Chap 1 Problem Solutions

1.1 If $v = 141.4\sin(\omega t + 30°)$ V and $i = 11.31\cos(\omega t - 30°)$ A. find for each (a) the maximum value, (b) the rms value and (c) the phasor expression in polar and rectangular form if voltage is the "reference." Is the circuit inductive or capacitive?
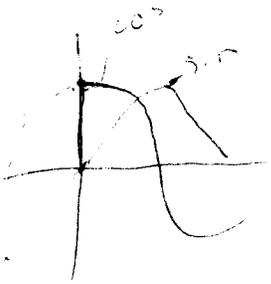
Solution:

$$v = 141.4\sin(\omega t + 30°)$$
$$= 141.4\cos(\omega t + 30° - 90°)$$

$8\underline{/-30°}$ A

$100\underline{/-60°}$

(a) Maximum values:
$$V_{max} = 141.4 \text{ V} \qquad I_{max} = 11.31 \text{ A}$$

(b) rms values:
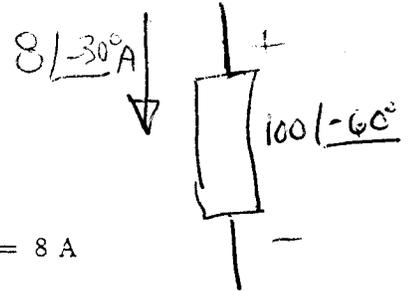$$|V| = \frac{141.4}{\sqrt{2}} = 100 \text{ V} \qquad |I| = \frac{11.31}{\sqrt{2}} = 8 \text{ A}$$

(c) Phasor expressions in polar and rectangular form:
$$V = 100\underline{/0°} = 100 + j0 \text{ V}$$
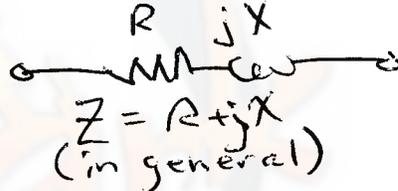$$I = 8\underline{/-60°} = 4 - j6.93 \text{ A}$$

$+30°$

Capacitive

The circuit is inductive as I *leads* V.

1.2 If the circuit of Prob. 1.1 consists of a purely resistive and a purely reactive element, find R and X, (a) if the elements are in series and (b) if the elements are in parallel.
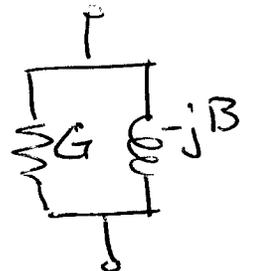
$R \quad jX$

$Z = R + jX$ (in general)

Solution:

(a) Elements in series:
$$Z = \frac{100\underline{/0°}}{8\underline{/-60°}} = 12.5\underline{/60°} \text{ }\Omega = 6.25 + j10.83 \text{ }\Omega$$
$$R = 6.25 \text{ }\Omega \qquad X_C = 10.83 \text{ }\Omega$$

$10.8\Omega \qquad\qquad 6.25\Omega$

$G \quad -jB$

(b) Elements in parallel:

$Y = \frac{1}{Z} = \frac{1}{R+jX} = G + jB$

Note: $R \neq \frac{1}{G}$; $X \neq \frac{1}{B}$

$.0693 + j.04$

$$Y = \frac{1}{Z} = 0.08\underline{/-60°} = 0.04 - j0.0693$$
$$R = \frac{1}{0.04} = 25 \text{ }\Omega \qquad X_C = \frac{1}{0.0693} = 14.43 \text{ }\Omega$$

$14.43\Omega \qquad\qquad 25\Omega$

$Y = G + jB$ (in general)

1.3 In a single-phase circuit $V_a = 120\underline{/45°}$ V and $V_b = 100\underline{/-15°}$ V with respect to a reference node o. Find V_{ba} in polar form.

Solution:

$$V_{ba} = V_{bo} - V_{ao}$$
$$V_{ba} = 100\underline{/-15°} - 120\underline{/45°} = 96.59 - j25.88 - (84.85 + j84.85)$$
$$= 11.74 - j110.73 = 111.35\underline{/-83.95°} \text{ V}$$

1.4 A single-phase ac voltage of 240 V is applied to a series circuit whose impedance is $10\angle 60°$ Ω. Find R, X, P, Q and the power factor of the circuit.

Solution:

$$
\begin{aligned}
R &= 10\cos 60° = 5.0 \ \Omega \\
X &= 10\sin 60° = 8.66 \ \Omega \\
I &= \frac{240\angle 0°}{10\angle 60°} = 24\angle -60° \ \text{A} \\
P &= (24)^2 \times 5 = 2880 \ \text{W} \\
Q &= (24)^2 \times 8.66 = 4988 \ \text{var} \\
\text{p.f.} &= \cos\left(\tan^{-1}\frac{4988}{2880}\right) = 0.50 \\
\text{or} \quad & \cos\left(\tan^{-1}\frac{X}{R}\right) = 0.50
\end{aligned}
$$

1.5 If a capacitor is connected in parallel with the circuit of Prob. 1.4 and if this capacitor supplies 1250 var, find the P and Q supplied by the 240-V source, and find the resultant power factor.

Solution:

$$
\begin{aligned}
P &= 2880 \ \text{W} \\
Q &= 4988 - 1250 = 3738 \ \text{var} \\
\text{p.f.} &= \cos\left(\tan^{-1}\frac{3738}{2880}\right) = 0.61
\end{aligned}
$$

1.6 A single-phase inductive load draws 10 MW at 0.6 power factor lagging. Draw the power triangle and determine the reactive power of a capacitor to be connected in parallel with the load to raise the power factor to 0.85.

Solution:

$$
\begin{aligned}
\frac{10}{0.6}\sin\left(\cos^{-1}0.6\right) &= 13.33 \\
\cos^{-1}0.85 &= 31.79° \\
10\tan 31.79° &= 6.2 \ \text{var} \\
Q_c &= -(13.33 - 6.2) \\
&= -7.13 \ \text{Mvar}
\end{aligned}
$$
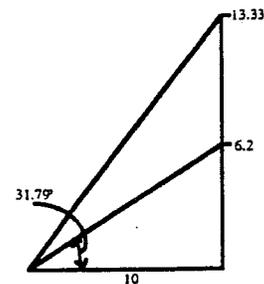
1.7 A single-phase induction motor is operating at a very light load during a large part of every day and draws 10 A from the supply. A device is proposed to "increase the efficiency" of the motor. During a demonstration the device is placed in parallel with the unloaded motor and the current drawn from the supply drops to 8 A. When two of the devices are placed in parallel the current drops to 6 A. What simple device will cause this drop in current? Discuss the advantages of the device. Is the efficiency of the motor increased by the device? (Recall that an induction motor draws lagging current).

Solution:

A capacitor will cause the drop in current in the line because the lagging component of current drawn by the motor will be partially offset by the leading current drawn by the capacitor. The current drawn by the motor, however, will be unchanged if the terminal voltage remains constant. So the motor efficiency will remain the same. Loss in the line supplying the motor will be less due to the lower line current. If the line to the motor from the supply bus is long, the voltage drop in the line will be reduced and this may be desirable.

1.8 If the impendance between machines 1 and 2 of Example 1.1 is $Z = 0 - j5 \ \Omega$ determine (a) whether each machine is generating or consuming power, (b) whether each machine is receiving or supplying positive reactor power and the amount, and (c) the value of P and Q absorbed by the impedance.

Solution:

$$I = \frac{100 + j0 - (86.6 + j50)}{-j5} = 10 + j2.68 = 10.35 \underline{/15°} \ \text{A}$$
$$E_1 I^* = 100(10 - j2.68) = 1000 - j268$$
$$E_2 I^* = (86.6 + j50)(10 - j2.68) = 1000 + j268$$

Machine 1 generates 1000 W, receives 268 var

Machine 2 absorbs 1000 W, receives 268 var

Capacitor in the line supplies $(10.35)^2 \times 5 = 536$ var

1.9 Repeat Problem 1.8 if $Z = 5 + j0 \ \Omega$.

Solution:

$$I = \frac{100 + j0 - (86.6 + j50)}{5} = 2.68 - j10 = 10.35 \underline{/-75°} \ \text{A}$$
$$E_1 I^* = 100(2.68 + j10) = 268 + j1000$$
$$E_2 I^* = (86.6 + j50)(2.68 + j10) = -268 + j1000$$

Machine 1 generates 268 W, delivers 1000 var

Machine 2 generates 268 W, receives 1000 var

Resistance in the line absorbs $(10.35)^2 \times 5 = 536$ W

Both machines are generators.

1.10 A voltage source $E_{an} = -120\angle 210°$ V and the current through the source is given by $I_{na} = 10\angle 60°$ A. Find the values of P and Q and state whether the source is delivering or receiving each.
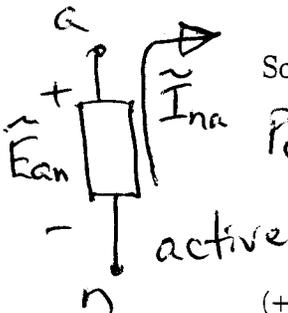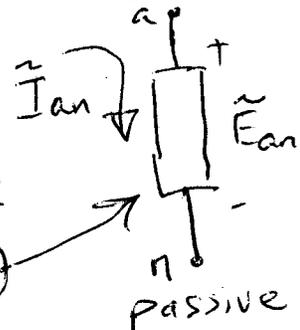
Solution:

$$
\begin{aligned}
P_{out} = E_{an}I_{na}^* &= -120\angle 210° \times 10\angle -60° = -1200\angle 150° = 1200\angle -30° \\
P &= 1039 \text{ W delivered} \\
Q &= -600 \text{ var delivered}
\end{aligned}
$$

$$P_{in} = \tilde{E}_{an}\tilde{I}_{an}^*$$

(+600 var absorbed by source, since I_{na} defines positive current from n to a and E_{an} defines point a at higher potential than n when e_{an} is positive.)

1.11 Solve Example 1.1 if $E_1 = 100\angle 0°$ V and $E_2 = 120\angle 30°$ V. Compare the results with Example 1.1 and form some conclusions about the effect of variation of the magnitude of E_2 in this circuit.

Solution:

$$
\begin{aligned}
I &= \frac{100 - (103.92 + j60)}{j5} = \frac{-3.92 - j60}{j5} = -12 + j0.78 \\
E_1 I^* &= 100\,(-12 - j0.78) = -1200 - j78 \\
E_2 I^* &= (103.92 + j60)\,(-12 - j0.78) = -1247 - j720 - j81 + 46.8 \\
&= -1200 - j801
\end{aligned}
$$

Machine 1 absorbs 1200 W and 78 var
Machine 2 delivers 1200 W and 801 var
$801 - 78 = 723$ var absorbed by line

In Example 2.1 the line received 536 var, half from each source. Raising $|E_2|$ caused some increase in power transfer and some increase in Q supplied to the line, but the significant fact is that raising $|E_2|$ caused that source to supply not only all the Q absorbed by the line but also 78 var delivered to the $|E_1|$ source.

1.12 Evaluate the following expressions in polar form:

(a) $a - 1$

(b) $1 - a^2 + a$

(c) $a^2 + a + j$

(d) $ja + a^2$

Solution:

(a) $a - 1 = -0.5 + j0.866 - 1 = 1.732 \angle 150°$

(b) $1 - a^2 + a = 1 - (-0.5 - j0.866) - 0.5 + j0.866 = 1 + j1.732 = 2.00 \angle 60°$

(c) $a^2 + a + j = -0.5 - j0.866 - 0.5 + j0.866 + j1 = -1 + j1 = 1.414 \angle 135°$

(d) $ja + a^2 = 1 \angle 210° + 1 \angle 240° = -0.866 - j0.5 - 0.5 - j0.866 = -1.366 - j1.366$
$= 1.932 \angle 225°$

1.13 Three identical impedances of $10 \angle -15°$ Ω are Y-connected to balanced three-phase line voltages of 208 V. Specify all the line and phase voltages and the currents as phasors in polar form with V_{ca} as reference for a phase sequence of abc.

Solution:
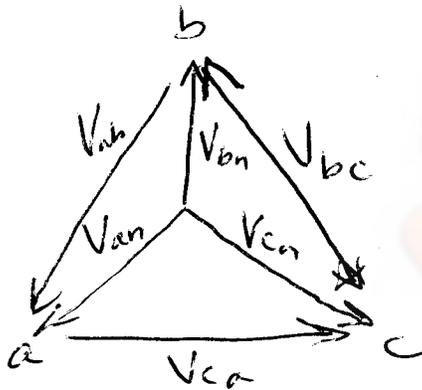
$$V_{an} = 120 \angle 210° \text{ V} \qquad V_{ab} = 208 \angle 240° \text{ V}$$
$$V_{bn} = 120 \angle 90° \text{ V} \qquad V_{bc} = 208 \angle 120° \text{ V}$$
$$V_{cn} = 120 \angle -30° \text{ V} \qquad V_{ca} = 208 \angle 0° \text{ V}$$

$$I_a = \frac{V_{an}}{Z} = \frac{120 \angle 210°}{10 \angle -15°} = 12 \angle 225° \text{ A}$$

$$I_b = \frac{V_{bn}}{Z} = \frac{120 \angle 90°}{10 \angle -15°} = 12 \angle 105° \text{ A}$$

$$I_c = \frac{V_{cn}}{Z} = \frac{120 \angle -30°}{10 \angle -15°} = 12 \angle -15° \text{ A}$$
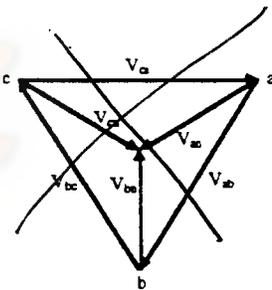
1.14 In a balanced three-phase system the Y-connected impedances are $10 \angle 30°$ Ω. If $V_{bc} = 416 \angle 90°$ V, specify I_{cn} in polar form.

given

Solution:

$$\frac{416}{\sqrt{3}} = 240 \text{ V}$$

$$V_{cn} = 240 \angle -60° \text{ V}$$

$$I_{cn} = \frac{240 \angle -60°}{10 \angle 30°} = 24 \angle -90° \text{ A}$$
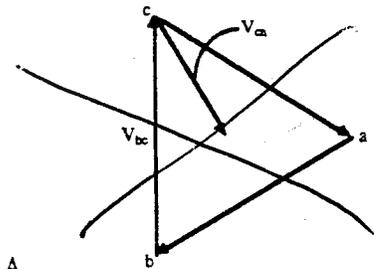
$$\Rightarrow \quad \tilde{V}_{cn} = 240 \angle -60°$$

$$\tilde{I}_{cn} = \frac{\tilde{V}_{cn}}{\tilde{Z}_{cn}} = \frac{240 \angle -60°}{10 \angle 30°} = 24 \angle -90° \text{ A}$$

1.15 The terminals of a three-phase supply are labeled a, b and c. Between any pair a voltmeter measures 115 V. A resistor of 100 Ω and a capacitor of 100 Ω at the frequency of the supply are connected in series from a to b with the resistor connected to a. The point of connection of the elements to each other is labeled n. Determine graphically the voltmeter reading between c and n if phase sequence is abc and if phase sequence is acb.

Solution:

Sequence a-b-c

$$\overline{nk} = \frac{115}{2} = 57.5 \text{ V}$$
$$\overline{kc} = 115\sin 60° = 99.6 \text{ V}$$
$$\text{meter reading} = 57.5 + 99.6 = 157.1 \text{ V}$$
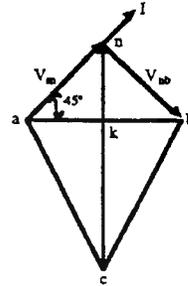
Sequence a-c-b

$$\overline{kc} = 99.6 \text{ V}$$
$$\overline{nk} = 57.5 \text{ V}$$
$$\text{meter reading} = 99.6 - 57.5 = 42.1 \text{ V}$$
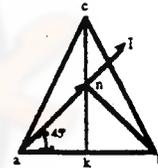
1.16 Determine the current drawn from a three-phase 440-V line by a three-phase 15-hp motor operating at full load, 90% efficiency and 80% power factor lagging. Find the values of P and Q drawn from the line.

Solution:

$$|I| = \frac{15 \times 746}{\sqrt{3} \times 440 \times 0.9 \times 0.8} = 20.39 \text{ A}$$
$$P = \sqrt{3} \times 440 \times 20.39 \times 0.8 = 12,431 \text{ W drawn from line}$$
$$Q = \sqrt{3} \times 440 \times 20.39 \times 0.6 = 9,324 \text{ var drawn from line}$$

1.17 If the impendance of each of the three lines connecting the motor of Prob. 1.16 to a bus is $0.3 + j1.0$ Ω, find the line-to-line voltage at the bus which supplies 440 V at the motor.

Solution:

$$I = 20.39\,(0.8 - j0.6) = 16.31 - j12.23 \ \ \text{A}$$

When the reference is voltage to neutral of the motor at the terminal where I is calculated, or $440/\sqrt{3} = 254\angle 0°$ V, the supply bus voltage to neutral is

$$254 + j0 + (0.3 + j1.0)(16.31 - j12.23) = 271.1 + j12.64$$

$$\text{Line-to-line voltage } |V| = \sqrt{3}\,|271.1 + j12.64| = 470 \ \ \text{V}$$

1.18 A balanced-Δ load consisting of pure resistances of 15 Ω per phase is in parallel with a balanced-Y load having phase impedances of $8 + j6$ Ω. Identical impedances of $2 + j5$ Ω are in each of the three lines connecting the combined loads to a 110-V three-phase supply. Find the current drawn from the supply and line voltage at the combined loads.

Solution:

Convert Δ to equivalent Y having $15/3 = 5$ Ω/phase

$$\frac{5\,(8 + j6)}{5 + 8 + j6} = \frac{40 + j30}{13 + j6} \times \frac{13 - j6}{13 - j6} = \frac{700 + j50}{205}$$

$$= 3.41 + j0.732 = 3.49\angle 12.1° \ \ \Omega$$

Current drawn at supply:

$$Z = 2 + j5 + 3.41 + j0.73 = 5.41 + j5.73 = 7.88\angle 46.65° \ \ \Omega$$

$$|I| = \frac{110/\sqrt{3}}{7.88} = 8.06 \ \ \text{A from supply}$$

Letting V_t equal voltage at the load, line-to-line voltage:

$$V_t = 8.06 \times 3.49 = 28.13 \ \ \text{V to neutral}$$

$$\text{Line-to-line } V_2 = \sqrt{3} \times 28.13 = 48.72 \ \ \text{V}$$

1.19 A three-phase load draws 250 kW at a power factor of 0.707 lagging from a 440-V line. In parallel with this load is a three-phase capacitor bank which draws 60 kVA. Find the total current and resultant power factor.

Solution:

Letting S_1 and S_2 represent the load and capacitor bank, respectively,

$$S_1 = 250 + j250$$

$$S_2 = 0 - j60$$

$$\text{where } S_1 + S_2 = 250 + j190 = 314\underline{/37.23°} \text{ kW}$$

$$|I| = \frac{314,000}{\sqrt{3} \times 440} = 412.0 \text{ A}$$

$$\text{p.f.} = \cos 37.23° = 0.796 \text{ lag}$$

1.20 A three-phase motor draws 20 kVA at 0.707 power factor lagging from a 220-V source. Determine the kilovoltampere rating of capacitors to make the combined power factor 0.90 lagging, and determine the line current before and after the capacitors are added.

Solution:

From the figure,

$$\theta = \cos^{-1} 0.9 = 25.84°$$

$$14.14 \tan 25.84° = 6.85$$

$$14.14 - 6.85 = 7.29 \text{ kvar}$$
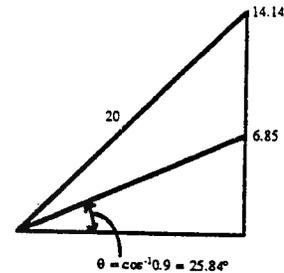
Without capacitors:

$$|I| = \frac{20,000}{\sqrt{3} \times 220} = 52.5 \text{ A}$$

With capacitors:

$$|I| = \frac{|14.14 + j6.85| \times 1000}{\sqrt{3} \times 220} = 41.2 \text{ A}$$

1.21 A coal mining "drag line" machine in an open-pit mine consumes 0.92 MVA at 0.8 power factor lagging when it digs coal, and it generates (delivers to the electric system) 0.10 MVA at 0.5 power factor leading when the loaded shovel swings away from the pit wall. At the end of the "dig" period, the change in supply current magnitude can cause tripping of a protective relay which is constructed of solid-state circuitry. Therefore it is desired to minimize the change in current magnitude. Consider the placement of capacitors at the machine terminals and find the amount of capacitive correction (in kvar) to eliminate the change in steady-state current magnitude. The machine is energized from a 36.5 kV, three-phase supply. Start the solution by letting Q be the total three-phase megavars of the capacitors connected across the machine terminals, and write an expression for the magnitude of the line current *drawn* by the machine in terms of Q for both the digging and generating operations.

Solution:

Assume line-to-line voltage $|V|$ is constant. Then constant current magnitude $|I|$ means constant $|S|$ where $|S| = \sqrt{3} |V| |I^*| \times 10^{-6}$ MVA.

Dig period:

$$|S| = |0.92(0.8 + j0.6) - jQ|$$
$$= |0.736 + j0.552 - jQ|$$
$$|S|^2 = 0.542 + 0.305 - 1.104Q + Q^2 = 0.847 - 1.104Q + Q^2$$

Swing period:

$$|S| = |-0.1\angle{-60°} - jQ| = |-0.05 + j0.0866 - jQ|$$
$$|S|^2 = (-0.05)^2 + (0.0866 - Q)^2 = 0.0025 + 0.0075 - 0.1732Q + Q^2$$
$$= 0.01 - 0.1732Q + Q^2$$

and equating $|S|^2$ for the dig and swing periods, we have

$$0.847 - 1.104Q + Q^2 = 0.01 - 0.1732Q + Q^2$$
$$0.937Q = 0.837$$
$$Q = 0.899 \text{ Mvar or 899 kvar}$$

1.22 A generator (which may be represented by an emf in series with an inductive reactance) is rated 500 MVA, 22 kV. Its Y-connected windings have a reactance of 1.1 per unit. Find the ohmic value of the reactance of the windings.

Solution:

$$\text{Base } Z = \frac{(22)^2}{500} = 0.968 \ \Omega$$
$$X = 1.1 \times 0.968 = 1.065 \ \Omega$$

1.23 The generator of Prob. 1.22 is in a circuit for which the bases are specified as 100 MVA, 20 kV. Starting with the per-unit value given in Prob. 1.22, find the per-unit value of reactance of the generator windings on the specified base.

Solution:

$$X = 1.065 \left(\frac{100}{500}\right)\left(\frac{22}{20}\right)^2 = 0.2577 \text{ per unit}$$

1.24 Draw the single-phase equivalent circuit for the motor (an emf in series with inductive reactance labeled Z_m) and its connection to the voltage supply described in Probs. 1.16 and 1.17. Show on the diagram the per-unit values of the line impedance and the voltage at the motor terminals on a base of 20 kVA, 440 V. Then using per-unit values find the supply voltage in per unit and convert the per-unit value of the supply voltage to volts.

Solution:

Per-unit base calculations:

$$\text{Base } Z = \frac{(0.44)^2 \times 1000}{20} = 9.68 \text{ per unit}$$

$$R = \frac{0.3}{9.68} = 0.031 \text{ per unit}$$

$$X = \frac{1.0}{9.68} = 0.1033 \text{ per unit}$$

$$\text{Base } I = \frac{20,000}{\sqrt{3} \times 440} = 26.24 \text{ A}$$

$$I = \frac{20.39}{26.24} = 0.777 \text{ per unit}$$

Voltage calculations:

$$V = 1.0 + 0.777\,(0.8 - j0.6)\,(0.031 + j0.1033)$$
$$= 1.0 + 0.777 \times 0.1079\,\underline{/\,36.43°}$$
$$= 1.0 + 0.0674 + j0.0498 = 1.0686\,\underline{/\,2.97°} \text{ per unit}$$
$$|V_{LL}| = 1.0686 \times 440 = 470 \text{ V}$$

1.25 Write the two nodal admittance equations, similar to Eqs. (1.57) and (1.58), for the voltages at nodes ② and ④ of the circuit of Fig. 1.23. Then arrange the nodal admittance equations for all four independent nodes of Fig. 1.23 into the $\mathbf{Y}_{bus}$ form of Eq. (1.61).

Solution:

$$\text{bus ②} \qquad (V_2 - V_3)\,Y_b + (V_2 - V_1)\,Y_d + (V_2 - V_4)\,Y_e = 0$$
$$\text{bus ④} \qquad V_4 Y_g + (V_4 - V_1)\,Y_f + (V_4 - V_2)\,Y_e = I_4$$

Rearranging equations for bus ② and bus ④ yields

$$\text{bus ②} \qquad -V_1 Y_d + V_2\,(Y_b + Y_d + Y_e) - V_3 Y_b - V_4 Y_e = 0$$
$$\text{bus ④} \qquad -V_1 Y_f - V_2 Y_e + V_4\,(Y_g + Y_f + Y_e) = I_4$$

The $\mathbf{Y}_{bus}$ form is

$$
\begin{array}{c}
 \\
① \\
② \\
③ \\
④
\end{array}
\begin{bmatrix}
(Y_c + Y_d + Y_f) & -Y_d & -Y_c & -Y_f \\
-Y_d & (Y_b + Y_d + Y_e) & -Y_b & -Y_e \\
-Y_c & -Y_b & (Y_a + Y_b + Y_c) & 0 \\
-Y_f & -Y_e & 0 & (Y_e + Y_f + Y_g)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
I_3 \\
I_4
\end{bmatrix}
$$

1.26 The values for the parameters of Fig. 1.23 are given in per unit as follows:

$$Y_a = -j0.8 \qquad Y_b = -j4.0 \qquad Y_c = -j4.0 \qquad Y_d = -j8.0 \qquad Y_e = -j5.0$$
$$Y_f = -j2.5 \qquad Y_g = -j0.8 \qquad I_3 = 1.0\,\underline{/\,-90°} \qquad I_4 = 0.68\,\underline{/\,-135°}$$

Substituting these values in the equations determined in Prob. 1.25, compute the voltages at the nodes of Fig. 1.23. Numerically determine the corresponding $\mathbf{Z}_{bus}$ matrix.

Solution:

Using the $\mathbf{Y}_{bus}$ solution of Problem 1.25, substitute the given admittance values:

$$
\begin{bmatrix}
-j14.5 & j8.0 & j4.0 & j2.5 \\
j8.0 & -j17.0 & j4.0 & j5.0 \\
j4.0 & j4.0 & -j8.8 & j0 \\
j2.5 & j5.0 & j0 & -j8.3
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
1.0\angle -90° \\
0.68\angle -135°
\end{bmatrix}
$$

Compute voltages:

$$\mathbf{Y}_{bus}\mathbf{V} = \mathbf{I}$$
$$\mathbf{Y}_{bus}^{-1}\mathbf{Y}_{bus}\mathbf{V} = \mathbf{Y}_{bus}^{-1}\mathbf{I}$$

where $\mathbf{Y}_{bus}^{-1} = \mathbf{Z}_{bus} = \begin{bmatrix}
j0.7187 & j0.6688 & j0.6307 & j0.6194 \\
j0.6688 & j0.7045 & j0.6242 & j0.6258 \\
j0.6307 & j0.7045 & j0.6840 & j0.5660 \\
j0.6194 & j0.6258 & j0.5660 & j0.6840
\end{bmatrix}$

$$\mathbf{V} = \mathbf{Y}_{bus}^{-1}\mathbf{I}$$

$$
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4
\end{bmatrix}
=
\begin{bmatrix}
j0.7187 & j0.6688 & j0.6307 & j0.6194 \\
j0.6688 & j0.7045 & j0.6242 & j0.6258 \\
j0.6307 & j0.7045 & j0.6840 & j0.5660 \\
j0.6194 & j0.6258 & j0.5660 & j0.6840
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1.0\angle -90° \\
0.68\angle -135°
\end{bmatrix}
$$

$$
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4
\end{bmatrix}
=
\begin{bmatrix}
0.9285 - j0.2978 \\
0.9251 - j0.3009 \\
0.9562 - j0.2721 \\
0.8949 - j0.3289
\end{bmatrix}
=
\begin{bmatrix}
0.9750\angle -17.78° \\
0.9728\angle -18.02° \\
0.9941\angle -15.89° \\
0.9534\angle -20.18°
\end{bmatrix}
$$

Chapter 2 Problem Solutions

2.1 A single-phase transformer rated 7.2 kVA, 1.2kV/120 V has a primary winding of 800 turns. Determine (a) the turns ratio and the number of turns in the secondary winding, (b) the currents carried by the two windings when the transformer delivers its rated kVA at rated voltages. Hence, verify Eq. (2.7).

Solution:

(a)

$$\frac{N_1}{N_2} = \frac{V_1}{V_2} = \frac{1.2 \times 10^3}{120} = 10$$

$$\text{Therefore,} \quad N_2 = \frac{N_1}{10} = \frac{800}{10} = 80$$

To: ee5200-l@mtu.edu
Subject: sign convention, conjugate, cosine reference

Glad to see this kind of exchange on the e-mail list,
that is what I have been hoping for.

Some comments that could help with the Ch.1 review problems:

1) As I mentioned in the first lecture when we discussed
Euler's identity, it is standard practice to define phasors
according to the cosine (real) component and this is termed
"cosine reference." Therefore, when converting from time domain
to phasor domain, we must first convert all sinusoidal functions to
equivalent cos functions. By sketching out a sine and a cosine
function, it becomes clear that a sine is just a cosine that has
been delayed by 90°. Therefore, $\sin(wt) = \cos(wt - 90°)$ or
$\cos(wt) = \sin(wt + 90°)$.

2) The negative sign associated with I is most likely due
to how I is defined on the circuit, i.e. the assumed reference
direction of current flow that is marked on the circuit.

3) To correctly calculate complex power consumed by (or flowing in to) a
circuit element, $S_{in} = VI^* = P + jQ$, where V and I have
reference polarity/direction according to passive sign convention.

4) Careful with conjugates: remember that the conjugate of a complex number
has the same magnitude, but the sign of its angle is changed. For
example, if $I = 10\underline{/30°}$ A, then $I^* = 10\underline{/-30°}$ A. Thus, negating a complex
number is not the same as taking its conjugate.

Thanks for the comments on the Ch.1 problems, I will try to go thru
these and then issue any corrections that may be required.

Looks like we are off to a good start, good to be thinking about these
details and doing some review/refreshing.

Dr. Mork

To: ee5200-l@mtu.edu
Subject: Solutions, Probs 1.1 and 1.10

Let's take a look at the first couple of items in question:

> 1.1, 1.2: v(t) is given as v(t)=141.4sin(wt+30) and i(t) is given as
> i(t)=11.31cos(wt-30). However, don't you need to put v(t) in terms of
> cosine, which becomes v(t)=141.4cos(wt+30-90)? Using v(t) as the reference,
> this give V=100@0 and I=8@30. Current is now leading the voltage and the
> circuit is capacitive. Does this sound right?

Yes, this is correct, we use cosine as the common basis/reference for
expressing all phasor angles.   Peak values must be divided by sqrt(2)
to get RMS values.

In prob. 1.2, the calculation method is correct, just update the values
according to the solution of problem 1.1.

> 1.10: I also assumed S=Ean(-Ian)* and got a different answer than the
> solutions.

The meaning of S=VI* really hinges on whether active or passive sign convention
is used. Use of double subscripts makes it much easier to explain and
to understand. The general equation S = VI* is clearly being implemented
here using ACTIVE sign convention, the subscripts clearly tell us.

Active sign convention (generator convention) calculates S produced:
==
Sout = (Ean)(Ina)* = (Ean)(-Ian)* = Pout + jQout

Passive sign convention (load convention) calculated S consumed:
==
Sin = (Ean)(Ian)* = (Ean)(-Ina)* = Pin + jQin.

The author's solution is therefore correct. From this we might agree
that it is indeed important it is to know the difference between active
and passive sign convention... Working at MS level, we need to
understand the concepts and details. These seemingly simple
problems bring that out.

Dr. Mork

To: ee5200-l@mtu.edu
Subject: Chap 1 - problems 1.8,9,11 and 1.23

> 1.8, 1.9, 1.11: I agree that they should use S1 = E1(-I)* like example 1.1. Also, check the P & Q being delivered versus the P & Q being absorbed/received, in 1.8 for example, where the Z=0-j5. The solution shows S1=1000-j268 and S2=1000+j268. If Z=0-j5, which is purely capacitive, how can all the Vars from machine 1 be transferred to machine 2, while both machines are absorbing 1000 Watts? If you use S1 = E1(-I)*, the transfer of P & Q makes more sense.

Here, I think that if you are careful with active and passive sign conventions, discussed in the previous e-mail,
you will calculate the correct Ps and Qs and all of the P and Q that are generated and consumed will balance out. Let me know if you have any subsequent questions.

> 1.23: In problem 1.22, you are given a per unit reactance of 1.1pu. On the given bases, this gives an actual reactance of 1.065 Ohms. In 1.23 you are asked to find the per-unit reactance using a change of base, which is illustrated in equation 1.56 on page 29. However, the solution uses the actual impedance of 1.065 Ohms from 1.22 in the equation and gets .2557pu as the answer. To get a per-unit reactance as the answer, wouldn't you need to use 1.1pu as Z-old in the equation?

yes, there is a typo, he should have started with 1.1 pu, not the value in ohms.

The "canned" equation given by Eqn. 1.56 is not a very intuitive one, and you have to be careful how you apply it. It is more intuitive if you think of a two-step process:

    a) multiply the given p.u. value by its Zbase to obtain the actual ohms.
    b) divide the actual ohms by the new Zbase to get the new p.u. impedance.

Therefore, it is more intuitive to express the equation as:

    Znew,pu = Zgiven,pu (kVbase,given^2 / MVAbase,given)   <== step a)

            / (kVbase,new^2 / MVAbase,new) <== step b)

If you rearrange these terms, you end up with what is given in Eqn 1.56.
To the author's credit, he suggests this two-step approach in the 2nd paragraph on p. 30.

A philosophical observation:
Studying at the MS level, our goal is not only to learn more advanced "stuff" but to also improve our understanding of the fundamentals and concepts.
Actually, encountering these errors and confusions in these review problems may have taught us more than if the solutions had all been totally correct -- we had to stop and question what's going on, go back to basic concepts,

and figure it out.

Any more points of uncertainty or possible errors? Please go ahead and start the discussion here, hopefully this is helpful.

See you all in class tomorrow morning, we will go through some more per unit things.

Dr Mork

To: ee5200-l@mtu.edu
Subject: Chapter one problem 1.7

At 12:02 PM 9/3/2003 -0500, you wrote:
| 1.7: I believe that they drew the correct conclusion about the efficiency
| but for the wrong reason. Anybody care to comment?

The author's rationalization seems to be sound:

Adding shunt capacitors (shunt compensation) reduces the
inductive component of the current being drawn from the
mains, i.e. flowing down the line, thus reducing the net
current flowing in the line. This reduces the I^2 R
line losses. The current flowing into the motor, however,
is unchanged (assuming the capacitor placement has not
changed the terminal voltage).

This is essentially a power factor correction situation, no
internal changes have been made to the motor, it is still
operating in the same way and with the same efficiency.
Since efficiency is related only to real power P, the Q
that is produced by the caps has no effect on motor
efficiency.

There are some devices, in cyclic loading applications,
that increase overall motor efficiency by reducing the source
voltage to the motor when the mechanical load on the motor
is removed/reduced, and then restores full voltage when the
motor is loaded down again. Not restoring full voltage, or
operating a loaded induction motor at reduced voltage
will draw excessive current, resulting in a very low efficiency
and extreme I^2 R heating of the armature windings, thus burning
it out. One basic type of motor protection is thus to trip the motor
off line if the voltage is too low and/or the current is too high,
and/or if the winding temperature gets too high.

Dr. Mork

Substituting these values in the equations determined in Prob. 1.25, compute the voltages at the nodes of Fig. 1.23. Numerically determine the corresponding Z_{bus} matrix.

Solution:

Using the Y_{bus} solution of Problem 1.25, substitute the given admittance values:

$$\begin{bmatrix} -j14.5 & j8.0 & j4.0 & j2.5 \\ j8.0 & -j17.0 & j4.0 & j5.0 \\ j4.0 & j4.0 & -j8.8 & j0 \\ j2.5 & j5.0 & j0 & -j8.3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.0\angle -90° \\ 0.68\angle -135° \end{bmatrix}$$

Compute voltages:

$$Y_{bus}V = I$$
$$Y_{bus}^{-1}Y_{bus}V = Y_{bus}^{-1}I$$

$$\text{where } Y_{bus}^{-1} = Z_{bus} = \begin{bmatrix} j0.7187 & j0.6688 & j0.6307 & j0.6194 \\ j0.6688 & j0.7045 & j0.6242 & j0.6258 \\ j0.6307 & j0.7045 & j0.6840 & j0.5660 \\ j0.6194 & j0.6258 & j0.5660 & j0.6840 \end{bmatrix}$$

$$V = Y_{bus}^{-1}I$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} j0.7187 & j0.6688 & j0.6307 & j0.6194 \\ j0.6688 & j0.7045 & j0.6242 & j0.6258 \\ j0.6307 & j0.7045 & j0.6840 & j0.5660 \\ j0.6194 & j0.6258 & j0.5660 & j0.6840 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1.0\angle -90° \\ 0.68\angle -135° \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.9285 - j0.2978 \\ 0.9251 - j0.3009 \\ 0.9562 - j0.2721 \\ 0.8949 - j0.3289 \end{bmatrix} = \begin{bmatrix} 0.9750\angle -17.78° \\ 0.9728\angle -18.02° \\ 0.9941\angle -15.89° \\ 0.9534\angle -20.18° \end{bmatrix}$$

Chapter 2 Problem Solutions

2.1 A single-phase transformer rated 7.2 kVA, 1.2kV/120 V has a primary winding of 800 turns. Determine (a) the turns ratio and the number of turns in the secondary winding, (b) the currents carried by the two windings when the transformer delivers its rated kVA at rated voltages. Hence, verify Eq. (2.7).

Solution:

(a)

$$\frac{N_1}{N_2} = \frac{V_1}{V_2} = \frac{1.2 \times 10^3}{120} = 10$$

$$\text{Therefore, } N_2 = \frac{N_1}{10} = \frac{800}{10} = 80$$

(b)

$$S_{rated} = |V_1|_{rated} |I_1|_{rated} = |V_2|_{rated} |I_2|_{rated}$$
$$7.2 \times 10^3 = 1.2 \times 10^3 |I_1|_{rated} = 120 |I_2|_{rated}$$
$$|I_1|_{rated} = \frac{7.2 \times 10^3}{1.2 \times 10^3} = 6 \text{ A}$$
$$|I_2|_{rated} = \frac{7.2 \times 10^3}{120} = 60 \text{ A}$$

Left-hand side of Eq. (2.7): $\quad \dfrac{I_1}{I_2} = \dfrac{6}{60} = 0.1$

Right-hand side of Eq. (2.7): $\quad \dfrac{N_2}{N_1} = \dfrac{1}{10} = 0.1$

Left-hand side of Eq. (2.7) $\quad=\quad$ Right-hand side of Eq. (2.7)

2.2 The transformer of Prob. 2.1 is delivering 6 kVA at its rated voltages and 0.8 power factor lagging. (a) Determine the impedance Z_2 connected across its secondary terminals. (b) What is the value of this impedance referred to the primary side (i.e. Z_2')? (c) Using the value of Z_2' obtained in part (b), determine the magnitude of the primary current and the kVA supplied by the source.

Solution:

(a)

$$S_2 = |S_2| \angle \theta = 6 \times 10^3 \angle 36.9° \text{ VA}$$
$$I_2 = \left(\frac{S_2}{V_2}\right)^*$$
$$Z_2 = \frac{V_2}{I_2} = \frac{V_2}{S_2^*/V_2^*} = \frac{|V_2|^2}{S_2^*}$$
$$= \frac{(120)^2}{6 \times 10^3 \angle -36.9°} \ \Omega$$
$$= 2.4 \angle 36.9° \ \Omega = (1.92 + j1.44) \ \Omega$$

(b)

$$Z_2' = \left(\frac{N_1}{N_2}\right)^2 Z_1 = \left(\frac{V_1}{V_2}\right)^2 Z_1 = 100 \times 2.4 \angle 36.9° \ \Omega$$
$$= 240 \angle 36.9° \ \Omega = 192 + j144 \ \Omega$$

(c)

$$|I_1| = \frac{|V_1|}{|Z_2'|} = \frac{1.2 \times 10^3}{240} \text{ A} = 5 \text{ A}$$
$$|S_1| = |V_1||I_1| = 1.2 \times 10^3 \times 5 \text{ VA} = 6 \text{ kVA}$$

2.3 With reference to Fig. 2.2, consider that the flux density inside the center-leg of the transformer core, as a function of time t, is $B(t) = B_m \sin(2\pi ft)$ where B_m is the peak value of the sinusoidal flux density and f is the operating frequency in Hz. If the flux density is uniformly distributed over the cross-sectional area A m^2 of the center-leg, determine

(a)   the instantaneous flux $\phi(t)$ in terms of B_m, f, A and t,

(b)   the instantaneous induced-voltage $e_1(t)$, according to Eq. (2.1).

(c)   Hence show that the rms magnitude of the induced voltage of the primary is given by $|E_1| = \sqrt{2}\pi f N_1 B_m A$.

(d)   If $A = 100$ cm^2, $f = 60$ Hz, $B_m = 1.5$ T and $N_1 = 1000$ turns, compute $|E_1|$.

Solution:

(a)

$$\phi(t) = B(t)A = B_m A \sin(2\pi ft)$$

(b)

$$e_1(t) = N_1 \frac{d\phi(t)}{dt} = N_1 B_m A \frac{d}{dt}\{\sin(2\pi ft)\} = 2\pi f N_1 B_m A \cos(2\pi ft)$$

(c)

$$E_1 = \frac{1}{\sqrt{2}}[e_1(t)]_{max} = \frac{2\pi f N_1 B_m A}{\sqrt{2}} = \sqrt{2}\pi f N_1 B_m A$$

(c) With given values,

$$E_1 = \sqrt{2}\pi \times 60 \times 1000 \times 1.5 \times 100 \times 10^{-4} \text{ V} = 4.0 \text{ kV}$$

2.4 For the pair of mutually coupled coils shown in Fig. 2.4, consider that $L_{11} = 1.9$ H, $L_{12} = L_{21} = 0.9$ H, $L_{22} = 0.5$ H and $r_1 = r_2 = 0$ Ω. The system is operated at 60 Hz.

(a)   Write the impedance form [Eq. (2.24)] of the system equations

(b)   Write the admittance form [Eq. (2.26)] of the system equations

(c)   Determine the primary voltage V_1 and the primary current I_1 when the secondary is

   (i)   open circuited and has the induced voltage $V_2 = 100\angle 0°$ V

   (ii)   short circuited and carries the current $I_2 = 2\angle 90°$ A

Solution:

(a) From Eq. (2.22) and (2.23),

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = j\omega \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = j120\pi \begin{bmatrix} 1.9 & 0.9 \\ 0.9 & 0.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$= j10^2 \begin{bmatrix} 7.163 & 3.393 \\ 3.393 & 1.885 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

(b) From Eq. (2.25),

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = -j10^{-2} \begin{bmatrix} 7.163 & 3.393 \\ 3.393 & 1.885 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$= -j10^{-2} \begin{bmatrix} 0.947 & -1.705 \\ -1.705 & 3.600 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

(c) (i)

$$\begin{bmatrix} V_1 \\ 100\angle 0° \end{bmatrix} = 100\angle 90° \begin{bmatrix} 7.163 \\ 3.393 \end{bmatrix} I_1$$
$$\text{hence } I_1 = 0.295\angle -90° \text{ A}$$
$$V_1 = 211.11\angle 0° \text{ V}$$

(ii)

$$\begin{bmatrix} I_1 \\ 2\angle 90° \end{bmatrix} = 10^{-2} \times 1\angle -90° \begin{bmatrix} 0.947 \\ -1.705 \end{bmatrix} V_1$$
$$\text{hence } V_1 = 117.30\angle 0° \text{ V}$$
$$I_1 = 1.11\angle -90° \text{ A}$$

2.5 For the pair of mutually coupled coils shown in Fig. 2.4, develop an equivalent-T network in the form of Fig. 2.5. Use the parameter values given in Prob. 2.4 and assume that the turns ratio a equals 2. What are the values of the leakage reactances of the windings and the magnetizing susceptance of the coupled coils?

Solution:

$$L_{1l} = L_{11} - aL_{21} = 1.9 - 2 \times 0.9 \text{ H} = 0.1 \text{ H}$$
$$L_{2l} = L_{22} - L_{12}/a = 0.5 - 2 \times 0.9/2 \text{ H} = 0.05 \text{ H}$$
$$a^2 L_{2l} = 4 \times 0.05 \text{ H} = 0.2 \text{ H}$$
$$L_m = aL_{21} = 2 \times 0.9 \text{ H} = 1.8 \text{ H}$$
$$\omega = 120\pi \text{ rad/sec}$$
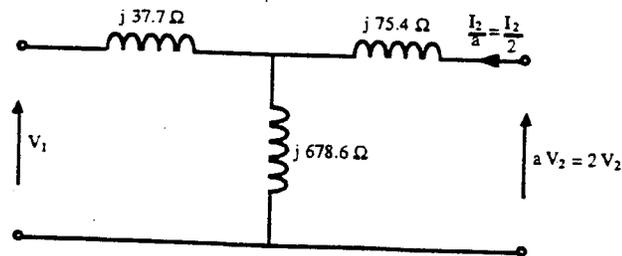
Leakage reactances: $x_1 = 37.7 \ \Omega$

$$x_2' = 75.4 \ \Omega$$

$$x_2 = \frac{75.4}{4} \ \Omega = 18.85 \ \Omega$$

Magnetizing susceptance: $\quad B_m = \dfrac{1}{\omega L_m} = \dfrac{1}{120\pi \times 1.8} \ S$

$$= 1.474 \times 10^{-3} \ S$$

2.6 A single-phase transformer rated 1.2 kV/120 V, 7.2 kVA has the following winding parameters: $r_1 = 0.8 \ \Omega$, $x_1 = 1.2 \ \Omega$, $r_2 = 0.01 \ \Omega$ and $x_2 = 0.01 \ \Omega$. Determine

(a) the combined winding resistance and leakage reactance referred to the primary side, as shown in Fig. 2.8,

(b) the values of the combined parameters referred to the secondary side

(c) the voltage regulation of the transformer when it is delivering 7.5 kVA to a load at 120 V and 0.8 power factor lagging.

Solution:

(a) With turns ratio $a = 1.2 \times 10^3/120 = 10$,

$$R_1 = r_1 + a^2 r_2 = 0.8 + 100 \times 0.01 \ \Omega = 1.8 \ \Omega$$

$$X_1 = x_1 + a^2 x_2 = 1.2 + 100 \times 0.01 \ \Omega = 2.2 \ \Omega$$

(b)

$$R_2 \triangleq R_1/a^2 = 1.8/100 \ \Omega = 0.018 \ \Omega$$

$$X_2 \triangleq X_1/a^2 = 2.2/100 \ \Omega = 0.022 \ \Omega$$
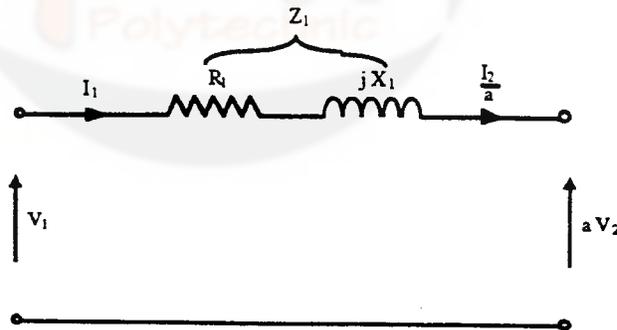
(c)

$$z_1 = (1.8 + j2.2) \ \Omega$$

$$I_{2,\text{FL}} = |S_2/V_2| \angle -\theta = \frac{7200}{120} \angle -36.9° \ A = 60 \angle -36.9° \ A$$

$$I_{1,\text{FL}} = \frac{I_{2,\text{FL}}}{a} = 6.0 \angle -36.9° \ A$$

$$aV_{2,\text{FL}} = 1200 \ V$$

$$V_{1,\text{FL}} = aV_{2,\text{FL}} + I_{1,\text{FL}} Z_1$$

$$
\begin{aligned}
&= 1200 + 6.0\underline{/-36.9^c}\,(1.8 + j2.2)\ \text{V} = 1216.57\underline{/0.19^c}\ \text{V} \\
|V_{2,\text{FL}}| &= 120\ \text{V} \\
|V_{2,\text{NL}}| &= V_{1,\text{FL}}/a = 121.66\ \text{V} \\
\%\ \text{Regulation} &= (121.66 - 120)/120 = 1.38\ \%
\end{aligned}
$$

2.7 A single-phase transformer is rated 440/220 V, 5.0 kVA. When the low-voltage side is short circuited and 35 V is applied to the high-voltage side, rated current flows in the windings and the power input is 100 W. Find the resistance and reactance of the high- and low-voltage windings if the power loss and ratio of reactance to resistance is the same in both windings.

Solution:

$$
\begin{aligned}
\text{Rated } I &= \frac{5000}{220} = 22.73\ \text{A (low voltage)} \\
&= \frac{5000}{440} = 11.36\ \text{A (high voltage)} \\
R &= \frac{100}{11.36^2} = 0.774\ \Omega \\
Z &= \frac{35}{11.36} = 3.08\ \Omega \quad (R,\ Z,\ X\ \text{high-voltage}) \\
X &= \sqrt{3.08^2 - 0.774^2} = 2.98\ \Omega \qquad \frac{X}{R} = \frac{2.98}{0.774} = 3.85
\end{aligned}
$$

For equal loss in high- and low-voltage windings,

$$
\begin{aligned}
\text{High voltage: } r &= \frac{0.774}{2} = 0.387\ \Omega \\
x &= 3.85 \times 0.387 = 1.49\ \Omega \\
\text{Low voltage: } r &= 0.387 \times \left(\frac{220}{440}\right)^2 = 0.097\ \Omega \\
x &= 1.49\left(\frac{220}{440}\right)^2 = 0.373\ \Omega
\end{aligned}
$$

2.8 A single-phase transformer rated 1.2 kV/120 V, 7.2 kVA yields the following test results:

Open-Circuit Test (Primary Open)

Voltage $V_2 = 120$ V; Current $I_2 = 1.2$ A; Power $W_2 = 40$ W

Short-Circuit Test (Secondary Shorted)

Voltage $V_1 = 20$ V; Current $I_1 = 6.0$ A; Power $W_1 = 36$ W

Determine

(a) the parameters $R_1 = r_1 + a^2 r_2$, $X_1 = x_1 + a^2 x_2$, G_c and B_m referred to the primary side, Fig. 2.7

(b) the values of the above parameters referred to the secondary side

(c) the efficiency of the transformer when it delivers 6 kVA at 120 V and 0.9 power factor.

Solution:

(a) From open-circuit test,

$$G'_c = W_2/V_2^2 = 40/120^2 \text{ S} = 2.78 \times 10^{-3} \text{ S}$$
$$|Y'_m| = I_2/V_2 = 1.2/120 \text{ S} = 0.01 \text{ S}$$
$$B'_m = \sqrt{|Y'_m|^2 - G'^2_c} = 9.606 \times 10^{-3} \text{ S}$$
$$a = 1.2 \times 10^3/120 = 10$$

Therefore,

$$G_c = G'_c/a^2 = 2.78 \times 10^{-5} \text{ S}$$
$$B_m = B'_m/a^2 = 9.606 \times 10^{-5} \text{ S}$$

From the short-circuit test,

$$R = W_1/I_1^2 = 36/6.0^2 \ \Omega = 1.0 \ \Omega$$
$$|Z| = V_1/I_1 = 20/6.0 \ \Omega = 3.33 \ \Omega$$
$$X = \sqrt{|Z|^2 - R^2} = 3.18 \ \Omega$$

(b)

$$R' = R/a^2 = 0.01 \ \Omega \qquad X' = X/a^2 = 0.0318 \ \Omega$$
$$G'_c = 2.78 \times 10^{-3} \text{ S} \qquad B'_m = 9.606 \times 10^{-3} \text{ S}$$

(c) When $S_2 = 6.0$ kVA and $V_2 = 120$ V,

$$I_2 = \frac{6 \times 10^3}{120} \text{ A} = 50 \text{ A}$$

$$\text{Core loss at } V_2 = 120 \text{ V} = 40 \text{ W}$$
$$\text{Winding loass at } I_2 = 50 \text{ A} = |I_2|^2 R' = 50^2 \times 0.01 \text{ W} = 25 \text{ W}$$
$$\text{Power output at } S_2 = 6.0 \text{ kVA at 0.9 p.f.} = 6 \times 10^3 \times 0.9 \text{ W} = 5400 \text{ W}$$
$$\eta = \frac{5400}{5400 + 40 + 25} = 98.81 \text{ \%}$$

2.9 A single-phase transformer rated 1.2 kV/120 V, 7.2 kVA has primary-referred parameters $R_1 = r_1 + a^2 r_2 = 1.0 \ \Omega$ and $X_1 = x_1 + a^2 x_2 = 4.0 \ \Omega$. At rated voltage its core loss may be assumed to be 40 W for all values of the load current.

(a) Determine the efficiency and regulation of the transformer when it delivers 7.2 kVA at $V_2 = 120$ V and power factor of (i) 0.8 lagging, (ii) 0.8 leading.

(b) For a given load voltage and power factor it can be shown that the efficiency of a transformer attains its maximum value at the kVA load level which makes the I^2R winding losses equal to the core loss. Using this result, determine the maximum efficiency of the above transformer at rated voltage and 0.8 power factor, and the kVA load level at which it occurs.

Solution:

(a) (i) $\cos\theta = 0.8$, lagging $\quad \theta = -36.9°$

$$V_2 = 120\angle 0° \text{ V}$$

$$I_2 = \frac{7200}{120}\angle{-36.9°} = 60\angle{-36.9°} \text{ A}$$

$$\text{Total loses} = 40 + 60^2 \times \frac{1.0}{100} \text{ W} = 76 \text{ W}$$

$$\text{Output power} = 7.2 \times 10^3 \times 0.8 \text{ W} = 5760 \text{ W}$$

$$\eta = \frac{5760}{5760 + 76} = 98.698 \text{ \%}$$

$$V_{1,FL} = aV_{2,FL} + \frac{I_{2,FL}}{a}(R_1 + jX_1)$$

$$V_{1,FL} = 120 \times 10\angle 0° + \frac{60}{10}\angle{-36.9°}(1.0 + j4.0) \text{ V} = 1219.3\angle 0.73°$$

$$|V_{2,FL}| = 120 \text{ V} \qquad |V_{2,FL}| = |V_{1,FL}|/a = 121.93 \text{ V}$$

$$\text{\% Regulation} = \frac{121.93 - 120}{120} = 1.61 \text{ \%}$$

(ii) $\cos\theta = 0.8$, leading $\quad \theta = 36.9°$

$\eta = 98.698\%$ because it does not depend on whether θ is leading or lagging.

$$V_{1,FL} = aV_{2,FL} + \frac{I_{2,FL}}{a}(R_1 + jX_1)$$

$$V_{1,FL} = 120 \times 10\angle 0° + \frac{60}{10}\angle 36.9°(1.0 + j4.0) \text{ V} = 1190.6\angle 1.1°$$

$$\text{\% Regulation} = \frac{119.06 - 120}{120} = -0.78 \text{ \%}$$

(b) Load current at which η is maximum is given by

$$|I_2^*|^2 \frac{R}{a^2} = P_{core}$$

Therefore,

$$|I_2^*| = \sqrt{\frac{40}{1.0/100}} \text{ A} = 63.245 \text{ A}$$

$$\text{Winding loss at } |I_2^*| = 40 \text{ W}$$

$$\text{Output} = 120 \times 63.245 \times 0.8 \text{ W} = 6071.57 \text{ W}$$

$$\eta_{max} = \frac{6071.57}{6071.57 + 40 + 40} = 98.700 \%$$

$$\text{Corresponding kVA level} = 120 \times 63.245 \text{ VA} = 7.589 \text{ kVA}$$

2.10 A single-phase system similar to that shown in Fig. 2.10 has two transformers A-B and B-C connected by a line B feeding a load at the receiving end C. The ratings and parameter values of the components are

Transformer A-B: 500 V/1.5 kV, 9.6 kVA, leakage reactance = 5%
Transformer B-C: 1.2 kV/120 V, 7.2 kVA, leakage reactance = 4%
Line B: series impedance = $(0.5 + j3.0)$ Ω
Load C: 120 V, 6 kVA at 0.8 power factor lagging

(a) Determine the value of the load impedance in ohms and the actual ohmic impedances of the two transformers referred to both their primary and secondary sides.

(b) Choosing 1.2 kV as the voltage base for circuit B and 10 kVA as the systemwide kVA base, express all system impedances in per unit.

(c) What value of sending-end voltage corresponds to the given loading conditions?

Solution:

(a) Ohmic impedances

Transformer A-B	Primary:	$\dfrac{500^2}{9.6 \times 10^3} \times j0.05$	$= j1.302$ Ω
	Secondary:	$\dfrac{1.5^2 \times 10^6}{9.6 \times 10^3} \times j0.05$	$= j11.719$ Ω
Transformer B-C	Primary:	$\dfrac{1.2^2 \times 10^6}{7.2 \times 10^3} \times j0.04$	$= j8.0$ Ω
	Secondary:	$\dfrac{120^2}{7.2 \times 10^3} \times j0.04$	$= j0.08$ Ω
Load:	$\dfrac{|V|^2}{|S|} \angle \theta$	$= \dfrac{120^2}{6 \times 10^3} \angle \cos^{-1} 0.8$	$= 2.4 \angle 36.9°$ Ω

(b) Impedance bases

$$\text{Circuit } B: \quad \frac{1.2^2 \times 10^6}{10 \times 10^3} \text{ Ω} = 144 \text{ Ω}$$

$$\text{Circuit } C: \quad \frac{120^2}{10 \times 10^3} \text{ Ω} = 1.44 \text{ Ω}$$

Per unit impedances on new bases:

Transformer A-B: $\quad j\dfrac{11.719}{144} \quad = j0.08138$ per unit

Transformer B-C: $\quad j\dfrac{8}{144} \quad = j0.0556$ per unit

Line B: $\quad \dfrac{(1.5 + j3.0)}{144} \quad = 0.0104 + j0.0208$ per unit

Load: $\quad \dfrac{2.4}{1.44}\underline{/36.9°} \quad = 1.667\underline{/36.9°}$ per unit

(c) Sending-end voltage calculations
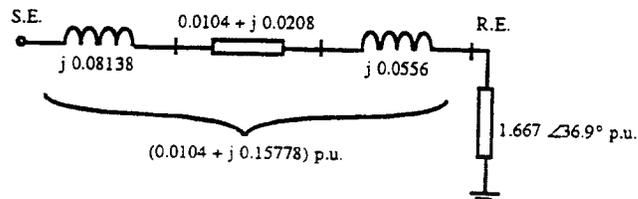
$$V_R = 120 \text{ V} = 1.0 \text{ per unit}$$

$$V_S = 1.0 \times \frac{1.667\underline{/36.9°} + (0.0104 + j0.15778)}{1.667\underline{/36.9°}} = 1.0642 \text{ per unit}$$

The sending-end voltage base is

$$V_{S,\text{base}} = \frac{500}{1.5 \times 10^3} \times 1.2 \times 10^3 = 400 \text{ V}$$

Therefore, the required sending-end voltage is

$$V_S = 400 \times 1.0642 = 425.69 \text{ V}$$

2.11 A balanced Δ-connected resistive load of 8000 kW is connected to the low-voltage, Δ-connected side of a Y-Δ transformer rated 10,000 kVA, 138/13.8 kV. Find the load resistance in ohms in each phase as measured from line to neutral on the high-voltage side of the transformer. Neglect transformer impedance and assume rated voltage is applied to the transformer primary.

Solution:

$$|I_\text{line}| = \frac{8,000}{\sqrt{3} \times 138} = 33.47 \text{ A}$$

$$R = \frac{138,000/\sqrt{3}}{33.47} = 2380 \ \Omega$$

2.12 Solve Prob. 2.11 if the same resistances are reconnected in Y.

Solution:

If the Δ-connected resistors are reconnected in Y, then the resistance to neutral will be three times as great and

$$R = 3 \times 2380 = 7140 \ \Omega$$

2.13 Three transformers, each rated 5 kVA, 220 V on the secondary side, are conected Δ-Δ and have been supplying a balanced 15 kW purely resistive load at 220 V. A change is made which reduces the load to 10 kW, still purely resistive and balanced. Someone suggests that, with two-thirds of the load, one transformer can be removed and the system can be operated open-Δ. Balanced three-phase voltages will still be supplied to the load since two of the line voltages (and thus also the third) will be unchanged.

To investigate further the suggestion

(a) Find each of the line currents (magnitude and angle) with the 10 kW load and the transformer between a and c removed. (Assume $V_{ab} = 220\angle 0°$ V, sequence $a\ b\ c$.)

(b) Find the kilovoltamperes supplied by each of the remaining transformers.

(c) What restriction must be placed on the load for open-Δ operation with these transformers?

(d) Think about why the individual transformer kilovoltampere values include a Q component when the load is purely resistive.

Solution:
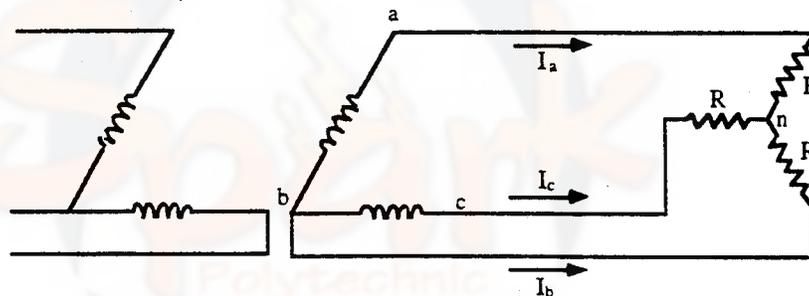
(a) V_{ab} and V_{bc} remain the same after removing the third transformer, so V_{ca} is also the same and we have a three-phase supply, and these voltages are: $V_{ab} = 220\angle 0°$ V, $V_{bc} = 220\angle 240°$ V and $V_{ca} = 220\angle 120°$ V. Then, $V_{an} = 127\angle -30°$ V, $V_{bn} = 127\angle 210°$ V and $V_{cn} = 127\angle 90°$ V. The line currents are

$$I_a = \frac{10,000}{\sqrt{3} \times 220}\angle -30° = 26.24\angle -30° \text{ A}$$
$$I_b = 26.24\angle 210° \text{ A}$$
$$I_c = 26.24\angle 90° \text{ A}$$

(b) kVA$_{\text{supplied}} = 220 \times 26.24 \times 10^{-3} = 5.772$ kVA

(c) The load must be reduced to $(5.0/5.772) \times 100 = 86.6\%$ or 4.33 kW for each transformer.

(d) The current and voltage in each of the remaining two transformers are not in phase. Output of each transformer before the reduction in load is,

$$S_1 = V_{ab}I_a^* = 220\angle 0° \times 26.24\angle 30° = 5000 + j2886 \text{ VA}$$
$$S_2 = V_{cb}I_c^* = 220\angle 60° \times 26.24\angle 270° = 5000 - j2886 \text{ VA}$$

Note that Q is equal in magnitude but opposite in sign. There is no Q output from the open delta. After the load reduction,

$$S_1 = 4333 + j2500 \text{ VA}$$
$$S_2 = 4333 - j2500 \text{ VA}$$

2.14 A transformer rated 200 MVA, 345Y/20.5Δ kV connects a balanced load rated 180 MVA, 22.5 kV, 0.8 power factor lag to a transmission line. Determine

(a) the rating of each of three single-phase transformers which when properly connected will be equivalent to the above three-phase transformer and

(b) the complex impedance of the load in per unit in the impedance diagram if the base in the transmission line is 100 MVA, 345 kV.

Solution:

(a) Each single-phase transformer is rated $200/3 = 66.7$ MVA. Voltage rating is $(345/\sqrt{3})/20.5$ or 199.2/20.5 kV.

(b)

$$\text{Load } Z = \frac{(22.5)^2}{180}\underline{/\cos^{-1}0.8} = 2.81\underline{/36.87°} \ \Omega \ \text{(low-voltage side)}$$

At the load,

$$\text{Base } V = 20.5 \text{ kV}$$
$$\text{Base } Z = \frac{(20.5)^2}{100} = 4.20 \ \Omega$$
$$\text{Load } Z = \frac{2.81}{4.20}\underline{/36.87°} = 0.669\underline{/36.87°} \ \text{per unit}$$

2.15 A three-phase transformer rated 5 MVA, 115/13.2 kV has per-phase series impedance of $(0.007 + j0.075)$ per unit. The transformer is connected to a short distribution line which can be represented by a series impedance per phase of $(0.02 + j0.10)$ per unit on a base of 10 MVA, 13.2 kV. The line supplies a balanced three-phase load rated 4 MVA, 13.2 kV, with lagging power factor 0.85.

(a) Draw an equivalent circuit of the system indicating all impedances in per unit. Choose 10 MVA, 13.2 kVA as the base at the load.

(b) With the voltage at the primary side of the transformer held constant at 115 kV, the load at the receiving end of the line is disconnected. Find the voltage regulation at the load.

Solution:

(a) Base voltages are shown on the single-line diagram.
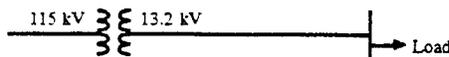
$$\text{Transformer } Z = \frac{10}{5}(0.007 + j0.075) = 0.014 + j0.150 \text{ per unit}$$

$$V_S = 1.0 \text{ per unit}$$

$$\text{Line } Z = 0.02 + j0.10 \text{ per unit}$$

$$\text{Load } |Z| = \frac{(13.2)^2 \times 1000}{3400/0.85} = 43.56 \ \Omega$$

$$\text{Base } Z \text{ at load} = \frac{(13.2)^2}{10} = 17.42 \ \Omega$$

$$\text{Load } Z = \frac{43.56}{17.42} \underline{/\cos^{-1} 0.85} = 2.50 \underline{/31.8°}$$

$$= 2.125 + j1.317 \text{ per unit}$$
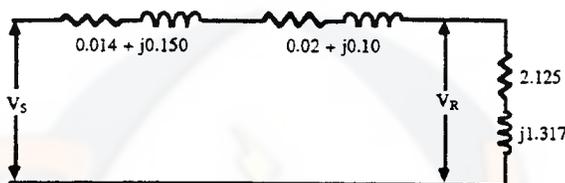
(values are in per unit)

(b) Voltage regulation calculations

$$I = \frac{1.0}{0.014 + 0.02 + 2.125 + j(0.150 + 0.10 + 1.317)} = \frac{1.0}{2.668 \underline{/35.97°}}$$

$$= 0.375 \underline{/-35.97°} \text{ per unit}$$

$$V_{R,FL} = 0.375 \underline{/-35.97°} \times 2.5 \underline{/31.8°} = 0.937 \underline{/-4.17°} \text{ per unit}$$

$$V_{R,NL} = V_S = 1.0$$

$$\text{V.R.} = \frac{1 - 0.937}{0.937} \times 100 = 6.72 \%$$

2.16 Three identical single-phase transformers, each rated 1.2 kV/120 V, 7.2 kVA and having a leakage reactance of 0.05 per unit, are connected together to form a three-phase bank. A balanced Y-connected load of 5 Ω per phase is connected across the secondary of the bank. Determine the Y-equivalent per-phase impedance (in ohms and in per unit) seen from the primary side when the transformer bank is connected (a) Y-Y, (b) Y-Δ, (c) Δ-Y and (d) Δ-Δ. Use Table 2.1.

Solution:

(a) Y-Y connection:

$$|V_{LL}| = 1.2 \times 10^3 \times \sqrt{3} \ \text{V}$$
$$|V_{ll}| = 120\sqrt{3} \ \text{V}$$
$$R'_L = 5 \times \left(\frac{1200\sqrt{3}}{120\sqrt{3}}\right)^2 = 500 \ \Omega$$
$$Z_b = \frac{(1.2\sqrt{3})^2 \times 10^6}{7.2 \times 10^3 \times 3} = 200 \ \Omega$$
$$X_l = 0.05 \ \text{per unit} = 200 \times 0.05 \ \Omega = 10 \ \Omega$$
$$Z'_L = (500 + j10) \ \Omega$$
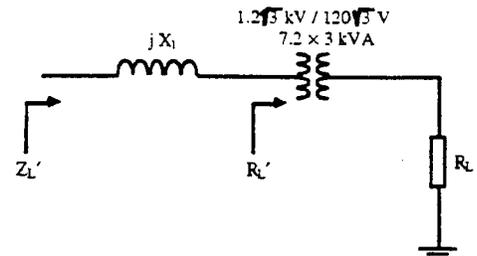
(b) Y-Δ connection:

$$|V_{LL}| = 1200 \times \sqrt{3} \ \text{V}$$
$$|V_{ll}| = 120 \ \text{V}$$
$$R'_L = 5 \times \left(\frac{1200\sqrt{3}}{120}\right)^2 = 1500 \ \Omega$$
$$X_l = 10 \ \Omega \ \text{from part} \ (a)$$
$$Z'_L = (1500 + j10) \ \Omega$$
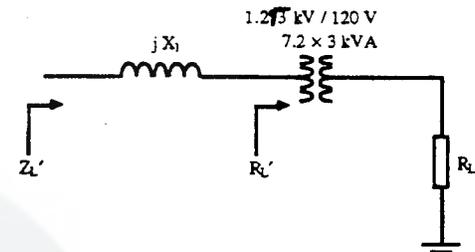
(c) Δ-Y connection:

$$|V_{LL}| = 1200 \ \text{V}$$
$$|V_{ll}| = 120\sqrt{3} \ \text{V}$$
$$R'_L = 5 \times \left(\frac{1200}{120\sqrt{3}}\right)^2 = \frac{500}{3} = 166.67 \ \Omega$$
$$Z_b = \frac{1200^2}{7.2 \times 3 \times 10^3} = 66.67 \ \Omega$$
$$X_l = 0.05 \ \text{per unit} = 66.67 \times 0.05 \ \Omega = 3.33 \ \Omega$$
$$Z'_L = (166.67 + j3.33) \ \Omega$$
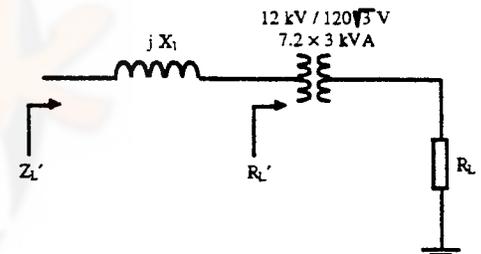
(d) Δ-Δ connection:

$$|V_{LL}| = 1200 \ \text{V}$$
$$|V_{ll}| = 120 \ \text{V}$$
$$R'_L = 5 \times \left(\frac{1200}{120}\right)^2 = 500 \ \Omega$$
$$X_l = 3.33 \ \Omega \ \text{from part} \ (c)$$
$$Z'_L = (500 + j3.33) \ \Omega$$
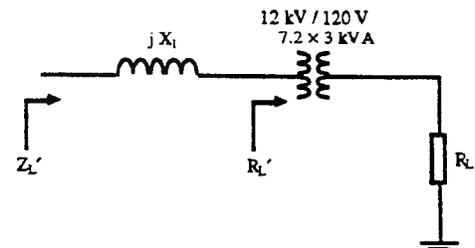
2.17 Figure 2.17a shows a three-phase generator supplying a load through a three-phase transformer rated 12 kVA/600 V Y, 600 kVA. The transformer has per-phase leakage reactance of 10%. The line-to-line voltage and the line current at

the generator terminals are 11.9 kV and 20 A, respectively. The power factor seen by the generator is 0.8 lagging and the phase sequence of supply is ABC.

(a) Determine the line current and the line-to-line voltage at the load, and the per-phase (equivalent-Y) impedance of the load.

(b) Using the line-to-neutral voltage V_A at the transformer primary as reference, draw complete per-phase phasor diagrams of all voltages and currents. Show the correct phase relations between primary and secondary quantities.

(c) Compute the real and reactive power supplied by the generator and consumed by the load.

Solution:

(a)
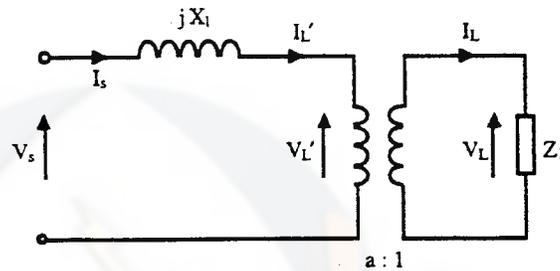
$$\text{Voltage ratio} = a = \frac{12 \times 10^3}{600}\angle 30° = 20\angle 30°$$

$$\text{Current ratio} = \frac{1}{a^*} = 0.05\angle 30°$$

$$X_l = \frac{(12 \times 10^3)^2}{600 \times 10^3} \times 0.1 = 24.0\ \Omega$$

$$\text{Let } V_S = \frac{11.9}{\sqrt{3}}\angle 0°\ \text{kV} = 6.87\ \text{kV}$$

$$\text{Then, } I_S = I_L' = 20\angle -36.9°\ \text{A}$$

$$I_L = I_L' a^* = 20 \times 20\angle -36.9° - 30°\ \text{A} = 400\angle -66.9°\ \text{A}$$

$$V_L' = V_S - jX_l I_S = 6.87\angle 0° - \left(\frac{24.0\angle 90° \times 20\angle -36.9°}{1000}\right)\ \text{kV}$$

$$= 6.593\angle -3.34°\ \text{kV}$$

$$V_L = V_L'/a = \frac{6.593\angle -3.34°}{20\angle 30°}\ \text{kV} = 329.65\angle -33.34°\ \text{V}$$

$$\text{Line voltage at the load} = \sqrt{3}|V_L| = 571\ \text{V}$$

$$\text{Line current at the load} = |I_L| = 400\ \text{A}$$

$$\text{Load impedance - } Z_L = V_L/I_L = \frac{329.65\angle -33.34°}{400\angle -66.9°}\ \Omega$$

$$= 0.824\angle 33.6°\ \Omega$$

(b)
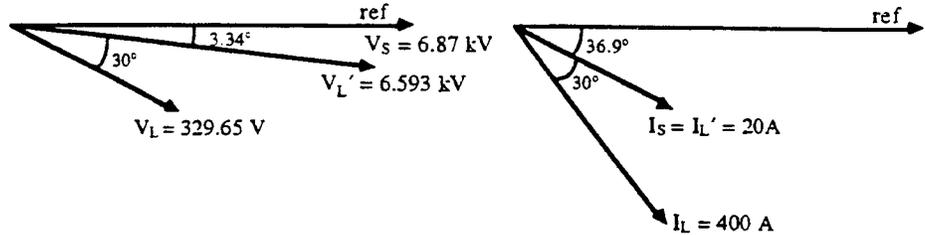
(c) $P_g + jQ_g$ from the generator is $3V_S I_S^*$, where

$$3V_S I_S^* = 3 \times 6.87 \angle 0° \times 20 \angle 36.9° \text{ kVA} = 412.2 \angle 36.9° \text{ kVA}$$
$$= 329.8 \text{ kW} + j247.3 \text{ kvar}$$

$P_L + jQ_L$ by the load is $3V_L I_L^*$, where

$$3V_L I_L^* = \frac{3 \times 329.65 \angle -33.34° \times 400 \angle 66.9°}{1000} \text{ kVA} = 395.6 \angle 33.56° \text{ kVA}$$
$$= 329.7 \text{ kW} + j218.7 \text{ kvar}$$

2.18 Solve Prob. 2.17 with phase sequence ACB.

Solution:

(a) Final answers remain the same except for the following intermediate results:

$$a = 20 \angle -30° \qquad 1/a^* = 0.05 \angle -30°$$
$$I_L = I_L' a^* = 400 \angle -36.9° + 30° \text{ A} = 400 \angle -6.9° \text{ A}$$
$$V_L = V_L'/a = 329.65 \angle -3.34° + 30° \text{ V} = 329.65 \angle 26.7° \text{ V}$$

(b)
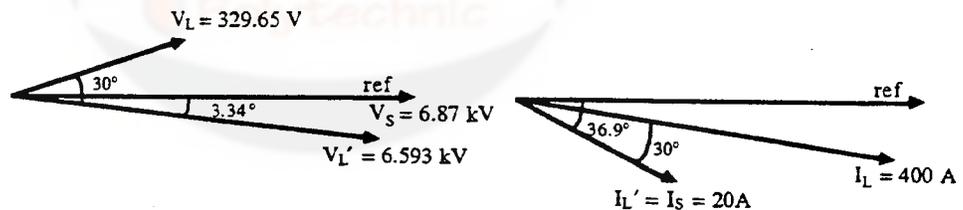
(c) Same results as in Problem 2.17.

2.19 A single-phase transformer rated 30 kVA, 1200/120 V is connected as an auto-transformer to supply 1320 V from a 1200 V bus.

(a) Draw a diagram of the transformer connections showing the polarity marks on the windings and directions chosen as positive for current in each winding so that the currents will be in phase.

(b) Mark on the diagram the values of rated current in the windings and at the input and output.

(c) Determine the rated kilovoltamperes of the unit as an autotransformer.

(d) If the efficiency of the transformer connected for 1200/120 V operation at rated load unity power factor is 97%, determine its efficiency as an autotransformer with rated current in the windings and operating at rated voltage to supply a load at unity power factor.

Solution:
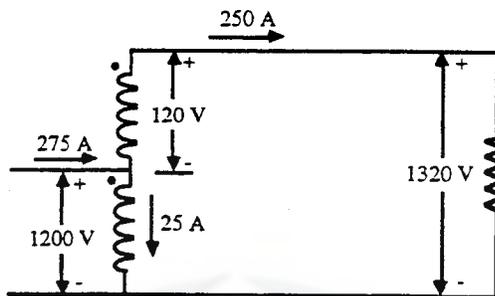
$$\text{rated } I_{HV} = \frac{30,000}{1200} = 25 \text{ A}$$

$$\text{rated } I_{LV} = \frac{30,000}{120} = 250 \text{ A}$$

Connected for 1200/120-V operation (regular transformer),

$$P_{out} = 30,000 \text{ W} \qquad P_{in} = 30,928 \text{ W}$$
$$\text{Loss} = 928 \text{ W}$$

Loss remains the same in the autotransformer because current in the windings and voltage across the windings are unchanged. For the autotransformer,

$$P_{out} = 250 \times 1320 = 330,000 \text{ W} \qquad P_{in} = 330,928 \text{ W}$$
$$\eta = \frac{330,000}{330,928} \times 100 = 99.7\% \qquad \text{Rated kVA} = 330,000$$

Note that, once we consider loss, we no longer have an ideal transformer; and both winding resistance and reactance as well as magnetizing current and core loss must be considered. The applied voltage and input current will be greater than the values shown to achieve rated output, in which case the equivalent circuit corresponding to Fig. 2.7 would be used.

2.20 Solve Prob. 2.19 if the transformer is to supply 1080 V from a 1200 V bus.

Solution:
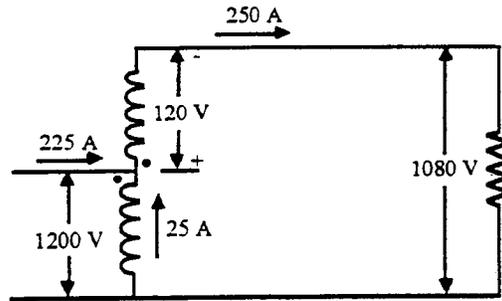
As in Prob. 2.19, Loss = 928 W. As an autotransformer,

$$P_{out} = 250 \times 1080 = 270,000 \text{ W}$$
$$P_{in} = 270,928 \text{ W}$$
$$\eta = \frac{270.000}{270,928} \times 100 = 99.7 \%$$

Rated kVA = 270,000, but see the note which accompanies the solution of Problem 2.19.

2.21 Two buses ⓐ and ⓑ are connected to each other through impedances $X_1 = 0.1$ and $X_2 = 0.2$ per unit in parallel. Bus b is a load bus supplying a current $I = 1.0 \angle -30°$ per unit. The per-unit bus voltage V_b is $1.0 \angle 0°$. Find P and Q into bus b through each of the parallel branches (a) in the circuit described, (b) if a regulating transformer is connected at bus b in the line of higher reactance to give a boost of 3% in voltage magnitude toward the load $(a = 1.03)$, and (c) if the regulating transformer advances the phase 2° $\left(a = \epsilon^{j\pi/90}\right)$. Use the circulating-current method for parts (b) and (c), and assume that V_a is adjusted for each part of the problem so that V_b remains constant. Figure 2.26 is the single-line diagram showing buses a and b of the system with the regulating transformer in place. Neglect the impedance of the transformer.

Solution:
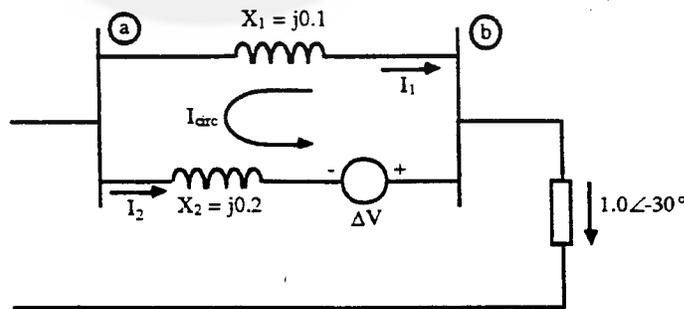
(a) Thru X_1 the current is $I_1 = \frac{2}{3} \times 1.0 \angle -30° = 0.577 - j0.333$ and thru X_2 the current is $I_2 = \frac{1}{3} \times 1.0 \angle -30° = 0.289 - j0.167$. Into bus ⓑ thru X_1,

$$P + jQ = V_b I_1^* = 0.577 + j0.333 \text{ per unit}$$

and into bus ⓑ thru X_2,

$$P + jQ = V_b I_2^* = 0.289 + j0.167 \quad \text{per unit}$$

(b) $\Delta V = 0.03; \quad I_{circ} = \dfrac{0.03}{j0.3} = -j0.1$

$$I_1 = 0.577 - j0.333 - (-j0.1) = 0.577 - j0.233$$
$$I_2 = 0.289 - j0.167 + (-j0.1) = 0.289 - j0.267$$

Into bus ⓑ thru X_1,

$$P + jQ = 0.577 + j0.233 \quad \text{per unit}$$

and into bus ⓑ thru X_2,

$$P + jQ = 0.289 + j0.267 \quad \text{per unit}$$

(c)

$$\Delta V = 1.0 \underline{/2°} - 1.0 = 0.9994 + j0.0349 - 1.0 = -0.0006 + j0.0349$$
$$I_{circ} = \frac{-0.0006 + j0.0349}{j0.3} = 0.116 + j0.002$$
$$I_1 = 0.577 - j0.333 - (0.116 + j0.002) = 0.461 - j0.335$$
$$I_2 = 0.289 - j0.167 + 0.116 + j0.002 = 0.405 - j0.165$$

Into bus ⓑ thru X_1,

$$P + jQ = V_b I^* = 0.461 + j0.335 \quad \text{per unit}$$

and into bus ⓑ thru X_2,

$$P + jQ = V_b I^* = 0.405 + j0.165 \quad \text{per unit}$$

Note: Compare P and Q found in parts (b) and (c) with part (a).

2.22 Two reactances $X_1 = 0.08$ and $X_2 = 0.12$ per unit are in parallel between two buses ⓐ and ⓑ in a power system. If $V_a = 1.05 \underline{/10°}$ and $V_b = 1.0 \underline{/0°}$ per unit, what should be the turns ratio of the regulating transformer to be inserted in series with X_2 at bus ⓑ so that no vars flow into bus ⓑ from the branch whose reactance is X_1? Use the circulating-current method, and neglect the reactance of the regulating transformer. P and Q of the load and V_b remain constant.

Solution:

In reactance X_1,

$$I_{ab} = \frac{1.05 \underline{/10°} - 1.0}{j0.08} = \frac{1.034 + j0.1823 - 1.0}{j0.08} = 2.279 - j0.425$$

To eliminate vars to bus ⓑ thru X_1, we need in the X_2 branch

$$I_{ab,circ} = -j0.425$$
$$\frac{\Delta V}{j0.8 + j0.12} = -j0.425$$
$$a - 1 = \Delta V = -j0.425(j0.08 + j0.12) = 0.0850$$
$$a = 1.085 \quad \text{turns ratio}$$

2.23 Two transformers each rated 115Y/13.2Δ kV operate in parallel to supply a load of 35 MVA, 13.2 kV at 0.8 power factor lagging. Transformer 1 is rated 20 MVA with $X = 0.09$ per unit, and transformer 2 is rated 15 MVA with $X = 0.07$ per unit. Find the magnitude of the current in per unit through each transformer, the megavoltampere output of each transformer, and the mega-voltamperes to which the total load must be limited so that neither transformer is overloaded. If the taps on transformer 1 are set at 111 kV to give a 3.6% boost in voltage toward the low-voltage side of that transformer compared to transformer 2, which remains on the 115 kV tap, find the megavoltampere output for the original 35 MVA total load and the maximum megavoltamperes of the total load which will not overload the transformers. Use a base of 35 MVA, 13.2 kV on the low-voltage side. The circulating-current method is satisfactory for this problem.

Solution:

Converting to the chosen base,

$$X_1 = 0.09 \times (35/20) = 0.1575 \text{ per unit}$$
$$X_2 = 0.07 \times (35/15) = 0.1633 \text{ per unit}$$
$$|S_1| = \frac{0.1633}{0.1575 + 0.1633} \times 35 = 17.8 \text{ MVA}$$
$$|S_2| = \frac{0.1575}{0.1575 + 0.1633} \times 35 = 17.2 \text{ MVA}$$

Unit #2 is overloaded, and therefore reduce load to $(15/17.2) \times 35 = 30.5$ MVA.

Currents with 35-MVA load:

$$I_1 = \frac{35}{17.8}(0.8 - j0.6) = 0.407 - j0.305$$
$$I_2 = \frac{35}{17.2}(0.8 - j0.6) = 0.393 - j0.295$$

With 3.6% magnitude boost,

$$I_{circ} = \frac{j(0.1633 + 0.1575)}{0.036} = -j0.112$$
$$|I_1 + I_{circ}| = |0.407 - j0.417| = 0.583$$
$$|I_2 - I_{circ}| = |0.393 - j0.185| = 0.434$$
$$|S_1| = 0.583 \times 1.0 \times 35 = 20.4 \text{ MVA}$$
$$|S_2| = 0.434 \times 1.0 \times 35 = 15.19 \text{ MVA}$$
$$\frac{20.4}{20} = 1.020 \qquad \frac{15.19}{15} = 1.013$$

Reduce load to $(35/1.02) = 34.3$ MVA.
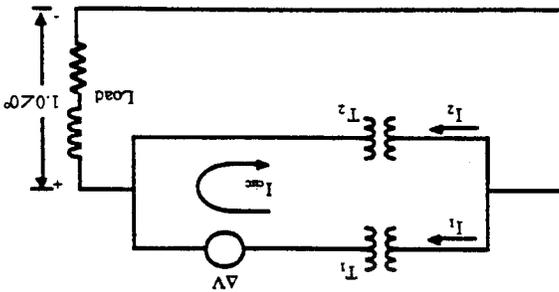
Chapter 3 Problem Solutions

3.1 Determine the highest speed at which two generators mounted on the same shaft can be driven so that the frequency of one generator is 60 Hz and the frequency of the other is 25 Hz. How many poles does each machine have?

Solution:

Let P = number of poles:

$$\text{speed} = \frac{2 \times 60 \times 60}{P_{60}} = \frac{2 \times 60 \times 25}{P_{25}}$$

$$\frac{P_{60}}{P_{25}} = \frac{60}{25}$$

P_{60} and P_{25} must be even integral numbers lowest value where $P_{60} = 2.4 P_{25}$. Thus,

$$P_{25} = 10 \qquad P_{60} = 24$$

3.2 The three-phase synchronous generator described in Example 3.1 is operated at 3600 rpm and supplies a unity power factor load. If the terminal voltage of the machine is 22 kV and the field current is 2500 A, determine the line current and the total power consumption of the load.

Solution:

Using the values in the solution of Example 3.1,

$$e_{a'_{\max}} = \frac{45855}{3838} \times 2500 \text{ V} = 29869.1 \text{ V}$$

Given:

$$V_{LL} = 22 \text{ kV}$$
$$v_{a_{\max}} = (\sqrt{2}/\sqrt{3}) \times 22000 \text{ V} = 17962.9 \text{ V}$$

If $v_a = 17962.9 \cos \omega t$, then $i_a = i_{a_{\max}} \cos \omega t$ and

$$e_{a'} = 17962.9 \cos \omega t - 4.1484 \times 10^{-3} \times 120\pi \times i_{a_{\max}} \sin \omega t$$
$$= 17962.9 \cos \omega t - 1.5639 \, i_{a_{\max}} \sin \omega t$$
$$e_{a'_{\max}} = \sqrt{17962.9^2 + (1.5639 i_{a_{\max}})^2} = 29869.1 \text{ V}$$

Hence, $\quad i_{a_{\max}} = 15259.4 \text{ A}$

$$I_a = i_{a_{\max}}/\sqrt{2} = 10.79 \text{ kA}$$
$$P_{3\phi} = \sqrt{3} \times 22 \times 10.79 \times 1 \text{ MW} = 411.2 \text{ MW}$$

3.3 A three-phase round-rotor synchronous generator has negligible armature resistance and a synchronous reactance X_d of 1.65 per unit. The machine is connected directly to an infinite bus of voltage $1.0 \angle 0°$ per unit. Find the internal voltage E_i of the machine when it delivers a current of (a) $1.0 \angle 30°$ per

unit, (b) $1.0\angle 0°$ per unit and (c) $1.0\angle -30°$ per unit to the infinite bus. Draw phasor diagrams depicting the operation of the machine in each case.

