12.0P_{26} + 0.0075P_{26}^2$$

The generator's real power limits are

| GENERATOR REAL POWER LIMITS | | |
|---|---|---|
| Gen. | Min. MW | Max. MW |
| 1 | 100 | 500 |
| 2 | 50 | 200 |
| 3 | 80 | 300 |
| 4 | 50 | 150 |
| 5 | 50 | 200 |
| 5 | 50 | 120 |

Write the necessary commands to obtain the optimal dispatch of generation using **dispatch**. Continue the optimization process until the difference (absolute value) between the scheduled slack generation, determined from the coordination equation, and the slack generation, obtained from the power flow solution, is within 0.001 MW.

We use the following commands:

```
clear
basemva = 100;  accuracy = 0.0001;  maxiter = 10;

%        Bus Bus  Voltage Angle  --Load-- --Generator---Injected
%        No  code Mag.    Degree MW  Mvar  MW   Mvar Qmin Qmax Mvar
busdata=[1    1   1.025    0.0   51   41    0    0     0    0    4
         2    2   1.020    0.0   22   15   79    0    40  250    0
         3    2   1.025    0.0   64   50   20    0    40  150    0
         4    2   1.050    0.0   25   10  100    0    25   80    2
         5    2   1.045    0.0   50   30  300    0    40  160    5
         6    0   1.00     0.0   76   29    0    0     0    0    2
         7    0   1.00     0.0    0    0    0    0     0    0    0
         8    0   1.00     0.0    0    0    0    0     0    0    0
         9    0   1.00     0.0   89   50    0    0     0    0    3
        10    0   1.00     0.0    0    0    0    0     0    0    0
        11    0   1.00     0.0   25   15    0    0     0    0  1.5
```

```
      12   0   1.00    0.0   89  48   0   0    0   0   2
      13   0   1.00    0.0   31  15   0   0    0   0   0
      14   0   1.00    0.0   24  12   0   0    0   0   0
      15   0   1.00    0.0   70  31   0   0    0   0 0.5
      16   0   1.00    0.0   55  27   0   0    0   0   0
      17   0   1.00    0.0   78  38   0   0    0   0   0
      18   0   1.00    0.0  153  67   0   0    0   0   0
      19   0   1.00    0.0   75  15   0   0    0   0   5
      20   0   1.00    0.0   48  27   0   0    0   0   0
      21   0   1.00    0.0   46  23   0   0    0   0   0
      22   0   1.00    0.0   45  22   0   0    0   0   0
      23   0   1.00    0.0   25  12   0   0    0   0   0
      24   0   1.00    0.0   54  27   0   0    0   0   0
      25   0   1.00    0.0   28  13   0   0    0   0   0
      26   2   1.015   0.0   40  20  60   0   15  50   0];

%         Bus bus    R       X        1/2 B  1 for lines code or
%         nl  nr     pu      pu        pu    tap setting value
linedata=[1    2   0.00055  0.00480  0.03000   1
          1   18   0.00130  0.01150  0.06000   1
          2    3   0.00146  0.05130  0.05000   0.96
          2    7   0.01030  0.05860  0.01800   1
          2    8   0.00740  0.03210  0.03900   1
          2   13   0.00357  0.09670  0.02500   0.96
          2   26   0.03230  0.19670  0.00000   1
          3   13   0.00070  0.00548  0.00050   1.017
          4    8   0.00080  0.02400  0.00010   1.050
          4   12   0.00160  0.02070  0.01500   1.050
