Information and Coding Theory

ENEE 5304

Problem Set 1

1. A source generates one of four symbols randomly every six microseconds. The probabilities of the symbols are 0.4, 0.3, 0.2, 0.1. Each emitted symbol is independent of the other symbols in the sequence
   a. What is the entropy H of the source
   b. What is the rate of information generated by this source (in bits/sec)

2. Television presents transmission of electron images (pictures) with rate 25 pictures/sec. A picture is composed of approximate 300,000 basic picture elements (about 600 picture elements in a horizontal line and about 500 horizontal lines per frame). Each of these elements can assume 10 distinguishable brightness levels (such as black and shades of gray) with equal probability.
   a. Find the information content I of a television picture frame.
   b. Find the rate of information R in TV transmission.

3. Find the entropy of the source having L output symbols with the probabilities $p_i = (1/2)^i \ , i = 1, 2, ..., L-1$.

4. Find the entropy of a geometrically distributed random variable $p_x = (1-p)^{x-1} p, \ x \geq 1$

5. Suppose that a binary stationary first-order Markov source has the state transition matrix

$$P_{ij} = \begin{bmatrix} p_{11} = 0.75 & p_{12} = 0.25 \\ p_{21} = 0.33 & p_{22} = 0.67 \end{bmatrix}.$$

   a. Draw the state diagram of the source.
   b. Find the steady state probabilities of the two states.
   c. Find the first order entropy assuming messages of length 1.
   d. Find the second order entropy assuming messages of length 2.
   e. Find the conditional entropy of each state.
   f. Find the source entropy.

6. Let X denote a binary random variable with $p(0) = 0.25$ and let Y be a binary random variable that depends on X through
$p(Y = 1 / X = 0) = p(Y = 0 / X = 1) = 0.1$.

   a. Find $H(Y)$ and $H(Y / X)$. Comment on the result

   b. Find $H(X, Y)$

7. Show that $\ln x \leq x - 1$, with equality occuring in $x = 1$. Using this result, show that $D(p_X(x) || q_X(x)) \geq 0$.

8. *Minimum entropy*. What is the minimum value of $H(p_1, \ldots, p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s that achieve this minimum.

9. Let $p(x, y)$ be given by

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find:

(a) $H(X)$, $H(Y)$.

(b) $H(X \mid Y)$, $H(Y \mid X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y \mid X)$.

10.

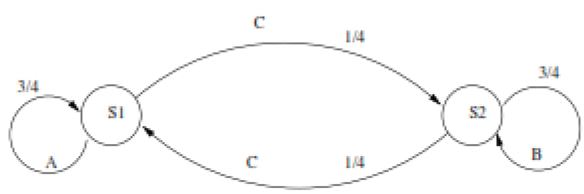
Relative entropy is not symmetric.

Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable:

Symbol	$p(x)$	$q(x)$
a	$\frac{1}{2}$	$\frac{1}{3}$
b	$\frac{1}{4}$	$\frac{1}{3}$
c	$\frac{1}{4}$	$\frac{1}{3}$

Calculate $H(p)$, $H(q)$, $D(p||q)$, and $D(q||p)$. Verify that in this case, $D(p||q) \neq D(q||p)$.

Consider an information source modeled bya first-order Markov process. The source has two states σ_1 and σ_2 and can emit three symbols A, B and C. The probability of emitting any of the thress symbols from each state is indicated in the figure below. The probability of



the states, $P(\sigma_1) = P(\sigma_2) = \frac{1}{2}$. Find the source entropy H. Consider a sequence m consisting of n symbols emitted from the source. We define the average information content per symbol in messages containing n symbols

$$G_n = \frac{1}{n} \sum_i P(m_i) \log_2 \frac{1}{P(m_i)}$$

where the sum is over all sequences m_i consisting of n symbols. Find G_1, G_2 and G_3.