Solution:

$$
\begin{aligned}
E_i\angle 0° &= V_a\angle 0° + I_a\angle\theta\ X_d\angle 90°\\
&= 1.0\angle 0° + 1.0\angle\theta \times 1.65\angle 90°\\
&= 1.0\angle 0° + 1.65\angle 90° + \theta
\end{aligned}
$$

(a)

$$
\begin{aligned}
\theta &= 30°\\
E_i\angle\delta &= 1.0\angle 0° + 1.65\angle 120°\\
&= 1.44\angle 83° \text{ per unit}
\end{aligned}
$$
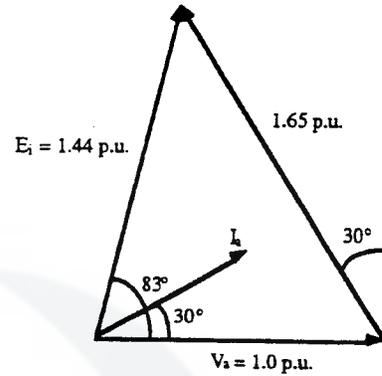
(b)

$$
\begin{aligned}
\theta &= 0°\\
E_i\angle\delta &= 1.0\angle 0° + 1.65\angle 90°\\
&= 1.93\angle 58.8° \text{ per unit}
\end{aligned}
$$
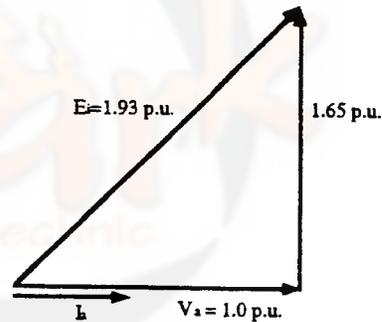
(c)

$$
\begin{aligned}
\theta &= -30°\\
E_i\angle\delta &= 1.0\angle 0° + 1.65\angle 60°\\
&= 2.32\angle 38° \text{ per unit}
\end{aligned}
$$
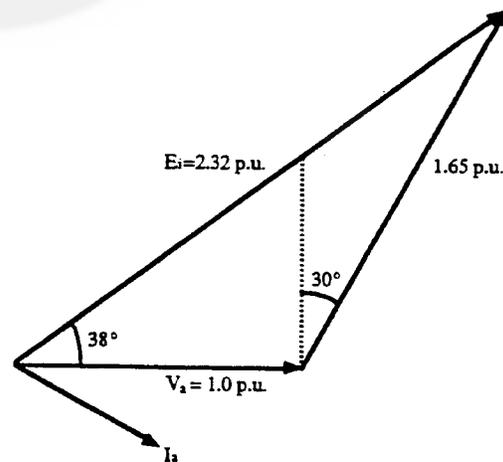
3.4 A three-phase round-rotor synchronous generator, rated 10 kV, 50 MVA has armature resistance R of 0.1 per unit and synchronous reactance X_d of 1.65 per unit. The machine operates on a 10 kV infinite bus delivering 2000 A at 0.9 power factor leading.

   (a)   Determine the internal voltage E_i and the power angle δ of the machine. Draw a phasor diagram depicting its operation.

   (b)   What is the open-circuit voltage of the machine at the same level of excitation?

   (c)   What is the *steady-state* short-circuit current at the same level of excitation? Neglect all saturation effects.

Solution:

   (a)   Choosing $V_b = 10$ kV and $MVA_b = 50$ MVA:

$$Z_d = (0.1 + j1.65) \text{ per unit} = 1.653\underline{/86.53°} \triangleq Z_d\underline{/\alpha}$$

$$V = \frac{10}{10} \text{ per unit} = 1.0 \text{ per unit}$$

$$I_b = \frac{50 \times 10^3}{\sqrt{3} \times 10} \text{ A} = 2886.75 \text{ A}$$

$$I_a = \frac{2000}{2886.75} \text{ per unit} = 0.693 \text{ per unit}$$

$$\theta = \cos^{-1} 0.9 = 25.8°$$

$$E_i\underline{/\delta} = V\underline{/0°} + I_a Z_d\underline{/\alpha + \theta}$$
$$= 1.0\underline{/0°} + 0.693 \times 1.6530\underline{/112.37°} \text{ per unit}$$
$$= 1.195\underline{/61.83°} = 11.95\underline{/61.83°} \text{ kV}$$
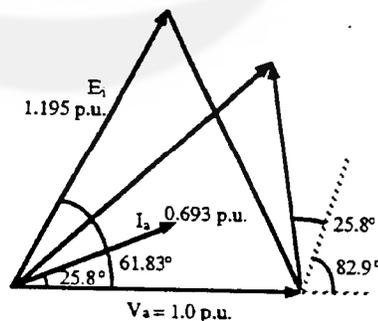
   (b)   Open-circuit voltage: $E_i = 11.95$ kV

   (c)   Short-circuit voltage:

$$\frac{E_i}{Z_d} = \frac{1.195}{1.653} \text{ per unit} = 0.7242 \text{ per unit} = 2090.7 \text{ A}$$

3.5 A three-phase round-rotor synchronous generator, rated 16 kV and 200 MVA, has negligible losses and synchronous reactance of 1.65 per unit. It is operated on an infinite bus having a voltage of 15 kV. The internal emf E_i and the power angle δ of the machine are found to be 24 kV (line-to-line) and 27.4°, respectively.

(a) Determine the line current and the three-phase real and reactive power being delivered to the system.

(b) If the mechanical power input and the field current of the generator are now changed so that the line current of the machine is reduced by 25% at the power factor of (a), find the new internal emf E_i and the power angle δ.

(c) While delivering the reduced line current of (b), the mechanical power input and the excitation are further adjusted so that the machine operates at unity power factor at its terminals. Calculate the new values of E_i and δ.

Solution:

(a) Using 16 kV, 200 MVA base;

$$V_a = 15/16 \text{ per unit} = 0.9375 \text{ per unit}$$

$$E_i\angle\delta = \frac{24}{16}\angle 27.4° \text{ per unit} = 1.5\angle 27.4° \text{ per unit}$$

$$E_i\angle\delta - V_a\angle 0° = I_a X_d\angle 90° - \theta$$

$$1.5\angle 27.4° - 0.9375\angle 0° = I_a \times 1.65\angle 90° - \theta$$

$$I_a\angle 90° - \theta = 0.4818\angle 60.27° \text{ per unit}$$

$$I_a\angle -\theta = 0.4818\angle -29.73° \text{ per unit}$$

$$\text{Base } I = \frac{200 \times 10^3}{\sqrt{3} \times 16} \text{ kA} = 7.217 \text{ kA}$$

$$\text{Therefore, } I_a = 0.4818 \times 7.217 \text{ kA} = 3.477 \text{ kA}$$

$$S = 0.9375 \times 0.4818 \text{ per unit} = 0.4517 \text{ per unit}$$

$$= 90.34 \text{ MVA}$$

$$\text{Thus, } P = 90.34 \cos 29.73° \text{ MW} = 78.45 \text{ MW}$$

$$Q = 90.34 \sin 29.73° \text{ Mvar} = 44.80 \text{ Mvar}$$

(b)

$$\text{New } I_a = 0.75 \times 0.4818 \text{ per unit} = 0.3614 \text{ per unit}$$

$$90 - \theta = 60.27°$$

$$E_i\angle\delta = V_a\angle 0° + I_a X_d\angle 90° - \theta$$

$$= 0.9375\angle 0° + 0.3614 \times 1.65\angle 60.27° = 1.337\angle 22.8° \text{ per unit}$$

$$= 21.4\angle 22.8° \text{ kV L-L}$$

(c)

$$\text{New } I_a = 0.3614 \text{ per unit} \qquad \theta = 0°$$

$$E_i \angle \delta = 0.9375 \angle 0° + 0.3614 \times 1.65 \angle 90°$$

$$= 1.111 \angle 32.5° \text{ per unit} = 17.8 \angle 32.5° \text{ kV}$$
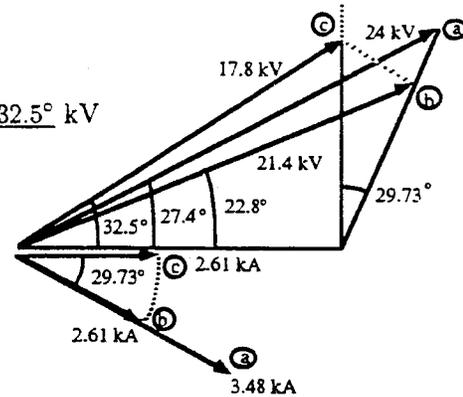
3.6 The three-phase synchronous generator of Prob. 3.5 is operated on an infinite bus of voltage 15 kV and delivers 100 MVA at 0.8 power factor lagging.

(a) Determine the internal voltage E_i, power angle δ and the line current of the machine.

(b) If the field current of the machine is reduced by 10%, while the mechanical power input to the machine is maintained constant, determine the new value of δ and the reactive power delivered to the system.

(c) The prime mover power is next adjusted without changing the excitation so that the machine delivers zero reactive power to the system. Determine the new power angle δ and the real power being delivered to the system.

(d) What is the maximum reactive power that the machine can deliver if the level of excitation is maintained as in (b) and (c)?

Draw a phasor diagram for the operation of the machine in cases (a), (b) and (c).

Solution:

(a) From Prob. 3.5,

$$V_a(\text{or } V_t) = 0.9375 \text{ per unit}$$

$$S = 0.5 \text{ per unit} \qquad X_d = 1.65 \text{ per unit} \qquad \theta = -36.9°$$

$$I_a = S/V_a = 0.5/0.9375 \text{ per unit}$$

$$E_i \angle \delta = V_a \angle 0° + I_a X_d \angle 90° + \theta$$

$$= 0.9375 \angle 0° + \frac{0.5}{0.9375} \times 1.65 \angle 90° - 36.9° \text{ per unit}$$

$$= 1.6258 \angle 25.7° \text{ per unit} = 26.0 \angle 25.7° \text{ kV}$$

(b)

$$\text{New } E_i = 0.9 \times 1.6258 \text{ per unit } = 1.46322 \text{ per unit}$$

$$P = 0.5 \times 0.8 \text{ per unit } = 0.4 \text{ per unit}$$

$$\delta = \sin^{-1}\left(\frac{PX_d}{V_t E_i}\right) = \sin^{-1}\left(\frac{0.4 \times 1.65}{0.9375 \times 1.46322}\right) = 28.76°$$

$$\text{New } Q = \frac{V_t}{X_d}(E_i \cos\delta - V_t)$$

$$= \frac{0.9375}{1.65}(1.46322 \cos 28.76° - 0.9375)$$

$$= 0.196 \text{ per unit } = 39.2 \text{ Mvar}$$

(c) When $Q = 0$,

$$\delta = \cos^{-1}\frac{V_t}{E_i} = \cos^{-1}\left(\frac{0.9375}{1.46322}\right) = 50.15°$$

$$P = \frac{E_i V_t}{X_d}\sin\delta = \frac{0.9375 \times 1.46322}{1.65}\sin 50.15° \text{ per unit}$$

$$= 0.638 \text{ per unit } = 127.65 \text{ MW}$$

(d) For V_t, E_i and X_d fixed, Q_{max} occurs when $\delta = 0°$. Hence,

$$Q_{max} = \frac{V_t}{X_d}(E_i - V_t) = \frac{0.9375}{1.65}(1.46322 - 0.9375) \text{ per unit}$$

$$= 0.2787 \text{ per unit } = 59.74 \text{ Mvar}$$
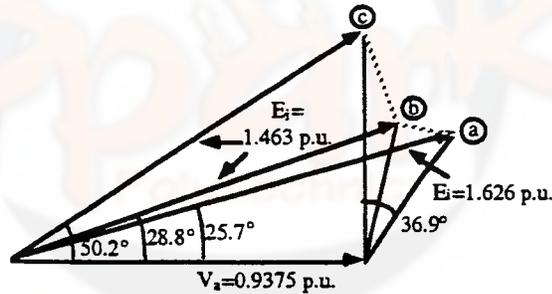
3.7 Starting with Eq. (3.31), modify Eq. (3.38) to show that

$$P = \frac{|V_t|}{R^2 + X_d^2}\left\{|E_i|(R\cos\delta + X_d\sin\delta) - |V_t|R\right\}$$

$$Q = \frac{|V_t|}{R^2 + X_d^2}\left\{X_d(|E_i|\cos\delta - |V_t|) - R|E_i|\sin\delta\right\}$$

when the synchronous generator has non-zero armature resistance R.

Solution:

From Eq. (3.55), $V = E_i + I_a(R + jX_d)$ and, therefore,

$$I_a = \frac{|E_i| \angle \delta - |V_t| \angle 0°}{(R + jX_d)}$$

$$I_a^* = \frac{|E_i| \angle -\delta - |V_t|}{(R - jX_d)}$$

$$S = P + jQ = V_t I_a^* = \frac{|V_t||E_i| \angle -\delta - |V_t|^2}{(R - jX_d)}$$

$$= \frac{|V_t||E_i|(\cos\delta - j\sin\delta) - |V_t|^2}{(R - jX_d)}$$

$$= \frac{|V_t||E_i|(\cos\delta - j\sin\delta)(R + jX_d) - |V_t|^2(R + jX_d)}{(R^2 + X_d^2)}$$

Separating real and imaginary parts,

$$P = \frac{|V_t||E_i|}{(R^2 + X_d^2)}\{R\cos\delta + X_d\sin\delta\} - \frac{|V_t|^2 R}{(R^2 + X_d^2)}$$

$$= \frac{|V_t|}{R^2 + X_d^2}\{|E_i|(R\cos\delta + X_d\sin\delta) - |V_t|R\}$$

$$Q = \frac{|V_t||E_i|}{(R^2 + X_d^2)}\{X_d\cos\delta - R\sin\delta\} - \frac{|V_t|^2 X_d}{(R^2 + X_d^2)}$$

$$= \frac{|V_t|}{R^2 + X_d^2}\{X_d(|E_i|\cos\delta - |V_t|) - R|E_i|\sin\delta\}$$

3.8 The three-phase synchronous generator described in Example 3.4 is now operated on a 25.2 kV infinite bus. It is found that the internal voltage magnitude $|E_i| = 49.5$ kV and that the power angle $\delta = 38.5°$. Using the loading capability diagram of Fig. 3.14, determine graphically the real and reactive power delivered to the system by the machine. Verify your answers using Eqs. (3.38).

Solution:

$$|V_t| = 25.2 \text{ kV} = \frac{25.2}{24} \text{ per unit} = 1.05 \text{ per unit}$$

$$|E_i| = 49.5 \text{ kV} = \frac{49.5}{24} \text{ per unit} = 2.0625 \text{ per unit}$$

The distance n-k on the loading capability diagram is

$$\frac{|E_i|}{|V_t|X_d} = \frac{2.0625}{1.05 \times 1.7241} \text{ units} = 1.1393 \text{ units}$$

The angle formed by points k-n-o is 38.5°. Hence, point k is marked as shown. By reading from the chart, $P_k = 0.7$ per unit and $Q_k = 0.31$ per unit.

$$P + jQ = |V_t|^2[P_k + jQ_k] = 1.05^2(0.7 + j0.31) \text{ per unit}$$

$$= 1.05^2(0.7 + j0.31) \times 635 \text{ MVA}$$

$$P = 490 \text{ MW} \qquad Q = 217 \text{ Mvar}$$

From Eq. (3.38),

$$P = \frac{|V_t||E_i|}{X_d}\sin\delta$$

$$= \frac{1.05 \times 2.0625}{1.7241}\sin(38.5°) \times 635 \text{ MW} = 496.5 \text{ MW}$$

$$Q = \frac{|V_t|}{X_d}(|E_i|\cos\delta - |V_t|)$$

$$= \frac{1.05}{1.7241}(2.0625\cos(38.5°) - 1.05) \times 635 \text{ Mvar} = 218.2 \text{ Mvar}$$

3.9 A three-phase salient-pole synchronous generator with negligible armature resistance has the following values for the inductance parameters specified in Table 3.1,

$$L_s = 2.7656 \text{ mH} \qquad M_f = 31.6950 \text{ mH} \qquad L_m = 0.3771 \text{ mH}$$
$$M_s = 1.3828 \text{ mH} \qquad L_{ff} = 433.6569 \text{ mH}$$

During balanced steady-state operation the field current and a-phase armature current of the machine have the respective values

$$i_f = 4000 \text{ A} \qquad i_a = 20,000\sin(\theta_d - 30°) \text{ A}$$

(a) Using Eq. (3.41), determine the instantaneous values of the flux linkages λ_a, λ_b, λ_c and λ_f when $\theta_d = 60°$.

(b) Using Park's Transformation given by Eqs. (3.42) and (3.43), determine the instantaneous values of the flux linkages λ_d, λ_q and λ_0, and the currents i_d, i_q and i_0 when $\theta_d = 60°$.

(c) Verify results using Eqs. (3.45) – (3.46)

Solution:

(a) From Table 3.1,

$$L_{abc} \triangleq \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} \\ L_{aa} & L_{ab} & L_{ac} \\ L_{aa} & L_{ab} & L_{ac} \end{bmatrix} = \begin{bmatrix} L_s & -M_s & -M_s \\ -M_s & L_s & -M_s \\ -M_s & -M_s & L_s \end{bmatrix}$$

$$+ L_m \begin{bmatrix} \cos 2\theta_d & -\cos 2(\theta_d + 30°) & -\cos 2(\theta_d + 150°) \\ -\cos 2(\theta_d + 30°) & \cos 2(\theta_d - 120°) & -\cos 2(\theta_d - 90°) \\ -\cos 2(\theta_d + 150°) & \cos 2(\theta_d - 90°) & \cos 2(\theta_d + 120°) \end{bmatrix}$$

$$L_{abc} = \begin{bmatrix} 2.7656 & -1.3828 & -1.3828 \\ -1.3828 & 2.7656 & -1.3828 \\ -1.3828 & -1.3828 & 2.7656 \end{bmatrix}$$

$$+ 0.3771 \begin{bmatrix} \cos 120° & -\cos 180° & -\cos 420° \\ -\cos 180° & \cos(-120°) & -\cos(-60°) \\ -\cos 420° & -\cos(-60°) & \cos 360° \end{bmatrix} \text{ mH}$$

$$= \begin{bmatrix} 2.57705 & -1.0057 & -1.57135 \\ -1.0057 & 2.57705 & -1.57135 \\ -1.57135 & -1.57135 & 3.1427 \end{bmatrix} \text{mH}$$

$$L_{abc,f} \triangleq \begin{bmatrix} L_{af} \\ L_{bf} \\ L_{cf} \end{bmatrix} = M_f \begin{bmatrix} \cos \theta_d \\ \cos (\theta_d - 120°) \\ \cos (\theta_d - 240°) \end{bmatrix}$$

$$= 31.695 \begin{bmatrix} \cos 60° \\ \cos (-60°) \\ \cos (-180°) \end{bmatrix} \text{mH} = \begin{bmatrix} 15.8475 \\ 15.8475 \\ -31.695 \end{bmatrix} \text{mH}$$

$$i_{abc} \triangleq \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = 20000 \begin{bmatrix} \sin (30°) \\ \sin (-90°) \\ \sin (-210°) \end{bmatrix} \text{A} = \begin{bmatrix} 10 \\ -20 \\ 10 \end{bmatrix} \text{kA}$$

With $i_f = 4$ kA and $L_{ff} = 433.6569$ mH,

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \\ \lambda_f \end{bmatrix} = \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} & L_{af} \\ L_{ba} & L_{bb} & L_{bc} & L_{bf} \\ L_{ca} & L_{cb} & L_{cc} & L_{cf} \\ L_{fa} & L_{fb} & L_{fc} & L_{ff} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_f \end{bmatrix}$$

$$= \begin{bmatrix} 2.57705 & -1.0057 & -1.57135 & 15.8475 \\ -1.0057 & 2.57705 & -1.57135 & 15.8475 \\ -1.57135 & -1.57135 & 3.1427 & -31.6950 \\ 15.8475 & 15.8475 & -31.6950 & 433.6569 \end{bmatrix} \begin{bmatrix} 10 \\ -20 \\ 10 \\ 4 \end{bmatrix} \text{Wb-T}$$

$$= \begin{bmatrix} 93.5610 \\ -13.9215 \\ -79.6395 \\ 1259.2026 \end{bmatrix} \text{Wb-T}$$

(b) When $\theta_d = 60°$,

$$P = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_d \\ \lambda_q \\ \lambda_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 93.5610 \\ -13.9215 \\ -79.6395 \end{bmatrix}}_{\lambda_{abc}} = \begin{bmatrix} 97.5381 \\ 76.0016 \\ 0 \end{bmatrix} \text{Wb-T}$$

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 10 \\ -20 \\ 10 \end{bmatrix}}_{i_{abc}} = \begin{bmatrix} -12.2474 \\ 21.2132 \\ 0 \end{bmatrix} \text{kA}$$

(c)

$$L_d = L_s + M_s + \tfrac{3}{2}L_m = 4.71405 \text{ mH}$$

$$
\begin{aligned}
L_q &= L_s + M_s - \tfrac{3}{2}L_m = 3.58275 \text{ mH} \\
\sqrt{\tfrac{3}{2}}M_f &= 38.8183 \text{ mH} \\
\lambda_d &= L_d i_d + \sqrt{\tfrac{3}{2}}M_f i_f = 4.71405 \times (-12.2474) + 38.8183 \times 4 \text{ Wb-T} \\
&= 97.5381 \text{ Wb-T} \\
\lambda_q &= L_q i_q = 3.58275 \times 21.2132 \text{ Wb-T} = 76.0016 \text{ Wb-T} \\
\lambda_0 &= L_0 i_0 = 0 \quad (\text{since } i_0 = 0) \\
\lambda_f &= \sqrt{\tfrac{3}{2}}M_f i_d + L_{ff}i_f = 38.8183 \times (-12.2474) + 433.6569 \times 4 \text{ Wb-T} \\
&= 1259.20 \text{ Wb-T}
\end{aligned}
$$

3.10 The armature of a three-phase salient-pole generator carries the currents

$$
\begin{aligned}
i_a &= \sqrt{2} \times 1000 \sin (\theta_d - \theta_a) \text{ A} \\
i_b &= \sqrt{2} \times 1000 \sin (\theta_d - 120° - \theta_a) \text{ A} \\
i_c &= \sqrt{2} \times 1000 \sin (\theta_d - 240° - \theta_a) \text{ A}
\end{aligned}
$$

(a) Using the **P**-Transformation matrix of Eq. (3.42), find the direct-axis current i_d and the quadrature-axis current i_q. What is the zero-sequence current i_0?

(b) Suppose that the armature currents are

$$
\begin{aligned}
i_a &= \sqrt{2} \times 1000 \sin (\theta_d - \theta_a) \text{ A} \\
i_b &= i_c = 0
\end{aligned}
$$

Determine i_d, i_q and i_0.

Solution:

(a)

$$
\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = P \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}
$$

$$
= \sqrt{\tfrac{2}{3}} \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} & -1 \\ \tfrac{\sqrt{3}}{2} & -\tfrac{\sqrt{3}}{2} & 0 \\ \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \end{bmatrix} \times 1000\sqrt{2} \begin{bmatrix} \sin(\theta_d - \theta_a) \\ \sin(\theta_d - \theta_a - 120°) \\ \sin(\theta_d - \theta_a - 240°) \end{bmatrix} \text{ A}
$$

$$
= 1000\sqrt{3} \begin{bmatrix} \sin(\theta_d - \theta_a - 60°) \\ \sin(\theta_d - \theta_a + 30°) \\ 0 \end{bmatrix} \text{ A}
$$

(b)

$$
\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \times 1000\sqrt{2} \begin{bmatrix} \sin(\theta_d - \theta_a) \\ 0 \\ 0 \end{bmatrix} \text{ A}
$$

$$
= \frac{2000}{\sqrt{3}} \begin{bmatrix} \frac{1}{2}\sin(\theta_d - \theta_a) \\ \frac{\sqrt{3}}{2}\sin(\theta_d - \theta_a) \\ \frac{1}{\sqrt{2}}\sin(\theta_d - \theta_a) \end{bmatrix} \text{ A}
$$

3.11 Calculate the direct-axis synchronous reactance X_d, the direct-axis transient reactance X'_d and the direct-axis subtransient reactance X''_d of the 60 Hz salient-pole synchronous machine with the following parameters:

$$
\begin{array}{lll}
L_s = 2.7656 \text{ mH} & L_{ff} = 433.6569 \text{ mH} & L_D = 4.2898 \text{ mH} \\
M_s = 1.3828 \text{ mH} & M_f = 31.6950 \text{ mH} & M_D = 3.1523 \text{ mH} \\
L_m = 0.3771 \text{ mH} & M_r = 37.0281 \text{ mH} &
\end{array}
$$

Solution:

$$
\begin{aligned}
L_d &= L_s + M_s - \frac{3}{2}L_m = 2.7656 + 1.3828 - \frac{3}{2} \times 0.3771 \text{ mH} = 4.71405 \text{ mH} \\
X_d &= 120\pi \times 4.71405 \times 10^{-3} \,\Omega = 1.777 \,\Omega \\
L'_d &= L_d - \frac{3}{2}\frac{M_f^2}{L_{ff}} = 4.71405 - \frac{3}{2} \times \frac{31.6950^2}{433.6569} \text{ mH} = 1.2393 \text{ mH} \\
X'_d &= 120\pi \times 1.2393 \times 10^{-3} \,\Omega = 0.467 \,\Omega \\
L''_d &= L_d - \frac{3}{2}\left(\frac{M_f^2 L_D + M_D^2 L_{ff} - 2M_f M_D M_r}{L_{ff}L_D - M_r^2}\right) \\
&= 4.71405 - \frac{3}{2}\left(\frac{31695^2 \times 4.2898 + 3.1523^2 \times 433.6569 - 2 \times 31.6950 \times 3.1523 \times 37.0281}{433.6569 \times 4.2898 - 37.0281^2}\right) \text{ mH} \\
&= 0.9748 \text{ mH} \\
X''_d &= 120\pi \times 0.9748 \times 10^{-3} \,\Omega = 0.367 \,\Omega
\end{aligned}
$$

3.12 The single-line diagram of an unloaded power system is shown in Fig. 3.22. Reactances of the two sections of transmission line are shown on the diagram. The generators and transformers are rated as follows:

Generator 1:   20 MVA, 13.8 kV, $X_d'' = 0.20$ per unit
Generator 2:   30 MVA, 18 kV, $X_d'' = 0.20$ per unit
Generator 3:   30 MVA, 20 kV, $X_d'' = 0.20$ per unit
Transformer T_1:   25 MVA, 220Y/13.8Δ kV, $X = 10\%$
Transformer T_2:   Single-phase units each rated 10 MVA, 127/18 kV, $X = 10\%$
Transformer T_3:   35 MVA, 220Y/22Y kV, $X = 10\%$

(a) Draw the impedance diagram with all reactances marked in per unit and with letters to indicate points corresponding to the single-line diagram. Choose a base of 50 MVA, 13.8 kV in the circuit of Generator 1.

(b) ·Suppose that the system is unloaded and that the voltage throughout the system is 1.0 per unit on bases chosen in part (a). If a three-phase short circuit occurs from bus C to ground, find the phasor value of the short-circuit current (in amperes) if each generator is represented by its subtransient reactance.

(c) Find the megavoltamperes supplied by each generator under the conditions of part (b).

Solution:

(a)
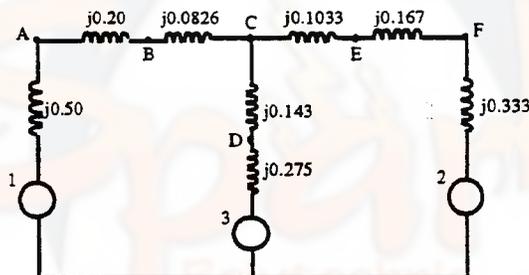
$$\text{Gen 1: } X'' = 0.2 \times \frac{50}{20} = 0.50 \text{ per unit}$$

$$3\phi \text{ rating } T_2 = 220/18 \text{ kV, 30 MVA}$$

$$\text{Base in trans. line:} \quad 220 \text{ kV, 50 MVA}$$

$$\text{Base for Gen 2} = 18 \text{ kV}$$

$$\text{Gen 2: } X'' = 0.2 \times \frac{50}{30} = 0.333 \text{ per unit}$$

$$\text{Base for Gen 3} = 22 \text{ kV}$$

$$\text{Gen 3: } X'' = 0.2 \left(\frac{20}{22}\right)^2 \times \frac{50}{30} = 0.275 \text{ per unit}$$

$$\text{Transformer } T_1: X = .01 \times \frac{50}{25} = 0.20 \text{ per unit}$$

$$\text{Transformer } T_2: X = .01 \times \frac{50}{30} = 0.167 \text{ per unit}$$

$$\text{Transformer } T_3: X = .01 \times \frac{50}{35} = 0.143 \text{ per unit}$$

Transmission lines:

$$\text{Base } Z = \frac{220^2}{50} = 968 \ \Omega$$

$$\frac{80}{968} = 0.0826 \text{ per unit} \qquad \frac{100}{968} = 0.1033 \text{ per unit}$$

(b)
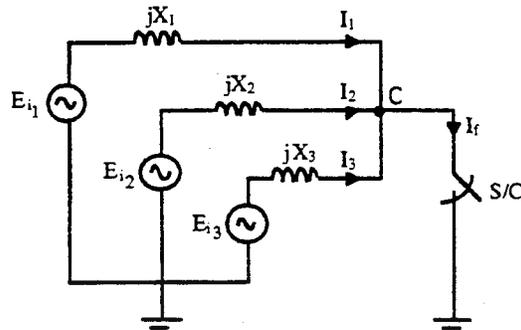
$$E_{i_1} = E_{i_2} = E_{i_3} = 1.0 \angle 0° \text{ per unit}$$

$$X_1 = 0.50 + 0.20 + 0.0826 \text{ per unit} = 0.7826 \text{ per unit}$$

$$X_2 = 0.333 + 0.167 + 0.1033 \text{ per unit} = 0.6033 \text{ per unit}$$

$$X_3 = 0.143 + 0.275 \text{ per unit} = 0.418 \text{ per unit}$$

$$I_1 = \frac{E_{i_1}}{jX_1} = \frac{1}{0.7826} \angle -90° \text{ per unit} = 1.278 \angle -90° \text{ per unit}$$

$$I_2 = \frac{E_{i_2}}{jX_2} = \frac{1}{0.6033} \angle -90° \text{ per unit} = 1.658 \angle -90° \text{ per unit}$$

$$I_3 = \frac{E_{i_3}}{jX_3} = \frac{1}{0.418} \angle -90° \text{ per unit} = 2.392 \angle -90° \text{ per unit}$$

$$I_f = I_1 + I_2 + I_3 = (1.278 + 1.658 + 2.392) \angle -90° \text{ per unit} = 5.328 \angle -90° \text{ per unit}$$

$$I_{\text{base}} \text{ at C} = \frac{50 \times 10^6}{\sqrt{3} \times 220 \times 10^3} \text{ A} = 131.22 \text{ A}$$

$$|I_f| = 5.328 \times 131.22 \text{ A} = 699 \text{ A}$$

(c)

$$|S_1| = E_{i_1} I_1 = 1.0 \times 1.278 \times 50 \text{ MVA} = 63.9 \text{ MVA}$$

$$|S_2| = E_{i_2} I_2 = 1.0 \times 1.658 \times 50 \text{ MVA} = 82.9 \text{ MVA}$$

$$|S_3| = E_{i_3} I_3 = 1.0 \times 2.392 \times 50 \text{ MVA} = 119.6 \text{ MVA}$$

3.13 The ratings of the generators, motors and transformers of Fig. 3.23 are:

Generator 1:	20 MVA, 18 kV, $X_d'' = 20\%$
Generator 2:	20 MVA, 18 kV, $X_d'' = 20\%$
Synchronous motor 3:	30 MVA, 13.8 kV, $X_d'' = 20\%$
Three-phase Y-Y transformers:	20 MVA, 138Y/20Y kV, $X = 10\%$
Three-phase Y-Δ transformers:	15 MVA, 138Y/13.8Δ kV, $X = 10\%$

(a) Draw the impedance diagram for the power system. Mark impedances in per unit. Neglect resistance and use a base of 50 MVA, 138 kV in the 40-Ω line.

(b) Suppose that the system is unloaded and that the voltage throughout the system is 1.0 per unit on bases chosen in part (a). If a three-phase short circuit occurs from bus C to ground, find the phasor value of the short-circuit current (in amperes) if each generator is represented by its subtransient reactance.

(c) Find the megavoltamperes supplied by each synchronous machine under the conditions of part (b).

Solution:

(a)
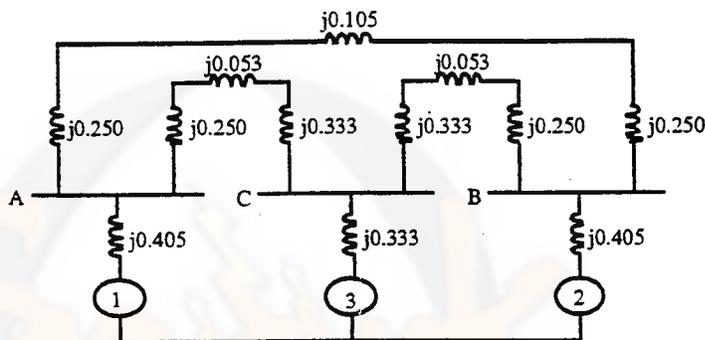
Base voltages are:

40 Ω lines	138 kV
20 Ω lines	138 kV
Gen. 1 & 2	20 kV
Motor 3	13.8 kV

$$\text{Base impedance in lines} \quad = \quad \frac{138^2}{50} \;=\; 381 \ \Omega$$

$$40 \ \Omega \text{ line: } Z \quad = \quad \frac{40}{381} \;=\; 0.105 \text{ per unit}$$

$$20 \ \Omega \text{ line: } Z \quad = \quad \frac{20}{381} \;=\; 0.053 \text{ per unit}$$

Transformers:

$$\text{Y-Y} \quad = \quad 0.1 \times \frac{50}{20} \;=\; 0.250 \text{ per unit}$$

$$\text{Y-}\Delta \quad = \quad 0.1 \times \frac{50}{15} \;=\; 0.333 \text{ per unit}$$

Gens. 1 & 2:

$$X'' \;=\; 0.20 \times \left(\frac{18}{20}\right)^2 \times \frac{50}{20} \;=\; 0.405 \text{ per unit}$$

Motor 3:

$$X'' = 0.20 \times \frac{50}{30} = 0.333 \text{ per unit}$$

(b) If a fault occurs at C, by symmetry equal currents are input from generators 1 and 2. Moreover, no current should exist between busses A and B through the $j0.105$ per unit branch. If this branch is omitted from the circuit, the system simplifies to
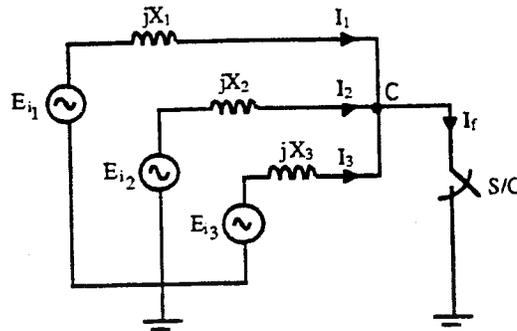
$$E_{i_1} = E_{i_2} = E_{i_3} = 1.0\angle 0° \text{ per unit}$$

$$X_1 = X_2 = 0.405 + 0.250 + 0.053 + 0.333 \text{ per unit} = 1.041 \text{ per unit}$$

$$X_3 = 0.333 \text{ per unit}$$

$$|I_1| = |I_2| = \frac{|E_{i_1}|}{|X_1|} = \frac{1.0}{1.041} \text{ per unit} = 0.9606 \text{ per unit}$$

$$|I_3| = \frac{|E_{i_3}|}{|X_3|} = \frac{1.0}{0.333} \text{ per unit} = 3.0 \text{ per unit}$$

$$|I_f| = 4.9212 \text{ per unit}$$

$$I_{\text{base}} \text{ at C} = \frac{50 \times 10^6}{\sqrt{3} \times 13.8 \times 10^3} \text{ A} = 2091.8 \text{ A}$$

$$|I_f| = 4.9212 \times 2091.8 \text{ A} = 10.29 \text{ kA}$$

(c)

$$|S_1| = |S_2| = 1.0 \times 50 \times 0.9606 \text{ MVA} = 48.03 \text{ MVA}$$

$$|S_3| = 1.0 \times 50 \times 3.0 \text{ MVA} = 150 \text{ MVA}$$

Chapter 4 Problem Solutions

4.1 The all-aluminum conductor identified by the code word *Bluebell* is composed of 37 strands each having a diameter of 0.1672 in. Tables of characteristics of all-aluminum conductors list an area of 1,033,500 cmil for this conductor (1 cmil = $(\pi/4) \times 10^{-6}$ in^2). Are these values consistent with each other? Find the overall area of the strands in square millimeters.

Solution:

$$\text{diameter} = 0.1672 \times 1000 = 167.2 \text{ mils/strand}$$

$$\text{cond. area} = (167.2)^2 \times 37 = 1,034,366 \text{ cmils}$$
$$(\text{compared to } 1,033,500 \text{ cmils})$$
$$\text{strand diam.} = 0.1672 \times 2.54 \times 10 = 4.24 \text{ mm}$$
$$\text{cond. area} = \frac{\pi}{4}(4.24)^2 \times 37 = 5224 \text{ mm}^2$$

4.2 Determine the dc resistance in ohms per km of *Bluebell* at 20° C by Eq. (4.2) and the information in Prob. 4.1, and check the result against the value listed in tables of 0.01678 Ω per 1000 ft. Compute the dc resistance in ohms per kilometer at 50° C and compare the result with the ac 60-Hz resistance of 0.1024 Ω/mi listed in tables for this conductor at 50° C. Explain any difference in values. Assume that the increase in resistance due to spiraling is 2%.

Solution:

$$R_{dc} = \frac{17.0 \times 1000}{1,033,500} = 0.01645$$

Corrected for stranding,

$$R_{dc} = 1.02 \times 0.01645 = 0.01678 \ \Omega/1000' \text{ at } 20°C$$

At 50° C,

$$R_{dc} = \frac{228 + 50}{228 + 20} \times 0.01678 \times 5.28 = 0.09932 \ \Omega/\text{mile}$$

This value does not account for skin effect and so is less than the 60-Hz value.

4.3 An all-aluminum conductor is composed of 37 strands each having a diameter of 0.333 cm. Compute the dc resistance in ohms per kilometer at 75° C. Assume that the increase in resistance due to spiraling is 2%.

Solution:

$$\text{Area} = \pi \frac{(0.333 \times 10^{-2})^2}{4} \times 37 = 3.222 \times 10^{-4} \text{ m}^2$$

$$R_{dc} = \frac{2.83 \times 10^{-8}}{3.222 \times 10^{-4}} \times 1000 = 0.0878 \ \Omega/\text{km at } 20°C$$

At 75°C, and corrected for stranding,

$$R_{dc} = 1.02 \times \frac{228 + 75}{228 + 20} \times 0.0878 = 0.1094 \ \Omega/\text{km, } 75°C$$

4.4 The energy density (that is, the energy per unit volume) at a point in a magnetic field can be shown to be $B^2/2\mu$ where B is the flux density and μ is the permeability. Using this result and Eq. (4.10) show that the total magnetic field energy stored within a unit length of solid circular conductor carrying current I is given by $\mu I^2/16\pi$. Neglect skin effect and thus verify Eq. (4.15).

Solution: From Eq. (4.10),

$$B_x = \frac{\mu x I}{2\pi r^2} \text{ Wb/m}^2$$

Energy stored in the tubular element of thickness dx, unit length and radius r:

$$dE = \frac{B_x^2}{2\mu} \cdot dV = \frac{B_x^2}{2\mu}(2\pi x \cdot 1 \cdot dx) \; \text{J}$$

$$= \frac{\mu^2 x^2 I^2}{4\pi^2 r^4} \cdot \frac{1}{2\mu} \cdot 2\pi x \, dx \; \text{J}$$

$$= \frac{\mu I^2 x^3}{4\pi r^4} \, dx \; \text{J}$$

Total energy per unit length is

$$E_{int} = \int_{x=0}^{x=r} dE = \frac{\mu I^2}{4\pi r^4} \int_0^r x^3 \, dx$$

$$= \frac{\mu I^2}{4\pi r^4} \cdot \frac{r^4}{4} = \frac{\mu I^2}{16\pi}$$

Since $E_{int} = \frac{1}{2} L_{int} I^2$,

$$L_{int} = 2 \times \frac{E_{int}}{I^2} = \frac{2}{I^2} \cdot \frac{\mu I^2}{16\pi} = \frac{\mu}{8\pi} \; \text{H/m}$$

$$= \frac{4\pi \times 10^{-7}}{8\pi} \; \text{H/m} = \frac{1}{2} \times 10^{-7} \; \text{H/m}$$

4.5 The conductor of a single-phase 60-Hz line is a solid round aluminum wire having a diameter of 0.412 cm. The conductor spacing is 3 m. Determine the inductance of the line in millihenrys per mile. How much of the inductance is due to internal flux linkages? Assume skin effect is negligible.

Solution:

$$r' = \frac{0.412}{2} \times 0.7788 = 0.1604 \; \text{cm}$$

$$L = 4 \times 10^{-7} \ln\left(\frac{3 \times 100}{0.1604}\right) 1609 \times 1000 = 4.85 \; \text{mH/mile}$$

Due to internal flux,

$$L_{int} = 2\left(\frac{1}{2} \times 10^{-7}\right) \times 1000 \times 1609 = 0.161 \; \text{mH/mile}$$

4.6 A single-phase 60-Hz overhead power line is symmetrically supported on a horizontal crossarm. Spacing between the centers of the conductors (say a and b) is 2.5 m. A telephone line is also symmetrically supported on a horizontal crossarm 1.8 m directly below the power line. Spacing between the centers of these conductors (say c and d) is 1.0 m.

(a) Using Eq. (4.36) show that the mutual inductance per unit length between circuit a-b and circuit c-d is given by

$$4 \times 10^{-7} \ln\sqrt{\frac{D_{ad}D_{bc}}{D_{ac}D_{bd}}} \; \text{H/m}$$

where, for example, D_{ad} denotes the distance in meters between conductors a and d.

(b) Hence compute the mutual inductance per kilometer between the power line and the telephone line.

(c) Find the 60 Hz voltage per kilometer induced in the telephone line when the power line carries 150 A.
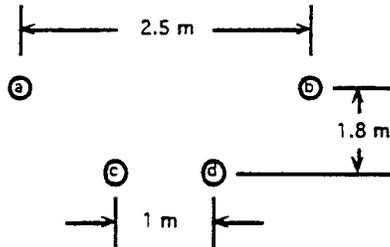
Solution:

(a) Let circuit a-b carry the current I, so that

$$I_a = -I_b = I \text{ A} \quad (\text{and } I_c = I_d = 0)$$

since $\sum I = 0$ for the group, Eq. (4.36) remains valid.

$$\lambda_c = 2 \times 10^{-7} \left(I_a \ln \frac{1}{D_{ac}} + I_b \ln \frac{1}{D_{bc}} + I_c \ln \frac{1}{r'_c} + I_a \ln \frac{1}{D_{dc}} \right)$$

Therefore,

$$\lambda_c = 2 \times 10^{-7} \times \ln \frac{D_{bc}}{D_{ac}} \text{ Wb-t/m}$$

Similarly,

$$\lambda_d = 2 \times 10^{-7} \times \ln \frac{D_{bd}}{D_{ad}} \text{ Wb-t/m}$$

$\lambda_{c\text{-}d}$ (linkage of the loop) is given by

$$\lambda_c - \lambda_d = 2 \times 10^{-7} \times \ln \frac{D_{bc} D_{ad}}{D_{ac} D_{bd}}$$

$$\text{Mutual Inductance} = \frac{\lambda_{c\text{-}d}}{I} = 2 \times 10^{-7} \times \ln \frac{D_{bc} D_{ad}}{D_{ac} D_{bd}}$$

$$= 4 \times 10^{-7} \ln \sqrt{\frac{D_{bc} D_{ad}}{D_{ac} D_{bd}}} \text{ H/m}$$

(b)

$$D_{ac} = \sqrt{(1.25 - 0.5)^2 + 1.8^2} = 1.95 \text{ m}$$

$$D_{ad} = \sqrt{(1.25 + 0.5)^2 + 1.8^2} = 2.51 \text{ m}$$

Flux linkages with c-d:

$$\left. \begin{array}{rcl} \text{due to } I_a \quad \varphi_{cd} &=& 2 \times 10^{-7} I_a \ln \frac{2.51}{1.95} \\ \text{due to } I_b \quad \varphi_{cd} &=& -2 \times 10^{-7} I_a \ln \frac{2.51}{1.95} \end{array} \right\}$$ Note that flux through c-d due to I_a is opposite that due to I_b

Note also that I_a and I_b are 180° out of phase. So, due to I_a and I_b,

$$\varphi_{cd} = 4 \times 10^{-7} I_a \ln \frac{2.51}{1.95}$$

$$M = 4 \times 10^{-7} \ln \frac{2.51}{1.95} = 1.01 \times 10^{-7} \text{ H/m}$$

(c) $V_{cd} = \omega M I = 377 \times 1.01 \times 10^{-7} \times 10^3 \times 150 = 5.71$ V/km

4.7 If the power line and the telephone line described in Prob. 4.6 are in the same horizontal plane and the distance between the nearest conductors of the two lines is 18 m, use the result of Prob. 4.6(a) to find the mutual inductance between the power and telephone circuits. Also find the 60 Hz voltage per kilometer induced in the telephone line when 150 A flows in the power line.

Solution:
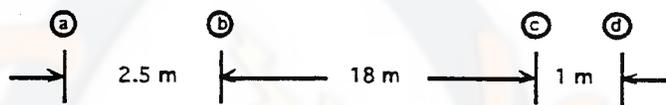
$$\text{due to } I_a \quad \phi_{cd} = 2 \times 10^{-7} I_a \ln \frac{21.5}{20.5}$$

$$\text{due to } I_b \quad \phi_{cd} = -2 \times 10^{-7} I_a \ln \frac{19}{18} \quad \text{since } I_b = -I_a$$

$$\text{due to } I_a \text{ and } I_b \quad \phi_{cd} = 2 \times 10^{-7} I_a \ln \frac{21.5 \times 18}{20.5 \times 19} = -0.01288 \times 10^{-7} I_a$$

$$M = \frac{\phi_{cd}}{I_a} = -0.01288 \times 10^{-7} \text{ H/m}$$

$$V_{cd} = \omega M I = 377 \times 150 M \times 10^3 = 0.0728 \text{ V/km}$$

4.8 Find the GMR of a three-strand conductor in terms of r of an individual strand.

Solution: Given this bundle:

$$\text{GMR} = \sqrt[9]{(0.779r \times 2r \times 2r)^3} = r \sqrt[3]{4 \times 0.779} = 1.46r$$

4.9 Find the GMR of each of the unconventional conductors shown in Fig. 4.15 in terms of the radius r of an individual strand.

Solution:

(a) Bundle:

$$\text{GMR} = \sqrt[16]{(0.779)^4 \left[\left(2 \times 2 \times 2\sqrt{2}\right) r^3\right]^4} = 1.723r$$

(b) Bundle:

$$\text{GMR} = \sqrt[16]{(0.779r)^4 \left(2 \times 2 \times 2\sqrt{3}\right)^2 \left(2 \times 2 \times 2\right)^2 r^{12}} = r\sqrt[4]{0.779 \times 8} \times 3^{1/16} = 1.692r$$

(c) Bundle:

$$\text{GMR} = \sqrt[9]{(0.779)^3 \times 8r^2 \times 8r^2 \times 4r^2} = 1.704r$$

(d) Bundle:

$$\text{GMR} = \sqrt[6]{0.779r} \times \sqrt[36]{\left(4 \times 4 \times 2 \times 2 \times 2\sqrt{3}\right)^3 \left(2^4 \times 2\sqrt{3}\right)^3 r^{30}}$$

$$= r \times 2 \times 3^{1/12} \times 0.779^{1/6} = 2.10r$$

4.10 The distance between conductors of a single-phase line is 10 ft. Each of its conductors is composed of six strands symmetrically placed around one center strand so that there are seven equal strands. The diameter of each strand is 0.1 in. Show that D_s of each conductor is 2.177 times the radius of each strand. Find the inductance of the line in mH/mile.

Solution:

Outside conductors are counter-clockwise numbered 1 through 6. The center conductor is number 7. Each radius is r and the distances between conductors are:

$$
\begin{aligned}
D_{12} &= 2r & D_{14} &= 4r \\
D_{13} &= \sqrt{D_{14}^2 - D_{34}^2} = 2r\sqrt{3} \\
D_s &= \sqrt[49]{(r')^7 \left(D_{12}^2 D_{13}^2 D_{14} D_{17}\right)^6 (2r)^6} = \sqrt[7]{(r')} \times \sqrt[49]{\left(2^2 r^2 \times 3 \times 2^2 r^2 \times 2^2 r^2 \times 2r \times 2r\right)^6} \\
&= \frac{2r \sqrt[7]{3\,(0.779)}}{\sqrt[49]{6}} = 2.177r \\
L &= 4 \times 10^{-7} \ln \frac{10 \times 12}{2.177 \times 0.05} \times 1000 \times 1609 = 4.51 \text{ mH/mi}
\end{aligned}
$$

4.11 Solve Example 4.2 for the case where side Y of the single-phase line is identical to side X and the two sides are 9 m apart as shown in Fig. 4.9.

Solution:
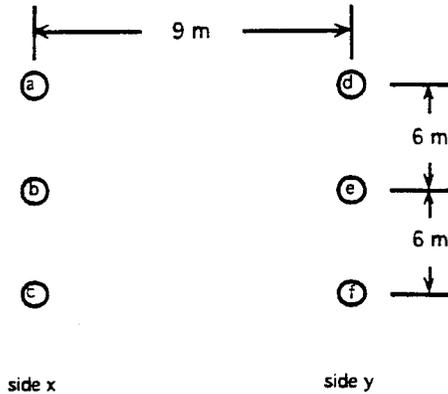
$$D_m = \sqrt[9]{D_{ad}D_{ae}D_{af}D_{bd}D_{be}D_{bf}D_{cd}D_{ce}D_{cf}}$$

$$D_{ad} = D_{be} = D_{cf} = 9\text{ m}$$

$$D_{ae} = D_{bd} = D_{bf} = D_{ce} = \sqrt{117}\text{ m}$$

$$D_{af} = D_{cd} = \sqrt{12^2 + 9^2}\text{ m} = 15\text{ m}$$

$$D_m = \sqrt[9]{9 \times \sqrt{117} \times 15 \times \sqrt{117} \times 9 \times \sqrt{117} \times 15 \times \sqrt{117} \times 9}\text{ m} = 10.940\text{ m}$$

$$D_s = 0.481 \quad \text{(from Example 4.2) for both sides}$$

$$L_x = L_y = 2 \times 10^{-7} \ln\frac{10.940}{0.481}\text{ H/m} = 6.249 \times 10^{-7}\text{ H/m}$$

$$L = L_x + L_y = 12.497 \times 10^{-7}\text{ H/m}$$

4.12 Find the inductive reactance of ACSR *Rail* in ohms per kilometer at 1-m spacing.

Solution:

From Table A.1 for *Rail* at 1-ft spacing:

$$D_s = 0.0386\text{ ft}$$

$$1\text{ ft} = 2.54 \times 12/100 = 0.3048\text{ m}$$

$$D_s = 0.3048 \times 0.0386 = 0.01177\text{ ft}$$

$$X_L = 2 \times 10^{-7}\left(\ln\frac{1}{0.01177}\right) \times 377 \times 1000 = 0.335\text{ }\Omega/\text{km at 1 m spacing}$$

4.13 Which conductor listed in Table A.3 has an inductive reactance at 7-ft. spacing of 0.651 Ω/mi?

Solution:

From Table A.3 at 7-ft spacing:

$$X_d = 0.2361\text{ }\Omega$$

$$0.651 - 0.2361 = 0.415\text{ }\Omega/\text{mi at 1-ft spacing}$$

The conductor is *Rook*.

4.14 A three-phase line has three equilaterally spaced conductors of ACSR *Dove*. If the conductors are 10 ft apart, determine the 60 Hz per-phase reactance of the line in Ω/km.

Solution:

For ACSR *Dove* conductors, $D_s = 0.0314$ ft. Given that $D = 10$ ft,

$$X_L = 2\pi \times 60 \times 2 \times 10^{-7} \ln \frac{10}{0.0314} \times 10^3 \ \Omega/\text{km} = 0.4346 \ \Omega/\text{km}$$

Alternatively, from Table A.3,

$$X_a = 0.420 \ \Omega/\text{mi} \qquad X_d = 0.2794 \ \Omega/\text{mi}$$
$$X_L = 0.420 + 0.2794 \ \Omega/\text{mi} = 0.6994 \ \Omega/\text{mi}$$
$$= 0.6994 \times 0.6214 \ \Omega/\text{mi} = 0.4346 \ \Omega/\text{mi}$$

4.15 A three-phase line is designed with equilateral spacing of 16 ft. It is decided to build the line with horizontal spacing ($D_{13} = 2D_{12} = 2D_{23}$). The conductors are transposed. What should be the spacing between adjacent conductors in order to obtain the same inductance as in the original design?

Solution:

$$\sqrt[3]{D \times D \times 2D} = \sqrt[3]{2}D = 16 \qquad D = 12.7 \text{ ft}$$

4.16 A three-phase 60-Hz transmission line has its conductors arranged in a triangular formation so that two of the distances between conductors are 25 ft and the third distance is 42 ft. The conductors are ACSR *Osprey*. Determine the inductance and inductive reactance per phase per mile.

Solution:

$$D_{eq} = \sqrt[3]{25 \times 25 \times 42} = 29.72 \text{ ft}$$
$$L = 2 \times 10^{-7} \ln \frac{29.72}{0.0284} \times 1000 \times 1609 = 2.24 \text{ mH/mi}$$
$$X_L = 0.377 \times 2.24 = 0.84 \ \Omega/\text{mi}$$

4.17 A three-phase 60-Hz line has flat horizontal spacing. The conductors have a GMR of 0.0133 m with 10 m between adjacent conductors. Determine the inductive reactance per phase in ohms per kilometer. What is the name of this conductor?

Solution:

$$D_{eq} = \sqrt[3]{10 \times 10 \times 20} = 12.6 \text{ ft}$$
$$X_L = 377 \times 2 \times 10^{-7} \ln \frac{12.3}{0.0133} \times 1000 = 5.17 \ \Omega/\text{km}$$
$$D_s = 0.0133/0.3048 = 0.0436 \text{ ft}$$

The conductor is *Finch*.

4.18 For short transmission lines if resistance is neglected the maximum power which can be transmitted per phase is equal to

$$\frac{|V_S| \times |V_R|}{|X|}$$

where V_S and V_R are the line-to-neutral voltages at the sending and receiving ends of the line and X is the inductive reactance of the line. This relationship will become apparent in the study of Chap. 6. If the magnitudes of V_S and V_R are held constant and if the cost of a conductor is proportional to its cross-sectional area, find the conductor in Table A.3 which has the maximum power-handling capacity per cost of conductor at a given geometric mean spacing.

> <u>Note to Instructor</u>: The purpose of this problem is to stimulate the student's examination of Table A.3 and is worthwhile in introducing class discussion of conductor selection.

Solution:

Power transmission capability per conductor cost if resistance is neglected is $|V_S||V_R|/(X \cdot A)$ based on our cost assumption where A is the cross-sectional area of the conductor. Therefore, the product $X \cdot A$ must be minimized. Assuming D_{eq} is fixed, examining the Table shows that in comparing any two conductors the percent difference in A is much greater than that of X. So, A is the controlling factor, and *Partridge* or *Waxwing* would be selected. *However*, resistance cannot be neglected. A conductor must be large enough in cross section that melt-down caused by $|I|^2 R$ loss will not occur under the most extreme operating conditions. The reference (Aluminum Electrical Conductor Handbook) gives information on thermal effects. If reactance causes too high a voltage drop on a line, double-circuit lines or bundled conductors must be provided. The reference (Analytical Development of Loadability Characteristics for EHV and UHV Transmission Lines) contains information on maximum transmission capability of lines.

4.19 A three-phase underground distribution line is operated at 23 kV. The three conductors are insulated with 0.5 cm solid black polyethylene insulation and lie flat, side by side, directly next to each other in a dirt trench. The conductor is circular in cross section and has 33 strands of aluminum. The diameter of the condutor is 1.46 cm. The manufacturer gives the GMR as 0.561 cm and the cross section of the conductor as 1.267 cm². The thermal rating of the line buried in normal soil whose maximum temperature is 30°C is 350 A. Find the dc and ac resistance at 50° C and the inductive reactance in ohms per kilometer. To decide whether to consider skin effect in calculating resistance determine the percent skin effect at 50° C in the ACSR conductor of size nearest that of the underground conductor. Note that the series impedance of the distribution line is dominated by R rather than X_L because of the very low inductance due to the close spacing of the conductors.

<u>Note to Instructor</u>: When assigning this problem, it may be advisable to outline part of the procedure.

Solution:

$$\frac{R_{50°,dc}}{R_{20°,dc}} = \frac{228+50}{228+20} = 1.121$$

$$R_{20°,dc} = \frac{\rho l}{A} = \frac{2.83 \times 10^{-8}}{1.267 \times 10^{-4}} = 0.223 \ \Omega/km$$

$$R_{50°,dc} = 1.121 \times 0.223 = 0.250 \ \Omega/km$$

Skin effect can be estimated from the values in Table A.3. The area 1.267 cm^2 is

$$1.267 \times \left(\frac{1}{2.54}\right)^2 \times \frac{4}{\pi} \times 10^6 = 250,000 \ \text{cmils}$$

Waxwing has an area of 266,800 cmils and for this conductor

$$\frac{R_{50°,ac}}{R_{20°,ac}} = \frac{0.3831}{0.0646 \times 5.28} = 1.123$$

Since temperature rise would account for a factor of 1.121, skin effect is only about 0.2%. With insulation thichness of 0.5 cm center-to-center conductor spacing is $2 \times 0.05 + 1.46 = 2.46$ cm. So,

$$D_{eq} = \sqrt[3]{2.46 \times 2.46 \times 2 \times 2.46} = 3.099$$

$$X_L = 377 \times 1000 \times 2 \times 10^{-7} \ln\frac{3.099}{0.561} = 0.129 \ \Omega/km$$

4.20 The single-phase power line of Prob. 4.6 is replaced by a three-phase line on a horizontal crossarm in the same position as that of the original single-phase line. Spacing of the conductors of the power line is $D_{13} = 2D_{12} = 2D_{23}$, and equivalent equilateral spacing is 3 m. The telephone line remains in the position described in Prob. 4.6. If the current in the power line is 150 A, find the voltage per kilometer induced in the telephone line. Discuss the phase relation of the induced voltage with respect to the power-line current.

Solution:

$$\sqrt[3]{D \times D \times 2D} = \sqrt[3]{2}\,D = 3$$

$$D = \frac{3}{\sqrt[3]{2}} = 2.38 \ \text{m}$$
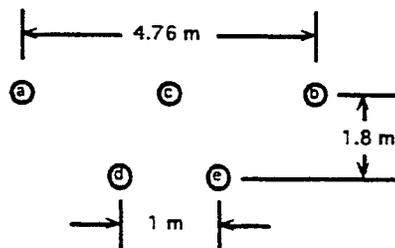
The center conductor of the 3-phase line causes no flux linkages with d-e since the conductor is at an equal distance from d and e.

$$D_{ad} = D_{be} = \sqrt[2]{1.8^2 + (2.38 - 0.5)^2} = 2.60 \text{ m}$$

$$D_{ae} = D_{bd} = \sqrt[2]{1.8^2 + (2.38 + 0.5)^2} = 3.40 \text{ m}$$

$$\text{due to } I_a, \quad \phi_{de} = 2 \times 10^{-7} I_a \ln \frac{3.40}{2.60}$$

$$\text{due to } I_b, \quad \phi_{de} = 2 \times 10^{-7} I_b \ln \frac{3.40}{2.60}$$

$$\text{Total flux linkages} = 2 \times 10^{-7} (I_a - I_b) \ln \frac{3.40}{2.60}$$

Since I_b lags I_a by 120°,
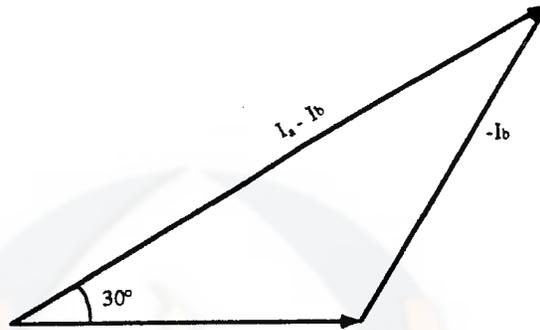
$$I_a - I_b = \sqrt{3} I_a \angle 30°$$

$$\phi_{de} = 2 \times 10^{-7} \sqrt{3} I_a \ln \frac{3.40}{2.60} \angle 30° \text{ W/m}$$

$$M = 9.29 \times 10^{-8} \text{ H/m}$$

$$V = \omega M \times 150 = 377 \times 10^{-8} \times 9.29 \times 150 \times 1000 = 5.25 \text{ V/km}$$

The induced voltage leads I_a by $90° + 30° = 120°$; that is, V is in phase with I_c.

4.21 A 60-Hz three-phase line composed of one ACSR *Bluejay* conductor per phase has flat horizontal spacing of 11 m between adjacent conductors. Compare the inductive reactance in ohms per kilometer per phase of this line with that of a line using a two-conductor bundle of ACSR 26/7 conductors having the same total cross-sectional area of aluminum as the single-conductor line and 11 m spacing measured from the center of the bundles. The spacing between conductors in the bundle is 40 cm.

Solution:

$$D_{eq} = \sqrt[3]{11 \times 11 \times 22} = 13.86 \text{ m}$$

Bluejay:

$$D_s = 0.0415 (2.54 \times 12 \times 10^{-2}) = 0.0126 \text{ m}$$

$$X = 2 \times 10^{-7} \times 10^3 \times 377 \ln \frac{13.86}{0.0126} = 0.528 \text{ } \Omega/\text{km}$$

Dove is the conductor for bundling:

$$D_s = 0.0314 \, (2.54 \times 12 \times 10^{-2}) = 0.00957 \text{ m}$$

$$D_s^b = \sqrt{0.00957 \times 0.4} = 0.0619 \text{ m}$$

$$X = 2 \times 10^{-7} \times 10^3 \times 377 \ln \frac{13.86}{0.0619} = 0.408 \text{ }\Omega/\text{km}$$

4.22 Calculate the inductive reactance in ohms per kilometer of a bundled 60-Hz three-phase line having three ACSR *Rail* conductors per bundle with 45 cm between conductors of the bundle. The spacing between bundle centers is 9, 9 and 18 m.

Solution:

$$D_{\text{eq}} = \sqrt[3]{9 \times 9 \times 18} = 11.34 \text{ m}$$

Rail:

$$D_s = 0.0386 \text{ ft} = 0.0386 \, (2.54 \times 12 \times 10^{-2}) = 0.0118 \text{ m}$$

$$D_s^b = \sqrt[3]{0.0118 \times 0.45 \times 0.45} = 0.1337 \text{ m}$$

$$X = 2 \times 10^{-7} \times 10^3 \times 377 \ln \frac{11.34}{0.1337} = 0.3348 \text{ }\Omega/\text{km}$$

Chapter 5 Problem Solutions

5.1 A three-phase transmission line has flat horizontal spacing with 2 m between adjacent conductors. At a certain instant the charge on one of the outside conductors is 60 μC/km, and the charge on the center conductor and on the other outside conductor is -30 μC/km. The radius of each condutor is 0.8 cm. Neglect the effect of the ground and find the voltage drop between the two identically charged conductors at the instant specified.

Solution:
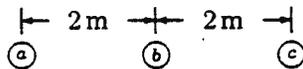
$$q_a = 60 \times 10^{-6} \text{ C/km}$$

$$q_b = q_c = -30 \times 10^{-6} \text{ C/km}$$

$$V_{bc} = \frac{10^{-6}}{2\pi k} \left(60 \ln \frac{4}{2} - 30 \ln \frac{2}{r} - 30 \ln \frac{r}{2} \right) \quad \text{where } r \text{ is in meters}$$

$$= \frac{10^{-6} \times 60}{2\pi \times 8.85 \times 10^{-9}} = 744.5 \text{ V}$$

5.2 The 60-Hz capacitive reactance to neutral of a solid conductor, which is one conductor of a single-phase line with 5 ft spacing , is 196.1 kΩ-mi. What value of reactance would be specified in a table listing the capacitive reactance in ohm-miles to neutral of the conductor at 1-ft spacing for 25 Hz? What is the cross-sectional area of the conductor in circular mils?

Solution:

At 5-ft spacing,

$$X_C = 2.965 \times 10^4 \ln \frac{5}{r} = 196,100 \ \Omega \cdot mi$$

$$\ln \frac{5}{r} = 6.614$$

$$r = 0.00670 \ ft, \ or \ 0.0805 \ in$$

$$= (2 \times 0.0805 \times 1000)^2 = 25,992 \ circ \ mils$$

From Eq. (5.12), at 1-ft spacing and 25 Hz,

$$X_C = \frac{1.779}{25} \times 10^6 \ln \frac{1}{0.00670} = 356,200 \ \Omega \cdot mi$$

5.3 Solve Example 5.1 for 50 Hz operation and 10 ft spacing.

Solution:

$$X_C = \frac{1.779 \times 10^6}{50} \ln \frac{10}{0.0268} \ \Omega \cdot mi = 0.2115 \ M\Omega \cdot mi$$

$$B_C = \frac{1}{X_C} = 4.728 \ \mu S/mi$$

$$X_a' = \frac{60}{50} \times 0.1074 \ M\Omega \cdot mi$$

$$X_d' = \frac{60}{50} \times 0.0683 \ M\Omega \cdot mi$$

$$X_C = \frac{60}{50}(0.1074 + 0.0683) \ M\Omega \cdot mi = 0.2109 \ M\Omega \cdot mi$$

$$B_C = 4.742 \ \mu S/mi$$

5.4 Using Eq. (5.23), determine the capacitance to neutral (in $\mu F/km$) of a three-phase line with three *Cardinal* ACSR conductors equilaterally spaced 20 ft apart. What is the charging current of the line (in A/km) at 60 Hz and 100 kV line to line?

Solution:

For *Cardinal* conductors,

$$r = \frac{1.196}{2} = \frac{1}{12} \ ft$$

$$C_n = \frac{2\pi \times 8.85 \times 10^{-12}}{\ln \frac{20}{(1.196)/24}} \ F/m = 9.276 \times 10^{-12} \ F/m = 9.276 \times 10^{-3} \ \mu F/km$$

$$I_{chg} = 2\pi \times 60 \times 9.276 \times 10^{-9} \times \frac{100 \times 10^3}{\sqrt{3}} \text{ A/km} = 0.202 \text{ A/km}$$

5.5 A three-phase 60-Hz transmission line has its conductors arranged in a triangular formation so that two of the distances between conductors are 25 ft and the third is 42 ft. The conductors are ACSR *Osprey*. Determine the capacitance to neutral in microfarads per mile and the capacitive reactance to neutral in ohm-miles. If the line is 150 mi long, find the capacitance to neutral and capacitive reactance of the line.

Solution:

$$Osprey \text{ diam.} = 0.879 \text{ in}$$
$$D_{eq} = \sqrt[3]{25 \times 25 \times 42} = 29.72 \text{ ft}$$
$$C_n = \frac{2\pi \times 8.85 \times 10^{-12}}{\ln \frac{29.72 \times 12}{(0.879)/2}} \text{ F/m}$$
$$= 8.301 \times 10^{-12} \text{ F/m} = 8.301 \times 10^{-6} \times 1.609 \ \mu\text{F/m} = 0.01336 \ \mu\text{F/mi}$$
$$X_C = \frac{10^6}{377 \times 0.01336} = 0.1985 \times 10^6 \ \Omega \cdot \text{mi}$$

From Table A.3, $X_a' = 0.0981$. Interpolation from Table A.4 yields $X_d' = 0.0999 + 0.72(0.1011 - 0.0999) = 0.1006$. From Table A.4, $X_C = 0.1987 \times 10^6 \ \Omega \cdot \text{mi}$.

For 150 miles,

$$C_n = 150 \times 0.01336 = 2.004 \ \mu\text{F}$$
$$X_C = \frac{0.1987}{150} \times 10^6 = 1325 \ \Omega$$

5.6 A three-phase 60-Hz line has flat horizontal spacing. The conductors have an outside diameter of 3.28 cm with 12 m between conductors. Determine the capacitive reactance to neutral in ohm-meters and the capacitive reactance of the line in ohms if its length is 125 mi.

Solution:

$$D_{eq} = \sqrt[3]{12 \times 12 \times 24} = 15.12 \text{ m}$$
$$r = 0.0328/2 = 0.0164$$
$$X_C = \frac{2.862}{60} \times 10^9 \ln \frac{15.12}{0.0164} = 3.256 \times 10^8 \ \Omega \cdot \text{m}$$

For 125 miles,

$$X_C = \frac{3.256 \times 10^8}{125 \times 1609} = 1619 \ \Omega$$

5.7 (a) Derive an equation for the capacitance to neutral in farads per meter of a single-phase line, taking into account the effect of ground. Use the same nomenclature as in the equation derived for the capacitance of a three-phase

line where the effect of ground is represented by image charges.

(b) Using the derived equation, calculate the capacitance to neutral in farads per meter of a single-phase line composed of two solid circular conductors each having a diameter of 0.229 in. The conductors are 10 ft apart and 25 ft above ground. Compare the result with the value obtained by applying Eq. (5.10).

Solution:
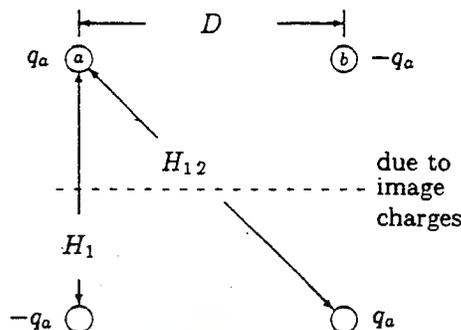
(a) Due to charges on a, b:

$$V_{ab} = \frac{1}{2\pi k} \left[q_a \ln \frac{D}{r} - q_a \ln \frac{r}{D} \right]$$

Due to image charges:

$$V_{ab} = \frac{1}{2\pi k} \left[-q_a \ln \frac{H_{12}}{H_1} + q_a \ln \frac{H_1}{H_{12}} \right]$$

Due to image and actual charges:

$$V_{ab} = \frac{q_a}{2\pi k} \left[\ln \frac{D^2}{r^2} - \ln \frac{H_{12}^2}{H_1^2} \right] = \frac{q_a}{\pi k} \left[\ln \frac{D}{r} - \ln \frac{H_{12}}{H_1} \right]$$

$$C_{an} = 2C_{ab} = \frac{2\pi k}{\ln \frac{D}{r} - \ln \frac{H_{12}}{H_1}} \text{ F/m}$$

(b) By Eq. (5.10),

$$C_n = \frac{2\pi \times 8.85 \times 10^{-12}}{\ln \left(\frac{10 \times 12}{0.229/2} \right)} = 7.996 \times 10^{-12} \text{ F/m}$$

And from part (a) above,

$$C_n = \frac{2\pi \times 8.85 \times 10^{-12}}{\ln \left(\frac{10 \times 12}{0.229/2} \right) - \ln \left(\frac{\sqrt{50^2 + 10^2}}{50} \right)} = 8.018 \times 10^{-12} \text{ F/m}$$

5.8 Solve Prob. 5.6 while taking account of the effect of ground. Assume that the conductors are horizontally placed 20 m above ground.

Solution:
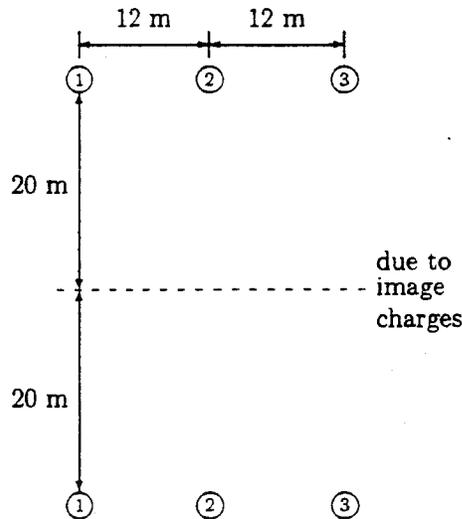
$$H_1 = H_2 = H_3 = 40 \text{ m}$$
$$H_{12} = H_{23} = \sqrt{40^2 + 12^2} \text{ m} = 41.761 \text{ m}$$
$$H_{31} = \sqrt{40^2 + 24^2} \text{ m} = 46.648 \text{ m}$$

$$D_{eq} = 15.12 \text{ m} \quad \text{and} \quad r = 0.0164 \text{ m}$$
$$X_C = \frac{2.862}{60} \times 10^9 \left[\ln \frac{D_{eq}}{r} - \frac{1}{3} \ln \frac{H_{12}H_{23}H_{31}}{H_1 H_2 H_3} \right] \Omega \cdot \text{m}$$
$$= 4.77 \times 10^7 \left[\ln \frac{15.12}{0.0164} - \frac{1}{3} \ln \frac{41.761 \times 41.761 \times 46.648}{40 \times 40 \times 40} \right] \Omega \cdot \text{m}$$
$$= 3.218 \times 10^8 \ \Omega \cdot \text{m}$$

For 125 miles,

$$X_C = \frac{3.218 \times 10^8}{125 \times 1609.34} \ \Omega = 1.60 \text{ k}\Omega$$

5.9 A 60-Hz three-phase line composed of one ACSR *Bluejay* conductor per phase has flat horizontal spaicng of 11 m between adjacent conductors. Compare the capacitive reactance in ohm-kilometers per phase of this line with that of a line using a two-conductor bundle of ACSR 26/7 conductors having the same total cross-sectional area of aluminum as the single-conductor line and the 11 m spacing measured between bundles. The spacing between conductors in the bundle is 40 cm.

Solution:

$$D_{eq} = \sqrt{11 \times 11 \times 22} = 13.86 \text{ m}$$

$$Bluejay: r = 1.259 \times 2.54/2 \times 10^{-2} = 0.016 \text{ m}$$

$$X_C = 4.77 \times 10^4 \ln \frac{13.86}{0.016} = 322,650 \ \Omega \cdot \text{km}$$

For *Dove*, 2-conductor bundle,

$$r = 0.927 \times 2.54/2 \times 10^{-2} = 0.01177$$
$$D_{sC}^b = \sqrt{rd} = \sqrt{0.0177 \times 0.4} = 0.0842 \text{ m}$$
$$X_C = 4.77 \times 10^4 \ln \frac{13.86}{0.0842} = 243,440 \ \Omega \cdot \text{km}$$

5.10 Calculate the capacitive reactance in ohm-kilometers of a bundled 60-Hz three-phase line having three ACSR *Rail* conductors per bundle with 45 cm between conductors of the bundle. The spacing between bundle centers is 9, 9 and 18 m.

Solution:

$$D_{eq} = \sqrt{9 \times 9 \times 18} = 11.34 \text{ m}$$
$$r = 1.165 \times 2.54/2 \times 10^{-2} = 0.0148 \text{ m}$$
$$D_{sC}^b = \sqrt[3]{0.0148 \times 0.45^2} = 0.1442 \text{ m}$$
$$X_C = 4.77 \times 10^4 \ln \frac{11.34}{0.1442} = 208,205 \ \Omega \cdot \text{km}$$

5.11 Six conductors of ACSR *Drake* constitute a 60-Hz double-circuit three-phase line arranged as shown in Fig. 5.11. The vertical spacing, however, is 14 ft; the longer horizontal distance is 32 ft; and the shorter horizontal distances are 25 ft. Find,

(*a*) the inductance per phase (in H/mi) and the inductive reactance in Ω/mi.

(*b*) the capacitive reactance to neutral in ohm-miles and the charging current A/mi per phase and per conductor at 138 kV.

Solution:
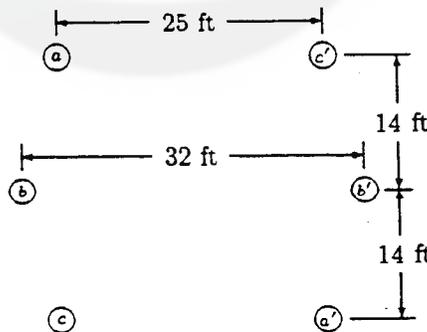
(*a*)

$$GMR = 0.0373 \ Drake$$

In original positions in the transposition cycle,

$$\text{distance } a\text{-}b = \sqrt{14^2 + 3.5^2} = 14.43 \text{ ft}$$

$$\text{distance } a\text{-}b' = \sqrt{14^2 + 28.5^2} = 31.75 \text{ ft}$$

$$\text{distance } a\text{-}a' = \sqrt{25^2 + 28^2} = 37.54 \text{ ft}$$

$$D_{ab}^p = D_{bc}^p = \sqrt[4]{14.43^2 \times 31.75^2} = 21.04 \text{ ft}$$

$$D_{ac} = \sqrt[4]{25^2 \times 28^2} = 26.46 \text{ ft}$$

$$D_{eq} = \sqrt[3]{21.04^2 \times 26.46} = 22.71 \text{ ft}$$

$$D_s = \left[\left(\sqrt{0.0373 \times 37.54} \right)^2 \sqrt{0.0373 \times 32} \right]^{\frac{1}{3}} = 1.152 \text{ ft}$$

$$L = 2 \times 10^{-7} \ln \frac{22.71}{1.152} = 5.693 \times 10^{-7} \text{ H/m}$$

$$= 5.963 \times 10^{-7} \times 10^3 \times 1609 = 0.959 \text{ mH/mi}$$

$$X_L = 377 \times 0.959 \times 10^{-3} = 0.362 \; \Omega/\text{mi/phase}$$

(b) $r = \frac{1.108}{2 \times 12} = 0.0462$ ft as in part (a) above, except that r is substituted for D_s:

$$D_{sC} = \left[\left(\sqrt{0.0462 \times 37.54} \right)^2 \sqrt{0.0462 \times 32} \right]^{\frac{1}{3}} = 1.282 \text{ ft}$$

From part (a) above, $D_{eq} = 22.71$ ft and

$$X_C = 2.965 \times 10^{-4} \ln \frac{22.71}{1.282} = 85,225 \; \Omega \cdot \text{mi/phase to neutral}$$

$$I_{chg} = \frac{138,000/\sqrt{3}}{85,225} = 0.935 \text{ A/mi/phase} = 0.467 \text{ A/mi/conductor}$$

Chapter 6 Problem Solutions

6.1 An 18-km 60-Hz single circuit three-phase line is composed of *Partridge* conductors equilaterally spaced with 1.6 m between centers. The line delivers 2500 kW at 11 kV to a balanced load. Assume a wire temperature of 50°C.

(a) Determine the per-phase series impedance of the line.

(b) What must be the sending-end voltage when the power factor is
(i) 80% lagging (ii) unity (iii) 90% leading?

(c) Determine the percent regulation of the line at the above power factors.

(d) Draw phasor diagrams depicting the operation of the line in each case.

Solution:

(a)

$$R = 0.3792 \times \frac{18}{1.609} = 4.242 \; \Omega$$

From Table A.3, $X_a = 0.465 \; \Omega/\text{mi}$

and since 1.6 m $= (1.6 \times 100)/(2.54 \times 12) = 5.25$ ft,

$$X_d = 0.2012 \quad \text{(Table A.4, 5'-3'')}$$
$$X = 0.465 + 0.2012 = 0.666 \ \Omega/\text{mi}$$
$$\text{For 18 km, } X = 18 \times \frac{0.666}{1.609} = 7.451 \ \Omega$$
$$Z = 4.242 + j7.451 = 8.57\underline{/60.35°} \ \Omega$$

(b) For power factor $= 1.0$,

$$I_R = \frac{2500}{\sqrt{3} \times 11} = 131.2 \text{ A} \qquad \frac{11,000}{\sqrt{3}} = 6350 \text{ V}$$
$$V_S = 6350 + 131.2\,(4.24 + j7.451)$$
$$= 6906 + j977.6 = 6975\underline{/8.06°}$$
$$\text{sending-end line voltage} = \sqrt{3}\,V_S = \sqrt{3} \times 6975 = 12,081 \text{ V}$$

For power factor $= 0.8$ lagging,

$$|I_R| = \frac{2500}{\sqrt{3} \times 11 \times 0.8} = 164 \text{ A}$$
$$V_S = 6350 + 164\underline{/-36.87°} \times 8.57\underline{/60.35°}$$
$$= 7639 + j5.60 = 7660\underline{/4.19°}$$
$$\text{sending-end line voltage} = \sqrt{3}\,V_S = \sqrt{3} \times 7660 = 13,268 \text{ V}$$

For power factor $= 0.9$ leading,

$$|I_R| = \frac{2500}{\sqrt{3} \times 11 \times 0.9} = 145.8 \text{ A}$$
$$V_S = 6350 + 145.8\underline{/25.84°} \times 8.57\underline{/60.35°}$$
$$= 6433 + j1247 = 6553\underline{/10.97°}$$
$$\text{sending-end line voltage} = \sqrt{3}\,V_S = \sqrt{3} \times 6553 = 11,350 \text{ V}$$

(c)

$$\% \text{ Regulation} = \frac{|V_S| - |V_R|}{|V_R|} \times 100\%$$
$$\text{at p.f.} = 0.8 \text{ lagging, } \% \text{ Reg.} = \frac{7660 - 6350}{6350} \times 100\% = 20.63\%$$
$$\text{at unity p.f., } \% \text{ Reg.} = \frac{6975 - 6350}{6350} \times 100\% = 9.84\%$$
$$\text{at p.f.} = 0.9 \text{ leading, } \% \text{ Reg.} = \frac{6553 - 6350}{6350} \times 100\% = 3.20\%$$

(d) For p.f. $= 0.8$ lagging,
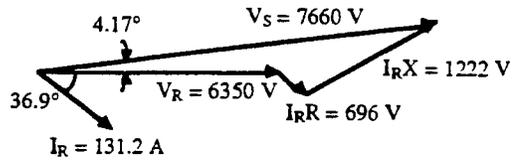
For unity p.f.,
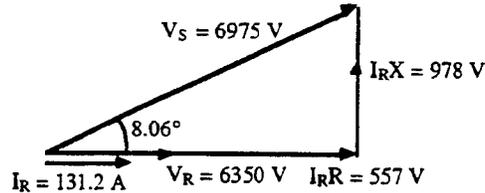
For p.f. = 0.9 leading,
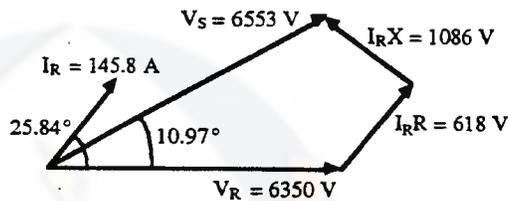
6.2 A 100-mi, single-circuit, three-phase transmission line delivers 55 MVA at 0.8 power factor lagging to the load at 132 kV (line-to-line). The line is composed of *Drake* conductors with flat horizontal spacing of 11.9 ft between adjacent conductors. Assume a wire temperature of 50°C. Determine

(a) the series impedance and the shunt admittance of the line.

(b) the ABCD constants of the line.

(c) the sending-end voltage, current real and reactive powers and the power factor.

(d) the percent regulation of the line.

Solution:

(a)

$$D_{eq} = \sqrt[3]{11.9 \times 11.9 \times 2 \times 11.9} = 15 \text{ ft}$$

$$\text{series impedance } Z = 100 \times (0.1284 + j0.399 + j0.3286)$$

$$= 12.84 + j72.76 = 73.88 \underline{/80°} \ \Omega$$

$$\frac{Y}{2} = j\frac{100}{2}\left(\frac{10^{-6}}{0.9012 + 0.0803}\right) = 2.915 \times 10^{-4} \underline{/90°} \text{ S}$$

$$\text{shunt admittance } Y = 5.83 \times 10^{-4} \underline{/90°} \text{ S}$$

(b)

$$A = D = 1 + \frac{ZY}{2}$$

$$= 1 + \frac{73.88 \times 5.83 \times 10^{-4}}{2} \angle 170° = 0.979 \angle 0.219°$$

$$B = Z = 73.88 \angle 80° \ \Omega$$

$$C = Y\left(1 + \frac{ZY}{4}\right)$$

$$= 5.83 \times 10^{-4}\left(1 + \frac{73.88 \times 5.83 \times 10^{-4}}{4} \angle 170°\right) S = 5.768 \times 10^{-4} \angle 90.108° \ S$$

(Check: $AD - BC = 1$ must be satisfied)

(c)

$$I_R = \frac{55,000}{\sqrt{3} \times 132}(0.8 - j0.6) = 192.4 - j144.3 \ A$$

Current in series arm:

$$I_{series} = 192.4 - j144.3 + j\frac{132,000}{\sqrt{3}} \times 2.915 \times 10^{-4}$$

$$= 192.4 - j122.1 = 227.9 \angle -32.40°$$

$$V_S = \frac{132,000}{\sqrt{3}} + 227.9 \angle -32.40° \times 73.88 \angle 80°$$

$$= 87,563 + j12,434 = 88,441 \angle 8.08° \ V \text{ to neutral}$$

$$|V_S| = \sqrt{3} \times 88,441 = 153.2 \ kV, \text{ line-to-line}$$

$$I_S = 192.4 - j122.4 + j2.915 \times 10^{-4} \times (87,563 + j12,434)$$

$$= 188.8 - j96.9 = 212 \angle -27.2°$$

$$|I_S| = 212 \ A$$

$$\theta_S = 8.08° - (-27.2°) = 35.28°$$

$$P_S = \left(\sqrt{3} \times 153.2 \times 212\right)\cos 35.28° \ kW = 45.92 \ MW$$

$$Q_S = \left(\sqrt{3} \times 153.2 \times 212\right)\sin 35.28° \ kvar = 32.49 \ Mvar$$

(sending-end) p.f. $= \cos 35.28° = 0.816$ lagging

(d)

$$\% \text{ Reg.} = \frac{|V_S|/|A| - |V_{R,FL}|}{|V_{R,FL}|} \times 100\%$$

$$= \frac{(153.2/0.979) - 132}{132} \times 100\% = 18.55\%$$

6.3 Find the $ABCD$ constants of a π circuit having a 600-Ω resistor for the shunt branch at the sending end, a 1-kΩ resistor for the shunt branch at the receiving end and an 80-Ω resistor for the series branch.

Solution:
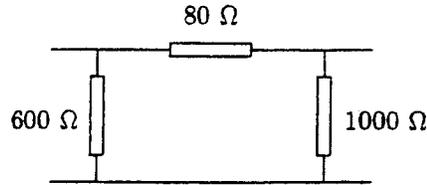
$$V_S = V_R + \left(I_R + \frac{V_R}{1000}\right) \times 80 = V_R + 80I_R + 0.08V_R$$
$$= 1.08V_R + 80I_R$$
$$I_S = I_R + \frac{V_R}{1000} + \frac{1.08V_R + 80I_R}{600} = 0.001V_R + 0.0018V_R + I_R + 0.133I_R$$
$$= 0.0028V_R + 1.133I_R$$

The $ABCD$ constants are

$$A = 1.08 \quad C = 0.0028 \text{ S}$$
$$B = 80 \text{ } \Omega \quad D = 1.133$$

6.4 The $ABCD$ constants of a three-phase transmission line are

$$A = D = 0.936 + j0.016 = 0.936\angle 0.98°$$
$$B = 33.5 + j138 = 142\angle 76.4° \text{ } \Omega$$
$$C = (-5.18 + j914) \times 10^{-6} \text{ S}$$

The load at the receiving end is 50 MW at 220 kV with a power factor of 0.9 lagging. Find the magnitude of the sending-end voltage and the voltage regulation. Assume the magnitude of the sending-end voltage remains constant.

Solution:

$$I_R = \frac{50,000}{\sqrt{3} \times 220 \times 0.9}\angle -25.84° = 145.8\angle -25.84° \text{ A}$$
$$V_R = \frac{220,000}{\sqrt{3}} = 127,000\angle 0° \text{ V}$$
$$V_S = 0.936\angle 0.98° \times 127,000\angle 0° + 142\angle 76.4° \times 145.8\angle -25.84°$$
$$= 118,855 + j2033 + 13,153 + j15,990 = 133.23\angle 7.77° \text{ kV}$$

With line-to-line sending-end voltage $|V_S| = \sqrt{3} \times 133.23 = 230.8$ kV,

$$|V_{R,NL}| = \frac{230.8}{0.936} = 246.5 \text{ kV}$$
$$\% \text{ Reg.} = \frac{246.5 - 220}{220} \times 100 = 12.0\%$$

6.5 A 70 mi, single-circuit, three-phase line composed of *Ostrich* conductors is arranged in flat horizontal spacing with 15 ft between adjacent conductors. The line delivers a load of 60 MW at 230 kV with 0.8 power factor lagging.

$$Q_S = 1.086 \times 0.742 \times \sin(2.125° - (-36.54°)) = 0.503 \text{ per unit}$$
$$= 100 \times 0.503 = 50.3 \text{ Mvar}$$

(c)

$$A = 1 + \frac{ZY}{2} = 1 + \frac{1}{2}(0.1166\underline{/67.5°} \times 0.192\underline{/90°}) = 0.990\underline{/0.248°}$$

$$\% \text{ Reg.} = \frac{|V_S|/|A| - |V_{R,FL}|}{|V_{R,FL}|} \times 100\%$$

$$= \frac{(1.086/0.990) - 1.0}{1.0} \times 100\% = 9.73\%$$

6.6 A single-circuit, three-phase transmission line is composed of *Parakeet* conductors with flat horizontal spacing of 19.85 ft between adjacent conductors. Determine the characteristic impedance and the propagation constant of the line at 60 Hz and 50°C temperature.

