# Dynamics


Sunday, November 1, 2020

جامعة بيرزيت
BIRZEIT UNIVERSITY
Dr. Mamon Horoub

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# ENME232: Dynamics

## CH 16: Planar kinematics of a rigid body

## Lecture 9: Sections 16.1-16.3

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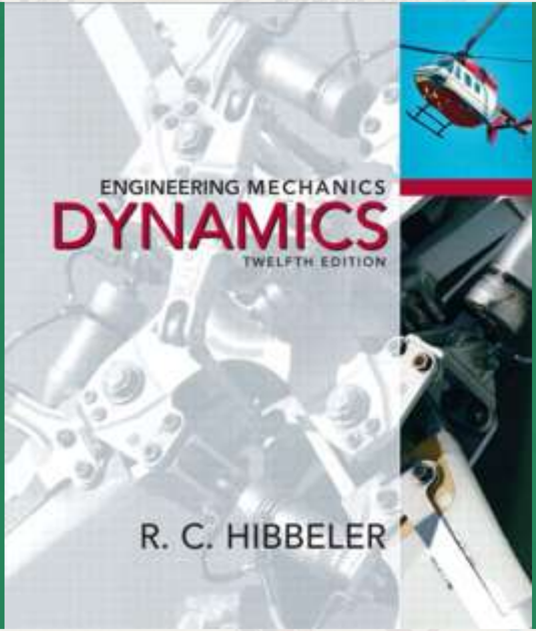
# Text books

**Engineering Mechanics: Dynamics**

C. Hibbeler, 12th Edition, Prentice Hall, 2010

…and probably some more…

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# Recap of the Previous Classes Agenda

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# Today's Class Agenda

1. Introduction
2. Planar Rigid-Body Motion
3. Translation
4. Rotation about a Fixed Axis

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# Part 1

# Introduction

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# Chapter Objectives

- To classify the various types of rigid-body planar motion.
- To investigate rigid body translation and analyze it.
- Study planar motion.
- Relative motion analysis using translating frame of reference.
- Find instantaneous center of zero velocity.
- Relative motion analysis using rotating frame of reference.

# Today's Objectives

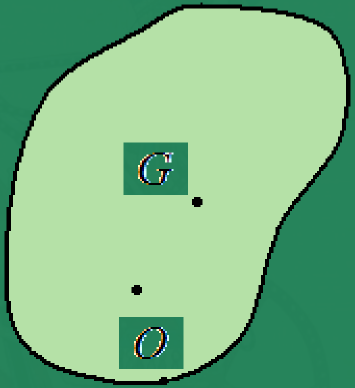
**Students should be able to:**

- Analyze the kinematics of a rigid body undergoing planar translation or rotation about a fixed axis

# Today's Objectives

## What is the rigid body?

In physics, a rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external force exerted on it. Even though such an object cannot physically exist due to relativity, objects can normally be assumed to be perfectly rigid if they are not moving near the speed of light.



# Applications

- Passengers on this amusement ride are subjected to curvilinear translation since the vehicle moves in a circular path but always remains upright.
- If the angular motion of the rotating arms is known, how can we determine the velocity and acceleration experienced by the passengers?
- Does each passenger feel the same acceleration?

# Applications (continued)

- Gears, pulleys and cams, which rotate about fixed axes, are often used in machinery to generate motion and transmit forces. The angular motion of these components must be understood to properly design the system.
- How can we relate the angular motions of contacting bodies that rotate about different fixed axes?

Part 2

# Planar Rigid body motion (section 16.1)

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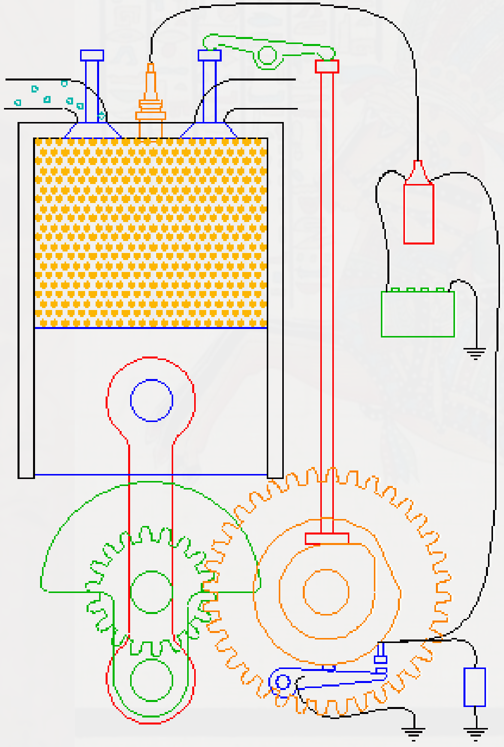
# Introduction

- There are cases where an object cannot be treated as a particle. In these cases the size or shape of the body must be considered. Also, rotation of the body about its center of mass requires a different approach.
- For example, in the design of gears, cams, and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.
- We will now start to study rigid body motion. The analysis will be limited to planar motion.

**A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.**

# Introduction

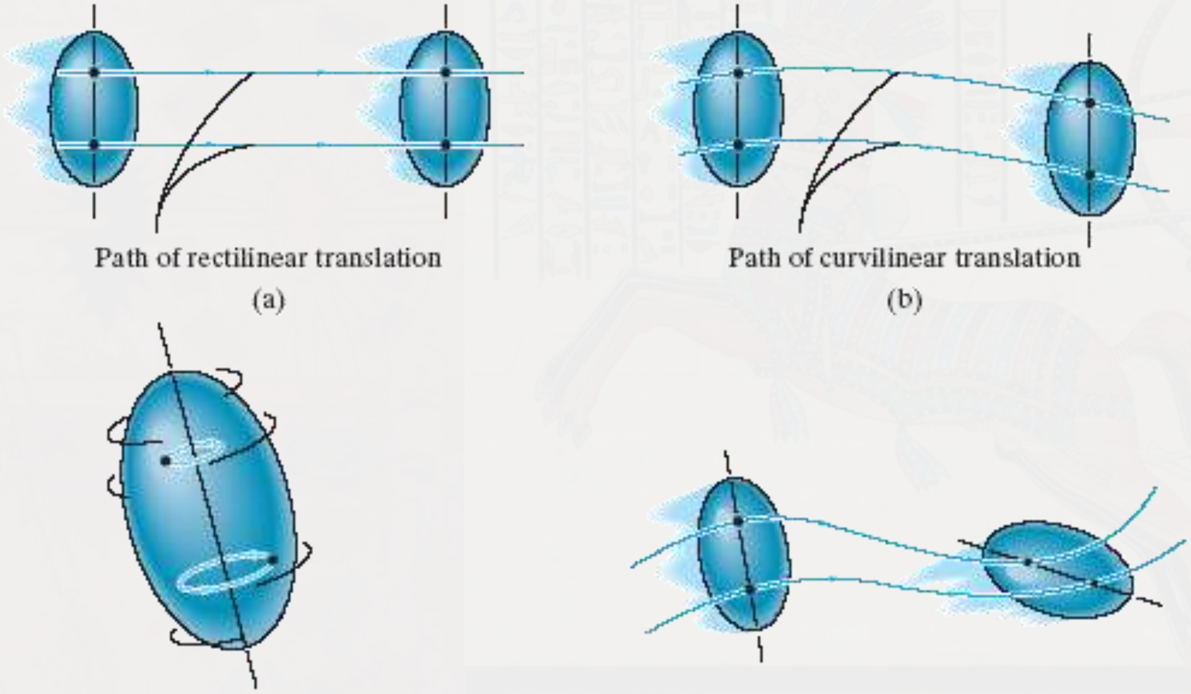
**Four stroke engine**




**Otto Cycle**

# Planar rigid body motion

There are three types of planar rigid body motion.