          5    6   0.00690  0.03000  0.09900   1
          6    7   0.00535  0.03060  0.00105   1
          6   11   0.00970  0.05700  0.00010   1
          6   18   0.00374  0.02220  0.00120   1
          6   19   0.00350  0.06600  0.04500   0.95
          6   21   0.00500  0.09000  0.02260   1
          7    8   0.00120  0.00693  0.00010   1
          7    9   0.00095  0.04290  0.02500   0.95
          8   12   0.00200  0.01800  0.02000   1
          9   10   0.00104  0.04930  0.00100   1
         10   12   0.00247  0.01320  0.01000   1
         10   19   0.05470  0.23600  0.00000   1
         10   20   0.00660  0.01600  0.00100   1
         10   22   0.00690  0.02980  0.00500   1
         11   25   0.09600  0.27000  0.01000   1
         11   26   0.01650  0.09700  0.00400   1
         12   14   0.03270  0.08020  0.00000   1
```





```
         12   15   0.01800  0.05980  0.00000   1
         13   14   0.00460  0.02710  0.00100   1
         13   15   0.01160  0.06100  0.00000   1
         13   16   0.01793  0.08880  0.00100   1
         14   15   0.00690  0.03820  0.00000   1
         15   16   0.02090  0.05120  0.00000   1
         16   17   0.09900  0.06000  0.00000   1
         16   20   0.02390  0.05850  0.00000   1
         17   18   0.00320  0.06000  0.03800   1
         17   21   0.22900  0.44500  0.00000   1
         19   23   0.03000  0.13100  0.00000   1
         19   24   0.03000  0.12500  0.00200   1
         19   25   0.11900  0.22490  0.00400   1
         20   21   0.06570  0.15700  0.00000   1
         20   22   0.01500  0.03660  0.00000   1
         21   24   0.04760  0.15100  0.00000   1
         22   23   0.02900  0.09900  0.00000   1
         22   24   0.03100  0.08800  0.00000   1
         23   25   0.09870  0.11680  0.00000   1];

cost = [240   7.0   0.0070
        200  10.0   0.0095
        220   8.5   0.0090
        200  11.0   0.0090
        220  10.5   0.0080
        190  12.0   0.0075];

mwlimits =[100  500
            50  200
            80  300
            50  150
            50  200
            50  120];

lfybus                  % Forms the bus admittance matrix
lfnewton   % Power flow solution by Newton-Raphson method
busout     % Prints the power flow solution on the screen
bloss             % Obtains the loss formula coefficients
gencost          % Computes the total generation cost $/h
dispatch         % Obtains optimum dispatch of generation
      % dpslack is the difference (absolute value) between
      % the scheduled slack generation determined from the
        % coordination equation, and the slack generation
               % obtained from the power flow solution.
```

```
while dpslack>.001%Repeat till dpslack is within tolerance
lfnewton                          % New power flow solution
bloss                       % Loss coefficients are updated
dispatch %Optimum dispatch of gen. with new B-coefficients
end
busout                  % Prints the final power flow solution
gencost  % Generation cost with optimum scheduling of gen.
```