Solution:

At 50°C and 60 Hz, from Table A.3, for *Parakeet* conductors,

$$r = 0.1832 \ \Omega/\text{mi} \qquad X_a = 0.423 \ \Omega/\text{mi}$$
$$D_{eq} = \sqrt[3]{19.85^3 \times 2} \text{ ft} = 25 \text{ ft}$$
$$\text{At 25 ft, } X_d(\text{inductive}) = 0.3906 \ \Omega/\text{mi}$$

Therefore,

$$z = 0.1832 + j(0.423 + 0.3906) \ \Omega/\text{mi}$$
$$= 0.834\underline{/77.31°} \ \Omega/\text{mi}$$
$$X_a' = 0.0969 \times 10^{-6} \ \Omega \cdot \text{mi}$$
$$X_d(\text{capacitive}) = 0.0955 \times 10^{-6} \ \Omega \cdot \text{mi}$$
$$y = \frac{j}{X_a' + X_d} = \frac{10^{-6}\underline{/90°}}{0.0969 + 0.0955} \ \text{S/mi}$$
$$= 5.1975 \times 10^{-6}\underline{/90°} \ \text{S/mi}$$

Characteristic impedance:

$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.834\underline{/77.31°}}{5.1975 \times 10^{-6}\underline{/90°}}} \ \Omega = 400.6\underline{/-6.345°} \ \Omega$$

Propagation constant:

$$\gamma = \sqrt{zy} = \sqrt{0.834 \times 5.1975 \times 10^{-6}\underline{/77.31° + 90°}} \ \text{mi}^{-1} = 2.08 \times 10^{-3} \ \text{mi}^{-1}$$

6.7 Using Eqs. (6.23) and (6.24) show that, if the receiving-end of a line is terminated by its characteristic impedance Z_c, then the impedance seen at the sending end of the line is also Z_c regardless of line length.

Solution:

If $Z_R = Z_c$, then $I_R = V_R/Z_c$, and $V_R - I_R Z_c = 0$.

From Eq. (6.23) $\quad V_S = \dfrac{V_R + I_R Z_c}{2} \epsilon^{\gamma L}$

From Eq. (6.24) $\quad I_S = \dfrac{V_R + I_R Z_c}{2 Z_c} \epsilon^{\gamma L}$

where L is the length of the line. Finally,

$$Z_s = V_S/I_S = Z_c \quad \text{(which is independent of } L)$$

6.8 A 200-mi transmission line has the following parameters at 60 Hz

$$\begin{aligned}
\text{resistance } r &= 0.21 \ \Omega/\text{mi per phase} \\
\text{series reactance } x &= 0.78 \ \Omega/\text{mi per phase} \\
\text{shunt susceptance } b &= 5.42 \times 10^{-6} \ \text{S/mi per phase}
\end{aligned}$$

(a) Determine the attenuation constant α, wavelength λ and the velocity of propagation of the line at 60 Hz.

(b) If the line is open circuited at the receiving end and the receiving-end voltage is maintained at 100 kV line-to-line use Eqs. (6.26) and (6.27) to determine the incident and reflected components of the sending-end voltage and current.

(c) Hence determine the sending-end voltage and current of the line.

Solution:

(a)

$$\begin{aligned}
r &= 0.21 \ \Omega/\text{mi} \qquad x_l = 0.78 \ \Omega/\text{mi} \\
z &= (0.21 + j0.78) \ \Omega/\text{mi} = 0.808 \underline{/77.31°} \ \Omega/\text{mi} \\
y &= 5.42 \times 10^{-6} \underline{/77.31°} \ \text{S/mi} \\
\gamma &= \sqrt{zy} = 2.092 \times 10^{-3} \underline{/82.47°} \ \text{mi}^{-1} \\
&= \alpha + j\beta = (2.744 \times 10^{-4} + j2.074 \times 10^{-3}) \ \text{mi}^{-1}
\end{aligned}$$

$$\begin{aligned}
\text{Attenuation-constant } \alpha &= 2.744 \times 10^{-4} \ \text{nepers/mi} \\
\text{Wavelength } \lambda &= \frac{2\pi}{\beta} = \frac{2\pi \times 10^3}{2.074} \ \text{mi} = 3030 \ \text{mi} \\
\text{Velocity of propagation } \lambda f &= \frac{2\pi f}{\beta} = \frac{120\pi \times 10^3}{2.074} \ \text{mi/s} = 181770 \ \text{mi/s}
\end{aligned}$$

(b)

$$\text{Characteristic impedance: } Z_c = \sqrt{\frac{z}{y}} = 386.05 \underline{/-7.53°} \ \Omega$$

When the receiving end is open circuited, $I_R = 0$. Then,

$$\text{from Eq. (6.26)} \quad V_S = \frac{V_R}{2}\epsilon^{\alpha L}\epsilon^{j\beta L} + \frac{V_R}{2}\epsilon^{-\alpha L}\epsilon^{-j\beta L}$$

$$\text{from Eq. (6.27)} \quad I_S = \underbrace{\frac{V_R}{2Z_c}\epsilon^{\alpha L}\epsilon^{j\beta L}}_{incident} - \underbrace{\frac{V_R}{2Z_c}\epsilon^{-\alpha L}\epsilon^{-j\beta L}}_{reflected}$$

$$\text{where} \quad L = 200 \text{ mi} = \text{length of the line}$$

$$\epsilon^{\alpha L} = 1.0564 \qquad \epsilon^{-\alpha L} = 0.9466$$
$$\beta L = 2.074 \times 10^{-3} \times 200 \times \frac{180}{\pi} \text{ deg} = 23.77°$$

Hence, at the sending end (taking the receiving-end line voltage as reference), the line-to-line voltages and currents are

$$\text{incident voltage } v_i = \frac{100}{2}\angle 0° \times 1.0564\angle 23.77° \text{ kV}$$
$$= 52.82\angle 23.77° \text{ kV}$$
$$\text{reflected voltage } v_r = \frac{100}{2}\angle 0° \times 0.9466\angle -23.77° \text{ kV}$$
$$= 47.33\angle -23.77° \text{ kV}$$
$$\text{incident current } I_i = \frac{100\angle 0°}{2 \times 386.05\angle -7.53°} \times \frac{1.0564\angle 23.77°}{\sqrt{3}\angle 30°} \text{ kA}$$
$$= 78.99\angle 1.3° \text{ A}$$
$$\text{reflected current } I_r = -\frac{100\angle 0°}{2 \times 386.05\angle -7.53°} \times \frac{0.9466\angle -23.77°}{\sqrt{3}\angle 30°} \text{ kA}$$
$$= 70.78\angle 133.76° \text{ A}$$

(The 30° angle in the denominator of the second fraction of the current equations above represents a phase/line V conversion.)

(c)

$$V_S = V_i + V_r = 52.82\angle 23.77° + 47.33\angle -23.77° \text{ kV} = 91.68\angle 1.38° \text{ kV}$$
$$I_S = I_i + I_r = 78.99\angle 1.3° + 70.78\angle 133.76° \text{ A} = 60.8370.78\angle 60.4° \text{ A}$$

where all angles are expressed with respect to receiving-end *line* voltage.

6.9 Evaluate $\cosh\theta$ and $\sinh\theta$ for $\theta = 0.5\angle 82°$.

Solution:

$$0.5\angle 82° = 0.0696 + j0.4951$$
$$0.4951 \text{ radian} = 28.37°$$
$$\cosh\theta = \frac{1}{2}\left(\epsilon^{0.0696}\angle 28.37° + \epsilon^{-0.0696}\angle -28.37°\right)$$
$$= \frac{1}{2}(0.9433 + j0.5094 + 0.8207 - j0.4432) = 0.8820 + j0.0331$$
$$\sinh\theta = \frac{1}{2}(0.9433 + j0.5094 - 0.8207 + j0.4432) = 0.0613 + j0.4763$$

6.10 Using Eqs. (6.1), (6.2), (6.10) and (6.37) show that the generalized circuit constants of all three transmission line models satisfy the condition that

$$AD - BC = 1$$

Solution:

Short-line model (from Eq. (6.1) and (6.2)):

$$A = D = 1 \qquad B = Z \qquad C = 0$$
$$AD - BC = 1 - Z \times 0 = 1$$

Medium-length line model (from Eq. (6.10)):

$$A = D = \left(1 + \frac{ZY}{2}\right) \qquad B = Z \qquad C = Y\left(1 + \frac{ZY}{4}\right)$$
$$AD - BC = \left(1 + \frac{ZY}{2}\right)^2 - Z \times Y\left(1 + \frac{ZY}{4}\right) = 1 + ZY + \frac{Z^2Y^2}{4} - ZY - \frac{Z^2Y^2}{4} = 1$$

Long-line model (from Eq. (6.37)):

$$A = D = \cosh \gamma l \qquad B = Z_c \sinh \gamma l$$
$$C = \frac{\sinh \gamma l}{Z_c}$$
$$AD - BC = \cosh^2 \gamma l - Z_c \sinh \gamma l \left(\frac{\sinh \gamma l}{Z_c}\right)$$
$$= \cosh^2 \gamma l - \sinh^2 \gamma l = \left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{2}\right)^2 - \left(\frac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{2}\right)^2$$
$$= \frac{\epsilon^{2\gamma l} + 2 + \epsilon^{-2\gamma l}}{4} - \frac{\epsilon^{2\gamma l} - 2 + \epsilon^{-2\gamma l}}{4} = 1$$

6.11 The sending-end voltage, current and power factor of the line described in Example 6.3 are found to be 260 kV (line-to-line), 300 A and 0.9 lagging, respectively. Find the corresponding receiving-end voltage, current and power factor.

Solution:

From Example 6.3,

$$A = D = \cosh \gamma l = 0.8904 \underline{/1.34°}$$
$$B = Z_c \sinh \gamma l = 406.4 \underline{/-5.48°} \times 0.4597 \underline{/84.93°} \ \Omega = 186.82 \underline{/79.45°} \ \Omega$$
$$C = \frac{\sinh \gamma l}{Z_c} = \frac{0.4597 \underline{/84.93°}}{406.4 \underline{/-5.48°}} \ S = 1.131 \times 10^{-3} \underline{/90.41°} \ S$$
$$\begin{bmatrix} V_R \\ I_R \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} V_S \\ I_S \end{bmatrix}$$
$$V_R = DV_S - BI_S \qquad I_R = -CV_S + AI_S$$
$$V_S = \frac{260}{\sqrt{3}} \underline{/0°} \ kV = 150.11 \underline{/0°} \ kV$$

$$I_S = 300\underline{/-\cos^{-1}0.9}\ \text{A} = 300\underline{/-25.84°}\ \text{A}$$

$$V_R = 0.8904\underline{/1.34°} \times 150.11\underline{/0°} - \frac{186.82\underline{/79.45°} \times 300\underline{/-25.84°}}{1000}\ \text{kV}$$

$$= 108.85\underline{/-22.76°}\ \text{kV}$$

$$|V_R| = \sqrt{3} \times 108.85\ \text{kV} = 188.5\ \text{kV line-to-line}$$

$$I_R = -1.131 \times 10^{-3}\underline{/90.41°} \times 150.11 \times 10^{-3}\underline{/0°} + 0.8904\underline{/1.34°} \times 300\underline{/-25.84°}\ \text{A}$$

$$= 372.0\underline{/-48.95°}\ \text{A}$$

$$|I_R| = 372\ \text{A}$$

The receiving-end power factor is then

$$\text{p.f.} = \cos(-22.76° + 48.95°) = 0.897\ \text{lagging}$$

6.12 A 60 Hz three-phase transmission is 175 mi long. It has a total series impedance of $35 + j140\ \Omega$ and a shunt admittance of $930 \times 10^{-6}\underline{/90°}$ S. It delivers 40 MW at 220 kV, with 90% power factor lagging. Find the voltage at the sending end by (a) the short-line approximation, (b) the nomial-π approximation and (c) the long-line equation.

Solution:

$$l = 175\ \text{mi}$$

$$Z = 35 + j40 = 144.3\underline{/75.96°}\ \Omega$$

$$Y = 930 \times 10^{-6}\ \text{S}$$

$$I_R = \frac{40,000}{\sqrt{3} \times 220 \times 0.9} = 116.6\underline{/-25.84°}\ \text{A}$$

(a) Using the short-line approximation,

$$V_S = 127,017 + 116.6\underline{/-25.84°} \times 144.3\underline{/75.96°} = 127,017 + 10,788 + j12,912$$

$$= 138,408\underline{/5.35°}\ \text{V}$$

$$|V_S| = \sqrt{3} \times 138,408 = 239.73\ \text{kV}$$

(b) Using the nominal-π approximation and Eq. (6.5),

$$V_S = 127,017\left(\frac{0.1342}{2}\underline{/165.96°} + 1\right) + 144.3\underline{/75.96°} \times 116.6\underline{/-25.84°}$$

$$= 127,017(0.935 + j0.0163) + 10,788 + j12,912 = 129,549 + j14,982$$

$$= 130,412\underline{/6.6°}$$

$$|V_S| = \sqrt{3} \times 130,412 = 225.88\ \text{kV}$$

(c) Using the long-line equation,

$$Z_c = \left(\frac{144.3\underline{/75.96°}}{930 \times 10^{-6}\underline{/90°}}\right)^{\frac{1}{2}} = 394\underline{/-7.02°}$$

$$\gamma l = \sqrt{144.3 \times 930 \times 10^{-6}\underline{/165.96°}} = 0.3663\underline{/83.0°} = 0.0448 + j0.364$$

$$\epsilon^{0.0448}\epsilon^{j0.364} = 1.0458\underline{/20.86°} = 0.9773 + j0.3724$$

$$\epsilon^{-0.0448}\epsilon^{-j0.364} = 0.9562\underline{/-20.86°} = 0.8935 - j0.3405$$

$$\cosh \gamma l = (0.9773 + j0.3724 + 0.8935 - j0.3405)/2 = 0.9354 + j0.0160$$

$$\sinh \gamma l = (0.9773 + j0.3724 - 0.8935 + j0.3405)/2 = 0.0419 + j0.3565$$

$$V_S = 127,017(0.9354 + j0.0160) + 116.6\underline{/-25.84°} \times 394\underline{/-7.02°}(0.0419 + j0.3565)$$

$$= 118,812 + j2,032 + 10,563 + j12,715 = 129,315 + j14,747$$

$$= 130,153\underline{/6.5°} \text{ V}$$

$$|V_S| = \sqrt{3} \times 130,153 = 225.4 \text{ kV}$$

6.13 Determine the voltage regulation for the line described in Prob. 6.12. Assume that the sending-end voltage remains constant.

Solution:

By Problem 6.12, volt-to-neutral results,

$$V_S = 130.15 \text{ kV} \qquad V_R = 127.02 \text{ kV}$$

For $I_R = 0$, $V_S = V_R \cosh \gamma l$,

$$|V_{R,NL}| = \frac{130.15}{|0.9354 + j0.0161|} = 139.12 \text{ kV}$$

$$\% \text{ Reg.} = \frac{139.12 - 127.02}{127.02} \times 100 = 9.53\%$$

6.14 A three-phase 60-Hz transmission line is 250 mi long. The voltage at the sending end is 220 kV. The parameters of the line are $R = 0.2 \ \Omega/\text{mi}$, $X = 0.8 \ \Omega/\text{mi}$ and $Y = 5.3 \ \mu\text{S}/\text{mi}$. Find the sending-end current when there is no load on the line.

Solution:

$$Z = (0.2 + j0.8) \times 250 = 206.1\underline{/75.96°}$$

$$Y = 250 \times 5.3 \times 10^{-6} = 1.325 \times 10^{-3}\underline{/90°}$$

$$\gamma l = \sqrt{ZY} = \sqrt{206.1 \times 1.325 \times 10^{-3}\underline{/165.96°}} = 0.5226\underline{/82.98°}$$

$$= 0.0639 + j0.5187$$

$$Z_c = \sqrt{Z/Y} = \sqrt{\frac{206.1\underline{/75.96°}}{1.325 \times 10^{-3}\underline{/90°}}} = 394\underline{/-7.02°} \ \Omega$$

By Eq. (6.39) for $I_R = 0$,

$$I_S = (V_S/Z_c)\frac{\sinh \gamma l}{\cosh \gamma l}$$

$$\beta l = 0.5187 \text{ rad} = 29.72°$$

$$\epsilon^{\alpha l}\epsilon^{j\beta l} = 0.9258 + j0.5285$$

$$\epsilon^{-\alpha l}\epsilon^{-j\beta l} = 0.8147 - j0.4651$$

$$\cosh \gamma l = \frac{1}{2}(0.9258 + 0.8147 + j0.5285 - j0.4651) = 0.8709\underline{/2.086°}$$

$$\sinh \gamma l = \frac{1}{2}[0.9258 - 0.8147 + j(0.5285 + 0.4651)] = 0.4999 \underline{/83.61°}$$

$$I_S = \frac{220.000/\sqrt{3}}{394 \underline{/-7.02°}} \times \frac{0.4999 \underline{/83.61°}}{0.8709 \underline{/2.086°}} = 185.0 \underline{/88.54°} \text{ A}$$

6.15 If the load on the line described in Prob. 6.14 is 80 MW at 220 kV, with unity power factor, calculate the current, voltage and power at the sending end. Assume that the sending-end voltage is held constant and calculate the voltage regulation of the line for the load specified above.

Solution:

$$V_R = \frac{220}{\sqrt{3}} = 127 \text{ kV} \qquad I_R = \frac{80,000}{\sqrt{3} \times 220} = 209.95 \text{ A}$$

With values of $\cosh \gamma l$ and $\sinh \gamma l$ from Problem 6.14,

$$\begin{aligned}
V_S &= 127,017(0.8703 + j0.0317) + 209.95 \times 394 \underline{/-7.02°} \times 0.4999 \underline{/83.61°} \\
&= 110,528 + j4,026 + 9,592 + j40,232 = 128,014 \underline{/20.23°} \text{ V to neutral} \\
|V_S| &= \sqrt{3} \times 128,014 = 221.7 \text{ kV} \\
I_S &= 209.95(0.8703 + j0.0317) + \frac{127,000}{394 \underline{/-7.02°}} \times 0.4999 \underline{/83.61°} \\
&= 182.72 + j6.66 - 1.77 + j161.13 = 246.8 \underline{/42.84°} \text{ A} \\
P_S &= \sqrt{3} \times 221.7 \times 246.8 \cos(20.3° - 42.84°) = 87,486 \text{ kW} \quad \text{(or 87.5 MW)}
\end{aligned}$$

At $I_R = 0$,

$$|V_R| = \frac{127,000}{0.8709} = 145,826 \text{ V to neutral}$$

$$\% \text{ Reg.} = \frac{145.8 - 127}{127} = 14.8 \%$$

6.16 A three-phase tranmission line is 300 mi long and serves a load of 400 MVA, 0.8 lagging power factor at 345 kV. The $ABCD$ constants of the line are

$$\begin{aligned}
A &= D = 0.8180 \underline{/1.3°} \\
B &= 172.2 \underline{/84.2°} \ \Omega \\
C &= 0.001933 \underline{/90.4°} \text{ S}
\end{aligned}$$

(a) Determine the sending-end line-to-neutral voltage, the sending-end current and the percent voltage drop at full load.

(b) Determine the receiving-end line-to-neutral voltage at no load, the sending-end current at no load and the voltage regulation.

Solution:

$$V_R = \frac{345,000}{\sqrt{3}} = 199,186 \underline{/0°} \text{ V} \qquad I_R = \frac{400,000}{\sqrt{3} \times 345} = 669.4 \underline{/-36.87°} \text{ A}$$

(a)

$$V_S = 0.8180\underline{/1.3°} \times 199,186\underline{/0°} + 172.2\underline{/84.2°} \times 669.4\underline{/-36.87°}$$
$$= 256,738\underline{/20.15°} \text{ V}$$
$$I_S = 0.001933\underline{/90.4°} \times 199,186\underline{/0°} + 0.8180\underline{/1.3°} \times 669.4\underline{/-36.87°}$$
$$= 447.7\underline{/8.54°} \text{ A}$$
$$\text{Voltage drop} = \frac{256,738 - 199,186}{256,738} \times 100 = 22.4\%$$

(b)

$$V_{R,NL} = \frac{256,738\underline{/20.15°}}{0.8180\underline{/1.3°}} = 313,861\underline{/18.85°} \text{ V}$$
$$I_{S,NL} = 0.001933\underline{/90.4°} \times 313,861\underline{/18.85°} = 606.7\underline{/109.25°}$$

(c)

$$\% \text{ Reg.} = \frac{313,861 - 199,186}{199,186} = 57.6\%$$

6.17 Justify Eq. (6.50) by substituting for the hyperbolic functions the equivalent exponential expressions.

Solution:

$$\text{left hand side} = \tanh\gamma\frac{l}{2} = \frac{\sinh\gamma l/2}{\cosh\gamma l/2} = \frac{\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2}}{\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2}}$$

$$\text{right hand side} = \frac{\cosh\gamma l - 1}{\sinh\gamma l} = \frac{\frac{1}{2}(\epsilon^{\gamma l} + \epsilon^{-\gamma l}) - 1}{\frac{1}{2}(\epsilon^{\gamma l} - \epsilon^{-\gamma l})} = \frac{\epsilon^{\gamma l} - 2 + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}$$

$$= \frac{(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})^2}{(\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2})(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})} = \frac{\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2}}{\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2}}$$

Therefore, left hand side = right hand side

6.18 Determine the equivalent-π circuit for the line of Prob. 6.12.

Solution:

By Eq. (6.46) and Problem 6.12,

$$\sinh\gamma l = 0.0419 + j0.3565 = 0.359\underline{/83.3°}$$
$$Z' = 144.3\underline{/75.96°} \times \frac{0.359\underline{/83.3°}}{0.3663\underline{/83.0°}} = 141.4\underline{/75.99°} \ \Omega$$

By Eq. (6.49) and Problem 6.12,

$$\frac{Y'}{2} = \frac{1}{394\underline{/-7.02°}} \times \frac{0.9354 + j0.016 - 1}{0.359\underline{/83.3°}} = 471 \times 10^{-6}\underline{/89.8°} \text{ S}$$

6.19 Use Eqs. (6.1) and (6.2) to simplify Eqs. (6.57) and (6.58) for the short transmission line with (a) series reactance X and resistance R and (b) series reactance X and negligible resistance.

Solution:

From Eq. (6.1) and (6.2), it follows that, for a short line

$$A = D = 1 \qquad B = Z = R + jX \triangleq |Z| \angle \phi \qquad C = 0$$

(a)

From Eq. (6.57): $P_R = \dfrac{|V_S||V_R|}{|Z|} \cos(\phi - \delta) - \dfrac{|V_R|^2}{|Z|} \cos\phi$

From Eq. (6.58): $Q_R = \dfrac{|V_S||V_R|}{|Z|} \sin(\phi - \delta) - \dfrac{|V_R|^2}{|Z|} \sin\phi$

(b) If $R = 0$, then $B = Z = X \angle 90°$ and

$$P_R = \frac{|V_S||V_R|}{X} \sin\delta$$

$$Q_R = \frac{|V_S||V_R|}{X} \cos\delta - \frac{|V_R|^2}{X}$$

6.20 Rights of way for transmission circuits are difficult to obtain in urban areas and existing lines are often upgraded by reconductoring the line with larger conductors or by reinsulating the line for operation at higher voltage. Thermal considerations and maximum power which the line can transmit are the important considerations. A 138-kV line is 50 km long and is composed of *Partridge* conductors with flat horizontal spacing of 5 m between adjacent conductors. Neglect resistance and find the percent increase in power which can be transmitted for constant $|V_S|$ and $|V_R|$ while δ is limited to 45°

(a) if the *Partridge* conductor is replaced by *Osprey* which has more than twice the area of aluminum in square millimeters,

(b) if a second *Partridge* conductor is placed in a two-conductor bundle 40 cm from the original conductor and a center-to-center distance between bundles of 5 m and

(c) if the voltage of the original line is raised to 230 kV with increased conductor spacing of 8 m.

Solution:

Length of 50 km is a short line and with resistance neglected the generalized circuit constants are $A = 1 \angle 0°$ and $B = X \angle 90°$. Then, since resistance is neglected conductor heating is disregarded; and from Eq. (6.57),

$$P_R = \frac{|V_S||V_R|}{X} \cos 45°$$

or, inversely proportional to X if we assume constant $|V_S|$ and $|V_R|$. Additionally,

$$D_{eq} = \sqrt[3]{5 \times 5 \times 10} = 6.30 \text{ m, or } 6.30/0.3048 = 20.67 \text{ ft}$$

(a)

$$\text{For } Partridge: X = 0.0754 \ln \frac{20.67}{0.0217} = 0.5172 \ \Omega/\text{km}$$

$$\text{For } Osprey: X = 0.0754 \ln \frac{20.67}{0.0284} = 0.4969 \ \Omega/\text{km}$$

Ratio of P_R (new/old):

$$\frac{0.5172}{0.4969} = 1.041 \quad (4.1\% \text{ increase})$$

(b)

$$D_s = \sqrt{0.0217 \times (0.4/0.3048)} = 0.1688 \text{ ft}$$

$$X = 0.0754 \ln \frac{20.67}{0.1688} = 0.3625 \ \Omega/\text{km}$$

$$\frac{0.5172}{0.3625} = 1.427 \quad (42.7\% \text{ increase})$$

(c) P_R increases by factor of $\left(\frac{230}{138}\right)^2 = 2.78$ due to increased V. P_R decreases due to increase of X.

$$D_{eq} = \sqrt[3]{8 \times 8 \times 15} = 33.07 \text{ ft}$$

$$X = 0.0754 \ln \frac{33.07}{0.0217} = 0.5526 \text{ km}$$

$$\text{Decrease factor} = \frac{0.5172}{0.5526}$$

$$\text{Resultant factor of increase} = 2.78 \times \frac{0.5172}{0.5526} = 2.602$$

$$\text{Increase} = 160.3\%$$

However, in addition to the increase in conductor spacing and insulation, larger conductors will probably be required since current will increase by a factor of about 230/138 and $|I|^2 R$ loss in the line by a factor of about 2.78 for the increase in load at the same power factor.

6.21 Construct a receiving-end power-circle diagram similar to Fig. 6.11 for the line of Prob. 6.12. Locate the point corresponding to the load of Prob. 6.12 and locate the center of circles for various values of $|V_S|$ if $|V_R| = 220$ kV. Draw the circle passing through the load point. From the measured radius of the latter circle determine $|V_S|$ and compare this value with the values calculated for Prob. 6.12.

Solution:

Use scale of $1'' = 50$ MVA. By comparing the work in Problem 6.12(c) with the equation $V_S = AV_R + BI_R$ we find

$$A = 0.9354 + j0.0160 = 0.936\underline{/0.98°}$$

$$B = 394\underline{/-7.02°}\,(0.0419 + j0.3565) = 141.4\underline{/76.28°}\ \Omega$$

$$\beta - \alpha = 76.28° - 0.98° = 75.3°$$

$$\frac{|A|\,|V_R|^2}{|B|} = \frac{0.9354 \times 220^2}{141.4} = 320.2 \text{ MVA}$$

Use above data to construct load line through origin at $\cos^{-1} 0.9 = 25.8°$ in the first quadrant. Draw a vertical line at 40 MW. The load point is at the intersection of this line and the load line. The radius of the circle through the load point is $7.05''$.

$$7.05 \times 50 = 352.5$$

$$\frac{|V_S|\,|V_R|}{|B|} = 352.5$$

$$|V_S| = \frac{352.5 \times 141.4}{220} = 226.5 \text{ kV}$$

6.22 A synchronous condenser is connected in parallel with the load described in Prob. 6.12 to improve the overall power factor at the receiving end. The sending-end voltage is always adjusted so as to maintain the receiving-end voltage fixed at 220 kV. Using the power-circle diagram constructed for Prob. 6.21, determine the sending-end voltage and the reactive power supplied by the synchronous condenser when the overall power factor at the receiving end is (a) unity (b) 0.9 leading.

Solution:

On the diagram for Problem 6.21 draw a new load line in the fourth quadrant at $\cos^{-1} 0.9$ with the horizontal axis. Draw power circles at radii $|V_S|\,|V_R|/|B| = 311,\ 327,\ 342,\ 358,\ 373$ and 389 MVA for $|V_S| = 200,\ 210,\ 220,\ 230,\ 240$ and 250 kV, respectively. This provides the power circle diagram that we can use for parts (a) and (b).

For p.f. = 1.0 read $|V_S| = 214$ kV at 40 MW on the horizontal axis. The vertical distance between the horizontal axis and the load line in the first quadrant respresents the kvar of the capacitors needed. The value is 19.3 kvar.

For p.f. = 0.9 leading, read $|V_S| = 202$ kV where the vertical line through 40 MW intersects the load line in the fourth quadrant. The vertical distance between the two load lines at 40 MW represents the kvar of capacitors needed. The value is 38.6 kvar.

6.23 A series capacitor bank having a reactance of 146.6 Ω is to be installed at the midpoint of the 300-mi line of Prob. 6.16. The $ABCD$ constants for each 150 mi portion of line are

$$A = D = 0.9534\underline{/0.3°}$$

$$B = 90.33\underline{/84.1°}\ \Omega$$

$$C = 0.001014\underline{/90.1°}\ \text{S}$$

(a) Determine the equivalent $ABCD$ constants of the cascade combination of the line-capacitor-line. (See Table A.6 in the Appendix.)

(b) Solve Prob. 6.16 using these equivalent $ABCD$ constants.

<u>Note to Instructor</u>: This problem is somewhat long, but the solution is interesting to show that the $ABCD$ constants of networks in series as given in Table A.6 can be calculated by matrix multiplication. The problem also shows the large reduction in voltage accomplished by series capacitors in the middle of the line. Compare results of Problems 6.16 and 6.23.

Solution:

(a)

$$\text{Let } \bar{A} = \begin{bmatrix} 0.9534\angle 0.3° & 90.33\angle 84.1° \\ 0.001014\angle 90.1° & 0.9534\angle 0.3° \end{bmatrix}$$

$$\bar{A} \times \begin{bmatrix} 1.0\angle 0° & 1.46\angle -90° \\ 0 & 1.0\angle 0° \end{bmatrix} \times \bar{A} =$$

$$\begin{bmatrix} 0.9534\angle 0.3° & 50.91\angle -78.65° \\ 0.001014\angle 90.1° & 1.1022\angle 0.27° \end{bmatrix} \times \begin{bmatrix} 0.9534\angle 0.3° & 90.33\angle 84.1° \\ 0.001014\angle 90.1° & 0.9534\angle 0.3° \end{bmatrix}$$

$$= \begin{bmatrix} 0.9597\angle 1.18° & 42.30\angle 64.5° \\ 0.002084\angle 90.4° & 0.9597\angle 1.18° \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{new}$$

(b) For V_R and I_R from Problem 6.16,

$$V_S = 0.9597\angle 1.18° \times 199,186\angle 0° + 42.30\angle 64.5° \times 669.4\angle -36.87°$$
$$= 216,870\angle 4.5°$$

$$I_S = 0.002084\angle 90.4° \times 199,186\angle 0° + 0.9597\angle 1.18° \times 669.4\angle -36.87°$$
$$= 520.4\angle 4.44°$$

$$\text{Voltage drop} = \frac{216,870 - 199,186}{216,870} \times 100 = 8.15\%$$

(Compare this voltage drop with that of Problem 6.16)

$$V_{R,NL} = \frac{216,870\angle 4.5°}{0.9597\angle 1.18°} = 225,977\angle 3.32° \text{ V}$$

$$I_{S,NL} = 0.002084\angle 90.4° \times 225,977\angle 3.32° = 470.9\angle 93.7° \text{ A}$$

$$\% \text{ Reg.} = \frac{225,977 - 199,186}{199,186} \times 100 = 13.45\%$$

(without capacitors 57.6%)

6.24 The shunt admittance of a 300-mi transmission line is

$$y_c = 0 + j6.87 \times 10^{-6} \text{ S/mi}$$

Determine the $ABCD$ constants of a shunt reactor that will compensate for 60% of the total shunt admittance.

Solution:

Capacitive susceptance:	$B_C = 6.87 \times 10^{-6} \times 300 = 0.002061$ S
Inductive susceptance:	$B_L = 0.6 \times 0.002061 = 0.001237$ S
For the shunt reactor,	$A = D = 1.0 \angle 0°$
	$B = 0$
	$C = -j0.001237$ S

6.25 A 250-Mvar, 345-kV shunt reactor whose admittance is $0.0021 \angle -90°$ S is connected to the receiving end of the 300-mi line of Prob. 6.16 at no load.

(a) Determine the equivalent $ABCD$ constants of the line in series with the shunt reactor. (See Table A.6 in the Appendix.)

(b) Rework part (b) of Prob. 6.16 using these equivalent $ABCD$ constants and the sending-end volage found in Prob. 6.16.

Solution:

For the shunt reactor

$$A = D = 1.0 \qquad B = 0 \qquad C = -j0.0021 \text{ S}$$

(a)

$$A_{eq} = 0.818 \angle 1.3° + 172.2 \angle 84.2° \times 0.0021 \angle -90° = 1.1777 \angle -0.88°$$
$$B_{eq} = 172.2 \angle 84.2° \qquad D_{eq} = 0.818 \angle 1.3°$$
$$C_{eq} = 0.001933 \angle 90.4° + 0.0021 \angle -90° \times 0.818 \angle 1.3° = 0.000217 \angle 83.25°$$

(b) From Problem 6.16, $V_S = 256,738 \angle 20.15°$. So,

$$V_{R,NL} = \frac{256,738 \angle 20.15°}{1.1777 \angle -0.88°} = 217,999 \angle 21.03° \text{ V}$$
$$I_{S,NL} = 0.000217 \angle 83.25° \times 217,999 \angle 21.03° = 47.3 \angle 104.28° \text{ A}$$

Recall that the shunt reactor is in the circuit only at no load. So, from Problem 6.16,

$$V_{R,FL} = 199,186 \angle 0° \text{ V}$$

and

$$\% \text{ Reg.} = \frac{217,999 - 199,186}{199,186} = 9.45\%$$

(compare with Problems 6.16 and 6.24)

6.26 Draw the lattice diagram for current and plot current versus time at the sending end of the line of Example 6.8 for the line terminated in (a) an open circuit (b) a short circuit.

Solution:

(a) $Z_R = \infty$, and for current

$$\rho_R = -\frac{Z_R - Z_c}{Z_R + Z_c} = -\frac{1 - \frac{Z_c}{Z_R}}{1 + \frac{Z_c}{Z_R}} = -1$$

$$\rho_s = -\frac{0 - Z_c}{0 + Z_c} = +1$$

Initially $i^+ = \frac{120}{30} = 4$ A
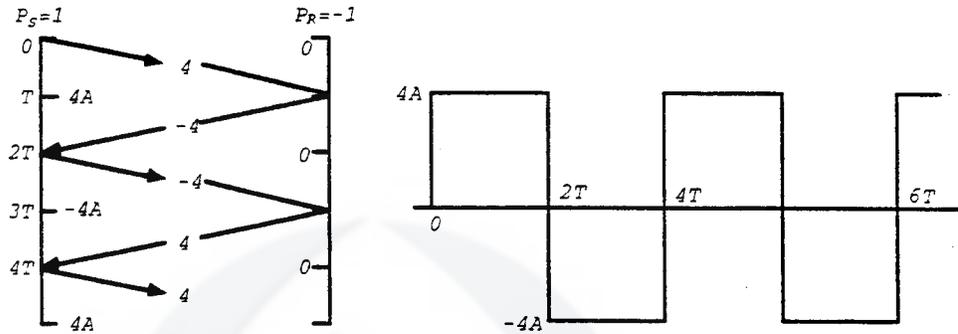
(b) $Z_R = 0$, and for current

$$\rho_R = -\frac{0 - Z_c}{0 + Z_c} = 1$$

$$\rho_s = -\frac{0 - Z_c}{0 + Z_c} = 1$$

Initially $i^+ = \frac{120}{30} = 4$ A
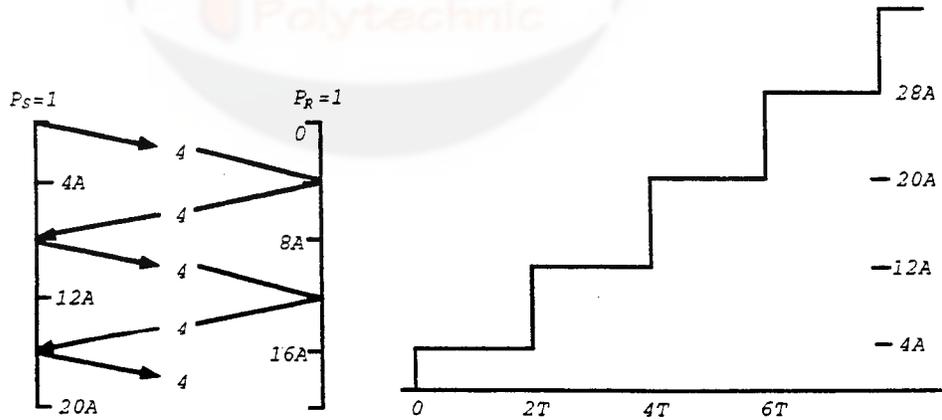
6.27 Plot voltage versus time for the line of Example 6.8 at a point distant from the sending end equal to one-fourth of the length of the line if the line is terminated in a resistance of 10 Ω.

Solution:

Imagine a vertical line on the diagram of Fig. 6.15(b) at one-fourth the line length from the sending end toward the receiving end. Intersections of this line and the slant lines occur at $T = 0.25T$, $1.75T$, $2.25T$, $3.75T$, etc. Changes in voltage occur at these times. The sum of the incident and reflected voltages are shown between slanted lines and determine the values plotted below.
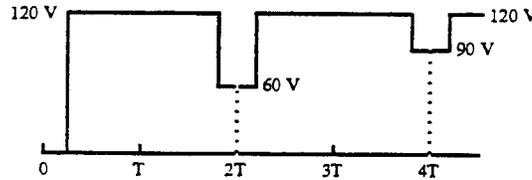
6.28 Solve Example 6.8 is a resistance of 54 Ω is in series with the source.

Solution:

For voltage,

$$\rho_s = \frac{54 - 30}{54 + 30} = \frac{2}{7}$$

$$\rho_R = \frac{90 - 30}{90 + 30} = \frac{1}{2}$$

Initial voltage impressed on line:

$$\frac{30}{30 + 54} \times 120 = 42.86 \text{ V}$$

Final value:

$$120 \times \frac{90}{90 + 54} = 75 \text{ V}$$
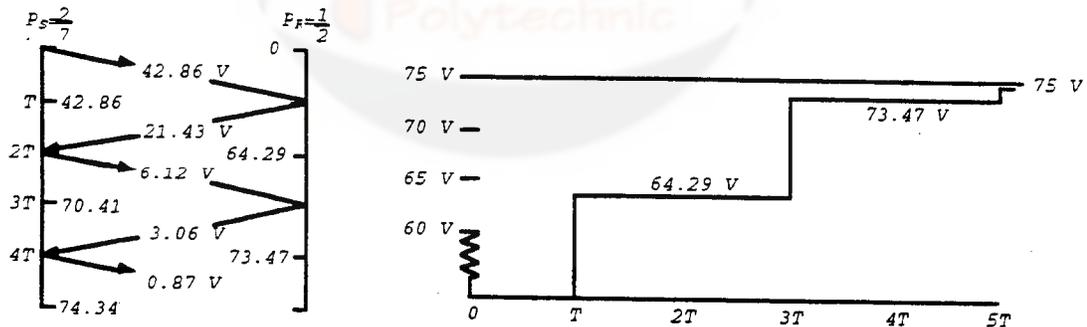
6.29 Voltage from a dc source is applied to an overhead transmission line by closing a switch. The end of the overhead line is connected to an underground cable. Assume both the line and the cable are lossless and that the initial voltage along the line is v^+. If the characteristic impedances of the line and cable are 400 Ω and 50 Ω, respectively, and the end of the cable is open-circuited, find in terms of v^+

(a)   the voltage at the junction of the line and cable immediately after the arrival of the incident wave and

(b)   the voltage at the open end of the cable immediately after arrival of the first voltage wave.

Solution:

(a) The initial wave of voltage v_0^+ arriving at the juction with the cable "sees" the Z_c of the cable. So, at the end of the overhead line:

$$\rho_R = \frac{50 - 400}{50 + 400} = -0.777$$

and the voltage at the juction is

$$(1 - 0.777)\, v^+ = 0.223 v_0^+$$

which is the refracted voltage wave travelling along the cable.

(b) At the end of the cable $\rho_R = 1.0$ and

$$v_R = (0.223 + 0.223)\, v^+ = 0.446 v_0^+$$

6.30 A dc source of voltage V_S and internal resistance R_S is connected through a switch to a lossless line having characteristic impedance R_c. The line is terminated in a resistance R_R. The travelling time of a voltage wave across the line is T. The switch closes at $t = 0$.

(a)   Draw a lattice diagram showing the voltage of the line during the period $t = 0$ to $t = 7T$. Indicate the voltage components in terms of V_S and the reflection coefficients ρ_R and ρ_s.

(b)   Determine the receiving-end voltage at $t = 0$, $2T$, $4T$ and $6T$, and hence at $t = 2nT$ where n is any non-negative integer.

(c)   Hence determine the steady state voltage at the receiving end of the line in terms of V_S, R_S, R_R and R_c.

(d)   Verify the result in Part (c) by analyzing the system as a simple dc circuit in the steady state. (Note that the line is lossless and remember how inductances and capacitances behave as short circuits and open circuits to dc.)

Solution:

(a)

$$V_f = V_S \left(\frac{R_c}{R_c + R_S} \right) \qquad \rho_R = \frac{R_R - R_c}{R_R + R_c} \qquad \rho_s = \frac{R_S - R_c}{R_S + R_c}$$
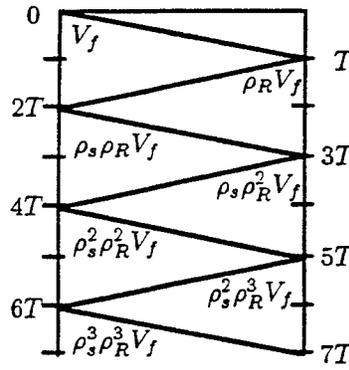
(b)

$$
\begin{aligned}
t = 0' \Rightarrow \quad V_R(0) &= 0 \\
t = 2T \Rightarrow V_R(2T) &= V_R(0) + V_f + \rho_R V_f = (1 + \rho_R)V_f \\
t = 4T \Rightarrow V_R(4T) &= V_R(2T) + \rho_s \rho_R V_f + \rho_s \rho_R^2 V_f \\
&= (1 + \rho_R)V_f + \rho_s \rho_R (1 + \rho_R)V_f \\
&= (1 + \rho_R)(1 + \rho_s \rho_R)V_f \\
t = 6T \Rightarrow V_R(6T) &= V_R(4T) + \rho_s^2 \rho_R^2 V_f + \rho_s^2 \rho_R^3 V_f \\
&= (1 + \rho_R)(1 + \rho_s \rho_R)V_f + \rho_s^2 \rho_R^2 (1 + \rho_R)V_f \\
&= (1 + \rho_R)\left[1 + \rho_s \rho_R + (\rho_s \rho_R)^2\right] V_f
\end{aligned}
$$

Hence at any given $t = 2nT$,

$$
V_R(2nT) = (1 + \rho_R)\left\{\sum_{j=0}^{n-1} (\rho_s \rho_R)^j\right\} V_f = (1 + \rho_R)\frac{1 - (\rho_s \rho_R)^n}{1 - \rho_s \rho_R} \cdot V_f
$$

(c) At the steady state, $n \to \infty$. If R_S or $R_R \neq 0$, $|\rho_s \rho_R| < 1$ and $(\rho_s \rho_R)^n \to 0$ as $n \to \infty$. Hence,

$$
\begin{aligned}
V_R(\infty) &= (1 + \rho_R)\frac{V_f}{1 - \rho_s \rho_R} \\
\text{Since, } \rho_R &= \frac{R_R - R_c}{R_R + R_c} \quad \text{and} \quad \rho_s = \frac{R_S - R_c}{R_S + R_c} \\
1 + \rho_R &= \frac{2R_R}{R_R + R_c} \\
\frac{1}{1 - \rho_s \rho_R} &= \frac{(R_S + R_c)(R_R + R_c)}{2R_c(R_S + R_R)} \\
V_f &= \frac{R_c}{R_S + R_c} \cdot V_S \\
V_R(\infty) &= \frac{2R_R}{R_R + R_c} \times \frac{(R_S + R_c)(R_R + R_c)}{2R_c(R_S + R_R)} \times \frac{R_c}{R_S + R_c} \cdot V_S = \frac{R_R}{R_S + R_R} \cdot V_S
\end{aligned}
$$

(d)

$$
V_R = V_S \left(\frac{R_R}{R_R + R_S}\right)
$$
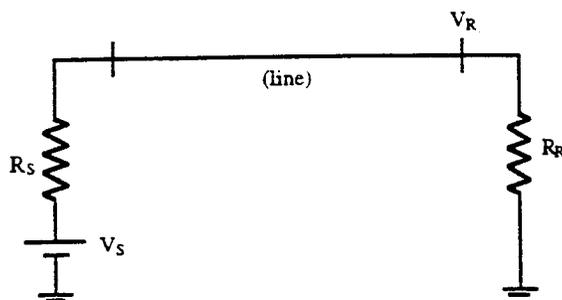
Chapter 7 Problem Solutions

7.1 Using the building-block procedure described in Sec. 7.1, determine $\mathbf{Y}_{bus}$ for the circuit of Fig. 7.18. Assume there is *no* mutual coupling between any of the branches.

Solution:

First the voltage sources are converted to current sources. Then, the building blocks are given as follows:

$$
\begin{array}{c}
①\\
①\ [\ 1\ ](-j1.0)
\end{array}
\qquad
\begin{array}{cc}
①\ \ \ \ ③\\
\begin{array}{c}①\\③\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j2.0)
\end{array}
\qquad
\begin{array}{cc}
①\ \ \ \ ②\\
\begin{array}{c}①\\②\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j2.5)
\end{array}
$$

$$
\begin{array}{cc}
②\ \ \ \ ③\\
\begin{array}{c}②\\③\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j4.0)
\end{array}
\qquad
\begin{array}{cc}
③\ \ \ \ ④\\
\begin{array}{c}③\\④\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j8.0)
\end{array}
\qquad
\begin{array}{cc}
②\ \ \ \ ⑤\\
\begin{array}{c}②\\⑤\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j5.0)
\end{array}
$$

$$
\begin{array}{cc}
④\ \ \ \ ⑤\\
\begin{array}{c}④\\⑤\end{array}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(-j2.0)
\end{array}
\qquad
\begin{array}{c}
⑤\\
⑤\ [\ 1\ ](-j8.0)
\end{array}
$$

Combining these together yields

$$
\begin{array}{c}
①\\②\\③\\④\\⑤
\end{array}
\begin{array}{ccccc}
①\quad\ \ \ \ ②\quad\ \ \ ③\quad\ \ \ ④\quad\ \ \ ⑤
\end{array}
$$

$$
\begin{array}{c}
①\\②\\③\\④\\⑤
\end{array}
\begin{bmatrix}
-j5.5 & j2.5 & j2 & j0 & j0\\
j2.5 & -j11.5 & j4 & j0 & j5\\
j2 & j4 & -j14 & j8 & j0\\
j0 & j0 & j8 & -j10 & j2\\
j0 & j5 & j0 & j2 & -j7.8
\end{bmatrix}
$$

7.2 Using the $\mathbf{Y_{bus}}$ modification procedure described in Sec. 7.4 and assuming no mutual coupling between branches, modify the $\mathbf{Y_{bus}}$ obtained in Prob. 7.1 to reflect removal of the two branches ①–③ and ②–⑤ from the circuit of Fig. 7.18.

Solution:

To remove branches ①–③ and ②–⑤, we add the following blocks to $\mathbf{Y_{bus}}$:

$$\begin{array}{c} \\ ① \\ ③ \end{array} \begin{array}{cc} ① & ③ \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}(j2.0) \qquad \begin{array}{c} \\ ② \\ ⑤ \end{array} \begin{array}{cc} ② & ⑤ \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}(j5.0)$$

This results in the following modified $\mathbf{Y_{bus}}$:

$$\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \\ \begin{bmatrix} -j3.5 & j2.5 & j0 & j0 & j0 \\ j2.5 & -j6.5 & j4 & j0 & j0 \\ j0 & j4 & -j12 & j8 & j0 \\ j0 & j0 & j8 & -j10 & j2 \\ j0 & j0 & j0 & j2 & -j2.8 \end{bmatrix} \end{array}$$

7.3 The circuit of Fig. 7.18 has the linear graph shown in Fig. 7.19 with arrows indicating directions assumed for the branches a to h. Disregarding all mutual coupling between branches

(a) determine the branch-to-node incidence matrix $\mathbf{A}$ for the circuit with node 0 as reference.

(b) find the circuit $\mathbf{Y_{bus}}$ using Eq. (7.37).

Solution:

(a) The branch-to-node incidence matrix is found to be

$$\mathbf{A} = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

(b) $\mathbf{Y}_{pr}$ is given by

$$\mathbf{Y}_{pr} = \begin{array}{c} \\ \text{ⓐ} \\ \text{ⓑ} \\ \text{ⓒ} \\ \text{ⓓ} \\ \text{ⓔ} \\ \text{ⓕ} \\ \text{ⓖ} \\ \text{ⓗ} \end{array} \begin{array}{cccccccc} \text{ⓐ} & \text{ⓑ} & \text{ⓒ} & \text{ⓓ} & \text{ⓔ} & \text{ⓕ} & \text{ⓖ} & \text{ⓗ} \\ \begin{bmatrix} -j2.5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -j2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -j4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -j5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -j8 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -j2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j0.8 \end{bmatrix} \end{array}$$

$$\mathbf{Y}_{bus} = \mathbf{A}^T\mathbf{Y}_{pr}\mathbf{A} = \begin{array}{c} \\ \text{①} \\ \text{②} \\ \text{③} \\ \text{④} \\ \text{⑤} \end{array} \begin{array}{ccccc} \text{①} & \text{②} & \text{③} & \text{④} & \text{⑤} \\ \begin{bmatrix} -j5.5 & j2.5 & j2 & j0 & j0 \\ j2.5 & -j11.5 & j4 & j0 & j5 \\ j2 & j4 & -j14 & j8 & j0 \\ j0 & j0 & j8 & -j10 & j2 \\ j0 & j5 & j0 & j2 & -j7.8 \end{bmatrix} \end{array}$$

7.4 Consider that *only* the two branches ①–③ and ②–③ in the circuit of Fig. 7.18 are mutually coupled as indicated by the dots beside them and that their mutual impedance is $j0.15$ per unit (that is, ignore the dot on branch ②-⑤). Determine the circuit $\mathbf{Y}_{bus}$ by the procedure described in Sec. 7.2.

Solution:

The primitive impedance matrix for the mutually coupled branches ①–③ and ②–③ is inverted as a single entity to yield the primitive admittance matrix

$$\begin{array}{c} \\ \text{①-③} \\ \text{②-③} \end{array} \begin{array}{cc} \text{①-③} & \text{②-③} \\ \begin{bmatrix} j0.5 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} \end{array} = \begin{array}{cc} \text{①-③} & \text{②-③} \\ \begin{bmatrix} -j2.43902 & j1.46341 \\ j1.46341 & -j4.87805 \end{bmatrix} \end{array}$$

Building blocks of the two mutually coupled branches ①–③ and ②–③ are

$$\begin{array}{c} \\ \text{①} \\ \text{③} \end{array} \begin{array}{cc} \text{①} & \text{③} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.43902) \end{array} \qquad \begin{array}{c} \\ \text{②} \\ \text{③} \end{array} \begin{array}{cc} \text{②} & \text{③} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.46341) \end{array}$$

$$\begin{array}{c} \\ \text{①} \\ \text{③} \end{array} \begin{array}{cc} \text{①} & \text{③} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.46341) \end{array} \qquad \begin{array}{c} \\ \text{②} \\ \text{③} \end{array} \begin{array}{cc} \text{②} & \text{③} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j4.87805) \end{array}$$

Building blocks of the remaining branches are determined as

$$\begin{array}{c} \\ \text{①} \end{array} \begin{array}{c} \text{①} \\ \begin{bmatrix} 1 \end{bmatrix}(-j1.0) \end{array} \qquad \begin{array}{c} \\ \text{①} \\ \text{②} \end{array} \begin{array}{cc} \text{①} & \text{②} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.5) \end{array} \qquad \begin{array}{c} \\ \text{③} \\ \text{④} \end{array} \begin{array}{cc} \text{③} & \text{④} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j8.0) \end{array}$$

$$\begin{array}{c} \\ \text{②} \\ \text{⑤} \end{array} \begin{array}{cc} \text{②} & \text{⑤} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j5.0) \end{array} \qquad \begin{array}{c} \\ \text{④} \\ \text{⑤} \end{array} \begin{array}{cc} \text{④} & \text{⑤} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.0) \end{array} \qquad \begin{array}{c} \\ \text{⑤} \end{array} \begin{array}{c} \text{⑤} \\ \begin{bmatrix} 1 \end{bmatrix}(-j8.0) \end{array}$$

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Combining all the above building blocks gives

$$
\begin{array}{c}
\begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\
\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array}
\begin{bmatrix}
-j5.93902 & j3.96341 & j0.97561 & j0 & j0 \\
j3.96341 & -j12.37805 & j3.41464 & j0 & j5 \\
j0.97561 & j3.41464 & -j12.39025 & j8 & j0 \\
j0 & j0 & j8 & -j10 & j2 \\
j0 & j5 & j0 & j2 & -j7.8
\end{bmatrix}
\end{array}
$$

7.5 Solve Prob. 7.4 using Eq. (7.37). Determine the branch-to-node incidence matrix **A** from the linear graph of Fig. 7.19 with node 0 as reference.

Solution:

The branch-to-node incidence matrix found in Prob. 7.3 can be used here. $\mathbf{Y}_{pr}$ is obtained by inverting $\mathbf{Z}_{pr}$ as follows.

$$
\mathbf{Y}_{pr} = \mathbf{Z}_{pr}^{-1} =
\begin{array}{c}
\begin{array}{cccccccc} ⓐ & ⓑ & ⓒ & ⓓ & ⓔ & ⓕ & ⓖ & ⓗ \end{array} \\
\begin{array}{c} ⓐ \\ ⓑ \\ ⓒ \\ ⓓ \\ ⓔ \\ ⓕ \\ ⓖ \\ ⓗ \end{array}
\begin{bmatrix}
j0.4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & j0.5 & j0.15 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & j0.15 & j0.25 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & j0.2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & j0.125 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & j0.5 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & j1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & j1.25
\end{bmatrix}^{-1}
\end{array}
$$

$$
=
\begin{array}{c}
\begin{array}{cccccccc} ⓐ & ⓑ & ⓒ & ⓓ & ⓔ & ⓕ & ⓖ & ⓗ \end{array} \\
\begin{array}{c} ⓐ \\ ⓑ \\ ⓒ \\ ⓓ \\ ⓔ \\ ⓕ \\ ⓖ \\ ⓗ \end{array}
\begin{bmatrix}
-j2.5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -j2.43902 & j1.46341 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & j1.46341 & -j4.87805 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -j5 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -j8 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -j2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j0.8
\end{bmatrix}
\end{array}
$$

Using Eq. (7.27), we have

$$
\begin{array}{c}
\begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\
\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array}
\begin{bmatrix}
-j5.93902 & j3.96341 & j0.97561 & 0 & 0 \\
j3.96341 & -j12.37805 & j3.41464 & 0 & j5 \\
j0.97561 & j3.41464 & -j12.39025 & j8 & 0 \\
0 & 0 & j8 & -j10 & j2 \\
0 & j5 & 0 & j2 & -j7.8
\end{bmatrix}
\end{array}
$$

(7.6) Using the modification procedure of Sec. 7.4, modify the Y$_{BUS}$ solution of Prob. 7.4 (or Prob. 7.5) to reflect removal of the branch 2-3 from the circuit.

Solution:

We first remove both branches (1-3) and (2-3) which are mutually coupled, and the new branch (1-3) is then reconnected.   Thus, the following building blocks are subtracted from Y$_{BUS}$:

$$
\begin{array}{c}
① \\
③
\end{array}
\begin{array}{cc}
① & ③ \\
\end{array}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
(-j2.43902)
\qquad
\begin{array}{c}
① \\
③
\end{array}
\begin{array}{cc}
② & ③ \\
\end{array}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
(j1.46341)
$$

$$
\begin{array}{c}
② \\
③
\end{array}
\begin{array}{cc}
① & ③ \\
\end{array}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
(j1.46341)
\qquad
\begin{array}{c}
② \\
③
\end{array}
\begin{array}{cc}
② & ③ \\
\end{array}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
(-j4.87805)
$$

And the following building block is added to Y$_{BUS}$:

$$
\begin{array}{c}
① \\
③
\end{array}
\begin{array}{cc}
① & ③ \\
\end{array}
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
(-j2.0)
$$

Giving Y_{BUS} =

$$
\begin{array}{c}
\quad\quad\quad ① \quad\quad ② \quad\quad ③ \quad\quad ④ \quad\quad ⑤ \\
\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array}
\begin{bmatrix}
-j5.5 & j2.5 & j2 & 0 & 0 \\
j2.5 & -j7.5 & 0 & 0 & j5 \\
j2 & 0 & -j10 & j8 & 0 \\
0 & 0 & j8 & -j10 & j2 \\
0 & j5 & 0 & j2 & -j7.8
\end{bmatrix}
\end{array}
$$

(7.7) Modify the Y_{BUS} determined in Example 7.3 to reflect removal of the mutually coupled branch 1-3 from the circuit of Fig. 7.11. Use the modification procedure of Sec. 7.4.

Solution:

First, both branches (1-3) and (2-3) are removed, and then branch (2-3) is reconnected. Thus, the following are subtracted from Y_{BUS}:

$$
\begin{array}{c} \\ ① \\ ③ \end{array}
\begin{array}{cc} ① & ③ \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j6.25)
\qquad
\begin{array}{c} \\ ① \\ ③ \end{array}
\begin{array}{cc} ② & ③ \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j3.75)
$$

$$
\begin{array}{c} \\ ② \\ ③ \end{array}
\begin{array}{cc} ① & ③ \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j3.75)
\qquad
\begin{array}{c} \\ ② \\ ③ \end{array}
\begin{array}{cc} ② & ③ \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j6.25)
$$

And the building block for the new branch (2-3) is added to Y_{BUS}:

$$
\begin{array}{c} \\ ② \\ ③ \end{array}
\begin{array}{cc} ② & ③ \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j4.0)
$$

Giving Y_{BUS} =

$$
\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array}
\begin{array}{cccc} ① & ② & ③ & ④ \end{array}
\begin{bmatrix}
-j10.5 & j8 & 0 & j2.5 \\
j8 & -j17 & j4 & j5 \\
0 & j4 & -j4.8 & 0 \\
j2.5 & j5 & 0 & -j8.3
\end{bmatrix}
$$

(7.8)   A new branch having a self-impedance of $j0.2$ per unit is added between nodes 2 and 3 in the circuit of Fig. 7.11. Mutual impedance of $j0.1$ per unit couples this new branch to the branch already existing between nodes 2 and 3. Modify the Y_{BUS} obtained in Example 7.3 to account for the addition of the new branch.

Solution:

Since the existing branch (2-3) is mutually coupled to branch (1-3), we remove these simultaneously. Then we reconnect these branches along with a new branch between nodes 2 and 3.

Thus, we subtract the following from Y_{BUS}:

$$
\begin{array}{c} \text{①} \\ \text{③} \end{array}
\begin{array}{cc} \text{①} & \text{③} \end{array}
\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] (-j6.25)
\qquad
\begin{array}{c} \text{①} \\ \text{③} \end{array}
\begin{array}{cc} \text{②} & \text{③} \end{array}
\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] (j3.75)
$$

$$
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{array}{cc} \text{①} & \text{③} \end{array}
\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] (j3.75)
\qquad
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{array}{cc} \text{②} & \text{③} \end{array}
\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] (-j6.25)
$$

And, to reconnect these branches, along with a new mutually coupled branch between nodes 2 and 3, the primitive admittance matrix is first found:

$$
\begin{array}{c} \text{1-3} \\ \text{2-3} \\ \text{2-3new} \end{array}
\left[\begin{array}{ccc} \text{1-3} & \text{2-3} & \text{2-3new} \\ j0.25 & j0.15 & 0 \\ j0.15 & j0.25 & j0.1 \\ 0 & j0.1 & j0.2 \end{array}\right]^{-1}
=
\left[\begin{array}{ccc} \text{1-3} & \text{2-3} & \text{2-3new} \\ -j7.27273 & j5.45455 & -j2.72727 \\ j5.45455 & -j9.09091 & j4.54545 \\ -j2.72727 & j4.54545 & -j7.27273 \end{array}\right]
$$

And from these, the additional building blocks:

$$
\begin{array}{c} \text{①} \quad \text{③} \end{array}
\begin{array}{c} \text{①} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j7.27273)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{①} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j5.45455)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{①} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j2.72727)
$$

$$
\begin{array}{c} \text{①} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j5.45455)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j9.09091)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j4.54545)
$$

$$
\begin{array}{c} \text{①} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j2.72727)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j4.54545)
\qquad
\begin{array}{c} \text{②} \quad \text{③} \end{array}
\begin{array}{c} \text{②} \\ \text{③} \end{array}
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j7.27273)
$$

Giving $Y_{BUS} =$

$$
\begin{bmatrix}
-j17.77273 & j10.72728 & j4.54545 & j2.5 \\
 & -j20.27274 & j4.54546 & j5.0 \\
 & & -j9.89091 & 0 \\
 & \text{Symmetric} & & -j8.3
\end{bmatrix}
$$

(7.9)   Suppose that mutual coupling exists pairwise between branches 1-3 and 2-3, and *also* between branches 2-3 and 2-5 of Fig. 7.18, as shown by the dots in that figure. The mutual impedance between the former pair of branches is $j0.15$ per unit (the same as in Prob. 7.4) and between the latter pair is $j0.1$ per unit. Use the procedure of Sec. 7.2 to find Y_{BUS} for the overall circuit including the *three* mutually coupled branches.

Solution:

The primitive impedance matrix is inverted as a single entity to yield:

$$
\begin{array}{c}
\quad\quad 3\text{-}1 \quad\quad 3\text{-}2 \quad\quad 5\text{-}2 \\
\begin{array}{c} 3\text{-}1 \\ 3\text{-}2 \\ 5\text{-}2 \end{array}
\begin{bmatrix}
j0.5 & j0.15 & 0 \\
j0.15 & j0.25 & j0.1 \\
0 & j0.1 & j0.2
\end{bmatrix}^{-1}
=
\begin{bmatrix}
-j2.58065 & j1.93548 & -j0.96774 \\
j1.93548 & -j6.45161 & j3.22581 \\
-j0.96774 & j3.22581 & -j6.61290
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{ccc}
3\text{-}1 & 3\text{-}2 & 5\text{-}2
\end{array}
$$

All other admittances are determined on an element-by-element basis.

The resulting admittance diagram is shown below:

The building blocks are:

$$\begin{array}{c} \quad ③ \quad ① \\ ③ \\ ① \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.58065) \qquad \begin{array}{c} \quad ③ \quad ② \\ ③ \\ ① \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.93548) \qquad \begin{array}{c} \quad ⑤ \quad ② \\ ③ \\ ① \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j0.96774)$$

$$\begin{array}{c} \quad ③ \quad ① \\ ③ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.93548) \qquad \begin{array}{c} \quad ③ \quad ② \\ ③ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j6.45161) \qquad \begin{array}{c} \quad ⑤ \quad ② \\ ③ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j3.22581)$$

$$\begin{array}{c} \quad ③ \quad ① \\ ⑤ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j0.96774) \qquad \begin{array}{c} \quad ③ \quad ② \\ ⑤ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(j3.22581) \qquad \begin{array}{c} \quad ⑤ \quad ② \\ ⑤ \\ ② \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j6.6129)$$

$$\begin{array}{c} \quad ② \quad ① \\ ② \\ ① \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.5) \qquad \begin{array}{c} \quad ③ \quad ④ \\ ③ \\ ④ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j8)$$

$$\begin{array}{c} \quad ⑤ \quad ④ \\ ⑤ \\ ④ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2) \qquad \begin{array}{c} ① \end{array}[\,1\,](-j1.0) \qquad \begin{array}{c} ⑤ \end{array}[\,1\,](-j0.8)$$

Giving Y_{BUS} =

$$\begin{bmatrix} -j6.08065 & j3.46774 & j0.64517 & 0 & j0.96774 \\ & -j9.11289 & j2.25806 & 0 & j3.38709 \\ & & -j13.16130 & j8 & j2.25807 \\ & \text{Symmetric} & & -j10 & j2 \\ & & & & -j9.41290 \end{bmatrix}$$

(7.10) Solve for the $\mathbf{Y}_{BUS}$ of Prob. 7.9 using Eq. (7.37). Use the linear graph of Fig. 7.19 with reference node 0 to determine the branch-to-node incidence matrix **A**.

Solution:

(a) The branch-to-node matrix is:

$$
A = \begin{array}{c} \\ ⓐ \\ ⓑ \\ ⓒ \\ ⓓ \\ ⓔ \\ ⓕ \\ ⓖ \\ ⓗ \end{array}
\begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \\ \end{array}
\left[\begin{array}{ccccc}
1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1
\end{array} \right]
$$

(b) The primitive admittance matrix is:

$$
Y_{pr} = Z_{pr}^{-1} =
\begin{array}{c} \\ ⓐ \\ ⓑ \\ ⓒ \\ ⓓ \\ ⓔ \\ ⓕ \\ ⓖ \\ ⓗ \end{array}
\begin{array}{cccccccc} ⓐ & ⓑ & ⓒ & ⓓ & ⓔ & ⓕ & ⓖ & ⓗ \\ \end{array}
\left[\begin{array}{cccccccc}
j0.4 & & & & & & & \\
& j0.5 & j0.15 & & & & & \\
& j0.15 & j0.25 & j0.1 & & & & \\
& & j0.1 & j0.2 & & & & \\
& & & & j0.125 & & & \\
& & & & & j0.5 & & \\
& & & & & & j1.0 & \\
& & & & & & & j1.25
\end{array} \right]^{-1}
$$

$$
=
\begin{array}{c}
ⓐ \\
ⓑ \\
ⓒ \\
ⓓ \\
ⓔ \\
ⓕ \\
ⓖ \\
ⓗ
\end{array}
\begin{bmatrix}
-j2.5 & & & & & & & \\
& -j2.58065 & j1.93548 & -j0.96774 & & & & \\
& j1.93548 & -j6.45161 & j3.22581 & & & & \\
& -j0.96774 & j3.22581 & -j6.61290 & & & & \\
& & & & -j8 & & & \\
& & & & & -j2 & & \\
& & & & & & -j1.0 & \\
& & & & & & & -j0.8
\end{bmatrix}
\begin{array}{c}
ⓐ \ ⓑ \ ⓒ \ ⓓ \ ⓔ \ ⓕ \ ⓖ \ ⓗ
\end{array}
$$

And $Y_{BUS} = A^T Y_{pr} A =$

$$
\begin{bmatrix}
-j6.08065 & j3.46774 & j0.64517 & 0 & j0.96774 \\
& -j9.11289 & j2.25806 & 0 & j3.38709 \\
& & -j13.16130 & j8 & j2.25807 \\
\text{Symmetric} & & & -j10 & j2 \\
& & & & -j9.41290
\end{bmatrix}
$$

(as in Prob. 7.9.)

(7.11). Suppose that the direction of branch *d* in Fig. 7.19 is reversed so that it is now directed from node 2 to node 5. Find the branch-to-node incidence matrix **A** of this modified graph and then solve for the Y$_{BUS}$ of Prob. 7.9 using Eq. (7.37).

Solution:
A is given by:

$$
\mathbf{A} = \begin{array}{c} \text{ⓐ} \\ \text{ⓑ} \\ \text{ⓒ} \\ \text{ⓓ} \\ \text{ⓔ} \\ \text{ⓕ} \\ \text{ⓖ} \\ \text{ⓗ} \end{array}
\begin{array}{ccccc}
\text{①} & \text{②} & \text{③} & \text{④} & \text{⑤} \\
\end{array}
\left[
\begin{array}{ccccc}
1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{array}
\right]
$$

The primitive impedance matrix can be written as follows where the sign of the mutual impedance between branches c and d is changed since the directions of the branch c to d does not match the polarities indicated by the dots of Fig. 7.18.

$$
Y_{pr} = Z_{pr}^{-1} =
\begin{array}{c}
 \\
ⓐ \\
ⓑ \\
ⓒ \\
ⓓ \\
ⓔ \\
ⓕ \\
ⓖ \\
ⓗ
\end{array}
\begin{bmatrix}
\overset{\textstyle ⓐ}{j0.4} & \overset{\textstyle ⓑ}{} & \overset{\textstyle ⓒ}{} & \overset{\textstyle ⓓ}{} & \overset{\textstyle ⓔ}{} & \overset{\textstyle ⓕ}{} & \overset{\textstyle ⓖ}{} & \overset{\textstyle ⓗ}{} \\
 & j0.5 & j0.15 & & & & & \\
 & j0.15 & j0.25 & -j0.1 & & & & \\
 & & -j0.1 & j0.2 & & & & \\
 & & & & j0.125 & & & \\
 & & & & & j0.5 & & \\
 & & & & & & j1.0 & \\
 & & & & & & & j1.25
\end{bmatrix}^{-1}
$$

$$
\begin{array}{c}
ⓐ \\
ⓑ \\
ⓒ \\
ⓓ \\
ⓔ \\
ⓕ \\
ⓖ \\
ⓗ
\end{array}
\begin{bmatrix}
\overset{\textstyle ⓐ}{-j2.5} & \overset{\textstyle ⓑ}{} & \overset{\textstyle ⓒ}{} & \overset{\textstyle ⓓ}{} & \overset{\textstyle ⓔ}{} & \overset{\textstyle ⓕ}{} & \overset{\textstyle ⓖ}{} & \overset{\textstyle ⓗ}{} \\
 & -j2.58065 & j1.93548 & j0.96774 & & & & \\
 & j1.93548 & -j6.45161 & -j3.22581 & & & & \\
 & j0.96774 & -j3.22581 & -j6.61290 & & & & \\
 & & & & -j8 & & & \\
 & & & & & -j2 & & \\
 & & & & & & -j1.0 & \\
 & & & & & & & -j0.8
\end{bmatrix}
$$

And $Y_{BUS} = A^T Y_{pr} A =$

$$
\begin{array}{c}
① \\
② \\
③ \\
④ \\
⑤
\end{array}
\begin{bmatrix}
\overset{\textstyle ①}{-j6.08065} & \overset{\textstyle ②}{j3.46774} & \overset{\textstyle ③}{j0.64517} & \overset{\textstyle ④}{0} & \overset{\textstyle ⑤}{j0.96774} \\
 & -j9.11289 & j2.25806 & 0 & j3.38709 \\
 & & -j13.16130 & j8 & j2.25807 \\
\text{Symmetric} & & & -j10 & j2 \\
 & & & & -j9.41290
\end{bmatrix}
$$

Note that regardless of the directions of the branches in the linear graph, a proper primitive impedance matrix can be chosen to give the correct result.

(7.12) Using the Y_{BUS} modification procedure described in Sec. 7.4, remove branch 2-3 from the Y_{BUS} solution obtained in Prob. 7.9 (or Prob. 7.10 or Prob. 7.11).

Solution:

Since line (2-3) is mutually coupled to lines (1-3) and (2-5), we first remove all three mutually coupled lines. The we reconnect lines (1-3) and (2-5) resulting in the removal of line (2-3) only.

The following building blocks are subtracted from Y_{BUS}:

$$
\begin{array}{c} ③ \\ ① \end{array}
\begin{bmatrix} ③ & ① \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2.58065)
\qquad
\begin{array}{c} ③ \\ ① \end{array}
\begin{bmatrix} ③ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.93548)
\qquad
\begin{array}{c} ③ \\ ① \end{array}
\begin{bmatrix} ⑤ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j0.96774)
$$

$$
\begin{array}{c} ③ \\ ② \end{array}
\begin{bmatrix} ③ & ① \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(j1.93548)
\qquad
\begin{array}{c} ③ \\ ② \end{array}
\begin{bmatrix} ③ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j6.45161)
\qquad
\begin{array}{c} ③ \\ ② \end{array}
\begin{bmatrix} ⑤ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(j3.22581)
$$

$$
\begin{array}{c} ⑤ \\ ② \end{array}
\begin{bmatrix} ③ & ① \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j0.96774)
\qquad
\begin{array}{c} ⑤ \\ ② \end{array}
\begin{bmatrix} ③ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(j3.22581)
\qquad
\begin{array}{c} ⑤ \\ ② \end{array}
\begin{bmatrix} ⑤ & ② \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j6.6129)
$$

And the following building blocks are added:

$$
\begin{array}{c} ① \\ ③ \end{array}
\begin{bmatrix} ① & ③ \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j2)
\qquad\qquad
\begin{array}{c} ① \\ ⑤ \end{array}
\begin{bmatrix} ② & ⑤ \\ 1 & -1 \\ -1 & 1 \end{bmatrix}(-j5)
$$

Giving:

$$
\begin{array}{ccccc}
 & ① & ② & ③ & ④ & ⑤ \\
\end{array}
$$

$$
\begin{array}{c}
① \\ ② \\ ③ \\ ④ \\ ⑤
\end{array}
\begin{bmatrix}
-j5.5 & j2.5 & j2 & 0 & 0 \\
j2.5 & -j7.5 & 0 & 0 & j5 \\
j2 & 0 & -j10 & j8 & 0 \\
0 & 0 & j8 & -j10 & j2 \\
0 & j5 & 0 & j2 & -j7.8
\end{bmatrix}
$$

(As in Prob. 7.6)

(7.13) Write nodal admittance equations for the circuit of Fig. 7.18 disregarding all mutual coupling. Solve the resultant equations for the bus voltages by the method of gaussian elimination.

Solution:

The nodal admittance equations are:

$$
\begin{bmatrix}
-j5.5 & j2.5 & j2 & 0 & 0 \\
j2.5 & -j11.5 & j4 & 0 & j5 \\
j2 & j4 & -j14 & j8 & 0 \\
0 & 0 & j8 & -j10 & j2 \\
0 & j5 & 0 & j2 & -j7.8
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
1.1\underline{/-90°} \\
0 \\
0 \\
0 \\
0.72\underline{/-120°}
\end{bmatrix}
$$

Forward elimination gives:

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & -j10.36364 & j4.90909 & 0 & j5 \\
0 & j4.90909 & -j13.27273 & j8 & 0 \\
0 & 0 & j8 & -j10 & j2 \\
0 & j5 & 0 & j2 & -j7.8
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
-j0.5 \\
-j0.4 \\
0 \\
-0.36-j0.62354
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & 1 & -0.47368 & 0 & -0.48246 \\
0 & 0 & -j10.94737 & j8 & j2.36842 \\
0 & 0 & j8 & -j10 & j2 \\
0 & 0 & j2.36842 & j2 & -j5.38772
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
0.04825 \\
-j0.63684 \\
0 \\
-0.36-j0.62354
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & 1 & -0.47368 & 0 & -0.48246 \\
0 & 0 & 1 & 0.73077 & -0.21635 \\
0 & 0 & 0 & -j4.15385 & j3.73077 \\
0 & 0 & 0 & j3.73077 & -j4.87532
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
0.04825 \\
0.05817 \\
-j0.46538 \\
-0.36-j0.62354
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & 1 & -0.47368 & 0 & -0.48246 \\
0 & 0 & 1 & -0.73077 & -0.21635 \\
0 & 0 & 0 & 1 & -0.89815 \\
0 & 0 & 0 & 0 & -j1.52454
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
0.04825 \\
0.05817 \\
0.11204 \\
-0.36-j0.62354
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & 1 & -0.47368 & 0 & -0.48246 \\
0 & 0 & 1 & -0.73077 & -0.21635 \\
0 & 0 & 0 & 1 & -0.89815 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
0.04825 \\
0.05817 \\
0.11204 \\
0.93178-j0.23614
\end{bmatrix}
$$

Back substitution yields the bus voltages:

$V_5 = 0.93178 - j0.23614 = 0.96124\underline{/-14.2210°}$

$V_4 = 0.94891 - j0.21209 = 0.97232\underline{/-12.5991°}$

$V_3 = 0.95319 - j0.20607 = 0.9752\underline{/-12.1990°}$

$V_2 = 0.94930 - j0.21154 = 0.97258\underline{/-12.5624°}$

$V_1 = 0.97812 - j0.17109 = 0.99297\underline{/-9.9216°}$

(7.14) Prove Eq. (7.69) based on Eq. (7.68).

Solution:

Consider the nodal admittance equations in the form:

$$
\begin{bmatrix}
Y_{11} & \cdots & Y_{1p} & \cdots & Y_{1n} \\
\vdots & & & & \\
Y_{p1} & \cdots & Y_{pp} & \cdots & Y_{pn} \\
\vdots & & & & \\
Y_{n1} & \cdots & Y_{np} & \cdots & Y_{nn}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\vdots \\
V_p \\
\vdots \\
V_n
\end{bmatrix}
=
\begin{bmatrix}
I_1 \\
\vdots \\
I_p \\
\vdots \\
I_n
\end{bmatrix}
$$

The pth equation is written as

$$
\sum_{k=1}^{n} Y_{pk}V_k = I_p
$$

If $I_p = 0$, V_p is given by:

$$
V_p = \frac{-1}{Y_{pp}}\left[\sum_{\substack{k=1 \\ k \ne p}}^{n} Y_{pk}V_k \right]
$$

By substitution of the above V_p into the jth equation, the jth equation becomes:

$$\sum_{\substack{k=1}}^{n} Y_{jk}V_k = \sum_{\substack{k=1 \\ k \neq p}}^{n} \left(Y_{jk} - \frac{Y_{jp}Y_{pk}}{Y_{pp}} \right)V_k$$

$$= \sum_{\substack{k=1 \\ k \neq p}}^{n} Y_{jk,\text{new}}\ V_k$$

In this reduced set of equations, the new coefficient is as defined in Eq. (7.69).

(7.15) Using the gaussian-elimination calculations of Prob. 7.13, find the triangular factors of Y_{BUS} for the circuit of Fig. 7.18.

Solution:
Elements of two matrices **L** and **U** are (refer to Prob. 7.14):

$$\mathbf{L} = \begin{bmatrix} -j5.5 & 0 & 0 & 0 & 0 \\ j2.5 & -j10.36364 & 0 & 0 & 0 \\ j2 & j4.90909 & -j10.94737 & 0 & 0 \\ 0 & 0 & j8 & -j4.15385 & 0 \\ 0 & j5 & j2.36842 & j3.73077 & -j1.52454 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & -0.45455 & -0.36364 & 0 & 0 \\ 0 & 1 & -0.47368 & 0 & -0.48246 \\ 0 & 0 & 1 & -0.73077 & -0.21635 \\ 0 & 0 & 0 & 1 & -0.89815 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(7.16) Use the triangular factors obtained in Prob. 7.15 to calculate new bus voltages for Fig. 7.18 when the voltage source at bus 5 is changed to $1.0\angle{-45°}$ per unit. Follow the procedure of Example 7.9.

Solution:

The new voltage source at bus 5 is converted to a current source, $1.0\angle{-45°} \div j1.25 = 0.8\angle{-125°}$ per unit. Using the L and U of Prob. 7.15, the equation to solve is

$$LUV = I = \begin{bmatrix} 1.1\angle{-90°} \\ 0 \\ 0 \\ 0 \\ 0.8\angle{-125°} \end{bmatrix}$$

We first let UV = V' and solve LV' = I for V' as follows:

$$\begin{bmatrix} -j5.5 & 0 & 0 & 0 & 0 \\ j2.5 & -j10.36364 & 0 & 0 & 0 \\ j2 & j4.90909 & -j10.94737 & 0 & 0 \\ 0 & 0 & j8 & -j4.15385 & 0 \\ 0 & j5 & j2.36842 & j3.73077 & -j1.52454 \end{bmatrix} \begin{bmatrix} V'_1 \\ V'_2 \\ V'_3 \\ V'_4 \\ V'_5 \end{bmatrix} = \begin{bmatrix} 1.1\angle{-90°} \\ 0 \\ 0 \\ 0 \\ 0.8\angle{-125°} \end{bmatrix}$$

Solving by back substitution:

$$V' = \begin{bmatrix} 0.2 \\ 0.04825 \\ 0.05817 \\ 0.11204 \\ 0.89383 - j0.37105 \end{bmatrix}$$

V is then determined from solution of **UV = V'**:

$$
\begin{bmatrix}
1 & -0.45455 & -0.36364 & 0 & 0 \\
0 & 1 & -0.47368 & 0 & -0.48246 \\
0 & 0 & 1 & -0.73077 & -0.21635 \\
0 & 0 & 0 & 1 & -0.89815 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{bmatrix}
=
\begin{bmatrix}
0.2 \\
0.04825 \\
0.05817 \\
0.11204 \\
0.89383 - j0.37105
\end{bmatrix}
$$

From which we get:

$$
V =
\begin{bmatrix}
0.95063 - j0.26884 \\
0.91531 - j0.33240 \\
0.92008 - j0.32381 \\
0.91483 - j0.33326 \\
0.89383 - j0.37105
\end{bmatrix}
=
\begin{bmatrix}
0.98791 \angle{-15.791°} \\
0.97380 \angle{-19.9588°} \\
0.97540 \angle{-19.3888°} \\
0.97364 \angle{-20.016°} \\
0.96779 \angle{-22.5445°}
\end{bmatrix}
$$

(7.17) Using the triangular factors obtained in Example 7.9, find the voltage at bus 3 of the circuit of Fig. 7.11 when an *additional* current of $0.2\angle{-120°}$ per unit is injected at bus 2. All other conditions of Fig. 7.11 are unchanged.

Solution:
The equations to be solved are:

$$
LUV = I =
\begin{bmatrix}
0 \\
0.2 \angle{-120°} \\
1.0 \angle{-90°} \\
0.68 \angle{-135°}
\end{bmatrix}
$$

L and U are given in Example 7.9. Let $UV = V'$; solve $LV' = I$ for V':

$$
\begin{bmatrix}
-j16.75 & 0 & 0 & 0 \\
j11.75 & -j11.00746 & 0 & 0 \\
j2.5 & j4.25373 & -j3.78305 & 0 \\
j2.5 & j6.75373 & j2.98305 & -j1.43082
\end{bmatrix}
\begin{bmatrix} V' \end{bmatrix}
=
\begin{bmatrix}
0 \\
0.2\underline{/-120°} \\
1.0\underline{/-90°} \\
0.68\underline{/-135°}
\end{bmatrix}
$$

Solving by back substitution:

$$
\begin{bmatrix} V' \end{bmatrix}
=
\begin{bmatrix}
0 \\
0.01574 - j0.00908 \\
0.28203 - j0.01022 \\
0.99832 - j0.40023
\end{bmatrix}
$$

V is next determined from solution of $UV = V'$:

$$
\begin{bmatrix}
1 & -0.70149 & -0.14925 & -0.14925 \\
0 & 1 & -0.38644 & -0.61356 \\
0 & 0 & 1 & -0.78853 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} V \end{bmatrix}
=
\begin{bmatrix}
0 \\
0.01574 - j0.00908 \\
0.28203 - j0.01022 \\
0.99832 - j0.40023
\end{bmatrix}
$$

From which we get:

$$
\begin{bmatrix} V \end{bmatrix}
=
\begin{bmatrix}
1.03916 - j0.37532 \\
1.04146 - j0.38055 \\
1.06924 - j0.32581 \\
0.99832 - j0.40023
\end{bmatrix}
=
\begin{bmatrix}
1.10486\underline{/-19.8585°} \\
1.10881\underline{/-20.0723°} \\
1.11778\underline{/-16.9466°} \\
1.07556\underline{/-21.8460°}
\end{bmatrix}
$$

(7.18) *(a)* Kron reduce Y_{BUS} of the circuit of Fig. 7.18 to reflect elimination of node 2. *(b)* Use the Y-Δ transformation of Table 1.2 to eliminate note 2 from the circuit of Fig. 7.18 and find Y_{BUS} for the resulting reduced network. Compare results of parts *(a)* and *(b)*.

Solution:

Y_{BUS} of the circuit of Fig. 7.18 is given by:

$$Y_{BUS} = j \begin{array}{c} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\ \begin{bmatrix} -5.5 & 2.5 & 2 & & \\ 2.5 & -11.5 & 4 & & 5 \\ 2 & 4 & -14 & 8 & \\ & & 8 & -10 & 2 \\ & 5 & & 2 & -7.8 \end{bmatrix} \end{array}$$

After Kron reduction of row 2 and column 2:

$$Y_{BUS}^{reduced} = j \begin{array}{c} \begin{array}{cccc} ① & ③ & ④ & ⑤ \end{array} \\ \begin{bmatrix} -4.95652 & 2.86957 & 0 & 1.08696 \\ 2.86957 & -12.60870 & 8 & 1.73913 \\ 0 & 8 & -10 & 2 \\ 1.08696 & 1.73913 & 2 & -5.62609 \end{bmatrix} \end{array}$$

(b) Node 2 is connected to nodes 1,3 and 5 as shown, all impedances are in per-unit.
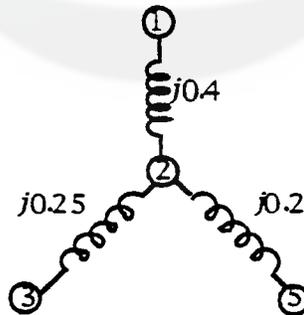
The Δ-equivalent circuit is:
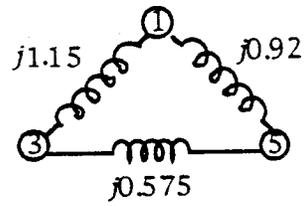
where:

$$Z_{15} = (j0.4)(j0.2)\frac{11.5}{j} = j0.92$$

$$Z_{13} = (j0.4)(j0.25)\frac{11.5}{j} = j1.15$$

$$Z_{35} = (j0.25)(j0.2)\frac{11.5}{j} = j0.575$$

When the Δ-equivalent circuit replaces the original star, the following results:

110
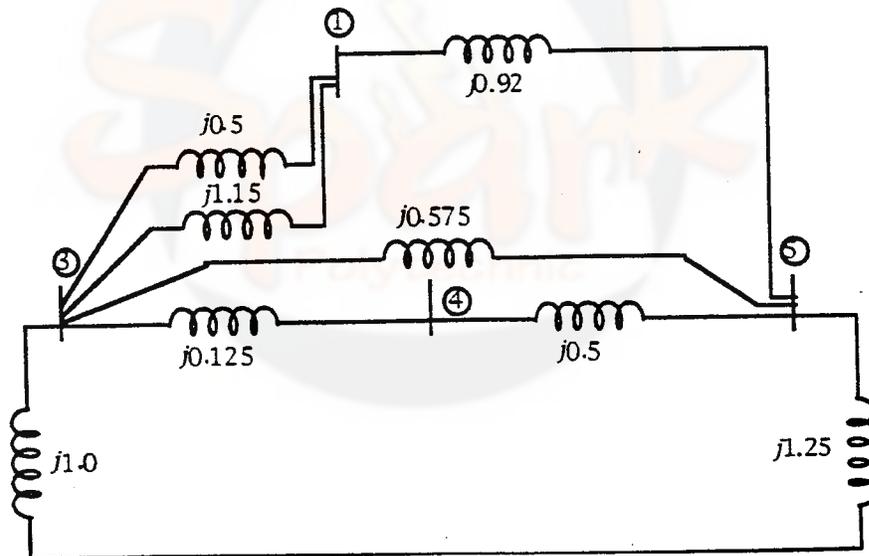
Giving :

$$\mathbf{Y}_{\text{BUS}} = j \begin{array}{cccc} \textcircled{1} & \textcircled{3} & \textcircled{4} & \textcircled{5} \\ \begin{bmatrix} -4.95652 & 2.86957 & 0 & 1.08696 \\ 2.86957 & -12.60870 & 8 & 1.73913 \\ 0 & 8 & -10 & 2 \\ 1.08696 & 1.73913 & 2 & -5.62609 \end{bmatrix} \end{array}$$

confirming the earlier result.

(7.19) Find the L and U triangular factors of the symmetric matrix

$$M = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

Verify the result using Eq. (7.75).

Solution:

Forward elimination yields the following:

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 9/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 5/9 \\ 0 & 0 & 10/9 \end{bmatrix}$$

L and U are:

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9/2 & 0 \\ 3 & 5/2 & 10/9 \end{bmatrix} \quad ; \quad U = \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 5/9 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$U^T D = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 5/9 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 9/2 & 0 \\ 0 & 0 & 10/9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 9/2 & 0 \\ 3 & 5/2 & 10/9 \end{bmatrix} = L$$

Which verifies Eq. (7.75).

Chapter 8 Problem Solutions

8.1 Form $\mathbf{Z_{bus}}$ for the circuit of Fig. 8.13 after removing node ⑤ by converting the voltage source to a current source. Determine the voltages with respect to reference node at each of the four other nodes when $V = 1.2\angle 0°$ and the load currents are $I_{L1} = -j0.1$, $I_{L2} = -j0.1$, $I_{L3} = -j0.2$ and $I_{L4} = -j0.2$, all in per unit.

Solution:

When the voltage source is converted to a current source and added as an injected current at that node, the voltages are

$$\mathbf{Z_{bus}I} = \begin{bmatrix} j0.2 & j0.2 & j0.2 & j0.2 \\ j0.2 & j0.6 & j0.2 & j0.6 \\ j0.2 & j0.2 & j0.8 & j0.2 \\ j0.2 & j0.6 & j0.2 & j1.1 \end{bmatrix} \begin{bmatrix} -j5.9 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix} \text{ per unit}$$

Alternatively, if the high-side of the voltage source is chosen as the reference:

$$\mathbf{Z_{bus}I} = \begin{bmatrix} j0.2 & j0.2 & j0.2 & j0.2 \\ j0.2 & j0.6 & j0.2 & j0.6 \\ j0.2 & j0.2 & j0.8 & j0.2 \\ j0.2 & j0.6 & j0.2 & j1.1 \end{bmatrix} \begin{bmatrix} j0.1 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} \text{ per unit}$$

and the bus voltages are

$$1.2 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix} \text{ per unit}$$

(8.2)   From the solution of Prob. 8.1, draw the Thévénin equivalent circuit at bus 4 of Fig. 8.13 and use it to determine the current drawn by a capacitor of reactance 5.4 per-unit connected between bus 4 and reference. Following the procedure of Example 8.2, calculate the voltage *changes* at each of the buses due to the capacitor.