Path of rectilinear translation
(a)

Path of curvilinear translation
(b)

Rotation about a fixed axis
(c)

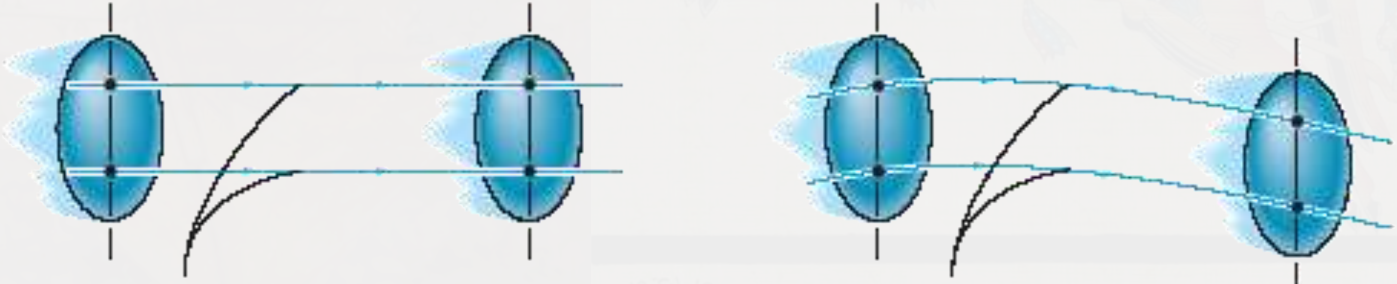
General plane motion
(d)

Fig. 16–1

# Planar rigid body motion

Translation: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called rectilinear translation.

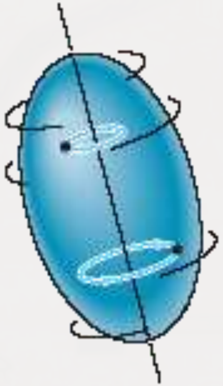
When the paths of motion are curved lines, the motion is called curvilinear translation.



Path of rectilinear translation

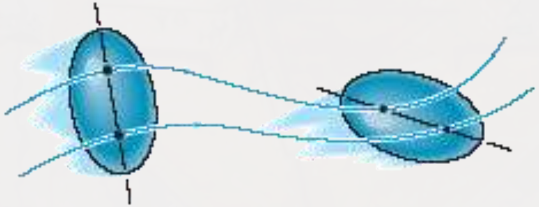
Path of curvilinear translation

# Planar rigid body motion



Rotation about a fixed axis

Rotation about a fixed axis: In this case, all the particles of the body, except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.

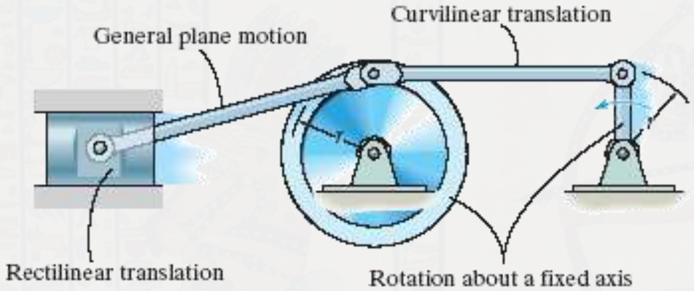


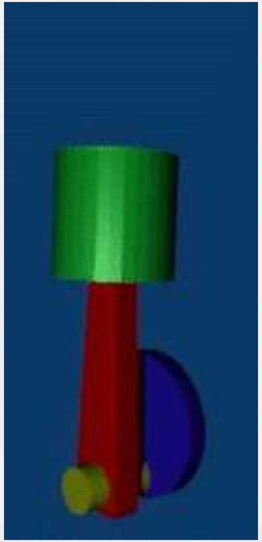
General plane motion

General plane motion: In this case, the body undergoes both translation and rotation. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.

# Planar rigid body motion

An example of bodies undergoing the three types of motion is shown in this mechanism.



- The wheel and crank undergo rotation about a fixed axis. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure.
- The piston undergoes rectilinear translation since it is constrained to slide in a straight line.
- The connecting rod undergoes curvilinear translation, since it will remain horizontal as it moves along a circular path.
- The connecting rod undergoes general plane motion, as it will both translate and rotate.

Part 3

# Rigid body motion – Translation (16.2)

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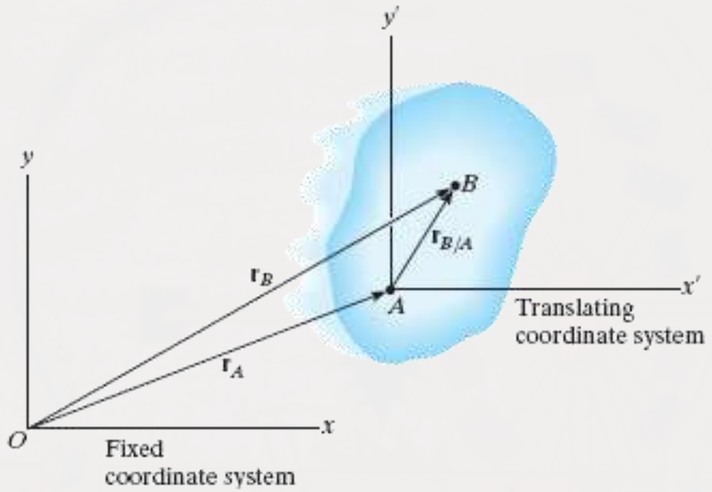
# Rigid body motion – Translation




The positions of two points A and B on a translating body can be related by

$$\boldsymbol{r}_{B} = \boldsymbol{r}_{A} + \boldsymbol{r}_{B/A}$$

Where $r_A$ & $r_B$ are the absolute position vectors defined from the fixed x-y coordinate system, and $r_{B/A}$ is the relative-position vector between B and A.

The velocity at B is $v_B = v_A + dr_{B/A}/dt$ .

Now $dr_{B/A}/dt = 0$ since $r_{B/A}$ is constant. So, $v_B = v_A$, and by following similar logic, $a_B = a_A$.

Note, all points in a rigid body subjected to translation move with the same velocity and acceleration.

# Part 4

# Rigid body motion – Rotation about a fixed axis (16.3)

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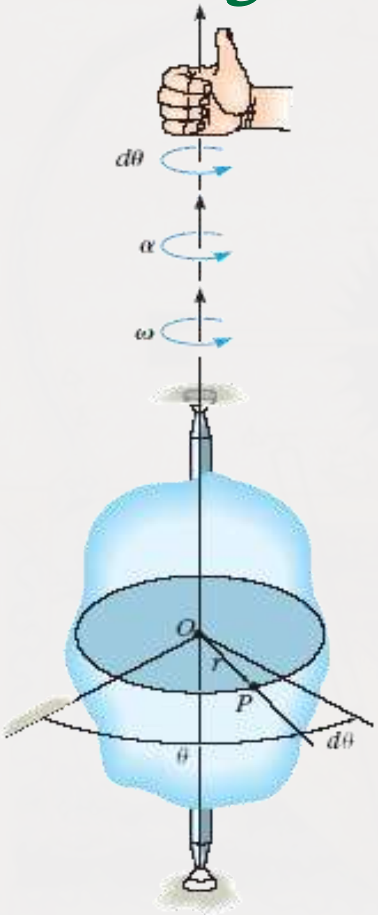
# Rigid body motion – Rotation about a fixed axis




When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by $\theta$.