The result is

```
Power Flow Solution by Newton-Raphson Method
   Maximum Power mismatch = 3.18289e-10
          No. of iterations = 6
```

| Bus No. | Voltage Mag. | Angle Degree | Load MW | Load Mvar | Generation MW | Generation Mvar | Injected Mvar |
|---|---|---|---|---|---|---|---|
| 1 | 1.025 | 0.000 | 51.000 | 41.000 | 719.534 | 224.011 | 4.00 |
| 2 | 1.020 | -0.931 | 22.000 | 15.000 | 79.000 | 125.354 | 0.00 |
| 3 | 1.035 | -4.213 | 64.000 | 50.000 | 20.000 | 63.030 | 0.00 |
| 4 | 1.050 | -3.582 | 25.000 | 10.000 | 100.000 | 49.223 | 2.00 |
| 5 | 1.045 | 1.129 | 50.000 | 30.000 | 300.000 | 124.466 | 5.00 |
| 6 | 0.999 | -2.573 | 76.000 | 29.000 | 0.000 | 0.000 | 2.00 |
| 7 | 0.994 | -3.204 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 8 | 0.997 | -3.299 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 9 | 1.009 | -5.393 | 89.000 | 50.000 | 0.000 | 0.000 | 3.00 |
| 10 | 0.989 | -5.561 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 11 | 0.997 | -3.218 | 25.000 | 15.000 | 0.000 | 0.000 | 1.50 |
| 12 | 0.993 | -4.692 | 89.000 | 48.000 | 0.000 | 0.000 | 2.00 |
| 13 | 1.014 | -4.430 | 31.000 | 15.000 | 0.000 | 0.000 | 0.00 |
| 14 | 1.000 | -5.040 | 24.000 | 12.000 | 0.000 | 0.000 | 0.00 |
| 15 | 0.991 | -5.538 | 70.000 | 31.000 | 0.000 | 0.000 | 0.50 |
| 16 | 0.983 | -5.882 | 55.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 17 | 0.987 | -4.985 | 78.000 | 38.000 | 0.000 | 0.000 | 0.00 |
| 18 | 1.007 | -1.866 | 153.000 | 67.000 | 0.000 | 0.000 | 0.00 |
| 19 | 1.004 | -6.397 | 75.000 | 15.000 | 0.000 | 0.000 | 5.00 |
| 20 | 0.980 | -6.025 | 48.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 21 | 0.977 | -5.778 | 46.000 | 23.000 | 0.000 | 0.000 | 0.00 |
| 22 | 0.978 | -6.437 | 45.000 | 22.000 | 0.000 | 0.000 | 0.00 |
| 23 | 0.976 | -7.087 | 25.000 | 12.000 | 0.000 | 0.000 | 0.00 |
| 24 | 0.968 | -7.347 | 54.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 25 | 0.974 | -6.775 | 28.000 | 13.000 | 0.000 | 0.000 | 0.00 |
| 26 | 1.015 | -1.803 | 40.000 | 20.000 | 60.000 | 32.706 | 0.00 |
| Total | | | 1263.000 | 637.000 | 1278.534 | 618.791 | 25.00 |





B =

| | | | | | |
|---|---|---|---|---|---|
| 0.0014 | 0.0015 | 0.0009 | -0.0001 | -0.0004 | -0.0002 |
| 0.0015 | 0.0043 | 0.0050 | 0.0001 | -0.0008 | -0.0003 |
| 0.0009 | 0.0050 | 0.0315 | -0.0000 | -0.0020 | -0.0016 |
| -0.0001 | 0.0001 | -0.0000 | 0.0029 | -0.0006 | -0.0009 |
| -0.0004 | -0.0008 | -0.0020 | -0.0006 | 0.0085 | -0.0001 |
| -0.0002 | -0.0003 | -0.0016 | -0.0009 | -0.0001 | 0.0176 |

B0 =

| | | | | | |
|---|---|---|---|---|---|
| -0.0002 | -0.0008 | 0.0067 | 0.0001 | 0.0000 | -0.0012 |

B00 =

0.0056

```
Total system loss = 15.53 MW

Total generation cost =    16760.73 $/h
Incremental cost of delivered power (system lambda) =
13.911780 $/MWh
Optimal Dispatch of Generation:

  474.1196
  173.7886
  190.9515
  150.0000
  196.7196
  103.5772

Absolute value of the slack bus real power mismatch,
dpslack = 2.4541 pu
```

In this example the final optimal dispatch of generation was obtained in three iterations. The results for final loss coefficients and final optimal dispatch of generation is presented below

B =

| | | | | | |
|---|---|---|---|---|---|
| 0.0017 | 0.0012 | 0.0007 | -0.0001 | -0.0005 | -0.0002 |
| 0.0012 | 0.0014 | 0.0009 | 0.0001 | -0.0006 | -0.0001 |
| 0.0007 | 0.0009 | 0.0031 | 0.0000 | -0.0010 | -0.0006 |
| -0.0001 | 0.0001 | 0.0000 | 0.0024 | -0.0006 | -0.0008 |
| -0.0005 | -0.0006 | -0.0010 | -0.0006 | 0.0129 | -0.0002 |
| -0.0002 | -0.0001 | -0.0006 | -0.0008 | -0.0002 | 0.0150 |

```
B0 =
    1.0e-03 *
   -0.3908   -0.1297    0.7047    0.0591    0.2161   -0.6635

B00 =
    0.0056

Total system loss = 12.807 MW
Incremental cost of delivered power (system lambda) =
13.538113 $/MWh
Optimal Dispatch of Generation:
  447.6919
  173.1938
  263.4859
  138.8142
  165.5884
   87.0260

Absolute value of the slack bus real power mismatch,
dpslack =  0.0008 pu

        Power Flow Solution by Newton-Raphson Method
           Maximum Power mismatch = 2.33783e-05
                  No. of iterations = 3
```