Solution:

The equivalent circuit is:
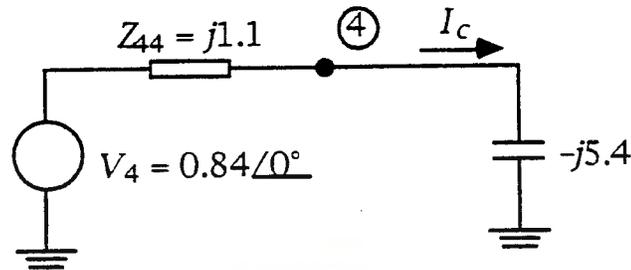
The capacitive current I_c is $0.84/j(1.1-5.4) = -j0.2$. The voltage changes are then:

$$\Delta V = Z_{BUS} \begin{bmatrix} 0 \\ 0 \\ 0 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.12 \\ 0.04 \\ 0.22 \end{bmatrix}$$

(8.3)   Modify Z_{BUS} of Prob. 8.1 to include a capacitor of reactance 5.4 per-unit connected between bus 4 and reference and then calculate the new bus voltages using the modified Z_{BUS}. Check your answers using the results of Probs. 8.1 and 8.2.

Solution: Z_{BUS} is augmented to add the shunt capacitor:

$$j\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 & 1.1 \\ \hline 0.2 & 0.6 & 0.2 & 1.1 & -4.3 \end{bmatrix}$$

After kron reduction, Z_{BUS} is given by:

$$j\begin{bmatrix} 0.20930 & 0.22791 & 0.20930 & 0.25116 \\ 0.22791 & 0.68372 & 0.22791 & 0.75349 \\ 0.20930 & 0.22791 & 0.80930 & 0.25116 \\ 0.25116 & 0.75349 & 0.25116 & 1.38140 \end{bmatrix}$$

And the voltages are:

$$j\begin{bmatrix} 0.20930 & 0.22791 & 0.20930 & 0.25116 \\ 0.22791 & 0.68372 & 0.22791 & 0.75349 \\ 0.20930 & 0.22791 & 0.80930 & 0.25116 \\ 0.25116 & 0.75349 & 0.25116 & 1.38140 \end{bmatrix}\begin{bmatrix} -j5.9 \\ j0.1 \\ j0.2 \\ j0.2 \end{bmatrix} = \begin{bmatrix} 1.12 \\ 1.08 \\ 1.00 \\ 1.08 \end{bmatrix}$$

Which is the sum of V and ΔV from Probs. 8.1 and 8.2.

(8.4)   Modify the Z_{BUS} determined in Example 8.4 for the circuit of Fig. 8.8 by adding a new node connected to Bus 3 through an impedance of $j0.5$ per-unit.

Solution: Let the new node be designated by node q. Z_{BUS} becomes:

$$
j\begin{bmatrix}
& ① & ② & ③ & ④ & ⑨ \\
0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.53340 \\
0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.64008 \\
0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.71660 \\
0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.66951 \\
0.53340 & 0.64008 & 0.71660 & 0.66951 & 1.21660
\end{bmatrix}
$$

(8.5) Modify the Z_{BUS} determined in Example 8.4 by adding a branch of impedance $j0.2$ between buses 1 and 4 of the circuit of Fig. 8.8.

Solution:
Z_{BUS} is augmented as follows:

$$
j\begin{bmatrix}
& ① & ② & ③ & ④ & ⑨ \\
0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.13611 \\
0.60992 & 0.73190 & 0.64008 & 0.69659 & -0.08667 \\
0.53340 & 0.64008 & 0.71660 & 0.66951 & -0.13611 \\
0.58049 & 0.69659 & 0.66951 & 0.76310 & -0.18261 \\
0.13611 & -0.08667 & -0.13611 & -0.18261 & 0.51872
\end{bmatrix}
$$

and by kron reduction, we get:

$$j \begin{bmatrix} 0.68089 & 0.63266 & 0.56911 & 0.62841 \\ 0.63266 & 0.71742 & 0.61734 & 0.66608 \\ 0.56911 & 0.61734 & 0.68089 & 0.62159 \\ 0.62841 & 0.66608 & 0.62159 & 0.69881 \end{bmatrix}$$

(8.6) Modify the Z_{BUS} determined in Example 8.4 by removing the impedance connected between buses 2 and 3 of the circuit of Fig. 8.8.

Solution:

To remove the branch between nodes 2 and 3, we add a second branch of impedance $-j0.4$ between the two nodes. Therefore, Z_{BUS} is augmented as follows:

$$j \begin{bmatrix} 0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.07652 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.09182 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & -0.07652 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.02708 \\ \hline 0.07652 & 0.09182 & -0.07652 & 0.02708 & -0.23166 \end{bmatrix}$$

and by kron reduction, we get:

$$j \begin{bmatrix} 0.74188 & 0.64025 & 0.50812 & 0.58943 \\ 0.64025 & 0.76829 & 0.60975 & 0.70732 \\ 0.50812 & 0.60975 & 0.74188 & 0.66057 \\ 0.58943 & 0.70732 & 0.66057 & 0.76627 \end{bmatrix}$$

(8.7) Find Z_{BUS} for the circuit of Fig. 7.18 by the Z_{BUS} building algorithm discussed in Sec. 8.4. Assume there is no mutual coupling between branches.

Solution:

$$0\text{-}1 \quad j\begin{array}{c}①\\[1]\end{array} \qquad 1\text{-}2 \quad j\begin{array}{cc}①&②\\\begin{bmatrix}1&1\\1&1.4\end{bmatrix}\end{array} \qquad 1\text{-}3 \quad j\begin{array}{ccc}①&②&③\\\begin{bmatrix}1&1&1\\1&1.4&1\\1&1&1.5\end{bmatrix}\end{array}$$

$$2\text{-}3 \quad j\begin{array}{cccc}①&②&③&④\\\left[\begin{array}{ccc|c}1&1&1&0\\1&1.4&1&0.4\\1&1&1.5&-0.5\\\hline0&0.4&-0.5&1.15\end{array}\right]\end{array} \; ;$$

After kron reduction: $\quad j\begin{array}{ccc}①&②&③\\\begin{bmatrix}1&1&1\\1&1.26087&1.17391\\1&1.17391&1.28261\end{bmatrix}\end{array}$

$$3\text{-}4 \quad j\begin{array}{cccc}①&②&③&④\\\begin{bmatrix}1&1&1&1\\1&1.26087&1.17391&1.17391\\1&1.17391&1.28261&1.28261\\1&1.17391&1.28261&1.40761\end{bmatrix}\end{array}$$

$$0\text{-}5 \quad j\begin{bmatrix}
\overset{①}{1} & \overset{②}{1} & \overset{③}{1} & \overset{④}{1} & \overset{⑤}{0} \\
1 & 1.26087 & 1.17391 & 1.17391 & 0 \\
1 & 1.17391 & 1.28261 & 1.28261 & 0 \\
1 & 1.17391 & 1.28261 & 1.40761 & 0 \\
0 & 0 & 0 & 0 & 1.25
\end{bmatrix} ;$$

$$2\text{-}5 \quad j\left[\begin{array}{ccccc|c}
\overset{①}{1} & \overset{②}{1} & \overset{③}{1} & \overset{④}{1} & \overset{⑤}{0} & \overset{ⓠ}{1} \\
1 & 1.26087 & 1.17391 & 1.17391 & 0 & 1.26087 \\
1 & 1.17391 & 1.28261 & 1.28261 & 0 & 1.17391 \\
1 & 1.17391 & 1.28261 & 1.40761 & 0 & 1.17391 \\
0 & 0 & 0 & 0 & 1.25 & -1.25 \\
\hline
1 & 1.26087 & 1.17391 & 1.17391 & -1.25 & 2.71087
\end{array}\right]$$

By kron reduction:

$$j\begin{bmatrix}
\overset{①}{0.63111} & \overset{②}{0.53488} & \overset{③}{0.56696} & \overset{④}{0.56696} & \overset{⑤}{0.46111} \\
0.53488 & 0.67442 & 0.62791 & 0.62791 & 0.58140 \\
0.56696 & 0.62791 & 0.77426 & 0.77426 & 0.54130 \\
0.56696 & 0.62791 & 0.77426 & 0.89926 & 0.54130 \\
0.46111 & 0.58140 & 0.54130 & 0.54130 & 0.67362
\end{bmatrix}$$

$$4\text{-}5 \quad j\left[\begin{array}{ccccc|c}
\overset{①}{0.63111} & \overset{②}{0.53488} & \overset{③}{0.56696} & \overset{④}{0.56696} & \overset{⑤}{0.46111} & \overset{ⓠ}{0.10585} \\
0.53488 & 0.67442 & 0.62791 & 0.62791 & 0.58140 & 0.04651 \\
0.56696 & 0.62791 & 0.77426 & 0.77426 & 0.54130 & 0.23296 \\
0.56696 & 0.62791 & 0.77426 & 0.89926 & 0.54130 & 0.35796 \\
0.46111 & 0.58140 & 0.54130 & 0.54130 & 0.67362 & -0.13232 \\
\hline
0.10585 & 0.04651 & 0.23296 & 0.35796 & -0.13232 & 0.99028
\end{array}\right]$$

After final kron reduction, Z_{BUS} becomes:

$$j\begin{array}{ccccc} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} \\ \begin{bmatrix} 0.61980 & 0.52991 & 0.54206 & 0.52870 & 0.47525 \\ & 0.67224 & 0.61697 & 0.61110 & 0.58761 \\ & & 0.71946 & 0.69005 & 0.57243 \\ & \text{Symmetric} & & 0.76987 & 0.58913 \\ & & & & 0.65594 \end{bmatrix} \end{array}$$

(8.8)   For the reactance network of Fig. 8.14, find
(a) Z_{BUS} by direct formation
(b) The voltage at each bus,
(c) The current drawn by a capacitor having a reactance of 5.0 per unit connected from bus 3 to neutral,
(d) the change in voltage at each bus when the capacitor is connected to bus 3, and
(e)   The voltage at each bus after connecting the capacitor. The magnitude and angle of each of the generated voltages is assumed to be constant.

Solution:

(a)

$$Z_{BUS} = j\begin{array}{ccc} \textcircled{1} & \textcircled{2} & \textcircled{3} \\ \begin{bmatrix} 0.411 & 0.310 & 0.354 \\ 0.310 & 0.446 & 0.333 \\ 0.354 & 0.333 & 0.450 \end{bmatrix} \end{array}$$

(b)

$$V = Z_{BUS}\begin{bmatrix} 1.6\angle{-90°} \\ 1.2\angle{-60°} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.980 + j0.186 \\ 0.959 + j0.268 \\ 0.912 + j0.200 \end{bmatrix} = \begin{bmatrix} 0.997\angle{10.75°} \\ 0.996\angle{15.61°} \\ 0.934\angle{12.37°} \end{bmatrix}$$

(c) The capacitive current is determined by using $Z_{th} = Z_{33}$ and V_3:

$$I_c = \frac{0.934\underline{/12.37°}}{j0.450 - j5.0} = 0.205\underline{/102.37°} \text{ p.u.}$$

(d) The changes in bus voltages due to I_c are:

$$\Delta V = Z_{BUS} \Delta I$$

$$= j \begin{bmatrix} 0.411 & 0.310 & 0.354 \\ 0.310 & 0.446 & 0.333 \\ 0.354 & 0.333 & 0.450 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.205\underline{/102.37°} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0726\underline{/12.37°} \\ 0.0683\underline{/12.37°} \\ 0.0923\underline{/12.37°} \end{bmatrix}$$

(e) The resulting voltages are:

$$V_{new} = V + \Delta V$$

$$= \begin{bmatrix} 0.997\underline{/10.75°} \\ 0.996\underline{/15.61°} \\ 0.934\underline{/12.37°} \end{bmatrix} + \begin{bmatrix} 0.0726\underline{/12.37°} \\ 0.0683\underline{/12.37°} \\ 0.0923\underline{/12.37°} \end{bmatrix}$$

$$= \begin{bmatrix} 1.070\underline{/10.86°} \\ 1.064\underline{/15.41°} \\ 1.026\underline{/12.37°} \end{bmatrix}$$

(8.9) Find Z_{BUS} for the three-bus circuit of Fig. 8.15 by using the Z_{BUS} building algorithm of Sec. 8.4.

Solution:

$$0\text{-}1 \quad j \overset{\textstyle ①}{\begin{bmatrix} 1.0 \end{bmatrix}} \qquad 0\text{-}2 \quad j \overset{\textstyle ① \quad ②}{\begin{bmatrix} 1.0 & 0 \\ 0 & 1.25 \end{bmatrix}}$$

$$1\text{-}3 \quad j \overset{\textstyle ① \quad ② \quad ③}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.25 & 1.25 \\ 0 & 1.25 & 1.3 \end{bmatrix}}$$

$$1\text{-}2 \quad j \overset{\textstyle ① \qquad ② \qquad ③ \qquad ⓠ}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1.25 & 1.25 & -1.25 \\ 0 & 1.25 & 1.3 & -1.25 \\ \hline 1 & -1.25 & -1.25 & 2.45 \end{array}\right]}$$

After kron reduction, Z_{BUS} is given by:

$$j \overset{\textstyle ① \qquad\quad ② \qquad\quad ③}{\begin{bmatrix} 0.5918 & 0.5102 & 0.5102 \\ 0.5102 & 0.6122 & 0.6122 \\ 0.5102 & 0.6122 & 0.6622 \end{bmatrix}}$$

(8.10) Find Z_{BUS} for the four-bus circuit of Fig. 7.12 which has per-unit *admittances* as marked.

Solution:

Z_{BUS} can be found by inverting the Y_{BUS} given in the text. $Z_{BUS} =$

$$j \begin{bmatrix} 0.7313 & 0.6914 & 0.6132 & 0.6368 \\ 0.6914 & 0.7197 & 0.6082 & 0.6418 \\ 0.6132 & 0.6082 & 0.6989 & 0.5511 \\ 0.6368 & 0.6418 & 0.5511 & 0.6989 \end{bmatrix}$$

(8.11) The three-bus circuit of Fig. 8.15 has per-unit *reactances* as marked. The symmetrical Y_{BUS} for the circuit has the triangular factors

$$L = \begin{bmatrix} -j6.0 & \cdot & \cdot \\ j5.0 & -j21.633333 & \cdot \\ 0 & j20.0 & -j1.510038 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -0.833333 & 0 \\ \cdot & 1 & -0.924499 \\ \cdot & \cdot & 1 \end{bmatrix}$$

Use L and U to calculate
(a) The elements Z_{12}, Z_{23}, and Z_{33} of the system Z_{BUS} and
(b) The Thévénin impedance $Z_{th,13}$ looking into the circuit of Fig. 8.15 between buses 1 and 3.

Solution:

(a) Using the method of section 8.5, the first column of Z_{BUS} is found from solution of:

$$L \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad U \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Solution of vector X gives:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} j0.1667 \\ j0.0385 \\ j0.5102 \end{bmatrix}$$

Solving the second equation involving U for column one of Z_{BUS}:

$$\begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{bmatrix} = \begin{bmatrix} j0.5918 \\ j0.5102 \\ j0.5102 \end{bmatrix}$$

And since in this circuit $Z_{12} = Z_{21}$, we have $Z_{12} = j0.5102$. Similarly, Z_{23}, and Z_{33} are found from:

$$L \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad U \begin{bmatrix} Z_{31} \\ Z_{32} \\ Z_{33} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Giving: And:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ j0.6622 \end{bmatrix} \qquad \begin{bmatrix} Z_{31} \\ Z_{32} \\ Z_{33} \end{bmatrix} = \begin{bmatrix} j0.5102 \\ j0.6122 \\ j0.6622 \end{bmatrix}$$

(b) The solution proceeds as follows:

$$
L \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad U \begin{bmatrix} Z_1^{(1-3)} \\ Z_2^{(1-3)} \\ Z_3^{(1-3)} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}
$$

Giving:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} j0.1667 \\ j0.0385 \\ -j0.1520 \end{bmatrix} \quad \text{And consequently:} \quad \begin{bmatrix} Z_1^{(1-3)} \\ Z_2^{(1-3)} \\ Z_3^{(1-3)} \end{bmatrix} = \begin{bmatrix} j0.0816 \\ -j0.1020 \\ -j0.1520 \end{bmatrix}
$$

and $Z_{th,13} = Z_1^{(1-3)} - Z_3^{(1-3)} = j0.2336$. Note that the difference is taken

in the same order, element 1 – element 3.

(8.12) Use the Y_{BUS} triangular factors of Prob. 8.11 to calculate the Thévénin impedance Z_{22} looking into the circuit of Fig. 8.15 between bus 2 and reference. Check your answer using the solution to Prob. 8.10.

Solution:

$$
L \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad U \begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}
$$

Giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0462 \\ j0.6122 \end{bmatrix} \quad \text{And consequently:} \quad \begin{bmatrix} Z_{21} \\ Z_{22} \\ Z_{23} \end{bmatrix} = \begin{bmatrix} j0.5102 \\ j0.6122 \\ j0.6122 \end{bmatrix}$$

with $Z_{22} = j0.6122$.
Check: $j1.2 \parallel j1.25 = j0.6122$.

(8.13) The Y_{BUS} for the circuit of Fig. 7.12 has triangular factors L and U given in Example 7.9. Use the triangular factors to calculate the Thévénin impedance $Z_{th,24}$ looking into the circuit of Fig. 7.12 between buses 2 and 4. Check your answer using the solution of Prob. 8.10.

Solution: Using L, solve for the intermediate variables X:

$$\begin{bmatrix} -j16.75 & & & \\ j11.75 & -j11.00746 & & \\ j2.5 & j4.25373 & -j3.78305 & \\ j2.5 & j6.75373 & j2.98305 & -j1.43082 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

giving:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0908 \\ j0.1022 \\ -j0.0571 \end{bmatrix}$$

Solving for the difference of Z_{BUS} columns:

$$\begin{bmatrix} 1 & -0.70149 & -0.14925 & -0.14925 \\ & 1 & -0.38644 & -0.61356 \\ & & 1 & -0.78853 \\ & & & 1 \end{bmatrix} \begin{bmatrix} Z_1^{(2-4)} \\ Z_2^{(2-4)} \\ Z_3^{(2-4)} \\ Z_4^{(2-4)} \end{bmatrix} = \begin{bmatrix} 0 \\ j0.0908 \\ j0.1022 \\ -j0.0571 \end{bmatrix}$$

gives:

$$\begin{bmatrix} Z_1^{(2-4)} \\ Z_2^{(2-4)} \\ Z_3^{(2-4)} \\ Z_4^{(2-4)} \end{bmatrix} = \begin{bmatrix} j0.0546 \\ j0.0779 \\ j0.0571 \\ -j0.0571 \end{bmatrix}$$

And $Z_{th,24} = = Z_2^{(2-4)} - Z_4^{(2-4)} = j0.1350$. Note that the difference is taken in the same order, element 2 – element 4.

Check: Using Z_{BUS}, $Z_{th,24} = Z_{22} + Z_{44} - 2Z_{24} = j(0.7197 + 0.6989 - 2 \times 0.6418) = j0.1350$.

(8.14) Using the notation of Sec. 8.6, prove that the total reactive power loss is given by the formula $Q_L = I^T X_{BUS} I^*$.

Solution:

$$S_L = P_L + jQ_L = I^T Z_{BUS} I^*$$
$$S_L^{*T} = P_L - jQ_L = I^T Z_{BUS}^{*T} I^*$$

Subtracting, we get,

$$\frac{S_L - S_L^{*T}}{2} = jQ_L = I^T \left[\frac{Z_{BUS} - Z_{BUS}^{*T}}{2} \right] I^*$$

and for symmetric $Z_{BUS} = R_{BUS} + jX_{BUS}$, we have:

$$Q_L = I^T [X_{BUS}] I^*$$

(8.15) Calculate the total reactive power loss in the system of Fig. 8.13 using Eq. (8.57).

Solution:

The vectors I and Z_{BUS} are given in Prob. 8.1.,

$$S = I^T Z_{BUS} I^* =$$

$$= j \begin{bmatrix} -j0.1 & j0.1 & j0.2 & j0.2 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix} \begin{bmatrix} j0.1 \\ -j0.1 \\ -j0.2 \\ -j0.2 \end{bmatrix}$$

$$= j0.152$$

(8.16) Using the procedure discussed in Sec. 8.6, modify the Z_{BUS} determined in Example 8.4 to reflect the choice of bus 2 of Fig. 8.8 as the reference.

Solution:

The transformation C that changes the reference from bus 0 to bus 2 is:

$$
C = \begin{array}{c} \text{①} \\ \text{②} \\ \text{③} \\ \text{④} \end{array}
\begin{array}{cccc} \text{①} & \text{⓪} & \text{③} & \text{④} \end{array}
\left[\begin{array}{cccc}
1 & & & \\
-1 & -1 & -1 & -1 \\
& & 1 & \\
& & & 1
\end{array}\right]
$$

$$Z_{BUS,new} = C^T Z_{BUS} C^* =$$

$$
\left[\begin{array}{cccc}
1 & -1 & & \\
& -1 & & \\
& -1 & 1 & \\
& -1 & & 1
\end{array}\right]
\left[\begin{array}{cccc}
j0.71660 & j0.60992 & j0.53340 & j0.58049 \\
j0.60992 & j0.73190 & j0.64008 & j0.69659 \\
j0.53340 & j0.64008 & j0.71660 & j0.66951 \\
j0.58049 & j0.69659 & j0.66951 & j0.76310
\end{array}\right]
\left[\begin{array}{cccc}
1 & & & \\
-1 & -1 & -1 & -1 \\
& & 1 & \\
& & & 1
\end{array}\right]
$$

$$
= \begin{array}{cccc} \text{①} & \text{⓪} & \text{③} & \text{④} \end{array}
\left[\begin{array}{cccc}
j0.22866 & j0.12198 & j0.01530 & j0.00588 \\
j0.12198 & j0.73190 & j0.09182 & j0.03531 \\
j0.01530 & j0.09182 & j0.16834 & j0.06474 \\
j0.00588 & j0.03531 & j0.06474 & j0.10182
\end{array}\right]
$$

(8.17)

(a) Find Z_{BUS} for the network of Fig. 8.13 using node 5 as reference. Change the reference from node 5 to node 4 and determine the new Z_{BUS} of the network using Eq. (8.60). Use the numerical values of the load currents I_{L1} of Prob. 8.1 to determine I_{new} by Eq. (8.55) and V_{new} by Eq. (8.56).

(b) Change the Z_{BUS} reference from node 4 back to node 5, using Eq. (8.63), determine the voltages at buses 1 and 4 relative to node 5. What are the values of these bus voltages with respect to the ground reference of Fig. 8.13?

Solution:

(a) From Prob. (8.1),

$$Z_{BUS} = j \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}$$

The transformation is:

$$C = \begin{array}{cc} & \begin{array}{cccc} ① & ② & ③ & ⑤ \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

$$Z_{BUS,new} = C^T Z_{BUS} C^* =$$

$$= j \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$$= j \begin{bmatrix} 0.9 & 0.5 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.9 & 0.5 & 1.5 & 0.9 \\ 0.9 & 0.5 & 0.9 & 1.1 \end{bmatrix}$$

(columns labelled ① ② ③ ⑤)

The new current vector must be determined from $\mathbf{I}=\mathbf{C}\mathbf{I}_{new}$,

$$\begin{bmatrix} I_{1,old} \\ I_{2,old} \\ I_{3,old} \\ I_{4,old} \end{bmatrix} = j \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_{1,new} \\ I_{2,new} \\ I_{3,new} \\ I_{5,new} \end{bmatrix}$$

giving $\mathbf{I}_{new}^T = j(0.1 \quad 0.1 \quad 0.2 \quad -0.6)$. And $\mathbf{V}_{new}$, the voltages with respect to node 4, are obtained from $\mathbf{Z}_{BUS,new}\mathbf{I}_{new}$:

$$\begin{bmatrix} V_{1,new} \\ V_{2,new} \\ V_{3,new} \\ V_{4,new} \end{bmatrix} = j \begin{bmatrix} 0.9 & 0.5 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.9 & 0.5 & 1.5 & 0.9 \\ 0.9 & 0.5 & 0.9 & 1.1 \end{bmatrix} \begin{bmatrix} j0.1 \\ j0.1 \\ j0.2 \\ -j0.6 \end{bmatrix} = \begin{bmatrix} 0.22 \\ 0.1 \\ 0.1 \\ 0.34 \end{bmatrix}$$

(b) $\mathbf{Z}_{BUS}$ is given by:

$$\mathbf{Z}_{BUS} = j \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.2 & 1.1 \end{bmatrix}$$

And the new voltages are given from solution of $V_{new} = C^{T*}V$; V comes from part (a):

$$V_{new} = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} 0.22 \\ 0.1 \\ 0.1 \\ 0.34 \end{bmatrix} = \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix}$$

And the voltage with respect to a ground reference is:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = 1.2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.12 \\ -0.24 \\ -0.24 \\ -0.34 \end{bmatrix} = \begin{bmatrix} 1.08 \\ 0.96 \\ 0.96 \\ 0.86 \end{bmatrix}$$

(8.18) A new branch having an impedance of $j0.25$ per unit is connected between nodes 3 and 4 of the circuit of Fig. 8.8 in parallel with the existing impedance of $j0.2$ per unit between the same two nodes. These two branches have mutual impedance of $j0.1$ per unit. Modify the Z_{BUS} determined in Example 8.4 to account for the addition of the new branch.

Solution:
First, add the mutually coupled line forming a new bus q:
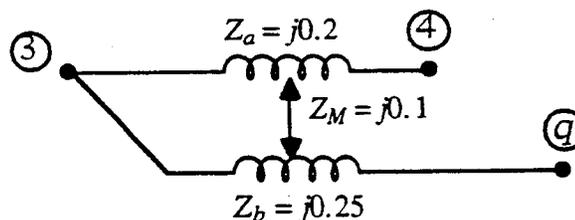
$$Z_{qj} = Z_{pj} - \frac{Z_M}{Z_a}(Z_{mj} - Z_{nj})$$

$$Z_{qq} = Z_{pq} - \frac{Z_M}{Z_a}(Z_{mq} - Z_{nq}) - (\frac{Z_M^2}{Z_a} - Z_b)$$

$$Z_{q1} = j0.53340 - \frac{j0.1}{j0.2}(j0.53340 - j0.58049) = j0.55695$$

$$Z_{q2} = j0.64008 - \frac{j0.1}{j0.2}(j0.64008 - j0.69659) = j0.66834$$

$$Z_{q3} = j0.71660 - \frac{j0.1}{j0.2}(j0.71660 - j0.66951) = j0.69306$$

$$Z_{q4} = j0.66951 - \frac{j0.1}{j0.2}(j0.66951 - j0.76310) = j0.71631$$

$$Z_{qq} = j0.69306 - \frac{j0.1}{j0.2}(j0.69306 - j0.71631) - (\frac{(j0.1)^2}{j0.2} - j0.25) = j0.90469$$

The augmented matrix is:

$$= j\begin{bmatrix} & ① & ② & ③ & ④ & ⑨ \\ 0.71660 & 0.60992 & 0.53340 & 0.58049 & 0.55695 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & 0.66834 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.69306 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & 0.71631 \\ \hline 0.55695 & 0.66834 & 0.69306 & 0.71631 & 0.90469 \end{bmatrix}$$

To parallel nodes 4 and q, row and column q is subtracted from row and column 4 to give:

$$j \begin{array}{c} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑨-④ \end{array} \\ \begin{bmatrix} 0.71660 & 0.60992 & 0.53340 & 0.58049 & -0.02354 \\ 0.60992 & 0.73190 & 0.64008 & 0.69659 & -0.02825 \\ 0.53340 & 0.64008 & 0.71660 & 0.66951 & 0.02355 \\ 0.58049 & 0.69659 & 0.66951 & 0.76310 & -0.04679 \\ \hline -0.02354 & -0.02825 & 0.02355 & -0.04679 & 0.23517 \end{bmatrix} \end{array}$$

After kron reduction, we have

$$j \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} 0.71424 & 0.60709 & 0.53576 & 0.57581 \\ 0.60709 & 0.72851 & 0.64291 & 0.69097 \\ 0.53576 & 0.64291 & 0.71424 & 0.67420 \\ 0.57581 & 0.69097 & 0.67420 & 0.75379 \end{bmatrix} \end{array}$$

(8.19) Derive Eqs. (8.95) and (8.96).

Solution:
These equations are a special case of the equations developed in Prob. 8.23.

(8.20) Modify the Z_{BUS} determined in Example 8.7 to remove the branch between buses 1 and 2 already coupled by mutual impedance $j0.15$ per unit to the branch between buses 1 and 4.

Solution:
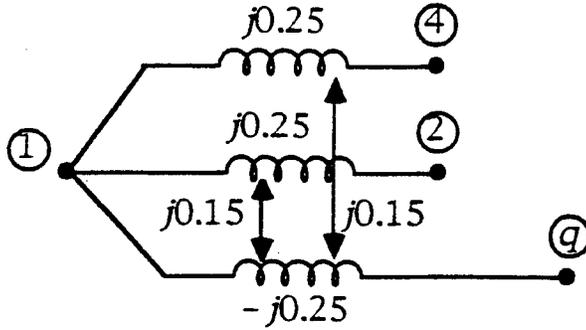
$$\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_B \end{bmatrix} = \begin{bmatrix} Z_a & Z_M \\ Z_M & Z_B \end{bmatrix}^{-1} = \begin{bmatrix} j0.25 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} = \begin{bmatrix} -j6.25 & j3.75 \\ j3.75 & -j6.25 \end{bmatrix}$$

$$Z_{qj} = Z_{1j} - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} Z_{1j} - Z_{4j} \\ Z_{1j} - Z_{2j} \end{bmatrix}$$

$$Z_{qq} = Z_{pq} - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} Z_{1q} - Z_{4q} \\ Z_{1q} - Z_{2q} \end{bmatrix} - (-j6.25(j0.15)^2 + j0.25)$$

Thus, we have:

$$Z_{q1} = j0.69890 - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} j0.69890 - j0.60822 \\ j0.69890 - j0.61323 \end{bmatrix} = j0.66208$$

$$Z_{q2} = j0.61323 - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} j0.61323 - j0.69140 \\ j0.61323 - j0.73128 \end{bmatrix} = j0.62011$$

$$Z_{q3} = j0.55110 - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} j0.55110 - j0.64178 \\ j0.55110 - j0.63677 \end{bmatrix} = j0.58792$$

$$Z_{q4} = j0.60822 - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} j0.60822 - j0.71966 \\ j0.60822 - j0.69140 \end{bmatrix} = j0.66591$$

$$Z_{qq} = j0.66208 - j0.15\begin{bmatrix} -j6.25 & j3.75 \end{bmatrix}\begin{bmatrix} j0.66208 - j0.66591 \\ j0.66208 - j0.62011 \end{bmatrix}$$
$$- (-j6.25(j0.15)^2 + j0.25) = j0.29865$$

The augmented matrix is:

$$j \begin{bmatrix} \overset{①}{0.69890} & \overset{②}{0.61323} & \overset{③}{0.55110} & \overset{④}{0.60822} & \overset{ⓠ}{0.66208} \\ 0.61323 & 0.73128 & 0.63677 & 0.69140 & 0.62011 \\ 0.55110 & 0.63677 & 0.69890 & 0.64178 & 0.58792 \\ 0.60822 & 0.69140 & 0.64178 & 0.71966 & 0.66591 \\ \hline 0.66208 & 0.62011 & 0.58792 & 0.66591 & 0.29865 \end{bmatrix}$$

Connecting buses 2 and q, we have:

$$j \begin{bmatrix} \overset{①}{0.69890} & \overset{②}{0.61323} & \overset{③}{0.55110} & \overset{④}{0.60822} & \overset{ⓠ-②}{0.04885} \\ 0.61323 & 0.73128 & 0.63677 & 0.69140 & -0.11117 \\ 0.55110 & 0.63677 & 0.69890 & 0.64178 & -0.04885 \\ 0.60822 & 0.69140 & 0.64178 & 0.71966 & -0.02549 \\ \hline 0.04885 & -0.11117 & -0.04885 & -0.02549 & -0.21029 \end{bmatrix}$$

After kron reduction:

$$j \begin{bmatrix} \overset{①}{0.71025} & \overset{②}{0.58741} & \overset{③}{0.53975} & \overset{④}{0.60230} \\ 0.58741 & 0.79005 & 0.66259 & 0.70488 \\ 0.53975 & 0.66259 & 0.71025 & 0.64770 \\ 0.60230 & 0.70488 & 0.64770 & 0.72275 \end{bmatrix}$$

(8.21) Assume that the two branches 1-3 and 2-3 in the circuit of Fig. 7.18 are the only mutually coupled branches (as indicated by the dots) with a mutual impedance of $j0.15$ per unit between them. Find Z_{BUS} for the circuit by the Z_{BUS} building algorithm.

Solution:

$$0\text{-}1: \quad \overset{\textstyle \textcircled{1}}{j\ [1]}$$

$$1\text{-}3: \quad j \begin{matrix} \textcircled{1} & \textcircled{3} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \end{bmatrix} \end{matrix}$$

3-2: Using the formula given in the text:

$$Z_{21} = j1 - \frac{j0.15}{j0.5}(j1 - j1) = j1$$

$$Z_{23} = j1.5 - \frac{j0.15}{j0.5}(j1.5 - j1) = j1.35$$

$$Z_{22} = j1.35 - \frac{j0.15}{j0.5}(j1.35 - j1) + j0.205 = j1.45$$

$$j \begin{matrix} \textcircled{1} & \textcircled{3} & \textcircled{2} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.35 \\ 1 & 1.35 & 1.45 \end{bmatrix} \end{matrix}$$

1-2: Form the augmented matrix:

$$j \begin{bmatrix}
\overset{\text{①}}{1} & \overset{\text{③}}{1} & \overset{\text{②}}{1} & 0 \\
1 & 1.5 & 1.35 & -0.35 \\
1 & 1.35 & 1.45 & -0.45 \\
\hline
0 & -0.35 & -0.45 & 0.85
\end{bmatrix}$$

and by kron reduction we get:

$$j \begin{bmatrix}
\overset{\text{①}}{1} & \overset{\text{③}}{1} & \overset{\text{②}}{1} \\
1 & 1.35588 & 1.16471 \\
1 & 1.16471 & 1.21176
\end{bmatrix}$$

3-4:

$$j \begin{bmatrix}
\overset{\text{①}}{1} & \overset{\text{③}}{1} & \overset{\text{②}}{1} & \overset{\text{④}}{1} \\
1 & 1.35588 & 1.16471 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088
\end{bmatrix}$$

4-5:

$$j \begin{bmatrix}
\overset{\text{①}}{1} & \overset{\text{③}}{1} & \overset{\text{②}}{1} & \overset{\text{④}}{1} & \overset{\text{⑤}}{1} \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088
\end{bmatrix}$$

0-5:

$$
j\begin{array}{c}
\;\;\;①\quad\;\;\;③\quad\;\;②\quad\;\;\;④\quad\;\;⑤\\
\left[\begin{array}{ccccc|c}
1 & 1 & 1 & 1 & 1 & 1\\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 & 1.35588\\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 & 1.16471\\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 & 1.48088\\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 1.98088\\
\hline
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 3.23088
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
j\begin{array}{c}
\;\;①\qquad③\qquad②\qquad④\qquad⑤\\
\left[\begin{array}{ccccc}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689\\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458\\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062\\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294\\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639
\end{array}\right]
\end{array}
$$

2-5: Using Case 4:

$$
j\begin{array}{c}
\;\;\;①\qquad\;\;③\qquad\;\;②\qquad\;\;④\qquad\;\;⑤\\
\left[\begin{array}{ccccc|c}
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & 0.25262\\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & 0.15134\\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & 0.34127\\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & 0.05792\\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & -0.31577\\
\hline
0.25262 & 0.15134 & 0.34127 & 0.05792 & -0.31577 & 0.85704
\end{array}\right]
\end{array}
$$

and after kron reduction:

$$
j\begin{bmatrix}
\overset{①}{0.61603} & \overset{③}{0.53573} & \overset{②}{0.53892} & \overset{④}{0.52458} & \overset{⑤}{0.47997} \\
 & 0.76015 & 0.61566 & 0.72418 & 0.58034 \\
 & & 0.65600 & 0.60780 & 0.57636 \\
 & \text{Symmetric} & & 0.79821 & 0.59428 \\
 & & & & 0.65005
\end{bmatrix}
$$

(Note node ordering.)

(8.22) Modify the Z_{BUS} obtained in Prob. 8.21 to remove branch 2-3 which is coupled to branch 1-3 through a mutual impedance of $j0.15$ per unit.

Solution:
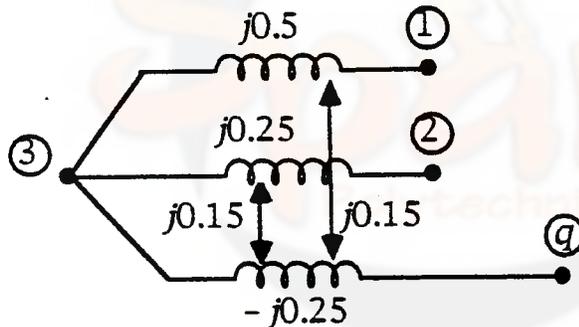
$$
\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_B \end{bmatrix} = \begin{bmatrix} Z_a & Z_M \\ Z_M & Z_B \end{bmatrix}^{-1} = \begin{bmatrix} j0.5 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} = \begin{bmatrix} -j2.43902 & j1.46341 \\ j1.46341 & -j4.87805 \end{bmatrix}
$$

$$Z_{q1} = j0.53573 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.53573 - j0.61603 \\ j0.53573 - j0.53892 \end{bmatrix} = j0.56441$$

$$Z_{q2} = j0.61566 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.61566 - j0.53892 \\ j0.61566 - j0.65600 \end{bmatrix} = j0.57873$$

$$Z_{q3} = j0.76015 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.76015 - j0.53573 \\ j0.76015 - j0.61566 \end{bmatrix} = j0.70976$$

$$Z_{q4} = j0.72418 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.72418 - j0.52458 \\ j0.72418 - j0.60780 \end{bmatrix} = j0.67670$$

$$Z_{q5} = j0.58034 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.58034 - j0.47997 \\ j0.58034 - j0.57636 \end{bmatrix} = j0.54449$$

$$Z_{qq} = j0.70976 - j0.15 \begin{bmatrix} -j2.43902 & j1.46341 \end{bmatrix} \begin{bmatrix} j0.70976 - j0.56441 \\ j0.70976 - j0.57873 \end{bmatrix}$$
$$- (-j2.43902(j0.15)^2 + j0.15) = j0.48047$$

The matrix is given by:

	①	③	②	④	⑤	⑨
	0.61603	0.53573	0.53892	0.52458	0.47997	0.56441
	0.53573	0.76015	0.61566	0.72418	0.58034	0.70976
j	0.53892	0.61566	0.65600	0.60780	0.57636	0.57873
	0.52458	0.72418	0.60780	0.79821	0.59428	0.67670
	0.47997	0.58034	0.57636	0.59428	0.65005	0.54449
	0.56441	0.70976	0.57873	0.67670	0.54449	0.48047

and node q is joined to node 2:

$$j\begin{bmatrix} & ① & ③ & ② & ④ & ⑤ & ⑨-② \\ & 0.61603 & 0.53573 & 0.53892 & 0.52458 & 0.47997 & 0.02549 \\ & 0.53573 & 0.76015 & 0.61566 & 0.72418 & 0.58034 & 0.09410 \\ & 0.53892 & 0.61566 & 0.65600 & 0.60780 & 0.57636 & -0.07727 \\ & 0.52458 & 0.72418 & 0.60780 & 0.79821 & 0.59428 & 0.06890 \\ & 0.47997 & 0.58034 & 0.57636 & 0.59428 & 0.65005 & -0.03187 \\ & 0.02549 & 0.09410 & -0.07727 & 0.06890 & -0.03187 & -0.02099 \end{bmatrix}$$

and after kron reduction:

$$j\begin{bmatrix} ① & ③ & ② & ④ & ⑤ \\ 0.64698 & 0.65000 & 0.44508 & 0.60825 & 0.44127 \\ & 1.18201 & 0.26925 & 1.03306 & 0.43746 \\ & & 0.94045 & 0.35416 & 0.69368 \\ \text{Symmetric} & & & 1.02438 & 0.48967 \\ & & & & 0.69844 \end{bmatrix}$$

(8.23) In Fig. 8.16 a new bus q is to be connected to an existing bus p through a new branch c. New branch c is mutually coupled to branches a and b, which are already coupled to one another as shown. The primitive impedance matrix defining the self- and mutual impedances of these three mutually coupled branches and its reciprocal, the primitive admittance matrix, have the forms:

$$\begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix}$$

To account for the addition of the new bus q, prove that the existing bus impedance matrix of the network must be augmented by a new row q and column q with elements given by:

$$Z_{qi} = Z_{pi} + \frac{1}{Y_{cc}} [\ Y_{ca} \quad Y_{ca}\ ] \begin{bmatrix} Z_{mi} - Z_{ni} \\ Z_{ji} - Z_{ki} \end{bmatrix}$$

$$Z_{qq} = Z_{pq} + \frac{1}{Y_{cc}} + \frac{1}{Y_{cc}} [\ Y_{ca} \quad Y_{ca}\ ] \begin{bmatrix} Z_{mq} - Z_{nq} \\ Z_{jq} - Z_{kq} \end{bmatrix}$$

Note that these equations are generalization of Eqs. (8.87) and (8.88).

Solution:

The voltage drop equations for the three mutually coupled branches are:

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix}^{-1} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} Y_{aa} & Y_{ab} & Y_{ac} \\ Y_{ba} & Y_{bb} & Y_{bc} \\ Y_{ca} & Y_{cb} & Y_{cc} \end{bmatrix}^{-1} \begin{bmatrix} V_m - V_n \\ V_j - V_k \\ V_p - V_q \end{bmatrix}$$

Since

$$I_q = -I_c = -Y_{ca}(V_m - V_n) - Y_{cb}(V_j - V_k) - Y_{cc}(V_p - V_q)$$

we have

$$V_q = V_p + \frac{1}{Y_{cc}} [\ Y_{ca} \quad Y_{cb}\ ] \begin{bmatrix} V_m - V_n \\ V_j - V_k \end{bmatrix} + \frac{1}{Y_{cc}} I_q$$

Using this equation and the expression for V_i^0 given in the text, we have for the qth row:

$$V_q^{(0)} + Z_{qq}I_q = V_p^{(0)} + Z_{pq}I_q + \frac{1}{Y_{cc}} \begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix} \begin{bmatrix} V_m^{(0)} - V_n^{(0)} + (Z_{mq} - Z_{nq})I_q \\ V_j^{(0)} - V_k^{(0)} + (Z_{jq} - Z_{kq})I_q \end{bmatrix} + \frac{1}{Y_{cc}}I_q$$

When $I_q = 0$, we have,

$$V_q^{(0)} = V_p^{(0)} + \frac{1}{Y_{cc}} \begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix} \begin{bmatrix} V_m^{(0)} - V_n^{(0)} \\ V_j^{(0)} - V_k^{(0)} \end{bmatrix}$$

Since

$$V_q^{(0)} = \sum_{i=1}^{N} Z_{qi}V_i \qquad i = 1, ..., N, q$$

we note that

$$Z_{qi} = Z_{pi} + \frac{1}{Y_{cc}} \begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix} \begin{bmatrix} Z_{mi} - Z_{ni} \\ Z_{ji} - Z_{ki} \end{bmatrix} \qquad \text{for } i = 1, ... N$$

Z_{qq} is determined by setting all other currents except I_q equal to zero:

$$Z_{qq} = Z_{pq} + \frac{1}{Y_{cc}} \begin{bmatrix} Y_{ca} & Y_{cb} \end{bmatrix} \begin{bmatrix} Z_{mq} - Z_{nq} \\ Z_{jq} - Z_{kq} \end{bmatrix} + \frac{1}{Y_{cc}}$$

(8.24) Branch 2-3 of the circuit of Fig. 7.18 is mutually coupled to two branches 1-3 and 2-5 through mutual impedances of $j0.15$ per unit and $j0.1$ per unit, respectively, as indicated by the dots. Using the formula given in Prob. 8.23, find the Z_{BUS} for the circuit by the Z_{BUS} building algorithm.

Solution:

$$0\text{-}1: \quad j \overset{\textcircled{1}}{[1]}$$

$$1\text{-}3: \quad j \begin{bmatrix} \overset{\textcircled{1}}{1} & \overset{\textcircled{3}}{1} \\ 1 & 1.5 \end{bmatrix}$$

3-2: Using the formula given in the text:

$$Z_{21} = j1 - \frac{j0.15}{j0.5}(j1 - j1) = j1$$

$$Z_{23} = j1.5 - \frac{j0.15}{j0.5}(j1.5 - j1) = j1.35$$

$$Z_{22} = j1.35 - \frac{j0.15}{j0.5}(j1.35 - j1) + j0.205 = j1.45$$

$$j \begin{bmatrix} \overset{\textcircled{1}}{1} & \overset{\textcircled{3}}{1} & \overset{\textcircled{2}}{1} \\ 1 & 1.5 & 1.35 \\ 1 & 1.35 & 1.45 \end{bmatrix}$$

1-2: Form the augmented matrix:

$$
j\left[\begin{array}{ccc|c}
 & ① & ③ & ② \\
1 & 1 & 1 & 0 \\
1 & 1.5 & 1.35 & -0.35 \\
1 & 1.35 & 1.45 & -0.45 \\
\hline
0 & -0.35 & -0.45 & 0.85
\end{array}\right]
$$

and by kron reduction we get:

$$
j\left[\begin{array}{ccc}
① & ③ & ② \\
1 & 1 & 1 \\
1 & 1.35588 & 1.16471 \\
1 & 1.16471 & 1.21176
\end{array}\right]
$$

3-4:

$$
j\left[\begin{array}{cccc}
① & ③ & ② & ④ \\
1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088
\end{array}\right]
$$

4-5:

$$
j\left[\begin{array}{ccccc}
① & ③ & ② & ④ & ⑤ \\
1 & 1 & 1 & 1 & 1 \\
1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 \\
1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 \\
1 & 1.35588 & 1.16471 & 1.48088 & 1.98088
\end{array}\right]
$$

0-5:

$$j\begin{bmatrix} & ① & ③ & ② & ④ & ⑤ & \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1.35588 & 1.16471 & 1.35588 & 1.35588 & 1.35588 \\ 1 & 1.16471 & 1.21176 & 1.16471 & 1.16471 & 1.16471 \\ 1 & 1.35588 & 1.16471 & 1.48088 & 1.48088 & 1.48088 \\ 1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 1.98088 \\ \hline 1 & 1.35588 & 1.16471 & 1.48088 & 1.98088 & 3.23088 \end{bmatrix}$$

and after kron reduction:

$$j\begin{bmatrix} ① & ③ & ② & ④ & ⑤ \\ 0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 \\ 0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 \\ 0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 \\ 0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 \\ 0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 \end{bmatrix}$$

2-5: The primitive admittance matrix of the three coupled branches is obtained as:

$$Y = \mathbf{Z}^{-1} = \begin{bmatrix} j0.5 & j0.15 & 0 \\ j0.15 & j0.25 & j0.1 \\ 0 & j0.1 & j0.2 \end{bmatrix}^{-1}$$

$$= j \begin{bmatrix} -2.58065 & 1.93548 & -0.96774 \\ 1.93548 & -6.45161 & 3.22581 \\ -0.96774 & 3.22581 & -j6.61290 \end{bmatrix}$$

From the equations given:

$$Z_{q1} = j0.38689 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.58034 - j0.69049 \\ j0.58034 - j0.63951 \end{bmatrix} = j0.39963$$

$$Z_{q2} = j0.45062 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.67592 - j0.63951 \\ j0.67592 - j0.79189 \end{bmatrix} = j0.51252$$

$$Z_{q3} = j0.52458 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.78687 - j0.58034 \\ j0.78687 - j0.67592 \end{bmatrix} = j0.50068$$

$$Z_{q4} = j0.57294 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.73441 - j0.54165 \\ j0.73441 - j0.63086 \end{bmatrix} = j0.55064$$

$$Z_{q5} = j0.76639 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.52458 - j0.38689 \\ j0.52458 - j0.45062 \end{bmatrix} = j0.75046$$

$$Z_{qq} = j0.75046 + \frac{1}{-j6.61290}\begin{bmatrix} -j0.96774 & j3.22581 \end{bmatrix}\begin{bmatrix} j0.50068 - j0.39963 \\ j0.50068 - j0.51252 \end{bmatrix}$$

$$+ \frac{1}{-j6.61290} = j0.92224$$

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the augmented matrix is

$$
j\begin{bmatrix}
 & ① & ③ & ② & ④ & ⑤ & Ⓠ \\
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & 0.39963 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & 0.50068 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & 0.51252 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & 0.55064 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & 0.75046 \\
0.39963 & 0.50068 & 0.51252 & 0.55064 & 0.75046 & 0.92224
\end{bmatrix}
$$

connecting buses 2 and q, we have:

$$
j\begin{bmatrix}
 ① & ③ & ② & ④ & ⑤ & Ⓠ-② \\
0.69049 & 0.58034 & 0.63951 & 0.54165 & 0.38689 & -0.23988 \\
0.58034 & 0.78687 & 0.67592 & 0.73441 & 0.52458 & -0.17524 \\
0.63951 & 0.67592 & 0.79189 & 0.63086 & 0.45062 & -0.27937 \\
0.54165 & 0.73441 & 0.63086 & 0.80212 & 0.57294 & -0.08022 \\
0.38689 & 0.52458 & 0.45062 & 0.57294 & 0.76639 & 0.29984 \\
-0.23988 & -0.17524 & -0.27937 & -0.08022 & 0.29984 & 0.68909
\end{bmatrix}
$$

and after kron reduction:

$$
j\begin{bmatrix}
 ① & ③ & ② & ④ & ⑤ \\
0.60699 & 0.51934 & 0.54226 & 0.51372 & 0.49127 \\
 & 0.74231 & 0.60487 & 0.71401 & 0.60083 \\
 & & 0.67863 & 0.59834 & 0.57218 \\
\text{Symmetric} & & & 0.79278 & 0.60785 \\
 & & & & 0.63592
\end{bmatrix}
$$

Chapter 9 Problem Solutions

9.1 In Example 9.3, suppose that the generator's maximum reactive power generation at bus ④ is limited to 125 Mvar. Re-compute the first-iteration value of the voltage at bus ④ using the Gauss-Seidel method.

Solution:

The net power injection found at bus ④ of Example 9.3 was

$$Q_4 = 1.654151 \text{ per unit} = 165.4151 \text{ Mvar}$$

Considering the reactive load of 49.58 Mvar at the bus, the required reactive power generation is $165.4151 + 49.58 = 214.9951$ Mvar, which exceeds the 125 Mvar limit specified. The bus is now regarded as a load bus, with total reactive power generation of 125 Mvar. So the net injected reactive power in this case is

$$125 - 49.58 = 75.42 \text{ Mvar} = 0.7521 \text{ per unit}$$

V_4 is now calculated as

$$
\begin{aligned}
V_4^{(1)} &= \frac{1}{Y_{44}} \left[\frac{P_{4,sch} - jQ_4^{(1)}}{V_4^{(0)*}} - \left(Y_{42}V_{2,acc}^{(1)} + Y_{43}V_{3,acc}^{(1)} \right) \right] \\
&= \frac{1}{8.193267 - j40.863838} \left[\frac{2.38 - j0.7542}{1.02} - (-5.573064 + j40.05939) \right] \\
&= 0.997117 - j0.006442 \text{ per unit}
\end{aligned}
$$

and using an acceleration factor of 1.6 yields

$$V_{4,acc}^{(1)} = 1.02 + 1.6 \left(0.997117 - j0.06442 - 1.02 \right) = 0.983387 - j0.0103073 \text{ per unit}$$

(9.2) For the system of Fig. 9.2, complete the second iteration of the Gauss-Seidel procedure using the first iteration value of the bus voltages obtained in Examples 9.2 and 9.3. Assume an acceleration factor of 1.6.

Solution:

$$V_2^{(2)} = \frac{1}{Y_{22}} \left[\frac{P_{2,sch} - jQ_{2,sch}}{V_2^{(1)*}} - (Y_{21}V_1^{(1)} + Y_{24}V_{4,acc}^{(1)}) \right]$$

$$= \frac{1}{Y_{22}} \left[\frac{-1.7 + j1.0535}{0.981113 + j0.031518} - \left\{ -3.815629 + j19.078144 \right. \right.$$

$$\left. \left. + (-5.169561 + j25.847809)(1.019922 + j0.012657) \right\} \right]$$

$$= \frac{7.718854 - j44.247184}{8.985190 - j44.835953}$$

$$= 0.9819338 - j0.0246233$$

$$V_{2,acc}^{(2)} = 0.981113 - j0.031518 + 1.6 (0.9819338 - j0.0246233$$
$$- 0.9819338 + j0.0246233)$$

$$= 0.982426 - j0.020486$$

$$V_3^{(2)} = \frac{1}{Y_{33}} \left[\frac{P_{3,\text{sch}} - jQ_{3,\text{sch}}}{V_3^{(1)*}} - \left(Y_{31} V_1^{(2)} + Y_{34} V_{4,\text{acc}}^{(1)} \right) \right]$$

$$= \frac{1}{Y_{33}} \left[\frac{-2 + j1.2394}{0.966597 + j0.040797} - \left\{ -5.16956 + j25.847809 \right. \right.$$

$$\left. \left. + (-3.023705 + j15.118528)(1.019922 + j0.012657) \right\} \right]$$

$$= \frac{6.433447 - j39.862133}{8.193267 - j40.863838}$$

$$= 0.9681332 - j0.0366761$$

$$V_{3,\text{acc}}^{(2)} = 0.966597 - j0.00.040797 + 1.6 \left(0.9681332 - j0.0366761 \right.$$
$$\left. - 0.966597 + j0.00.040797 \right)$$

$$= 0.969055 - j0.034195$$

$$Q_4^{(2)} = -\text{Im}\left\{ V_4^{(1)*} \left[Y_{42} V_2^{(2)} + Y_{43} V_3^{(2)} + Y_{44} V_4^{(1)} \right] \right\}$$

$$= -\text{Im}\{ (1.019922 - j0.012657)$$

$$\times [(-5.16956 + j25.847809)(0.982426 - j0.020486)$$
$$+ (-3.023705 + j15.118528)(0.969055 - j0.034195)$$
$$+ (8.193267 - j40.863837)(1.019922 + j0.012657)] \}$$

$$= -\text{Im}\{ 1.911362 - j1.320680 \} = 1.320680$$

$$V_4^{(2)} = \frac{1}{Y_{44}} \left[\frac{P_{3,\text{sch}} - jQ_4^{(2)}}{V_4^{(1)*}} - (Y_{42}V_2^{(2)} + Y_{43}V_3^{(2)}) \right]$$

$$= \frac{1}{Y_{44}} \left[\frac{2.38 - j1.320680}{1.019922 - j0.012657} \right.$$

$$- \left\{ (-5.16956 + j25.847809)(0.982426 - j0.020486) \right.$$

$$\left. + (-3.023705 + j15.118528)(0.969055 - j0.034195) \right\} \right]$$

$$= \frac{9.311570 - j41.519274}{8.193267 - j40.863838}$$

$$= 1.020695 + j0.023217$$

$$V_{4,\text{corr}}^{(2)} = \frac{1.02}{1.020959} (1.020695 + j0.023217)$$

$$= 1.019736 + j0.023195$$

(9.3) A synchronous condenser, whose reactive power capability is assumed to be unlimited, is installed at load bus 2 of the system of Example 9.2 to hold the bus-voltage magnitude at 0.99 per unit. Using the Gauss-Seidel method, find the voltage at buses 2 and 3 for the first iteration.

Solution:

$$Q_2 = -\text{Im}\{\, V_2{}^* \left[Y_{21} V_1 + Y_{22} V_2 + Y_{23} V_3 + Y_{24} V_4 \right] \}$$

$$= -\text{Im}\{\, (0.99)$$

$$\times [\, (1.0)(\,-3.815629 + j19.078144)$$
$$+ (0.99)(\,8.985190 - j44.835953)$$
$$+ (1.02)(\,-5.169561 + j25.847809)] \}$$

$$= -1.0447626$$

Using the above value of Q_2, V_2 is computed:

$$V_2 = \frac{1}{Y_{22}} \left[\frac{P_{2,\text{sch}} - jQ_2}{V_2^{(0)}{}^*} - (Y_{21} V_1 + Y_{24} V_4) \right]$$

$$= \frac{1}{Y_{22}} \left[\frac{-1.7 + j1.0447626}{0.99} \right.$$

$$- \{\, (1.0)(-3.815629 + j19.078144)$$

$$\left. + (1.02)(-5.169561 + j25.847809) \} \right]$$

$$= \frac{7.3714095 - j44.387593}{8.985190 - j44.835953}$$

$$= 0.9834515 - j0.0326767$$

The magnitude of V_2 is now corrected to 0.99:

$$V_{2,\text{corr}} = \frac{0.99}{0.9839942}(0.9834515 - j0.0326767)$$

$$= 0.9894539 - j0.0328761$$

At bus 3, we have,

$$V_3 = \frac{1}{Y_{33}}\left[\frac{P_{3,\text{sch}} - jQ_{3,\text{sch}}}{V_3^*} - (Y_{31}V_1 + Y_{32}V_2 + Y_{34}V_4)\right]$$

$$= \frac{1}{Y_{33}}\left[\frac{-2 + j1.2394}{1.0}\right.$$

$$- \left\{(1.0)(-5.169561 + j25.847809)\right.$$

$$+ (1.02)(-3.023705 + j15.118528)\Big\}\Big]$$

$$= \frac{6.2537401 - j40.029308}{8.193267 - j40.863838}$$

$$= 0.9712184 - j0.0416924$$

If desired, an acceleration factor may be used.

(9.4) Take Fig. 9.12 as the equivalent-π representation of the transmission line between bus 3 and 4 of the system of Fig. 9.2. Using the power-flow solution given in Fig. 9.4, determine and indicate on Fig. 9.12 the values of (a) P and Q leaving buses 3 and 4 on line 3-4, (b) charging megavars of the equivalent π of the line 3-4, and (c) P and Q at both ends of the series part of the equivalent π of the line 3-4.

Solution:

(a)
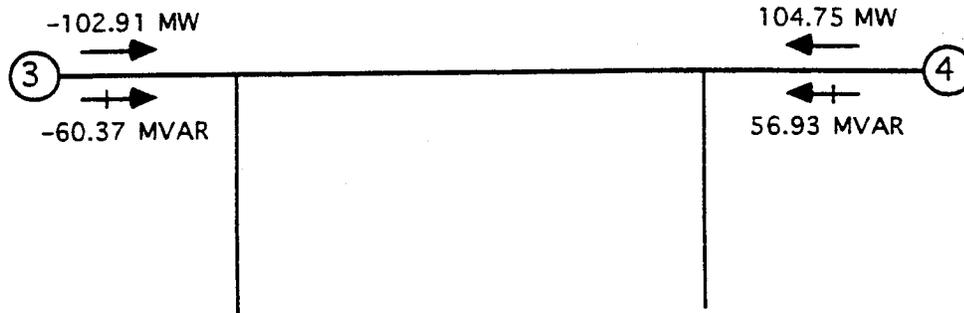
(b) Charging MVAR varies as the square of the voltage:

At bus 3: $\dfrac{12.75}{2}(0.969)^2 = 5.99$ MVAR

At bus 4: $\dfrac{12.75}{2}(1.02)^2 = 6.63$ MVAR

(c) These are the sums of (a) and (b):
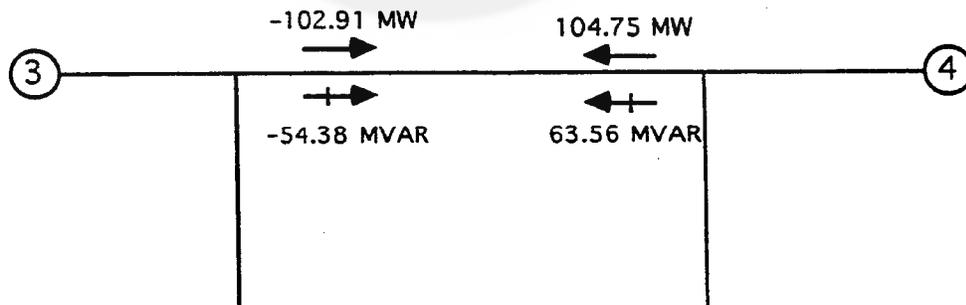
(9.5) From the line-flow information of the power flow solution given in Fig. 9.4, determine I^2R loss in each of the four transmission lines, and verify that the sum of these line losses is equal to the total system loss of 4.81 MW.

Solution:

Line 1-2:

$= 38.69 - 38.46 = 0.23$ MW

Line 1-3:

$= 98.12 - 97.09 = 1.03$ MW

Line 2-4:

$= 133.25 - 131.54 = 1.71$ MW

Line 3-4:

$= 104.75 - 102.91 = 1.84$ MW

Summing up these four line loss components, we get the total loss is 4.8 MW.

(9.6) Suppose that a shunt capacitor bank rated 18 MVAR is connected between bus 3 and the reference node in the system of Example 9.5. Modify the Y_{BUS} given in Table 9.4 to account for this capacitor, and *estimate* the actual megavar reactive power injected into this system from this capacitor.

Solution:

Only the diagonal element corresponding to bus 3 needs to be modified:

$$Y_{33}^{new} = (8.193267 - j40.863838) + j0.18$$

$$= 8.193267 - j40.683838$$

Using the voltage at bus 3 given in Fig. 9.4, the approximate power injection is

$$18 \times (0.969)^2 = 16.9 \text{ MVAR}.$$

(9.7) For the system of Example 9.5 augmented with a synchronous condenser as described in Prob. 9.3, find the jacobian calculated at the initial estimates. *Hint:* It would be simpler to modify the jacobian matrix shown in Sec. 9.4 following Example 9.5 than to start calculations from the beginning.

Solution:
Since bus 2 is now a *PV* bus, the mismatch equations can be written in the following form:

$$
\begin{array}{c}
 \\
\begin{array}{c}②\\③\\④\\③\end{array}
\end{array}
\begin{bmatrix}
M_{22} & 0 & M_{24} & 0 \\
0 & M_{33} & M_{34} & N_{33}+2|V_3|^2 G_{33} \\
M_{42} & M_{43} & M_{44} & -N_{43} \\
\hline
0 & N_{33} & N_{34} & -M_{33}-2|V_3|^2 B_{33}
\end{bmatrix}
\begin{bmatrix}
\Delta\delta_2 \\
\Delta\delta_3 \\
\Delta\delta_4 \\
\hline
\Delta|V_3|/|V_3|
\end{bmatrix}
=
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta P_4 \\
\hline
\Delta Q_3
\end{bmatrix}
$$

Note that the values are given in Sec. 9.4. To determine the first iteration value of the above, we use the same initial voltages as before, except $|V_2| = 0.99$ p.u. Only the terms involving V_2 change. The M_{22}, M_{24} and M_{42} terms need only be corrected by 0.99:

$$M_{22} = 45.442909 * 0.99 = 44.98848$$
$$M_{24} = -26.364763 * 0.99 = -26.101115$$
$$M_{42} = -26.364763 * 0.99 = -26.101115$$

And M_{44} is recalculated as follows:

$$M_{44} = \sum_{\substack{n=1 \\ n\neq 4}}^{4} |V_4 V_n Y_{4n}| \sin(\theta_{4n} + \delta_n - \delta_4)$$

$$= 1.02 \times 0.99 \times 26.359696 \times \sin(101.30993°)$$
$$+ 1.02 \times 1 \times 15.41793 \times \sin(101.30993°)$$
$$= 26.101117 + 15.420899$$
$$= 41.522016$$

Giving the jacobian:

$$
\begin{array}{c}
\quad\quad ② \quad\quad\quad\quad ③ \quad\quad\quad\quad ④ \quad\quad\quad ③ \\
\begin{array}{c} ② \\ ③ \\ ④ \\ ③ \end{array}
\left[
\begin{array}{ccc|c}
44.98848 & 0 & -26.101115 & 0 \\
0 & 41.268707 & -15.420898 & 8.132792 \\
-26.101115 & -15.420898 & 41.522016 & -3.084180 \\
\hline
0 & -8.25374 & -3.084180 & 40.458969
\end{array}
\right]
\end{array}
$$

(9.8) Suppose that in Fig. 9.7 the tap is on the side of node i so that the transformation ratio is $t:1$. Find the elements of Y_{BUS} similar to those in Eq. (9.74), and draw the equivalent-π representation similar to Fig. 9.8.

Solution:

$$
S_i = V_i I_i^* \quad ; \quad S_j = \frac{1}{t} V_j I_j^*
$$

Since $S_i + S_j = 0$,

$$
I_i = \frac{-1}{t^*} I_j
$$

$$
I_j = \left(V_j - \frac{V_i}{t} \right) Y = -\frac{Y}{t} V_i + Y V_j
$$

and

$$
-t^* I_i = -\frac{Y}{t} V_i + Y V_j
$$

from which:

$$
I_i = -\frac{Y}{t t^*} V_i + \frac{Y}{t^*} V_j
$$

giving:

$$\begin{bmatrix} I_i \\ \\ I_j \end{bmatrix} = \begin{bmatrix} \dfrac{Y}{|t|^2} & \dfrac{-Y}{t^*} \\ \\ \dfrac{-Y}{t} & Y \end{bmatrix} \begin{bmatrix} V_i \\ \\ V_j \end{bmatrix}$$

For real t, the equivalent network is:
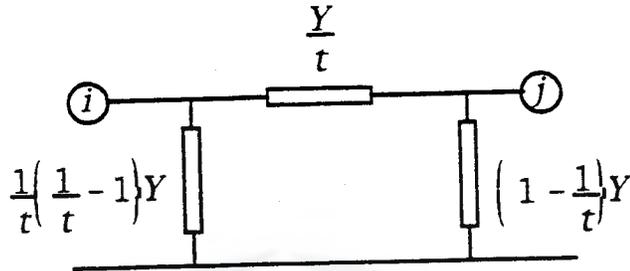
(9.9) In the four-bus system of Example 9.5, suppose that a magnitude-regulating transformer with 0.2 per-unit reactance is inserted between the load and the bus at bus 3, as shown in Fig. 9.10. The variable tap is on the load side of the transformer. If the voltage magnitude at the new load bus 5 is prespecified, and therefore is not a state variable, the tap t of the transformer should be regarded as a state variable. The Newton-Raphson method is to be applied to the solution of the power-flow equations.

(a) Write mismatch equations for this problem in symbolic form similar to Eq. (9.45)

(b) Write equations of the jacobian elements of the column corresponding to the variable t (that is, partial derivatives with respect to t), and evaluate them using the initial voltage estimates shown in Table 9.3 and assuming that the voltage magnitude at bus 5 is specified to be 0.97. The initial estimate of δ_5 is 0.

(c) Write equations of P and Q mismatches at bus 5 and evaluate them for the first iteration. Assume the initial estimate of variable t is 1.0.

Note to instructor: In future printings, this value of 0.2 per unit for the transformer reactance should be changed to a value of 0.02 per unit.

Solution:

(a)

$$
\begin{bmatrix}
\dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \dfrac{\partial P_2}{\partial \delta_4} & \dfrac{\partial P_2}{\partial \delta_5} & |V_2|\dfrac{\partial P_2}{\partial |V_2|} & |V_3|\dfrac{\partial P_2}{\partial |V_3|} & \dfrac{\partial P_2}{\partial t} \\[2mm]
\dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \cdots & & \cdots & |V_3|\dfrac{\partial P_3}{\partial |V_3|} & \dfrac{\partial P_3}{\partial t} \\[2mm]
\dfrac{\partial P_4}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \ddots & & & & \dfrac{\partial P_4}{\partial t} \\[2mm]
\dfrac{\partial P_5}{\partial \delta_2} & \dfrac{\partial P_5}{\partial \delta_3} & \dfrac{\partial P_5}{\partial \delta_4} & & & & \dfrac{\partial P_5}{\partial t} \\[2mm]
\hline
\dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_2} & \cdots & & \cdots & |V_3|\dfrac{\partial Q_2}{\partial |V_3|} & \dfrac{\partial Q_2}{\partial t} \\[2mm]
\dfrac{\partial Q_3}{\partial \delta_2} & \dfrac{\partial Q_3}{\partial \delta_2} & \ddots & & & \cdots & \dfrac{\partial Q_3}{\partial t} \\[2mm]
\dfrac{\partial Q_5}{\partial \delta_2} & \cdots & & & & \cdots & \dfrac{\partial Q_5}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_2 \\[2mm]
\Delta \delta_3 \\[2mm]
\Delta \delta_4 \\[2mm]
\Delta \delta_5 \\[2mm]
\hline
\dfrac{\Delta |V_2|}{|V_2|} \\[2mm]
\dfrac{\Delta |V_3|}{|V_3|} \\[2mm]
t
\end{bmatrix}
=
\begin{bmatrix}
\Delta P_2 \\[2mm]
\Delta P_3 \\[2mm]
\Delta P_4 \\[2mm]
\Delta P_5 \\[2mm]
\hline
\Delta Q_2 \\[2mm]
\Delta Q_3 \\[2mm]
\Delta Q_5
\end{bmatrix}
$$

(b) For the last column:

$$\frac{\partial P_2}{\partial t} = 0$$

$$\frac{\partial P_3}{\partial t} = \frac{\partial}{\partial t}\left\{ |V_3|^2 G_{33} + \sum_{\substack{n=1 \\ n \neq 3}}^{5} |V_3 V_n Y_{3n}| \cos(\theta_{3n} + \delta_n - \delta_3) \right\}$$

$$= \frac{\partial}{\partial t}\left\{ |V_3 V_5(-tY)| \cos(\theta_{35} + \delta_5 - \delta_3) \right\}$$

$$= -|V_3 V_5 Y|\cos(90°) = 0$$

$$\frac{\partial P_4}{\partial t} = 0$$

$$\frac{\partial P_5}{\partial t} = \frac{\partial}{\partial t}\left\{ |V_5|^2 G_{55} + \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \cos(\theta_{5n} + \delta_n - \delta_5) \right\}$$

$$= \frac{\partial}{\partial t}\left\{ |V_5 V_3(-tY)| \cos(\theta_{53} + \delta_3 - \delta_5) \right\}$$

$$= -|V_5 V_3 Y|\cos(90°) = 0$$

$$\frac{\partial Q_2}{\partial t} = 0$$

$$\frac{\partial Q_3}{\partial t} = \frac{\partial}{\partial t}\left\{ -|V_3|^2 B_{33} - \sum_{\substack{n=1 \\ n \neq 3}}^{5} |V_3 V_n Y_{3n}| \sin(\theta_{3n} + \delta_n - \delta_3) \right\}$$

$$= \frac{\partial}{\partial t}\left\{ -|V_3|^2(B_{33,\text{fixed}} - t^2|Y|) - |V_5 V_3(-tY)| \sin(\theta_{53} + \delta_3 - \delta_5) \right\}$$

$$= 2t|V_3|^2|Y| - |V_5 V_3 Y| \sin(90°) =$$

$$= \frac{2}{0.2} - \frac{0.97}{0.2} = 5.15$$

$$\frac{\partial Q_4}{\partial t} = 0$$

$$\frac{\partial Q_5}{\partial t} = \frac{\partial}{\partial t}\left\{ -|V_5|^2 B_{55} - \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \sin(\theta_{5n} + \delta_n - \delta_5) \right\}$$

$$= \frac{\partial}{\partial t}\left\{ -|V_5 V_3(-tY)| \sin(\theta_{53} + \delta_3 - \delta_5) \right\}$$

$$= -|V_5 V_3 Y| \sin(90°) = -\frac{0.97}{0.2} = -4.85$$

(*c*)

$$P_{5,\text{calc}}^{(0)} = |V_5|^2 G_{55} + \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \cos(\theta_{5n} + \delta_n - \delta_5)$$

$$= 0 + |V_5 V_3 Y_{53}| \cos(\theta_{53} + \delta_3 - \delta_5)$$

$$= 0 + 0.97 \times 5 \times \cos(90°) = 0$$

$$\Delta P_5^{(0)} = P_{5,\text{sch}} - P_{5,\text{calc}}^{(0)}$$

$$= -2 - 0 = -2$$

$$Q_{5,\text{calc}}^{(0)} = -|V_5|^2 B_{55} - \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \sin(\theta_{5n} + \delta_n - \delta_5)$$

$$= -|V_5|^2 B_{55} - |V_5 V_3 Y_{53}| \sin(\theta_{53} + \delta_3 - \delta_5)$$

$$= -(0.97)^2(-5) - 0.97 \times 5 \times \sin(90°) =$$

$$= 4.7045 - 4.85$$

$$= -0.1455$$

$$\Delta Q_5^{(0)} = Q_{5,\text{sch}} - Q_{5,\text{calc}}^{(0)}$$

$$= -1.2394 - (-0.1455) = -1.0939$$

(9.10) If the tap setting of the transformer of Prob. 9.9 is prespecified instead of the voltage magnitude at bus 5, then V_5 should be regarded as a state variable. Suppose that the tap setting t is specified to be 1.05.

- (a) In this case write mismatch equations in symbolic form similar to Eq. (9.45)
- (b) Write equations of the jacobian elements which are partial derivatives with respect to $|V_5|$, and evaluate them using the initial estimates. The initial estimate of V_5 is $1.0\underline{/0°}$.
- (c) Write equations of P and Q mismatches at bus 5 and evaluate them for the first iteration.

Solution:

(a)

$$
\begin{bmatrix}
\dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \dfrac{\partial P_2}{\partial \delta_4} & \dfrac{\partial P_2}{\partial \delta_5} & |V_2|\dfrac{\partial P_2}{\partial |V_2|} |V_3|\dfrac{\partial P_2}{\partial |V_3|} |V_5|\dfrac{\partial P_2}{\partial |V_5|} \\
\dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \cdots & \cdots & |V_3|\dfrac{\partial P_3}{\partial |V_3|} |V_5|\dfrac{\partial P_3}{\partial |V_5|} \\
\dfrac{\partial P_4}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \ddots & & |V_5|\dfrac{\partial P_4}{\partial |V_5|} \\
\dfrac{\partial P_5}{\partial \delta_2} & \dfrac{\partial P_5}{\partial \delta_3} & \dfrac{\partial P_5}{\partial \delta_4} & & |V_5|\dfrac{\partial P_5}{\partial |V_5|} \\
\hline
\dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_2} & \cdots & \cdots & |V_3|\dfrac{\partial Q_2}{\partial |V_3|} |V_5|\dfrac{\partial Q_2}{\partial |V_5|} \\
\dfrac{\partial Q_3}{\partial \delta_2} & \dfrac{\partial Q_3}{\partial \delta_2} & \ddots & & \cdots |V_5|\dfrac{\partial Q_3}{\partial |V_5|} \\
\dfrac{\partial Q_5}{\partial \delta_2} & \cdots & & & \cdots |V_5|\dfrac{\partial Q_5}{\partial |V_5|}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_2 \\
\Delta \delta_3 \\
\Delta \delta_4 \\
\Delta \delta_5 \\
\hline
\dfrac{\Delta |V_2|}{|V_2|} \\
\dfrac{\Delta |V_3|}{|V_3|} \\
\dfrac{\Delta |V_5|}{|V_5|}
\end{bmatrix}
=
\begin{bmatrix}
\Delta P_2 \\
\Delta P_3 \\
\Delta P_4 \\
\Delta P_5 \\
\hline
\Delta Q_2 \\
\Delta Q_3 \\
\Delta Q_5
\end{bmatrix}
$$

(b)

$$|V_5| \frac{\partial P_2}{\partial |V_5|} = |V_5|\ |V_2 Y_{25}|\ \cos(\ \theta_{52} + \delta_5 - \delta_2\ ) = 0$$

$$|V_5| \frac{\partial P_3}{\partial |V_5|} = |V_5|\ |V_3 Y_{35}|\ \cos(\ \theta_{53} + \delta_5 - \delta_3\ )$$

$$= |V_5|\ |V_3(-tY)|\ \cos(90°\ ) = 0$$

$$|V_5| \frac{\partial P_4}{\partial |V_5|} = |V_5|\ |V_4 Y_{45}|\ \cos(\ \theta_{54} + \delta_5 - \delta_4\ ) = 0$$

$$|V_5| \frac{\partial P_5}{\partial |V_5|} = |V_5| \left\{ 2|V_5|G_{55} + \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_n Y_{5n}|\ \cos(\ \theta_{5n} + \delta_n - \delta_5\ ) \right\}$$

$$= |V_5|\ |V_3(-tY)|\ \cos(90°\ ) = 0$$

$$|V_5| \frac{\partial Q_2}{\partial |V_5|} = -|V_5|\ |V_2 Y_{52}|\ \sin(\ \theta_{52} + \delta_5 - \delta_2\ ) = 0$$

$$|V_5| \frac{\partial Q_3}{\partial |V_5|} = -|V_5|\ |V_3 Y_{53}|\ \sin(\ \theta_{53} + \delta_5 - \delta_3\ )$$

$$= -|V_5|\ |V_3(-tY)|\ \sin(90°\ )$$

$$= -1 \times \frac{1.05}{0.2} = -5.25$$

$$|V_5| \frac{\partial Q_5}{\partial |V_5|} = -2|V_5|^2 B_{55} - \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \sin(\theta_{5n} + \delta_n - \delta_5)$$

$$= -2|V_5|^2 B_{55} - |V_5| |V_3(-tY)| \sin(90°)$$

$$= -2\left(\frac{-1}{0.2}\right) - \frac{1.05}{0.2} = 4.75$$

(c)

$$P_{5,calc}^{(0)} = |V_5|^2 G_{55} + \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \cos(\theta_{5n} + \delta_n - \delta_5)$$

$$= 0 + |V_5 V_3 Y_{53}| \cos(\theta_{53} + \delta_3 - \delta_5)$$
$$= \frac{1.05}{0.2} \cos(90°) = 0$$

Therefore,

$$\Delta P_5^{(0)} = P_{5,sch} - P_{5,calc}^{(0)}$$

$$= -2 - 0 = -2$$

and,

$$Q_{5,\text{calc}}^{(0)} = -|V_5|^2 B_{55} - \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \sin(\theta_{5n} + \delta_n - \delta_5)$$

$$= -|V_5|^2 B_{55} - |V_5 V_3 Y_{53}| \sin(\theta_{53} + \delta_3 - \delta_5)$$

$$= -1 \times (-5) - \frac{1.05}{0.2} \times \sin(90°) =$$

$$= 5 - 5.25$$

$$= -0.25$$

$$\Delta Q_5^{(0)} = Q_{5,\text{sch}} - Q_{5,\text{calc}}^{(0)}$$

$$= -1.2394 - (-0.25) = -0.9894$$

(9.12) The generator at bus 4 of the system of Example 9.5 is to be represented by a generator connected to bus 4 through a generator step-up transformer as shown in Fig. 9.13. The reactance of this transformer is 0.02 per unit; the tap is on the high-voltage side of the transformer with the off-nominal turns ratio of 1.05. Evaluate the jacobian elements of the rows corresponding to buses 4 and 5.

Solution:
Note to instructor: In this problem it is assumed that the new bus 5 becomes the regulated bus with $|V_5| = 1.02$ p.u., and that bus 4 is a load bus with an initial voltage value of $|V_4| = 1.0$ p.u.

الإتجاه الإسلامي

The transformer contribution to Y_{BUS} is:

$$
\begin{array}{cc}
\textcircled{4} & \textcircled{5} \\
\begin{bmatrix} Y & -tY \\ -tY & t^2Y \end{bmatrix}
\end{array}
$$

The augmented Y_{BUS} is:

$$
\begin{array}{c|ccccc}
 & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} \\
\textcircled{1} & Y_{11} & Y_{12} & Y_{13} & 0 & 0 \\
\textcircled{2} & Y_{21} & Y_{22} & 0 & Y_{24} & 0 \\
\textcircled{3} & Y_{31} & 0 & Y_{33} & Y_{34} & 0 \\
\textcircled{4} & 0 & Y_{42} & Y_{43} & Y_{44}+Y & -tY \\
\textcircled{5} & 0 & 0 & 0 & -tY & t^2Y
\end{array}
$$

where $Y = -j50$ and $t = 1.05$. Some of the desired jacobian entries corresponding to bus 4 are:

$$
\begin{aligned}
\frac{\partial P_4}{\partial \delta_4} &= \sum_{\substack{n=1 \\ n \neq 4}}^{5} |V_4 V_n Y_{4n}| \sin(\theta_{4n} + \delta_n - \delta_4) \\
&= |V_4 V_2 Y_{42}| \sin(\theta_{42} + \delta_2 - \delta_4) \\
&\quad + |V_4 V_3 Y_{43}| \sin(\theta_{43} + \delta_3 - \delta_4) \\
&\quad + |V_4 V_5 Y_{45}| \sin(\theta_{45} + \delta_5 - \delta_4) \\
&= |1.0 \times 1.0 \times 26.359696| \sin(101.30993°) \\
&\quad + |1.0 \times 1.0 \times 15.417934| \sin(101.30993°) \\
&\quad + |1.0 \times 1.02 \times 1.05 \times 50| \sin(90°) \\
&= 94.516337
\end{aligned}
$$

$$\frac{\partial P_4}{\partial \delta_5} = -|V_4 V_5 Y_{45}| \sin(\theta_{45} + \delta_5 - \delta_4)$$

$$= -|1 \times 1.02 \times 1.05 \times 50| \sin(90°)$$

$$= -53.55$$

$$\frac{\partial Q_4}{\partial \delta_2} = -|V_4 V_2 Y_{42}| \cos(\theta_{42} + \delta_2 - \delta_4)$$

$$= -|1 \times 1 \times 26.359696| \cos(101.30993°)$$

$$= 5.1695606$$

$$|V_2|\frac{\partial Q_4}{\partial |V_2|} = -|V_2 V_4 Y_{42}| \sin(\theta_{42} + \delta_2 - \delta_4)$$

$$= -| 1.0 \times 1.0 \times 26.359696| \sin(101.30993°)$$

$$= -25.847809$$

The remaining terms corresponding to bus 4 are calculated in a similar manner. The terms corresponding to bus 5 are:

$$\frac{\partial P_5}{\partial \delta_4} = - |V_5 V_4 Y_{54}| \sin(\theta_{54} + \delta_4 + \delta_5)$$

$$= - | 1.02 \times 1.0 \times 1.05 \times 50| \sin(90°)$$

$$= -53.55$$

$$\frac{\partial P_5}{\partial \delta_5} = \sum_{\substack{n=1 \\ n \neq 5}}^{5} |V_5 V_n Y_{5n}| \sin(\theta_{5n} + \delta_n - \delta_5)$$

$$= |1.02 \times 1.0 \times 1.05 \times 50| \sin(90°)$$

$$= 53.55$$

$$|V_4| \frac{\partial P_5}{\partial |V_4|} = |V_5 V_4 Y_{54}| \cos(\theta_{54} + \delta_4 - \delta_5)$$

$$= |1.02 \times 1.0 \times 1.05 \times 50| \cos(90°) = 0$$

Since bus 5 is a regulated bus, there are no terms involving Q_5 or $|V_5|$.

(9.13) For the system of Prob. 9.12, find the matrices **B'** and **B''** for use in the decoupled power-flow method.

Solution:

Y_{BUS} is:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \\
\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} &
\left[\begin{array}{ccccc}
Y_{11} & Y_{12} & Y_{13} & 0 & 0 \\
Y_{21} & Y_{22} & 0 & Y_{24} & 0 \\
Y_{31} & 0 & Y_{33} & Y_{34} & 0 \\
0 & Y_{42} & Y_{43} & Y_{44}+Y & -tY \\
0 & 0 & 0 & -tY & t^2Y
\end{array}\right]
\end{array}
$$

where $Y = -j50$ and the remaining entries are given in Table 9.4. The imaginary part of Y_{BUS} is called **B** and is given by:

$$
\mathbf{B} = \left[\begin{array}{ccccc}
B_{11} & B_{12} & B_{13} & 0 & 0 \\
B_{21} & B_{22} & 0 & B_{24} & 0 \\
B_{31} & 0 & B_{33} & B_{34} & 0 \\
0 & B_{42} & B_{43} & B_{44} & B_{45} \\
0 & 0 & 0 & B_{54} & B_{55}
\end{array}\right]
$$

with:

$$B_{44} = -40.863838 - 50$$
$$B_{45} = 50t$$
$$B_{54} = 50t$$
$$B_{55} = -50t^2$$

and the remaining B_{ij}'s are obtained from Table 9.4.

Using $t = 1$, changing the signs, and deleting the first row and column, we get **B'**:

$$\mathbf{B'} = - \begin{bmatrix} B_{22} & 0 & B_{24} & 0 \\ 0 & B_{33} & B_{34} & 0 \\ B_{42} & B_{43} & B_{44} & B_{45} \\ 0 & 0 & B_{54} & B_{55} \end{bmatrix}$$

or

$$\mathbf{B'} = \begin{array}{cccc} \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} \\ \begin{bmatrix} 44.835953 & 0 & -25.847809 & 0 \\ 0 & 40.863838 & -15.118528 & 0 \\ -25.847809 & -15.118528 & 90.863838 & -50 \\ 0 & 0 & -50 & 50 \end{bmatrix} \end{array}$$

B'' is obtained from **B** by deleting the rows and columns corresponding to the slack bus (bus 1) and the *PV* bus (bus 5), while t is set to 1.05 as specified:

$$\mathbf{B''} = \begin{array}{ccc} \textcircled{2} & \textcircled{3} & \textcircled{4} \\ \begin{bmatrix} 44.835953 & 0 & -25.847809 \\ 0 & 40.863838 & -15.118528 \\ -25.847809 & -15.118528 & 90.863838 \end{bmatrix} \end{array}$$

(9.14) A five-bus power system is shown in Fig. 9.14. The line, bus, transformer, and capacitor data are given in Tables 9.6, 9.7, 9.8, and 9.9, respectively. Use the Gauss-Seidel method to find the bus voltages for the first iteration.

Solution:

For t = 0.975:

$$-j25t = -j24.375$$
$$t(t-1)(-j25) = j0.609375$$
$$(1-t)(-j25) = -j0.625$$

And **Y**$_{BUS}$ is:

$$
\begin{bmatrix}
5-j24.947 & -2.5+j15 & 0 & -2.5+j10 & 0 \\
-2.5+j15 & 7.5-j59.932 & j24.375 & 0 & -5+j20 \\
0 & j24.375 & 4-j39.546 & 0 & -4+j16 \\
-2.5+j10 & 0 & 0 & 6.5-j27.8 & -4+j18 \\
0 & -5+j20 & -4+j16 & -4+j18 & 13-53.895
\end{bmatrix}
$$

$$V_2^{(1)} = \frac{1}{Y_{22}}\left[\frac{P_{2,sch} - jQ_{2,sch}}{V_2^{(0)*}} - \left(Y_{21}V_1^{(0)} + Y_{23}V_3^{(0)} + Y_{25}V_5^{(0)}\right)\right]$$

$$= \frac{1}{Y_{22}}\left[\frac{-0.6+j0.35}{1.0} - \Big\{(-2.5+j15)(1.01)\right.$$

$$\left. + (j24.375)(1.0) + (-5+j20)(1.0))\Big\}\right]$$

$$= \frac{6.925 - j59.175}{7.5 - j59.932}$$

$$= 0.986382 - j0.007890$$

$$V_{2,\text{acc}}^{(1)} = 1 + 1.6 \ (0.986382 - j0.007890 - 1)$$

$$= 0.978211 - j0.012624$$

$$V_3^{(1)} = \frac{1}{Y_{33}} \left[\frac{P_{3,\text{sch}} - jQ_{3,\text{sch}}}{V_3^{(0)}*} - (Y_{32}V_2^{(1)} + Y_{35}V_5^{(0)}) \right]$$

$$= \frac{1}{Y_{33}} \left[\frac{-0.7 + j0.42}{1.0} - \Big\{ (j24.375)(0.978211 - j0.012624) \right.$$

$$\left. + (-4 + j16)(1.0) \Big\} \right]$$

$$= \frac{2.99229 - j39.423893}{4 - j39.546}$$

$$= 0.994392 - j0.024915$$

$$V_{3,\text{acc}}^{(1)} = 1 + 1.6 \ (0.994392 - j0.024915 - 1)$$

$$= 0.991028 - j0.039864$$

$$V_4^{(1)} = \frac{1}{Y_{44}} \left[\frac{P_{4,sch} - jQ_{4,sch}}{V_4^{(0)*}} - (Y_{41} V_1^{(1)} + Y_{45} V_5^{(0)}) \right]$$

$$= \frac{1}{Y_{44}} \left[\frac{-0.8 + j0.5}{1.0} - \{ (-2.5 + j10)(1.01) \right.$$

$$\left. + (-4 + j18)(1.0) \} \right]$$

$$= \frac{5.725 - j27.6}{6.5 - j27.8}$$

$$= 0.9869984 - j0.0248378$$

$$V_{4,acc}^{(1)} = 1 + 1.6 \, (\, 0.9869984 - j0.0248378 - 1)$$

$$= 0.979197 - j0.039740$$

$$Q_5^{(1)} = - \, \text{Im} \left\{ V_5^{(0)*} \left[Y_{52} V_2^{(1)} + Y_{53} V_3^{(1)} + Y_{54} V_4^{(1)} + Y_{55} V_5^{(0)} \right] \right\}$$

$$= - \, \text{Im} \{ (1.0) \times [(-5 + j20)(0.978211 - j0.012624)$$
$$+ (-4 + j16)(0.991028 - j0.039864)$$
$$+ (-4 + j18)(0.979197 - j0.039740)$$
$$+ (13 - j53.895)(1.0)]\}$$

$$= - \, \text{Im} \{ 1.833669 - j0.46725 \} = 0.46725$$

$$V_5^{(1)} = \frac{1}{Y_{55}} \left[\frac{P_{5,\text{sch}} - jQ_5^{(1)}}{V_4^{(0)*}} - (Y_{52}V_2^{(1)} + Y_{53}V_3^{(1)} + Y_{54}V_4^{(1)}) \right]$$

$$= \frac{1}{Y_{55}} \left[\frac{1.25 - j0.46725}{1.0} - \left\{ (-11.166331 + j53.427750)(1.01) \right\} \right]$$

$$= \frac{12.527994 - j54.429278}{13 - j53.895}$$

$$= 1.007372 - j0.010536$$

and,

$$V_{5,\text{corr}}^{(1)} = \frac{1.0}{1.007427}(1.007372 - j0.010536)$$

$$= 0.999903 - j0.010458$$

(9.15) To apply the Newton-Raphson method to the power-flow solution of the system of Fig. 9.14, determine (a) Y_{BUS} of the system, (b) the mismatch equation at bus 5 evaluated at the initial voltage estimates of Table 9.7 for the first iteration, and (c) write mismatch equations in a form similar to Eq. (9.45).