The change in angular position, $d\theta$, is called the angular displacement, with units of either radians or revolutions. They are related by

1 revolution = $2\pi$ radians

Angular velocity, $\omega$, is obtained by taking the time derivative of angular displacement:

$\omega = d\theta/dt$ (rad/s) +

Similarly, angular acceleration is

$\alpha = d^2\theta/dt^2 = d\omega/dt$ or $\alpha = \omega(d\omega/d\theta)$ + rad/s$^2$

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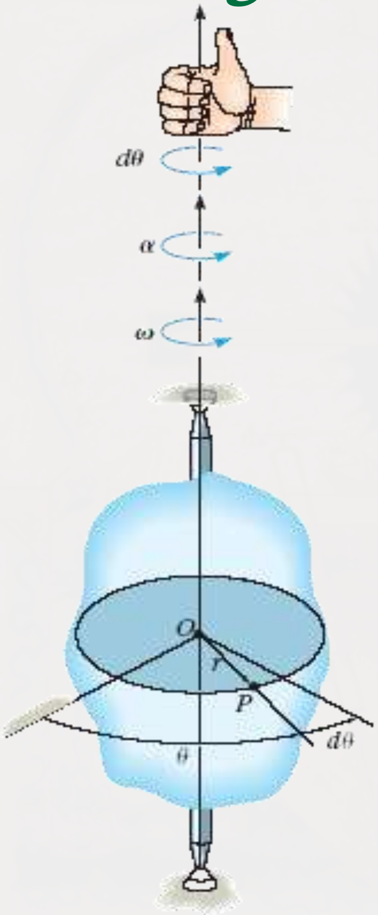
# Rigid body motion – Rotation about a fixed axis




If the angular acceleration of the body is constant, $\alpha = \alpha_C$, the equations for angular velocity and acceleration can be integrated to yield the set of algebraic equations below.

$$\omega = \omega_0 + \alpha_C t$$

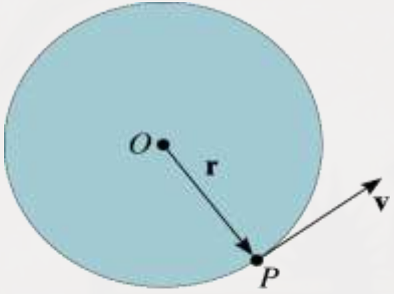
$$\theta = \theta_0 + \omega_0 t + 0.5\alpha_C t^2$$

$$\omega^2 = (\omega_0)^2 + 2\alpha_C (\theta - \theta_0)$$

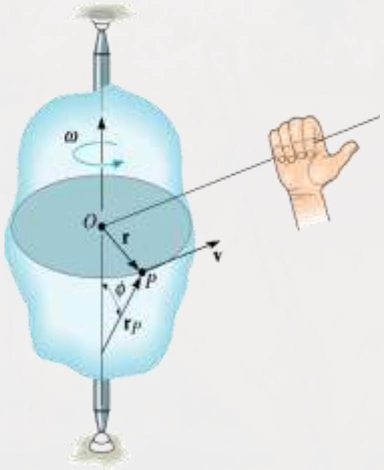
$\theta_0$ and $\omega_0$ are the initial values of the body's angular position and angular velocity.

Note: these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.

# Rigid body rotation – Velocity of point P




The magnitude of the velocity of P is equal to ωr (the text provides the derivation). The velocity's direction is tangent to the circular path of P.

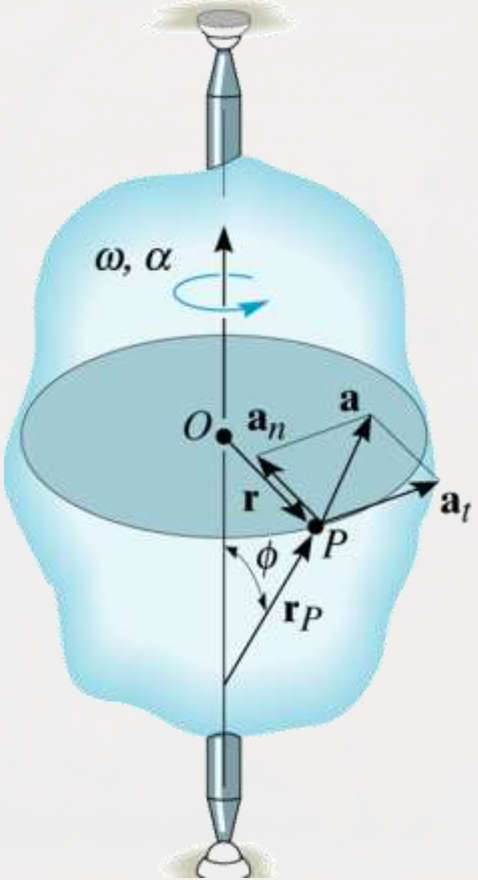



In the vector formulation, the magnitude and direction of $\boldsymbol{v}$ can be determined from the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_p$. Here $\boldsymbol{r}_p$ is a vector from any point on the axis of rotation to P.

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_p = \boldsymbol{\omega} \times \boldsymbol{r}$$


The direction of $\boldsymbol{v}$ is determined by the right-hand rule.

# Rigid body rotation – Acceleration of point P




The acceleration of P is expressed in terms of its normal (***a***$_n$) and tangential (***a***$_t$) components.
In scalar form, these are $a_t = \alpha\, r$ and $a_n = \omega^2 r$.

The tangential component, ***a***$_t$, represents the time rate of change in the velocity's magnitude. It is directed tangent to the path of motion.

The normal component, ***a***$_n$, represents the time rate of change in the velocity's direction. It is directed toward the center of the circular path.

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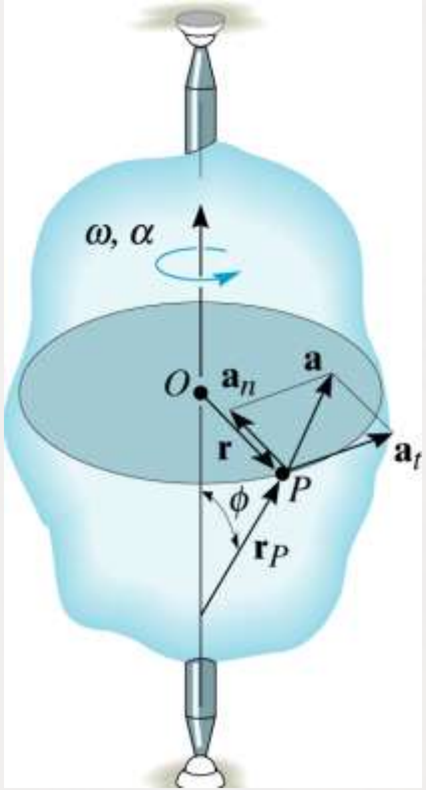
# Rigid body rotation – Acceleration of point P




Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

$$\boldsymbol{a} = d\mathbf{v}/dt = d\boldsymbol{\omega}/dt \times \boldsymbol{r}_P + \boldsymbol{\omega} \times d\boldsymbol{r}_P/dt$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P)$$

It can be shown that this equation reduces to

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

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# Rotation about a fixed axis - Procedure

- Establish a sign convention along the axis of rotation.
- If a relationship is known between any two of the variables ($\alpha$, $\omega$, $\theta$, or t), the other variables can be determined from the equations: $\omega = d\theta/dt$ $\quad \alpha = d\omega/dt$ $\quad \alpha \, d\theta = \omega \, d\omega$
- If $\alpha$ is constant, use the equations for constant angular acceleration.
- To determine the motion of a point, the scalar equations $v = \omega r$, $\quad a_t = \alpha r$, $a_n = \omega^2 r$ , and $a = \sqrt{(a_t)^2 + (a_n)^2}$ can be used.
- Alternatively, the vector form of the equations can be used (with ***i***, ***j***, ***k*** components).