| Bus No. | Voltage Mag. | Angle Degree | -----Load----- MW | Mvar | --Generation-- MW | Mvar | Injected Mvar |
|---|---|---|---|---|---|---|---|
| 1 | 1.025 | 0.000 | 51.000 | 41.000 | 447.611 | 250.582 | 4.00 |
| 2 | 1.020 | -0.200 | 22.000 | 15.000 | 173.087 | 57.303 | 0.00 |
| 3 | 1.045 | -0.639 | 64.000 | 50.000 | 263.363 | 78.280 | 0.00 |
| 4 | 1.050 | -2.101 | 25.000 | 10.000 | 138.716 | 33.449 | 2.00 |
| 5 | 1.045 | -1.453 | 50.000 | 30.000 | 166.099 | 142.890 | 5.00 |
| 6 | 1.001 | -2.874 | 76.000 | 29.000 | 0.000 | 0.000 | 2.00 |
| 7 | 0.995 | -2.406 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 8 | 0.998 | -2.278 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 9 | 1.010 | -4.387 | 89.000 | 50.000 | 0.000 | 0.000 | 3.00 |
| 10 | 0.991 | -4.311 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 |
| 11 | 0.998 | -2.824 | 25.000 | 15.000 | 0.000 | 0.000 | 1.50 |
| 12 | 0.994 | -3.282 | 89.000 | 48.000 | 0.000 | 0.000 | 2.00 |
| 13 | 1.022 | -1.261 | 31.000 | 15.000 | 0.000 | 0.000 | 0.00 |
| 14 | 1.008 | -2.445 | 24.000 | 12.000 | 0.000 | 0.000 | 0.00 |
| 15 | 0.999 | -3.229 | 70.000 | 31.000 | 0.000 | 0.000 | 0.50 |
| 16 | 0.990 | -3.990 | 55.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 17 | 0.983 | -4.366 | 78.000 | 38.000 | 0.000 | 0.000 | 0.00 |
| 18 | 1.007 | -1.884 | 153.000 | 67.000 | 0.000 | 0.000 | 0.00 |
| 19 | 1.005 | -6.074 | 75.000 | 15.000 | 0.000 | 0.000 | 5.00 |
| 20 | 0.983 | -4.759 | 48.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 21 | 0.977 | -5.411 | 46.000 | 23.000 | 0.000 | 0.000 | 0.00 |
| 22 | 0.980 | -5.325 | 45.000 | 22.000 | 0.000 | 0.000 | 0.00 |
| 23 | 0.978 | -6.388 | 25.000 | 12.000 | 0.000 | 0.000 | 0.00 |
| 24 | 0.969 | -6.672 | 54.000 | 27.000 | 0.000 | 0.000 | 0.00 |
| 25 | 0.975 | -6.256 | 28.000 | 13.000 | 0.000 | 0.000 | 0.00 |
| 26 | 1.015 | -0.284 | 40.000 | 20.000 | 86.939 | 27.892 | 0.00 |
| Total | | | 1263.000 | 637.000 | 1275.800 | 590.396 | 25.00 |





```
Total generation cost =    15447.72 $/h
```

The total generation cost for the initial operating condition is 16,760.73 \$/h and the total generation cost with optimal dispatch of generation is 15,447.72 \$/h. This results in a savings of 1,313.01 \$/h. That is, with this loading, the total annual savings is over \$11 million.

## PROBLEMS

**7.1.** Find a rectangle of maximum perimeter that can be inscribed in a circle of unit radius given by

$$g(x,y) = x^2 + y^2 - 1 = 0$$

Check the eigenvalues for sufficient conditions.