Solution:
(a) For t = 0.975:

$$-j25t = -j24.375$$
$$t(t-1)(-j25) = j0.609375$$
$$(1-t)(-j25) = -j0.625$$

And Y_{BUS} is:

$$\begin{bmatrix} 5-j24.947 & -2.5+j15 & 0 & -2.5+j10 & 0 \\ -2.5+j15 & 7.5-j59.932 & j24.375 & 0 & -5+j20 \\ 0 & j24.375 & 4-j39.546 & 0 & -4+j16 \\ -2.5+j10 & 0 & 0 & 6.5-j27.8 & -4+j18 \\ 0 & -5+j20 & -4+j16 & -4+j18 & 13-53.895 \end{bmatrix}$$

(b)

$$\frac{\partial P_5}{\partial \delta_2} = -|V_5 V_2 Y_{52}| \sin(\theta_{52} + \delta_2 - \delta_5)$$

$$= -20.616 \sin(104.04°) = -20$$

$$\frac{\partial P_5}{\partial \delta_3} = -|V_5 V_3 Y_{53}| \sin(\theta_{53} + \delta_3 - \delta_5)$$

$$= -16.492 \sin(104.04°) = -16$$

$$\frac{\partial P_5}{\partial \delta_4} = -|V_5 V_4 Y_{54}| \sin(\theta_{54} + \delta_4 - \delta_5)$$

$$= -18.439 \sin(102.53°) = -18$$

$$\frac{\partial P_5}{\partial \delta_5} = -\left(\frac{\partial P_5}{\partial \delta_2} + \frac{\partial P_5}{\partial \delta_3} + \frac{\partial P_5}{\partial \delta_4}\right)$$

$$= -(-20 - 16 - 18) = 54.0$$

$$|V_2|\frac{\partial P_5}{\partial |V_2|} = |V_5 V_2 Y_{52}| \cos(\theta_{52} + \delta_2 - \delta_5)$$

$$= 20.616 \cos(104.04°) = -5.0$$

$$|V_3|\frac{\partial P_5}{\partial |V_3|} = |V_5 V_3 Y_{53}| \cos(\theta_{53} + \delta_3 - \delta_5)$$

$$= 16.492 \cos(104.04°) = -4.0$$

$$|V_4|\frac{\partial P_5}{\partial |V_4|} = |V_5 V_4 Y_{54}| \cos(\theta_{54} + \delta_4 - \delta_5)$$

$$= 18.439 \cos(102.53°) = -4.0$$

$$P^{(0)}_{5,\text{calc}} = 1.0^2 \times 13 + 20.616 \cos(104.04) + 16.492 \cos(104.04)$$
$$+ 18.439 \cos(104.04)$$
$$= 13 - 5 - 4 - 4 = 0$$

$$\Delta P^{(0)}_5 = P_{5,\text{sch}} - P^{(0)}_{5,\text{calc}}$$
$$= (1.9 - 0.65) - 0 = 1.25$$

The mismatch equation at bus 5 is:

$$-20\Delta\delta_2 - 16\Delta\delta_3 - 18\Delta\delta_4 + 54\Delta\delta_5 - 5\Delta|V_2| - 4\Delta|V_3| - 4\Delta|V_4| = 0$$

(c) For this system, we have:

$$
\begin{bmatrix}
\dfrac{\partial P_2}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \dfrac{\partial P_2}{\partial \delta_4} & \dfrac{\partial P_2}{\partial \delta_5} & |V_2|\dfrac{\partial P_2}{\partial |V_2|} & |V_3|\dfrac{\partial P_2}{\partial |V_3|} & |V_4|\dfrac{\partial P_2}{\partial |V_4|} \\
\dfrac{\partial P_3}{\partial \delta_2} & \dfrac{\partial P_3}{\partial \delta_3} & \cdots & & \cdots & |V_3|\dfrac{\partial P_3}{\partial |V_3|} & |V_4|\dfrac{\partial P_3}{\partial |V_4|} \\
\dfrac{\partial P_4}{\partial \delta_2} & \dfrac{\partial P_2}{\partial \delta_3} & \ddots & & & & |V_4|\dfrac{\partial P_4}{\partial |V_4|} \\
\dfrac{\partial P_5}{\partial \delta_2} & \dfrac{\partial P_5}{\partial \delta_3} & \dfrac{\partial P_5}{\partial \delta_4} & & & & |V_4|\dfrac{\partial P_5}{\partial |V_4|} \\
\dfrac{\partial Q_2}{\partial \delta_2} & \dfrac{\partial Q_2}{\partial \delta_2} & \cdots & & \cdots & |V_3|\dfrac{\partial Q_2}{\partial |V_3|} & |V_4|\dfrac{\partial Q_2}{\partial |V_4|} \\
\dfrac{\partial Q_3}{\partial \delta_2} & \dfrac{\partial Q_3}{\partial \delta_2} & \ddots & & \cdots & & |V_4|\dfrac{\partial Q_3}{\partial |V_4|} \\
\dfrac{\partial Q_4}{\partial \delta_2} & \cdots & & & \cdots & & |V_4|\dfrac{\partial Q_4}{\partial |V_4|}
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_2 \\ \Delta \delta_3 \\ \Delta \delta_4 \\ \Delta \delta_5 \\ \dfrac{\Delta |V_2|}{|V_2|} \\ \dfrac{\Delta |V_3|}{|V_3|} \\ \dfrac{\Delta |V_4|}{|V_4|}
\end{bmatrix}
=
\begin{bmatrix}
\Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \\ \Delta P_5 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4
\end{bmatrix}
$$

(9.16) For the system of Fig. 9.14, find matrices **B'** and **B''** for use in the decoupled power-flow method. Also, determine the first iteration P and Q mismatch equations at bus 4, and find the voltage magnitude at bus 4 at the end of the first iteration.

Solution:
To form **B'**, the capacitors are neglected and transformer tap t is set equal to 1.

$$
\begin{array}{cccc}
② & ③ & ④ & ⑤
\end{array}
$$
$$
\begin{bmatrix}
59.932 & -25 & 0 & -20 \\
-25 & 40.96 & 0 & -16 \\
0 & 0 & 27.95 & -18 \\
-20 & -16 & -18 & 53.895
\end{bmatrix}
$$

where:

$$\begin{bmatrix} & & \\ & B' & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \\ \Delta\delta_5 \end{bmatrix} = \begin{bmatrix} \Delta P_2/|V_2| \\ \Delta P_3/|V_3| \\ \Delta P_4/|V_4| \\ \Delta P_5/|V_5| \end{bmatrix}$$

In the formation of **B''**, the capacitors and off-nominal tap settings are considered. Bus 5 is deleted since it is a regulated bus.

$$\begin{matrix} ② & ③ & ④ \end{matrix}$$
$$\begin{bmatrix} 59.932 & -24.375 & 0 \\ -24.375 & 39.546 & 0 \\ 0 & 0 & 27.8 \end{bmatrix}$$

where:

$$\begin{bmatrix} & & \\ & B'' & \\ & & \end{bmatrix} \begin{bmatrix} \Delta|V_2| \\ \Delta|V_3| \\ \Delta|V_4| \end{bmatrix} = \begin{bmatrix} \Delta Q_2/|V_2| \\ \Delta Q_3/|V_3| \\ \Delta Q_4/|V_4| \end{bmatrix}$$

$$P_{4,calc}^{(0)} = |V_4|^2 G_{44} + \sum_{\substack{n=1 \\ n \neq 4}}^{5} |V_4 V_n Y_{4n}| \cos(\theta_{4n} + \delta_n - \delta_4)$$

$$= 6.5 + 1 \times 1.01 \times 10.308 \cos(104.04°) + 18.439 \cos(102.53°)$$

$$= 6.5 - 1.01 \times 2.5 - 4 = -0.025$$

$$\Delta P_4^{(0)} = P_{4,sch} - P_{4,calc}^{(0)} = -0.8 - (-0.025)$$

$$= -0.775 \text{ p.u.}$$

$$Q_{4,calc}^{(0)} = -|V_4|^2 B_{44} - \sum_{\substack{n=1 \\ n \neq 4}}^{5} |V_4 V_n Y_{4n}| \sin(\theta_{4n} + \delta_n - \delta_4)$$

$$= -(-27.8) - 1 \times 1.01 \times 10.308 \sin(104.04°) - 18.439 \sin(102.53°)$$

$$= 27.8 - 1.01 \times 10 - 18 = -0.3$$

$$\Delta Q_4^{(0)} = Q_{4,sch} - Q_{4,calc}^{(0)} = -0.5 - (-0.3)$$

$$= -0.2 \text{ p.u.}$$

The P- and Q-equations at bus 4 are:

$$27.95 \, \Delta\delta_4 - 18\Delta\delta_5 = -0.775/|V_4| = -0.775$$

$$27.8 \, \Delta|V_4| = -0.2/|V_4| = -0.2$$

Therefore, the voltage magnitude at bus 4 after the first iteration is:

$$|V_4^{(1)}| = |V_4^{(0)}| + \Delta|V_4| = 1 + \frac{-0.2}{27.8} = 0.9928$$

(9.17) Suppose that in Fig. 9.14 the transformer between buses 2 and 3 is a phase shifter where t is now the complex variable and is $1.0\underline{/-2°}$. (a) Find Y_{BUS} of this system, (b) When compared with the power-flow solution of Prob. 9.15, will the real power in the line from bus 5 to bus 3 increase or decrease? What about the reactive power flow? Explain why qualitatively.

Solution:

(a) The following is used in construction of Y_{BUS} where $t = 1.0\underline{/-2°}$ and $Y = 1/j0.04$:

$$
\begin{bmatrix}
Y & -tY \\
-t^*Y & |t|^2Y
\end{bmatrix}
=
\begin{bmatrix}
-j25 & j25\underline{/-2°} \\
j25\underline{/+2°} & -j25
\end{bmatrix}
=
\begin{bmatrix}
-j25 & 0.873+j24.985 \\
-0.873+j24.985 & -j25
\end{bmatrix}
$$

Y_{BUS} is then given by:

$$
\begin{bmatrix}
5-j24.947 & -2.5+j15 & 0 & -2.5+j10 & 0 \\
-2.5+j15 & 7.5-j59.932 & 0.873+j24.985 & 0 & -5+j20 \\
0 & -0.873+j24.985 & 4-j40.78 & 0 & -4+j16 \\
-2.5+j10 & 0 & 0 & 6.5-j27.8 & -4+j18 \\
0 & -5+j20 & -4+j16 & -4+j18 & 13-53.895
\end{bmatrix}
$$

Compare this with that obtained in Prob. 9.15.

(b) Because of the phase angle of –2°, more real power will flow from bus 2 to 3. Consequently, less real power will flow from bus 5 to bus 3. Since the magnitude of t is 1.0, the voltage magnitude at bus 3 will drop compared to that of Prob. 9.15. As a result, more reactive power will flow from bus 5 to bus 3.

(9.18) To apply the decoupled power-flow method to the system of Prob. 9.17, find matrices **B'** and **B''**.

Solution:

To form **B'**, only capacitors are to be neglected. Using the result of Y_{BUS} determined in Prob. 9.17, **B'** is:

$$
\begin{array}{cccc}
② & ③ & ④ & ⑤
\end{array}
$$

$$
\begin{bmatrix}
59.932 & -24.985 & 0 & -20 \\
-24.985 & 40.96 & 0 & -16 \\
0 & 0 & 27.95 & -18 \\
-20 & -16 & -18 & 53.895
\end{bmatrix}
$$

In determining **B''**, the phase shifter angle is neglected and the rows and columns corresponding to regulated bus 5 are deleted. Thus, **B''** is given by:

$$
\begin{array}{ccc}
② & ③ & ④
\end{array}
$$

$$
\begin{bmatrix}
59.932 & -25 & 0 \\
-25 & 40.78 & 0 \\
0 & 0 & 27.8
\end{bmatrix}
$$

(9.19) Redo Example 9.10 when an 18-MVAR shunt capacitor bank is added to bus 3.

Solution:

The **B** matrix (the imaginary part of Y_{BUS}) is modified to account for the 18 MVAR capacitor. This is done by adding a 0.18 p.u. value to B_{33}. All other elements are unchanged.

Thus:

$$
B = \begin{array}{ccc} \quad\quad ② & \quad\quad ③ & \quad\quad ④ \\ \begin{bmatrix} -44.835953 & 0 & 25.847809 \\ 0 & -40.863838+0.18 & 15.118528 \\ 25.847809 & 15.118528 & -40.863838 \end{bmatrix} \end{array}
$$

The capacitor is ignored in the P-equations, and so

$$
B' = \begin{bmatrix} 44.835953 & 0 & -25.847809 \\ 0 & 40.863838 & -15.118528 \\ -25.847809 & -15.118528 & 40.863838 \end{bmatrix}
$$

Which is the same as in the example. Thus, the initial angle corrections for the first iteration will be the same as calculated in the example.

The matrix B'' is:

$$
B'' = \begin{array}{cc} \quad\quad ② & \quad\quad ③ \\ \begin{bmatrix} 44.835953 & 0 \\ 0 & 40.683838 \end{bmatrix} \end{array}
$$

The Q-equations to be solved are:

$$
\begin{bmatrix} 44.835953 & 0 \\ 0 & 40.683838 \end{bmatrix} \begin{bmatrix} \Delta|V_2| \\ \Delta|V_3| \end{bmatrix} = \begin{bmatrix} \Delta Q_2/\Delta|V_2| \\ \Delta Q_3/\Delta|V_3| \end{bmatrix}
$$

The value calculated for $\Delta Q_3 / |V_3|$ must be modified to reflect the change in B_{33}. To do this, we add an additional term as follows:

$$\frac{\Delta Q_3}{\Delta |V_3|} = -1.27684 + \frac{1}{|1.0|} 1.0^2 (0.18) = -1.09684$$

Solution of the Q-equations gives:

$$\Delta |V_2| = -0.01793 \quad \text{and} \quad \Delta |V_3| = -0.02696$$

And the new voltage magnitudes at buses 2 and 3 are:

$$|V_2| = 0.98207 \quad \text{and} \quad |V_3| = 0.97304$$

(9.20) In applying the Newton-Raphson method, if the amount of reactive power required to maintain the specified voltage at a *PV* bus exceeds the maximum limit of its reactive power generation capacity, the reactive power at that bus is set to that limit and the type of bus becomes a load bus. Suppose the maximum reactive power generation at bus 4 is limited to 150 MVAR in the system of Example 9.5. Using the first-iteration result given in Sec. 9.4 following Example 9.5, determine whether or not the type of bus 4 should be converted to a load bus at the start of the second iteration. If so, calculate the reactive power mismatch at bus 4 that should be used in the second-iteration mismatch equation.

Solution:

$$Q_{4,calc}^{(0)} = -|V_4|^2 B_{44} - \sum_{\substack{n=1 \\ n \neq 4}}^{4} |V_4 V_n Y_{4n}| \sin(\theta_{4n} + \delta_n - \delta_4)$$

$$= -1.02^2(-40.863838) - 0$$
$$- 1.02 \times 0.98335 \times 26.359696 \sin(101.30993 - 0.93094 - 1.54383°)$$
$$- 1.02 \times 0.97095 \times 15.417934 \sin(101.30993 - 1.78790 - 1.54383°)$$

$$= 42.514737 - 26.125504 - 15.12165$$
$$= 1.2675831$$

When the reactive load of 49.58 MVAR is added to this reactive power requirement of 126.76 MVAR, the actual reactive power generation is 176.34 MVAR which exceeds the limit of 150 MVAR. Therefore, bus 4 should be considered to be a load bus. The net reactive power becomes:

$$150 \text{ MVAR} - 49.58 = 100.42 \text{ MVAR} = 1.0042 \text{ p.u.}$$

Thus the scheduled reactive power at this bus is 1.0042 p.u. Since the calculated reactive power is 1.2676 p.u., the reactive power mismatch at bus 4 is 1.0042 - 1.2676 = -0.2634 p.u.

Chapter 10 Problem Solutions

10.1 A 60-Hz alternating voltage having a rms value of 100 V is applied to a series RL circuit by closing a switch. The resistance is 15 Ω and the inductance is 0.12 H.

(a) Find the value of the dc component of current upon closing the switch if the instantaneous value of the voltage is 50 V at that time.

(b) What is the instantaneous value of the voltage which will produce the maximum dc component of current upon closing the switch?

(c) What is the instantaneous value of the voltage which will result in the absence of any dc component of current upon closing the switch?

(d) If the switch is closed when the instantaneous voltage is zero, find the instantaneous current 0.5, 1.5 and 5.5 cycles later.

Solution:

(a)

$$v = V_m \sin(\omega t + \alpha)$$

$$\text{For } t = 0 \quad 50 = \sqrt{2} \times 100 \sin\alpha$$

$$\alpha = 20.70° \text{ or } 159.30°$$

$$Z = 15 + j2\pi \times 60 \times 0.12 = 47.66 \underline{/71.66°}$$

$$\text{At } t = 0 \quad i_{dc} = -\frac{100 \times \sqrt{2}}{47.66} \sin(20.7° - 71.66°) = 2.305 \text{ A}$$

$$\text{or} \quad i_{dc} = -\frac{100 \times \sqrt{2}}{47.66} \sin(159.3° - 71.66°) = -2.965 \text{ A} \quad \text{(max.)}$$

(b) Maximum dc component occurs when $\sin(\alpha - \theta) = \pm 1$ or when $(\alpha - \theta) = \pm 90°$ when $\alpha = 161.66°$ or $-18.34°$.

$$v = 100\sqrt{2}\sin 161.66° = 100\sqrt{2}\sin -18.34° = \pm 44.5 \text{ V}$$

(c) No dc component will occur when $\alpha - \theta = 0$, or $180°$, i.e. when $\alpha = 71.66°$ or $251.66°$.

$$v = 100\sqrt{2}\sin 71.66° = 100\sqrt{2}\sin 251.66° = \pm 134.24 \text{ V}$$

(d) For $v = 0$ when $t = 0$ and $\alpha = 0$. 0.5 cycles later $\omega t = \pi$ rad.

$$t = \frac{\pi}{2\pi 60} = 0.008333 \text{ s}$$

$$i = \frac{100\sqrt{2}}{47.66}\left[\sin(180° - 71.66°) - \epsilon^{\frac{-15}{0.12}(0.008333)} \times \sin(-71.66°)\right]$$

$$= \frac{100\sqrt{2}}{47.66}\left(1 + \epsilon^{-1.0461}\right)\sin(-71.66°) = 3.810 \text{ A}$$

Similarly,

$$1.5 \text{ cycles later:} \quad \omega t = 3\pi$$
$$t = 0.025 \text{ s}$$
$$i = 2.940 \text{ A}$$

and

$$5.5 \text{ cycles later:} \quad \omega t = 11\pi$$
$$t = 0.09167 \text{ s}$$
$$i = 2.817 \text{ A}$$

Note that the dc component has essentially disappeared after 5.5 cycles.
(5 time constants = 0.04 s).

10.2 A generator connected through a 5-cycle circuit breaker to a transformer is rated 100 MVA, 18 kV, with reactances of $X_d'' = 19\%$, $X_d' = 26\%$ and $X_d = 130\%$. It is operating at no load and rated voltage when a three-phase short circuit occurs between the breaker and the transformer. Find (a) the sustained short-circuit current in the breaker, (b) the initial symmetrical rms current in the breaker and (c) the maximum possible dc component of the short-circuit current in the breaker.

Solution:

$$\text{Base current} = \frac{100,000}{\sqrt{3} \times 18} = 3207.5 \text{ A}$$

$$(a) \quad \frac{1}{j1.3} \times 3207.5 = 2,467 \text{ A}$$

$$(b) \quad \frac{1}{j0.19} \times 3207.5 = 16,882 \text{ A}$$

$$(c) \quad \sqrt{2} \times 16,882 = 23,874 \text{ A}$$

10.3 The three-phase transformer connected to the generator described in Prob. 10.2 is rated 100 MVA, 240Y/18Δ kV, $X = 10\%$. If a three-phase short circuit occurs on the high-voltage side of the transformer at rated voltage and no load, find (a) the initial symmetrical rms current in the transformer windings on the high-voltage side and (b) the initial symmetrical rms current in the line on the low-voltage side.

Solution:

$$I'' = \frac{1.0}{j(0.19 + 0.10)} = -j3.448 \text{ per unit}$$

$$\text{Base } I_{\text{HV}} = \frac{100,000}{\sqrt{3} \times 240} = 240.6 \text{ A}$$

$$\text{Base } I_{\text{LV}} = \frac{100,000}{\sqrt{3} \times 18} = 3207.5 \text{ A}$$

(a) $3.448 \times 240.6 = 829.5$ A

(b) $3.448 \times 3207.5 = 11,060$ A

10.4 A 60-Hz generator is rated 500 MVA, 20 kV, with $X_d'' = 0.20$ per unit. It supplies a purely resistive load of 400 MW at 20 kV. The load is connected directly across the terminals of the generator. If all three phases of the load are short-circuited simultaneously, find the initial symmetrical rms current in the generator in per unit on a base of 500 MVA, 20 kV.

Solution:

$$I_{\text{Load}} = \frac{400}{500} = 0.8 \text{ per unit}$$

$$E_g'' = 1.0 + 0.8 \times j0.20 = 1.0 + j0.16 \text{ per unit}$$

$$I_g'' = \frac{1 + j0.16}{j0.20} = 0.8 - j5.0 \text{ per unit or } 5.06 \text{ per unit}$$

10.5 A generator is connected through a transformer to a synchronous motor. Reduced to the same base, the per-unit subtransient reactances of the generator and motor are 0.15 and 0.35, respectively, and the leakage reactance of the transformer is 0.10 per unit. A three-phase fault occurs at the terminals of the motor when the terminal voltage of the generator is 0.9 per unit and the output current of the generator is 1.0 per unit at 0.8 power factor leading. Find the subtransient current in per unit in the fault, in the generator and in the motor. Use the terminal voltage of the generator as the reference phasor and obtain the solution (a) by computing the voltages behind subtransient reactance in the generator and motor and (b) by using Thévenin's theorem.

Solution:
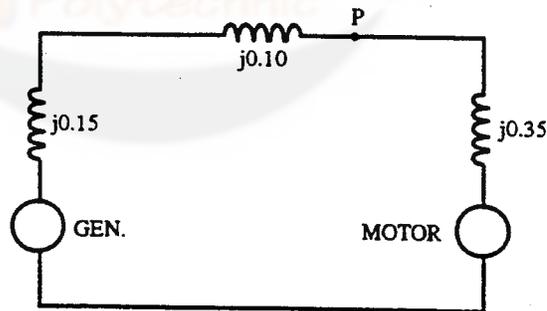
P: fault point

(a)

$$E_g'' = 0.9 + (0.8 + j0.6)(j0.15) = 0.81 + j0.12 \text{ per unit}$$

$$E_m'' = 0.9 - (0.8 + j0.6)(j0.45) = 1.17 - j0.36 \text{ per unit}$$

$$I_g'' = \frac{0.81 + j0.12}{j0.25} = 0.48 - j3.24 \text{ per unit}$$

$$I''_m = \frac{1.17 - j0.36}{j0.35} = -1.03 - j3.34 \text{ per unit}$$

$$I''_f = I''_g + I''_m = -0.55 - j6.58 \text{ per unit}$$

(b)

$$V_f = 0.9 - (0.8 + j0.6)(j0.1) = 0.96 - j0.08 \text{ per unit}$$

$$Z_{\text{th}} = \frac{j0.25 \times j0.35}{j0.60} = j0.146 \text{ per unit}$$

$$I''_f = \frac{0.96 - j0.08}{j0.146} = -0.55 - j6.58 \text{ per unit}$$

By replacing I''_f by a current source and then applying the principle of superposition,

$$I''_g = 0.8 + j0.6 + \frac{j0.35}{j0.60}(-0.55 - j6.58) = 0.48 - j3.24 \text{ per unit}$$

$$I''_m = -0.8 - j0.6 + \frac{j0.25}{j0.60}(-0.55 - j6.58) = -1.03 - j3.34 \text{ per unit}$$

10.6 Two synchronous motors having subtransient reactances of 0.80 and 0.25 per unit, respectively, on a base of 480 V, 2000 kVA are connected to a bus. This motor is connected by a line having a reactance of 0.023 Ω to a bus of a power system. At the power-system bus the short-circuit megavoltamperes of the power system are 9.6 MVA for a nominal voltage of 480 V. When the voltage at the motor bus is 440 V, neglect load current and find the initial symmetrical rms current in a three-phase fault at the motor bus.

Solution:
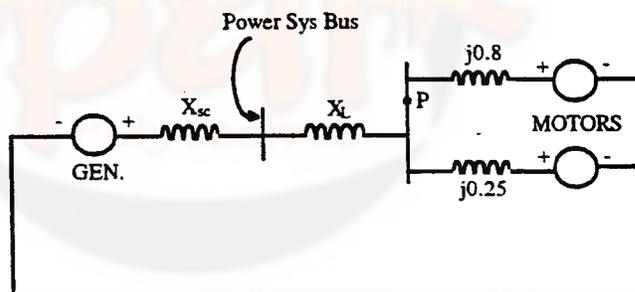
P: fault point

$$\text{Base } Z = \frac{0.48^2}{2} = 0.1152 \ \Omega$$

$$X_L = \frac{0.023}{0.1152} = 0.20 \text{ per unit}$$

$$X_{SC} = \frac{2}{96} = 0.208 \text{ per unit}$$

$$X_{\text{th}} = \frac{1}{\frac{1}{0.8} + \frac{1}{0.25} + \frac{1}{0.408}} = 0.130 \text{ per unit}$$

$$I''_f = \frac{440/480}{j1.30} = 7.05 \text{ per unit}$$

$$\text{or} \quad 7.05 \times \frac{2000}{\sqrt{3} \times 0.48} = 17,000 \text{ A}$$

10.7 The bus impedance matrix of a four-bus network with values in per unit is

$$\mathbf{Z_{bus}} = \begin{bmatrix} j0.15 & j0.08 & j0.04 & j0.07 \\ j0.08 & j0.15 & j0.06 & j0.09 \\ j0.04 & j0.06 & j0.13 & j0.05 \\ j0.07 & j0.09 & j0.05 & j0.12 \end{bmatrix}$$

Generators connected to buses ① and ② have their subtransient reactances included in $\mathbf{Z_{bus}}$. If prefault current is neglected, find the subtransient current in per unit in the fault for a three-phase fault on bus ④. Assume the voltage at the fault is $1.0 \angle 0°$ per unit before the fault occurs. Find also the per-unit current from generator 2 whose subtransient reactance is 0.2 per unit.

<u>Note to Instructor</u>: This short problem is easily varied by assuming the fault to occur on other buses.

Solution:

At bus ④,

$$I''_f = \frac{1}{j0.12} = -j8.33 \text{ per unit}$$

$$V_2 = 1.0 - \frac{0.09}{0.12} = 0.25 \text{ per unit}$$

From generator 2,

$$I''_2 = \frac{1 - 0.25}{j0.2} = -3.75 \text{ per unit}$$

10.8 For the network shown in Fig. 10.17, find the subtransient current in per unit from generator 1 and in line ①-② and the voltages at buses ① and ③ for a three-phase fault on bus ②. Assume that no current is flowing prior to the fault and that the prefault voltage at bus ② is $1.0\angle 0°$ per unit. Use the bus impedance matrix in the calculations.

Solution:
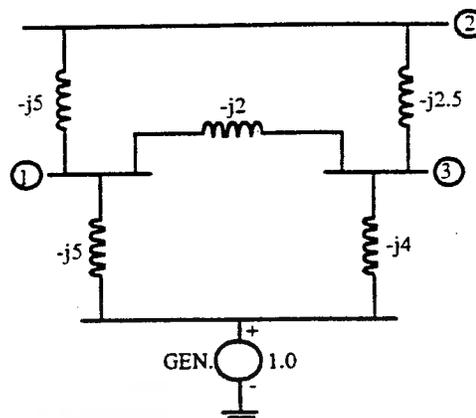
Thevenin Network
(Admittances marked in per unit)

$$\mathbf{Y}_{bus} = \begin{bmatrix} -j12 & j5 & j2 \\ j5 & -j7.5 & j2.5 \\ j2 & j2.5 & -j8.5 \end{bmatrix}$$

$$\Delta = \frac{1}{-j}\{12(7.5 \times 8.5 - 2.5 \times 2.5) + 5(-5 \times 8.5 - 2 \times 2.5) - 2[-5(-25) - (-2 \times 7.5)]\}$$

$$= j397.5$$

For the fault at bus ② the impedances needed are

$$Z_{12} = \frac{\Delta_{21}}{\Delta} = -\frac{j5(-j8.5) - j2.5(j2)}{j397.5} = \frac{-42.5 - j5}{j397.5} = j0.1195$$

$$Z_{22} = \frac{\Delta_{21}}{\Delta} = \frac{-j12(-j8.5) - j2(j2)}{j397.5} = \frac{-102 + 4}{j397.5} = j0.2465$$

$$Z_{32} = \frac{\Delta_{23}}{\Delta} = -\frac{-j12(j2.5) - (j5)(j2)}{j397.5} = \frac{-30 - 10}{j397.5} = j0.1006$$

$$I''_f = \frac{1.0}{-j0.2465} = -j4.056 \text{ per unit}$$

$$V_1 = 1 - \frac{j0.1195}{j0.2465} = 0.515 \text{ per unit}$$

$$V_3 = 1 - \frac{j0.1006}{j0.2465} = 0.592 \text{ per unit}$$

$$I''_{12} = \frac{0.515}{j0.2} = -j2.58 \text{ per unit}$$

From generator 1,

$$I''_g = \frac{1 - 0.515}{j0.2} = -j2.43 \text{ per unit}$$

10.9 For the network shown in Fig. 10.17 determine $\mathbf{Y_{bus}}$ and its triangular factors. Use the triangular factors to generate the elements of $\mathbf{Z_{bus}}$ needed to solve Prob. 10.8.

Solution:

$$\mathbf{Y_{bus}} = \begin{bmatrix} -j12 & j5 & j2 \\ j5 & -j7.5 & j2.5 \\ j2 & j2.5 & -j8.5 \end{bmatrix} \text{ per unit}$$

$$= \underbrace{\begin{bmatrix} -j12 & \cdot & \cdot \\ j5 & -j5.4167 & \cdot \\ j2 & j3.3333 & -j6.1154 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 1 & -0.4167 & -0.1667 \\ \cdot & 1 & -0.6154 \\ \cdot & \cdot & 1 \end{bmatrix}}_{\mathbf{U}}$$

$$\mathbf{Y_{bus}}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} \quad \text{where}$$

$$\mathbf{U}^{-1} = \begin{bmatrix} 1 & 0.4167 & 0.4231 \\ \cdot & 1 & 0.6154 \\ \cdot & \cdot & 1 \end{bmatrix} \qquad \mathbf{L}^{-1} = \begin{bmatrix} j0.0833 & & \\ j0.0769 & j0.1846 & \\ j0.0692 & j0.1006 & j0.1635 \end{bmatrix}$$

Hence,

$$\mathbf{Z_{bus}} = \mathbf{Y_{bus}}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} = \begin{bmatrix} j0.1447 & j0.1195 & j0.0692 \\ j0.1195 & j0.2465 & j0.1006 \\ j0.0692 & j0.1006 & j0.1635 \end{bmatrix} \text{ per unit}$$

10.10 If a three-phase fault occurs at bus ① of the network of Fig. 10.5 when there is no load (all bus voltages equal $1.0\underline{/0°}$ per unit), find the subtransient current in the fault, the voltages at buses ②, ③ and ④, and the current from the generator connected to bus ④. Use equivalent circuits based on $\mathbf{Z_{bus}}$ of Example 10.3 and similar to those of Fig. 10.7 to illustrate your calculations.

Solution:

$$I_f'' = \frac{1.0\underline{/0°}}{Z_{11}} = \frac{1.0\underline{/0°}}{j0.2436} = -j4.105 \text{ per unit}$$
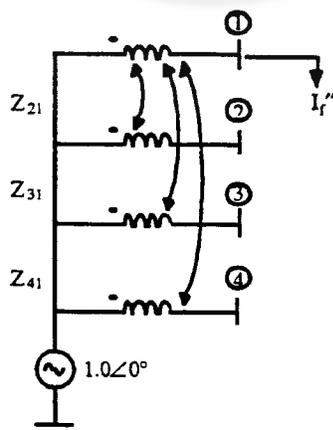
During the fault,

$$\begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - I_f'' \begin{bmatrix} Z_{21} \\ Z_{31} \\ Z_{41} \end{bmatrix}$$

$$= 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - (-j4.105) \begin{bmatrix} j0.1938 \\ j0.1544 \\ j0.1456 \end{bmatrix} = \begin{bmatrix} 0.2444\angle 0° \\ 0.3662\angle 0° \\ 0.4023\angle 0° \end{bmatrix} \text{ per unit}$$

Current from generator at bus ④ is calculated to be

$$\frac{(E_b'' - V_4)}{j0.30} = \frac{(1.0 - 0.4023)}{j0.30} = -j1.992 \text{ per unit}$$
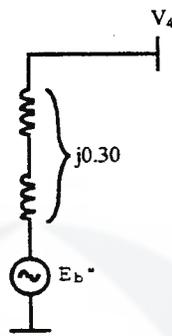
10.11 The network of Fig. 10.8 has the bus impedance matrix given in Example 10.4. If a short-circuit fault occurs at bus ② of the network when there is no load (all bus voltages equal $1.0\angle 0°$ per unit), find the subtransient current in the fault, the voltages at buses ① and ③, and the current from the generator connected to bus ①. Use equivalent circuits based on Z_{bus} and similar to those of Fig. 10.7 to illustrate your calculations.

Solution:

$$I_f'' = \frac{1.0\angle 0°}{Z_{22}} = \frac{1.0\angle 0°}{j0.1338} = -j7.474 \text{ per unit}$$
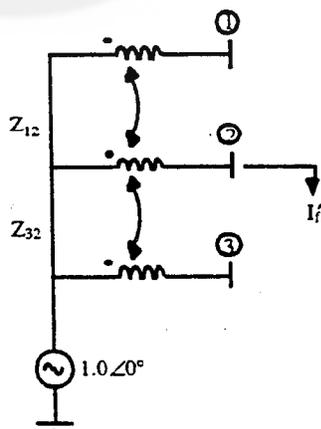
During the fault,

$$\begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \end{bmatrix} - I_f'' \begin{bmatrix} Z_{12} \\ Z_{32} \end{bmatrix}$$

$$= 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (-j7.474) \begin{bmatrix} j0.0793 \\ j0.0664 \end{bmatrix} = \begin{bmatrix} 0.5830\angle 0° \\ 0.5037\angle 0° \end{bmatrix} \text{ per unit}$$

Current from generator at bus ① is calculated to be

$$\frac{(1.0\angle 0° - V_1)}{j0.1111} = \frac{(1.0 - 0.5830)}{j0.1111} = -j3.753 \text{ per unit}$$
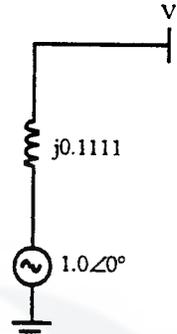
10.12 $\mathbf{Z_{bus}}$ for the network of Fig. 10.8 is given in Example 10.4. If a tail-end short-circuit fault occurs on line ③–⑤ of the network on the line side of the breaker at bus ③, calculate the subtransient current in the fault when only the near-end breaker at bus ③ has opened. Use the equivalent circuit approach of Fig. 10.11.

Solution:
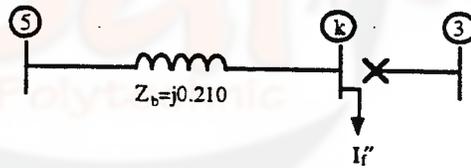
$$I_f'' = \frac{1.0\angle 0°}{Z_{kk,\text{new}}} \quad \text{where}$$

$$Z_{kk,\text{new}} = Z_{55} + Z_b - \frac{(Z_{55} - Z_{35})^2}{Z_{\text{th},53} - Z_b}$$

$$Z_{\text{th},53} = Z_{55} + Z_{33} - 2Z_{53}$$

Hence,

$$Z_{\text{th},53} = j[0.1301 + 0.0875 - 2 \times 0.0603] = j0.097 \text{ per unit}$$

$$Z_{kk,\text{new}} = j\left[0.1301 + 0.210 - \frac{(0.1301 - 0.0603)^2}{0.097 - 0.210}\right] = j0.2970 \text{ per unit}$$

$$I_f'' = \frac{1.0\angle 0°}{j0.2970} = -j3.367 \text{ per unit}$$

10.13 Figure 9.2 shows the one-line diagram of a single power network which has the line data given in Table 9.2. Each generator connected to buses ① and ④ has a subtransient reactance of 0.25 per unit. Making the usual fault-study assumptions, summarized in Sec. 10.6, determine for the network (a) Y_{bus}, (b) Z_{bus}, (c) the subtransient current in per unit in a three-phase fault on bus ③ and (d) the contributions to the fault current from line ①–③ and from line ④–③.

Solution:
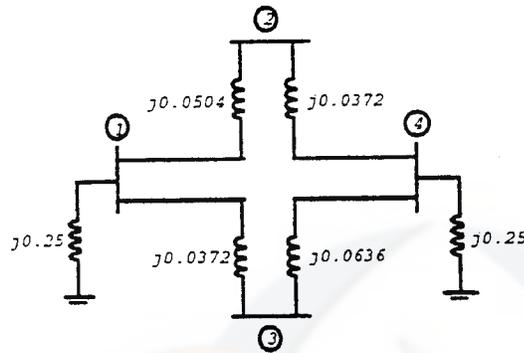
Reactance diagram: Admittance diagram:

(a)
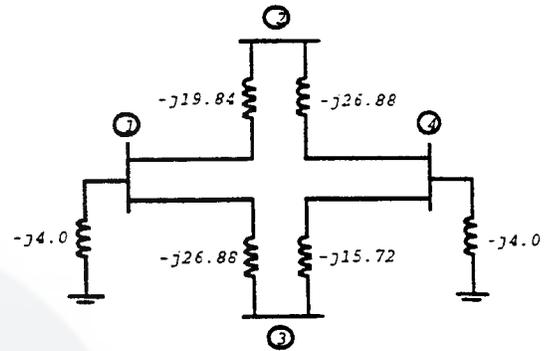
$$Y_{bus} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{cccc} ① & ② & ③ & ④ \\ \left[\begin{array}{cccc} -j50.72 & j19.84 & j26.88 & j0 \\ j19.84 & -j46.72 & j0 & j26.88 \\ j26.88 & j0 & -j42.60 & j15.72 \\ j0 & j26.88 & j15.72 & -j46.60 \end{array}\right] \end{array}$$

(b)

$$Z_{bus} = Y_{bus}^{-1} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{cccc} ① & ② & ③ & ④ \\ \left[\begin{array}{cccc} j0.1357 & j0.1234 & j0.1278 & j0.1143 \\ j0.1234 & j0.1466 & j0.1246 & j0.1266 \\ j0.1278 & j0.1246 & j0.1492 & j0.1222 \\ j0.1143 & j0.1266 & j0.1222 & j0.1357 \end{array}\right] \end{array}$$

(c) From a fault at bus ③,

$$I_f'' = \frac{1.0\angle 0°}{Z_{33}} = \frac{1.0\angle 0°}{j0.1492} = -j6.702 \text{ per unit}$$

(d) During the fault,

$$\begin{bmatrix} V_1 \\ V_4 \end{bmatrix} = 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \end{bmatrix} - I_f'' \begin{bmatrix} Z_{13} \\ Z_{43} \end{bmatrix}$$

$$= 1.0\angle 0° \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (-j6.702) \begin{bmatrix} j0.1278 \\ j0.1222 \end{bmatrix} = \begin{bmatrix} 0.1435\angle 0° \\ 0.1810\angle 0° \end{bmatrix} \text{ per unit}$$

Current flow in line ①–③ is calculated to be

$$\frac{(V_1 - V_3)}{jX_{13}} = \frac{(0.1435 - 0)}{j0.0372} = -j3.857 \text{ per unit}$$

Current flow in line ④–③ is calculated to be

$$\frac{(V_4 - V_3)}{jX_{43}} = \frac{(0.1810 - 0)}{j0.0638} = -j2.846 \text{ per unit}$$

where the sum of these currents is $-j6.703 (\cong I_f'')$.

10.14 A 625-kV generator with $X_d'' = 0.20$ per unit is connected to a bus through a circuit breaker as shown in Fig. 10.18. Connected through circuit breakers to the same bus are three synchronous motors rated 250 hp, 2.4 kV, 1.0 power factor, 90% efficiency, with $X_d'' = 0.20$ per unit. The motors are operating at full load, unity power factor and rated voltage, with the load equally divided between the machines.

(a) Draw the impedance diagram with the impedances marked in per unit on a base of 625 kVA, 2.4 kV.

(b) Find the symmetrical short-circuit current in amperes which must be interrupted by breakers A and B for a three-phase fault at point P. Simplify the calculations by neglecting the prefault current.

(c) Repeat part (b) for a three-phase fault at point Q.

(d) Repeat part (b) for a three-phase fault at point R.

Solution:

(a)
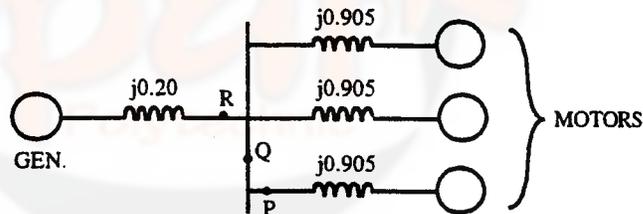
$$\text{Motor input} = \frac{250 \times 0.746}{0.9} = 207.2 \text{ kVA}$$

$$X_m'' = 0.2 \times \frac{625}{207.2} = 0.603 \text{ per unit}$$

For interrupting current use

$$1.5X_m = 1.5 \times 0.603 = 0.905 \text{ per unit}$$

$$\text{Base I} = \frac{625}{\sqrt{3} \times 2.4} = 150.4 \text{ per unit}$$

$$Z_{\text{th}} = \frac{(j0.905/3)j0.2}{j0.905/3 + j0.2} = j0.1203 \text{ per unit}$$

$$I_f'' = \frac{1}{j0.1203} = -j8.315 \text{ per unit}$$

From the generator:

$$I = -j8.315 \left(\frac{j0.905/3}{j0.905/3 + j0.2} \right) = -j5.000 \text{ per unit, or } 752 \text{ A}$$

From each motor:

$$I = \frac{[-j8.315 - (-j5.0)]}{3} = -j1.105 \text{ per unit, or } 166.2 \text{ A}$$

(b) Fault at P

Thru A: $I = 752$ A (gen. only)

Thru B: $I = -j5.0 + 2(-j1.105) = -j7.210$ per unit or 1084 A

(c) Fault at Q

Thru A: $I = 752$ A (gen. only)

Thru B: $I = 166.2$ A (one motor)

(d) Fault at R

Thru A: $I = 3(166.2) = 493.6$ A

Thru B: $I = 166.2$ A

Maximum currents to be interrupted by A and B are 752 A and 1084 A, respectively.

10.15 A circuit breaker having a nominal rating of 34.5 kV and a continuous current rating of 1500 A has a voltage range factor K of 1.65. Rated maximum voltage is 38 kV and the rated short-circuit current at that voltage is 22 kA. Find (a) the voltage below which rated short-circuit current does not increase as operating voltage decreases and the value of that current and (b) rated short-circuit current at 34.5 kV.

> Note to Instructor: The attention of the student should be directed to the paragraph just preceding Example 10.7. Students wishing to learn about circuit breaker applications should review Application Guide for AC High-Voltage Circuit Breakers Rated on a Symmetrical Current Basis, ANSI C37.010-1979, American National Standards Institute, New York. This publication is also IEEE Std 320-1979.

Solution:

(a) The voltage below which rated short-circuit current does not increase as operating voltage decreases and the value of that current are

$$V = \frac{38}{1.65} = 23.0 \text{ kV}$$
$$I = 1.65 \times 22,000 = 36,300 \text{ A}$$

(b) The rated short-circuit current at 34.5 kV is

$$I = \frac{22,000 \times 38}{34.5} = 24,232 \text{ A}$$

Chapter 11 Problem Solutions

11.1 If $V_{an}^{(1)} = 50\angle 0°$, $V_{an}^{(2)} = 20\angle 90°$ and $V_{an}^{(0)} = 10\angle 180°$ V, determine analytically the voltages to neutral V_{an}, V_{bn} and V_{cn}, and also show graphically the sum of the given symmetrical components which determine the line-to-neutral voltages.

Solution:

$$
\begin{aligned}
V_{an} &= 50 + j20 - 10 = 40 + j20 = 44.72\angle 26.6° \text{ V} \\
V_{bn} &= 50\angle 240° + 20\angle 210° - 10 = -25 - j43.33 - 17.32 - j10 - 10 \\
&= -52.32 - j53.33 = 7.47\angle -134.4° \text{ V} \\
V_{cn} &= 50\angle 120° + 20\angle 330° - 10 = -25 + j43.33 + 17.32 - j10 - 10 \\
&= -17.68 + j33.3 = 37.7\angle 117°
\end{aligned}
$$
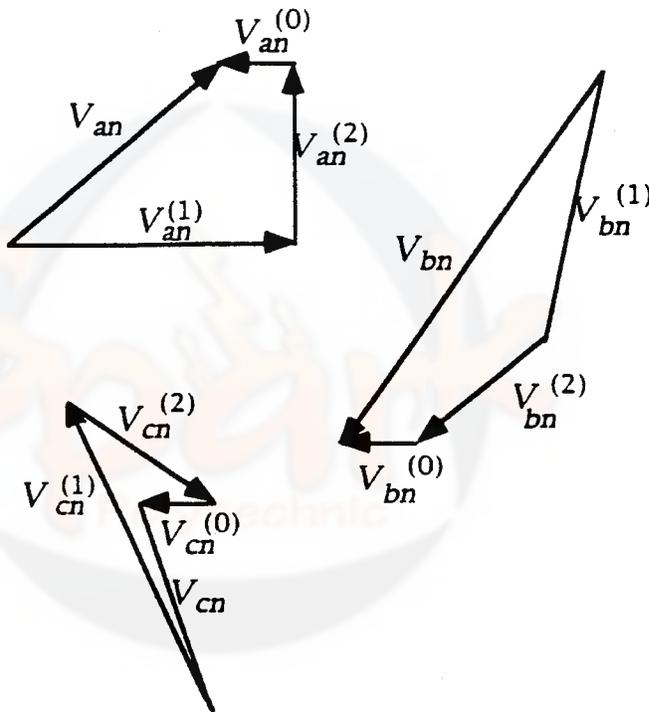
(11.2) When a generator has terminal a open and the other two terminals are connected to each other with a short circuit from this connection to ground, typical values for the symmetrical components of current in phase a are $I_a^{(1)} = 600\underline{/-90°}$, $I_a^{(2)} = 250\underline{/90°}$, and $I_a^{(0)} = 350\underline{/90°}$ A. Find the current into the ground and the current in each phase of the generator.

Solution:

$$I_a = -j600 + j250 + j350 = 0 \text{ A}$$

$$I_b^{(1)} = 600\underline{/150°} = -519.6 + j300$$

$$I_b^{(2)} = 250\underline{/210°} = -216.5 - j125$$

$$I_b^{(0)} = 350\underline{/90°} = 0 + j350$$

$$I_b = -736.1 + j525 = 904.1\underline{/144.5°} \text{ A}$$

$$I_c^{(1)} = 600\underline{/30°} = 519.6 + j300$$

$$I_c^{(2)} = 250\underline{/330°} = 216.5 - j125$$

$$I_c^{(0)} = 350\underline{/90°} = 0 + j350$$

$$I_c = 736.1 + j525 = 904.1\underline{/35.5°} \text{ A}$$

$$I_n = I_b + I_c = j1050 \text{ A}$$
$$\text{or } I_n = 3I_a^{(0)} = 3 \times j350 = j1050 \text{ A}$$

(11.3) Determine the symmetrical components of the three currents
$I_a = 10\underline{/0°}$, $I_b = 10\underline{/230°}$, and $I_c = 10\underline{/130°}$ A.

Solution:

$$I_a^{(1)} = \tfrac{1}{3}(10\underline{/0°} + 10\underline{/350°} + 10\underline{/370°})$$

$$= \tfrac{1}{3}(10 + 9.848 - j1.736 + 9.848 + j1.736)$$

$$= 9.899\underline{/0°} \text{ A}$$

$$I_a^{(2)} = \tfrac{1}{3}(10\underline{/0°} + 10\underline{/470°} + 10\underline{/250°})$$

$$= \tfrac{1}{3}(10 - 3.420 + j9.397 - 3.420 - j9.397)$$

$$= 1.053\underline{/0°} \text{ A}$$

$$I_a^{(0)} = \tfrac{1}{3}(10\underline{/0°} + 10\underline{/230°} + 10\underline{/130°})$$

$$= \tfrac{1}{3}(10 - 6.428 - j7.66 - 6.428 + j7.66)$$

$$= 0.952\underline{/180°} \text{ A}$$

The components of I_b and I_c are easily found from $I_a^{(1)}$, $I_a^{(2)}$ and $I_a^{(0)}$.

Check: $I_a = 9.899 + 1.053 - 0.952 = 10.00$.

(11.4) The currents flowing in the lines toward a balanced load connected in Δ are $I_a = 100\underline{/0°}$, $I_b = 141.4\underline{/225°}$, and $I_c = 100\underline{/90°}$. Find the symmetrical components of the given line currents and draw phasor diagrams of the positive- and negative-sequence line and phase currents. What is I_{ab} in amperes?

Solution:
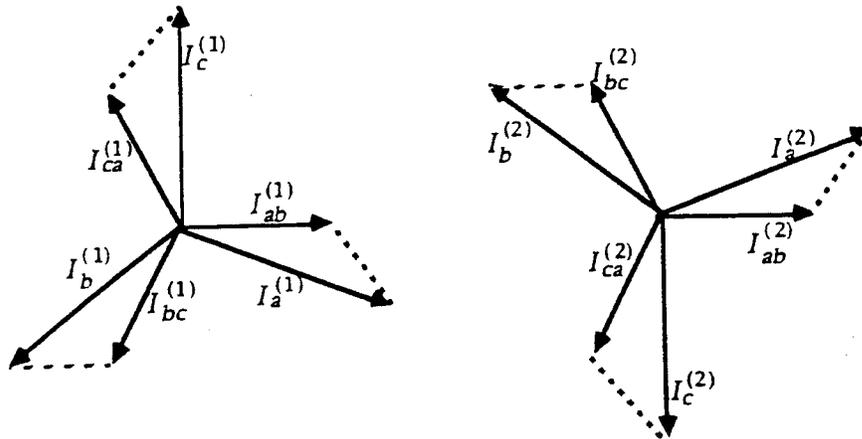
The phasor diagrams for positive- and negative-sequence currents in the lines and in the Δ-connected load show the desired relations, namely:

$$I_{ab}^{(1)} = \frac{I_a^{(1)}}{\sqrt{3}} \angle +30° \quad \text{and} \quad I_{ab}^{(2)} = \frac{I_a^{(2)}}{\sqrt{3}} \angle -30°$$

For the given currents, we find

$$I_a^{(1)} = \frac{1}{3}(100 + 141.4\angle 345° + 100\angle 330°)$$
$$= 107.7 - j28.9 = 111.5\angle -15° \text{ A}$$

$$I_a^{(2)} = \frac{1}{3}(100 + 141.4\angle 105° + 100\angle 210°)$$
$$= -7.73 + j28.9 = 29.9\angle 105° \text{ A}$$

$$I_a^{(0)} = \frac{1}{3}(100 - 100 - j100 + j100)$$
$$= 0 \text{ (since zero-sequence cannot flow into the Δ).}$$

and,

$$I_{ab}^{(1)} = \frac{111.5}{\sqrt{3}} \angle -15° +30° = 64.4\angle 15° = 62.2 + j16.66$$

$$I_{ab}^{(2)} = \frac{29.9}{\sqrt{3}} \angle 105° -30° = 17.26\angle 75° = 4.47 + j16.67$$

$$I_{ab} = 66.67 + j33.33 = 74.5\angle 26.6° \text{ A}$$

(11.5) The voltages at the terminals of a balanced load consisting of three 10-Ω resistors connected in Y are $V_{ab} = 100\angle 0°$ $V_{bc} = 80.8\angle -121.44°$, and $V_{ca} = 90\angle 130°$ V. Assuming that there is no connection to the neutral of the load, find the line currents from the symmetrical components of the given line voltages.

Solution:

Phasor diagrams for the positive and negative sequence voltages are:
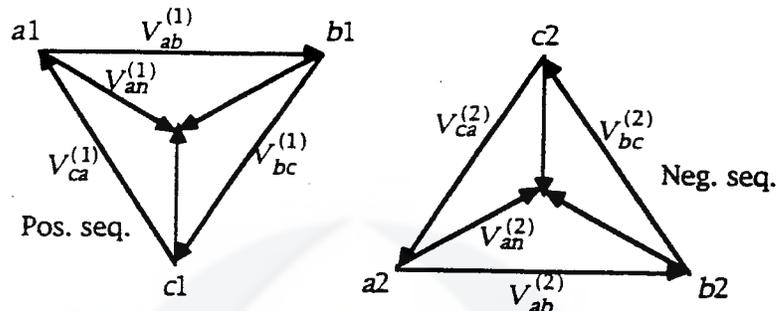
$$V_{an}^{(1)} = \frac{V_{ab}^{(1)}}{\sqrt{3}} \angle -30° \quad \text{and} \quad V_{an}^{(2)} = \frac{V_{ab}^{(2)}}{\sqrt{3}} \angle +30°$$

(no zero-sequence components)

$$V_{ab}^{(1)} = \frac{1}{3} (100\angle 0° + 80.8\angle -1.44° + 90\angle 370°)$$
$$= \frac{1}{3} (100 + 80.77 - j2.03 + 88.63 + j15.63)$$
$$= 89.8 + j4.53 = 89.91\angle 2.89°$$

$$V_{ab}^{(2)} = \frac{1}{3} (100\angle 0° + 80.8\angle 118.56° + 90\angle 250°)$$
$$= \frac{1}{3} (100 - 38.63 + j70.97 - 30.78 - j84.57)$$
$$= 10.2 - j4.53 = 11.16\angle -23.95°$$

$$V_{an}^{(1)} = \frac{89.91}{\sqrt{3}} \underline{/-27.11°} = 46.21 - j23.66$$

$$V_{an}^{(2)} = \frac{11.16}{\sqrt{3}} \underline{/6.05°} = 6.41 + j0.68$$

$$V_{an} = 52.62 - j22.98 = 57.42\underline{/-23.59°}$$

$$I_a = \frac{57.42\underline{/-23.59°}}{10} = 5.74\underline{/-23.59°} \text{ A}$$

$$V_{bn}^{(1)} = \frac{89.91}{\sqrt{3}} \underline{/212.89°} = -43.59 - j28.19$$

$$V_{bn}^{(2)} = \frac{11.16}{\sqrt{3}} \underline{/126.05°} = -3.79 + j5.27$$

$$V_{bn} = -47.38 - j22.98 = 52.66\underline{/-154.13°}$$

$$I_b = \frac{52.66\underline{/-154.13°}}{10} = 5.27\underline{/-154.13°} \text{ A}$$

$$V_{cn}^{(1)} = \frac{89.91}{\sqrt{3}} \underline{/92.89°} = -2.62 + j51.84$$

$$V_{cn}^{(2)} = \frac{11.16}{\sqrt{3}} \underline{/246.05°} = -2.62 - j5.89$$

$$V_{cn} = -5.24 + j45.95 = 46.25\underline{/96.51°}$$

$$I_c = \frac{46.25\underline{/96.51°}}{10} = 4.63\underline{/96.51°} \text{ A}$$

Note that $I_a + I_b + I_c = 0$.

(11.6) Find the power expended in the three 10-Ω resistors of Prob. 11.5 from the symmetrical components of currents and voltages. Check the answer.

Solution:

From Prob. 11.5,

$$V_{an}^{(1)} = 51.91\underline{/-27.11°}$$
$$V_{an}^{(2)} = 6.44\underline{/6.05°}$$
$$V_{an}^{(0)} = 0$$

$$I_{an}^{(1)} = \frac{V_{an}^{(1)}}{10} = 5.19\underline{/-27.11°}$$
$$I_{an}^{(2)} = \frac{V_{an}^{(2)}}{10} = 0.644\underline{/6.05°}$$
$$I_{an}^{(0)} = 0$$

$$S = 3 \begin{bmatrix} 51.91\underline{/-27.11°} & 6.44\underline{/6.05°} \end{bmatrix} \begin{bmatrix} 5.19\underline{/-27.11°} \\ 0.644\underline{/6.05°} \end{bmatrix}^*$$
$$S = 3 (51.91 \times 5.19 + 6.44 \times 0.644) = 820.7 \text{ W}$$

Check:
$(5.74)^2 \times 10 + (5.27)^2 \times 10 + (4.63)^2 \times 10 = 821.6 \text{ W}.$

(11.7) If there is impedance in the neutral connection to ground of a Y-connected load, then show that the voltages V_a, V_b, and V_c of Eq. (11.26) must be interpreted as voltages with respect to ground.

Solution:
Power supplied by the source is:

$$S = V_a I_a^* + V_b I_b^* + V_c I_c^*$$

Where the voltages V_a,, V_b and V_c are voltages with respect to ground.

Likewise, the power consumed by the load is:

$$S = V_{an}I_a^* + V_{bn}I_b^* + V_{cn}I_c^* + V_n(I_a^* + I_b^* + I_c^*)$$
$$= (V_{an} + V_n)\,I_a^* + (V_{bn} + V_n)\,I_b^* + (V_{cn} + V_n)\,I_c^*$$

But $(V_{an} + V_n)$, $(V_{bn} + V_n)$ and $(V_{cn} + V_n)$ are voltages with respect to ground.

(11.8)   A balanced three-phase load consists of Δ-connected impedances Z_Δ in parallel with solidly grounded Y-connected impedances Z_Y.

   (a)   Express the currents I_a, I_b, and I_c flowing in the lines from the supply source toward the load in terms of the source voltages V_a, V_b, and V_c.

   (b)   Transform the expressions of part (a) into their symmetrical component equivalents, and thus express $I_a^{(0)}$, $I_a^{(1)}$, and $I_a^{(2)}$ in terms of $V_a^{(0)}$, $V_a^{(1)}$ and $V_a^{(2)}$.

   (c)   Hence, draw the sequence circuit for the combined load.

Solution:

(a)
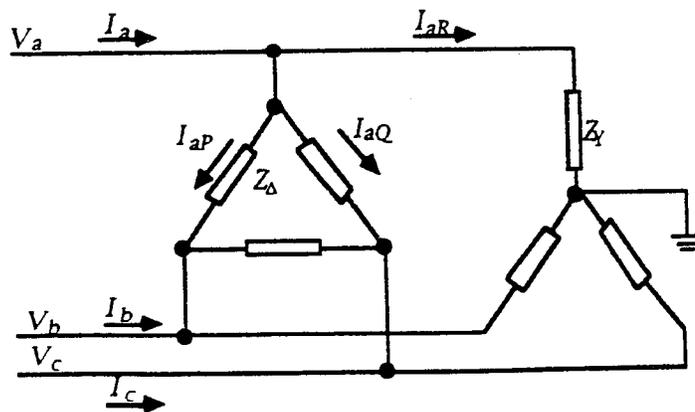
$$I_a = I_{aP} + I_{aQ} + I_{aR} = \frac{V_a - V_b}{Z_\Delta} + \frac{V_a - V_c}{Z_\Delta} + \frac{V_a}{Z_Y}$$

$$= \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right)V_a - \frac{1}{Z_\Delta}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

Writing similar equations for each of the three currents, and rearranging in matrix form, we have:

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right)\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} - \frac{1}{Z_\Delta}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$$

where the matrix of ones is henceforth called **P**.

(b)

$$\mathbf{I_{abc}} = \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right)\mathbf{V_{abc}} - \frac{1}{Z_\Delta}\mathbf{P}\,\mathbf{V_{abc}}$$

by premultiplying with A^{-1} and using the identity $V_{abc} = AV_{012}$, we have:

$$I_{0\,12} = \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right) V_{0\,12} - \frac{1}{Z_\Delta} A^{-1} P A V_{012}$$

$$= \begin{bmatrix} 1/Z_Y & & \\ & 3/Z_\Delta + 1/Z_Y & \\ & & 3/Z_\Delta + 1/Z_Y \end{bmatrix} V_{012}$$

(c)
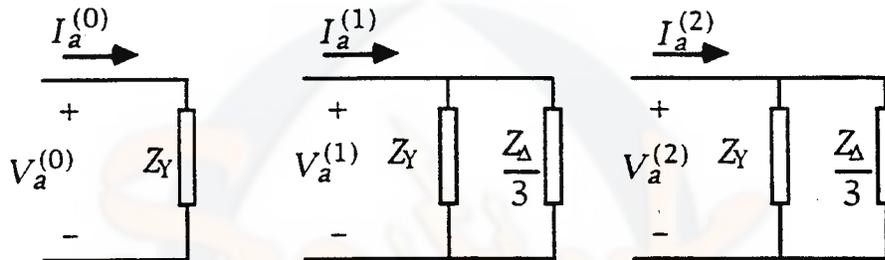
(11.9) The Y-connected impedances in parallel with the Δ-connected impedances Z_Δ of Prob. 11.8 are now grounded through an impedance Z_g.

- (a) Express the currents I_a, I_b, and I_c flowing in the lines from the supply source toward the load in terms of the source voltage V_a, V_b, and V_c the voltage V_n of the neutral point.

- (b) Expressing V_n in terms of $I_a^{(0)}$, $I_a^{(1)}$, $I_a^{(2)}$, and Z_g, find the equations for these currents in terms of $V_a^{(0)}$, $V_a^{(1)}$, and $V_a^{(2)}$.

(c)   Hence, draw the sequence circuit for the combined load.

Solution:

(a)
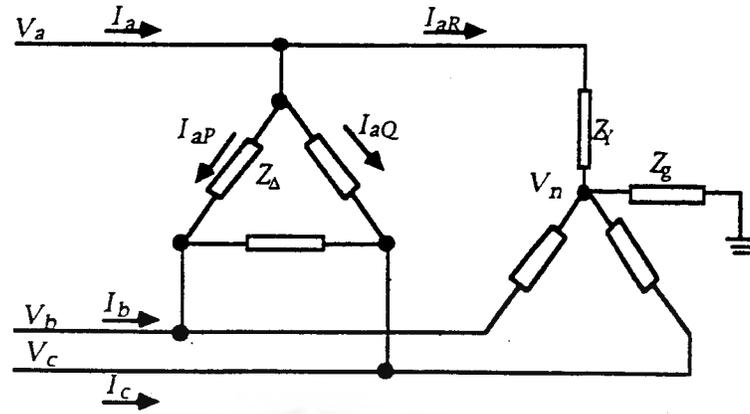
$$I_a = I_{aP} + I_{aQ} + I_{aR} = \frac{V_a - V_b}{Z_\Delta} + \frac{V_a - V_c}{Z_\Delta} + \frac{V_a - V_n}{Z_Y}$$

$$= \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right) V_a - \frac{1}{Z_\Delta} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} - \frac{V_n}{Z_Y}$$

Writing similar equations for each of the three currents, we get:

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right) \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} - \frac{1}{Z_\Delta} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} - \frac{1}{Z_Y} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_n$$

(b)   With the matrix of ones called P, the above equation is premultiplied with A^{-1} and after using the identity $V_{abc} = AV_{012}$, we have:

$$I_{012} = \left(\frac{3}{Z_\Delta} + \frac{1}{Z_Y}\right) V_{012} - \frac{1}{Z_\Delta} A^{-1} P \, AV_{012} - \frac{1}{Z_Y} A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} V_n$$

$$= \begin{bmatrix} 1/Z_Y & & \\ & 3/Z_\Delta + 1/Z_Y & \\ & & 3/Z_\Delta + 1/Z_Y \end{bmatrix} V_{012} - \frac{1}{Z_Y} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V_n$$

Since $V_n = Z_g (I_a + I_b + I_c) = 3I_a^{(0)}$,

$$I_a^{(0)} = \frac{1}{Z_Y} V_a^{(0)} - \frac{3Z_g}{Z_Y} I_a^{(0)}$$

$$I_a^{(0)} = \frac{V_a^{(0)}}{Z_Y + 3Z_g}$$

Therefore, the zero-sequence admittance is

$$Y_0 = \frac{1}{Z_Y + 3Z_g}$$

and all others are unchanged.

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(c)
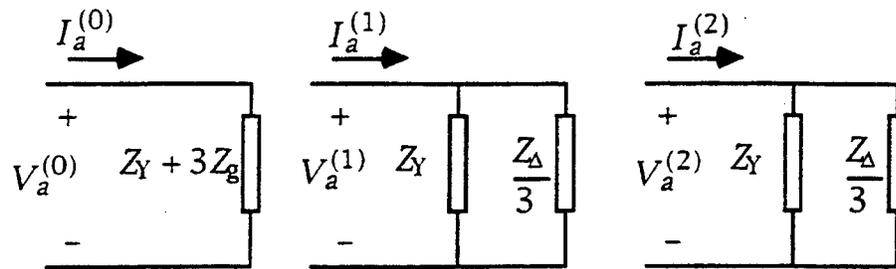
(11.10) Suppose that the line-to-neutral voltages at the sending end of the line described in Example 11.5 can be maintained constant at 200-kV and that a single-phase inductive load of 420 Ω is connected between phase a and neutral at the receiving end

(a) Use Eqs. (11.51) to express numerically the receiving-end sequence voltages $V_{a'n'}^{(0)}$, $V_{a'n'}^{(1)}$, and $V_{a'n'}^{(2)}$ in terms of the load current I_L and the sequence impedances Z_0, Z_1, and Z_2 of the line.

(b) Hence, determine the line current I_L in amperes.

(c) Determine the open-circuit voltages to neutral of phases b and c at the receiving end.

(d) Verify your answer to part (c) without using symmetrical components.

Solution:

(a)

$$\begin{bmatrix} V_{an}^{(0)} \\ V_{an}^{(1)} \\ V_{an}^{(2)} \end{bmatrix} - \begin{bmatrix} V_{a'n'}^{(0)} \\ V_{a'n'}^{(1)} \\ V_{a'n'}^{(2)} \end{bmatrix} = \begin{bmatrix} Z_0 I_a^{(0)} \\ Z_1 I_a^{(1)} \\ Z_2 I_a^{(2)} \end{bmatrix}$$

$I_a = I_L$, $I_b = I_c = 0$ gives $I_a^{(0)} = I_a^{(1)} = I_a^{(2)} = \dfrac{I_L}{3}$. Thus,

$$\begin{bmatrix} V_{a'n'}^{(0)} \\[2mm] V_{a'n'}^{(1)} \\[2mm] V_{a'n'}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\[2mm] 200{\times}10^3 \\[2mm] 0 \end{bmatrix} - I_L \begin{bmatrix} \dfrac{Z_0}{3} \\[2mm] \dfrac{Z_1}{3} \\[2mm] \dfrac{Z_2}{3} \end{bmatrix} = \begin{bmatrix} -I_L \times j\dfrac{160}{3} \\[2mm] 200{\times}10^3 - I_L \times j\dfrac{40}{3} \\[2mm] -I_L \times j\dfrac{40}{3} \end{bmatrix}$$

(b) Since $V_{a'n'} = V_{a'n'}^{(0)} + V_{a'n'}^{(1)} + V_{a'n'}^{(2)}$,

$$V_{a'n'} = 200{\times}10^3 - I_L \times \frac{j}{3}(160 + 40 + 40)$$

also, $\quad V_{a'n'} = I_L \times j420$

Therefore,

$$I_L = \frac{200{\times}10^3}{j500} \text{ A} = 400\underline{/-90^\circ} \text{ A}$$

(c)

$$\begin{bmatrix} V_{a'n'}^{(0)} \\[2mm] V_{a'n'}^{(1)} \\[2mm] V_{a'n'}^{(2)} \end{bmatrix} = \begin{bmatrix} -400\underline{/-90^\circ} \times \dfrac{160\underline{/90^\circ}}{3} \\[2mm] 200{\times}10^3 - 400\underline{/-90^\circ} \times \dfrac{40\underline{/90^\circ}}{3} \\[2mm] -400\underline{/-90^\circ} \times \dfrac{40\underline{/90^\circ}}{3} \end{bmatrix} \text{V} = \begin{bmatrix} -\dfrac{64\underline{/0^\circ}}{3} \\[2mm] \dfrac{584\underline{/0^\circ}}{3} \\[2mm] -\dfrac{16\underline{/0^\circ}}{3} \end{bmatrix} \text{kV}$$

$$\begin{aligned} V_{b'n'} &= V_{a'n'}^{(0)} + 1\underline{/240^\circ}\, V_{a'n'}^{(1)} + 1\underline{/120^\circ}\, V_{a'n'}^{(2)} \\ &= -\frac{64}{3} + \frac{584}{3}\underline{/240^\circ} - \frac{16}{3}\underline{/120^\circ} \text{ kV} \\ &= 208.46\underline{/-123.8^\circ} \text{ kV} \end{aligned}$$

$$V_{c'n'} = V_{a'n'}^{(0)} + 1\underline{/120°}\,V_{a'n'}^{(1)} + 1\underline{/240°}\,V_{a'n'}^{(2)}$$
$$= -\frac{64}{3} + \frac{584}{3}\underline{/120°} - \frac{16}{3}\underline{/240°}\ \text{kV}$$
$$= 208.46\underline{/123.8°}\ \text{kV}$$

(d) From Eq. (11.49),

$$V_{bb'} = V_{cc'} = (Z_{ab} - Z_{an})I_L - (Z_{an} - Z_{nn})I_L$$
$$= (Z_{ab} + Z_{nn} - 2Z_{an})I_L$$
$$= j(20 + 80 - 60) \times 400\underline{/-90°}\ \text{V}$$
$$= 16.0\ \text{kV}$$

$$V_{b'n'} = V_{bn} - V_{bb'} = 200\underline{/-120°}\ \text{kV} - 16\underline{/0°}\ \text{kV}$$
$$= 208.46\underline{/-123.8°}\ \text{kV}$$

$$V_{c'n'} = V_{cn} - V_{cc'} = 200\underline{/120°}\ \text{kV} - 16\underline{/0°}\ \text{kV}$$
$$= 208.46\underline{/123.8°}\ \text{kV}$$

(11.11) Solve Prob. 11.10 if the same 420-Ω inductive load is connected between phases a and b at the receiving end. In part (c) find the open-circuit voltage of phase c only.

Solution:

(a)

$$\begin{bmatrix} V_{an}^{(0)} \\ V_{an}^{(1)} \\ V_{an}^{(2)} \end{bmatrix} - \begin{bmatrix} V_{a'n'}^{(0)} \\ V_{a'n'}^{(1)} \\ V_{a'n'}^{(2)} \end{bmatrix} = \begin{bmatrix} Z_0 I_a^{(0)} \\ Z_1 I_a^{(1)} \\ Z_2 I_a^{(2)} \end{bmatrix}$$

$I_a = -I_b = I_L$, $I_c = 0$ gives

$$I_a^{(0)} = 0; \quad I_a^{(1)} = \frac{I_L}{\sqrt{3}} \angle{-30°}; \quad I_a^{(2)} = \frac{I_L}{\sqrt{3}} \angle{+30°}$$

Thus,

$$
\begin{bmatrix} V_{a'n'}^{(0)} \\ V_{a'n'}^{(1)} \\ V_{a'n'}^{(2)} \end{bmatrix} =
\begin{bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{bmatrix} - I_L
\begin{bmatrix} 0 \times Z_0 \\ \frac{Z_1}{\sqrt{3}} \angle{-30°} \\ \frac{Z_2}{\sqrt{3}} \angle{+30°} \end{bmatrix} =
\begin{bmatrix} 0 \\ 200 \times 10^3 - \frac{40 I_L}{\sqrt{3}} \angle{60°} \\ -\frac{40 I_L}{\sqrt{3}} \angle{120°} \end{bmatrix} \text{V}
$$

(b)   Therefore,

$$
\begin{aligned}
V_{a'b'} &= V_{a'n'} - V_{b'n'} \\
&= (V_{a'n'}^{(0)} + V_{a'n'}^{(1)} + V_{a'n'}^{(2)}) - (V_{a'n'}^{(0)} + a^2 V_{a'n'}^{(1)} + a V_{a'n'}^{(2)}) \\
&= \sqrt{3} \, V_{a'n'}^{(1)} \angle{+30°} + \sqrt{3} \, V_{a'n'}^{(2)} \angle{-30°} \\
&= 200 \times 10^3 \sqrt{3} \angle{+30°} - 80 I_L \angle{90°}
\end{aligned}
$$

also,

$$V_{a'b'} = j420 \times I_L$$

and so

$$I_L = \frac{200 \times 10^3 \sqrt{3} \angle{30°}}{500 \angle{90°}} = 400 \sqrt{3} \angle{-60°} \text{ A}$$

(c)

$$
\begin{bmatrix} V_{a'n'}^{(0)} \\ V_{a'n'}^{(1)} \\ V_{a'n'}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 200\times10^3 - \dfrac{40}{\sqrt{3}}\angle{+60°}\times400\sqrt{3}\angle{-60°} \\ -\dfrac{40}{\sqrt{3}}\angle{120°}\times400\sqrt{3}\angle{-60°} \end{bmatrix} V = \begin{bmatrix} 0 \\ 184\angle{0°} \\ -16\angle{60°} \end{bmatrix} kV
$$

$$
\begin{aligned}
V_{c'n'} &= V_{a'n'}^{(0)} + V_{a'n'}^{(1)}\angle{120°} + V_{a'n'}^{(2)}\angle{240°} \\
&= 0 + 184\angle{120°} - 16\angle{300°} \\
&= 200\angle{120°} \ kV
\end{aligned}
$$

(d) From Eq. (11.49),

$$
\begin{aligned}
V_{cc'} &= (Z_{aa} - Z_{an})\times 0 + (Z_{ab} - Z_{an})(I_L - I_L) + (Z_{an} - Z_{nn})\times 0 \\
&= 0
\end{aligned}
$$

Therefore,
$$
V_{c'n'} = V_{cn} = 200\angle{120°} \ kV
$$

(11.12) A Y-connected synchronous generator has sequence reactances $X_0 = 0.09$, $X_1 = 0.22$, and $X_2 = 0.36$, all in per unit. The neutral point of the machine is grounded through a reactance of 0.09 per unit. The machine is running on no load with rated terminal voltage when it suffers and unbalanced fault. The fault currents out of the machine are $I_a = 0$, $I_b = 3.75\angle{150°}$, and $I_c = 3.75\angle{30°}$, all in per unit with respect to phase a line-to-neutral voltage. Determine

    (a)   The terminal voltages in each phase of the machine with respect to ground,

    (b)   The voltage of the neutral point of the machine with respect to ground, and

    (c)   The nature (type) of the fault from the results of part (a).

Solution:

(a)

$$Z_1 = j0.22 \text{ p.u.,} \quad Z_2 = 0.36 \text{ p.u.,}$$
$$Z_0 = Z_{g0} + 3Z_n = j0.09 + 3 \times j0.09 = 0.36 \text{ p.u.}$$

$$I_a = 0, \; I_b = 3.75\underline{/0°} \text{ p.u., and } I_c = 3.75\underline{/0°} \text{ p.u.}$$

$$\begin{bmatrix} I_a^{(0)} \\ I_a^{(1)} \\ I_a^{(2)} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 0 \\ 3.75\underline{/150°} \\ 3.75\underline{/30°} \end{bmatrix} = \begin{bmatrix} j1.25 \\ -j2.5 \\ j1.25 \end{bmatrix}$$

Hence,

$$V_a^{(0)} = -I_a^{(0)}Z_0 = -j1.25 \times j0.36 = 0.45 \text{ p.u.}$$

$$V_a^{(1)} = E_{an} - I_a^{(1)}Z_1 = 1\underline{/0°} - (-j2.5 \times j0.22)$$
$$= 0.45 \text{ p.u.}$$

$$V_a^{(2)} = -I_a^{(2)}Z_2 = -j1.25 \times j0.36 = 0.45 \text{ p.u.}$$

and,

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}\begin{bmatrix} 0.45\underline{/0°} \\ 0.45\underline{/0°} \\ 0.45\underline{/0°} \end{bmatrix} = \begin{bmatrix} 1.35\underline{/0°} \\ 0 \\ 0 \end{bmatrix} \text{ p.u.}$$

(b)

$$V_n = -3I_a^{(0)} \times j0.09 \text{ p.u.}$$
$$= -3 \times j1.25 \times j0.09 \text{ p.u.}$$
$$= 0.3375 \text{ p.u.}$$

(c) since $V_b = V_c = 0$, it is a double-line-to-ground fault.

(11.13) Solve Prob. 11.12 if the fault currents in per unit are $I_a = 0$, $I_b = -2.986\underline{/0°}$, and $I_c = 2.986\underline{/0°}$.

Solution:

(a)

$$\begin{bmatrix} I_a^{(0)} \\ I_a^{(1)} \\ I_a^{(2)} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} 0 \\ -2.986 \\ 2.986 \end{bmatrix} = \begin{bmatrix} 0 \\ -j1.724 \\ j1.724 \end{bmatrix}$$

$$V_a^{(0)} = -I_a^{(0)} Z_0 = 0$$

$$V_a^{(1)} = E_{an} - I_a^{(1)} Z_1 = 1\underline{/0°} - (-j1.724)(j0.22)$$
$$= 0.621 \text{ p.u.}$$

$$V_a^{(2)} = - I_a^{(2)} Z_2 = - (-j1.724)(j0.36)$$
$$= 0.621 \text{ p.u.}$$

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.621 \\ 0.621 \end{bmatrix} = \begin{bmatrix} 1.242\underline{/0°} \\ -0.621\underline{/0°} \\ -0.621\underline{/0°} \end{bmatrix} \text{ p.u.}$$

(b) Since $I_a^{(0)} = 0$, $V_n = 0$.

(c) Since $V_b = V_c$, it is a line-to-line fault.

(11.14) Assume that the currents specified in Prob. 11.4 are flowing toward a load from lines connected to the Y side of a Δ-Y transformer rated 10 MVA, 13.2Δ/66Y kV. Determine the currents flowing in the lines on the Δ side by converting the symmetrical components of the currents to per unit on the base of the transformer rating and by shifting the components according to Eq. (11.88). Check the results by computing the currents in each phase of the Δ windings in amperes directly from the currents on the Y side by multiplying by

the turns ratio of the windings. Complete the check by computing the line currents from the phase currents on the Δ side.

Solution:

Capital letters are here used for currents to the load since the load is on the high tension side of the transformer.

Base line currents are:

Y-side: $\dfrac{10,000}{66\times\sqrt{3}} = 87.5$ A

Δ-side: $\dfrac{10,000}{13.2\times\sqrt{3}} = 437.4$ A

Line-currents on Y-side are:

$$I_A^{(1)} = \frac{111.5\underline{/-15°}}{87.5} = 1.274\underline{/-15°} \text{ p.u.}$$

$$I_A^{(2)} = \frac{29.9\underline{/105°}}{87.5} = 0.342\underline{/105°} \text{ p.u}$$

and on the Δ-side:

$$I_a^{(1)} = jI_A^{(1)} = 1.274\underline{/75°} = 0.330 + j1.231 \text{ p.u.}$$

$$I_a^{(2)} = -jI_A^{(2)} = 0.342\underline{/15°} = 0.330 + j0.089 \text{ p.u}$$

thus $I_a = 0.660 + j1.320 = 1.476\underline{/63.4°}$ p.u.

$$I_b^{(1)} = I_a^{(1)}\underline{/240°} = 1.274\underline{/315°} = 0.901 - j0.901 \text{ p.u.}$$

$$I_b^{(2)} = I_a^{(2)}\underline{/120°} = -0.242 + j0.242 \text{ p.u}$$

thus $I_b = 0.659 - j0.659 = 0.932\underline{/45°}$ p.u. and

$$I_c^{(1)} = I_a^{(1)}\underline{/120°} = -1.231 - j0.330$$
$$I_c^{(2)} = I_a^{(2)}\underline{/240°} = -0.089 - j0.330$$

and $I_c = -1.320 - j0.660 = 1.476\underline{/206.6°}$ p.u.

In terms of actual currents:
$$I_a = 437.4 \times 1.476\underline{/63.4°} = 645.6\underline{/63.4°} \text{ A}$$
$$I_b = 407.7\underline{/-45°} \text{ A}$$
$$I_c = 645.6\underline{/206.6°} \text{ A}$$

The turns ratio of the transformer is $(66/\sqrt{3})/13.2 = 2.89$. Check:

$$I_{ab} = 2.89I_C = 2.89 \times 100\underline{/90°} = j289$$
$$I_{bc} = 2.89I_A = 2.89 \times 100\underline{/0°} = 289$$
$$I_{ca} = 2.89I_B = 2.89 \times 141.4\underline{/225°} = -289 - j289$$

$$I_a = I_{ab} - I_{ca} = 289 + j578 = 646\underline{/63.4°}$$
$$I_b = I_{bc} - I_{ab} = 289 - j289 = 408.7\underline{/-45°}$$
$$I_c = I_{ca} - I_{bc} = -578 - j289 = 646\underline{/206.6°}$$

(11.15) Three single-phase transformers are connected as shown in Fig. 11.29 to form a Y-Δ transformer. The high-voltage windings are Y-connected with polarity marks as indicated. Magnetically coupled windings are drawn in parallel directions. Determine the correct placement of polarity marks on the low-voltage windings. Identify the numbered terminals on the low-voltage side (a) with the letters a, b, and c, where $I_A^{(1)}$ leads $I_a^{(1)}$ by 30°, and (b) with the letters a', b', and c' so that $I_a^{(1)}$ is 90° out of phase with $I_A^{(1)}$.

Solution:

The correct voltage diagram if V_{AB} is reference are shown.
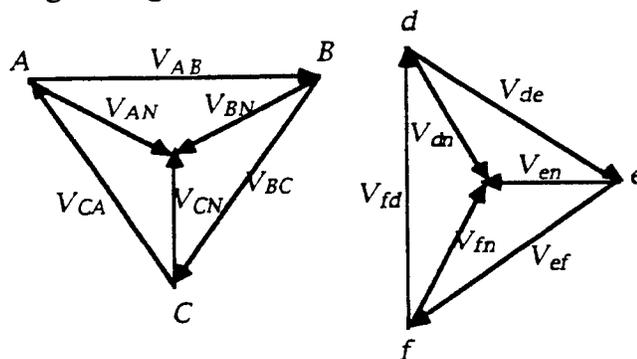
The solutions for both cases are:
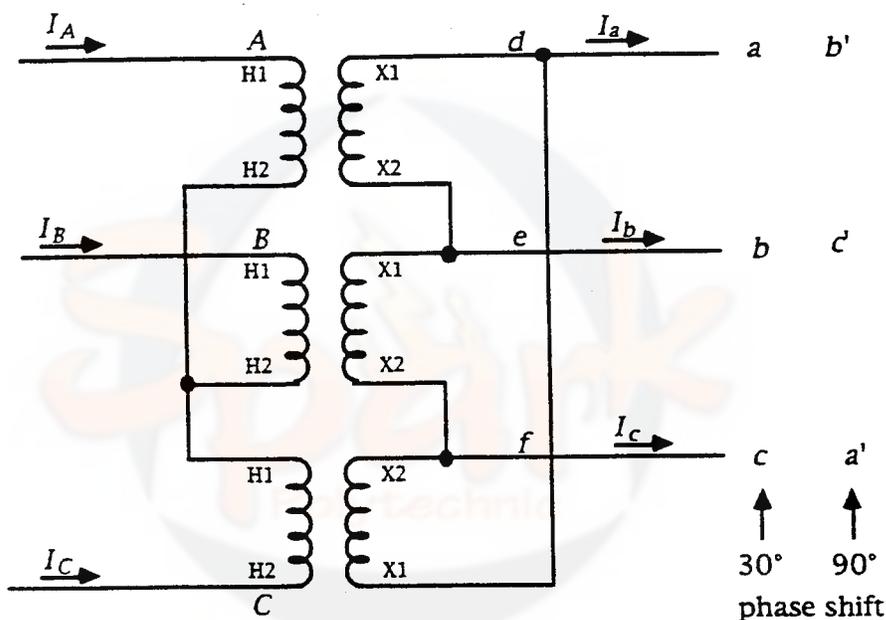
(11.16) Balanced three-phase voltages of 100 V line to line are applied to a Y-connected load consisting of three resistors. The neutral of the load is not grounded. The resistance in phase a is 10 Ω, in phase b is 20 Ω, and in phase c is 30 Ω. Select voltage to neutral of the three-phase line as reference and determine the current in phase a and the voltage V_{an}.

Solution:

If an ideal generator represents the source, the circuit diagram is:
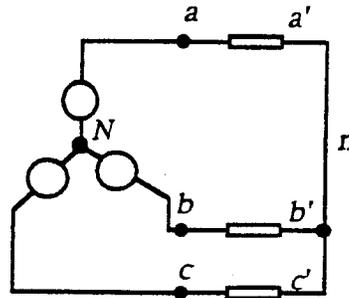
We see that $V_{aN} = V_{an} + V_{Nn}$. Voltage V_{Nn} can only be a zero-sequence voltage, and therefore $V_{aa'} = V_{an}$ has no negative sequence components. Also, $I_a^{(0)} = 0$ since node n is isolated.

With V_{aN} as reference,

$$V_{aa'}^{(1)} = \frac{100}{\sqrt{3}} \angle 0° = 57.7 \angle 0° ;$$

$$V_{aa'}^{(2)} = 0$$

In order to calculate the voltage drops in the unbalanced network, the coupled sequence-impedances are determined:

$$\frac{1}{3}(Z_a + Z_b + Z_c) = 20 \angle 0° \ \Omega$$

$$\frac{1}{3}(Z_a + aZ_b + a^2Z_c) = 5.77 \angle 210° \ \Omega$$

$$\frac{1}{3}(Z_a + a^2Z_b + aZ_c) = 5.77 \angle 150° \ \Omega$$

The voltage drops due to only the positive- and negative-sequence currents are:

$$\begin{bmatrix} 57.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 20\underline{/0°} & 5.77\underline{/150°} \\ 5.77\underline{/210°} & 20\underline{/0°} \end{bmatrix} \begin{bmatrix} I_a^{(1)} \\ I_a^{(2)} \end{bmatrix}$$

And so the currents are calculated as

$$\begin{bmatrix} I_a^{(1)} \\ I_a^{(2)} \end{bmatrix} = \begin{bmatrix} 20\underline{/0°} & 5.77\underline{/150°} \\ 5.77\underline{/210°} & 20\underline{/0°} \end{bmatrix}^{-1} \begin{bmatrix} 57.7 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.147 \\ 0.908\underline{/30°} \end{bmatrix} A$$

Giving I_a = 3.933 + j0.454 A = 3.959$\underline{/6.58°}$ A. The value of V_n is obtained from

$$V_{aa'}^{(0)} = 3.147 \times 5.77\underline{/150°} + 0.908\underline{/30°} + 5.77\underline{/210°}$$
$$= -18.346 + j4.542$$

$$V_n = V_{aa'}^{(1)} + V_{aa'}^{(0)} = 57.7 - 18.346 + j4.4542$$
$$= 39.61\underline{/6.58°} \text{ V}$$

or by using $I_a Z_a$.

(11.17) Draw the negative- and zero-sequence impedance networks for the power system of Prob. 3.12. Mark the values of all reactances in per unit on a base of 50 MVA, 13.8 kV in the circuit of generator 1. Letter the networks to correspond to the single-line diagram. The neutrals of generators 1 and 3 are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Each generator has negative- and zero-sequence reactances of 20 and 5%, respectively, on its own rating as base. The zero-sequence reactance of the transmission line is 210 Ω from B to C and 250 Ω from C to E.

Solution:

The negative-sequence network is the same as the positive-sequence network for the power system of Prob. 3.12 *with the EMF's short circuited* since the values for Z_2 are the same as Z_1 in all parts of the network. Consequently, it will not be drawn here. Z_0 for the transformers is the same as Z_1. The new values needed are:

Gen. 1: $\quad Z_{0g} = j0.05 \times \dfrac{50}{20} = j0.125$ p.u. ;

$\qquad\quad 3Z_n = 3 \times j0.05 \times \dfrac{50}{20} = j0.375$ p.u.

Gen. 2: $\quad Z_{0g} = j0.05 \times \dfrac{50}{30} = j0.0833$ p.u. ;

Gen. 3: $\quad Z_{0g} = j0.05 \times \dfrac{50}{30} \times \left(\dfrac{20}{22}\right)^2 = j0.0689$ p.u. ;

$\qquad\quad 3Z_n = 3 \times j0.05 \times \dfrac{50}{30} \times \left(\dfrac{20}{22}\right)^2 = j0.2066$ p.u.

And the zero-sequence reactance diagram is:
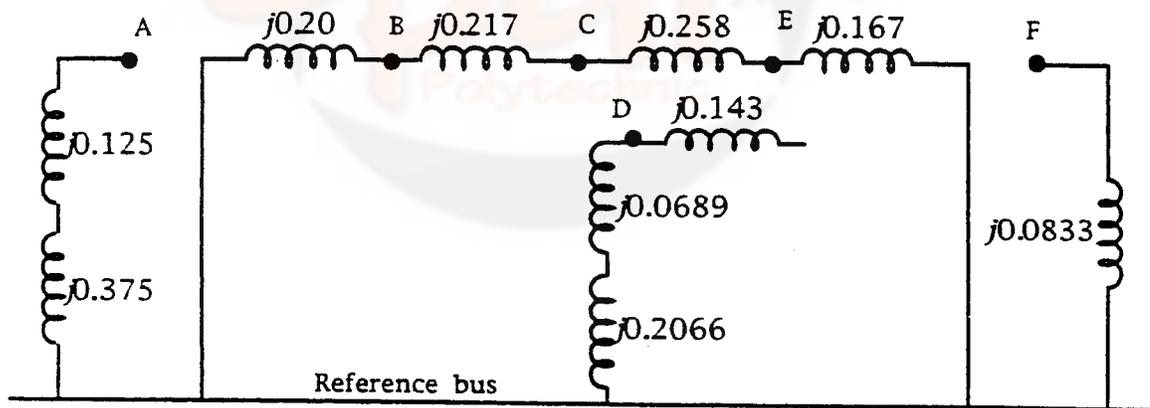
(11.18) Draw the negative- and zero-sequence impedance networks for the power system of Prob. 3.13. Choose a base of 50 MVA, 138 kV in the 40-Ω transmission line and mark all reactances in per unit. The negative-sequence reactance of each synchronous machine is equal to its subtransient reactance. The zero-sequence reactance of each machine is 8% based on its own rating. The neutrals of the machines are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Assume that the zero-sequence reactances of the transmission lines are 300% of their positive-sequence reactances.