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_P = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{a} = \boldsymbol{a}_t + \boldsymbol{a}_n = \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P) = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r}$$

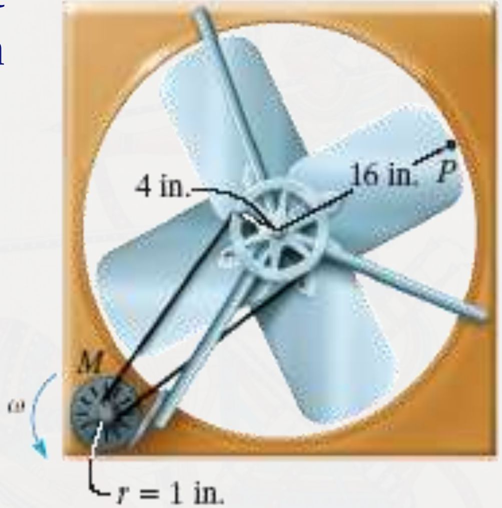
# Example

**Given:** The motor M begins rotating at $\omega = 4(1 - e^{-t})$ rad/s, where t is in seconds. The radii of the motor, fan pulleys, and fan blades are 1 in, 4 in, and 16 in, respectively.

**Find:** The magnitudes of the velocity and acceleration at point P on the fan blade when t = 0.5 s.

**Plan:**

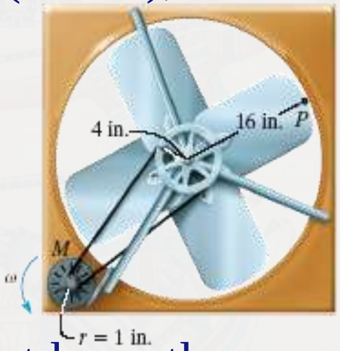
1) Determine the angular velocity and acceleration of the motor using kinematics of angular motion.
2) Assuming the belt does not slip, the angular velocity and acceleration of the fan are related to the motor's values by the belt.
3) The magnitudes of the velocity and acceleration of point P can be determined from the scalar equations of motion for a point on a rotating body.



# Example

**Solution:**

1) Since the angular velocity is given as a function of time, $\omega_m = 4(1 - e^{-t})$, the angular acceleration can be found by differentiation.




$$\alpha_m = d\omega_m/dt = 4e^{-t} \ \ \text{rad/s}^2$$

When t = 0.5 s,

$\omega_m = 4(1 - e^{-0.5}) = 1.5739$ rad/s, $\alpha_m = 4e^{-0.5} = 2.4261$ rad/s$^2$

2) Since the belt does not slip (and is assumed inextensible), it must have the same speed and tangential component of acceleration at all points. Thus the pulleys must have the same speed and tangential acceleration at their contact points with the belt. Therefore, the angular velocities of the motor ($\omega_m$) and fan ($\omega_f$) are related as

$$v = \omega_m r_m = \omega_f r_f \Rightarrow (1.5739)(1) = \omega_f(4) \Rightarrow \omega_f = 0.3935 \text{ rad/s}$$

# Example

**Solution:**

3) Similarly, the tangential accelerations are related as

$$a_t = \alpha_m r_m = \alpha_f r_f \Rightarrow (2.4261)(1) = \alpha_f(4) \Rightarrow \alpha_f = 0.6065 \text{ rad/s}^2$$

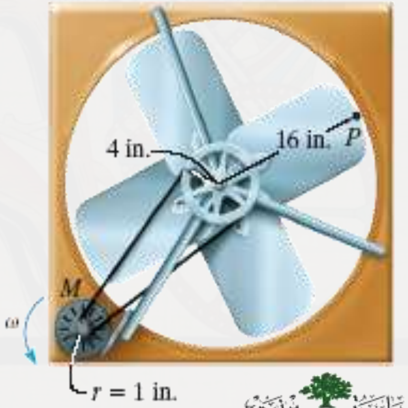
4) The speed of point P on the fan, at a radius of 16 in, is now determined as

$$v_P = \omega_f r_P = (0.3935)(16) = 6.30 \text{ in/s}$$

The normal and tangential components of acceleration of point P are calculated as

$$a_n = (\omega_f)^2 r_P = (0.3935)^2 (16) = 2.477 \text{ in/s}^2$$

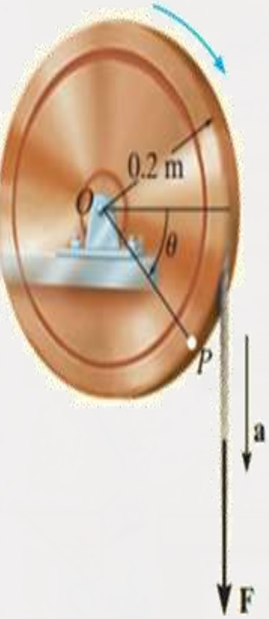
$$a_t = \alpha_f r_P = (0.6065)(16) = 9.704 \text{ in/s}^2$$




The magnitude of the acceleration of P can be determined by

$$a_P = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(2.477)^2 + (9.704)^2} = 10.0 \text{ in/s}^2$$

# Example




A cord is wrapped around a wheel in Fig. 16–5, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \text{m/s}^2$, where $t$ is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line $OP$ in radians.

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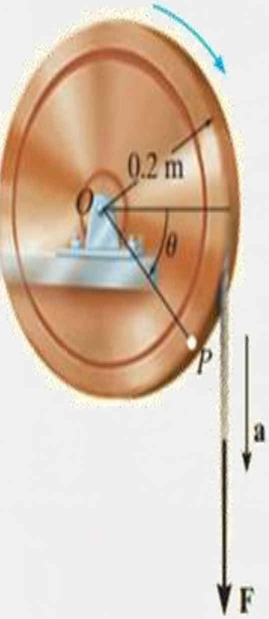
# Example




A cord is wrapped around a wheel in Fig. 16–5, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \text{m/s}^2$, where $t$ is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line $OP$ in radians.

SOLUTION

**Part (a).** The wheel is subjected to rotation about a fixed axis passing through point $O$. Thus, point $P$ on the wheel has motion about a circular path, and the acceleration of this point has *both* tangential and normal components. The tangential component is $(a_P)_t = (4t)\ \text{m/s}^2$, since the cord is wrapped around the wheel and moves *tangent* to it. Hence the angular acceleration of the wheel is

$$(a_P)_t = \alpha r$$

$$(4t)\ \text{m/s}^2 = \alpha(0.2\ \text{m}) \qquad (\circlearrowright +)$$

$$\alpha = (20t)\ \text{rad/s}^2 \circlearrowright$$

Using this result, the wheel's angular velocity $\omega$ can now be determined from $\alpha = d\omega/dt$, since this equation relates $\alpha$, $t$, and $\omega$. Integrating, with the initial condition that $\omega = 0$ when $t = 0$, yields

$$\alpha = \frac{d\omega}{dt} = (20t)\ \text{rad/s}^2 \qquad (\circlearrowright +)$$

$$\int_0^\omega d\omega = \int_0^t 20t\, dt$$

$$\omega = 10t^2\ \text{rad/s} \circlearrowright$$

**Part (b).** Using this result, the angular position $\theta$ of $OP$ can be found from $\omega = d\theta/dt$, since this equation relates $\theta$, $\omega$, and $t$. Integrating, with the initial condition $\theta = 0$ when $t = 0$, we have

$$\frac{d\theta}{dt} = \omega = (10t^2)\ \text{rad/s} \qquad (\circlearrowright +)$$

$$\int_0^\theta d\theta = \int_0^t 10t^2\, dt$$

$$\theta = 3.33t^3\ \text{rad}$$

**NOTE:** We cannot use the equation of constant angular acceleration, since $\alpha$ is a function of time.

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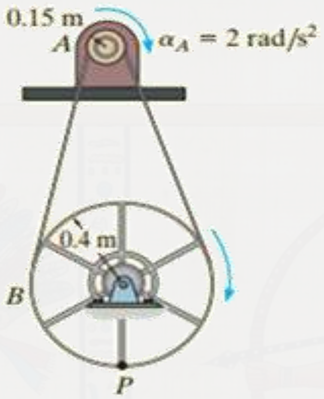
# Example

The motor shown in the photo is used to turn a wheel and attached blower contained within the housing. The details of the design are shown in Fig. 16–6*a*. If the pulley *A* connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$, determine the magnitudes of the velocity and acceleration of point *P* on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.