**7.2.** Find the minimum of the function

$$f(x,y) = x^2 + 2y^2$$

subject to the equality constraint

$$g(x,y) = x + 2y + 4 = 0$$

Check for the sufficient conditions.

**7.3.** Use the Lagrangian multiplier method for solving constrained parameter optimization problems to determine an isosceles triangle of maximum area that may be inscribed in a circle of radius 1.

**7.4.** For a second-order bandpass filter with transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

determine the values of the damping ratio and natural frequency, $\zeta$ and $\omega_n$, corresponding to a Bode plot whose peak occurs at 7071.07 radians/sec and whose half-power bandwidth is 12,720.2 radians/sec.

**7.5.** Find the minimum value of the function

$$f(x, y) = x^2 + y^2$$

subject to the equality constraint

$$g(x, y) = x^2 - 6x - y^2 + 17 = 0$$

**7.6.** Find the minimum value of the function

$$f(x, y) = x^2 + y^2$$

subject to one equality constraint

$$g(x, y) = x^2 - 5x - y^2 + 20 = 0$$

and one inequality constraint

$$u(x, y) = 2x + y \geq 6$$

**7.7.** The fuel-cost functions in \$/h for two 800 MW thermal plants are given by

$$C_1 = 400 + 6.0P_1 + 0.004P_1^2$$
$$C_2 = 500 + \beta P_2 + \gamma P_2^2$$

where $P_1$ and $P_2$ are in MW.

(a) The incremental cost of power $\lambda$ is \$8/MWh when the total power demand is 550 MW. Neglecting losses, determine the optimal generation of each plant.

(b) The incremental cost of power $\lambda$ is \$10/MWh when the total power demand is 1300 MW. Neglecting losses, determine the optimal generation of each plant.

(c) From the results of (a) and (b) find the fuel-cost coefficients $\beta$ and $\gamma$ of the second plant.

**7.8.** The fuel-cost functions in \$/h for three thermal plants are given by

$$C_1 = 350 + 7.20P_1 + 0.0040P_1^2$$
$$C_2 = 500 + 7.30P_2 + 0.0025P_2^2$$
$$C_3 = 600 + 6.74P_3 + 0.0030P_3^2$$





where $P_1$, $P_2$, and $P_3$ are in MW. The governors are set such that generators share the load equally. Neglecting line losses and generator limits, find the total cost in \$/h when the total load is

(i) $P_D =$ 450 MW
(ii) $P_D =$ 745 MW
(iii) $P_D =$ 1335 MW

**7.9.** Neglecting line losses and generator limits, determine the optimal scheduling of generation for each loading condition in Problem 7.8
(a) by analytical technique, using (7.33) and (7.31).
(b) using Iterative method. Start with an initial estimate of $\lambda = 7.5$ \$/MWh.
(c) find the savings in \$/h for each case compared to the costs in Problem 7.8 when the generators shared load equally.
Use the **dispatch** program to check your results.

**7.10.** Repeat Problem 7.9 (a) and (b), but this time consider the following generator limits (in MW)

$$122 \leq P_1 \leq 400$$
$$260 \leq P_2 \leq 600$$
$$50 \leq P_3 \leq 445$$

Use the **dispatch** program to check your results.

**7.11.** The fuel-cost function in \$/h of two thermal plants are

$$C_1 = 320 + 6.2P_1 + 0.004P_1^2$$
$$C_2 = 200 + 6.0P_2 + 0.003P_2^2$$

where $P_1$ and $P_2$ are in MW. Plant outputs are subject to the following limits (in MW)

$$50 \leq P_1 \leq 250$$
$$50 \leq P_2 \leq 350$$

The per-unit system real power loss with generation expressed in per unit on a 100-MVA base is given by

$$P_{L(pu)} = 0.0125P_{1(pu)}^2 + 0.00625P_{2(pu)}^2$$

The total load is 412.35 MW. Determine the optimal dispatch of generation. Start with an initial estimate of $\lambda = 7$ \$/MWh. Use the **dispatch** program to check your results.