Solution:

The negative-sequence network is the same as the positive-sequence network for the power system of Prob. 3.13 *with the EMF's short circuited* since the values for Z_2 are the same as Z_1 in all parts of the network. Consequently, it will not be drawn here. Calculations for the zero sequence network require:

G_1: $3Z_n = 3j\, 0.05 \left(\dfrac{50}{20}\right)\left(\dfrac{18}{20}\right)^2 = j0.304$ p.u.

G_2: $3Z_n = j0.304$ p.u.

M_3: $3Z_n = 3j\, 0.05 \left(\dfrac{50}{30}\right) = j0.25$ p.u.

X_0 is three times the positive sequence reactance in the lines;

G_1 & G_2: $Z_{0g} = j\, 0.08 \left(\dfrac{50}{20}\right)\left(\dfrac{18}{20}\right)^2 = j0.162$ p.u.

M_3: $Z_{0g} = j\, 0.08 \left(\dfrac{50}{30}\right) = j0.1333$ p.u.

The zero-sequence reactance diagram is then:
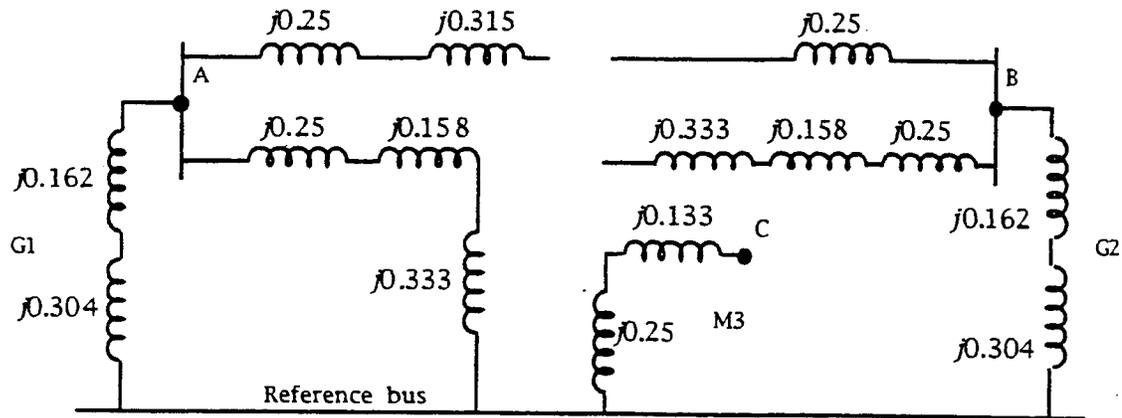
(11.19) Determine the zero-sequence Thévenin impedance seen at bus *c* of the system described in Prob. 11.17 if transformer T_3 has (a) one ungrounded and one solidly grounded neutral, as shown in Fig. 3.23, and (b) both neutrals are solidly grounded.

Solution:

The zero-sequence network is:
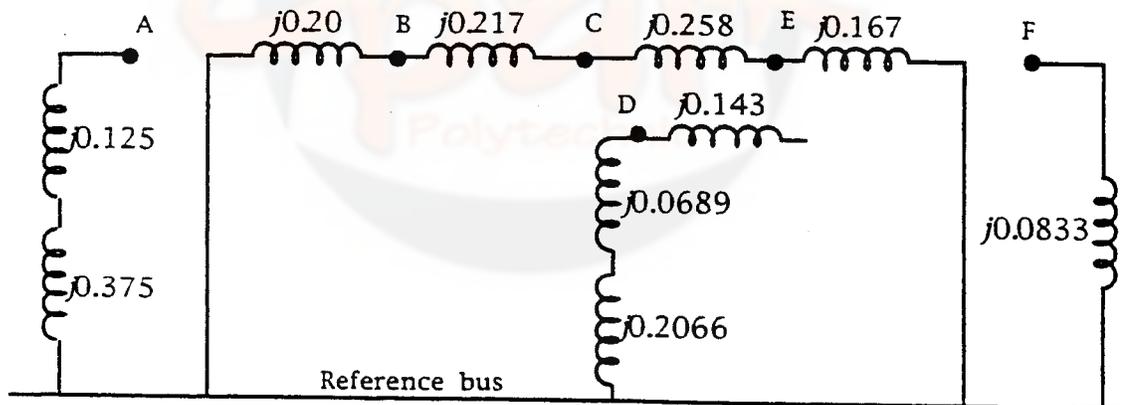
(a) The zero-sequence Thévenin impedance is:

$$Z = (j0.20 + j0.217) \parallel (j0.258 + j0.167)$$
$$= j0.417 \parallel j0.425 = j0.210 \text{ p.u.}$$

(*b*) If both neutrals of T3 are solidly grounded, then the zero-sequence network becomes:
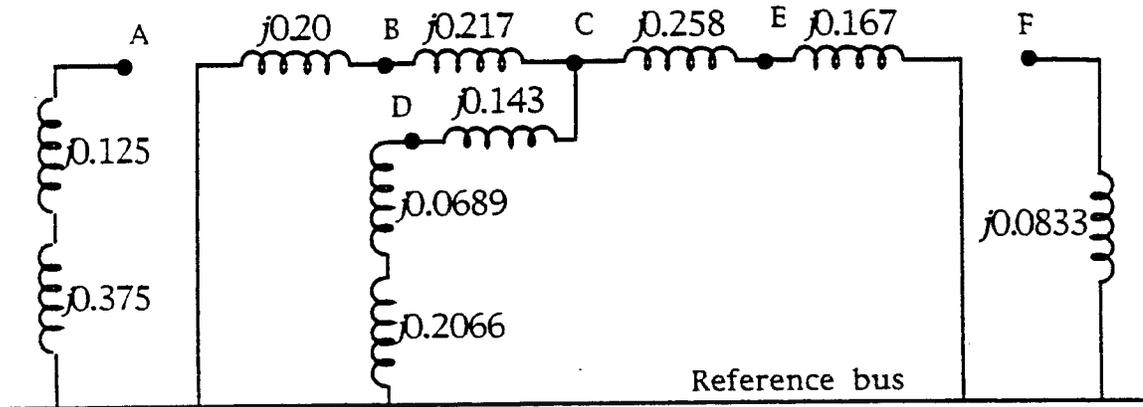
and the zero-sequence Thévénin impedance becomes:

$$Z = j0.417 \parallel j0.425 \parallel (j0.143 + j0.0689 + j0.2066)$$
$$= j0.417 \parallel j0.425 \parallel j0.425 = j0.140 \text{ p.u.}$$
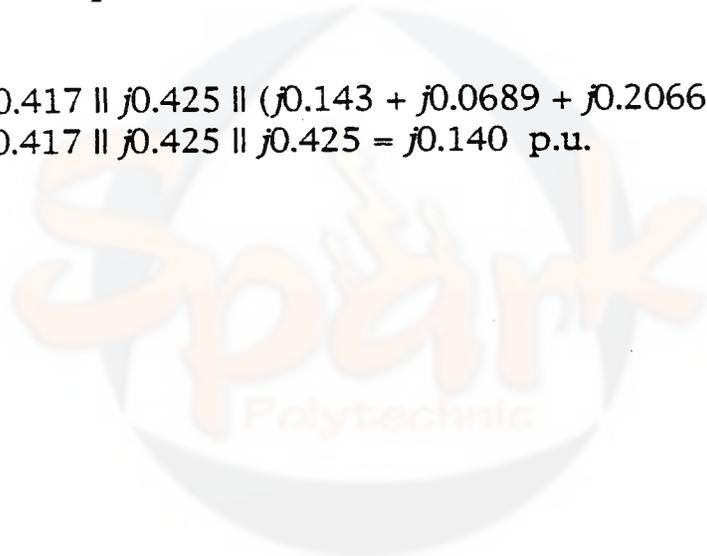
Chapter 12 Problem Solutions

12.1 A 60-Hz turbogenerator is rated 500 MVA, 22 kV. It is Y-connected and solidly grounded and is operating at rated voltage at no load. It is disconnected from the rest of the system. Its reactances are $X_d'' = X_1 = X_2 = 0.15$ and $X_0 = 0.05$ per unit. Find the ratio of the subtransient line current for a single line-to-ground fault to the subtransient line current for a symmetrical three-phase fault.

Solution:

Single line-to-ground fault:

$$I_a^{(1)} = \frac{1}{j0.15 + j0.15 + j0.15} = -j2.857 \text{ per unit}$$
$$I_a = 3I_a^{(1)} = -j8.571 \text{ per unit}$$

Three-phase fault:

$$I_a = \frac{1}{j0.15} = -j6.667 \text{ per unit}$$

The ratio is $8.571/6.667 = 1.286/1$.

12.2 Find the ratio of the subtransient line current for a line-to-line fault to the subtransient current for a symmetrical three-phase fault on the generator of Prob. 12.1.

Solution:

Line-to-line fault:

$$I_a^{(1)} = \frac{1}{j0.15 + j0.15} = -j3.333 \text{ per unit}$$
$$I_a^{(2)} = -I_a^{(1)} = j3.333 \text{ per unit}$$
$$I_b^{(1)} = a^2 I_a^{(1)} = 3.333 \underline{/150°} \text{ per unit}$$
$$I_{b2} = a I_a^{(2)} = 3.333 \underline{/210°} \text{ per unit}$$
$$I_b = I_b^{(1)} + I_{b2} = -5.773 \text{ per unit}$$

Using the three-phase fault value for I_a from Prob. 12.1, the ratio is now

$$5.773/6.667 = 0.866/1$$

12.3 Determine the inductive reactance in ohms to be inserted in the neutral connection of the generator of Prob. 12.1 to limit the subtransient line current for a single line-to-ground fault to that for a three-phase fault.

Solution:

From Prob. 12.1, $I_a = -j6.667$ per unit for a three-phase fault. Let x be the inductive reactance in per unit to be inserted. Then, for a single line-to-ground fault,

$$I_a = 3I_a^{(1)} = \frac{3}{j(0.15 + 0.15 + 0.05 + 3x)}$$

For a three-phase fault, $I_a = 1/j0.15 = -j6.667$ per unit. Equating the values for I_a, we have

$$3 = -j^2(0.35 + 3x)(6.667)$$
$$x = 0.0333 \text{ per unit}$$
$$\text{Base } Z = \frac{(22)^2}{500} = 0.968 \ \Omega$$
$$x = 0.0333 \times 0.968 = 0.3226 \ \Omega$$

12.4 With the inductive reactance found in Prob. 12.3 inserted in the neutral of the generator of Prob. 12.1, find the ratios of the subtransient line currents for the following faults to the subtransient line current for a three-phase fault:
(a) single line-to-ground fault, (b) line-to-line fault and (c) double line-to-ground fault.

Solution:

(a) The ratio equals one since reactance was added to achieve this ratio.

(b) The ratio is 0.866 (see Prob. 12.2; this is because the fault current in the line-to-line fault is not affected by the reactance on the neutral).

(c) For a double line-to-ground fault,

$$Z_0 = j(0.05 + 3 \times 0.0333) = j0.15$$
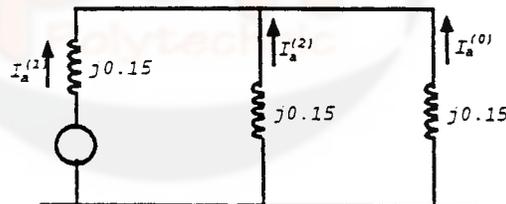
$$I_a^{(1)} = \frac{1.0}{j0.15 + \frac{j0.15 \times j0.15}{j0.15 + j0.15}} = \frac{1}{j0.225} = -j4.44 \text{ per unit}$$

$$I_a^{(2)} = I_a^{(0)} = j4.44\left(\frac{0.15}{0.30}\right) = j2.22 \text{ per unit}$$

$$I_f = 3I_a^{(0)} = j6.67 \text{ per unit}$$

$$I_b = 4.44\underline{/-90° + 240°} + 2.22\underline{/90° + 120°} + 2.22\underline{/90°}$$

$$= -3.85 + j2.22 - 1.923 - j1.11 + j2.22 = -5.773 + j3.33 = 6.67\underline{/150°} \text{ per unit}$$

For a three-phase fault,

$$I_a = \frac{1}{j0.15} = -j6.67 \text{ per unit} \qquad \text{Ratio} = \frac{6.67}{6.67} = 1.0$$

12.5 How many ohms of resistance in the neutral connection of the generator of Prob. 12.1 would limit the subtransient line current for a single line-to-ground fault to that for a three-phase fault?

Solution:

For a three-phase fault:

$$I_a = \frac{1}{j0.15} = -j6.667 \text{ per unit} \qquad\qquad R = 0.0943 \text{ per unit}$$

$$|-6.667| = 3\left|\frac{1}{j(0.15 + 0.15 + 0.05) + 3R}\right| \qquad \text{Base } Z = \frac{(22)^2}{500} = 0.968 \ \Omega$$

$$2.222 = \frac{1}{\sqrt{9R^2 + (0.35)^2}} \qquad\qquad R = 0.0943 \times 0.968 = 0.0913 \ \Omega$$

$$4.9383 = \frac{1}{9R^2 + 0.1225}$$

12.6 A generator rated 100 MVA, 20 kV has $X_d'' = X_1 = X_2 = 20\%$ and $X_0 = 5\%$. Its neutral is grounded through a reactor of 0.32 Ω. The generator is operating at rated voltage without load and is disconnected from the system when a single line-to-ground fault occurs at its terminals. Find the subtransient current in the faulted phase.

Solution:

$$\text{Base } Z = \frac{(20)^2}{100} = 4.0 \ \Omega \qquad\qquad I_a = 3I_a^{(1)} = -j4.347 \text{ per unit}$$

$$X_n = \frac{0.32}{4} = 0.08 \text{ per unit} \qquad\qquad \text{Base } I = \frac{100,000}{\sqrt{3} \times 20} = 2887 \text{ A}$$

$$Z_0 = j0.05 + 3 \times j0.08 = j0.29 \text{ per unit} \qquad |I_a''| = 4.347 \times 2887 = 12,550 \text{ A}$$

$$I_a^{(1)} = \frac{1}{j0.2 + j0.2 + j0.29} = -j1.449 \text{ per unit}$$

12.7 A 100-MVA 18-kV turbogenerator having $X_d'' = X_1 = X_2 = 20\%$ and $X_0 = 5\%$ is about to be connected to a power system. The generator has a current-limiting reactor of 0.162 Ω in the neutral. Before the generator is connected to the system, its voltage is adjusted to 16 kV when a double line-to-ground fault develops at terminals b and c. Find the initial symmetrical rms current in the ground and in line b.

> Note to Instructor: Since $Z_1 = Z_2 = Z_0$ we see from Prob. 12.4 that $|I_f| = |I_b| = |I_c|$ and fewer calculations are needed. Of course, the problem may be worked in the usual manner, but here we will take advantage of Prob. 12.4. A good variation is to make $Z_n = 0.243 \ \Omega$.

Solution:

$$\text{Base } Z = \frac{(18)^2}{100} = 3.24 \ \Omega \qquad\qquad Z_0 = j0.05 + 3 \times j0.05 = j0.20 \text{ per unit}$$

$$X_n = \frac{0.162}{3.24} = 0.05 \text{ per unit} \qquad\qquad E_{g\cdot} = \frac{16}{18} = 0.8889 \text{ per unit}$$

At this point we may recognize that Prob. 12.4 has shown that when $Z_1 = Z_2 = Z_0$, both $|I_b|$ and $|I_f|$ for a double line-to-ground fault are equal to $|I_a|$ for a single line-to-ground fault, which is

$$\frac{0.8889}{0.20} = 4.444 \text{ per unit}$$

where

$$\text{Base } I = \frac{100,000}{\sqrt{3} \times 18} = 3207.5 \text{ A}$$

$$|I_b| = |I_f| = 4.444 \times 3207.5 = 14,254 \text{ A}$$

Note that the above calculation is valid only because $Z_1 = Z_2 = Z_0$. If this were not the case, I_b must be found by first determining $I_a^{(1)}$, $I_a^{(2)}$ and $I_a^{(0)}$.

12.8 The reactances of a generator rated 100 MVA, 20 kV, are $X_d'' = X_1 = X_2 = 20\%$ and $X_0 = 5\%$. the generator is connected to a Δ-Y transformer rated 100 MVA, 20Δ-230Y kV, with a reactance of 10%. The neutral of the transformer is solidly grounded. The terminal voltage of the generator is 20 kV when a single line-to-ground fault occurs on the open-circuited, high-voltage side of the transformer. Find the initial symmetrical rms current in all phases of the generator.

Solution:
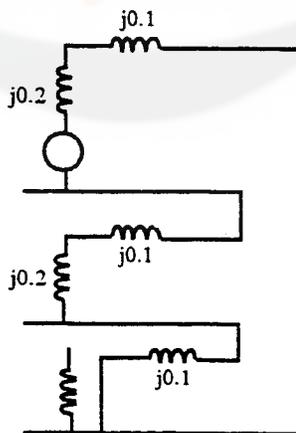
On the high-voltage side,

$$I_A^{(2)} = I_A^{(0)} = I_A^{(1)} = \frac{1}{j0.3 + j0.3 + j0.1} = -j1.429 = 1.429 \underline{/-90°} \text{ per unit}$$

In the generator, $I_a^{(0)} = 0$ and

$$I_a^{(1)} = I_A^{(1)} \angle -30° = 1.429 \angle -120° \text{ per unit}$$

$$I_a^{(2)} = I_A^{(2)} \angle 30° = 1.429 \angle -60° \text{ per unit}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.429 \angle -120° \\ 1.429 \angle -60° \end{bmatrix} = \begin{bmatrix} 2.475 \angle -90° \\ 2.475 \angle 90° \\ 0 \end{bmatrix} \text{ per unit}$$

Calculating current values we have

$$\text{Base } I = \frac{100,000}{\sqrt{3} \times 20} = 2887 \text{ A}$$

$$|I_a| = |I_b| = 2.475 \times 2887 = 7145 \text{ A}$$

$$|I_c| = 0 \text{ A}$$

12.9 A generator supplies a motor through a Y-Δ transformer. The generator is connected to the Y side of the transformer. A fault occurs between the motor terminals and the transformer. The symmetrical components of the subtransient current in the motor flowing toward the fault are

$$I_a^{(1)} = -0.8 - j2.6 \text{ per unit}$$
$$I_a^{(2)} = -j2.0 \text{ per unit}$$
$$I_a^{(0)} = -j3.0 \text{ per unit}$$

From the transformer toward the fault

$$I_a^{(1)} = 0.8 - j0.4 \text{ per unit}$$
$$I_a^{(2)} = -j1.0 \text{ per unit}$$
$$I_a^{(0)} = 0 \text{ per unit}$$

Assume $X_d'' = X_1 = X_2$ for both the motor and the generator. Describe the type of fault. Find (a) the prefault current, if any, in line a, (b) the subtransient fault current in per unit and (c) the subtransient current in each phase of the generator in per unit.

Solution:
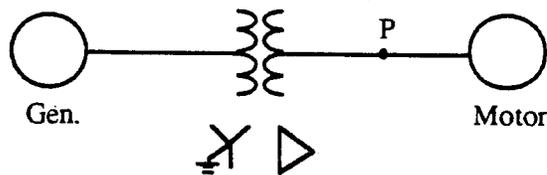
The fault occurs at point P in the above diagram. The sum of $I_a^{(1)}$ from the motor and from the transformer gives $I_a^{(1)}$ in the fault. So in the fault,

$$I_a^{(1)} = -0.8 - j2.6 + 0.8 - j0.4 = -j3.0 \text{ per unit}$$

Similarly,

$$I_a^{(2)} = 0 - j1.0 + 0 - j2.0 = -j3.0 \text{ per unit}$$
$$I_a^{(0)} = -j3.0 \text{ per unit}$$

which means a single line-to-ground fault. The connection of the sequence networks is shown as
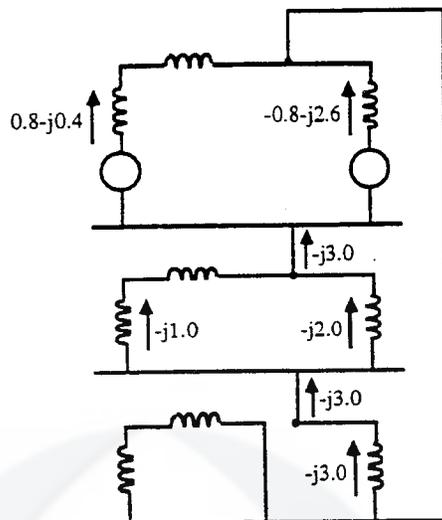
(a) Currents are marked on the network diagram. The division of the current due to the fault between the branches of the positive-sequence network is the same as between the branches of the negative-sequence network since all X_1 values equal the X_2 values in corresponding branches. The prefault current in line a toward the motor is

$$0.8 - j0.4 - (-j1.0) = 0.8 + j0.6 \text{ per unit}$$
$$\text{or} \quad -[-0.8 - j2.6 - (-j2.0)] = 0.8 + j0.6 \text{ per unit}$$

(b) $I_f'' = 3(-j3.0) = -j9.0$ per unit

(c) Assume that the generator is on the high-voltage side of the transformer such that

$$I_A^{(1)} = I_a^{(1)} e^{j30°} = (0.8 - j0.4)e^{j30°} = 0.894 \underline{/3.43°} \text{ per unit}$$
$$I_A^{(2)} = I_a^{(2)} e^{-j30°} = (-j1.0)e^{-j30°} = 1.0 \underline{/-120°} \text{ per unit}$$
$$I_A^{(0)} = 0$$

$$\begin{bmatrix} I_A \\ I_B \\ I_C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.894 \underline{/3.43°} \\ 1.0 \underline{/-120°} \end{bmatrix} = \begin{bmatrix} 0.903 \underline{/-64.3°} \\ 1.0 \underline{/-53.1°} \\ 1.893 \underline{/121.6°} \end{bmatrix} \text{ per unit}$$

12.10 Using Fig. 12.18, calculate the bus impedance matrices $\mathbf{Z}_{bus}^{(1)}$, $\mathbf{Z}_{bus}^{(2)}$ and $\mathbf{Z}_{bus}^{(0)}$ for the network of Example 12.6.

Solution:

From Fig. 12.18,

$$\mathbf{Y}_{bus}^{(1)} = \mathbf{Y}_{bus}^{(2)} = \begin{bmatrix} -j40/3 & j10 \\ j10 & -j20 \end{bmatrix} \text{ per unit}$$

Hence,

$$\mathbf{Z}_{\text{bus}}^{(1)} = \mathbf{Z}_{\text{bus}}^{(2)} = \begin{bmatrix} -j40/3 & j10 \\ j10 & -j20 \end{bmatrix}^{-1} = \begin{bmatrix} j0.12 & j0.06 \\ j0.06 & j0.08 \end{bmatrix} \text{ per unit}$$

By inspection,

$$\mathbf{Z}_{\text{bus}}^{(0)} = \begin{bmatrix} j(0.09 + 0.06) & j0 \\ j0 & \frac{j(0.2 \times 0.1)}{(0.2 + 0.1)} \end{bmatrix} = \begin{bmatrix} j0.15 & j0 \\ j0 & j0.0667 \end{bmatrix} \text{ per unit}$$

12.11 Solve for the subtransient current in a single line-to-ground fault first on bus ① and then on bus ② of the network of Example 12.6. Use the bus impedance matrices of Prob. 12.10. Also find the voltages to neutral at bus ② with the fault at bus ①.

Solution:

From Eq. (12.12), under single line-to-ground fault conditions at bus ⓚ,

$$I_{fa}^{(0)} = I_{fa}^{(1)} = I_{fa}^{(2)} = \frac{V_f}{Z_{kk}^{(1)} + Z_{kk}^{(2)} + Z_{kk}^{(0)} + 3Z_f}$$

Fault at bus ①:

$$\begin{aligned} I_{fa}^{(0)} = I_{fa}^{(1)} = I_{fa}^{(2)} &= \frac{V_f}{Z_{11}^{(1)} + Z_{11}^{(2)} + Z_{11}^{(0)} + 3Z_f} \\ &= \frac{1.0 \angle 0°}{j(0.12 + 0.12 + 0.15 + 0)} = -j2.564 \text{ per unit} \end{aligned}$$

Fault current calculation yields

$$I_{fa} = I_{fa}^{(0)} + I_{fa}^{(1)} + I_{fa}^{(2)} = 3 \times -j2.564 = -j7.692 \text{ per unit}$$

Fault at bus ②:

$$\begin{aligned} I_{fa}^{(0)} = I_{fa}^{(1)} = I_{fa}^{(2)} &= \frac{V_f}{Z_{22}^{(1)} + Z_{22}^{(2)} + Z_{22}^{(0)} + 3Z_f} \\ &= \frac{1.0 \angle 0°}{j(0.08 + 0.08 + 0.0667 + 0)} = -j4.411 \text{ per unit} \end{aligned}$$

Fault current calculation yields

$$I_{fa} = I_{fa}^{(0)} + I_{fa}^{(1)} + I_{fa}^{(2)} = 3 \times -j4.411 = -j13.233 \text{ per unit}$$

Voltage at bus ② with fault at bus ①:

$$\begin{aligned} V_a^{(1)} &= 1.0 \angle 0° - I_{fa}^{(1)} Z_{21}^{(1)} = 1.0 - (-j2.564)(j0.06) = 0.84616 \text{ per unit} \\ V_a^{(2)} &= -I_{fa}^{(2)} Z_{21}^{(2)} = -(-j2.564)(j0.06) = -0.15384 \text{ per unit} \\ V_a^{(0)} &= -I_{fa}^{(0)} Z_{21}^{(0)} = 0 \text{ per unit} \end{aligned}$$

$$
\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} V_a^{(0)} \\ V_b^{(1)} \\ V_c^{(2)} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.84616 \\ -0.15384 \end{bmatrix} = \begin{bmatrix} 0.692\angle 0° \\ 0.933\angle{-111.8°} \\ 0.933\angle 111.8° \end{bmatrix} \text{ per unit}
$$

12.12 Calculate the subtransient currents in all parts of the system of Example 12.6 with prefault current neglected if the fault on the low-voltage side of the transformer is a line-to-line fault. Use $\mathbf{Z}_{bus}^{(1)}$, $\mathbf{Z}_{bus}^{(2)}$ and $\mathbf{Z}_{bus}^{(0)}$ of Prob. 12.10.

Solution:

With a double line fault (between phases b and c) at bus ①,

$$
I_{fa}^{(1)} = -I_{fa}^{(2)} = \frac{V_f}{Z_{11}^{(1)} + Z_{11}^{(2)}} = \frac{1.0}{j(0.12 + 0.12)} = -j4.167 \text{ per unit}
$$

$$
I_{fa}^{(0)} = 0
$$

$$
\begin{bmatrix} I_{fa} \\ I_{fb} \\ I_{fc} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ -j4.167 \\ j4.167 \end{bmatrix} = \begin{bmatrix} 0 \\ -7.217 \\ 7.217 \end{bmatrix} \text{ per unit}
$$

During this type of fault, no zero-sequence current flows through the system.

(Positive Sequence)   (Negative Sequence)

(i) No current flows in any of the grounding connections.

(ii) In the low-voltage side of the transformer:

$$
I_a^{(0)} = 0 \qquad I_a^{(1)} = -j2.50 \text{ per unit} \qquad I_a^{(2)} = j2.50 \text{ per unit}
$$

$$
\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 0 \\ -j\sqrt{3} \\ j\sqrt{3} \end{bmatrix} (-j2.50) = \begin{bmatrix} 0 \\ -4.33 \\ 4.33 \end{bmatrix} \text{ per unit}
$$

(iii) In the generator and the high-voltage side of the transformer:

The Y-Δ connection introduces a 30° phase lead to positive-sequence currents and a 30° phase lag to negative-sequence currents. Therefore,

$$
I_a^{(1)} = 2.50\angle{-60°} \text{ per unit}
$$

$$
I_a^{(2)} = 2.50\angle 60° \text{ per unit}
$$

$$I_a^{(0)} = 0$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 2.50\underline{/-60°} \\ 2.50\underline{/60°} \end{bmatrix} = \begin{bmatrix} 2.50 \\ -5.0 \\ 2.50 \end{bmatrix} \text{ per unit}$$

(iv) In the motor section (towards the fault):

$$I_a^{(0)} = 0 \qquad I_a^{(1)} = -j1.667 \text{ per unit} \qquad I_a^{(2)} = j1.667 \text{ per unit}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 0 \\ -j\sqrt{3} \\ j\sqrt{3} \end{bmatrix} (-j1.667) = \begin{bmatrix} 0 \\ -2.88 \\ 2.88 \end{bmatrix} \text{ per unit}$$

(Each motor supplies 1/4 of these currents)

12.13 Repeat Prob. 12.12 for a double line-to-ground fault.

Solution:

Under double line-to-ground (phases b, c and ground) fault conditions,

$$I_{fa}^{(1)} = \frac{V_f}{Z_{11}^{(1)} + \frac{Z_{11}^{(2)} Z_{11}^{(0)}}{Z_{11}^{(2)} + Z_{11}^{(0)}}} = \frac{1.0\underline{/0°}}{j\left(0.12 + \frac{0.15 \times 0.15}{0.12 + 0.15}\right)} = -j5.357 \text{ per unit}$$

$$I_{fa}^{(2)} = -I_{fa}^{(1)} \frac{Z_{11}^{(0)}}{Z_{11}^{(0)} + Z_{11}^{(2)}} = j2.976 \text{ per unit}$$

$$I_{fa}^{(0)} = -I_{fa}^{(1)} - I_{fa}^{(2)} = j2.381 \text{ per unit}$$

Currents toward the fault from the three phases are

$$\begin{bmatrix} I_{fa} \\ I_{fb} \\ I_{fc} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} j2.381 \\ -j5.357 \\ j2.976 \end{bmatrix} = \begin{bmatrix} 0 \\ 8.052\underline{/153.7°} \\ 8.052\underline{/26.3°} \end{bmatrix} \text{ per unit}$$

Sequence current flows through various parts of the network as shown below:
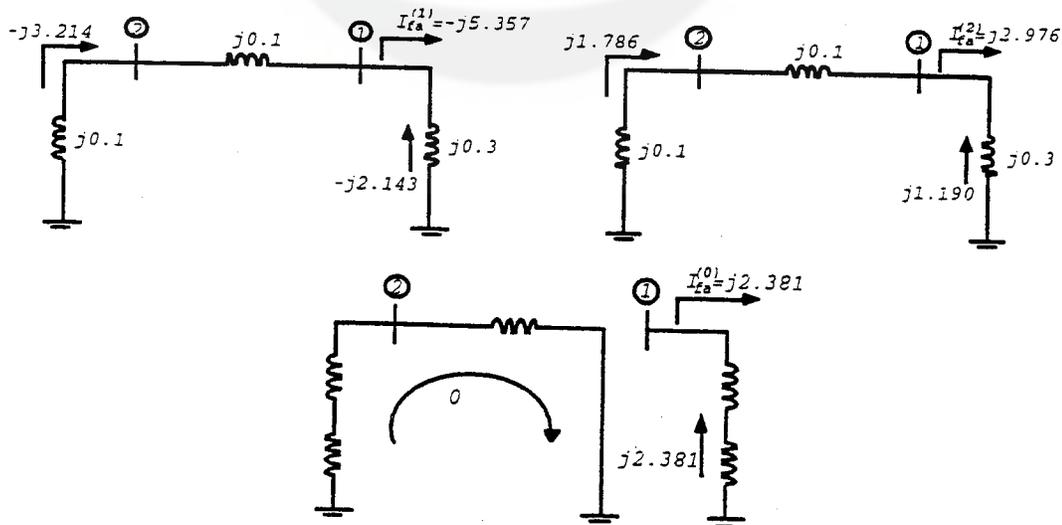
(i) In the generator grounding connection, no current flows since zero-sequence current in that part of the network is zero.

(ii) Current in the motor grounding section (toward motors):

$$3I_a^{(0)} = 3 \times j2.381 = j7.143 \text{ per unit}$$

(Each motor handles 1/4 of this current)

(iii) In the low-voltage side of the transformer:

$$I_a^{(0)} = 0 \qquad I_a^{(1)} = -j3.214 \text{ per unit} \qquad I_a^{(2)} = j1.786 \text{ per unit}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ -j3.214 \\ j1.786 \end{bmatrix} = \begin{bmatrix} 1.429\angle -90° \\ 4.388\angle 170.6° \\ 4.388\angle 9.4° \end{bmatrix} \text{ per unit}$$

(iv) In the generator and high-voltage side of the transformer:

As in Prob. 12.12,

$$\begin{aligned} I_a^{(1)} &= 3.214\angle -60° \text{ per unit} \\ I_a^{(2)} &= 1.786\angle 60° \text{ per unit} \\ I_a^{(0)} &= 0 \end{aligned}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 3.214\angle -60° \\ 1.786\angle 60° \end{bmatrix} = \begin{bmatrix} 2.789\angle -26.3° \\ 5.0\angle 180° \\ 2.789\angle 26.3° \end{bmatrix} \text{ per unit}$$

(v) In the motor section:

$$I_a^{(0)} = j2.381 \text{ per unit} \qquad I_a^{(1)} = -j2.143 \text{ per unit} \qquad I_a^{(2)} = j1.190 \text{ per unit}$$

$$\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} j2.381 \\ -j2.143 \\ j1.190 \end{bmatrix} = \begin{bmatrix} 1.429\angle 90° \\ 4.062\angle 135.3° \\ 4.062\angle 44.7° \end{bmatrix} \text{ per unit}$$

(Each motor carries 1/4 of these currents)

12.14 The machines connected to the two high-voltage buses shown in the single-line diagram of Fig. 12.32 are each rated 100 MVA, 20 kV with reactances of $X_d'' = X_1 = X_2 = 20\%$ and $X_0 = 4\%$. Each three-phase transformer is rated 100 MVA, 345Y/20Δ kV, with leakage reactance of 8%. On a base of 100 MVA, 345 kV the reactances of the transmission line are $X_1 = X_2 = 15\%$ and $X_0 = 50\%$. Find the 2×2 bus impedance matrix for each of the three sequence networks. If no prefault current is flowing in the network, find the subtransient current to ground for a double line-to-ground fault on lines B and C at bus ①. Repeat for a fault at bus ②. When the fault is at bus ②, determine the current in phase b of machine 2 if the lines are so named that $V_A^{(1)}$ leads $V_a^{(1)}$ by 30°. If the phases are named so that $I_a^{(1)}$ leads $I_A^{(1)}$ by 30°, what letter (a, b or c) would identify the phase of machine 2 which would carry the current found for phase b above?

<u>Note to Instructor</u>: Although this very simple two-node problem is solved just as easily by not resorting to the $\mathbf{Z}_{bus}$ method, the problem is useful to point out the great advantage of $\mathbf{Z}_{bus}$ where fault calculations are needed at many nodes beacuse the computer takes the labor out of finding $\mathbf{Z}_{bus}$. With $\mathbf{Z}_{bus}$, the problem is easily extended to find the voltage of unfaulted buses.

Solution:

The negative-sequence network shown below is identical to the positive-sequence network with emfs short-circuited.
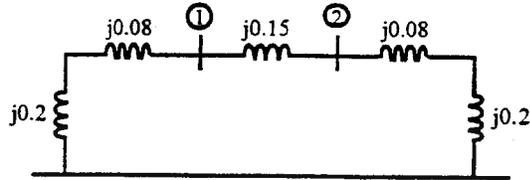
$$Y_{11}^{(2)} = Y_{11}^{(1)} = \frac{1}{j0.28} + \frac{1}{j0.15} = -j10.24 \text{ per unit}$$

$$Y_{22}^{(2)} = Y_{11}^{(2)} = Y_{11}^{(1)} = -j10.24 \text{ per unit}$$

$$Y_{12}^{(1)} = Y_{12}^{(2)} = \frac{-1}{j0.15} = j6.67 \text{ per unit}$$

The zero-sequence network is shown as
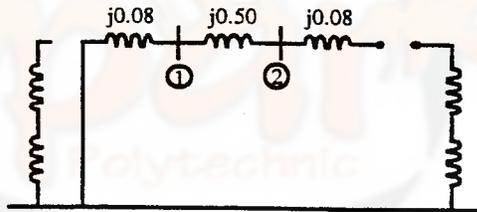
Since the impedance $j0.08$ connected to bus ② is not connected to any other bus, it is not included in $Y_{22}^{(0)}$.

$$Y_{11}^{(0)} = \frac{1}{j0.08} + \frac{1}{j0.50} = -j14.50 \text{ per unit}$$

$$Y_{12}^{(0)} = Y_{22}^{(0)} = \frac{-1}{j0.5} = j2.0 \text{ per unit}$$

$$\mathbf{Z}_{bus}^{(1)} = \mathbf{Z}_{bus}^{(2)} = \begin{bmatrix} -j10.24 & j6.67 \\ j6.67 & -j10.24 \end{bmatrix}^{-1} = \begin{bmatrix} j0.170 & j0.110 \\ j0.110 & j0.170 \end{bmatrix} \text{ per unit}$$

$$\mathbf{Z}_{bus}^{(0)} = \begin{bmatrix} -j14.50 & j2.00 \\ j2.00 & -j2.00 \end{bmatrix}^{-1} = \begin{bmatrix} j0.08 & j0.08 \\ j0.08 & j0.58 \end{bmatrix} \text{ per unit}$$

$\mathbf{Z}_{bus}$ equivalent networks are shown below (mutual impedances not indicated) for a double line-to-ground fault on bus ①:
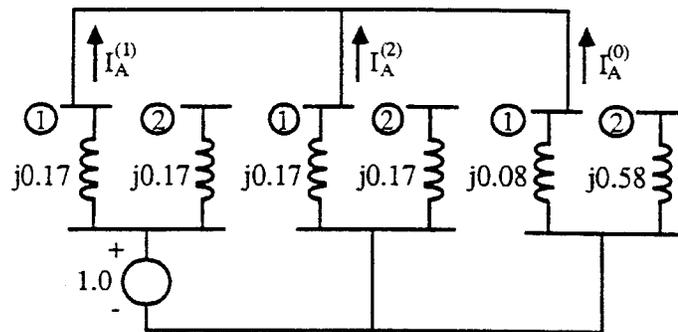
For the fault on bus ① (as shown):

$$I_A^{(1)} = \frac{1}{j\left(0.170 + \frac{0.170 \times 0.08}{0.170 + 0.08}\right)} = -j4.456 \text{ per unit}$$

$$V_A^{(1)} = 1.0 - j0.170(-j4.456) = 0.2425 \text{ per unit}$$

$$I_A^{(2)} = \frac{-0.2425}{j0.170} = j1.426 \text{ per unit}$$

$$I_A^{(0)} = \frac{-0.2425}{j0.08} = j3.031 \text{ per unit}$$

$$I_n = 3I_A^{(0)} = 3(j3.031) = j9.093 \text{ per unit}$$

Calculations for magnitude of the fault current yields

$$\text{Base } I = \frac{100,000}{\sqrt{3} \times 345} = 167.3 \text{ A}$$

$$|I_n| = 167.3 \times 9.093 = 1521 \text{ A}$$

For the fault on bus ②:

$$I_A^{(1)} = \frac{1}{j\left(0.170 + \frac{0.170 \times 0.58}{0.170 + 0.58}\right)} = -j3.317 \text{ per unit}$$

$$V_A^{(1)} = 1.0 - j0.170(-j3.317) = 0.4361 \text{ per unit}$$

$$I_A^{(2)} = \frac{-0.4361}{j0.170} = j2.565 \text{ per unit}$$

$$I_A^{(0)} = \frac{-0.4361}{j0.58} = j0.7519 \text{ per unit}$$

$$I_n = 3I_A^{(0)} = j2.256 \text{ per unit}$$

The magnitude of the fault current is

$$|I_n| = 167.3 \times 2.256 = 377 \text{ A}$$

From the transformer at bus ② to the fault at bus ②:

$$I_A^{(1)} = -j3.317 \times \frac{j0.28 + j0.15}{j0.28 + j0.28 + j0.15} = -j2.009 \text{ per unit}$$

$$I_A^{(2)} = j2.565 \times \frac{j0.28 + j0.15}{j0.28 + j0.28 + j0.15} = j1.553 \text{ per unit}$$

In generator 2:

$$I_a^{(0)} = 0$$

$$I_a^{(1)} = I_A^{(1)}e^{-j30°} = 2.009\underline{/-120°} \text{ per unit}$$

$$I_a^{(2)} = I_A^{(2)}e^{j30°} = 1.553\underline{/120°} \text{ per unit}$$

$$I_b = \begin{bmatrix} 1 & a^2 & a \end{bmatrix}\begin{bmatrix} 0 \\ 2.009\underline{/-120°} \\ 1.553\underline{/120°} \end{bmatrix} = 1.824\underline{/167.5°} \text{ per unit}$$

Calculating for the fault current value yields

$$\text{Base } I = \frac{100,000}{\sqrt{3}\times 20} = 2887 \text{ A}$$

$$|I_b| = 2887\times 1.824 = 5266 \text{ A}$$

12.15 Two generators G_1 and G_2 are connected respectively through transformers T_1 and T_2 to a high-voltage bus which supplies a transmission line. The line is open at the far end at which point F a fault occurs. The prefault voltage at point F is 515 kV. Apparatus ratings and reactances are

G_1 1000 MVA, 20 kV, $X_s = 100\%$ $X_d'' = X_1 = X_2 = 10\%$ $X_0 = 5\%$
G_2 800 MVA, 22 kV, $X_s = 120\%$ $X_d'' = X_1 = X_2 = 15\%$ $X_0 = 8\%$
T_1 1000 MVA, 500Y/20Δ kV, $X = 17.5\%$
T_2 800 MVA, 500Y/22Y kV, $X = 16.0\%$
Line $X_1 = 15\%$, $X_0 = 40\%$ on a base of 1500 MVA, 500 kV

The neutral of G_1 is grounded through a reactance of 0.04 Ω. The neutral of G_2 is not grounded. Neutrals of all transformers are solidly grounded. Work on a base of 1000 MVA, 500 kV in the transmission line. Neglect prefault current and find subtransient current (a) in phase c of G_1 for a three-phase fault at F, (b) in phase B at F for a line-to-line fault on lines B and C, (c) in phase A at F for a line-to-ground fault on line A and (d) in phase c of G_2 for a line-to-ground fault on line A. Assume $V_A^{(1)}$ leads $V_a^{(1)}$ by 30° in T_1.

Solution:

Base kV for
Gen. 1 = 20 kV
Gen. 2 = 22 kV
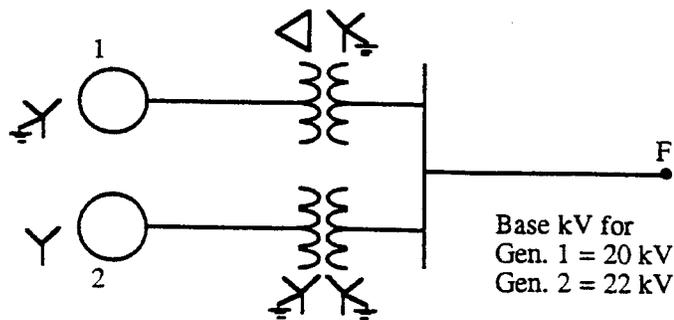
The base currents are calculated as

$$\text{Line:} \quad \frac{1,000,000}{\sqrt{3} \times 500} = 1155 \text{ A}$$

$$\text{Gen. 1:} \quad \frac{1,000,000}{\sqrt{3} \times 20} = 28,868 \text{ A}$$

$$\text{Gen. 2:} \quad \frac{1,000,000}{\sqrt{3} \times 22} = 26,243 \text{ A}$$

Impedances in per unit are

$$\text{Gen. 1:} \quad X_d'' = X_2 = 0.10 \qquad X_0 = 0.05$$

$$\text{Gen. 2:} \quad X_d'' = X_2 = 0.15 \times \frac{1000}{800} = 0.1875$$

$$T_1: \quad X = 0.175 \qquad T_2: \quad X = 0.16 \times \frac{1000}{800} = 0.20$$

$$\text{Line:} \quad X_1 = X_2 = 0.15 \times \frac{1000}{1500} = 0.10$$

$$X_0 = 0.40 \times \frac{1000}{1500} = 0.267$$

$$\text{Operating voltage} = \frac{515}{500} = 1.03 \text{ per unit}$$

Positive Sequence Network
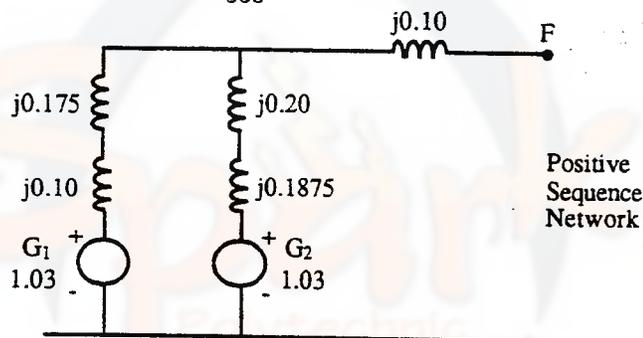
The above network reduces to

Zero-sequence Network
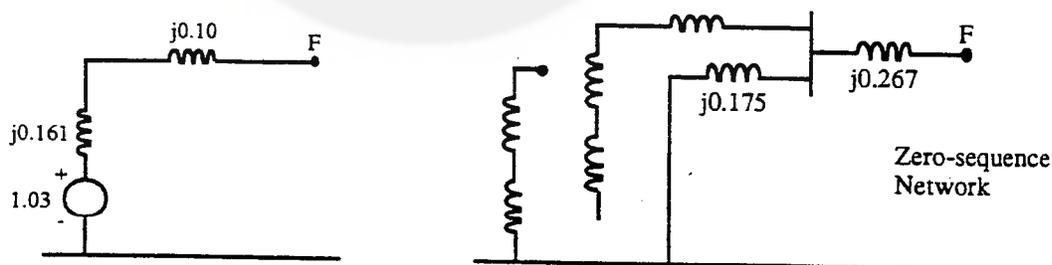
(a) Three-phase fault at F

$$I_A^{(1)} = \frac{1.03}{j0.261} = -j3.946 \text{ per unit}$$

$$\text{In Gen. 1:} \quad I_a^{(1)} = \left(-j3.946 \frac{j0.3875}{j0.275 + j0.3875} \right) e^{-j30°} = 2.308 \angle{-120°} \text{ per unit}$$

$$I_c = 2.308 \angle{0°} \text{ per unit}$$

In all three phases $|I_f''| = 2.308 \times 28.868 = 66,630$ A.

(b) Line-to-line fault at F ($Z_1 = Z_2$)

$$I_A^{(1)} = -I_A^{(2)} = \frac{1.03}{2(j0.261)} = -j1.973 \text{ per unit}$$

$$I_B = (a^2 - a)I_A^{(1)} = -j\sqrt{3}(-j1.973) = 3.417\angle 180° \text{ per unit}$$

$$|I_B''| = 3.417 \times 1155 = 3947 \text{ A}$$

(c) Single line-to-ground fault at F

$$I_A^{(1)} = \frac{1.03}{j0.261 + j0.261 + j0.175 + j0.267} = -j1.068 \text{ per unit}$$

$$|I_A''| = 3(1.068) \times 1155 = 3700 \text{ A}$$

(d) Single line-to-ground fault at F

In Gen. 2: $\quad I_a^{(1)} = I_a^{(2)} \qquad I_a^{(0)} = 0$

$$I_a^{(1)} = -j1.068\frac{j0.275}{j0.275 + j0.3875} = -j0.443 \text{ per unit}$$

$$I_c = 0.443\angle 120° - 90° + 0.443\angle 240° - 90°$$

$$= 0.384 + j0.222 - 0.384 + j0.222 = j0.444 \text{ per unit}$$

$$|I_c''| = 0.444 \times 26,243 = 11,652 \text{ A}$$

12.16 In the network shown in Fig. 10.17, Y-Y connected transformers, each with grounded neutrals, are at the ends of each transmission line not terminating at bus ③. The transformers connecting the lines to bus ③ are Y-Δ with the neutral of the Y solidly grounded and the Δ sides connected to bus ③. All line reactances shown in Fig. 10.17 between buses include the reactances of the transformers. Zero-sequence values for these lines including transformers are 2.0 times those shown in Fig. 10.17.

Both generators are Y-connected. Zero-sequence reactances of the generators connected to buses ① and ③ are 0.04 and 0.08 per unit, respectively. The neutral of the generator at bus ① is connected to ground through a reactor of 0.02 per unit; the generator at bus ③ has a solidly grounded neutral.

Find the bus impedance matrices $Z_{bus}^{(1)}$, $Z_{bus}^{(2)}$ and $Z_{bus}^{(0)}$ for the given network and then compute the subtransient current in per unit (a) in a single line-to-ground fault on bus ② and (b) in the faulted phase of line ①-②. Assume no prefault current is flowing and all prefault voltages at all the buses is $1.0\angle 0°$ per unit.

Solution:

The schematic diagram is shown as
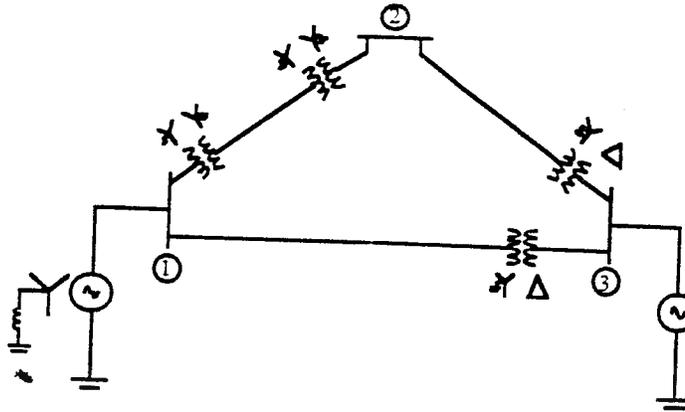
The positive- and negative-sequence network is shown as
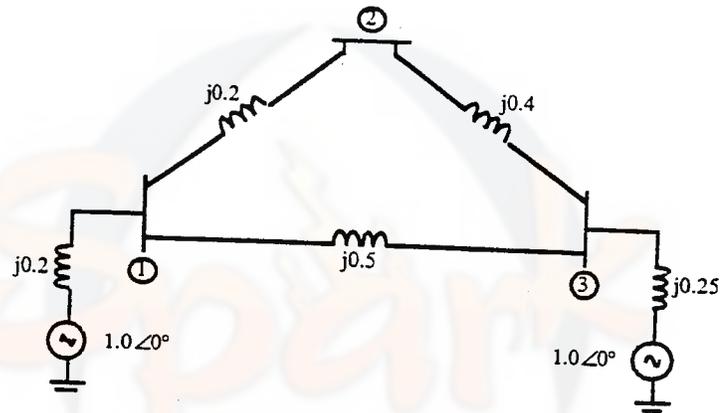
The zero-sequence network is shown as
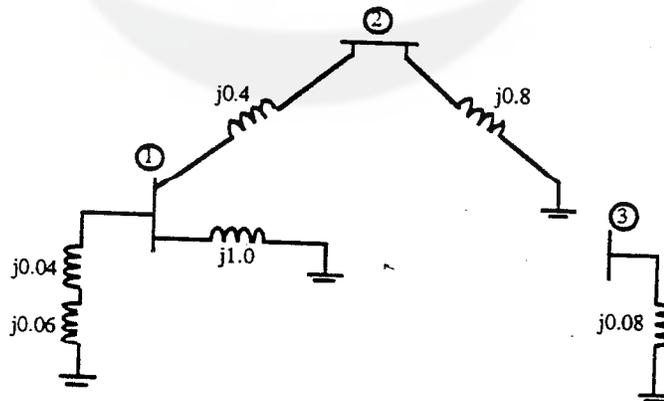
From Prob. 10.9,

$$\mathbf{Z}_{bus}^{(1)} = \mathbf{Z}_{bus}^{(2)} = \begin{bmatrix} j0.1447 & j0.1195 & j0.0692 \\ j0.1195 & j0.2465 & j0.1006 \\ j0.0692 & j0.1006 & j0.1635 \end{bmatrix} \text{ per unit}$$

By examining the zero-sequence network,

$$\mathbf{Y}_{bus}^{(0)} = \begin{bmatrix} -j13.5 & j2.5 & j0 \\ j2.5 & -j3.75 & j0 \\ j0 & j0 & -j12.5 \end{bmatrix} \text{ per unit}$$

$$\mathbf{Z}_{bus}^{(0)} = \left[\mathbf{Y}_{bus}^{(0)} \right]^{-1} = \begin{bmatrix} j0.0845 & j0.0563 & j0 \\ j0.0563 & j0.3042 & j0 \\ j0 & j0 & j0.08 \end{bmatrix} \text{ per unit}$$

For a fault at bus ②,

$$I_{fa}^{(0)} = I_{fa}^{(1)} = I_{fa}^{(2)} = \frac{V_f}{Z_{22}^{(0)} + Z_{22}^{(1)} + Z_{22}^{(2)}}$$

$$= \frac{1.0 \underline{/0°}}{j(0.2465 + 0.2465 + 0.3042)} = -j1.254 \text{ per unit}$$

$$I_{fa}'' = I_{fa}^{(0)} + I_{fa}^{(1)} + I_{fa}^{(2)} = -j3.762 \text{ per unit}$$

Currents through line ①–②:

$$I_{12,a}^{(1)} = \frac{\left(Z_{12}^{(1)} - Z_{22}^{(1)} \right) \left(-I_{fa}^{(1)} \right)}{z_{12}^{(1)}} = \frac{j(0.1195 - 0.2465)(j1.254)}{j0.2} = -j0.796 \text{ per unit}$$

$$I_{12,a}^{(2)} = I_{12,a}^{(1)} = -j0.796 \text{ per unit}$$

$$I_{12,a}^{(0)} = \frac{\left(Z_{12}^{(0)} - Z_{22}^{(0)} \right) \left(-I_{fa}^{(0)} \right)}{z_{12}^{(0)}} = \frac{j(0.0563 - 0.3042)(j1.254)}{j0.4} = -j0.777 \text{ per unit}$$

$$I_{12,a} = I_{12,a}^{(0)} + I_{12,a}^{(1)} + I_{12,a}^{(2)} = -j2.369 \text{ per unit}$$

12.17 The network of Fig. 9.2 has the line data specified in Table 9.2. The two generators connected to buses ① and ④ have $X_d'' = X_1 = X_2 = 0.25$ per unit each. Making the usual simplifying assumptions of Sec. 10.6, determine the sequence matrices $\mathbf{Z}_{bus}^{(1)} = \mathbf{Z}_{bus}^{(2)}$ and use them to calculate

(a) the subtransient current in per unit in a line-to-line fault on bus ② of the network

(b) the fault current contributions from line ①–② and line ④–②.

Assume that lines ①–② and ④–② are connected to bus ② directly (not through transformers) and that all positive- and negative-sequence reactances are identical.

> Note to Instructor: The text requests for current contributions from line ③–② while it does not exist in Fig. 9.2. The solution below uses line ④–②.

Solution:

Z_{bus} is shown in the solution for Prob. 10.13 and

$$Z_{bus}^{(1)} = Z_{bus}^{(2)} = Z_{bus}$$

(a) For a fault at bus ②:

$$I_{fa}^{(1)} = -I_{fa}^{(2)} = \frac{V_f}{Z_{22}^{(1)} + Z_{22}^{(2)}} = \frac{1.0\angle 0°}{j0.1466 + j0.1466} = -j3.411 \text{ per unit}$$

$$\begin{bmatrix} I_{fa} \\ I_{fb} \\ I_{fc} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ I_{fa}^{(1)} \\ -I_{fa}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -j\sqrt{3}I_{fa}^{(1)} \\ j\sqrt{3}I_{fa}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -5.908 \\ 5.908 \end{bmatrix} \text{ per unit}$$

(b) Currents through line ①-②:

$$I_{12,a}^{(1)} = \frac{\left(Z_{12}^{(1)} - Z_{22}^{(1)}\right)\left(-I_{fa}^{(1)}\right)}{z_{12}^{(1)}} = \frac{j(0.1234 - 0.1466)(j3.411)}{j0.0504} = -j1.570 \text{ per unit}$$

$$\begin{bmatrix} I_{12,a} \\ I_{12,b} \\ I_{12,c} \end{bmatrix} = \begin{bmatrix} 0 \\ -j\sqrt{3}I_{12,a}^{(1)} \\ j\sqrt{3}I_{12,a}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -2.720 \\ 2.720 \end{bmatrix} \text{ per unit}$$

Currents through line ④-②:

$$I_{42,a}^{(1)} = \frac{\left(Z_{42}^{(1)} - Z_{22}^{(1)}\right)\left(-I_{fa}^{(1)}\right)}{z_{42}^{(1)}} = \frac{j(0.1266 - 0.1466)(j3.411)}{j0.0372} = -j1.834 \text{ per unit}$$

$$\begin{bmatrix} I_{42,a} \\ I_{42,b} \\ I_{42,c} \end{bmatrix} = \begin{bmatrix} 0 \\ -j\sqrt{3}I_{42,a}^{(1)} \\ j\sqrt{3}I_{42,a}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -3.176 \\ 3.176 \end{bmatrix} \text{ per unit}$$

12.18 In the system of Fig. 12.9a consider that Machine 2 is a motor drawing a load equivalent to 80 MVA at 0.85 power factor lagging and nominal system voltage of 345 kV at bus ③. Determine the change in voltage at bus ③ when the transmission line undergoes (a) a one-open-conductor fault and (b) a two-open-conductor fault along its span between buses ② and ③. Choose a base of 100 MVA, 345 kV in the transmission line. Consult Examples 12.1 and 12.2 for $Z_{bus}^{(0)}$, $Z_{bus}^{(1)}$ and $Z_{bus}^{(2)}$.

Solution:

For line ②-③:

$$Z_1 = j0.15 \text{ per unit} \qquad Z_2 = j0.15 \text{ per unit} \qquad Z_0 = j0.50 \text{ per unit}$$

$$I_{23} = \left|\frac{S}{V}\right|\angle -\theta = \frac{80/100}{1.0}\angle -\cos^{-1}0.85 = 0.8\angle -31.79° \text{ per unit}$$

From Eqs. (12.28) and (12.32),

$$Z_{pp'}^{(1)} = Z_{pp'}^{(2)} = \frac{-Z_1^2}{Z_{22}^{(1)} + Z_{33}^{(1)} - 2Z_{23}^{(1)} - Z_1} = \frac{-(j0.15)^2}{j(0.1696 + 0.1696 - 2 \times 0.1104 - 0.15)}$$
$$= j0.7120 \text{ per unit}$$

$$Z_{pp'}^{(0)} = \frac{-Z_0^2}{Z_{22}^{(0)} + Z_{33}^{(0)} - 2Z_{23}^{(0)} - Z_0} = \frac{-(j0.50)^2}{j(0.1999 + 0.1999 - 2 \times 0.0701 - 0.50)}$$
$$= j1.0399 \text{ per unit}$$

(a) One open conductor fault:

From Eq. (12.38),

$$V_a^{(0)} = V_a^{(1)} = V_a^{(2)} = \frac{I_{23}}{\frac{1}{Z_{pp'}^{(0)}} + \frac{1}{Z_{pp'}^{(1)}} + \frac{1}{Z_{pp'}^{(2)}}}$$
$$= \frac{0.8 \angle -31.79°}{\frac{1}{j0.7120} + \frac{1}{j0.7120} + \frac{1}{j1.0399}} = 0.2122 \angle 58.21° \text{ per unit}$$

From Eq. (12.27),

$$\Delta V_3^{(1)} = \Delta V_3^{(2)} = \frac{Z_{32}^{(1)} - Z_{33}^{(1)}}{Z_1} V_a^{(1)}$$
$$= \frac{j0.1104 - j0.1696}{j0.15} \times 0.2122 \angle 58.21° = 0.0837 \angle -121.79° \text{ per unit}$$

$$\Delta V_3^{(0)} = \frac{Z_{32}^{(0)} - Z_{33}^{(0)}}{Z_0} V_a^{(0)}$$
$$= \frac{j0.0701 - j0.1999}{j0.50} \times 0.2122 \angle 58.21° = 0.0551 \angle -121.79° \text{ per unit}$$

Change in a-phase voltage at bus ③ is

$$\Delta V_3 = \Delta V_3^{(1)} + \Delta V_3^{(2)} + \Delta V_3^{(0)} = (2 \times 0.0837 + 0.0551) \angle -121.79° = 0.2225 \angle -121.79° \text{ per unit}$$

(b) Two open conductor fault:

From Eq. (12.43),

$$V_a^{(1)} = I_{23} \frac{Z_{pp'}^{(1)} \left(Z_{pp'}^{(2)} + Z_{pp'}^{(0)} \right)}{Z_{pp'}^{(1)} + Z_{pp'}^{(2)} + Z_{pp'}^{(0)}}$$
$$= 0.8 \angle -31.79° \times \frac{j0.7120 \times (j0.7120 + j1.0399)}{j0.7120 + j0.7120 + j1.0399} = 0.4050 \angle 58.21° \text{ per unit}$$

$$V_a^{(2)} = I_{23} \frac{-Z_{pp'}^{(1)} Z_{pp'}^{(2)}}{Z_{pp'}^{(1)} + Z_{pp'}^{(2)} + Z_{pp'}^{(0)}}$$
$$= 0.8 \angle -31.79° \times \frac{-j0.7120 \times j0.7120}{j0.7120 + j0.7120 + j1.0399} = 0.1646 \angle -121.79° \text{ per unit}$$

$$V_a^{(0)} = I_{23} \frac{-Z_{pp'}^{(1)} Z_{pp'}^{(0)}}{Z_{pp'}^{(1)} + Z_{pp'}^{(2)} + Z_{pp'}^{(0)}}$$
$$= 0.8 \angle -31.79° \times \frac{-j0.7120 \times j1.0399}{j0.7120 + j0.7120 + j1.0399} = 0.2404 \angle -121.79° \text{ per unit}$$

From Eq. (12.27):

$$\Delta V_3^{(1)} = \frac{j0.1104 - j0.1696}{j0.15} \times 0.4050 \underline{/58.21°} = 0.1598 \underline{/-121.79°} \text{ per unit}$$

$$\Delta V_3^{(2)} = \frac{j0.1104 - j0.1696}{j0.15} \times 0.1646 \underline{/-121.79°} = 0.0650 \underline{/58.21°} \text{ per unit}$$

$$\Delta V_3^{(0)} = \frac{j0.0701 - j0.1999}{j0.50} \times 0.2404 \underline{/-121.79°} = 0.0624 \underline{/58.21°} \text{ per unit}$$

Change in a-phase voltage at bus ③ is

$$\Delta V_3 = \Delta V_3^{(1)} + \Delta V_3^{(2)} + \Delta V_3^{(0)} = (0.1598 - 0.0650 - 0.8624) \underline{/-121.79°} = 0.0324 \underline{/-121.79°} \text{ per unit}$$

Chapter 13 Problem Solutions

13.1 For a generating unit the fuel input in millions of Btu/h is expressed as a function of output P_g in megawatts by $0.032P_g^2 + 5.8P_g + 120$. Determine

   (a) the equation for incremental fuel cost in dollars per megawatthour as a function of P_g in megawatts based on a fuel cost of \$2 per million Btu.

   (b) the average cost of fuel per megawatthour when $P_g = 200$ MW.

   (c) the approximate additional fuel cost per hour to raise the output of the unit from 200 MW to 201 MW. Also find this additional cost accurately and compare it with the approximate value.

Solution:

   (a) The input-output curve in dollars per MWh is

$$f = (0.032P_g^2 + 5.8P_g + 120) \times 2$$
$$= 0.064P_g^2 + 11.6P_g + 240 \text{ \$/MWh}$$

The incremental fuel cost is

$$\frac{df}{dP_g} = 0.128P_g + 11.6 \text{ \$/MWh}$$

   (b) The average cost of fuel when $P_g = 200$ MW is

$$\left.\frac{f}{P_g}\right|_{P_g=200} = \frac{0.064(202)^2 + 11.6(200) + 240}{200} = 25.6 \text{ \$/MWh}$$

   (c) The approximate incremental cost for an additional 1 MW generation when $P_g = 200$ MW is

$$\left.\frac{df}{dP_g}\right|_{P_g=200} = 0.128(200) + 11.6 = 37.2 \text{ \$/h}$$

The additional cost per hour to raise the output from 200 MW to 201 MW can be calculated accurately as follows:

$$\int_{200}^{201} (0.128P_g + 11.6)\, P_g = 0.064P_g^2 + 11.6P_g \Big|_{200}^{201} = 37.264 \text{ \$/h}$$

13.2 The incremental fuel costs in \$/MWh for four units of a plant are

$$\lambda_1 = \frac{df_1}{dP_{g1}} = 0.012P_{g1} + 9.0 \qquad \lambda_2 = \frac{df_2}{dP_{g2}} = 0.0096P_{g2} + 6.0$$

$$\lambda_3 = \frac{df_3}{dP_{g3}} = 0.008P_{g3} + 8.0 \qquad \lambda_4 = \frac{df_4}{dP_{g4}} = 0.0068P_{g4} + 10.0$$

Assuming that all four units operate to meet the total plant load of 800 MW, find the incremental fuel cost λ of the plant and the required output of each unit for economic dispatch.

Solution:

$$a_T = \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right)^{-1}$$

$$= \left(\frac{1}{0.012} + \frac{1}{0.0096} + \frac{1}{0.008} + \frac{1}{0.0068}\right)^{-1} = 2.176 \times 10^{-3}$$

$$b_T = a_T \left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \frac{b_4}{a_4}\right)$$

$$= 2.176 \times 10^{-3} \left(\frac{9}{0.012} + \frac{6}{0.0096} + \frac{8}{0.008} + \frac{10}{0.0068}\right) = 8.368$$

$$P_{gT} = 800 \text{ MW}$$

The λ of the plant is given by

$$\lambda = a_T P_{gT} + b_T = 2.176 \times 10^{-3} \times 800 + 8.368 = 10.1088 \text{ \$/MWh}$$

Using Eq. (13.4), for each unit we have

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{10.1088 - 9}{0.012} = 92.4 \text{ MW}$$

$$P_{g2} = \frac{\lambda - b_2}{a_2} = \frac{10.1088 - 6}{0.0096} = 428 \text{ MW}$$

$$P_{g3} = \frac{\lambda - b_3}{a_3} = \frac{10.1088 - 8}{0.008} = 263.6 \text{ MW}$$

$$P_{g4} = \frac{\lambda - b_4}{a_4} = \frac{10.1088 - 10}{0.0068} = 16 \text{ MW}$$

13.3 Assume that maximum load on each of the four units described in Prob. 13.2 is 200 MW, 400 MW, 270 MW and 300 MW, respectively, and that minimum load on each unit is 50 MW, 100 MW, 80 MW and 110 MW, respectively. With

these maximum and minimum output limits, find the plant λ and MW output of each unit for economic dispatch.

Solution:

The solution to Prob. 13.2 shows that each unit's output would be 92.4 MW, 428 MW, 263.6 MW and 16 MW, respectively, if there were no maximum and minimum limits on unit outputs. It is seen that the output of Unit 2 violates its upper limit, and the output of Unit 4 violates its lower limit. This fact does not necessarily mean that the outputs of Units 2 and 4 should be set at their upper and lower limits, respectively. In fact, these limits should be checked individually.

First, assume that Unit 2 is operating at its upper limit of 400 MW. Using the remaining Units 1, 2 and 3, we calculate the plant λ as follows:

$$a_T = \left(\frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4}\right)^{-1} = \left(\frac{1}{0.012} + \frac{1}{0.008} + \frac{1}{0.0068}\right)^{-1} = 2.813793 \times 10^{-3}$$

$$b_T = a_T\left(\frac{b_1}{a_1} + \frac{b_3}{a_3} + \frac{b_4}{a_4}\right) = 2.813793 \times 10^{-3}\left(\frac{9}{0.012} + \frac{8}{0.008} + \frac{10}{0.0068}\right) = 9.062069$$

Since $P_{g2} = 400$ MW, the total output of Units 1, 3 and 4 should be 400 MW. Therefore,

$$\lambda = a_T P_{gT} + b_T = 2.813793 \times 10^{-3} \times 400 + 9.062069 = 10.187586 \text{ \$/MWh}$$

Using this plant λ, each unit's output is

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{10.187586 - 9}{0.012} = 98.9655 \text{ MW}$$

$$P_{g3} = \frac{\lambda - b_3}{a_3} = \frac{10.187586 - 8}{0.008} = 273.4483 \text{ MW}$$

$$P_{g4} = \frac{\lambda - b_4}{a_4} = \frac{10.187586 - 10}{0.0068} = 27.5862 \text{ MW}$$

$$P_{g2} \triangleq 400 \text{ MW}$$

It is seen that the outputs of Units 3 and 4 violate their respective upper and lower limits. Consequently it is concluded that other units besides Unit 2 need be operating at their limits if the output of Unit 2 is specified to be 400 MW.

This time assume that Unit 4 is operating at its lower limit of 110 MW. Using Units 1, 2 and 3 only, the plant λ is calculated as follows:

$$a_T = \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)^{-1} = \left(\frac{1}{0.012} + \frac{1}{0.0096} + \frac{1}{0.008}\right)^{-1} = 3.2 \times 10^{-3}$$

$$b_T = a_T\left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3}\right) = 3.2 \times 10^{-3}\left(\frac{9}{0.012} + \frac{6}{0.0096} + \frac{8}{0.008}\right) = 7.6$$

Since $P_{g4} = 110$ MW, the total output of Units 1, 2 and 3 should be 690 MW. Therefore,

$$\lambda = a_T P_{gT} + b_T = 3.2 \times 10^{-3} \times 690 + 7.6 = 9.808 \text{ \$/MWh}$$

Using this plant λ, each unit's output is

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{9.808 - 9}{0.012} = 67.3333 \text{ MW}$$

$$P_{g2} = \frac{\lambda - b_2}{a_2} = \frac{9.808 - 6}{0.0096} = 396.6667 \text{ MW}$$

$$P_{g3} = \frac{\lambda - b_3}{a_3} = \frac{9.808 - 8}{0.008} = 226 \text{ MW}$$

$$P_{g4} \triangleq 110 \text{ MW}$$

Apparently there are no limit violations here. Therefore, economic dispatch requires that the output of Unit 4 be set to its lower limit of 110 MW and that the outputs of the remaining units be those obtained above.

13.4 Solve Prob. 13.3 when the minimum load on Unit 4 is 50 MW rather than 110 MW.

Solution:

It was shown in Prob. 13.3 that if the output of Unit 2 is set to its maximum limit of 400 MW, some other units will also have to be operating at their limits. We now examine whether load limit constraints will be violated if Unit 4 is operating at its new lower limit of 50 MW.

Using Units 1, 2 and 3, the plant λ is calculated as follows:

$$a_T = \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)^{-1} = \left(\frac{1}{0.012} + \frac{1}{0.0096} + \frac{1}{0.008}\right)^{-1} = 3.2 \times 10^{-3}$$

$$b_T = a_T\left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3}\right) = 3.2 \times 10^{-3}\left(\frac{9}{0.012} + \frac{6}{0.0096} + \frac{8}{0.008}\right) = 7.6$$

Since $P_{g4} = 50$ MW, the total output of Units 1,2 and 3 should be 750 MW. Therefore,

$$\lambda = a_T P_{gT} + b_T = 3.2 \times 10^{-3} \times 750 + 7.6 = 10 \text{ \$/MWh}$$

Each unit's output is calculated as follows:

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{10 - 9}{0.012} = 83.3333 \text{ MW}$$

$$P_{g2} = \frac{\lambda - b_2}{a_2} = \frac{10 - 6}{0.0096} = 416.6667 \text{ MW}$$

$$P_{g3} = \frac{\lambda - b_3}{a_3} = \frac{10 - 8}{0.008} = 250 \text{ MW}$$

It is noted that the output of Unit 2 exceeds its maximum load limit.

It follows from the above analysis that both Units 2 and 4 should be operating at their upper and lower limits, respectively. Therefore, let $P_{g2} \triangleq 400$ MW and $P_{g4} \triangleq 50$ MW, and find the plant λ as follows:

$$a_T = \left(\frac{1}{a_1} + \frac{1}{a_3}\right)^{-1} = \left(\frac{1}{0.012} + \frac{1}{0.008}\right)^{-1} = 4.8 \times 10^{-3}$$

$$b_T = a_T\left(\frac{b_1}{a_1} + \frac{b_3}{a_3}\right) = 4.8 \times 10^{-3}\left(\frac{9}{0.012} + \frac{8}{0.008}\right) = 8.4$$

The plant λ in this case is

$$\lambda = a_T P_{gT} + b_T = 4.8 \times 10^{-3} \times 350 + 8.4 = 10.08 \text{ \$/MWh}$$

The outputs of Units 1 and 3 are calculated to be

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{1008 - 9}{0.012} = 90 \text{ MW}$$

$$P_{g3} = \frac{\lambda - b_3}{a_3} = \frac{1008 - 8}{0.008} = 260 \text{ MW}$$

In summary,

$$
\begin{array}{ll}
P_{g1} = 90 \text{ MW} & P_{g3} = 260 \text{ MW} \\
P_{g2} = 400 \text{ MW} & P_{g4} = 50 \text{ MW}
\end{array}
$$

13.5 The incremental fuel costs for two units of a plant are

$$\lambda_1 = \frac{df_1}{dP_{g1}} = 0.012P_{g1} + 8.0 \qquad \lambda_2 = \frac{df_2}{dP_{g2}} = 0.008P_{g2} + 9.6$$

where f is in dollars per hour and P_g is in megawatts. If both units operate at all times and maximum and minimum loads on each unit are 550 MW and 100 MW, plot λ of the plant in \$/MWh versus plant output in MW for economic dispatch as total load varies from 200 to 1100 MW.

Solution:

At their lower limit of 100 MW, the incremental costs of the units are calculated as

$$\frac{df_1}{dP_{g1}} = 0.012P_{g1} + 8.0 \Big|_{P_{g1}=100} = 9.2$$

$$\frac{df_2}{dP_{g2}} = 0.008P_{g2} + 9.6 \Big|_{P_{g2}=100} = 10.4$$

As the plant output exceeds 200 MW, initially the incremental fuel cost λ of the plant is determined by Unit 1 alone and the additional power should come from Unit 1. This will continue until the incremental fuel cost of Unit 1 becomes \$10.4/MWh, (i.e., $0.012P_{g1}+8.0 = 10.4$) from which the values of $P_{g1} = 200$ MW. Therefore, for $200 \le P_{gT} \le 300$,

$$\lambda = 0.012P_{g1} + 8.0 = 0.012(P_{gT} - 100) + 8.0 = 0.012P_{gT} + 6.8$$

For $P_{gT} > 300$, both units will increase their outputs simultaneously. To determine which unit will reach its upper limit first, we calculate incremental costs at the upper limit as follows:

$$\frac{df_1}{dP_{g1}}\Big|_{P_{g1}=550} = 0.012P_{g1} + 8.0 \Big|_{P_{g1}=550} = 14.6$$

$$\frac{df_2}{dP_{g2}}\Big|_{P_{g2}=550} = 0.008P_{g2} + 9.6 \Big|_{P_{g2}=550} = 14.0$$

The result shows that Unit 2 will reach its maximum load limit earlier than Unit 1. The value of P_{g1} for which the incremental cost becomes \$14.0/MWh is computed from $0.012P_{g1}+8.0 = 14.0$ which yields $P_{g1} = 500$ MW. For $300 \le P_{gT} \le 1050$, the plant λ is calculated. Since the incremental fuel costs of Units 1 and 2 should be the same, we have

$$0.012P_{g1} + 8.0 = 0.008P_{g2} + 9.6$$

from which $P_{g2} = 1.5P_{g1} - 200$. Since $P_{g1} + P_{g2} = P_{gT}$, P_{g1} can be represented in terms of P_{gT} as $P_{g1} + 1.5P_{g1} - 200 = P_{gT}$ from which $P_{g1} = 0.4P_{gT} + 80$. The plant λ is then given by

$$\lambda = 0.012P_{g1} + 8.0 = 0.012(0.4P_{gT} + 80) + 8.0 = 0.0048P_{gT} + 8.96$$

For $P_{gT} > 1050$, only Unit 1 will have an excess capacity, and the plant λ is determined by Unit 1 alone as

$$\lambda = 0.012P_{g1} + 8.0 = 0.012(P_{gT} - 550) + 8.0 = 0.012P_{gT} + 1.4$$

The results are summarized as follows:

$$
\begin{aligned}
\text{For} \quad 200 \leq P_{gT} \leq 300 \qquad & \lambda = 0.012P_{gT} + 6.8 \\
\text{For} \quad 300 \leq P_{gT} \leq 1050 \qquad & \lambda = 0.0048P_{gT} + 8.96 \\
\text{For} \quad 1050 \leq P_{gT} \leq 1100 \qquad & \lambda = 0.012P_{gT} + 1.4
\end{aligned}
$$

13.6 Find the savings in \$/h for economic dispatch of load between the units of Prob. 13.5 compared with their sharing the output equally when the total plant output is 600 MW.

Solution:

Economic dispatch for $P_{gT} = 600$ MW requires that

$$0.012P_{g1} + 8.0 = 0.008P_{g2} + 9.6 \quad \text{and} \quad P_{g1} + P_{g2} = 600$$

Solving the two equations for P_{g1} and P_{g2} yields

$$P_{g1} = 320 \text{ MW} \qquad P_{g2} = 280 \text{ MW}$$

The savings calculations are as follows:

$$
\begin{aligned}
\text{Net savings} &= \int_{320}^{300} \frac{df_1}{dP_{g1}} dP_{g1} + \int_{280}^{300} \frac{df_2}{dP_{g2}} dP_{g2} \\
&= \int_{320}^{300} (0.012P_{g1} + 8) dP_{g1} + \int_{280}^{300} (0.008P_{g2} + 9.6) dP_{g2} \\
&= \left. (0.006P_{g1}^2 + 8P_{g1}) \right|_{320}^{300} + \left. (0.004P_{g2}^2 + 9.6P_{g2}) \right|_{280}^{300} \\
&= -234.4 + 238.4 = 4.0 \text{ \$/h}
\end{aligned}
$$

13.7 A power system is supplied by three plants, all of which are operating on economic dispatch. At the bus of Plant 1 the incremental cost is \$10.0 per MWh, at Plant 2 it is \$9.0 per MWh and at Plant 3 it is \$11.0 per MWh. Which plant has the highest penalty factor and which one has the lowest penalty factor? Find the penalty factor of Plant 1 if the cost per hour to increase the total delivered load by 1 MW is \$12.0.

Solution:

Since the system λ should satisfy the equation

$$\lambda = 10.0L_1 = 9.0L_2 = 11.0L_3$$

it must be the case that $L_2 > L_1 > L_3$. Given the system λ of \$12.0/MWh, the penalty factor of plant 1, L_1, can be calculated from $10.0L_1 = 12.0$ from which we have

$$L_1 = 1.2$$

13.8 A power system has two generating plants and B-coefficients corresponding to Eq. (13.37) which are given in per unit on a 100 MVA base by

$$\left[\begin{array}{cc|c} 5.0 & -0.03 & 0.15 \\ -0.03 & 8.0 & 0.20 \\ \hline 0.15 & 0.20 & 0.06 \end{array}\right] \times 10^{-3}$$

The incremental fuel costs in \$/MWh of the generating units at the two plants are

$$\lambda_1 = \frac{df_1}{dP_{g1}} = 0.012P_{g1} + 6.6 \qquad \lambda_2 = \frac{df_2}{dP_{g2}} = 0.0096P_{g2} + 6.0$$

If Plant 1 presently supplies 200 MW and Plant 2 supplies 300 MW, find the penalty factors of each plant. Is the present dispatch most economical? If not, which plant output should be increased and which one should be decreased? Explain why.

Solution:

The power loss P_L is given by the equation

$$P_L = \left[\begin{array}{cc|c} P_{g1} & P_{g2} & 1 \end{array}\right] \left[\begin{array}{cc|c} 5 \times 10^{-3} & -0.03 \times 10^{-3} & 0.15 \times 10^{-3} \\ -0.03 \times 10^{-3} & 8 \times 10^{-3} & 0.2 \times 10^{-3} \\ \hline 0.15 \times 10^{-3} & 0.2 \times 10^{-3} & 0.06 \times 10^{-3} \end{array}\right] \left[\begin{array}{c} P_{g1} \\ P_{g2} \\ \hline 1 \end{array}\right]$$

$$= 5 \times 10^{-3}P_{g1}^2 - 2\left(0.03 \times 10^{-3}\right)P_{g1}P_{g2} + 8 \times 10^{-3}P_{g2}^2 + 0.15 \times 10^{-3}P_{g1}$$
$$+ 0.2 \times 10^{-3}P_{g2} + 0.06 \times 10^{-3}$$

where P_{g1} and P_{g2} are in per unit on the 100 MVA base. Penalty factors are calculated as

$$L_1 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g1}}} = \frac{1}{1 - \left\{2 \times \left(5 \times 10^{-3}\right)P_{g1} - 2\left(0.03 \times 10^{-3}\right)P_{g2} + 0.15 \times 10^{-3}\right\}\Big|_{\substack{P_{g1}=2 \\ P_{g2}=3}}}$$

$$= \frac{1}{1 - 19.97 \times 10^{-3}} = 1.0203769$$

$$L_2 = \frac{1}{1 - \frac{\partial P_L}{\partial P_{g2}}} = \frac{1}{1 - \left\{2 \times \left(8 \times 10^{-3}\right)P_{g2} - 2\left(0.03 \times 10^{-3}\right)P_{g1} + 0.2 \times 10^{-3}\right\}\Big|_{\substack{P_{g1}=2 \\ P_{g2}=3}}}$$

$$= \frac{1}{1 - 48.08 \times 10^{-3}} = 1.0505084$$

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The incremental fuel costs at the two plant buses are calculated to be

$$\left.\frac{df_1}{dP_{g1}}\right|_{P_{g1}=200} = 0.012 \times 200 + 6.6 = 9.0,$$

$$\left.\frac{df_2}{dP_{g2}}\right|_{P_{g2}=300} = 0.0096 \times 300 + 6.0 = 8.88$$

when penalty factors are incorporated into the incremental fuel costs, we have

$$L_1 \frac{df_1}{dP_{g1}} = 9.1833921 \qquad L_2 \frac{df_2}{dP_{g2}} = 9.3283146$$

Since $L_1 (df_1/dP_{g1})$ is smaller than $L_2 (df_2/dP_{g2})$, the output of plant 1 should be increased while that of plant 2 should be decreased to achieve economic dispatch.

13.9 Using \$10.0/MWh as the starting value of system λ in Example 13.4, perform the necessary calculations during the first iteration to obtain an updated λ.

Solution:

Step 1: $\overline{P_D = 5}$ per unit.

Step 2: $\lambda^{(1)} = 10.0$ is given.

Step 3: It follows from Eq. (13.43) that the following two equations should be solved for P_{g1} and P_{g2}.

$$(a_1 P_{g1} + b_1) - \lambda + \lambda (2B_{11}P_{g1} + 2B_{12}P_{g2} + B_{10}) = 0$$
$$(a_2 P_{g2} + b_2) - \lambda + \lambda (2B_{22}P_{g2} + 2B_{21}P_{g1} + B_{20}) = 0$$

which are rewritten as

$$(a_1 + 2\lambda B_{11}) P_{g1} + (2\lambda B_{12}) P_{g2} = (-b_1 + \lambda - \lambda B_{10})$$
$$(2\lambda B_{21}) P_{g1} + (a_2 + 2\lambda B_{22}) P_{g2} = (-b_2 + \lambda - \lambda B_{20})$$

Upon substituting proper values, we have

$$(0.8 + 2 \times 10 \times 8.383183 \times 10^{-3}) P_{g1} + (2 \times 10 \times (-0.049448) \times 10^{-3}) P_{g2} = -8 + 10 - 10 \times 0.750164 \times 10^{-3}$$

$$(2 \times 10 \times (-0.049448) \times 10^{-3}) P_{g1} + (0.96 + 2 \times 10 \times 5.963568 \times 10^{-3}) P_{g2} = -6.4 + 10 - 10 \times 0.38994 \times 10^{-3}$$

which are rewritten as

$$0.9676636 P_{g1} - 0.0009890 P_{g2} = 1.9924984$$
$$-0.0009890 P_{g1} + 1.0792714 P_{g2} = 3.5961006$$

The values of P_{g1} and P_{g2} which solve the above equations are

$$P_{g1} = 2.062489 \qquad P_{g2} = 3.333861$$

Step 4: The transmission loss is computed to be

$$
\begin{aligned}
P_L &= B_{11}P_{g1}^2 + 2B_{12}P_{g1}P_{g2} + B_{22}P_{g2}^2 + B_{10}P_{g1} + B_{20}P_{g2} + B_{00} \\
&= 8.383183 \times 10^{-3} \times (2.062489)^2 + 2 \times (-0.049448) \times 10^{-3} \times 2.062489 \times 3.333861 \\
&\quad + 5.963568 \times 10^{-3} \times (3.333861)^2 + 0.750164 \times 10^{-3} \times 2.062489 \\
&\quad + 0.38994 \times 10^{-3} \times 3.333861 + 0.090121 \times 10^{-3} = 0.104201 \text{ per unit}
\end{aligned}
$$

Step 5: Since $P_D = 5$ per unit,

$$
\begin{aligned}
P_D + P_L - \left(P_{g1}^{(1)} + P_{g2}^{(1)} \right) &= 5 + 0.104201 - (2.062489 + 3.333861) \\
&= -0.292148 \text{ per unit}
\end{aligned}
$$

The incremental change in λ is calculated from Eq. (13.58) as follows:

$$
\begin{aligned}
\Delta\lambda^{(1)} &= \left(\lambda^{(1)} - \lambda^{(0)} \right) \frac{P_D + P_L^{(1)} - \left(P_{g1}^{(1)} + P_{g2}^{(1)} \right)}{\left(P_{g1}^{(1)} + P_{g2}^{(1)} \right) - \left(P_{g1}^{(0)} + P_{g2}^{(0)} \right)} \\
&= (10 - 0)\frac{-0.292149}{5.396350 - 0} = -0.5413814
\end{aligned}
$$

and the updated λ becomes

$$
\lambda^{(2)} = \lambda^{(1)} + \Delta\lambda^{(1)} = 10 - 0.5413826 = 9.4586174
$$

13.10 Suppose that bus ② of a four-bus system is a generator bus and at the same time a load bus. By defining both a generation current and a load current at bus ② as shown in Fig. 13.5c, find the transformation matrix **C** for this case in the form shown in Eq. (13.31).

Solution:

Let the generator and load currents at bus ② be denoted by I_2^g and I_2^d, respectively. The total system load current is given by

$$
I_D = I_2^d + I_3 + I_4
$$

Constants d_2, d_3 and d_4 are then obtained to be

$$
d_2 = \frac{I_2^d}{I_D} \qquad d_3 = \frac{I_3}{I_D} \qquad d_4 = \frac{I_4}{I_D}
$$

Since the net current injection at bus ②, I_2, is $I_2^g + I_2^d$, from Eq. (13.23) we have

$$
\begin{aligned}
V_{1n} &= Z_{11}I_1 + Z_{12}\left(I_2^g + I_2^d \right) + Z_{13}I_3 + Z_{14}I_4 \\
&= Z_{11}I_1 + Z_{12}I_2^g + (d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}) I_D
\end{aligned}
$$

from which we also get

$$
\begin{aligned}
I_D &= \frac{-Z_{11}}{d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}} I_1 + \frac{-Z_{12}}{d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}} I_2^g + \frac{-Z_{11}}{d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}} I_n^0 \\
&\triangleq -t_1 I_1 - t_2 I_2^g - t_1 I_n^0
\end{aligned}
$$

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where $I_n^0 = -V_{1n}/Z_{11}$. Now the load currents can be represented in terms of generator currents and the no-load current as follows:

$$I_2^d = -d_2t_1I_1 - d_2t_2I_2^g - d_2t_1I_n^0$$
$$I_3 = -d_3t_1I_1 - d_3t_2I_2^g - d_3t_1I_n^0$$
$$I_4 = -d_4t_1I_1 - d_4t_2I_2^g - d_4t_1I_n^0$$

The transformation C of old currents I_1, $I_2^g + I_2^d$, I_3 and I_4 to the generator and no-load currents I_1, I_2^g and I_n^0 is defined as

$$
\begin{bmatrix} I_1 \\ I_2^g + I_2^d \\ I_3 \\ I_4 \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 \\
-d_2t_1 & 1 - d_2t_2 & -d_2t_1 \\
-d_3t_1 & d_3t_2 & -d_3t_1 \\
-d_4t_1 & d_4t_2 & -d_4t_1
\end{bmatrix}
\begin{bmatrix} I_1 \\ I_2^g \\ I_n^0 \end{bmatrix}
\triangleq \mathbf{C}
\begin{bmatrix} I_1 \\ I_2^g \\ I_n^0 \end{bmatrix}
$$

13.11 The four-bus system depicted in Fig. 13.5 has bus and line data given in Table 13.2. Suppose that the bus data is slightly modified such that at bus ②, P-generation is 4.68 per unit, and P-load and Q-load are 1.5 per unit and 0.9296 per unit, respectively. Using the results of Table 13.3, find the power-flow solution corresponding to this modified bus data. Using the solution to Prob. 13.10, also find the B-coefficients of this modified problem in which there is load as well as generation at bus ②.