SOLUTION

**Angular Motion.**

First we will convert the two revolutions to radians. Since there are $2\pi$ rad in one revolution, then

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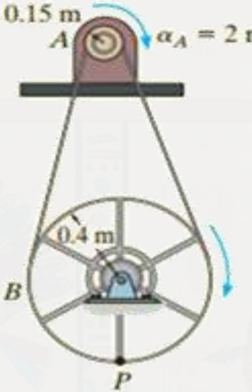
# Example

The motor shown in the photo is used to turn a wheel and attached blower contained within the housing. The details of the design are shown in Fig. 16–6*a*. If the pulley $A$ connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2\ \text{rad/s}^2$, determine the magnitudes of the velocity and acceleration of point $P$ on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.




SOLUTION

**Angular Motion.** First we will convert the two revolutions to radians. Since there are $2\pi$ rad in one revolution, then

$$\theta_A = 2\ \text{rev}\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 12.57\ \text{rad}$$

Since $\alpha_A$ is constant, the angular velocity of pulley $A$ is therefore

$$(\circlearrowleft +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$\omega_A^2 = 0 + 2(2\ \text{rad/s}^2)(12.57\ \text{rad} - 0)$$

$$\omega_A = 7.090\ \text{rad/s}$$

The belt has the same speed and tangential component of acceleration as it passes over the pulley and wheel. Thus,

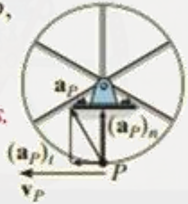
$$v = \omega_A r_A = \omega_B r_B;\quad 7.090\ \text{rad/s}\ (0.15\ \text{m}) = \omega_B(0.4\ \text{m})$$

$$\omega_B = 2.659\ \text{rad/s}$$

$$a_t = \alpha_A r_A = \alpha_B r_B;\quad 2\ \text{rad/s}^2\ (0.15\ \text{m}) = \alpha_B(0.4\ \text{m})$$

$$\alpha_B = 0.750\ \text{rad/s}^2$$

**Motion of *P*.** As shown on the kinematic diagram in Fig. 16–6*b*, we have




$$v_P = \omega_B r_B = 2.659\ \text{rad/s}\ (0.4\ \text{m}) = 1.06\ \text{m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha_B r_B = 0.750\ \text{rad/s}^2\ (0.4\ \text{m}) = 0.3\ \text{m/s}^2$$

$$(a_P)_n = \omega_B^2 r_B = (2.659\ \text{rad/s})^2(0.4\ \text{m}) = 2.827\ \text{m/s}^2$$

Thus $$a_P = \sqrt{(0.3\ \text{m/s}^2)^2 + (2.827\ \text{m/s}^2)^2} = 2.84\ \text{m/s}^2 \qquad \textit{Ans.}$$

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# Planar kinematics of a rigid body:

- Absolute motion analysis, Relative motion analysis: Velocity, Instantaneous centre of zero velocity, Relative motion analysis: Acceleration
- 16.4-16.7

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# Sections' Objectives

**Students should be able to:**

- Determine the velocity and acceleration of a rigid body undergoing general plane motion using the absolute motion analysis (16.4)
- Describe the velocity of a rigid body in terms of translation and rotation components (16.5)
- Perform a relative-motion velocity analysis of a point on the body (16.5)
- Locate the instantaneous center of zero velocity.
- Use the instantaneous center to determine the velocity of any point on a rigid body in general plane motion (16.6)
- Resolve the acceleration of a point on a body into components of translation and rotation (16.7)
- Determine the acceleration of a point on a body by using a relative acceleration analysis (16.7)

# Part 5

# Absolute motion analysis (16.4)

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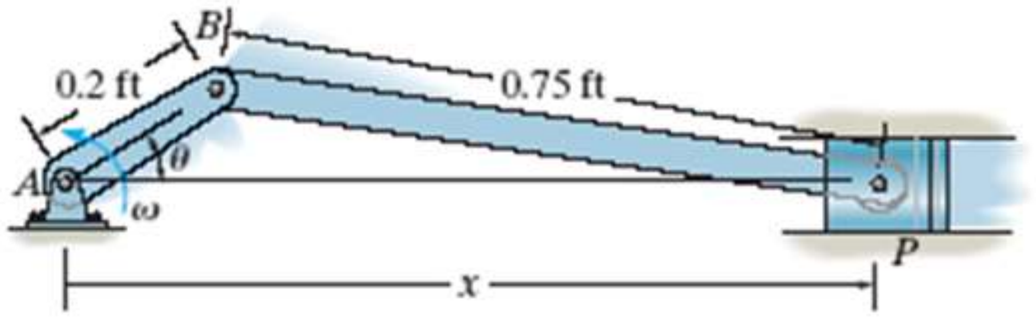
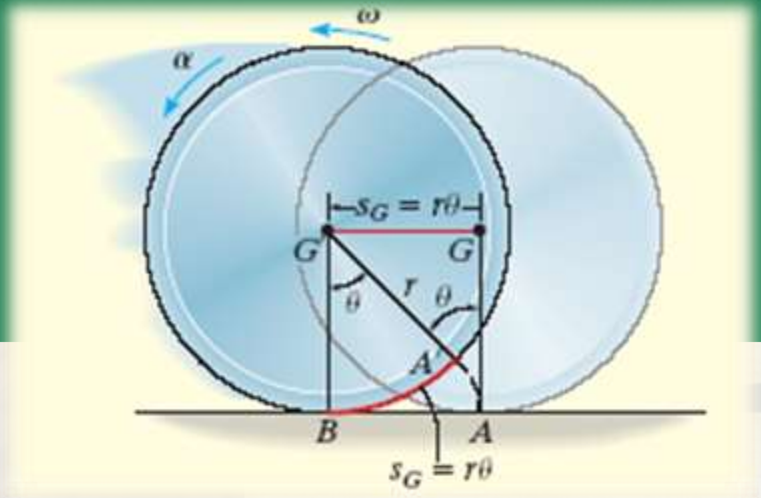
# Applications for absolute motion analysis (16.4)

- The position of the piston, x, can be defined as a function of the angular position of the crank, θ. By differentiating x with respect to time, the velocity of the piston can be related to the angular velocity, ω, of the crank.



# Applications for absolute motion analysis (16.4)

- The rolling of a cylinder is an example of general plane motion.
- During this motion, the cylinder rotates counter clockwise while it translates to the left.

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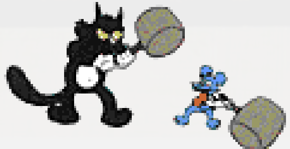
# Procedure for Analysis

The absolute motion analysis method (also called the parametric method) relates the position of a point, P, on a rigid body undergoing rectilinear motion to the angular position, θ (parameter), of a line contained in the body. (Often this line is a link in a machine.)

Once a relationship in the form of $s_P = f(\theta)$ is established, the velocity and acceleration of point P are obtained in terms of the angular velocity, ω, and angular acceleration, α, of the rigid body by taking the first and second time derivatives of the position function. Usually the chain rule must be used when taking the derivatives of the position coordinate equation.