**7.12.** The 9-bus power system network of an Electric Utility Company is shown in Figure 7.9. The load data is tabulated below. Voltage magnitude, generation schedule and the reactive power limits for the regulated buses are also tabulated below. Bus 1, whose voltage is specified as $V_1 = 1.03\angle 0°$, is taken as the slack bus.

| LOAD DATA | | |
|---|---|---|
| Bus | Load | |
| No. | MW | Mvar |
| 1 | 0 | 0 |
| 2 | 20 | 10 |
| 3 | 25 | 15 |
| 4 | 10 | 5 |
| 5 | 40 | 20 |
| 6 | 60 | 40 |
| 7 | 10 | 5 |
| 8 | 80 | 60 |
| 9 | 100 | 80 |

| GENERATION DATA | | | | |
|---|---|---|---|---|
| Bus | Voltage | Generation | Mvar Limits | |
| No. | Mag. | MW | Min. | Max. |
| 1 | 1.03 | | | |
| 2 | 1.04 | 80 | 0 | 250 |
| 7 | 1.01 | 120 | 0 | 100 |

The Mvar of the shunt capacitors installed at substations are given below

| SHUNT CAPACITORS | |
|---|---|
| Bus No. | Mvar |
| 3 | 1.0 |
| 4 | 3.0 |

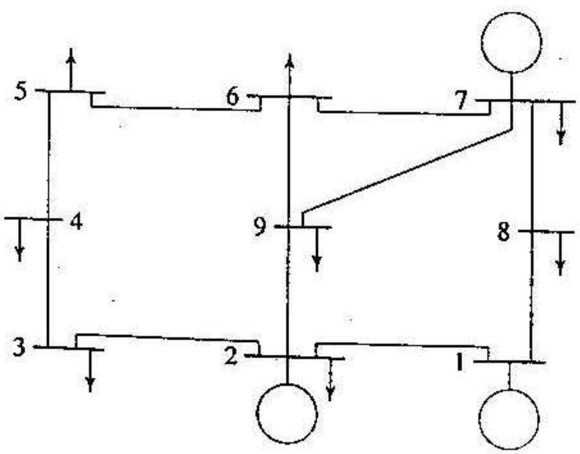



**FIGURE 7.9**
One-line diagram for Problem 7.12.





The line data containing the series resistance and reactance in per unit, and one-half of the total capacitance in per unit susceptance on a 100 MVA base is tabulated below.

| LINE DATA | | | | |
|---|---|---|---|---|
| Bus No. | Bus No. | R, PU | X, PU | $\frac{1}{2}$ B, PU |
| 1 | 2 | 0.018 | 0.054 | 0.0045 |
| 1 | 8 | 0.014 | 0.036 | 0.0030 |
| 2 | 9 | 0.006 | 0.030 | 0.0028 |
| 2 | 3 | 0.013 | 0.036 | 0.0030 |
| 3 | 4 | 0.010 | 0.050 | 0.0000 |
| 4 | 5 | 0.018 | 0.056 | 0.0000 |
| 5 | 6 | 0.020 | 0.060 | 0.0000 |
| 6 | 7 | 0.015 | 0.045 | 0.0038 |
| 6 | 9 | 0.002 | 0.066 | 0.0000 |
| 7 | 8 | 0.032 | 0.076 | 0.0000 |
| 7 | 9 | 0.022 | 0.065 | 0.0000 |

The generator's operating costs in $/h are as follows:

$$C_1 = 240 + 6.7P_1 + 0.009P_1^2$$
$$C_2 = 220 + 6.1P_2 + 0.005P_2^2$$
$$C_7 = 240 + 6.5P_7 + 0.008P_7^2$$

The generator's real power limits are

| GENERATOR REAL POWER LIMITS | | |
|---|---|---|
| Gen. | Min. MW | Max. MW |
| 1 | 50 | 200 |
| 2 | 50 | 200 |
| 7 | 50 | 100 |

Write the necessary commands to obtain the optimal dispatch of generation using **dispatch**. Continue the optimization process until the difference (absolute value) between the scheduled slack generation, determined from the coordination equation, and the slack generation, obtained from the power flow solution, is within 0.001 MW.