Solution:

The power flow solution should be the same as that of Table 13.3 except that P- and Q-generation at bus ② needs to be modified to account for the load at that bus. Using P- and Q-generation of 3.8 per unit and 1.325439 per unit as shown in Table 13.3, and adding to those P- and Q-load at bus ②, we get

$$P_{g2} = 3.18 + 1.5 = 4.68 \text{ per unit}$$
$$Q_{g2} = 1.325439 + 0.9296 = 2.255039 \text{ per unit}$$

The bus voltages and P- and Q-generation at bus ① should remain the same. The load currents are calculated from the power-flow results as follows:

$$I_2^d = \frac{P_2^d - jQ_2^d}{V_2^*} = \frac{-1.5 + j0.9296}{1.0 \angle -2.43995°} = -1.5382150 + j0.8648990$$
$$I_3 = \frac{P_3 - jQ_3}{V_3^*} = \frac{-2.2 + j1.36340}{0.96051 \angle 1.07932°} = -2.2633193 + j1.4623529$$
$$I_4 = \frac{P_4 - jQ_4}{V_4^*} = \frac{-2.8 + j1.73520}{0.94304 \angle 2.62658°} = -2.881685 + j1.9741431$$

Constants d_2, d_3 and d_4 are then found to be

$$d_2 = \frac{I_2^d}{I_2^d + I_3 + I_4} = 0.2216412 + j0.0132372$$
$$d_3 = \frac{I_3}{I_2^d + I_3 + I_4} = 0.3390421 - j0.0005983$$
$$d_4 = \frac{I_4}{I_2^d + I_3 + I_4} = 0.4393167 - j0.0126389$$

Constants t_1 and t_2 defined in Prob. 13.10 are calculated to be

$$t_1 = \frac{Z_{11}}{d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}} = 0.9930664 + j0.0013435$$

$$t_2 = \frac{Z_{12}}{d_2 Z_{12} + d_3 Z_{13} + d_4 Z_{14}} = 1.0020780 - j0.0004610$$

Using the constants d_2, d_3, d_4, t_1 and t_2, the transformation **C** defined in Prob. 13.10 is obtained:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ -0.2200866 - j0.0134432 & 0.7778921 - j0.0131625 & -0.2200866 - j0.0134432 \\ -0.3366921 + j0.0001386 & -0.3397464 + j0.0007558 & -0.3366921 + j0.0001386 \\ -0.4362876 + j0.0119610 & -0.4402238 + j0.0128677 & -0.4362876 + j0.0119610 \end{bmatrix}$$

Using $\mathbf{R}_{bus}$ given in Example 13.3, we then find

$$\mathbf{C}^T \mathbf{R}_{bus} \mathbf{C}^* = \begin{bmatrix} 4.543134 + j0 & -0.892927 - j0.076535 & 0.938793 - j0.023045 \\ -0.892927 + j0.076535 & 3.075414 + j0 & 0.194991 + j0.054548 \\ 0.938793 + j0.023045 & 0.194991 - j0.054548 & 0.246415 + j0 \end{bmatrix} \times 10^{-3}$$

The no-load current is calculated, as given in Example 13.3, to be

$$I_n^0 = \frac{-V_1}{Z_{11}} = -0.000436 - j0.387164$$

Also using the power-flow solution, we have

$$\alpha_1 = \frac{1 - js_1}{V_1^*} = \frac{1 - j\left(\frac{1.872240}{1.913152}\right)}{1.0 \angle 0°} = 1.0 - j0.978615$$

$$\alpha_2 = \frac{1 - js_2}{V_2^*} = \frac{1 - j\left(\frac{2.255039}{4.68}\right)}{1.0 \angle -2.43995°} = 1.0196070 - j0.4388369$$

The matrix $\mathbf{T}_\alpha$ of Eq. (13.36) is then calculated to be

$$\mathbf{T}_\alpha = \begin{bmatrix} 8.894036 + j0 & -1.336685 + j0.388211 & 0.364217 + j0.355146 \\ -1.336685 - j0.388211 & 3.789452 + j0 & 0.011499 + j0.086254 \\ 0.364217 - j0.355146 & 0.011499 - j0.086254 & 0.036937 + j0 \end{bmatrix} \times 10^{-3}$$

The B-coefficients are the real parts of the matrix $\mathbf{T}_\alpha$. Finally, the power loss is calculated as follows:

$$P_L = \begin{bmatrix} 1.913152 & 4.68 & | & 1 \end{bmatrix} \begin{bmatrix} \mathbf{B} \end{bmatrix} \begin{bmatrix} 1.913152 \\ 4.68 \\ \hline 1 \end{bmatrix} = 0.093153 \text{ per unit}$$

13.12 Three generating units operating in parallel at 60 Hz have ratings of 300 MW, 500 MW, and 600 MW and have speed-droop characteristics of 5%, 4% and 3%, respectively. Due to a change in load, an increase in system frequency of 0.3 Hz is experienced before any supplementary control action occurs. Determine the amount of the change in system load, and also the amount of the change in generation of each unit to absorb the load change.

Solution:

Using Eq. (13.65), the change in the system load is calculated to be

$$\Delta P = -\left(\frac{S_{R1}}{R_{1u}} + \frac{S_{R2}}{R_{2u}} + \frac{S_{R3}}{R_{3u}}\right)\frac{\Delta f}{f_R}$$

$$= -\left(\frac{300}{0.05} + \frac{500}{0.04} + \frac{600}{0.03}\right)\frac{0.3}{60} = -192.5 \text{ MW}$$

The change in the outputs of the units can be calculated by Eq. (13.63) or Eq. (13.67) as

$$\Delta P_{g1} = -\frac{S_{R1}}{R_{1u}}\frac{\Delta f}{f_R} = -\frac{300}{0.05}\frac{0.3}{60} = -30 \text{ MW}$$

$$\Delta P_{g2} = -\frac{S_{R2}}{R_{2u}}\frac{\Delta f}{f_R} = -\frac{500}{0.04}\frac{0.3}{60} = -62.5 \text{ MW}$$

$$\Delta P_{g3} = -\frac{S_{R3}}{R_{3u}}\frac{\Delta f}{f_R} = -\frac{600}{0.03}\frac{0.3}{60} = -100 \text{ MW}$$

13.13 A 60-Hz system consisting of the three generating units described in Prob. 13.12 is connected to a neighboring system via a tie line. Suppose that a generator in the neighboring system is forced out of service, and that the tie-line flow is observed to increase from the scheduled value of 400 MW to 631 MW. Determine the amount of the increase in generation of each of the three units and find the ACE of this system whose frequency-bias setting is −58 MW/0.1 Hz.

Solution:

The increase in the total generation resulted in the increase in the tie line flow by 231 MW. Therefore, it follows from Eq. (13.65) that

$$231 = -\left(\frac{300}{0.05} + \frac{500}{0.04} + \frac{600}{0.03}\right)\frac{\Delta f}{60}$$

from which we have

$$\Delta f = -231 \times 60 = -0.36 \text{ Hz}$$

The three units should have increased their outputs according to Eq. (13.63) as follows:

$$\Delta P_{g1} = -\frac{300}{0.05}\frac{(-0.36)}{60} = 36 \text{ MW}$$

$$\Delta P_{g2} = -\frac{500}{0.04}\frac{(-0.36)}{60} = 75 \text{ MW}$$

$$\Delta P_{g3} = -\frac{600}{0.03}\frac{(-0.36)}{60} = 120 \text{ MW}$$

The ACE of the system is determined from Eq. (13.68) as

$$ACE = (631 - 400) - 10(-58)(-0.36) = 22.2 \text{ MW}$$

13.14 Suppose that it takes 5 minutes for the AGC of the power system of Prob. 13.13 to command the three units to increase their generation to restore system fre-

quency to 60 Hz. What is the time error in seconds incurred during this 5-minute period? Assume that the initial frequency deviation remains the same throughout this restoration period.

Solution:

The frequency error in per unit is

$$\frac{-0.36}{60} = -6 \times 10^{-3} \text{ per unit}$$

Therefore, the time error incurred during the 5-minute period is

$$-6 \times 10^{-3} \times 5 \times 60 = -1.8 \text{ s}$$

13.15 Solve Example 13.8 when the system load level is 1300 MW.

Solution:

Among the four combinations x_1, x_2, x_3 and x_9, combination x_9 is infeasible since the total generation from units 1 and 2 cannot exceed 1250 MW.

(i) Combination x_3:

With units 1, 2 and 4 operating, we use Eqs. (13.7) and (13.8) to calculate the coefficients

$$a_T = \left(a_1^{-1} + a_2^{-1} + a_4^{-1}\right)^{-1} = \left(0.008^{-1} + 0.0096^{-1} + 0.011^{-1}\right)^{-1} = 3.1243 \times 10^{-3}$$

$$b_T = a_t\left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_4}{a_4}\right) = a_T\left(\frac{8}{0.008} + \frac{6.4}{0.0096} + \frac{7.5}{0.011}\right) = 7.3374$$

The incremental fuel cost at the load level of 1300 MW is

$$\lambda = a_T P_{gT} + b_T = 3.1243 \times 10^{-3} \times 1300 + 7.3374 = 11.3990$$

The corresponding economic dispatch outputs are

$$P_{g1} = \frac{\lambda - b_1}{a_1} = \frac{11.3990 - 8.0}{0.008} = 424.88 \cong 425 \text{ MW}$$

$$P_{g2} = \frac{\lambda - b_2}{a_2} = \frac{11.3990 - 6.4}{0.0096} = 520.73 \cong 521 \text{ MW}$$

$$P_{g4} = \frac{\lambda - b_4}{a_4} = \frac{11.3990 - 7.5}{0.011} = 354.45 \cong 354 \text{ MW}$$

The hourly production costs of the three units are calculated to be

$$f_1 = \left. 0.004P_{g1}^2 + 8.0P_{g1} + 500 \right|_{P_{g1}=425} = 4623 \text{ \$/h}$$

$$f_2 = \left. 0.0048P_{g2}^2 + 6.4P_{g2} + 400 \right|_{P_{g2}=521} = 5037 \text{ \$/h}$$

$$f_4 = \left. 0.0055P_{g4}^2 + 7.5P_{g4} + 400 \right|_{P_{g4}=354} = 3744 \text{ \$/h}$$

(ii) Combination x_2:

Using the values of a_T and b_T found in Example 13.8, λ is calculated to be

$$\lambda = a_T P_{gT} + b_T = 3.038 \times 10^{-3} \times 1350 + 7.4634 = 11.4128$$

The corresponding economic dispatch outputs are

$$P_{g\,1} = \frac{\lambda - b_1}{a_1} = \frac{11.4128 - 8.0}{0.008} = 426.60 \cong 427 \text{ MW}$$

$$P_{g\,2} = \frac{\lambda - b_2}{a_2} = \frac{11.4128 - 6.4}{0.0096} = 522.17 \cong 522 \text{ MW}$$

$$P_{g\,3} = \frac{\lambda - b_3}{a_3} = \frac{11.4128 - 7.9}{0.01} = 351.28 \cong 351 \text{ MW}$$

The hourly production costs of the three units are calculated to be

$$f_1 = 0.004 P_{g\,1}^2 + 8.0 P_{g\,1} + 500 \Big|_{P_{g\,1}=427} = 4645 \text{ \$/h}$$

$$f_2 = 0.0048 P_{g\,2}^2 + 6.4 P_{g\,2} + 400 \Big|_{P_{g\,2}=522} = 5049 \text{ \$/h}$$

$$f_3 = 0.005 P_{g\,3}^2 + 7.9 P_{g\,3} + 600 \Big|_{P_{g\,3}=351} = 3989 \text{ \$/h}$$

(iii) Combination x_1:

Using the values of a_T and b_T found in Example 13.8, λ is calculated to be

$$\lambda = a_T P_{gT} + b_T = 2.3805 \times 10^{-3} \times 1300 + 7.4712 = 10.56585$$

The corresponding economic dispatch outputs are

$$P_{g\,1} = \frac{\lambda - b_1}{a_1} = \frac{10.56585 - 8.0}{0.008} = 320.73 \cong 321 \text{ MW}$$

$$P_{g\,2} = \frac{\lambda - b_2}{a_2} = \frac{10.56585 - 6.4}{0.0096} = 433.94 \cong 434 \text{ MW}$$

$$P_{g\,3} = \frac{\lambda - b_3}{a_3} = \frac{10.56585 - 7.9}{0.01} = 266.59 \cong 266 \text{ MW}$$

$$P_{g\,4} = \frac{\lambda - b_4}{a_4} = \frac{10.56585 - 7.5}{0.11} = 278.71 \cong 279 \text{ MW}$$

The hourly production costs of the four units are calculated to be

$$f_1 = 0.004 P_{g\,1}^2 + 8.0 P_{g\,1} + 500 \Big|_{P_{g\,1}=321} = 3480 \text{ \$/h}$$

$$f_2 = 0.0048 P_{g\,2}^2 + 6.4 P_{g\,2} + 400 \Big|_{P_{g\,2}=434} = 4082 \text{ \$/h}$$

$$f_3 = 0.005 P_{g\,3}^2 + 7.9 P_{g\,3} + 600 \Big|_{P_{g\,3}=266} = 3055 \text{ \$/h}$$

$$f_4 = 0.0055 P_{g\,4}^2 + 7.5 P_{g\,4} + 400 \Big|_{P_{g\,4}=279} = 2921 \text{ \$/h}$$

13.16 If the start-up costs of the four units of Example 13.9 are changed to \$2500, \$3000, \$3400, and \$2600, and the shut-down costs are changed to \$1500, \$1200, \$1000, and \$1400, respectively, find the optimal unit commitment policy. Assume that all other conditions remain unchanged.

Solution:

At stage 6, the condition remains the same. Therefore,

$$F_9(6) = \$45,868$$

At stage 5, by using different transition costs T, we have

$$F_1(5) = \{P_1(5) + T_{1,9}(5) + F_9(6)\} = [58,428 + 1000 + 1400 + 45,868] = \$106,696$$
$$F_2(5) = \{P_2(5) + T_{2,9}(5) + F_9(6)\} = [59,356 + 1000 + 45,868] = \$106,224$$
$$F_3(5) = \{P_3(5) + T_{3,9}(5) + F_9(6)\} = [58,236 + 1400 + 45,868] = \$105,504$$

At stage 4,

$$
\begin{aligned}
F_1(4) &= \min [P_1(4) + T_{1,1}(4) + F_1(5); P_1(4) + T_{1,2}(4) + F_2(5); P_1(4) + T_{1,3}(4) + F_3(5)] \\
&= \min [76,472 + 0 + 106,696; 76,472 + 1400 + 106,224; 76,472 + 1000 + 105,504] \\
&= \min [183,168; 184,096; 182,976] = \$182,976 \\
F_2(4) &= \min [P_2(4) + T_{2,1}(4) + F_1(5); P_2(4) + T_{2,2}(4) + F_2(5); P_2(4) + T_{2,3}(4) + F_3(5)] \\
&= \min [79,184 + 2600 + 106,696; 79,184 + 0 + 106,224; 79,184 + 1000 + 2600 + 105,504] \\
&= \min [188,480; 185,408; 188,288] = \$185,408
\end{aligned}
$$

At stage 3,

$$
\begin{aligned}
F_1(3) &= \min [P_1(3) + T_{1,1}(3) + F_1(4); P_1(3) + T_{1,2}(3) + F_2(4)] \\
&= \min [70,908 + 0 + 182,976; 70,908 + 1400 + 185,408] \\
&= \min [253,884; 257,716] = \$253,884 \\
F_2(3) &= \min [P_2(3) + T_{2,1}(3) + F_1(4); P_2(3) + T_{2,2}(3) + F_2(4)] \\
&= \min [68,976 + 2600 + 182,976; 68,976 + 0 + 185,408] \\
&= \min [254,552; 254,384] = \$254,384 \\
F_3(3) &= \min [P_3(3) + T_{3,1}(3) + F_1(4); P_3(3) + T_{3,2}(3) + F_2(4)] \\
&= \min [67,856 + 3400 + 182,976; 67,856 + 3400 + 1400 + 185,408] \\
&= \min [254,232; 258,064] = \$254,232
\end{aligned}
$$

At stage 2,

$$
\begin{aligned}
F_1(2) &= \min [P_1(2) + T_{1,1}(2) + F_1(3); P_1(2) + T_{1,2}(2) + F_2(3); P_1(2) + T_{1,3}(2) + F_3(3)] \\
&= \min [58,428 + 0 + 253,884; 58,428 + 1400 + 254,384; 58,428 + 1000 + 254,232] \\
&= \min [312,312; 314,212; 313,660] = \$312,312 \\
F_2(2) &= \min [P_2(2) + T_{2,1}(2) + F_1(3); P_2(2) + T_{2,2}(2) + F_2(3); P_2(2) + T_{2,3}(2) + F_3(3)] \\
&= \min [59,356 + 2600 + 253,884; 59,356 + 0 + 254,384; 59,356 + 1000 + 2600 + 254,232] \\
&= \min [315,840; 313,740; 317,188] = \$313,740 \\
F_3(2) &= \min [P_3(2) + T_{3,1}(2) + F_1(3); P_3(2) + T_{3,2}(2) + F_2(3); P_3(2) + T_{3,3}(2) + F_3(3)] \\
&= \min [58,236 + 3400 + 253,884; 58,236 + 3400 + 1400 + 254,384; 58,236 + 0 + 254,232] \\
&= \min [315,520; 317,420; 312,468] = \$312,468
\end{aligned}
$$

At stage 1,

$$F_9(1) = \min[P_9(1) + T_{9,1}(1) + F_1(2); P_9(1) + T_{9,2}(1) + F_2(2); P_9(1) + T_{9,3}(1) + F_3(2)]$$
$$= \min[45,868 + 3400 + 2600 + 312,312; 45,868 + 3400 + 313,740; 45,868 + 2600 + 312,468]$$
$$= \min[364,180; 363,008; 360,936] = \$360,936$$

When the least cost path is retraced, the optimal unit commitment is found to be the same as that of Example 13.9. The associated total operating cost in this case is \$360,936, which is \$600 less than \$361,536 obtained in Example 13.9.

13.17 Due to a 400 MW short-term purchase request from the neighboring utility, the demand during the second interval of the day is expected to increase from 1400 MW to 1800 MW for the system described in Example 13.9. Assuming that other conditions remain unchanged, find the optimal unit commitment policy and the associated total operating cost for the day.

Solution:

In applying dynamic programming to this problem, the process up to stage 3 should be the same as that given in Example 13.9. At stage 2 no other combinations besides x_1 and x_2 have sufficient capacity to serve the increased load of 1800 MW; therefore, we only have to consider combinations x_1 and x_2. It was found at stage 4 that to serve the load of 1800 MW, the minimum production costs of combinations x_1 and x_2 would be \$76,472 and \$79,184, respectively. Stage 2 can now be handled as follows:

$$F_1(2) = \min[P_1(2) + T_{1,1}(2) + F_1(3); P_1(2) + T_{1,2}(2) + F_2(3); P_1(2) + T_{1,3}(2) + F_3(3)]$$
$$= \min[76,472 + 0 + 254,484; 76,472 + 1500 + 254,884; 76,472 + 1500 + 254,432]$$
$$= \min[330,956; 332,856; 332,404] = \$330,956$$

$$F_2(2) = \min[P_2(2) + T_{2,1}(2) + F_1(3); P_2(2) + T_{2,2}(2) + F_2(3); P_2(2) + T_{2,3}(2) + F_3(3)]$$
$$= \min[79,184 + 3000 + 254,484; 79,184 + 0 + 254,884; 79,184 + 4500 + 254,432]$$
$$= \min[336,668; 334,068; 338,116] = \$334,068$$

At stage 1,

$$F_9(1) = \min[P_9(1) + T_{9,1}(1) + F_1(2); P_9(1) + T_{9,2}(1) + F_2(2)]$$
$$= \min[45,868 + 6000 + 330,956; 45,868 + 3000 + 334,068]$$
$$= \min[382,824; 382,936] = \$382,824$$

The optimal unit commitment is found by retracing the least cost path in the forward direction, and is given by

Stage	Optimal combination	Load level
1	x_9	1100 MW
2	x_1	1800 MW
3	x_1	1600 MW
4	x_1	1800 MW
5	x_3	1400 MW
6	x_9	1100 MW

From $F_9(1)$, the total operating cost is found to be \$382,824.

13.18 Suppose Unit 4 of Example 13.9 will have to be taken off line for 8 hours beginning at the fifth interval of the day to undergo minor repair work. Determine the optimal unit commitment policy to serve the system load of Fig. 13.11 and the increase in the operating cost for the day.

Solution:

Combinations x_1 or x_3 are no longer a viable option during the 5th and 6th intervals. Therefore, the only $F_i(k)$ that needs to be evaluated at stage 5 is $F_2(5) = 106,724$ which was given in Figure 13.14. Since x_2 is the only combination available at stage 5, $F_i(4)$, for $i = 1$ and $i = 2$, are to be recalculated (using the already available information in Figure 13.14) as

$$F_1(4) = 184,696 \qquad F_2(4) = 185,908$$

Now, recalculation of $F_i(k)$ is required from stage 3 onwards. At stage 3,

$$
\begin{aligned}
F_1(3) &= \min\left[P_1(3) + T_{1,1}(3) + F_1(4); P_1(3) + T_{1,2}(3) + F_2(4)\right] \\
&= \min\left[70,908 + 0 + 184,696; 70,908 + 1500 + 185,908\right] \\
&= \min\left[255,604; 258,316\right] = \$255,604 \\
F_2(3) &= \min\left[P_2(3) + T_{2,1}(3) + F_1(4); P_2(3) + T_{2,2}(3) + F_2(4)\right] \\
&= \min\left[68,976 + 3000 + 184,696; 68,976 + 0 + 185,908\right] \\
&= \min\left[256,672; 254,884\right] = \$254,884 \\
F_3(3) &= \min\left[P_3(3) + T_{3,1}(3) + F_1(4); P_3(3) + T_{3,2}(3) + F_2(4)\right] \\
&= \min\left[67,856 + 3000 + 184,696; 67,856 + 4500 + 185,908\right] \\
&= \min\left[255,552; 258,264\right] = \$255,552
\end{aligned}
$$

At stage 2,

$$
\begin{aligned}
F_1(2) &= \min\left[P_1(2) + T_{1,1}(2) + F_1(3); P_1(2) + T_{1,2}(2) + F_2(3); P_1(2) + T_{1,3}(2) + F_3(3)\right] \\
&= \min\left[58,428 + 0 + 255,604; 58,428 + 1500 + 254,884; 58,428 + 1500 + 255,552\right] \\
&= \min\left[314,032; 314,812; 315,480\right] = \$314,032 \\
F_2(2) &= \min\left[P_2(2) + T_{2,1}(2) + F_1(3); P_2(2) + T_{2,2}(2) + F_2(3); P_2(2) + T_{2,3}(2) + F_3(3)\right] \\
&= \min\left[59,356 + 3000 + 255,604; 59,356 + 0 + 254,884; 59,356 + 4500 + 255,552\right] \\
&= \min\left[317,960; 314,240; 319,408\right] = \$314,240 \\
F_3(2) &= \min\left[P_3(2) + T_{3,1}(2) + F_1(3); P_3(2) + T_{3,2}(2) + F_2(3); P_3(2) + T_{3,3}(2) + F_3(3)\right] \\
&= \min\left[58,236 + 3000 + 255,604; 58,236 + 4500 + 254,884; 59,236 + 0 + 255,552\right] \\
&= \min\left[316,840; 317,620; 313,788\right] = \$313,788
\end{aligned}
$$

At stage 1,

$$
\begin{aligned}
F_9(1) &= \min\left[P_9(1) + T_{9,1}(1) + F_1(2); P_9(1) + T_{9,2}(1) + F_2(2); P_9(1) + T_{9,3}(1) + F_3(2)\right] \\
&= \min\left[45,868 + 6000 + 314,032; 45,868 + 3000 + 314,240; 45,868 + 3000 + 313,788\right] \\
&= \min\left[365,900; 363,108; 362,656\right] = \$362,656
\end{aligned}
$$

Tracing the process in the forward direction, we find the optimal policy as $(x_9\ x_3\ x_3\ x_1\ x_2\ x_9)$ from stage 1 to stage 6. The increase in the operating cost is

$$362,656 - 361,536 = \$1120$$

13.19 A diagram similar to Fig. 13.14 is shown in Fig. 13.15 in which directed branches represent transitions from one state, represented by a node, to another. Associated with each directed branch (i, j) is the cost $f_{ij}(k)$, as defined in Eq. (13.72). The values of $f_{ij}(k)$ are given in Table 13.7. Note that index k of $f_{ij}(k)$ does not play any role here, and consequently will now be omitted. If the value of f_{ij} is interpreted to be the distance between states i and j, then the unit commitment problem becomes that of finding the *shortest path* from the origin, represented by node ①, to the destination, represented by node ⑩. The problem of this nature is called the *stagecoach problem*. Write the backward recurrence equation similar to Eq. (13.75), and solve the problem by commencing calculations at the destination and then moving toward the origin.

In forward recurrence the process starts with the origin and moves toward the destination. Write the forward recurrence equation, solve the problem and check the result with that of the backward dynamic programming procedure.

Table 13.7 Matrix of costs (or distances) f_{ij} between states (or nodes) ⓘ and ⓙ of Fig. 13.15

		②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
	①	20	15	17						
	②				35	31	38			
	③				39	42	33			
	④				36	40	34			
ⓘ	⑤							26	22	
	⑥							29	25	
	⑦							41	44	
	⑧									15
	⑨									18

(column header ⓙ spans columns ② through ⑩)

Solution:

The backward recurrence equation can be written as

$$F_i(N - 1) = \min_j \{f_{ij} + F_j(N)\}$$

Starting with node 10, the destination, we have at stage 5,

$$F_{10}(5) = 0$$

At stage 4,

$$F_8(4) \;=\; \min[f_{8\,10} + F_{10}(5)] \;=\; \min[15 + 0] \;=\; 15$$
$$F_9(4) \;=\; \min[f_{9\,10} + F_{10}(5)] \;=\; \min[18 + 0] \;=\; 18$$

At stage 3,

$$F_5(3) \;=\; \min[f_{58} + F_8(4); f_{59} + F_9(4)] \;=\; \min[26 + 15; 22 + 18]$$
$$\;=\; \min[41; 40] \;=\; 40$$
$$F_6(3) \;=\; \min[f_{68} + F_8(4); f_{69} + F_9(4)] \;=\; \min[29 + 15; 25 + 18]$$
$$\;=\; \min[44; 43] \;=\; 43$$
$$F_7(3) \;=\; \min[f_{78} + F_8(4); f_{79} + F_9(4)] \;=\; \min[41 + 15; 44 + 18]$$
$$\;=\; \min[56; 62] \;=\; 56$$

At stage 2,

$$F_2(2) \;=\; \min[f_{25} + F_5(3); f_{26} + F_6(3); f_{27} + F_7(3)] \;=\; \min[35 + 40; 31 + 43; 38 + 56]$$
$$\;=\; \min[75; 74; 94] \;=\; 74$$
$$F_3(2) \;=\; \min[f_{35} + F_5(3); f_{36} + F_6(3); f_{37} + F_7(3)] \;=\; \min[39 + 40; 42 + 43; 33 + 56]$$
$$\;=\; \min[79; 85; 89] \;=\; 79$$
$$F_4(2) \;=\; \min[f_{45} + F_5(3); f_{46} + F_6(3); f_{47} + F_7(3)] \;=\; \min[36 + 40; 40 + 43; 34 + 56]$$
$$\;=\; \min[76; 83; 90] \;=\; 76$$

At stage 1,

$$F_1(1) \;=\; \min[f_{12} + F_2(2); f_{13} + F_3(2); f_{14} + F_4(2)] \;=\; \min[20 + 74; 15 + 79; 17 + 76]$$
$$\;=\; \min[94; 94; 93] \;=\; 93$$

Retracing the path in the forward direction, the shortest path is found to be

$$1 \rightarrow 4 \rightarrow 5 \rightarrow 9 \rightarrow 10$$

and the shortest distance is 93.

The forward recurrence equation can be written as

$$F_j(N) \;=\; \min_i \{f_{ij} + F_i(N-1)\}$$

Starting with node 1, we have at stage 1,

$$F_1(1) \;=\; 0$$

At stage 2,

$$F_2(2) \;=\; \min[f_{12} + F_1(1)] \;=\; \min[20 + 0] \;=\; 20$$
$$F_3(2) \;=\; \min[f_{13} + F_1(1)] \;=\; \min[15 + 0] \;=\; 15$$
$$F_4(2) \;=\; \min[f_{14} + F_1(1)] \;=\; \min[17 + 0] \;=\; 17$$

At stage 3,

$$F_5(3) \;=\; \min[f_{25} + F_2(2); f_{35} + F_3(2); f_{45} + F_4(2)]$$
$$\;=\; \min[35 + 20; 39 + 15; 36 + 17] \;=\; \min[55; 54; 53] \;=\; 53$$
$$F_6(3) \;=\; \min[f_{26} + F_2(2); f_{36} + F_3(2); f_{46} + F_4(2)]$$
$$\;=\; \min[31 + 20; 42 + 15; 40 + 17] \;=\; \min[51; 57; 57] \;=\; 51$$
$$F_7(3) \;=\; \min[f_{27} + F_2(2); f_{37} + F_3(2); f_{47} + F_4(2)]$$
$$\;=\; \min[38 + 20; 33 + 15; 34 + 17] \;=\; \min[58; 58; 51] \;=\; 51$$

At stage 4,

$$F_8(4) = \min\left[f_{58} + F_5(3); f_{68} + F_6(3); f_{78} + F_7(3)\right]$$
$$= \min[26 + 53; 29 + 51; 41 + 51] = \min[79; 80; 92] = 79$$
$$F_9(4) = \min\left[f_{59} + F_5(3); f_{69} + F_6(3); f_{79} + F_7(3)\right]$$
$$= \min[22 + 53; 25 + 51; 44 + 51] = \min[75; 76; 95] = 75$$

At stage 5,

$$F_{10}(5) = \min\left[f_{8\,10} + F_8(4); f_{9\,10} + F_9(4)\right]$$
$$= \min[15 + 79; 18 + 75] = \min[94; 93] = 93$$

Retracing the process from node 10 in the backward direction, the shortest path is identified to be

$$10 \rightarrow 9 \rightarrow 5 \rightarrow 4 \rightarrow 1$$

and the distance of the corresponding path is 93. This result is identical to that by the backward dynamic programming procedure.

Chapter 14 Problem Solutions

14.1 A four-bus system with $\mathbf{Z_{bus}}$ given in per unit by

$$
\begin{array}{c}
\begin{array}{cccc} \quad\;① & \quad② & \quad③ & \quad④ \end{array}\\
\begin{array}{c}①\\②\\③\\④\end{array}
\begin{bmatrix}
j0.041 & j0.031 & j0.027 & j0.018\\
j0.031 & j0.256 & j0.035 & j0.038\\
j0.027 & j0.035 & j0.158 & j0.045\\
j0.018 & j0.038 & j0.045 & j0.063
\end{bmatrix}
\end{array}
$$

has bus voltages $V_1 = 1.0\angle 0°$, $V_2 = 0.98\angle 0°$, $V_3 = 0.96\angle 0°$ and $V_4 = 1.04\angle 0°$. Using the compensation current method, determine the change in voltage at bus ② due to the outage of line ①–③ with series impedance $j0.3$ per unit.

Solution:

$$
\mathbf{Z}_{bus}^{(1-3)} = \mathbf{Z_{bus}}
\begin{bmatrix} 1\\ 0\\ -1\\ 0 \end{bmatrix}
=
\begin{bmatrix}
j(0.041 - 0.027)\\
j(0.031 - 0.035)\\
j(0.027 - 0.158)\\
j(0.018 - 0.045)
\end{bmatrix}
=
\begin{bmatrix}
j0.014\\
-j0.004\\
-j0.131\\
-j0.027
\end{bmatrix}
$$

By Eq. (14.14),

$$I_{13} = \frac{(V_1 - V_3)}{Z_{\text{th},13} + z_{13}} = \frac{(1.0 - 0.96)}{j0.014 - j(-0.131) - j0.3} = \frac{2.581}{-j}$$

By Eq. (14.7),

$$\Delta V_2 = -\left(\frac{2.581}{-j}\right) \times (-j0.004) = -0.0103 \text{ per unit}$$

14.2 Solve Prob. 14.1 when the outage involves (a) only line ①-④ of series impedance $j0.2$ per unit and (b) both line ①-④ and line ①-③ of series impedance $j0.3$ per unit.

Solution:

(a)

$$\mathbf{Z}_{bus}^{(1-4)} = \mathbf{Z}_{bus} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} j(0.041 - 0.018) \\ j(0.031 - 0.038) \\ j(0.027 - 0.045) \\ j(0.018 - 0.063) \end{bmatrix} = \begin{bmatrix} j0.023 \\ -j0.007 \\ -j0.018 \\ -j0.045 \end{bmatrix}$$

By Eq. (14.14),

$$I_{14} = \frac{(V_1 - V_4)}{Z_{th,14} + z_{14}} = \frac{(1.0 - 1.04)}{j0.023 - (-j0.045) - j0.2} = \frac{-0.04}{-j0.132} = \frac{0.303}{j}$$

By Eq. (14.7),

$$\Delta V_2 = -\frac{0.303}{j} \times (-j0.007) = 0.0021 \text{ per unit}$$

(b) Setting $m = 1$; $n = 4$; $p = 1$; $q = 3$ in $\mathbf{Z}$ of Eq. (14.14), we obtain

$$\mathbf{Z}_{bus} = \begin{bmatrix} (Z_{11} - Z_{14}) - (Z_{41} - Z_{44}) + Z_a & (Z_{11} - Z_{13}) - (Z_{41} - Z_{43}) \\ \hline (Z_{11} - Z_{14}) - (Z_{31} - Z_{34}) & (Z_{11} - Z_{13}) - (Z_{31} - Z_{33}) + Z_b \end{bmatrix}$$

$$= \begin{bmatrix} (\text{row 1} - \text{row 4}) \text{ of } \mathbf{Z}_{bus}^{(1-4)} + (-z_{14}) & (\text{row 1} - \text{row 4}) \text{ of } \mathbf{Z}_{bus}^{(1-3)} \\ \hline (\text{row 1} - \text{row 3}) \text{ of } \mathbf{Z}_{bus}^{(1-4)} & (\text{row 1} - \text{row 3}) \text{ of } \mathbf{Z}_{bus}^{(1-3)} + (-z_{13}) \end{bmatrix}$$

$$\mathbf{Z}_{bus}^{(1-3)} = \begin{bmatrix} j0.014 \\ -j0.004 \\ -j0.131 \\ -j0.027 \end{bmatrix} \qquad \mathbf{Z}_{bus}^{(1-4)} = \begin{bmatrix} j0.023 \\ -j0.007 \\ -j0.018 \\ -j0.045 \end{bmatrix}$$

Hence,

$$\mathbf{Z} = \begin{bmatrix} -j0.132 & j0.041 \\ j0.041 & -j0.155 \end{bmatrix} \qquad \mathbf{Z}^{-1} = \begin{bmatrix} j8.25390 & j2.18329 \\ j2.18329 & j7.02913 \end{bmatrix}$$

$$\begin{bmatrix} I_a = I_{14} \\ I_b = I_{13} \end{bmatrix} = \begin{bmatrix} j8.25390 & j2.18329 \\ j2.18329 & j7.02913 \end{bmatrix} \begin{bmatrix} V_1 - V_4 = -0.04 \\ V_1 - V_3 = 0.04 \end{bmatrix}$$

$$\begin{bmatrix} I_{14} \\ I_{13} \end{bmatrix} = \begin{bmatrix} -j0.24282 \\ j0.19359 \end{bmatrix}$$

$$\Delta V_2 = -\left(\text{row 2 of } \mathbf{Z}_{bus}^{(1-4)}\right) I_{14} - \left(\text{row 2 of } \mathbf{Z}_{bus}^{(1-3)}\right) I_{13}$$

$$= -(-j0.007)(-j0.24282) - (-j0.004)(j0.19359) = 9.254 \times 10^{-4} \text{ per unit}$$

14.3 Suppose that line ③-④ of the system of Prob. 14.1 is actually a double-circuit line of combined impedance $j0.2$ per unit and that one of the circuits of series impedance $j0.4$ per unit is to be removed. Using the compensation current method, determine the change in voltage at bus ② due to this outage.

Solution:

$$\mathbf{Z}_{\text{bus}}^{(3-4)} = \begin{bmatrix} j(0.027 - 0.018) \\ j(0.035 - 0.038) \\ j(0.158 - 0.045) \\ j(0.045 - 0.063) \end{bmatrix} = \begin{bmatrix} j0.009 \\ -j0.003 \\ j0.113 \\ -j0.018 \end{bmatrix}$$

The branch to be removed has impedance $z_{34} = j0.4$ and, therefore,

$$I_{34} = \frac{(V_3 - V_4)}{(\text{row } 3 - \text{row } 4) \text{ of } \mathbf{Z}_{\text{bus}}^{(3-4)} + (-z_{34})} = \frac{(0.96 - 1.04)}{j(0.113 + 0.018) + (-j0.4)} = \frac{0.14870}{j}$$

By Eq. (14.7),

$$\Delta V_2 = -\left(\frac{0.14870}{j}\right) \times (-j0.003) = 4.461 \times 10^{-4} \text{ per unit}$$

14.4 Consider a portion of a large power system, whose $\mathbf{Z}_{\text{bus}}$ elements corresponding to the selected buses ① to ⑤ are given in per unit by

$$\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \\ \begin{bmatrix} j0.038 & j0.034 & j0.036 & j0.018 & j0.014 \\ j0.034 & j0.057 & j0.044 & j0.019 & j0.013 \\ j0.036 & j0.044 & j0.062 & j0.018 & j0.014 \\ j0.018 & j0.019 & j0.018 & j0.028 & j0.010 \\ j0.014 & j0.013 & j0.014 & j0.010 & j0.018 \end{bmatrix} \end{array}$$

The base-case bus voltages at those selected buses are $V_1 = 1.0\angle 0°$, $V_2 = 1.1\angle 0°$, $V_3 = 0.98\angle 0°$, $V_4 = 1.0\angle 0°$ and $V_5 = 0.99\angle 0°$, all in per unit. Using the compensating current method, determine the change in voltage at bus ① when line ②-③ of series impedance $j0.05$ per unit and line ④-⑤ of series impedance $j0.08$ per unit are both outaged.

Solution:

Set $m = 2$; $n = 3$; $p = 4$; $q = 5$ in $\mathbf{Z}$ of Eq. (14.14) to give

$$\mathbf{Z} = \begin{bmatrix} Z_{\text{th}, 23} + (-z_{23}) & (\text{row } 2 - \text{row } 3) \text{ of } \mathbf{Z}_{\text{bus}}^{(4-5)} \\ \hline (\text{row } 4 - \text{row } 5) \text{ of } \mathbf{Z}_{\text{bus}}^{(2-3)} & Z_{\text{th}, 45} + (-z_{45}) \end{bmatrix}$$

$$= \begin{bmatrix} j0.013 - (-j0.018) + (-j0.05) & j0.006 - j(0.004) \\ \hline j0.001 - (-j0.001) & j0.018 - (-j0.008) + (-j0.08) \end{bmatrix}$$

$$= \begin{bmatrix} -j0.019 & j0.002 \\ j0.002 & -j0.054 \end{bmatrix} \quad \text{where } \mathbf{Z}^{-1} = \begin{bmatrix} j52.83757 & j1.95695 \\ j1.95695 & j18.59100 \end{bmatrix}$$

$$\begin{bmatrix} I_{23} \\ I_{45} \end{bmatrix} = \mathbf{Z}^{-1} \begin{bmatrix} (V_2 - V_3) \\ (V_4 - V_5) \end{bmatrix} = \begin{bmatrix} j52.83757 & j1.95695 \\ j1.95695 & j18.59100 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.01 \end{bmatrix} = \begin{bmatrix} j6.36008 \\ j0.42074 \end{bmatrix}$$

$$\Delta V_1 = -(\text{row 1 of } \mathbf{Z}_{\text{bus}}^{(2-3)})I_{23} - (\text{row 1 of } \mathbf{Z}_{\text{bus}}^{(4-5)})I_{45}$$

$$= -(-j0.002)(j6.36008) - (j0.004)(j0.42074) = -0.011037 \text{ per unit}$$

14.5 Re-do Prob. 14.4 when line ②-③ and line ①-④ of series impedance $j0.10$ per unit are outaged.

Solution:

$$\mathbf{Z}_{\text{bus}}^{(2-3)} = \begin{bmatrix} -j0.002 \\ j0.013 \\ -j0.018 \\ j0.001 \\ -j0.001 \end{bmatrix} \qquad \mathbf{Z}_{\text{bus}}^{(1-4)} = \begin{bmatrix} j0.020 \\ j0.015 \\ j0.018 \\ j0.010 \\ -j0.004 \end{bmatrix}$$

$$\mathbf{Z} = \left[\begin{array}{c|c} (\text{row 2} - \text{row 3}) \text{ of } \mathbf{Z}_{\text{bus}}^{(2-3)} + (-z_{23}) & (\text{row 2} - \text{row 3}) \text{ of } \mathbf{Z}_{\text{bus}}^{(1-4)} \\ \hline (\text{row 1} - \text{row 4}) \text{ of } \mathbf{Z}_{\text{bus}}^{(2-3)} & (\text{row 1} - \text{row 4}) \text{ of } \mathbf{Z}_{\text{bus}}^{(1-4)} + (-z_{14}) \end{array} \right]$$

$$= \left[\begin{array}{c|c} j0.013 - (-j0.018) - j0.05) & j0.015 - j0.018 \\ \hline -j0.002 - j0.001 & j0.020 - j0.010 + (-j0.10) \end{array} \right]$$

$$= \begin{bmatrix} -j0.019 & -j0.003 \\ -j0.003 & -j0.090 \end{bmatrix} \quad \text{where} \quad \mathbf{Z}^{-1} = \begin{bmatrix} j52.91005 & -j1.76367 \\ -j1.76367 & j11.16990 \end{bmatrix}$$

$$\begin{bmatrix} I_{23} \\ I_{14} \end{bmatrix} = \mathbf{Z}^{-1} \begin{bmatrix} (V_2 - V_3) \\ (V_1 - V_4) \end{bmatrix} = \begin{bmatrix} j52.91005 & -j1.76367 \\ -j1.76367 & j11.16990 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0 \end{bmatrix} = \begin{bmatrix} j6.34921 \\ -j0.21164 \end{bmatrix}$$

$$\Delta V_1 = -(\text{row 1 of } \mathbf{Z}_{\text{bus}}^{(2-3)})I_{23} - (\text{row 1 of } \mathbf{Z}_{\text{bus}}^{(1-4)})I_{14}$$

$$= -(-j0.002)(j6.34921) - (j0.020)(-j0.21164) = -0.01693 \text{ per unit}$$

14.6 Two systems are connected by three branches a, b and c in the network of Fig. 14.15. Using the sequential rules of Sec. 14.2, write the elements of the loop impedance matrix $\mathbf{Z}$ in symbolic form. Assume that System I contains buses ① through ④ and System II contains buses ⑤ through ⑦. Use superscripts [1] and [2] to denote elements of $\mathbf{Z}_I$ and $\mathbf{Z}_{II}$, respectively.

Solution:

$$\mathbf{Z} = \underbrace{\begin{array}{c} ⓐ \\ ⓑ \\ ⓒ \end{array} \begin{bmatrix} Z_{22}^{(1)} & Z_{23}^{(1)} & Z_{24}^{(1)} \\ Z_{32}^{(1)} & Z_{33}^{(1)} & Z_{34}^{(1)} \\ Z_{42}^{(1)} & Z_{43}^{(1)} & Z_{44}^{(1)} \end{bmatrix}}_{\text{System I}} + \begin{bmatrix} Z_a & \cdot & \cdot \\ \cdot & Z_b & \cdot \\ \cdot & \cdot & Z_c \end{bmatrix} + \underbrace{\begin{array}{c} ⓐ \\ ⓑ \\ ⓒ \end{array} \begin{bmatrix} Z_{55}^{(2)} & Z_{56}^{(2)} & Z_{56}^{(2)} \\ Z_{55}^{(2)} & Z_{55}^{(2)} & Z_{56}^{(2)} \\ Z_{65}^{(2)} & Z_{65}^{(2)} & Z_{66}^{(2)} \end{bmatrix}}_{\text{System II}}$$

$$= \left[\begin{array}{c|c|c} Z_{22}^{(1)} + Z_{55}^{(2)} + Z_a & Z_{23}^{(1)} + Z_{55}^{(2)} & Z_{24}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{32}^{(1)} + Z_{55}^{(2)} & Z_{33}^{(1)} + Z_{55}^{(2)} + Z_b & Z_{34}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{42}^{(1)} + Z_{65}^{(2)} & Z_{43}^{(1)} + Z_{65}^{(2)} & Z_{44}^{(1)} + Z_{66}^{(2)} + Z_c \end{array} \right]$$

14.7 Form the branch-to-node incidence matrix $\mathbf{A}_c$ for branches a, b and c of the network of Fig. 14.15. Then, for the network determine the matrix $\mathbf{Z}$ defined in Eq. (14.9). Compare your answer with that of Prob. 14.6.

Solution:

$$\mathbf{A}_c = \left[\ \mathbf{A}_{c\,\mathrm{I}}\ |\ \mathbf{A}_{c\,\mathrm{II}}\ \right] = \begin{array}{c} \\ a \\ b \\ c \end{array} \overset{\begin{array}{cccccc} ① & ② & ③ & ④ & ⑤ & ⑥ \end{array}}{\left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right]}$$

$$\underbrace{\hspace{3cm}}_{\mathbf{A}_{c\,\mathrm{I}}} \quad \underbrace{\hspace{1.5cm}}_{\mathbf{A}_{c\,\mathrm{II}}}$$

$$\mathbf{Z} = \left[\begin{array}{ccc} Z_a & \cdot & \cdot \\ \cdot & Z_b & \cdot \\ \cdot & \cdot & Z_c \end{array} \right] + \mathbf{Z}' \qquad \text{where}\ \ \mathbf{Z}' = \mathbf{A}_c \mathbf{Z}_{\mathrm{bus}} \mathbf{A}_c^T$$

$$\mathbf{Z}' = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array} \right] \left[\begin{array}{cccc|cc} Z_{11}^{(1)} & Z_{12}^{(1)} & Z_{13}^{(1)} & Z_{14}^{(1)} & & \\ Z_{21}^{(1)} & Z_{22}^{(1)} & Z_{23}^{(1)} & Z_{24}^{(1)} & & 0 \\ Z_{31}^{(1)} & Z_{32}^{(1)} & Z_{33}^{(1)} & Z_{34}^{(1)} & & \\ Z_{41}^{(1)} & Z_{42}^{(1)} & Z_{43}^{(1)} & Z_{44}^{(1)} & & \\ \hline & 0 & & & Z_{55}^{(2)} & Z_{56}^{(2)} \\ & & & & Z_{65}^{(2)} & Z_{66}^{(2)} \end{array} \right] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right]$$

$$= \left[\begin{array}{c|c|c} Z_{22}^{(1)} + Z_{55}^{(2)} & Z_{23}^{(1)} + Z_{55}^{(2)} & Z_{24}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{32}^{(1)} + Z_{55}^{(2)} & Z_{33}^{(1)} + Z_{55}^{(2)} & Z_{34}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{42}^{(1)} + Z_{65}^{(2)} & Z_{43}^{(1)} + Z_{65}^{(2)} & Z_{44}^{(1)} + Z_{66}^{(2)} \end{array} \right]$$

Therefore,

$$\mathbf{Z} = \left[\begin{array}{c|c|c} Z_{22}^{(1)} + Z_{55}^{(2)} + Z_a & Z_{23}^{(1)} + Z_{55}^{(2)} & Z_{24}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{32}^{(1)} + Z_{55}^{(2)} & Z_{33}^{(1)} + Z_{55}^{(2)} + Z_b & Z_{34}^{(1)} + Z_{56}^{(2)} \\ \hline Z_{42}^{(1)} + Z_{65}^{(2)} & Z_{43}^{(1)} + Z_{65}^{(2)} & Z_{44}^{(1)} + Z_{66}^{(2)} + Z_c \end{array} \right]$$

which is the same answer as for Prob. 14.6.

14.8 Use the piecewise method to determine the bus voltages for the reactance network of Fig. 14.16 where System I and System II are defined. Current injections I_1 to I_4 are each equal to $1.0\underline{/-90°}$ per unit and current injections I_5 and I_6 are each equal to $4.0\underline{/-90°}$ per unit.

Solution:

$\mathbf{Z}_I$ and $\mathbf{Z}_{II}$ are found to be

$$
\mathbf{Z}_I = \begin{matrix} \text{①} & \text{②} & \text{③} & \text{④} \\ \begin{bmatrix} j0.2 & j0.2 & j0.2 & j0.2 \\ j0.2 & j0.43684 & j0.30526 & j0.2 \\ j0.2 & j0.30526 & j0.35789 & j0.2 \\ j0.2 & j0.2 & j0.2 & j0.6 \end{bmatrix} \end{matrix}
\qquad
\mathbf{Z}_{II} = \begin{matrix} \text{⑤} & \text{⑥} \\ \begin{bmatrix} j0.16 & j0.08 \\ j0.08 & 0.24 \end{bmatrix} \end{matrix}
$$

Using Eq. (14.19), $\mathbf{V}$ is found to be

$$
\mathbf{V} = \left[\begin{array}{c|c} \mathbf{Z}_I & \\ \hline & \mathbf{Z}_{II} \end{array} \right] \begin{bmatrix} -j1 \\ -j1 \\ -j1 \\ -j1 \\ \hline -j4 \\ -j4 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1.14210 \\ 1.06315 \\ 1.2 \\ \hline 0.96 \\ 1.28 \end{bmatrix}
$$

By the sequential rules of Sec. 14.2 we obtain

$$
\mathbf{Z} = \left[\begin{array}{c|c|c} Z_{22}^{(1)}+Z_{55}^{(2)}+Z_a & Z_{23}^{(1)}+Z_{55}^{(2)} & Z_{24}^{(1)}+Z_{56}^{(2)} \\ \hline Z_{32}^{(1)}+Z_{55}^{(2)} & Z_{33}^{(1)}+Z_{55}^{(2)}+Z_b & Z_{34}^{(1)}+Z_{56}^{(2)} \\ \hline Z_{42}^{(1)}+Z_{65}^{(2)} & Z_{43}^{(1)}+Z_{65}^{(2)} & Z_{44}^{(1)}+Z_{66}^{(2)}+Z_c \end{array} \right] = \begin{bmatrix} j0.89684 & j0.46526 & j0.28 \\ j0.46526 & j0.91789 & j0.28 \\ j0.28 & j0.28 & j1.09 \end{bmatrix}
$$

Currents I_a, I_b and I_c are found from

$$
\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \mathbf{Z}^{-1} \begin{bmatrix} (V_2-V_5) \\ (V_3-V_5) \\ (V_4-V_6) \end{bmatrix} = \mathbf{Z}^{-1} \begin{bmatrix} (1.14210-0.96) \\ (1.06315-0.96) \\ (1.2-1.28) \end{bmatrix} = \begin{bmatrix} 0.22606 \\ 0.04112 \\ -0.14203 \end{bmatrix}
$$

From Eq. (14.7), $\Delta \mathbf{V}$ is found to be

$$
\Delta \mathbf{V} = -\left[\begin{array}{c|c} \mathbf{Z}_I & \\ \hline & \mathbf{Z}_{II} \end{array} \right] \mathbf{A}_c^T \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}
$$

$$
= \left[\begin{array}{cccc|cc} j0.2 & j0.2 & j0.2 & j0.2 & & \\ j0.2 & j0.43684 & j0.30526 & j0.2 & & \\ j0.2 & j0.30526 & j0.35789 & j0.2 & & \\ j0.2 & j0.2 & j0.2 & j0.6 & & \\ \hline & & & & j0.16 & j0.08 \\ & & & & j0.08 & 0.24 \end{array} \right] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right] \begin{bmatrix} 0.22606 \\ 0.04112 \\ -0.14203 \end{bmatrix}
$$

$$
= \begin{bmatrix} -0.02503 \\ -0.08290 \\ -0.19217 \\ 0.03139 \\ -0.01271 \end{bmatrix} \text{ per unit}
$$

These voltage changes can now be added to $\mathbf{V}$ to obtain overall system voltages.

14.9 Consider the four-bus system of Fig. 14.17, in which line impedances are shown in per unit. Using a generation-shift distribution factor based on the dc power-flow method, find the change in power flow in line ①–② when the power generation at bus ④ is incrementally increased by 0.1 per unit. Note: An appropriate bus should be selected as reference to find Z_{bus} of the system.

Solution:

Suppose we choose bus ① as reference. Then using the Z_{bus} building algorithm, we find

$$Z_{bus} = \begin{array}{ccc} ② & ③ & ④ \end{array}$$
$$Z_{bus} = \begin{bmatrix} j0.19522 & j0.19304 & j0.19130 \\ j0.19304 & j0.46258 & j0.27825 \\ j0.19130 & j0.27825 & j0.34779 \end{bmatrix}$$

The incremental current injection into bus ④ causes current flow change in line ①–② as follows (See Eqs. (14.29) and (14.30))

$$\Delta I_{12} = K_{12,4}\Delta I_4 = \frac{(Z_{14} - Z_{24})}{Z_c}\Delta I_4 = \frac{(0 - j0.19130)}{j0.2}(0.1) = -0.09565 \text{ per unit}$$

Thus the current flow *increases* from bus ② to bus ①.

14.10 Consider the power system of Fig. 14.12 discussed in Example 14.4. Using distribution factors, predict the current in line ①–④ when line ④–⑤ is outaged. Assume that the operating condition specified remains the same as in the example.

Solution:

$$L_{14,45} = \frac{\Delta I_{14}}{I_{45}} = -1.0 \quad \text{(from Table 14.3)}$$
$$\therefore I'_{14} = I^a_{14} + \Delta I_{14} = I^a_{14} - I^a_{45}$$

From Table 14.2 we have

$$I^a_{14} = \frac{(V_1 - V_4)}{z_{14}} = \frac{(1 + j0) - (0.993653 - j0.045583)}{j0.05} = 0.91166 - j0.12694 \text{ per unit}$$

From Example 14.5,

$$I^a_{45} = 0.115150 + j0.060563 \text{ per unit}$$

Hence,

$$I'_{14} = (0.91166 - j0.12694) - (0.115150 + j0.060563) = 0.79651 - j0.18750 \text{ per unit}$$

14.11 The base-case bus voltages at buses ⓜ and ⓝ in a large power system are $1.02\angle 0°$ and $0.99\angle 0°$ per unit, respectively. Suppose that line ⓜ–ⓝ has been

removed from service and that the following selected elements are extracted from the $\mathbf{Z}_{bus}$ of the system with line $\textcircled{m}$-$\textcircled{n}$ excluded.

$$
\begin{array}{c}
\quad\quad\quad \textcircled{i} \quad\quad\quad \textcircled{j} \quad\quad\quad \textcircled{m} \quad\quad\quad \textcircled{n} \\
\begin{array}{c} \textcircled{i} \\ \textcircled{j} \\ \textcircled{m} \\ \textcircled{n} \end{array}
\begin{bmatrix}
j0.019 & j0.015 & j0.017 & j0.014 \\
j0.015 & j0.044 & j0.025 & j0.030 \\
j0.017 & j0.025 & j0.075 & j0.052 \\
j0.014 & j0.030 & j0.052 & j0.064
\end{bmatrix}
\end{array}
$$

The impedances of lines $\textcircled{i}$-$\textcircled{j}$ and $\textcircled{m}$-$\textcircled{n}$ are $j0.05$ and $j0.1$ per unit, respectively. Using the principle of superposition, determine the change in the line current of line $\textcircled{i}$-$\textcircled{j}$ due to an outage of line $\textcircled{m}$-$\textcircled{n}$.

Solution:

$$
K_{ij,mn} = \frac{(Z_{im} - Z_{jm}) - (Z_{in} - Z_{jn})}{z_{ij}} = \frac{(0.017 - 0.025) - (0.014 - 0.030)}{0.05} = 0.16
$$

$$
I^0_{mn} = \frac{(1.02 - 0.99)}{j0.1} = \frac{0.3}{j}
$$

$$
\Delta I_{ij} = \frac{0.3}{j} \times 0.16 = -j0.48 \text{ per unit}
$$

14.12 Solve Example 14.6 for the case where only lines $\textcircled{1}$-$\textcircled{2}$ and $\textcircled{2}$-$\textcircled{5}$ are simultaneously outaged.

Solution:

In Eq. 14.56, let $i = 5$, $j = 3$; $m = 1$, $n = 2$; $p = 2$, $q = 5$; then,

$$
\Delta I_{53} = L'_{53,52} I^a_{12} + L'_{53,25} I^a_{25}
$$

where

$$
L'_{53,12} = \frac{L_{53,12} + L_{53,25} L_{25,12}}{1 - L_{25,12} L_{12,25}} \qquad L'_{53,25} = \frac{L_{53,25} + L_{53,12} L_{12,25}}{1 - L_{12,25} L_{25,12}}
$$

From Table 14.3, we obtain

$$
L'_{53,12} = \frac{0.262295 + (-0.671533)(-0.737705)}{1 - (-0.737705)(-0.328467)} = 1
$$

$$
L'_{53,25} = -1 \quad \text{(from Example 14.6)}
$$

$$
I^a_{12} = \frac{(V_1 - V_2)}{z_{12}} = \frac{(1.0 + j0.0) - (0.986301 - j0.083834)}{j0.1} = 0.83834 - j0.13699 \text{ per unit}
$$

From Example 14.4,

$$
I^a_{25} = -(0.725975 - j0.304925) \text{ per unit}
$$

Hence,

$$
\begin{aligned}
\Delta I_{53} &= (1)I^a_{12} + (-1)I^a_{25} \\
&= (0.83834 - j0.13699) + (0.725975 - j0.304925) = 1.564315 - j0.441915 \text{ per unit}
\end{aligned}
$$

The current I_{53}^a is known from Example 14.6. So,

$$I_{53}'' = I_{53}^a + \Delta I_{53}$$
$$= (0.806260 - j0.274180) + (1.564315 - j0.441915) = 2.370575 - j0.716095 \text{ per unit}$$

An ac power flow solution for the system with both lines ①-② and ②-⑤ outaged would compare with this approximate result.

14.13 In the five-bus system of Fig. 14.12, consider that line ④-⑤ has been taken out of service when an additional incremental load of $1 + j0$ at bus ③ has to be met by an additional 1 MW generation at bus ⑤. Use Eq. (14.44) to predict the change in current flow from bus ② to bus ③.

Solution:

By definition in Eq. (14.30),

$$K_{ij,m} = \frac{Z_{im} - Z_{jm}}{Z_c}$$

Equation (14.44) gives for this case

$$\Delta I_{23}^{sh} = (K_{23,3} + L_{23,45}K_{45,3})\Delta I_3 + (K_{23,5} + L_{23,45}K_{45,5})\Delta I_5$$

From Table 14.1, we can calculate

$$K_{23,3} = \frac{Z_{23} - Z_{33}}{z_{23}} = \frac{5.000 - 5.035850}{0.0625} = -0.5736$$

$$K_{45,3} = \frac{Z_{43} - Z_{53}}{z_{45}} = \frac{5.009476 - 5.014529}{0.08} = -0.063163$$

From Table 14.3, we have

$$K_{23,5} = -0.232470 \qquad L_{23,45} = 0.262295$$
$$K_{45,5} = -0.113708$$

Hence,

$$\Delta I_{23}^{sh} = [-0.5736 + 0.262295 \times (-0.063163)](-1) + [-0.232470 + 0.262295 \times (-0.113708)](1)$$
$$= (0.327872 + j0) \text{ MVA}$$

14.14 A five-bus system consisting of buses ① through ⑤ is connected to a partially-depicted large power system through three tie lines, as shown in Fig. 14.18 in which line admittances are indicated in per unit. The equivalent current injections at those five buses are $I_1 = 1.5 + j0$, $I_2 = -0.8 + j0$, $I_3 = 0.5 + j0$, $I_4 = -1.2 + j0$ and $I_5 = 0.4 + j0$, all in per unit. By gaussian elimination, reduce the Y_{bus} representation of the five-bus system down to the boundary buses ④ and ⑤, and draw the corresponding Ward equivalent network.

Solution:

By inspection of Fig. 14.18 we find

$$
\begin{array}{ccccc}
 & \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \cdots
\end{array}
$$

$$
\begin{array}{c}
\textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \\ \textcircled{5} \\ \vdots
\end{array}
\left[\begin{array}{ccc|cc}
-j5 & j4 & j0 & j1 & j0 & \cdots \\
j4 & -j5 & j1 & j0 & j0 & \cdots \\
j0 & j1 & -j3 & j0 & j2 & \cdots \\
\hline
j1 & j0 & j0 & -j5 & j0 & \cdots \\
j0 & j0 & j2 & j0 & -j5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
=
\left[\begin{array}{c|c|c}
\mathbf{Y}_{EE} & \mathbf{Y}_{EB} & 0 \\
\hline
\mathbf{Y}_{BE} & \mathbf{Y}_{BB} & \mathbf{Y}_{BI} \\
\hline
0 & \mathbf{Y}_{IB} & \mathbf{Y}_{II}
\end{array}\right]
$$

After elimination of bus ①, we obtain

$$
\begin{array}{cccc}
 & \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5}
\end{array}
$$

$$
\begin{array}{c}
\textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \\ \textcircled{5} \\ \vdots
\end{array}
\left[\begin{array}{cccc}
-j9/5 & j1 & j4/5 & j0 & \cdots \\
j1 & -j3 & j0 & j2 & \cdots \\
j4/5 & j0 & -j24/5 & j0 & \cdots \\
j0 & j2 & j0 & -j5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

After dlimination of bus ②, we obtain

$$
\begin{array}{ccc}
 & \textcircled{3} & \textcircled{4} & \textcircled{5}
\end{array}
$$

$$
\begin{array}{c}
\textcircled{3} \\ \textcircled{4} \\ \textcircled{5} \\ \vdots
\end{array}
\left[\begin{array}{ccc}
-j22/9 & j4/9 & j2 & \cdots \\
j4/9 & -j40/9 & j0 & \cdots \\
j2 & j0 & -j5 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

After elimination of bus ③, we obtain the equivalent admittance matrix

$$
\begin{array}{cc}
 & \textcircled{4} & \textcircled{5}
\end{array}
$$

$$
\begin{array}{c}
\textcircled{4} \\ \textcircled{5}
\end{array}
\left[\begin{array}{cc}
-j48/11 & j4/11 \\
j4/11 & -j37/11
\end{array}\right]
$$

Given:

$$
\begin{array}{ll}
I_1 = 1.5 + j0 & I_4 = -1.2 + j0 \\
I_2 = -0.8 + j0 & I_5 = 0.4 + j0 \\
I_3 = 0.5 + j0 &
\end{array}
$$

After elimination of bus ①, we obtain current injections

$$
I_2' = I_2 - \frac{Y_{21}}{Y_{11}} I_1 = (-0.8 + j0) - \frac{4}{-5}(1.5 + j0) = 0.4 + j0
$$

$$
I_3' = I_3 - \frac{Y_{31}}{Y_{11}} I_1 = (0.5 + j0) - \frac{0}{-5}(1.5 + j0) = 0.5 + j0
$$

$$
I_4' = I_4 - \frac{Y_{41}}{Y_{11}} I_1 = (-1.2 + j0) - \frac{1}{-5}(1.5 + j0) = -0.9 + j0
$$

$$
I_5' = I_5 - \frac{Y_{51}}{Y_{11}} I_1 = (0.4 + j0) - \frac{0}{-5}(1.5 + j0) = 0.4 + j0
$$

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After elimination of bus ②, we obtain current injections

$$I_3'' = I_3' - \frac{Y_{32}^{(1)}}{Y_{22}^{(1)}}I_2' = (0.5 + j0) - \frac{1}{-9/5}(0.4 + j0) = \frac{6.5}{9} + j0$$

$$I_4'' = I_4' - \frac{Y_{42}^{(1)}}{Y_{22}^{(1)}}I_2' = (-0.9 + j0) - \frac{-4/5}{-9/5}(0.4 + j0) = \frac{-6.5}{9} + j0$$

$$I_5'' = I_5' - \frac{Y_{52}^{(1)}}{Y_{22}^{(1)}}I_2' = (0.4 + j0) - 0(0.4 + j0) = 0.4 + j0$$

After elimination of bus ③, we obtain current injections

$$I_4''' = I_4'' - \frac{Y_{43}^{(2)}}{Y_{33}^{(2)}}I_3'' = \left(\frac{-6.5}{9} + j0\right) - \frac{4/9}{-22/9}\left(\frac{6.5}{9} + j0\right) = \frac{-6.5}{11} + j0$$

$$I_5''' = I_5'' - \frac{Y_{53}^{(2)}}{Y_{33}^{(2)}}I_3'' = (0.4 + j0) - \frac{2}{-22/9}\left(\frac{6.5}{9} + j0\right) = \frac{10.9}{11} + j0$$

So, at buses ④ and ⑤ the Ward equivalent of the external system is
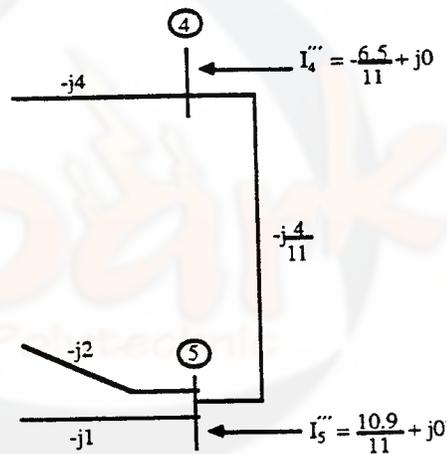
14.15 Re-do Prob. 14.14 using matrix inversion as in Eq. (14.69). Compare the results with the solution to Prob. 14.14. Now using gaussian elimination, continue the elimination of all five buses in the external system, and then find the resulting Ward equivalent at buses ⑥, ⑦ and ⑧. Draw an equivalent admittance circuit and mark the values of all quantities.

Solution:

From the solution to Prob. 14.14, we find

$$\mathbf{Y}_{EE} = \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{array}{ccc} ① & ② & ③ \end{array} \\ \left[\begin{array}{ccc} -j5 & j4 & j0 \\ j4 & -j5 & j1 \\ j0 & j1 & -j3 \end{array} \right]$$

Therefore,

$$
\mathbf{Y}_{EE}^{-1} = \frac{1}{22} \times
\begin{array}{c}
 \\
① \\ ② \\ ③
\end{array}
\begin{bmatrix}
j14 & j12 & j4 \\
j12 & j15 & j5 \\
j4 & j5 & j9
\end{bmatrix}
\qquad
\mathbf{Y}_{BB} =
\begin{array}{c}
 \\
④ \\ ⑤
\end{array}
\begin{bmatrix}
-j5 & j0 \\
j0 & -j5
\end{bmatrix}
$$

$$
\mathbf{Y}_{EB} =
\begin{array}{c}
 \\
① \\ ② \\ ③
\end{array}
\begin{bmatrix}
j1 & j0 \\
j0 & j0 \\
j0 & j2
\end{bmatrix}
\qquad
\mathbf{Y}_{BE} =
\begin{array}{c}
 \\
④ \\ ⑤
\end{array}
\begin{bmatrix}
j1 & j0 & j0 \\
j0 & j0 & j2
\end{bmatrix}
$$

$$
\therefore \ \mathbf{Y}_{BE}\mathbf{Y}_{EE}^{-1}\mathbf{Y}_{EB} =
\begin{bmatrix}
j1 & j0 & j0 \\
j0 & j0 & j2
\end{bmatrix}
\frac{1}{22}
\begin{bmatrix}
j14 & j12 & j4 \\
j12 & j15 & j5 \\
j4 & j5 & j9
\end{bmatrix}
\begin{bmatrix}
j1 & j0 \\
j0 & j0 \\
j0 & j2
\end{bmatrix}
$$

$$
= -\frac{1}{22}
\begin{bmatrix}
j14 & j8 \\
j8 & j36
\end{bmatrix}
$$

$$
\therefore \ \mathbf{Y}_{BB} - \mathbf{Y}_{BE}\mathbf{Y}_{EE}^{-1}\mathbf{Y}_{EB} = \mathbf{Y}_{BB} =
\begin{array}{c}
 \\
④ \\ ⑤
\end{array}
\begin{bmatrix}
-j5 & j0 \\
j0 & -j5
\end{bmatrix}
- \frac{-1}{22}
\begin{bmatrix}
j14 & j8 \\
j8 & j36
\end{bmatrix}
$$

$$
= \frac{1}{11} \times
\begin{array}{c}
 \\
④ \\ ⑤
\end{array}
\begin{bmatrix}
-j48 & j4 \\
j4 & -j37
\end{bmatrix}
$$

which is the same result as in Prob. 14.14. Equivalent current injections are found as follows:

$$
\mathbf{I}_B' = \mathbf{I}_B - \mathbf{Y}_{BE}\mathbf{Y}_{EE}^{-1}\mathbf{I}_E
$$

$$
=
\begin{bmatrix}
-1.2 + j0 \\
0.4 + j0
\end{bmatrix}
-
\begin{bmatrix}
j1 & j0 & j0 \\
j0 & j0 & j2
\end{bmatrix}
\frac{1}{22}
\begin{bmatrix}
j14 & j12 & j4 \\
j12 & j15 & j5 \\
j4 & j5 & j9
\end{bmatrix}
\begin{bmatrix}
1.5 + j0 \\
-0.8 + j0 \\
0.5 + j0
\end{bmatrix}
$$

$$
= \frac{1}{11}
\begin{bmatrix}
-6.5 + j0 \\
10.9 + j0
\end{bmatrix}
\text{per unit}
$$

which are the same net injections as in Prob. 14.14. We now continue to eliminate buses ④ and ⑤:

$$
\begin{array}{c}
④ \\ ⑤ \\ ⑥ \\ ⑦ \\ ⑧
\end{array}
\begin{bmatrix}
-j(48/11) & j(4/11) & j4 & j0 & j0 & \cdots \\
j(4/11) & -j(37/11) & j0 & j2 & j1 & \cdots \\
j4 & j0 & -j7 & j1 & j0 & \cdots \\
j0 & j2 & j1 & -j5 & j2 & \cdots \\
j0 & j1 & j0 & j2 & -j7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

After elimination of bus ④, we have

$$
\begin{array}{c}
⑤ \\ ⑥ \\ ⑦ \\ ⑧
\end{array}
\begin{bmatrix}
-j(10/3) & j(1/3) & j2 & j1 & \cdots \\
j(1/3) & -j(10/3) & j1 & j0 & \cdots \\
j2 & j1 & -j5 & j2 & \cdots \\
j1 & j0 & j2 & -j7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

After elimination of bus ⑤, we have

$$
\begin{array}{c}
\begin{array}{ccc} ⑥ & ⑦ & ⑧ \end{array} \\
\begin{array}{c} ⑥ \\ ⑦ \\ ⑧ \end{array}
\left[\begin{array}{cccc}
-j3.3 & j1.2 & j0.1 & \cdots \\
j1.2 & -j3.8 & j2.6 & \cdots \\
j0.1 & j2.6 & -j6.7 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}
$$

The Ward equivalent at buses ⑥, ⑦ and ⑧ is shown below:
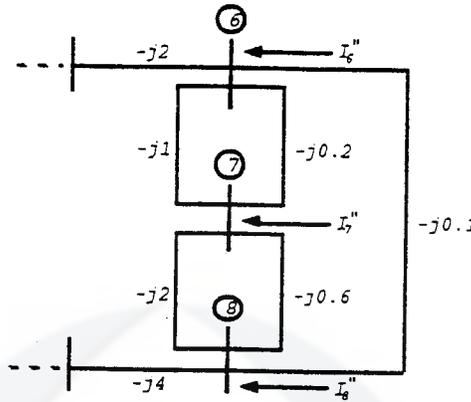
Calculations for the equivalent current injections into buses ⑥, ⑦ and ⑧ now follow. From the previous calculations we have

$$
\begin{aligned}
I_4 &= -(6.5/11) + j0 \\
I_5 &= (10.9/11) + j0
\end{aligned}
$$

Note that $I_6 = I_7 = I_8 = 0$ since no currents are specified at buses ⑥, ⑦ and ⑧. By eliminating bus ④, we find

$$
\begin{aligned}
I_5' &= I_5 - \frac{Y_{54}}{Y_{44}}I_4 = \left(\frac{10.9}{11} + j0\right) - \frac{j4/11}{-j48/11}\left(\frac{-6.5}{11} + j0\right) = \frac{11.3}{12} + j0 \\
I_6' &= I_6 - \frac{Y_{64}}{Y_{44}}I_4 = 0 - \frac{j4}{-j48/11}\left(\frac{-6.5}{11} + j0\right) = \frac{-6.5}{12} + j0 \\
I_7' &= I_8' = 0
\end{aligned}
$$

By eliminating bus ⑤, we obtain new currents

$$
\begin{aligned}
I_6'' &= I_6' - \frac{Y_{65}^{(1)}}{Y_{55}^{(1)}}I_5' = \left(\frac{-6.5}{12} + j0\right) - \frac{j1/3}{-j10/3}\left(\frac{11.3}{12} + j0\right) = -0.4475 + j0 \\
I_7'' &= I_7' - \frac{Y_{75}^{(1)}}{Y_{55}^{(1)}}I_5' = 0 - \frac{j2}{-j10/3}\left(\frac{11.3}{12} + j0\right) = 0.565 + j0 \\
I_8'' &= I_8' - \frac{Y_{85}^{(1)}}{Y_{55}^{(1)}}I_5' = 0 - \frac{j1}{-j10/3}\left(\frac{11.3}{12} + j0\right) = 0.2825 + j0
\end{aligned}
$$

1. Note that the net current into the *internal* system is

$$
(-0.4475 + j0) + (0.565 + j0) + (0.2825 + j0) = 0.4 + j0
$$

2. Note that the net current into buses ④ and ⑤ is

$$\left(\frac{-6.5}{11} + j0\right) + \left(\frac{10.9}{11} + j0\right) = 0.4 + j0$$

3. Note that the net current *specified* into buses ①, ②, ③, ④ and ⑤ is the sum of I_1, I_2, I_3, I_4 and I_5:

$$(1.5 + j0) + (-0.8 + j0) + (0.5 + j0) + (-1.2 + j0) + (0.4 + j0) = 0.4 + j0$$

Chapter 15 Problem Solutions

15.1 The circuit of Fig. 15.1 is redrawn in Fig. 15.7, in which three loop current variables are identified as x_1, x_2 and x_3. Although not shown, ammeters and voltmeters with the same accuracy are assumed to be installed as in Fig. 15.1, and the meter readings are also assumed to be the same as those in Example 15.1. Determine the weighted least-squares estimates of the three loop currents. Using the estimated loop currents, determine the source voltages V_1 and V_2, and compare the results with those of Example 15.1.

Solution:

The meter readings z_1 to z_4 are related to the three loop currents as follows:

$$
\begin{aligned}
z_1 &= x_1 + e_1 \\
z_2 &= x_2 + e_2 \\
z_3 &= x_1 - x_3 + e_3 \\
z_4 &= x_2 + x_3 + e_4
\end{aligned}
$$

The matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The inverse of the gain matrix becomes

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T \mathbf{W} \mathbf{H}\right)^{-1} = \begin{bmatrix} 0.00833 & -0.00167 & 0.00500 \\ -0.00167 & 0.00833 & -0.00500 \\ 0.00500 & -0.00500 & 0.01500 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 50 \end{bmatrix}$$

The estimates of the loop currents are calculated from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 9.01 \\ 3.02 \\ 6.98 \\ 5.01 \end{bmatrix} = \begin{bmatrix} 9.0033 \text{ A} \\ 3.0133 \text{ A} \\ 2.0100 \text{ A} \end{bmatrix}$$

Using the loop currents and elementary circuit analysis, the source voltages are determined to be

$$
\begin{aligned}
V_1 &= 2\hat{x}_1 - \hat{x}_3 = 15.9966 \text{ V} \\
V_2 &= 2\hat{x}_2 + \hat{x}_3 = 8.0366 \text{ V}
\end{aligned}
$$

Note that these values are different from those obtained in Example 15.1 because the quantities to be estimated here are the loop currents, while the source voltages were sought before. Another method to find the source voltages is as follows:

$$
\begin{aligned}
V_1 &= \hat{z}_3 + \hat{z}_1 \times 1 \\
V_2 &= \hat{z}_4 + \hat{z}_2 \times 1
\end{aligned}
$$

where $\hat{z}_1$ to $\hat{z}_4$ are the estimated measurements such that $\hat{z} = \mathbf{H}\hat{x}$. This will yield the same result.

15.2 Show that $E\left[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\right] = \mathbf{G}^{-1}$ where $\mathbf{G}$ is the gain matrix.

Solution:

$$
\begin{aligned}
E\left[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\right] &= E\left[(\mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{e})\left(\mathbf{e}^T\mathbf{W}\mathbf{H}\mathbf{G}^{-1^T}\right)\right] \\
&= \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}E\left[\mathbf{e}\mathbf{e}^T\right]\mathbf{W}\mathbf{H}\mathbf{G}^{-1^T} \\
&= \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{W}\mathbf{H}\mathbf{G}^{-1^T} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{H}\mathbf{G}^{-1^T} \\
&= \mathbf{G}^{-1}\mathbf{G}\mathbf{G}^{-1^T} = \mathbf{G}^{-1}
\end{aligned}
$$

15.3 Show that the sum of the diagonal elements in the matrix $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}$ in Eq. (15.40) is numerically equal to the number of state variables.

Solution:

The sum of the diagonal elements of $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}$ is

$$
\text{trace}\left[\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}\right] = \text{trace}\left[\underbrace{\mathbf{H}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}}_{\mathbf{X}}\underbrace{\mathbf{H}^T\mathbf{R}^{-1}}_{\mathbf{Y}}\right]
$$

since the trace $(\mathbf{XY}) = \text{trace}(\mathbf{YX})$, when such products are meaningful,

$$
\begin{aligned}
\text{trace}\left[\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}\right] &= \text{trace}\left[\underbrace{\mathbf{H}^T\mathbf{R}^{-1}}_{\mathbf{Y}}\underbrace{\mathbf{H}\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}}_{\mathbf{X}}\right] \\
&= \text{trace}\left[\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\right)^{-1}\right] = \text{trace}\left[\mathbf{I}\right] = N_s
\end{aligned}
$$

Note that matrix $\mathbf{I}$ size is $N_s \times N_s$.

15.4 Prove Eq. (15.47).

Solution:

$$
\begin{aligned}
\hat{f} &= \hat{\mathbf{e}}^T \mathbf{R}^{-1} \hat{\mathbf{e}} \\
&= \left\{ \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \mathbf{e} \right\}^T \mathbf{R}^{-1} \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \mathbf{e} \\
&= \mathbf{e}^T \left\{ \mathbf{R}^{-1} - \mathbf{R}^{-1^T} \mathbf{H} \mathbf{G}^{-1^T} \mathbf{H}^T \mathbf{R}^{-1} \right\} \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \mathbf{e} \\
&= \mathbf{e}^T \mathbf{R}^{-1} \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1^T} \mathbf{H}^T \mathbf{R}^{-1} \right] \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \mathbf{e} \\
&= \mathbf{e}^T \mathbf{R}^{-1} \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \mathbf{e}
\end{aligned}
$$

(The matrix in the bracket is an idempotent matrix)

Note that

$$
E\left[e_i^2\right] = \sigma_i^2 = \frac{1}{R_{ii}}
$$

Since measurements i and j are uncorrelated, $E[e_i e_j] = 0$. Therefore,

$$
\begin{aligned}
E\left[\hat{f}\right] &= \text{trace} \left[\mathbf{I} - \mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \\
&= \text{trace} \left[\mathbf{I} \right] - \text{trace} \left[\mathbf{H} \mathbf{G}^{-1} \mathbf{H}^T \mathbf{R}^{-1} \right] \\
&= N_m - \text{trace} \left\{ \left[\mathbf{H} \left(\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} \right)^{-1} \right] \left[\mathbf{H}^T \mathbf{R}^{-1} \right] \right\} \\
&= N_m - \text{trace} \left\{ \left[\mathbf{H}^T \mathbf{R}^{-1} \right] \left[\mathbf{H} \left(\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} \right)^{-1} \right] \right\} = N_m - N_s
\end{aligned}
$$

15.5 Consider the voltages at the two nodes labeled ① and ② in the circuit of Fig. 15.7 to be state variables. Using the ammeters and voltmeters connected as shown in Fig. 15.1 and their readings given in Example 15.1, determine the weighted least-squares estimates of these node voltages. Using the result, determine the source voltages V_1 and V_2, and compare the results with those of Example 15.1. Also calculate the expected value of the sum-of-squares of the measurement residuals using Eq. (15.46), and check your answer using Eq. (15.47).

Solution:

The meter readings are related to the node voltages x_1 and x_2 by

$$
\begin{aligned}
z_1 &= x_1 + x_1 - x_2 + e_1 = 2x_1 - x_2 + e_1 \\
z_2 &= x_2 + x_2 - x_1 + e_2 = -x_1 + 2x_2 + e_2 \\
z_3 &= x_1 + e_3 \\
z_4 &= x_2 + e_4
\end{aligned}
$$

The matrix $\mathbf{H}$ is

$$
\mathbf{H} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.00386 & 0.00281 \\ 0.00281 & 0.00386 \end{bmatrix}$$

The estimates of the node voltages are found as

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z}$$

$$= \mathbf{G}^{-1} \mathbf{H}^T \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} 9.01 \\ 3.02 \\ 6.98 \\ 5.01 \end{bmatrix} = \begin{bmatrix} 7.0060 \text{ V} \\ 5.0107 \text{ V} \end{bmatrix}$$

Using the node voltages found, the source voltages are

$$V_1 = \hat{x}_1 + \hat{z}_1 = 7.0060 + 9.0012 = 16.0072 \text{ V}$$
$$V_2 = \hat{x}_2 + \hat{z}_2 = 5.0107 + 3.0154 = 8.0261 \text{ V}$$

where $\hat{\mathbf{z}} = \mathbf{H}\hat{\mathbf{x}}$. It is seen that the estimated source voltages are the same as those of Example 15.1. The expected value of the sum of squares of the measurement residuals is

$$E\left[\hat{f}\right] = \sum_{j=1}^{N_m} \frac{R'_{jj}}{\sigma_j^2}$$

$$= \frac{0.00193}{(0.1)^2} + \frac{0.00193}{(0.1)^2} + \frac{0.01614}{(\sqrt{2}/10)^2} + \frac{0.01614}{(\sqrt{2}/10)^2} = 2$$

which can be checked by

$$N_m - N_s = 4 - 2 = 2 \quad \checkmark$$

15.6 Five ammeters numbered A_1 to A_5 are used in the dc circuit of Fig. 15.8 to determine the two unknown source currents I_1 and I_2. The standard deviations of the meter errors are 0.2 A for meters A_2 and A_5, and 0.1 A for the other three meters. The readings of the five meters are 0.12 A, 1.18 A, 3.7 A, 0.81 A and 7.1 A, respectively.

(a) Determine the weighted least-squares estimates of the source currents I_1 and I_2.

(b) Using the chi-square test of Eq. (15.49) for $\alpha = 0.01$, check for the presence of bad data in the measurements.

(c) Eliminate any bad data detected in (b) and find the weighted least-squares estimates of the source currents using the reduced data set.

(d) Apply the chi-square test for $\alpha = 0.01$ to the results of (c) to check if the result is statistically acceptable.

Solution:

(a) Let the unknown source currents be denoted by x_1 and x_2, and the meter readings by z_1 through z_5. It follows from circuit analysis that

$$z_1 = \frac{7}{40}x_1 - \frac{3}{40}x_2 + e_1$$

$$z_2 = \frac{-3}{40}x_1 + \frac{7}{40}x_2 + e_2$$

$$z_3 = \frac{33}{40}x_1 + \frac{3}{40}x_2 + e_3$$

$$z_4 = \frac{4}{40}x_1 + \frac{4}{40}x_2 + e_4$$

$$z_5 = \frac{3}{40}x_1 + \frac{33}{40}x_2 + e_5$$

from which the matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{bmatrix} 0.175 & -0.075 \\ -0.075 & 0.175 \\ 0.825 & 0.075 \\ 0.100 & 0.100 \\ 0.075 & 0.825 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T \mathbf{W} \mathbf{H}\right)^{-1} = \begin{bmatrix} 0.01431 & -0.00510 \\ -0.00510 & 0.05205 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix}$$

The estimates of the source currents are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.7218 \text{ A} \\ 8.0451 \text{ A} \end{bmatrix}$$

(b)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.04793 \\ 1.12877 \\ 3.67385 \\ 1.17669 \\ 6.91638 \end{bmatrix} = \begin{bmatrix} 0.07207 \\ 0.05123 \\ 0.02615 \\ -0.36669 \\ 0.18362 \end{bmatrix}$$

$$\hat{f} = \sum_{j=1}^{5} \frac{\hat{e}_j^2}{\sigma_j^2}$$

$$= 100 \times 0.07207^2 + 25 \times 0.05123^2 + 100 \times 0.02615^2 + 100 \times (-0.36669)^2 + 25 \times 0.18362^2$$

$$= 14.9427$$

Since $N_m = 5$ and $N_s = 2$, $k = N_m - N_s = 3$. For $\alpha = 0.01$, $\chi^2_{3,0.01} = 11.35$. There is at least one bad measurement by observing that $\hat{f} > 11.35$.

(c) To find the standardized error-estimates, the diagonal elements of the matrix $\mathbf{R}'$ are first calculated using $\mathbf{H}$ and $\mathbf{G}^{-1}$ obtained above. Thus,

$$\mathbf{R}' = \left(\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}\right)\mathbf{R}$$

$$= \begin{bmatrix} 0.00914 & \times & \times & \times & \times \\ \times & 0.03819 & \times & \times & \times \\ \times & \times & 0.00060 & \times & \times \\ \times & \times & \times & 0.00944 & \times \\ \times & \times & \times & \times & 0.00512 \end{bmatrix}$$

where $\mathbf{R} = \mathbf{W}^{-1}$

The standardized errors are calculated from

$$\frac{\hat{e}_1}{\sqrt{R'_{11}}} = \frac{0.07207}{\sqrt{0.00914}} = 0.75410$$

$$\frac{\hat{e}_2}{\sqrt{R'_{22}}} = \frac{0.05123}{\sqrt{0.03819}} = 0.26216$$

$$\frac{\hat{e}_3}{\sqrt{R'_{33}}} = \frac{0.02615}{\sqrt{0.00060}} = 1.06923$$

$$\frac{\hat{e}_4}{\sqrt{R'_{44}}} = \frac{-0.36669}{\sqrt{0.00944}} = -3.77445$$

$$\frac{\hat{e}_5}{\sqrt{R'_{55}}} = \frac{0.18362}{\sqrt{0.00512}} = 2.56567$$

from which z_4 is identified as the bad measurement. To perform state estimation without z_4, the 4th row of $\mathbf{H}$ is eliminated and the 4th row and 4th column of $\mathbf{W}$ are also eliminated. The inverse of the new gain matrix becomes

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.01440 & -0.00464 \\ -0.00464 & 0.05439 \end{bmatrix}$$

where

$$\mathbf{H} = \begin{bmatrix} 0.175 & -0.075 \\ -0.075 & 0.175 \\ 0.825 & 0.075 \\ 0.075 & 0.825 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$$

The new estimates of the source currents are then given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.75756 \text{ A} \\ 8.22756 \text{ A} \end{bmatrix}$$

(d)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.04051 \\ 1.15801 \\ 3.71706 \\ 7.06956 \end{bmatrix} = \begin{bmatrix} 0.07949 \\ 0.02199 \\ -0.01706 \\ 0.03044 \end{bmatrix}$$

$$\hat{f} = \sum_{\substack{j=1 \\ j \neq 4}}^{5} \frac{\hat{e}_j^2}{\sigma_j^2}$$

$$= 100 \times 0.07949^2 + 25 \times 0.02199^2 + 100 \times (-0.01706)^2 + 25 \times 0.03044^2$$

$$= 0.69628$$

Note that in this case $k = N_m - N_s = 4 - 2 = 2$. For $\alpha = 0.01$, $\chi^2_{2,0.01} = 9.21$. Since $\hat{f} < 9.21$, it is concluded that the set of remaining four measurements does not have any bad measurements.

15.7 Re-do Prob. 15.6 when the unknowns to be determined are not the source currents, but the voltages at the three nodes labeled ①, ② and ③ in Fig. 15.8.

Solution:

Let the voltages at the three nodes identified as ①, ② and ③ in Fig. 15.8 be denoted by x_1, x_2 and x_3, respectively. From circuit analysis,

$$z_1 = (x_1 - x_2)/3 + e_1$$
$$z_2 = (x_3 - x_2)/3 + e_2$$
$$z_3 = x_1 + e_3$$
$$z_4 = x_2/3 + e_4$$
$$z_5 = x_3/1 + e_5$$

from which matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ 0 & -1/3 & 1/3 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.00942 & 0.00424 & 0.00042 \\ 0.00424 & 0.04235 & 0.00424 \\ 0.00042 & 0.00424 & 0.03642 \end{bmatrix}$$

where

$$\mathbf{W} = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{bmatrix}$$

The estimates of the node voltages are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 3.65951 \text{ V} \\ 2.93507 \text{ V} \\ 7.03751 \text{ V} \end{bmatrix}$$

To check for the presence of bad data,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \\ 0.81 \\ 7.1 \end{bmatrix} - \begin{bmatrix} 0.24148 \\ 1.36747 \\ 3.65951 \\ 0.97835 \\ 7.03751 \end{bmatrix} = \begin{bmatrix} -0.12148 \\ -0.18747 \\ 0.04049 \\ -0.16835 \\ 0.06249 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^{5} \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times (-0.12148)^2 + 25 \times (-0.18747)^2 + 100 \times 0.04049^2 + 100 \times (-0.16835)^2 + 25 \times 0.06249^2 \\ &= 5.44991 \end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi^2_{2,\,0.01} = 9.21$. Since $\hat{f} < 9.21$, the set of measurements has no bad data for the specified confidence level.

15.8 Consider the circuit of Fig. 15.8 for which accuracy of the ammeters and their readings are the same as those specified in Prob. 15.6. As in Prob. 15.7, the voltages at the three nodes labeled ①, ② and ③ are to be estimated without first finding the source currents.

(a) Suppose that meters A_4 and A_5 are found to be out of order, and therefore, only three measurements $z_1 = 0.12$, $z_2 = 1.18$ and $z_3 = 3.7$ are available. Determine the weighted least-squares estimates of the nodal voltages, and the estimated errors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$.