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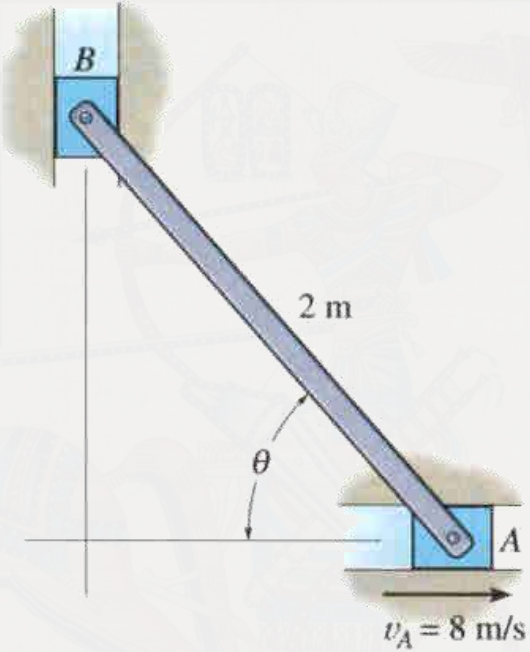
# Problem 1 (16.4)

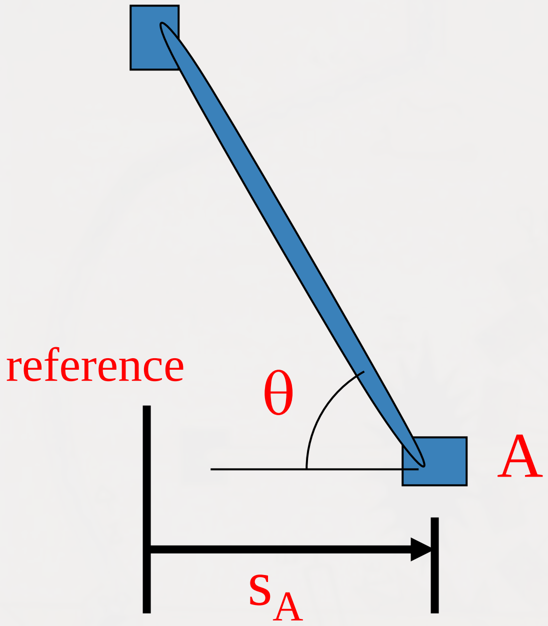



**Given:** Two slider blocks are connected by a rod of length 2 m. Also, $v_A = 8$ m/s and $a_A = 0$.

**Find:** Angular velocity, ω, and angular acceleration, α, of the rod when θ = 60°.

**Plan:** Choose a fixed reference point and define the position of the slider A in terms of the parameter θ. Notice from the position vector of A, positive angular position θ is measured clockwise.

# Problem 1 (16.4)




**Solution:**

By geometry, $s_A = 2 \cos \theta$

By differentiating with respect to time,

$$v_A = -2\,\omega \sin \theta$$

Using $\theta = 60°$ and $v_A = 8$ m/s and solving for $\omega$:

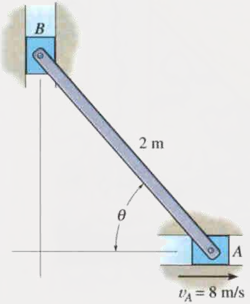
$$\omega = 8/(-2 \sin 60°) = -\,4.62 \text{ rad/s}$$

(The negative sign means the rod rotates counterclockwise as point A goes to the right.)

Differentiating $v_A$ and solving for a,

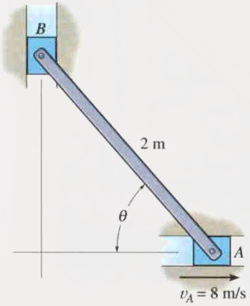
$$a_A = -2\alpha \sin \theta - 2\omega^2 \cos \theta = 0$$

$$\alpha = -\,\omega^2/\tan \theta = -12.32 \text{ rad/s}^2$$



# Problem 1 (16.4)

## Solution:



# Problem 2 (16.4)

The end of rod $R$ shown in Fig. 16–7 maintains contact with the cam by means of a spring. If the cam rotates about an axis passing through point $O$ with an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, determine the velocity and acceleration of the rod when the cam is in the arbitrary position $\theta$.

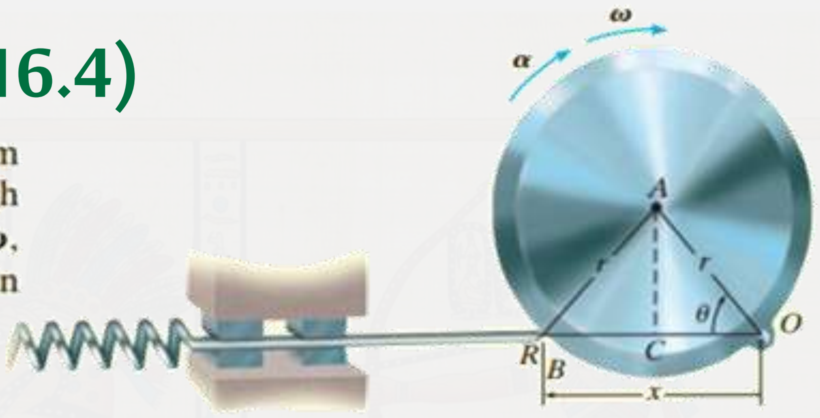



Fig. 16–7

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# Problem 2 (16.4)

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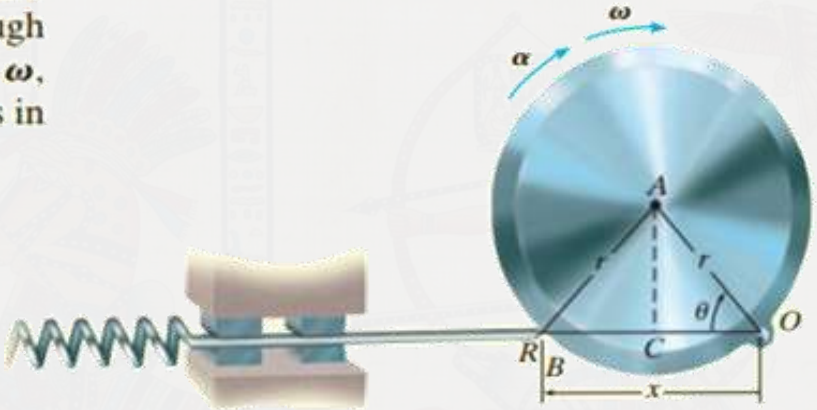

Fig. 16–7

SOLUTION

**Position Coordinate Equation.** Coordinates $\theta$ and $x$ are chosen in order to relate the *rotational motion* of the line segment $OA$ on the cam to the *rectilinear translation* of the rod. These coordinates are measured from the *fixed point* $O$ and can be related to each other using trigonometry. Since $OC = CB = r\cos\theta$, Fig. 16–7, then

$$x = 2r\cos\theta$$

**Time Derivatives.** Using the chain rule of calculus, we have

$$\frac{dx}{dt} = -2r(\sin\theta)\frac{d\theta}{dt}$$

$$v = -2r\omega\sin\theta \qquad \textit{Ans.}$$

$$\frac{dv}{dt} = -2r\left(\frac{d\omega}{dt}\right)\sin\theta - 2r\omega(\cos\theta)\frac{d\theta}{dt}$$

$$a = -2r(\alpha\sin\theta + \omega^2\cos\theta) \qquad \textit{Ans.}$$

**NOTE:** The negative signs indicate that $v$ and $a$ are opposite to the direction of positive $x$. This seems reasonable when you visualize the motion.

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# Problem 3 (16.4)

The large window in Fig. 16–9 is opened using a hydraulic cylinder *AB*. If the cylinder extends at a constant rate of 0.5 m/s, determine the angular velocity and angular acceleration of the window at the instant $\theta = 30°$.

*Ans.*

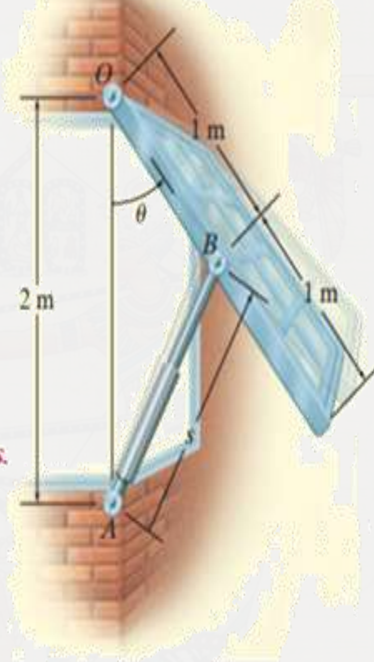


Fig. 16–9

# Problem 3 (16.4)

The large window in Fig. 16–9 is opened using a hydraulic cylinder $AB$. If the cylinder extends at a constant rate of 0.5 m/s, determine the angular velocity and angular acceleration of the window at the instant $\theta = 30°$.

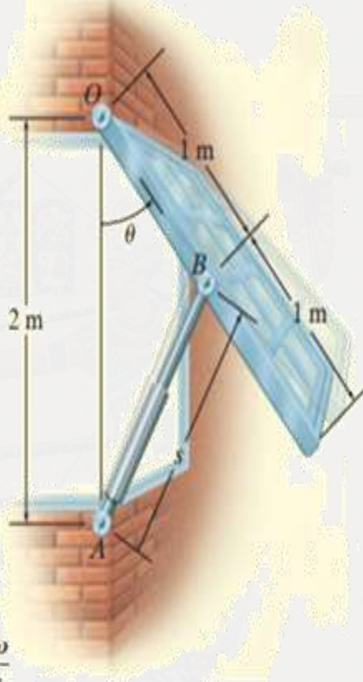


Fig. 16–9

SOLUTION

**Position Coordinate Equation.** The angular motion of the window can be obtained using the coordinate $\theta$, whereas the extension or motion *along the hydraulic cylinder* is defined using a coordinate $s$, which measures its length from the fixed point $A$ to the moving point $B$. These coordinates can be related using the law of cosines, namely,

$$s^2 = (2\text{ m})^2 + (1\text{ m})^2 - 2(2\text{ m})(1\text{ m})\cos\theta$$

$$s^2 = 5 - 4\cos\theta \qquad (1)$$

When $\theta = 30°$,

$$s = 1.239\text{ m}$$

**Time Derivatives.**

Taking the time derivatives of Eq. 1, we have

$$2s\frac{ds}{dt} = 0 - 4(-\sin\theta)\frac{d\theta}{dt}$$

$$s(v_s) = 2(\sin\theta)\omega \qquad (2)$$

Since $v_s = 0.5$ m/s, then at $\theta = 30°$,

$$(1.239\text{ m})(0.5\text{ m/s}) = 2\sin 30°\omega$$

$$\omega = 0.6197\text{ rad/s} = 0.620\text{ rad/s} \qquad \textit{Ans.}$$

Taking the time derivative of Eq. 2 yields

$$\frac{ds}{dt}v_s + s\frac{dv_s}{dt} = 2(\cos\theta)\frac{d\theta}{dt}\omega + 2(\sin\theta)\frac{d\omega}{dt}$$

$$v_s^2 + sa_s = 2(\cos\theta)\omega^2 + 2(\sin\theta)\alpha$$

Since $a_s = dv_s/dt = 0$, then

$$(0.5\text{ m/s})^2 + 0 = 2\cos 30°(0.6197\text{ rad/s})^2 + 2\sin 30°\alpha$$

$$\alpha = -0.415\text{ rad/s}^2 \qquad \textit{Ans.}$$

Because the result is negative, it indicates the window has an angular deceleration.

# Part 6

# Relative motion analysis: Velocity (16.5)

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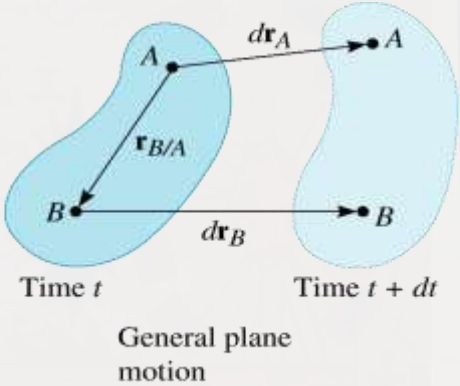
# Applications for Relative motion analysis: Velocity (16.5)

As the slider block A moves horizontally to the left with $\boldsymbol{v}_A$, it causes the link CB to rotate counterclockwise. Thus $\boldsymbol{v}_B$ is directed tangent to its circular path.

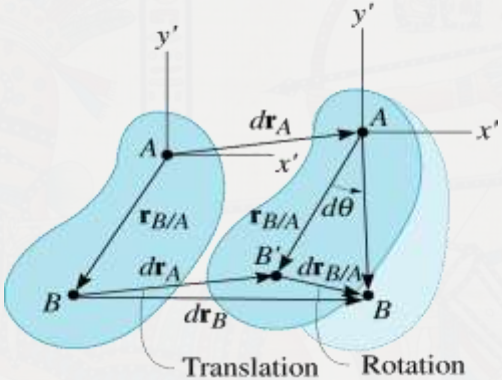
# Relative motion analysis (16.5)

When a body is subjected to general plane motion, it undergoes a combination of translation and rotation.




General plane motion

=




Point A is called the base point in this analysis. It is generally has a known motion. The x'-y' frame translates with the body, but does not rotate. The displacement of point B can be written:

$$d\boldsymbol{r}_B = d\boldsymbol{r}_A + d\boldsymbol{r}_{B/A}$$

Disp. due to translation and rotation

Disp. due to translation

Disp. due to rotation

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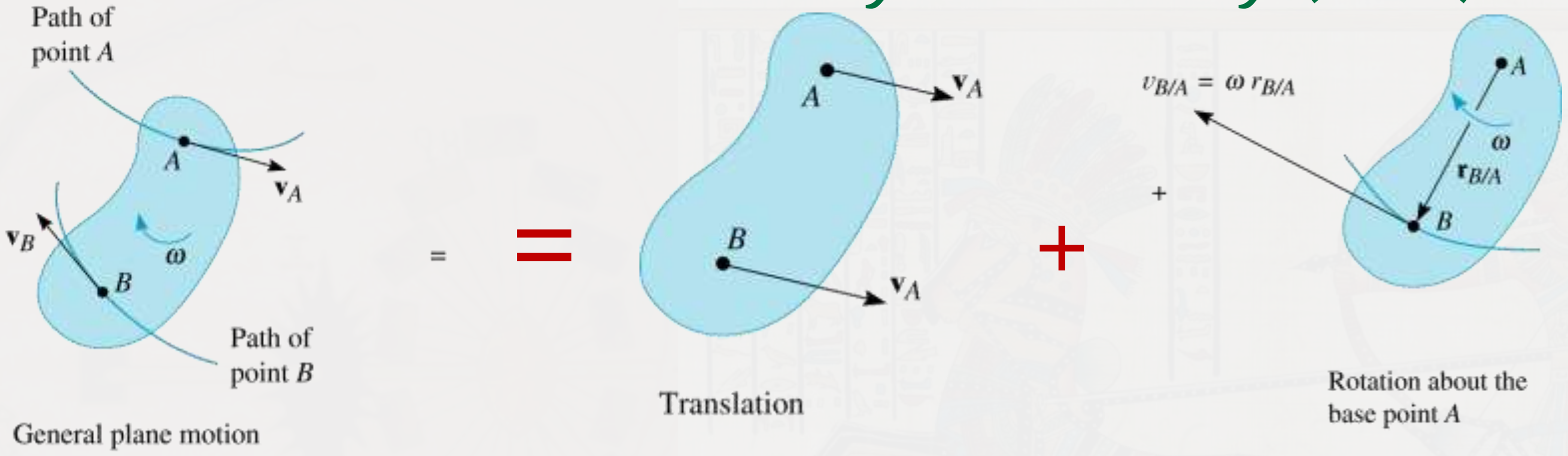
# Relative motion analysis: Velocity (16.5)




The velocity at B is given as : $(d\boldsymbol{r}_B/dt) = (d\boldsymbol{r}_A/dt) + (d\boldsymbol{r}_{B/A}/dt)$ or

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

Since the body is taken as rotating about A,

$$\boldsymbol{v}_{B/A} = d\boldsymbol{r}_{B/A}/dt = \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$$

Here $\boldsymbol{\omega}$ will only have a $\boldsymbol{k}$ component since the axis of rotation is perpendicular to the plane of translation.