(b) This time suppose meters A_2 and A_5 are now out of order and the remaining three meters are working. Using three measurements $z_1 = 0.12$, $z_3 = 3.7$ and $z_4 = 0.81$, can the nodal voltages be estimated without finding the source currents first? Explain why be examining the matrix $\mathbf{G}$.

Solution:

(a) By eliminating the 4th and 5th rows from $\mathbf{H}$ obtained in Prob. 15.7, we have

$$\mathbf{H} = \begin{bmatrix} 1/3 & -1/3 & 0 \\ 0 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.10 & 0.10 \\ 0.01 & 0.10 & 0.46 \end{bmatrix}$$

The estimates of the node voltages are determined to be

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \end{bmatrix} = \begin{bmatrix} 3.7 \text{ V} \\ 3.34 \text{ V} \\ 6.88 \text{ V} \end{bmatrix}$$

It is easy to check that

$$\hat{\mathbf{e}} \;=\; \mathbf{z} - \hat{\mathbf{z}} \begin{bmatrix} 0.12 \\ 1.18 \\ 3.70 \end{bmatrix} - \mathbf{H}\hat{\mathbf{x}} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the degree of freedom is zero. Therefore, in this case, the states of the system are completely defined without any redundancy in measurements.

(b) If only z_1, z_3 and z_4 are used, node ③ cannot be represented at all and, consequently, its nodal voltage cannot be estimated. This can also be checked by examining the matrix $\mathbf{H}$ for this case:

$$\mathbf{H} \;=\; \begin{bmatrix} 1/3 & -1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

Note that all the elements in the 3rd column are 0, indicating that x_3(the voltage at node ③) cannot affect the measurements. One can also note that the gain matrix becomes

$$\mathbf{G} \;=\; \mathbf{H}^T\mathbf{W}\mathbf{H} \;=\; \begin{bmatrix} 1/3 & 1 & 0 \\ -1/3 & 0 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

$$\;=\; \begin{bmatrix} 0 & -100/9 & 0 \\ -100/9 & 200/9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which cannot be inverted.

15.9 Suppose that the two voltage sources in Example 15.1 have been replaced with new ones, and the meter readings now show $z_1 = 2.9$ A, $z_2 = 10.2$ A, $z_3 = 5.1$ V and $z_4 = 7.2$ V.

(a) Determine the weighted least-squares estimates of the new source voltages.

(b) Using the chi-square test for $\alpha = 0.005$, detect bad data.

(c) Eliminate the bad data and determine again the weighted least-squares estimates of the source voltages.

(d) Check your result in (c) again using the chi-square test.

Solution:

(a) The estimates of the new source voltages are obtained from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \;=\; \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} \;=\; \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 2.9 \\ 10.2 \\ 5.1 \\ 7.2 \end{bmatrix} \;=\; \begin{bmatrix} 8.00175\ \text{V} \\ 17.66491\ \text{V} \end{bmatrix}$$

where $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{W}$ are all specified in Example 15.1.

(b) To verify the presence of bad data,

$$\hat{e} = z - \hat{z} = z - H\hat{x} = \begin{bmatrix} 2.9 \\ 10.2 \\ 5.1 \\ 7.2 \end{bmatrix} - \begin{bmatrix} 2.79298 \\ 10.04035 \\ 5.20877 \\ 7.62456 \end{bmatrix} = \begin{bmatrix} 0.10702 \\ 0.15965 \\ -0.10877 \\ -0.42456 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^{4} \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 100 \times 0.10702^2 + 100 \times 0.15965^2 + 50 \times (-0.10877)^2 + 50 \times (-0.42456)^2 \\ &= 13.29826 \end{aligned}$$

Note that in this case $k = N_m - N_s = 4 - 2 = 2$. For $\alpha = 0.005$, $\chi^2_{2,\,0.005} = 10.60$. Since $\hat{f} > 10.60$, there is at least one bad measurement.

(c) The diagonal elements of matrix R' are first found as

$$R' = (I - HG^{-1}H^T R^{-1})\, R$$

$$= \begin{bmatrix} 0.00193 & \times & \times & \times \\ \times & 0.00193 & \times & \times \\ \times & \times & 0.01614 & \times \\ \times & \times & \times & 0.01614 \end{bmatrix}$$

where $R = W^{-1}$

The standardized errors are calculated from

$$\frac{\hat{e}_1}{\sqrt{R'_{11}}} = \frac{0.10702}{\sqrt{0.00193}} = 2.43611$$

$$\frac{\hat{e}_2}{\sqrt{R'_{22}}} = \frac{0.15965}{\sqrt{0.00193}} = 3.63421$$

$$\frac{\hat{e}_3}{\sqrt{R'_{33}}} = \frac{-0.10877}{\sqrt{0.01614}} = -0.85617$$

$$\frac{\hat{e}_4}{\sqrt{R'_{44}}} = \frac{-0.42456}{\sqrt{0.01614}} = -3.34183$$

from which z_2 is identified as the bad measurement. To perform state estimation without z_2, the 2nd row of H is eliminated and the 2nd row and 2nd column of W are also eliminated. The inverse of the new gain matrix becomes

$$G^{-1} = (H^T W H)^{-1} = \begin{bmatrix} 0.02182 & 0.00727 \\ 0.00727 & 0.10909 \end{bmatrix}$$

where

$$H = \begin{bmatrix} 0.625 & -0.125 \\ 0.375 & 0.125 \\ 0.125 & 0.375 \end{bmatrix} \qquad W = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The new estimates of the source voltages are then given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = G^{-1} H^T W z = G^{-1} H^T W \begin{bmatrix} 2.9 \\ 5.1 \\ 7.2 \end{bmatrix} = \begin{bmatrix} 7.97273 \text{ A} \\ 16.59091 \text{ A} \end{bmatrix}$$

(d)

$$\hat{e} = z - \hat{z} = z - H\hat{x} = \begin{bmatrix} 2.9 \\ 5.1 \\ 7.2 \end{bmatrix} - \begin{bmatrix} 2.90909 \\ 5.06364 \\ 7.21818 \end{bmatrix} = \begin{bmatrix} -0.00909 \\ 0.03636 \\ -0.01818 \end{bmatrix}$$

$$\begin{aligned}
\hat{f} &= \sum_{j=1}^{3} \frac{\hat{e}_j^2}{\sigma_j^2} \\
&= 100 \times (-0.00909)^2 + 50 \times 0.03636^2 + 50 \times (-0.01818)^2 \\
&= 0.09091
\end{aligned}$$

Note that $k = N_m - N_s = 3 - 2 = 1$. For $\alpha = 0.005$, $\chi^2_{1,\,0.005} = 7.88$. Since $\hat{f} < 7.88$, no more bad data exists.

15.10 Five wattmeters are installed on the four-bus system of Fig. 15.9 to measure line real power flows, where per unit reactances of the lines are $X_{12} = 0.05$, $X_{13} = 0.1$, $X_{23} = 0.04$, $X_{24} = 0.0625$ and $X_{34} = 0.08$. Suppose that the meter readings show that

$$\begin{aligned}
z_1 &= P_{12} = \quad 0.34 \text{ per unit} \\
z_2 &= P_{13} = \quad 0.26 \text{ per unit} \\
z_3 &= P_{23} = \quad 0.17 \text{ per unit} \\
z_4 &= P_{24} = -0.24 \text{ per unit} \\
z_5 &= P_{34} = -0.22 \text{ per unit}
\end{aligned}$$

where the variances of the measurement errors in per unit are given by

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = (0.01)^2$$

(a) Apply the dc power-flow method of Sec. 9.7 to this network with bus ① as reference, and determine the corresponding H matrix. Then, compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.

(b) Using the chi-square test for $\alpha = 0.01$, identify two bad measurements. Between the two bad measurements, one is not worse than the other as far as accuracy is concerned. Explain why. If both bad measurements are eliminated simultaneously, would it be possible to estimate the states of the system?

(c) For the two bad measurements identified in (b), determine the relationship between the two error estimates in terms of the reactances of the corresponding two lines.

(d) Eliminate one of the bad measurements identified in (b), and determine the weighted least-squares estimates of the phase angles of the bus voltages using the reduced data set. Do the same for the other bad measurement. By comparing the two results, identify the buses at which the estimated phase angles are equal in the two cases.

Solution:

(a) Let x_1, x_2 and x_3 represent the phase angles of the bus voltages at buses ②, ③ and ④, respectively. With bus ① as the reference (with the phase angle of 0 radians), a dc power flow analysis would yield

$$z_1 = \frac{1}{0.05}(0 - x_1) + e_1$$

$$z_2 = \frac{1}{0.1}(0 - x_2) + e_2$$

$$z_3 = \frac{1}{0.04}(x_1 - x_2) + e_3$$

$$z_4 = \frac{1}{0.0625}(x_1 - x_3) + e_4$$

$$z_5 = \frac{1}{0.08}(x_2 - x_3) + e_5$$

from which matrix **H** is

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1} = \begin{bmatrix} 0.20499 & 0.18005 & 0.19554 \\ 0.18005 & 0.27980 & 0.21786 \\ 0.19554 & 0.21786 & 0.44657 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 10000 \end{bmatrix}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01750 \\ -0.02401 \\ -0.00398 \end{bmatrix}$$

(b)

$$\hat{\mathbf{e}} \ = \ \mathbf{z} - \hat{\mathbf{z}} \ = \ \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} \ = \ \begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} - \begin{bmatrix} 0.34996 \\ 0.24009 \\ 0.16278 \\ -0.21328 \\ -0.25036 \end{bmatrix} \ = \ \begin{bmatrix} -0.00996 \\ 0.01991 \\ 0.00722 \\ -0.02372 \\ 0.03036 \end{bmatrix}$$

$$\hat{f} \ = \ \sum_{j=1}^{5} \frac{\hat{e}_j^2}{\sigma_j^2}$$

$$= \ 10000 \times \left[(-0.00996)^2 + 0.01991^2 + 0.00722^2 + (-0.02372)^2 + 0.03036^2 \right]$$

$$= \ 20.31951$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} > 9.21$, there exists at least one bad measurement. To find the bad measurement, the diagonal elements of $\mathbf{R}'$ are computed:

$$\mathbf{R}' \ = \ \left(\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1} \right) \mathbf{R}$$

$$= \ \begin{bmatrix} 0.18005 & \times & \times & \times & \times \\ \times & 0.72020 & \times & \times & \times \\ \times & \times & 0.22073 & \times & \times \\ \times & \times & \times & 0.33317 & \times \\ \times & \times & \times & \times & 0.54586 \end{bmatrix} \times 10^{-4}$$

The standardized errors are calculated from

$$\frac{\hat{e}_1}{\sqrt{R_{11}'}} \ = \ \frac{-0.00996}{\sqrt{0.18005 \times 10^{-4}}} \ = \ -2.34601$$

$$\frac{\hat{e}_2}{\sqrt{R_{22}'}} \ = \ \frac{0.01991}{\sqrt{0.72020 \times 10^{-4}}} \ = \ 2.34601$$

$$\frac{\hat{e}_3}{\sqrt{R_{33}'}} \ = \ \frac{0.00722}{\sqrt{0.22073 \times 10^{-4}}} \ = \ 1.53608$$

$$\frac{\hat{e}_4}{\sqrt{R_{44}'}} \ = \ \frac{-0.02372}{\sqrt{0.33317 \times 10^{-4}}} \ = \ -4.10936$$

$$\frac{\hat{e}_5}{\sqrt{R_{55}'}} \ = \ \frac{0.03036}{\sqrt{0.54586 \times 10^{-4}}} \ = \ 4.10936$$

Note that the standardized errors for the 4th and 5th measurements are equally bad. This can be expected since both measurements z_4 and z_5 are equally affected by the voltage phase angle at bus ④. Note that the standardized errors for the first and second measurements are also equal. If both z_4 and z_5 are discarded, bus ④ will virtually be disconnected from the system, making state estimation impossible. The elimination of both z_4 and z_5 is also equivalent to deleting the 4th and 5th rows from $\mathbf{H}$. It is easy to check that the resulting gain matrix cannot be inverted.

(c) From (b) above, we have

$$\hat{e}_4 \ = \ z_4 - \hat{z}_4 \ = \ -0.02372$$

$$\hat{e}_5 \ = \ z_5 - \hat{z}_5 \ = \ 0.03036$$

Note that in the equations relating x to z, the absolute magnitude of the coefficients of x_3 in the z_4 and z_5 equations are $1/0.0625$ and $1/0.08$, respectively, as can be seen in **H**. This means that z_4 and z_5 are sensitive to changes in x_3 in the ratio of $0.08 : 0.0625$. Consequently, the error estimates will be inversely proportional to this ratio ($0.0625 : 0.08$). This can be verified by observing that

$$\frac{|\hat{e}_4|}{|\hat{e}_5|} = \frac{0.02372}{0.03036} = \frac{0.0625}{0.08} = \frac{X_{24}}{X_{34}}$$

(d) By deleting z_4, the reduced matrix **H** becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the new gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.17731 \\ 0.17731 & 0.29078 & 0.29078 \\ 0.17731 & 0.29078 & 0.93078 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 10000 \end{bmatrix}$$

The new estimates of the phase angles of the bus voltages, in the absence of z_4, are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01739 \\ -0.02444 \\ -0.00684 \end{bmatrix}$$

If z_5 is deleted instead, the reduced matrix **H** becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \end{bmatrix}$$

In this case, the inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.20567 \\ 0.17731 & 0.29078 & 0.17731 \\ 0.20567 & 0.17731 & 0.59630 \end{bmatrix} \times 10^{-6}$$

In this case, the new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \end{bmatrix} = \begin{bmatrix} -0.01739 \\ -0.02444 \\ -0.00239 \end{bmatrix}$$

From the above two results, the phase angles are equal for the two cases at buses ①, ② and ③, respectively. This could be expected because the state estimation procedure, in the absence of either z_4 or z_5, will minimize $\hat{f}$ with respect to the line measurements connecting buses ①, ② and ③. Note that the phase angle at bus ④ becomes a dependent state variable if no measurement is made on either line ②–④ or line ③–④.

15.11 In the four-bus system of Prob. 15.10, suppose that the variance of the measurement error for z_5 is $(0.05)^2$ and that all the other data remain the same. Qualitatively describe how the newly estimated values z_4 and z_5 of the measurements will differ from those obtained in Prob. 15.10. Verify your answer by recalculating the weighted least-squares estimates of the phase angles (in radians) of the bus voltages and the corresponding $\hat{z}$.

Solution:

Since $\sigma_5^2 = 0.05^2 > \sigma_4^2 = 0.01^2$, $\hat{x}_3$ will be determined such that the corresponding $\hat{z}_4$ is much closer to the measurement z_4 than $\hat{z}_5$ is to z_5. The **H** matrix is the same as that found in Prob. 15.10.

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

However, **W** is different and is given by

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 400 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.20563 & 0.17750 & 0.20496 \\ 0.17750 & 0.29000 & 0.18018 \\ 0.20496 & 0.18018 & 0.58568 \end{bmatrix} \times 10^{-6}$$

The estimates of the phase angles of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\begin{bmatrix} 0.34 \\ 0.26 \\ 0.17 \\ -0.24 \\ -0.22 \end{bmatrix} = \begin{bmatrix} -0.01740 \\ -0.02441 \\ -0.00250 \end{bmatrix}$$

The estimated values of z_i's are calculated from

$$\hat{z} = \mathbf{H}\hat{x} = \begin{bmatrix} 0.34795 \\ 0.24409 \\ 0.17529 \\ -0.23832 \\ -0.27383 \end{bmatrix}$$

For the fourth and fifth measurements,

$$\hat{e}_4 = z_4 - \hat{z}_4 = -0.24 - (-0.23832) = -0.00168$$
$$\hat{e}_5 = z_5 - \hat{z}_5 = -0.22 - (-0.27383) = 0.05383$$

Note that the weighted least squares estimation procedure regards z_4 much more accurately than z_5.

15.12 Suppose that a line of impedance $j0.025$ per unit is added between buses ① and ④ in the network of Fig. 15.9, and that a wattmeter is installed on this line at bus ①. The variance of the measurement error for this added wattmeter is assumed to be the same as that of the others. The meter readings now show

$$z_1 = P_{12} = 0.32 \text{ per unit}$$
$$z_2 = P_{13} = 0.24 \text{ per unit}$$
$$z_3 = P_{23} = 0.16 \text{ per unit}$$
$$z_4 = P_{24} = -0.29 \text{ per unit}$$
$$z_5 = P_{34} = -0.27 \text{ per unit}$$
$$z_6 = P_{14} = 0.05 \text{ per unit}$$

(a) Find the **H** matrix that describes the dc power flow with bus ① as reference, and compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.

(b) Using the chi-square test for $\alpha = 0.01$, eliminate any bad data and recompute the weighted least-squares estimates of the phase angles of the bus voltages. Check your result again using the chi-square test for $\alpha = 0.01$.

Solution:

(a) Let x_1, x_2 and x_3 denote the phase angles of the voltages at buses ②, ③ and ④, respectively. With the phase angle of the voltage at bus ① specified to be 0, dc power flow analysis provides

$$z_1 = \frac{1}{0.05}(0 - x_1) + e_1$$
$$z_2 = \frac{1}{0.1}(0 - x_2) + e_2$$
$$z_3 = \frac{1}{0.04}(x_1 - x_2) + e_3$$
$$z_4 = \frac{1}{0.0625}(x_1 - x_3) + e_4$$
$$z_5 = \frac{1}{0.08}(x_2 - x_3) + e_5$$
$$z_6 = \frac{1}{0.025}(0 - x_3) + e_6$$

from which matrix **H** is

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \\ 0 & 0 & -40 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.12988 & 0.09637 & 0.02401 \\ 0.09637 & 0.18657 & 0.02675 \\ 0.02401 & 0.02675 & 0.05483 \end{bmatrix} \times 10^{-6}$$

where

$$\mathbf{W} = \begin{bmatrix} 10000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10000 \end{bmatrix}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \\ 0.05 \end{bmatrix} = \begin{bmatrix} -0.01712 \\ -0.02341 \\ -0.00101 \end{bmatrix}$$

(b)

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \\ 0.05 \end{bmatrix} - \begin{bmatrix} 0.34240 \\ 0.23412 \\ 0.15732 \\ -0.25781 \\ -0.28007 \\ 0.04027 \end{bmatrix} = \begin{bmatrix} -0.02240 \\ 0.00588 \\ 0.00268 \\ -0.03219 \\ 0.01007 \\ 0.00973 \end{bmatrix}$$

$$\begin{aligned} \hat{f} &= \sum_{j=1}^{6} \frac{\hat{e}_j^2}{\sigma_j^2} \\ &= 10000 \times \left[(-0.02240)^2 + 0.00588^2 + 0.00268^2 + (-0.03219)^2 + 0.01007^2 + 0.00973^2 \right] \\ &= 17.7568 \end{aligned}$$

Note that $k = N_m - N_s = 6 - 3 = 3$. For $\alpha = 0.01$, $\chi^2_{3,\,0.01} = 11.35$. Since $\hat{f} > 11.35$, there exists at least one bad measurement. To identify the bad measurement, the diagonal elements of matrix $\mathbf{R}'$ are computed from

$$\mathbf{R}' = \left(\mathbf{I} - \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\mathbf{R}^{-1}\right)\mathbf{R}$$

$$= \begin{bmatrix} 0.48048 & \times & \times & \times & \times & \times \\ \times & 0.81344 & \times & \times & \times & \times \\ \times & \times & 0.22684 & \times & \times & \times \\ \times & \times & \times & 0.65006 & \times & \times \\ \times & \times & \times & \times & 0.70641 & \times \\ \times & \times & \times & \times & \times & 0.12277 \end{bmatrix} \times 10^{-4}$$

The standardized error-estimates are calculated from

$$\frac{\hat{e}_1}{\sqrt{R'_{11}}} = \frac{-0.02240}{\sqrt{0.48048 \times 10^{-4}}} = -3.23106$$

$$\frac{\hat{e}_2}{\sqrt{R'_{22}}} = \frac{0.00588}{\sqrt{0.81344 \times 10^{-4}}} = 0.65148$$

$$\frac{\hat{e}_3}{\sqrt{R'_{33}}} = \frac{0.22684}{\sqrt{0.22684 \times 10^{-4}}} = 0.56377$$

$$\frac{\hat{e}_4}{\sqrt{R'_{44}}} = \frac{-0.03219}{\sqrt{0.65006 \times 10^{-4}}} = -3.99261$$

$$\frac{\hat{e}_5}{\sqrt{R'_{55}}} = \frac{0.01007}{\sqrt{0.70641 \times 10^{-4}}} = 1.19820$$

$$\frac{\hat{e}_6}{\sqrt{R'_{66}}} = \frac{0.00973}{\sqrt{0.12277 \times 10^{-4}}} = 2.77670$$

from which z_4 is identified as the bad measurement. After deleting z_4, the reduced **H** matrix becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \\ 0 & 0 & -40 \end{bmatrix}$$

The new inverse gain matrix is

$$\mathbf{G}^{-1} = (\mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} = \begin{bmatrix} 0.17402 & 0.12540 & 0.01116 \\ 0.12540 & 0.20566 & 0.01830 \\ 0.01116 & 0.01830 & 0.05857 \end{bmatrix} \times 10^{-6}$$

where **W** is now a 5×5 matrix which is obtained by deleting the 4th row and 4th column from the **W** matrix specified in (a) above. The new estimates of the phase angles of the bus voltages are determined as

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \mathbf{z} = \mathbf{G}^{-1} \mathbf{H}^T \mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \\ 0.05 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02286 \\ -0.00125 \end{bmatrix}$$

Conducting a Chi-square test,

$$\hat{\mathbf{e}} = \mathbf{z} - \hat{\mathbf{z}} = \mathbf{z} - \mathbf{H}\hat{\mathbf{x}} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \\ 0.05 \end{bmatrix} - \begin{bmatrix} 0.32562 \\ 0.22861 \\ 0.16450 \\ -0.27012 \\ 0.05004 \end{bmatrix} = \begin{bmatrix} -0.00562 \\ 0.01139 \\ -0.00450 \\ 0.00012 \\ -0.00004 \end{bmatrix}$$

$$\hat{f} = \sum_{j=1}^{5} \frac{\hat{e}_j^2}{\sigma_j^2}$$

$$= 10000 \times \left[(-0.00562)^2 + 0.01139^2 + (-0.00450)^2 + 0.00012^2 + (-0.00004)^2\right]$$

$$= 1.81584$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,\,0.01}^2 = 9.21$. Since $\hat{f} < 1.81584$, we conclude that no more bad measurement exists.

15.13 In the four-bus system described in Prob. 15.12, suppose that the wattmeter on line ①–④ is out of order and that the readings of the remaining five wattmeters are the same as those specified in Prob. 15.12.

(a) Apply the dc power flow analysis with bus ① as reference, and determine the **H** matrix. Then, compute the weighted least-squares estimates of the phase angles of the bus voltages in radians.

(b) Using the chi-square test for $\alpha = 0.01$, identify two bad measurements. Eliminate one of them and compute the weighted least-squares estimates of the bus voltage phase angles. Restore the eliminated bad measurement and remove the second one before recomputing the estimates of the bus voltage phase angles. Compare the two sets of results, and identify the buses at the estimated angles are equal in the two cases. Does the presence of line ①–④ (but with no line measurement) affect the identification of those buses? Compare the identified buses with those identified in Prob. 15.10(d).

Solution:

(a) Let x_1, x_2 and x_3 denote the phase angles of the voltages at buses ②, ③ and ④, respectively. With the phase angle at bus ① equal to 0, dc power flow equations are determined to be the same as those of Prob. 15.10(a), and the corresponding **H** matrix is given by

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T \mathbf{W} \mathbf{H}\right)^{-1} = \begin{bmatrix} 0.20499 & 0.18005 & 0.19554 \\ 0.18005 & 0.27980 & 0.21786 \\ 0.19554 & 0.21786 & 0.44657 \end{bmatrix} \times 10^{-6}$$

The estimates of the phase angles (in radians) of the voltages at buses ②, ③ and ④ are determined from

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \end{bmatrix} = \begin{bmatrix} -0.01636 \\ -0.02257 \\ 0.00073 \end{bmatrix}$$

(b) Conducting a Chi-square test,

$$\hat{e} = z - \hat{z} = z - H\hat{x} = \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \\ -0.27 \end{bmatrix} - \begin{bmatrix} 0.32718 \\ 0.22565 \\ 0.15514 \\ -0.27344 \\ -0.29120 \end{bmatrix} = \begin{bmatrix} -0.00718 \\ 0.01435 \\ 0.00486 \\ -0.01656 \\ 0.02120 \end{bmatrix}$$

$$\begin{aligned}
\hat{f} &= \sum_{j=1}^{5} \frac{\hat{e}_j^2}{\sigma_j^2} \\
&= 10000 \times \left[(-0.00718)^2 + 0.01435^2 + 0.00486^2 + (-0.01656)^2 + 0.02120^2 \right] \\
&= 10.0467
\end{aligned}$$

Note that $k = N_m - N_s = 5 - 3 = 2$. For $\alpha = 0.01$, $\chi_{2,0.01}^2 = 9.21$. Since $\hat{f} > 9.21$, at least one bad measurement exists. To find the bad measurement, the diagonal elements of the matrix R' are computed from

$$\begin{aligned}
R' &= \left(I - HG^{-1}H^T R^{-1} \right) R \\
&= \begin{bmatrix} 0.18005 & \times & \times & \times & \times \\ \times & 0.72020 & \times & \times & \times \\ \times & \times & 0.22073 & \times & \times \\ \times & \times & \times & 0.33317 & \times \\ \times & \times & \times & \times & 0.54586 \end{bmatrix} \times 10^{-4}
\end{aligned}$$

The standardized error-estimates are calculated from

$$\frac{\hat{e}_1}{\sqrt{R'_{11}}} = \frac{-0.00718}{\sqrt{0.18005 \times 10^{-4}}} = -1.69141$$

$$\frac{\hat{e}_2}{\sqrt{R'_{22}}} = \frac{0.01435}{\sqrt{0.72020 \times 10^{-4}}} = 1.69141$$

$$\frac{\hat{e}_3}{\sqrt{R'_{33}}} = \frac{0.00486}{\sqrt{0.22073 \times 10^{-4}}} = 1.03374$$

$$\frac{\hat{e}_4}{\sqrt{R'_{44}}} = \frac{-0.01656}{\sqrt{0.33317 \times 10^{-4}}} = -2.86900$$

$$\frac{\hat{e}_5}{\sqrt{R'_{55}}} = \frac{0.02120}{\sqrt{0.54586 \times 10^{-4}}} = 2.86900$$

The fourth and fifth measurements are found to be equally bad. These two measurements are discarded respectively in the following two cases.

Case A: z_4 is deleted. The reduced H matrix becomes

$$H = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 0 & 12.5 & -12.5 \end{bmatrix}$$

The new inverse of the gain matrix is

$$G^{-1} = \left(H^T W H \right)^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.17731 \\ 0.17731 & 0.29078 & 0.29078 \\ 0.17731 & 0.29078 & 0.93078 \end{bmatrix} \times 10^{-6}$$

The new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.27 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02287 \\ -0.00127 \end{bmatrix}$$

<u>Case B</u>: z_5 is deleted. The reduced $\mathbf{H}$ matrix becomes

$$\mathbf{H} = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -10 & 0 \\ 25 & -25 & 0 \\ 16 & 0 & -16 \end{bmatrix}$$

The new inverse of the gain matrix is

$$\mathbf{G}^{-1} = \left(\mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.20567 & 0.17731 & 0.20567 \\ 0.17731 & 0.29078 & 0.17731 \\ 0.20567 & 0.17731 & 0.59630 \end{bmatrix} \times 10^{-6}$$

The new estimates of the phase angles of the bus voltages are

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W}\mathbf{z} = \mathbf{G}^{-1}\mathbf{H}^T\mathbf{W} \begin{bmatrix} 0.32 \\ 0.24 \\ 0.16 \\ -0.29 \end{bmatrix} = \begin{bmatrix} -0.01628 \\ -0.02287 \\ 0.00184 \end{bmatrix}$$

From the above results, the phase angles are equal for the two cases at buses ①, ② and ③, respectively. The presence of line ①-④ does not change these identified buses as long as its measurement is not accounted for in state estimation. Note that the phase angles are found to be equal at buses ①, ② and ③ in Prob. 15.10(d). Note that further calculation shows that the weighted sum-of-squares of the errors will be the same in Cases A and B above.

15.14 Three voltmeters and four wattmeters are installed on the three-bus system of Fig. 15.10, where per-unit reactances of the lines are $X_{12} = 0.1$, $X_{13} = 0.08$ and $X_{23} = 0.05$. The per unit values of the three voltmeter measurements are $z_1 = |V_1| = 1.01$, $z_2 = |V_2| = 1.02$ and $z_3 = |V_3| = 0.98$. The readings of the two wattmeters measuring MW generation at buses ① and ② are $z_4 = 0.48$ per unit and $z_5 = 0.33$ per unit, respectively. The measurement of the wattmeter on line ①-③ at bus ① shows $z_6 = 0.41$ per unit, and that of the wattmeter on line ②-③ at bus ② is $z_7 = 0.38$ per unit. The variances of the measurement errors are given in per unit as

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (0.02)^2$$
$$\sigma_4^2 = \sigma_5^2 = \sigma_6^2 = \sigma_7^2 = (0.05)^2$$

(a) Use bus ① as reference to find expressions for the elements of the matrix $\mathbf{H}_{\mathbf{x}}^{(k)}$ and those of the measurement errors $e_i^{(k)}$ in terms of state variables, as done in Example 15.5.

(b) Using the initial value of $1.0 \angle 0°$ per unit for all bus voltages, find the values of the state variables that will be obtained at the end of the first iteration of the weighted least-squares state estimation process.

Solution:

(a) Five state variables are defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_1| \\ |V_2| \\ |V_3| \end{bmatrix}$$

The expressions of the measurement errors are

$$e_1 = z_1 - h_1 = z_1 - |V_1| = z_1 - x_3$$

$$e_2 = z_2 - h_2 = z_2 - |V_2| = z_2 - x_4$$

$$e_3 = z_3 - h_3 = z_3 - |V_3| = z_3 - x_5$$

$$e_4 = z_4 - h_4 = z_4 - \left[\frac{|V_1||V_2|}{|Z_{12}|} \sin(\delta_1 - \delta_2) + \frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right]$$
$$= z_4 - \left[\frac{x_3 x_4}{0.1} \sin(0 - x_1) + \frac{x_3 x_5}{0.08} \sin(0 - x_2) \right]$$

$$e_5 = z_5 - h_5 = z_5 - \left[\frac{|V_2||V_1|}{|Z_{12}|} \sin(\delta_2 - \delta_1) + \frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right]$$
$$= z_5 - \left[\frac{x_3 x_4}{0.1} \sin(x_1 - 0) + \frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right]$$

$$e_6 = z_6 - h_6 = z_6 - \left[\frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right]$$
$$= z_6 - \left[\frac{x_3 x_5}{0.08} \sin(0 - x_2) \right]$$

$$e_7 = z_7 - h_7 = z_7 - \left[\frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right]$$
$$= z_7 - \left[\frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right]$$

The jacobian matrix $\mathbf{H_x}$ is now written as

$$\mathbf{H_x} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\partial P_1}{\partial \delta_2} & \frac{\partial P_1}{\partial \delta_3} & \frac{\partial P_1}{\partial |V_1|} & \frac{\partial P_1}{\partial |V_2|} & \frac{\partial P_1}{\partial |V_3|} \\ \frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_1|} & \frac{\partial P_2}{\partial |V_2|} & \frac{\partial P_2}{\partial |V_3|} \\ \frac{\partial P_{13}}{\partial \delta_2} & \frac{\partial P_{13}}{\partial \delta_3} & \frac{\partial P_{13}}{\partial |V_1|} & \frac{\partial P_{13}}{\partial |V_2|} & \frac{\partial P_{13}}{\partial |V_3|} \\ \frac{\partial P_{23}}{\partial \delta_2} & \frac{\partial P_{23}}{\partial \delta_3} & \frac{\partial P_{23}}{\partial |V_1|} & \frac{\partial P_{23}}{\partial |V_2|} & \frac{\partial P_{23}}{\partial |V_3|} \end{bmatrix}$$

$$\mathbf{H_X} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\frac{-|V_1||V_2|\cos(\delta_1-\delta_2)}{|Z_{12}|} & \frac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \left(\begin{array}{c}\frac{|V_2|\sin(\delta_1-\delta_2)}{|Z_{12}|} \\ +\frac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|}\end{array}\right) & \frac{|V_1|\sin(\delta_1-\delta_2)}{|Z_{12}|} & \frac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\
\left(\begin{array}{c}\frac{|V_2||V_1|\cos(\delta_2-\delta_1)}{|Z_{12}|} \\ -\frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|}\end{array}\right) & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_1)}{|Z_{12}|} & \left(\begin{array}{c}\frac{|V_2|\sin(\delta_2-\delta_1)}{|Z_{12}|} \\ +\frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|}\end{array}\right) & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\
0 & \frac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \frac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} & 0 & \frac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\
\frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|}
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-10x_3x_4\cos(-x_1) & -12.5x_3x_5\cos(-x_2) & \left(\begin{array}{c}10x_4\sin(-x_1) \\ +12.5x_5\sin(-x_2)\end{array}\right) & 10x_3\sin(-x_1) & 12.5x_3\sin(-x_2) \\
\left(\begin{array}{c}10x_3x_4\cos(x_1) \\ +20x_4x_5\cos(x_1-x_2)\end{array}\right) & -20x_4x_5\cos(x_1-x_2) & 10x_4\sin(x_1) & \left(\begin{array}{c}10x_4\sin(x_1) \\ +20x_5\sin(x_1-x_2)\end{array}\right) & 20x_4\sin(x_1-x_2) \\
0 & -12.5x_3x_5\cos(-x_2) & 12.5x_5\sin(-x_2) & 0 & 12.5x_3\sin(-x_2) \\
20x_4x_5\cos(x_1-x_2) & -20x_4x_5\cos(x_1-x_2) & 0 & 20x_5\sin(x_1-x_2) & 20x_4\sin(x_1-x_2)
\end{bmatrix}$$

(b) Using flat-start values,

$$\mathbf{H_X}^{(0)} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-10 & -12.5 & 0 & 0 & 0 \\
30 & -20 & 0 & 0 & 0 \\
0 & -12.5 & 0 & 0 & 0 \\
20 & -20 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{aligned}
e_1^{(0)} &= z_1 - h_1^{(0)} = 1.01 - 1.0 = 0.01 \\
e_2^{(0)} &= z_2 - h_2^{(0)} = 1.02 - 1.0 = 0.02 \\
e_3^{(0)} &= z_3 - h_3^{(0)} = 0.98 - 1.0 = -0.02 \\
e_4^{(0)} &= z_4 - h_4^{(0)} = 0.48 - 0 = 0.48 \\
e_5^{(0)} &= z_5 - h_5^{(0)} = 0.33 - 0 = 0.33 \\
e_6^{(0)} &= z_6 - h_6^{(0)} = 0.41 - 0 = 0.41
\end{aligned}$$

$$e_7^{(0)} \quad = \quad z_7 - h_7^{(0)} = 0.38 - 0 \quad = \quad 0.38$$

Note that

$$\mathbf{R}^{-1} = \begin{bmatrix} \frac{1}{0.02^2} & \times & \times & \times & \times & \times & \times \\ \times & \frac{1}{0.02^2} & \times & \times & \times & \times & \times \\ \times & \times & \frac{1}{0.02^2} & \times & \times & \times & \times \\ \times & \times & \times & \frac{1}{0.05^2} & \times & \times & \times \\ \times & \times & \times & \times & \frac{1}{0.05^2} & \times & \times \\ \times & \times & \times & \times & \times & \frac{1}{0.05^2} & \times \\ \times & \times & \times & \times & \times & \times & \frac{1}{0.05^2} \end{bmatrix}$$

Using $\mathbf{R}^{-1}$ and $\mathbf{H_X}^{(0)}$ obtained above,

$$\left[\mathbf{H_X}^{(0)^T} \mathbf{R}^{-1} \mathbf{H_X}^{(0)} \right]^{-1} =$$

$$\begin{bmatrix} 0.3512 \times 10^{-5} & 0.2762 \times 10^{-5} & -0.3843 \times 10^{-10} & 0.3297 \times 10^{-10} & 0.3553 \times 10^{-18} \\ 0.2762 \times 10^{-5} & 0.4420 \times 10^{-5} & -0.5400 \times 10^{-10} & -0.2318 \times 10^{-10} & -0.1748 \times 10^{-10} \\ -0.3843 \times 10^{-10} & -0.5400 \times 10^{-10} & 0.4000 \times 10^{-3} & -0.3382 \times 10^{-16} & 0.5217 \times 10^{-23} \\ 0.3297 \times 10^{-10} & -0.2318 \times 10^{-10} & -0.3382 \times 10^{-16} & 0.4000 \times 10^{-3} & -0.2413 \times 10^{-22} \\ 0.3553 \times 10^{-18} & -0.1748 \times 10^{-10} & -0.5217 \times 10^{-23} & -0.2413 \times 10^{-22} & 0.4000 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \\ x_5^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \\ x_5^{(0)} \end{bmatrix} + \left(\mathbf{H_X}^{(0)^T} \mathbf{R}^{-1} \mathbf{H_X}^{(0)} \right)^{-1} \mathbf{H_X}^{(0)^T} \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.01014 \\ -0.03074 \\ 0.01000 \\ 0.02000 \\ -0.02000 \end{bmatrix} = \begin{bmatrix} -0.01014 \text{ radian} \\ -0.03074 \text{ radian} \\ 1.01 \text{ per unit} \\ 1.02 \text{ per unit} \\ 0.98 \text{ per unit} \end{bmatrix}$$

15.15 Application of the weighted least-squares state estimation to the three-bus system with all the measurements described in Prob. 15.14 yields the following estimates of the states

$$\begin{array}{ll} |V_1| = 1.0109 \text{ per unit} & \\ |V_2| = 1.0187 \text{ per unit} & \delta_2 = -0.0101 \text{ radians} \\ |V_3| = 0.9804 \text{ per unit} & \delta_3 = -0.0308 \text{ radians} \end{array}$$

The diagonal elements in the covariance matrix $\mathbf{R}'$ are 0.8637×10^{-6}, 0.1882×10^{-5}, 0.2189×10^{-6}, 0.7591×10^{-3}, 0.8786×10^{-3}, 0.1812×10^{-2} and 0.1532×10^{-2}. Find the estimates of the measurement errors $\hat{e}_i$, and the corresponding standardized errors.

Solution:

Refer to the solution of Prob. 15.14 for expressions for $h(x)$. The estimates of the measurement errors are

$$\hat{e}_1 = z_1 - h_1(x) = z_1 - |V_1| = 1.01 - 1.0109 = -0.0009$$

$$\hat{e}_2 = z_2 - h_2(x) = z_2 - |V_2| = 1.02 - 1.0187 = 0.0013$$

$$\hat{e}_3 = z_3 - h_3(x) = z_3 - |V_3| = 0.98 - 0.9804 = -0.0004$$

$$\hat{e}_4 = z_4 - h_4(x) = 0.48 - [10 \times 1.0109 \times 1.0187 \times \sin(0.0101)$$
$$+ 12.5 \times 1.0109 \times 0.9804 \times \sin(0.0308)]$$
$$= 0.48 - 0.1040 - 0.3815 = -0.0055$$

$$\hat{e}_5 = z_5 - h_5(x) = 0.33 - [10 \times 1.0187 \times 1.0109 \times \sin(-0.0101)$$
$$+ 20 \times 1.0187 \times 0.9804 \times \sin(-0.0101 + 0.0308)]$$
$$= 0.33 + 0.1040 - 0.4134 = 0.0206$$

$$\hat{e}_6 = z_6 - h_6(x) = 0.41 - [12.5 \times 1.0109 \times 0.9804 \times \sin(0.0308)]$$
$$= 0.41 - 0.3815 = 0.0285$$

$$\hat{e}_7 = z_7 - h_7(x) = 0.38 - [20 \times 1.0187 \times 0.9804 \times \sin(-0.0101 + 0.0308)]$$
$$= 0.38 - 0.4134 = -0.0334$$

The standardized errors become

$$\frac{\hat{e}_1}{\sqrt{R'_{11}}} = \frac{-0.0009}{\sqrt{0.8637 \times 10^{-6}}} = -0.9684$$

$$\frac{\hat{e}_2}{\sqrt{R'_{22}}} = \frac{0.0013}{\sqrt{0.1882 \times 10^{-5}}} = 0.9476$$

$$\frac{\hat{e}_3}{\sqrt{R'_{33}}} = \frac{-0.0004}{\sqrt{0.2189 \times 10^{-6}}} = -0.8549$$

$$\frac{\hat{e}_4}{\sqrt{R'_{44}}} = \frac{-0.0055}{\sqrt{0.7591 \times 10^{-3}}} = -0.1996$$

$$\frac{\hat{e}_5}{\sqrt{R'_{55}}} = \frac{0.0206}{\sqrt{0.8786 \times 10^{-3}}} = 0.6950$$

$$\frac{\hat{e}_6}{\sqrt{R'_{66}}} = \frac{0.0285}{\sqrt{0.1812 \times 10^{-2}}} = 0.6695$$

$$\frac{\hat{e}_7}{\sqrt{R'_{77}}} = \frac{-0.0334}{\sqrt{0.1532 \times 10^{-2}}} = -0.8533$$

15.16 Solve Prob. 15.14 when the two wattmeters installed on lines ①–③ and ②–③ are replaced with two varmeters and their readings are 0.08 and 0.24 per unit, respectively.

Solution:

(a) Five state variables are defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_1| \\ |V_2| \\ |V_3| \end{bmatrix}$$

The expressions of the measurement errors are

$$e_1 = z_1 - h_1 = z_1 - |V_1| = z_1 - x_3$$
$$e_2 = z_2 - h_2 = z_2 - |V_2| = z_2 - x_4$$
$$e_3 = z_3 - h_3 = z_3 - |V_3| = z_3 - x_5$$
$$e_4 = z_4 - h_4 = z_4 - \left[\frac{|V_1||V_2|}{|Z_{12}|} \sin(\delta_1 - \delta_2) + \frac{|V_1||V_3|}{|Z_{13}|} \sin(\delta_1 - \delta_3) \right]$$
$$= z_4 - \left[\frac{x_3 x_4}{0.1} \sin(0 - x_1) + \frac{x_3 x_5}{0.08} \sin(0 - x_2) \right]$$
$$e_5 = z_5 - h_5 = z_5 - \left[\frac{|V_2||V_1|}{|Z_{12}|} \sin(\delta_2 - \delta_1) + \frac{|V_2||V_3|}{|Z_{23}|} \sin(\delta_2 - \delta_3) \right]$$
$$= z_5 - \left[\frac{x_3 x_4}{0.1} \sin(x_1 - 0) + \frac{x_4 x_5}{0.05} \sin(x_1 - x_2) \right]$$
$$e_6 = z_6 - h_6 = z_6 - \left[\frac{|V_1|^2}{|Z_{13}|} - \frac{|V_1||V_3|}{|Z_{13}|} \cos(\delta_1 - \delta_3) \right]$$
$$= z_6 - \left[\frac{(x_3)^2}{0.08} - \frac{x_3 x_5}{0.08} \cos(0 - x_2) \right]$$
$$e_7 = z_7 - h_7 = z_7 - \left[\frac{|V_2|^2}{|Z_{23}|} - \frac{|V_2||V_3|}{|Z_{23}|} \cos(\delta_2 - \delta_3) \right]$$
$$= z_7 - \left[\frac{(x_4)^2}{0.05} - \frac{x_4 x_5}{0.05} \cos(x_1 - x_2) \right]$$

The jacobian matrix $\mathbf{H_x}$ is now written as

$$\mathbf{H_x} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\partial P_1}{\partial \delta_2} & \frac{\partial P_1}{\partial \delta_3} & \frac{\partial P_1}{\partial |V_1|} & \frac{\partial P_1}{\partial |V_2|} & \frac{\partial P_1}{\partial |V_3|} \\ \frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_1|} & \frac{\partial P_2}{\partial |V_2|} & \frac{\partial P_2}{\partial |V_3|} \\ \frac{\partial Q_{13}}{\partial \delta_2} & \frac{\partial Q_{13}}{\partial \delta_3} & \frac{\partial Q_{13}}{\partial |V_1|} & \frac{\partial Q_{13}}{\partial |V_2|} & \frac{\partial Q_{13}}{\partial |V_3|} \\ \frac{\partial Q_{23}}{\partial \delta_2} & \frac{\partial Q_{23}}{\partial \delta_3} & \frac{\partial Q_{23}}{\partial |V_1|} & \frac{\partial Q_{23}}{\partial |V_2|} & \frac{\partial Q_{23}}{\partial |V_3|} \end{bmatrix}$$

$$
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\dfrac{-|V_1||V_2|\cos(\delta_1-\delta_2)}{|Z_{12}|} & \dfrac{-|V_1||V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & \left(\dfrac{|V_2|\sin(\delta_1-\delta_2)}{|Z_{12}|}+\dfrac{|V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|}\right) & \dfrac{|V_1|\sin(\delta_1-\delta_2)}{|Z_{12}|} & \dfrac{|V_1|\sin(\delta_1-\delta_3)}{|Z_{13}|} \\
\left(\dfrac{|V_2||V_1|\cos(\delta_2-\delta_1)}{|Z_{12}|}+\dfrac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|}\right) & \dfrac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \dfrac{|V_2|\sin(\delta_2-\delta_1)}{|Z_{12}|} & \left(\dfrac{|V_1|\sin(\delta_2-\delta_1)}{|Z_{12}|}+\dfrac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|}\right) & \dfrac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\
0 & \dfrac{-|V_1||V_3|\sin(\delta_1-\delta_3)}{|Z_{13}|} & \dfrac{2|V_1|}{Z_{13}}-\dfrac{|V_3|\cos(\delta_1-\delta_3)}{|Z_{13}|} & 0 & \dfrac{-|V_1|\cos(\delta_1-\delta_3)}{|Z_{13}|} \\
\dfrac{|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \dfrac{-|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \dfrac{2|V_2|}{Z_{23}}-\dfrac{|V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \dfrac{-|V_2|\cos(\delta_2-\delta_3)}{|Z_{23}|}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-10x_3x_4\cos(-x_1) & -12.5x_3x_5\cos(-x_2) & \left(\begin{array}{c}10x_4\sin(-x_1)\\ +12.5x_5\sin(-x_2)\end{array}\right) & 10x_3\sin(-x_1) & 12.5x_3\sin(-x_2) \\
\left(\begin{array}{c}10x_3x_4\cos(x_1)\\ +20x_4x_5\cos(x_1-x_2)\end{array}\right) & -20x_4x_5\cos(x_1-x_2) & 10x_4\sin(x_1) & \left(\begin{array}{c}10x_4\sin(x_1)\\ +20x_5\sin(x_1-x_2)\end{array}\right) & 20x_4\sin(x_1-x_2) \\
0 & -12.5x_3x_5\sin(-x_2) & 25x_3-12.5x_5\cos(-x_2) & 0 & -12.5x_3\cos(-x_2) \\
20x_4x_5\sin(x_1-x_2) & -20x_4x_5\sin(x_1-x_2) & 0 & 40x_4-20x_5\cos(x_1-x_2) & -20x_4\cos(x_1-x_2)
\end{bmatrix}
$$

(b) Using flat-start values,

$$
\mathbf{H_x^{(0)}} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-10 & -12.5 & 0 & 0 & 0 \\
30 & -20 & 0 & 0 & 0 \\
0 & 0 & 12.5 & 0 & -12.5 \\
0 & 0 & 0 & 20 & -20
\end{bmatrix}
$$

$$
\begin{aligned}
e_1^{(0)} &= z_1 - h_1^{(0)} = 1.01 - 1.0 = 0.01 \\
e_2^{(0)} &= z_2 - h_2^{(0)} = 1.02 - 1.0 = 0.02 \\
e_3^{(0)} &= z_3 - h_3^{(0)} = 0.98 - 1.0 = -0.02 \\
e_4^{(0)} &= z_4 - h_4^{(0)} = 0.48 - 0 = 0.48 \\
e_5^{(0)} &= z_5 - h_5^{(0)} = 0.33 - 0 = 0.33 \\
e_6^{(0)} &= z_6 - h_6^{(0)} = 0.08 - 0 = 0.08
\end{aligned}
$$

$$e_7^{(0)} = z_7 - h_7^{(0)} = 0.24 - 0 = 0.24$$

Note that

$$\mathbf{R}^{-1} = \begin{bmatrix} \frac{1}{0.02^2} & \times & \times & \times & \times & \times & \times \\ \times & \frac{1}{0.02^2} & \times & \times & \times & \times & \times \\ \times & \times & \frac{1}{0.02^2} & \times & \times & \times & \times \\ \times & \times & \times & \frac{1}{0.05^2} & \times & \times & \times \\ \times & \times & \times & \times & \frac{1}{0.05^2} & \times & \times \\ \times & \times & \times & \times & \times & \frac{1}{0.05^2} & \times \\ \times & \times & \times & \times & \times & \times & \frac{1}{0.05^2} \end{bmatrix}$$

Using $\mathbf{R}^{-1}$ and $\mathbf{H_X^{(0)}}$ obtained above,

$$\left[\mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \mathbf{H_X^{(0)}} \right]^{-1} =$$

$$\begin{bmatrix} 0.4206 \times 10^{-5} & 0.3592 \times 10^{-5} & -0.4194 \times 10^{-11} & -0.2214 \times 10^{-11} & -0.2715 \times 10^{-11} \\ 0.3592 \times 10^{-5} & 0.7561 \times 10^{-5} & -0.4035 \times 10^{-10} & -0.3918 \times 10^{-10} & -0.3905 \times 10^{-10} \\ -0.4194 \times 10^{-11} & -0.4035 \times 10^{-10} & 0.1409 \times 10^{-3} & 0.1285 \times 10^{-3} & 0.1305 \times 10^{-3} \\ -0.2214 \times 10^{-11} & -0.3918 \times 10^{-10} & 0.1285 \times 10^{-3} & 0.1378 \times 10^{-3} & 0.1337 \times 10^{-3} \\ -0.2715 \times 10^{-11} & -0.3905 \times 10^{-10} & 0.1305 \times 10^{-3} & 0.1337 \times 10^{-3} & 0.1358 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \\ x_5^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \\ x_5^{(0)} \end{bmatrix} + \left(\mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \mathbf{H_X^{(0)}} \right)^{-1} \mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.00952 \\ -0.03078 \\ 0.00371 \\ 0.00923 \\ -0.00294 \end{bmatrix} = \begin{bmatrix} -0.00952 \text{ radian} \\ -0.03078 \text{ radian} \\ 1.00371 \text{ per unit} \\ 1.00923 \text{ per unit} \\ 0.99706 \text{ per unit} \end{bmatrix}$$

15.17 Suppose that real and reactive power flows are measured at both ends of each of the five lines in the four-bus system of Fig. 15.9 using ten wattmeters and ten varmeters. The voltage magnitude is measured at bus ② only, and bus injected powers are not measured at all.

(a) Determine the structure of $\mathbf{H_x}$ by writing the partial derivative form a its non-zero elements, as shown in Example 15.8. Assume that line flow measurements are ordered in the following sequence: ①-②, ①-③, ②-③, ②-④ and ③-④ (and the same sequence also in reverse directions).

(b) Suppose that the elements of the $\mathbf{Y_{bus}}$ of the network is given by

$$Y_{ij} = G_{ij} + jB_{ij} = |Y_{ij}| \underline{/\theta_{ij}}$$

and that the total charging susceptance of line $\textcircled{i}$–$\textcircled{j}$ is B'_{ij}. Write out nonlinear functions which express the measured quantities P_{21} and Q_{21} in terms of state variables.

(c) Write out the expressions, similar to those given in Example 15.8, for the non-zero elements in the rows of the matrix $\mathbf{H_x}$ corresponding to measurements P_{21} and Q_{21} in terms of state variables.

Solution:

(a)

$$
\mathbf{H_x} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\[4pt]
\frac{\partial P_{12}}{\partial \delta_2} & 0 & 0 & \frac{\partial P_{12}}{\partial |V_1|} & \frac{\partial P_{12}}{\partial |V_2|} & 0 & 0 \\[4pt]
0 & \frac{\partial P_{13}}{\partial \delta_3} & 0 & \frac{\partial P_{13}}{\partial |V_1|} & 0 & \frac{\partial P_{13}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial P_{23}}{\partial \delta_2} & \frac{\partial P_{23}}{\partial \delta_3} & 0 & 0 & \frac{\partial P_{23}}{\partial |V_2|} & \frac{\partial P_{23}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial P_{24}}{\partial \delta_2} & 0 & \frac{\partial P_{24}}{\partial \delta_4} & 0 & \frac{\partial P_{24}}{\partial |V_2|} & 0 & \frac{\partial P_{24}}{\partial |V_4|} \\[4pt]
0 & \frac{\partial P_{34}}{\partial \delta_3} & \frac{\partial P_{34}}{\partial \delta_4} & 0 & 0 & \frac{\partial P_{34}}{\partial |V_3|} & \frac{\partial P_{34}}{\partial |V_4|} \\[4pt]
\frac{\partial P_{21}}{\partial \delta_2} & 0 & 0 & \frac{\partial P_{21}}{\partial |V_1|} & \frac{\partial P_{21}}{\partial |V_2|} & 0 & 0 \\[4pt]
0 & \frac{\partial P_{31}}{\partial \delta_3} & 0 & \frac{\partial P_{31}}{\partial |V_1|} & 0 & \frac{\partial P_{31}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial P_{32}}{\partial \delta_2} & \frac{\partial P_{32}}{\partial \delta_3} & 0 & 0 & \frac{\partial P_{32}}{\partial |V_2|} & \frac{\partial P_{32}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial P_{42}}{\partial \delta_2} & 0 & \frac{\partial P_{42}}{\partial \delta_4} & 0 & \frac{\partial P_{42}}{\partial |V_2|} & 0 & \frac{\partial P_{42}}{\partial |V_4|} \\[4pt]
0 & \frac{\partial P_{43}}{\partial \delta_3} & \frac{\partial P_{43}}{\partial \delta_4} & 0 & 0 & \frac{\partial P_{43}}{\partial |V_3|} & \frac{\partial P_{43}}{\partial |V_4|} \\[4pt]
\frac{\partial Q_{12}}{\partial \delta_2} & 0 & 0 & \frac{\partial Q_{12}}{\partial |V_1|} & \frac{\partial Q_{12}}{\partial |V_2|} & 0 & 0 \\[4pt]
0 & \frac{\partial Q_{13}}{\partial \delta_3} & 0 & \frac{\partial Q_{13}}{\partial |V_1|} & 0 & \frac{\partial Q_{13}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial Q_{23}}{\partial \delta_2} & \frac{\partial Q_{23}}{\partial \delta_3} & 0 & 0 & \frac{\partial Q_{23}}{\partial |V_2|} & \frac{\partial Q_{23}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial Q_{24}}{\partial \delta_2} & 0 & \frac{\partial Q_{24}}{\partial \delta_4} & 0 & \frac{\partial Q_{24}}{\partial |V_2|} & 0 & \frac{\partial Q_{24}}{\partial |V_4|} \\[4pt]
0 & \frac{\partial Q_{34}}{\partial \delta_3} & \frac{\partial Q_{34}}{\partial \delta_4} & 0 & 0 & \frac{\partial Q_{34}}{\partial |V_3|} & \frac{\partial Q_{34}}{\partial |V_4|} \\[4pt]
\frac{\partial Q_{21}}{\partial \delta_2} & 0 & 0 & \frac{\partial Q_{21}}{\partial |V_1|} & \frac{\partial Q_{21}}{\partial |V_2|} & 0 & 0 \\[4pt]
0 & \frac{\partial Q_{31}}{\partial \delta_3} & 0 & \frac{\partial Q_{31}}{\partial |V_1|} & 0 & \frac{\partial Q_{31}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial Q_{32}}{\partial \delta_2} & \frac{\partial Q_{32}}{\partial \delta_3} & 0 & 0 & \frac{\partial Q_{32}}{\partial |V_2|} & \frac{\partial Q_{32}}{\partial |V_3|} & 0 \\[4pt]
\frac{\partial Q_{42}}{\partial \delta_2} & 0 & \frac{\partial Q_{42}}{\partial \delta_4} & 0 & \frac{\partial Q_{42}}{\partial |V_2|} & 0 & \frac{\partial Q_{42}}{\partial |V_4|} \\[4pt]
0 & \frac{\partial Q_{43}}{\partial \delta_3} & \frac{\partial Q_{43}}{\partial \delta_4} & 0 & 0 & \frac{\partial Q_{43}}{\partial |V_3|} & \frac{\partial Q_{43}}{\partial |V_4|}
\end{bmatrix}
$$

(b)

$$P_{21} = -|V_2|^2 G_{21} + |V_2||V_1||Y_{21}|\cos(\theta_{21} + \delta_1 - \delta_2)$$

$$Q_{21} = |V_2|^2 B_{21} - |V_2|^2\left(\frac{B'_{21}}{2}\right) - |V_2||V_1||Y_{21}|\sin(\theta_{21} + \delta_1 - \delta_2)$$

(c)

$$\frac{\partial P_{21}}{\partial \delta_2} = |V_2||V_1||Y_{21}|\sin(\theta_{21} - \delta_2)$$

$$\frac{\partial P_{21}}{\partial |V_1|} = |V_2||Y_{21}|\cos(\theta_{21} - \delta_2)$$

$$\frac{\partial P_{21}}{\partial |V_2|} = -2|V_2|G_{21} + |V_1||Y_{21}|\cos(\theta_{21} - \delta_2)$$

$$\frac{\partial Q_{21}}{\partial \delta_2} = |V_2||V_1||Y_{21}|\cos(\theta_{21} - \delta_2)$$

$$\frac{\partial Q_{21}}{\partial |V_1|} = -|V_2||Y_{21}|\sin(\theta_{21} - \delta_2)$$

$$\frac{\partial Q_{21}}{\partial |V_2|} = 2|V_2|B_{21} - |V_2|B'_{21} - |V_1||Y_{21}|\sin(\theta_{21} - \delta_2)$$

$$\text{where } \delta_1 \triangleq 0$$

15.18 The method of Example 15.8 based on measurements of only line flows (plus a voltage measurement at one bus) is applied to the three-bus system of Fig. 15.10 using three wattmeters and three varmeters. The per unit values of the measurements are

$$z_1 = |V_1| = 1.0 \qquad z_5 = Q_{12} = -0.101$$
$$z_2 = P_{12} = 0.097 \qquad z_6 = Q_{13} = 0.048$$
$$z_3 = P_{13} = 0.383 \qquad z_7 = Q_{23} = 0.276$$
$$z_4 = P_{23} = 0.427$$

where the variances of all the measurements are 0.02^2. The per-unit reactances of the lines are as specified in Prob. 15.14. Using bus ① as reference and flat-start values, find the values of the state variables that will be obtained at the end of the first iteration of the weighted least-squares state estimation.

Solution:

The measurement errors are

$$e_1 = z_1 - h_1 = z_1 - |V_1|$$

$$e_2 = z_2 - h_2 = z_2 - P_{12} = z_2 - \frac{|V_1||V_2|}{|Z_{12}|}\sin(\delta_1 - \delta_2)$$

$$e_3 = z_3 - h_3 = z_3 - P_{13} = z_3 - \frac{|V_1||V_3|}{|Z_{13}|}\sin(\delta_1 - \delta_3)$$

$$e_4 = z_4 - h_4 = z_4 - P_{23} = z_4 - \frac{|V_2||V_3|}{|Z_{23}|}\sin(\delta_2 - \delta_3)$$

$$e_5 = z_5 - h_5 = z_5 - Q_{12} = z_5 - \left[\frac{|V_1|^2}{|Z_{12}|} - \frac{|V_1||V_2|}{|Z_{12}|}\cos(\delta_1 - \delta_2)\right]$$

$$e_6 = z_6 - h_6 = z_6 - Q_{13} = z_6 - \left[\frac{|V_1|^2}{|Z_{13}|} - \frac{|V_1||V_3|}{|Z_{13}|}\cos(\delta_1 - \delta_3)\right]$$

$$e_7 = z_7 - h_7 = z_7 - Q_{23} = z_7 - \left[\frac{|V_2|^2}{|Z_{23}|} - \frac{|V_2||V_3|}{|Z_{23}|}\cos(\delta_2 - \delta_3)\right]$$

Using the flat-start values, the initial measurement errors become

$$e_1^{(0)} = 1.0 - 1.0 = 0$$

$$e_2^{(0)} = 0.097 - 0 = 0.097$$

$$e_3^{(0)} = 0.383 - 0 = 0.383$$

$$e_4^{(0)} = 0.427 - 0 = 0.427$$

$$e_5^{(0)} = -0.101 - \left[\frac{1}{0.1} - \frac{1}{0.1}\right] = -0.101$$

$$e_6^{(0)} = 0.048 - \left[\frac{1}{0.08} - \frac{1}{0.08}\right] = 0.048$$

$$e_7^{(0)} = 0.276 - \left[\frac{1}{0.05} - \frac{1}{0.05}\right] = 0.276$$

The jacobian matrix $\mathbf{H_X}$ is given by

$$
\mathbf{H_X} =
\begin{array}{ccccc}
\delta_2 & \delta_3 & |V_1| & |V_2| & |V_3| \\
\end{array}
$$

$$
\mathbf{H_X} =
\left[
\begin{array}{ccccc}
0 & 0 & 1 & 0 & 0 \\
\frac{-|V_1||V_2|\cos(-\delta_2)}{|Z_{12}|} & 0 & \frac{|V_2|\sin(-\delta_2)}{|Z_{12}|} & \frac{|V_1|\sin(-\delta_2)}{|Z_{12}|} & 0 \\
0 & \frac{-|V_1||V_3|\cos(-\delta_3)}{|Z_{13}|} & \frac{|V_3|\sin(-\delta_3)}{|Z_{13}|} & 0 & \frac{|V_1|\sin(-\delta_3)}{|Z_{13}|} \\
\frac{|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \frac{|V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{|V_2|\sin(\delta_2-\delta_3)}{|Z_{23}|} \\
\frac{-|V_1||V_2|\sin(-\delta_2)}{|Z_{12}|} & 0 & \left(\frac{2|V_1|}{|Z_{12}|} - \frac{|V_2|\cos(-\delta_2)}{|Z_{12}|}\right) & \frac{-|V_1|\cos(-\delta_2)}{|Z_{12}|} & 0 \\
0 & \frac{-|V_1||V_3|\sin(-\delta_3)}{|Z_{13}|} & \left(\frac{2|V_1|}{|Z_{13}|} - \frac{|V_3|\cos(-\delta_3)}{|Z_{13}|}\right) & 0 & \frac{-|V_1|\cos(-\delta_3)}{|Z_{13}|} \\
\frac{|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & \frac{-|V_2||V_3|\sin(\delta_2-\delta_3)}{|Z_{23}|} & 0 & \left(\frac{2|V_2|}{|Z_{23}|} - \frac{|V_3|\cos(\delta_2-\delta_3)}{|Z_{23}|}\right) & \frac{-|V_2|\cos(\delta_2-\delta_3)}{|Z_{23}|}
\end{array}
\right]
$$

where $\delta_1 \triangleq 0$.

$$
\mathbf{H_X}^{(0)} =
\left[
\begin{array}{ccccc}
0 & 0 & 1 & 0 & 0 \\
-10 & 0 & 0 & 0 & 0 \\
0 & -12.5 & 0 & 0 & 0 \\
20 & -20 & 0 & 0 & 0 \\
0 & 0 & 10 & -10 & 0 \\
0 & 0 & 12.5 & 0 & -12.5 \\
0 & 0 & 0 & 20 & -20
\end{array}
\right]
$$

Note that

$$
\mathbf{R}^{-1} =
\left[
\begin{array}{ccccccc}
\frac{1}{0.02^2} & \times & \times & \times & \times & \times & \times \\
\times & \frac{1}{0.02^2} & \times & \times & \times & \times & \times \\
\times & \times & \frac{1}{0.02^2} & \times & \times & \times & \times \\
\times & \times & \times & \frac{1}{0.02^2} & \times & \times & \times \\
\times & \times & \times & \times & \frac{1}{0.02^2} & \times & \times \\
\times & \times & \times & \times & \times & \frac{1}{0.02^2} & \times \\
\times & \times & \times & \times & \times & \times & \frac{1}{0.02^2}
\end{array}
\right]
$$

Using $\mathbf{H_X^{(0)}}$ obtained above, we have

$$\left[\mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \mathbf{H_X^{(0)}}\right]^{-1} =$$

$$\begin{bmatrix} 0.1884 \times 10^{-5} & 0.1355 \times 10^{-5} & 0.5578 \times 10^{-18} & 0.5571 \times 10^{-18} & 0.5583 \times 10^{-18} \\ & 0.1693 \times 10^{-5} & 0.4179 \times 10^{-18} & 0.4184 \times 10^{-18} & 0.4176 \times 10^{-18} \\ & & 0.4000 \times 10^{-3} & 0.4000 \times 10^{-3} & 0.4000 \times 10^{-3} \\ & & & 0.4019 \times 10^{-3} & 0.4014 \times 10^{-3} \\ & & & & 0.4017 \times 10^{-3} \end{bmatrix}$$

Finally, we have

$$\begin{bmatrix} \delta_2^{(1)} \\ \delta_3^{(1)} \\ |V_1|^{(1)} \\ |V_2|^{(1)} \\ |V_3|^{(1)} \end{bmatrix} = \begin{bmatrix} \delta_2^{(0)} \\ \delta_3^{(0)} \\ |V_1|^{(0)} \\ |V_2|^{(0)} \\ |V_3|^{(0)} \end{bmatrix} + \left(\mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \mathbf{H_X^{(0)}}\right)^{-1} \mathbf{H_X^{(0)}}^T \mathbf{R}^{-1} \begin{bmatrix} e_1^{(0)} \\ e_2^{(0)} \\ e_3^{(0)} \\ e_4^{(0)} \\ e_5^{(0)} \\ e_6^{(0)} \\ e_7^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.00948 \\ -0.03078 \\ 0.0 \\ 0.01003 \\ -0.00379 \end{bmatrix} = \begin{bmatrix} -0.00948 \ \text{radian} \\ -0.03078 \ \text{radian} \\ 1.0 \qquad \text{per unit} \\ 1.01003 \ \text{per unit} \\ 0.99621 \ \text{per unit} \end{bmatrix}$$

Chapter 16 Problem Solutions

16.1 A 60-Hz four-pole turbogenerator rated 500 MVA, 22 kV has an inertia constant of $H = 7.5$ MJ/MVA. Find (a) the kinetic energy stored in the rotor at synchronous speed and (b) the angular acceleration if the electrical power developed is 400 MW when the input less the rotational losses is 740,000 hp.

Solution:

(a) Kinetic energy $= 500 \times 7.5 = 3750$ MJ

(b) Input power $= 740,000 \times 746 \times 10^{-6} = 552$ MW. By Eq. (16.14),

$$\text{Input power} - \text{rotational loss} = \frac{7.5}{180 \times 60} \frac{d^2\delta}{dt^2} = \frac{552 - 400}{500}$$

$$\frac{d^2\delta}{dt^2} = 437.8 \ \text{elec. degrees/s}^2$$

For a four-pole machine,

$$\frac{d^2\delta}{dt^2} = \frac{437.8}{2} = 218.9 \ \text{mech. degrees/s}^2$$

$$\text{or } 60 \times \frac{218.9}{360} = 36.5 \ \text{rpm/s}^2$$

16.2 If the acceleration computed for the generator described in Prob. 16.1 is constant for a period of 15 cycles, find the change in δ in electrical degrees in that period and the speed in revolutions per minute at the end of 15 cycles. Assume that the generator is synchronized with a large system and has no accelerating torque before the 15-cycle period begins.

Solution:

$$\text{duration of acceleration} = \frac{15}{60} = 0.25 \text{ s}$$

$$\text{acceleration} = 437.8 \text{ elec. degrees/s}^2 = 36.5 \text{ rpm/s}$$

$$\text{change in } \delta \text{ in 15 cycles} = \frac{1}{2}(437.8)(0.25)^2 = 13.68 \text{ elec. degrees}$$

$$\text{synchronous speed} = \frac{120 \times 60}{4} = 1800 \text{ rpm}$$

$$\text{After 15 cycles, speed} = 1800 + 0.25 \times 36.5 = 1809.1 \text{ rpm}$$

16.3 The generator of Prob. 16.1 is delivering rated megavolt-amperes at 0.8 power factor lag when a fault reduces the electric power output by 40%. Determine the accelerating torque in newton-meters at the time the fault occurs. Neglect losses and assume constant power input to the shaft.

Solution:

$$P_a = \omega_m T_a = 0.8 \times 500 - 0.6 \times 0.8 \times 500 = 160 \text{ MW}$$

$$\omega_m = \frac{2\pi f}{2} \text{ mech. radians/s}$$

$$T_a = \frac{160 \times 10^6}{2\pi f/2} = 848,826 \text{ N·m}$$

16.4 Determine the WR^2 of the generator of Prob. 16.1.

Solution:

$$WR^2 = \frac{S_{mach} \times H \times 10^{10}}{2.31(\text{rpm})^2} = \frac{500 \times 7.5 \times 10^{10}}{2.31(1800)^2} = 5,010,422 \text{ lb-ft}^2$$

16.5 A generator having $H = 6$ MJ/MVA is connected to a synchronous motor having $H = 4$ MJ/MVA through a network of reactances. The generator is delivering power of 1.0 per unit to the motor when a fault occurs which reduces the delivered power. At the time when the reduced power delivered is 0.6 per unit determine the angular acceleration of the generator with respect to the motor.

Solution:

$$\frac{6 \times 4}{6 + 4} \times \frac{1}{180f} \frac{d^2\delta_{12}}{dt^2} = 1.0 - 0.6$$

$$\frac{d^2\delta_{12}}{dt^2} = 1800 \text{ elec. degrees/s}^2$$

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16.6 A power system is identical to that of Example 16.3 except that the impedance of each of the parallel transmission lines is $j0.5$ and the delivered power is 0.8 per unit when both the terminal voltage of the machine and the voltage of the infinite bus are 1.0 per unit. Determine the power-angle equation for the system during the specified operating conditions.

Solution:
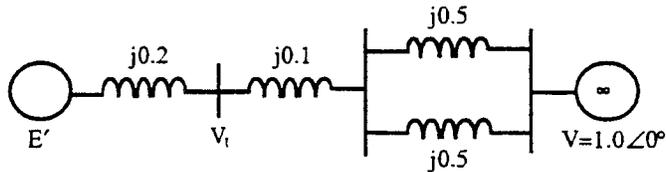
X between V_t and V is

$$j0.1 + \frac{j0.5}{2} = j0.35 \text{ per unit}$$

If $V_t = 1.0 \angle \alpha$,

$$\frac{1.0 \times 1.0}{j0.35} \sin \alpha = 0.8, \quad \alpha = 16.26°$$

$$I = \frac{1.0 \angle 16.26° - 1.0 \angle 0°}{0.35 \angle 90°} = \frac{0.96 + j0.28 - 1.0}{j0.35}$$

$$= 0.8 + j0.1143 = 0.8081 \angle 8.13°$$

$$E' = 1.0 \angle 16.26° + 0.808 \angle 8.13° \times 0.2 \angle 90°$$

$$= 0.96 + j0.28 - 0.023 + j0.16 = 1.0352 \angle 25.15°$$

$$P_e = \frac{1.0352 \times 1.0}{0.35 + 0.20} \sin \delta = 1.882 \sin \delta$$

16.7 If a three-phase fault occurs on the power system of Prob. 16.6 at a point on one of the transmission lines at a distance of 30% of the line length away from the sending-end terminal of the line, determine (*a*) the power-angle equation during the fault and (*b*) the swing equation. Assume the system is operating under the conditions specified in Prob. 16.6 when the fault occurs. Let $H = 5.0$ MJ/MVA as in Example 16.4.

Solution:

The circuit diagram with admittances marked in per unit and the fault as described is shown below:
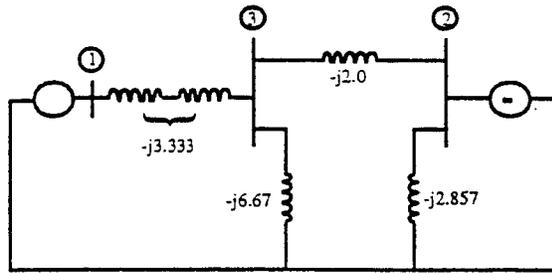
$$\mathbf{Y}_{\text{bus}} = \begin{bmatrix} -j3.333 & 0 & j3.333 \\ 0 & -j4.857 & j2.0 \\ j3.333 & j2.0 & -j12.0 \end{bmatrix}$$

After elimination of node 3 by the usual method, in row 1, column 2 of the new $\mathbf{Y}_{\text{bus}}$ matrix,

$$Y_{12} = \frac{j2.0 \times j3.333}{-j12} = j0.556$$

$$P_e = 1.0352 \times 1.0 \times 0.556 \sin\delta = 0.575 \sin\delta$$

$$\frac{5}{180f} \frac{d^2\delta}{dt^2} = 0.8 - 0.575 \sin\delta$$

$$\frac{d^2\delta}{dt^2} = 36f(0.8 - 0.575 \sin\delta)$$

16.8 Series resistance in the transmission network results in positive values for P_c and γ in Eq. (16.80). For a given electrical power output, show the effects of resistance on the synchronizing coefficient S_p, the frequency of rotor oscillations and the damping of these oscillations.

Solution:

Equation (16.80) is $P_e = P_c + P_{max} \sin(\delta - \gamma)$ and Eq. (16.47) defines

$$S_p = \left.\frac{dP_e}{d\delta}\right|_{\delta=\delta_0}$$

So, if the network is resistive

$$S_p = P_{max} \cos(\delta_0 - \gamma)$$

This S_p is greater than that for a purely reactive network where $\gamma = 0$. Hence, by Eq. (16.50) which shows

$$f_n = \sqrt{\frac{S_p \omega_s}{2H}}$$

wherein f_n is correspondingly larger. We now define $\delta' = \delta - \gamma$ and $P'_m = P_m - P_c$ so that the swing equation becomes

$$\frac{2H}{\omega_s} \times \frac{d^2\delta'}{dt^2} = P'_m - P_{max} \sin\delta'$$

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which must have a solution reflecting undamped oscillations (see footnote in Sec. 16.5) as in a purely reactive network. Consequently, series resistance cannot introduce damping of mechanical oscillations.

16.9 A generator having $H = 6.0$ MJ/MVA is delivering power of 1.0 per unit to an infinite bus through a purely reactive network when the occurrence of a fault reduces the generator output power to zero. The maximum power that could be delivered is 2.5 per unit. When the fault is cleared the original network conditions again exist. Determine the critical clearing angle and critical clearing time.

Solution:

$$
\begin{aligned}
2.5 \sin \delta_0 &= 1.0 \\
\delta_0 &= 23.58° \text{ or } 0.4115 \text{ rad} \\
\text{By Eq. (16.70), } \delta_{cr} &= \cos^{-1} \left[(\pi - 0.823) \sin 23.58° - \cos 23.58° \right] \\
&= \cos^{-1} (0.9275 - 0.9165) = 89.27° = 1.560 \text{ rad} \\
\text{By Eq. (16.72), } t_{cr} &= \sqrt{\frac{4 \times 6 (1.395 - 0.4115)}{2\pi 60 \times 1.0}} = 0.270 \text{ s}
\end{aligned}
$$

16.10 A 60-Hz generator is supplying 60% of P_{max} to an infinite bus through a reactive network. A fault occurs which increases the reactance of the network between the generator internal voltage and the infinite bus by 400%. When the fault is cleared the maximum power that can be delivered is 80% of the original maximum value. Determine the critical clearing angle for the condition described.

Solution:

$$
\begin{aligned}
P_{max} \sin \delta_0 &= 0.6 P_{max} \\
\delta_0 &= 36.87°, \ 0.6435 \text{ rad} \\
r_1 &= 0.25 \qquad r_2 = 0.8 \\
r_2 P_{max} \sin \delta_{max} &= P_m \quad \text{(Fig. 16.11)} \\
\frac{P_m}{P_{max}} &= 0.6 \quad \text{(given)} \\
\sin \delta_{max} &= \frac{0.6}{0.8} = 0.75 \\
\delta_{max} &= 180° - 48.59° = 131.41° = 2.294 \text{ rad} \\
\cos \delta_{cr} &= \frac{0.6(2.294 - 0.6435) + 0.8 \cos 131.4° - 0.25 \cos 36.87°}{0.8 - 0.25} = 0.475 \\
\delta_{cr} &= \cos^{-1} 0.475 = 61.64°
\end{aligned}
$$

16.11 If the generator of Prob. 16.10 has an inertia constant of $H = 6$ MJ/MVA and P_m (equal to 0.6 P_{max}) is 1.0 per-unit power, find the critical clearing time for

the condition of Prob. 16.10. Use $\Delta t = 0.05$ to plot the necessary swing curve.

Solution:

From Prob. 16.10, $\delta_{cr} = 61.64°$ and t_{cr} can be read from the swing curve for a sustained fault

$$P_{max} = \frac{1.0}{0.6} = 1.667 \text{ per unit}$$

$$P_e = 1.667/4 = 0.4167 \text{ during fault}$$

$$k = \frac{180 \times 60}{6}(0.05)^2 = 4.5$$

$$\delta_0 = 36.87° \qquad P_m = 1.0 \qquad P_c = 0 \qquad Y = 0$$

Values in the table below were found by a computer program and rounded off only for tabulation.

t	P_e	P_a	kP_a	$\Delta\delta_n$	δ_n
0−	1.0	0			36.87°
0+	0.250	0.75			36.87°
0 av		0.375	1.688		36.87°
				1.688°	
0.05	0.260	0.740	3.331		38.56°
				5.019°	
0.10	0.287	0.713	3.207		43.58°
				8.226°	
0.15	0.328	0.673	3.026		51.81°
				11.252°	
0.20					63.05°

Problem 16.11 Solution Data

By linear interpolation,

$$t_c \cong 0.15 + 0.05 \left(\frac{61.64 - 51.81}{63.05 - 51.81}\right)$$

$$\cong 0.15 + 0.044 = 0.194 \text{ s or } 11.6 \text{ cycles}$$

16.12 For the system and fault conditions described in Probs. 16.6 and 16.7 determine the power-angle equation if the fault is cleared by the simultaneous opening of breakers at both ends of the faulted line at 4.5 cycles after the fault occurs. Then plot the swing curve of the generator through $t = 0.25$ s.

Solution:

From Prob. 16.6 and 16.7 $E' = 1.0352\underline{/25.15°}$ per unit and before the fault

$$P_e = 1.882\sin\delta \qquad P_m = 0.8 \qquad \delta_0 = 25.15°$$

During the fault,

$$P_e = 0.575\sin\delta$$

after clearing,

$$Y_{12} = \frac{1}{j0.3 + j0.5} = -j1.25 \text{ per unit}$$

and

$$P_e = 1.0352 \times 1.0 \times 1.25 \sin \delta = 1.294 \sin \delta$$
$$k = \frac{180 \times 60}{5}(0.05)^2 = 5.4$$
$$4.5 \text{ cycles} = 0.075 \text{ s} \quad (\text{middle of interval})$$

Values in the table below were found by a computer program and rounded off only for tabulation.

t	P_e	P_a	kP_a	$\Delta\delta_n$	δ_n
0−	0.8	0.0			25.15°
0+	0.244	0.556	3.000		25.15°
0 av			1.500		25.15°
				1.500°	
0.05	0.258	0.542	2.927		26.65°
				4.427°	
0.10	0.668	0.132	0.713		31.08°
				5.140°	
0.15	0.765	0.035	0.191		36.22°
				5.332°	
0.20	0.858	−0.058	−0.315		41.55°
				5.017°	
0.25					46.57°

Problem 16.12 Solution Data

Note: If the table is continued a maximum value of δ will be found equal to 56.20° at $t = 0.45$ s. At 0.55 s, $\delta = 52.56°$.

16.13 Extend Table 16.6 to find δ at $t = 1.00$ s.

Solution:

Continuing the computer program used to generate Table 16.6 and tabulating values only to the fourth decimal place we obtain:

t	$\delta_n - \gamma$	$P_{max}\sin(\delta_n - \gamma)$	P_a	kP_a	$\Delta\delta_n$	δ_n
0.85	16.9591	1.8940	−0.2244	−0.7575		17.8061°
					−3.2292°	
0.90	13.7299	1.5412	0.1284	0.4334		14.5769°
					−2.7957°	
0.95	10.9342	1.2317	0.4379	1.4780		11.7812°
					−1.3177°	
1.0						10.4634°

Problem 16.13 Solution Data

<u>Note</u>: At $t = 1.05$, $\delta = 11.1196°$.

Sample calculation (at $t = 0.85$ s):

$$\delta_n - \gamma = 17.8061 - 0.847 = 16.9591°$$
$$P_{max} \sin(\delta - \gamma) = 6.4934 \sin 16.9591° = 1.8940$$
$$P_a = P_m - P_c - P_{max} \sin(\delta - \gamma) = 1.6696 - 1.8940 = -0.2244$$
$$kP_a = -0.7574$$
$$\Delta\delta_n = \Delta\delta_{n-1} - kP_a = -2.4716 - (-0.7574) = -3.2292°$$

16.14 Calculate the swing curve for machine 2 of Examples 16.9 − 16.11 for fault clearing at 0.05 s by the method described in Sec. 16.9. Compare the results with the values obtained by the production-type program and listed in Table 16.7.

Solution:

Using the computer programmed to obtain δ vs. t showing intermediate steps in the calculation and rounding off only for tabulation we have

t	$\delta_n - \gamma$	$P_{max}\sin(\delta_n - \gamma)$	P_a	kP_a	$\Delta\delta_n$	δ_n
0−			0.000			16.19°
0+	15.435	1.4644	0.2310			16.19°
0 av			0.1155	0.3898		16.19°
					0.3898°	
0.05−	15.8248	1.5005	0.1950			16.5798°
0.05+	15.7328	1.7607	−0.0911			
0.05 av			0.0520	0.1753		
					0.5653°	
0.10	16.2983	1.8223	−0.1527	−0.5153		17.1453°
					0.0500°	
0.15	16.3483	1.8227	−0.1581	−0.5337		17.1953°
					−0.4837°	
0.20	15.8685	1.7751	−0.1055	−0.3559		16.7155°
					−0.8396°	
0.25	15.0249	1.6833	−0.0137	−0.0464		15.8719°
					−0.8860°	
0.30	14.1389	1.5862	0.0834	0.2816		14.9859°
					−0.6044°	
0.35	13.5345	1.5197	0.1499	0.5061		14.3815°
					−0.0983°	
0.40	13.4361	1.5088	0.1608	0.5427		14.2831°
					0.4443°	
0.45	13.8804	1.5577	0.1119	0.3775		14.7274°
					0.8218°	
0.50						15.5493°
0.55						16.444°
0.60						17.0813°
0.65						17.2267°

Problem 16.14 Solution Data

Note: Collecting student prepared computer programs is suggested.

16.15 If the three-phase fault on the system of Example 16.9 occurs on line ④–⑤ at bus ⑤ and is cleared by simultaneous opening of breakers at both ends of the line at 4.5 cycles after the fault occurs prepare a table like that of Table 16.6 to plot the swing curve of machine 2 through $t = 0.30$ s.

Solution:

Before the fault and after clearing, the conditions are the same as in Examples 16.9 and 16.11. During the fault P_m is still 1.85 per unit for machine 2, but $P_e = 0$. So, $P_a = 1.85$ per unit. After clearing, $P_m - P_c = 1.6696$, $P_{max} = 6.4934$. $Y = 0.847°$. Clearing in 4.5 cycles, or $t = 0.075$ s. Values in the table below were obtained by a computer program and rounded off for tabulation only.

t	$P_{max}\sin(\delta_n - \gamma)$	P_a	kP_a	$\Delta\delta_n$	δ_n
0−	1.85	0	0	0°	16.19°
0+	0	1.850	6.244		16.19°
0 av		0.925	3.122		16.19°
				3.122°	
0.05	0	1.85	6.244		19.31°
				9.366°	
0.10	3.031	−1.362	−4.596		28.68°
				4.769°	
0.15	3.498	−1.829	−6.172		33.45°
				−1.403°	
0.20	3.363	−1.694	−5.717		32.04°
				−7.120°	
0.25	2.649	−0.979	−3.306		24.92°
				−10.425°	
0.30	1.533	0.137	0.463		14.50°
				−9.963°	
0.35	0.418	1.252	4.225		4.54°
				−5.738°	
0.40	−0.232	1.902	6.419		−1.20°
				0.681°	
0.45	−0.155	1.825	6.158		−0.52°
				6.839°	
0.50	0.619	1.051	3.546		6.32°
				10.385°	
0.55					16.70°

Problem 16.15 Solution Data

Note: Although the problem does not ask for values beyond $t = 0.30$ s, the table has been extended to show the extent of the variation of δ.

16.16 By applying the equal-area criterion to the swing curves obtained in Examples 16.9 and 16.10 for machine 1, (a) derive an equation for the critical clearing angle, (b) solve the equation by trial and error to evaluate δ_{cr} and (c) use Eq. (16.72) to find the critical clearing time.

Solution:

Note: Students may need guidance in starting this problem which determines the critical clearing time for machine 1 for the fault specified in Example 16.9. This time must, of course, be less than 0.225 s as is evident from examination of Fig. 16.15 and Table 16.7.
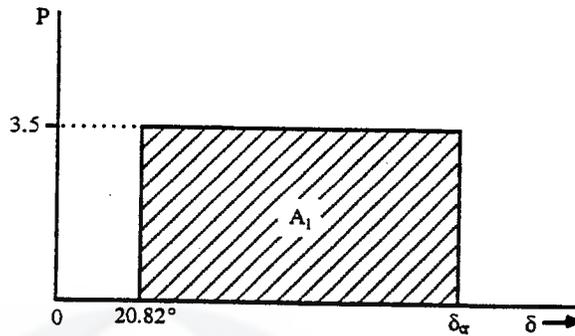
(a) From Example 16.9 for machine 1:

$$P_m = 3.5 \text{ per unit} \quad \text{(Table 16.3)}$$
$$E_1' = 1.100 \underline{/20.82°}$$
$$\text{Thus, } \delta_0 = 20.82° = 0.3634 \text{ rad}$$

Since the impedance between E_1' and the three-phase fault is pure inductive reactance, $P_e = 0$ during the fault and $P_a = P_m - P_e = 3.5$. The area A_1 for the equal-area criterion is shown below.
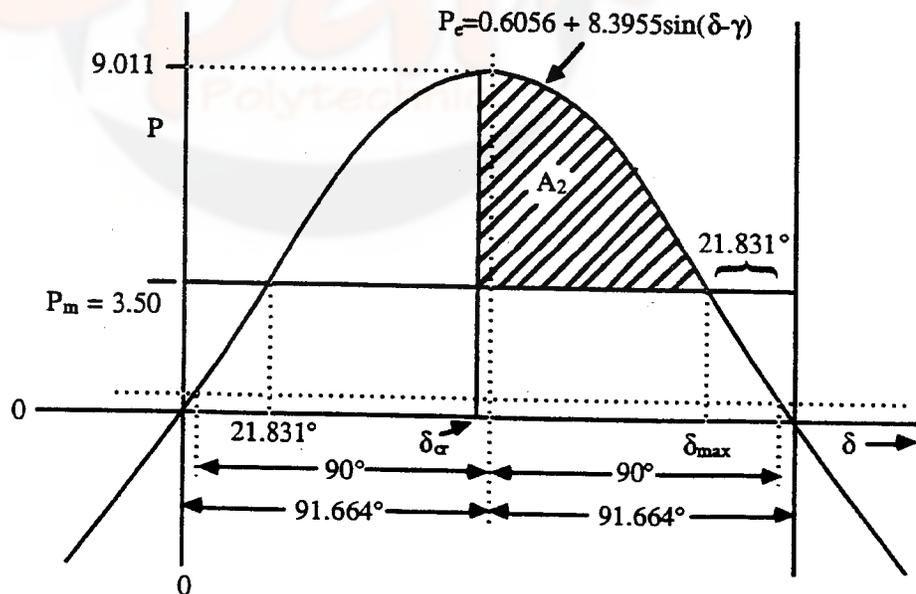

$$\text{where} \quad A_1 = 3.5(\delta_{cr} - 0.3634) = 3.5\delta_{cr} - 1.2719$$

From Example 16.10, the post-fault power-angle curve is given by

$$P_e = 0.6056 + 8.3955\sin(\delta - 1.664°)$$

The curve, P_e vs. δ, is shown below:

Where P_m intercepts the fault curve,

$$3.5 = 0.6056 + 8.3955\sin(\delta - 1.664°)$$

$$\delta = 21.8309°$$

$$P_{e,\,max} = 0.6056 + 8.3955 \quad \text{where } \delta = 90° + 1.664° = 91.664°$$

$$\delta_{max} = 2 \times 91.664° - 21.8309° = 161.497° = 2.8187 \text{ rad}$$

$$\text{Area } A_2 = \int_{\delta_{cr}}^{\delta_{max}} \left[0.6056 + 8.3955\sin(\delta - 1.664°)\right] d\delta - 3.50\,(\delta_{max} - \delta_{cr})$$

$$= (0.6056 - 3.5)\,(\delta_{max} - \delta_{cr}) + 8.3955\left[\cos\left(\delta_{cr} - 1.664°\right) - \cos\left(\delta_{max} - 1.664°\right)\right]$$

$$= -2.8944\,(2.8187 - \delta_{cr}) + 8.3955\left[\cos\left(\delta_{cr} - 1.664°\right) - \cos\left(161.497° - 1.664°\right)\right]$$

$$= -0.2776 + 2.8944\delta_{cr} + 8.3955\cos\left(\delta_{cr} - 1.664°\right)$$

Equating A_1 and A_2 yields

$$0.6056\delta_{cr} - 8.3955\cos\left(\delta_{cr} - 1.664°\right) = 0.9943$$

(b) By trial and error we find

$$\delta_{cr} \cong 91.83° = 1.6027 \text{ rad}$$

(c) The critical clearing time can be found from Eq. (16.72) since $P_e = 0$ during the fault:

$$t_{cr} = \sqrt{\frac{4 \times 11.2\,(1.6027 - 0.3644)}{377 \times 3.5}} = 0.205 \text{ s}$$