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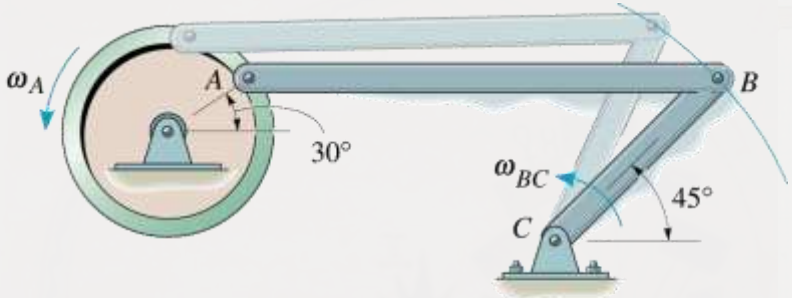
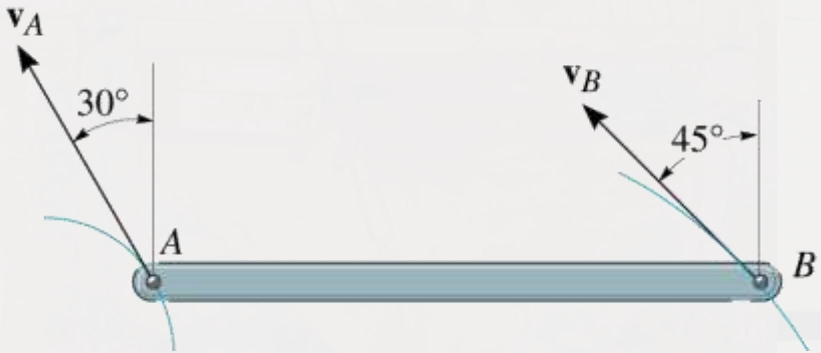
# Relative motion analysis: Velocity (16.5)



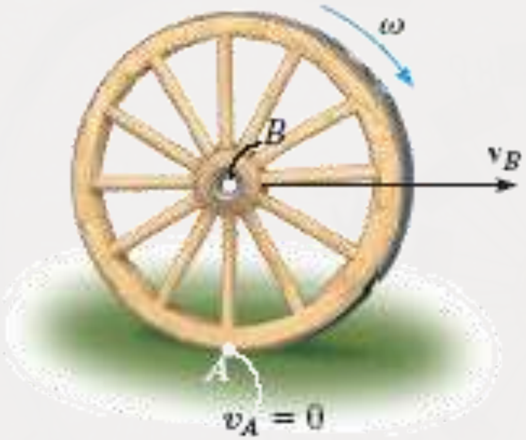

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$$

When using the relative velocity equation, points A and B should generally be points on the body with a known motion. Often these points are pin connections in linkages.

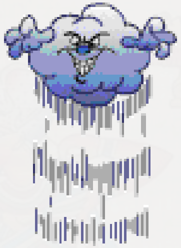



Here both points A and B have circular motion since the disk and link BC move in circular paths. The directions of $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are known since they are always tangent to the circular path of motion.

# Relative motion analysis: Velocity (16.5)




$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$$



When a wheel rolls without slipping, point A is often selected to be at the point of contact with the ground. Since there is no slipping, point A has zero velocity.

Furthermore, point B at the center of the wheel moves along a horizontal path. Thus, $\boldsymbol{v}_B$ has a known direction, e.g., parallel to the surface.

# Relative motion analysis: Analysis Procedure

The relative velocity equation can be applied using either a Cartesian vector analysis or by writing scalar x and y component equations directly.

## Scalar Analysis:

1. Establish the fixed x-y coordinate directions and draw a kinematic diagram for the body. Then, establish the magnitude and direction of the relative velocity vector $\boldsymbol{v}_{B/A}$.
2. Write the equation $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$ and by using the kinematic diagram, underneath each term represent the vectors graphically by showing their magnitudes and directions.
3. Write the scalar equations from the x and y components of these graphical representations of the vectors. Solve for the unknowns.

# Relative motion analysis: Analysis Procedure

The relative velocity equation can be applied using either a Cartesian vector analysis or by writing scalar x and y component equations directly.

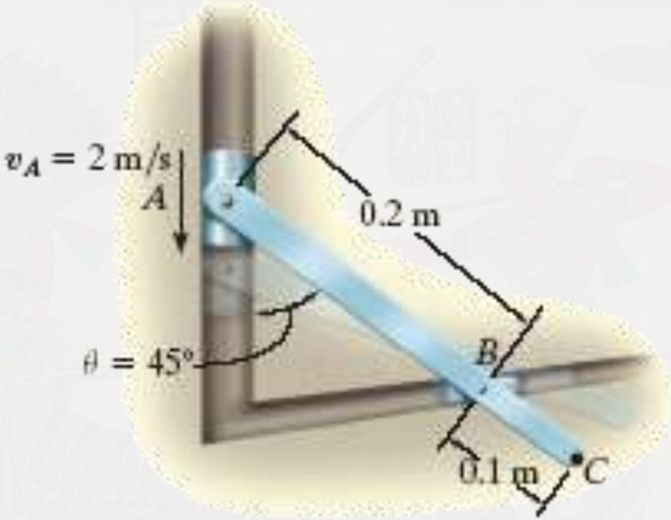
Vector Analysis:

1. Establish the fixed x-y coordinate directions and draw the kinematic diagram of the body, showing the vectors $\boldsymbol{v}_A$, $\boldsymbol{v}_B$, $\boldsymbol{r}_{B/A}$ and $\boldsymbol{\omega}$. If the magnitudes are unknown, the sense of direction may be assumed.

2. Express the vectors in Cartesian vector form and substitute into $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$. Evaluate the cross product and equate respective $\boldsymbol{i}$ and $\boldsymbol{j}$ components to obtain two scalar equations.

3. If the solution yields a negative answer, the sense of direction of the vector is opposite to that assumed.

# Relative motion analysis: Velocity (16.5) Problem 1




**Given:** Block A is moving down at 2 m/s.

**Find:** The velocity of B at the instant $\theta = 45°$.

**Plan:**
1. Establish the fixed x-y directions and draw a kinematic diagram.
2. Express each of the velocity vectors in terms of their ***i***, ***j***, ***k*** components and solve $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$.

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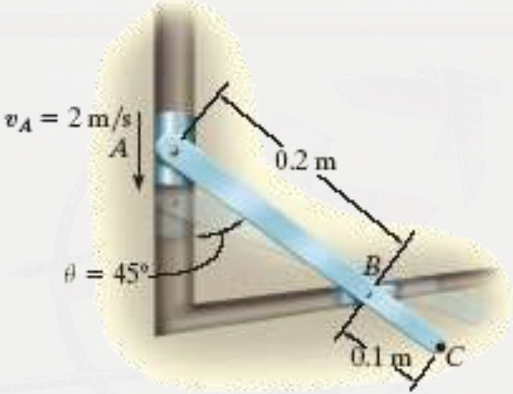
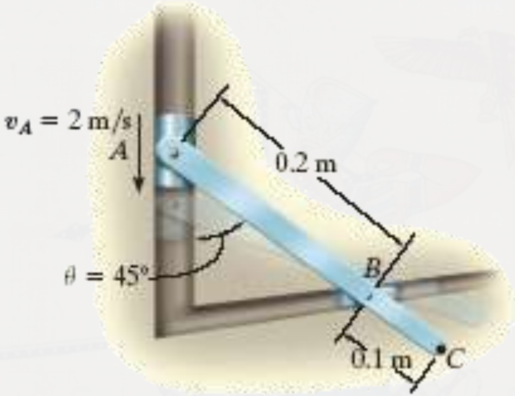
# Relative motion analysis: Velocity (16.5) Problem 1

**Solution:**



# Relative motion analysis: Velocity (16.5) Problem 1

**Solution:**




$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega}_{AB} \times \boldsymbol{r}_{B/A}$$

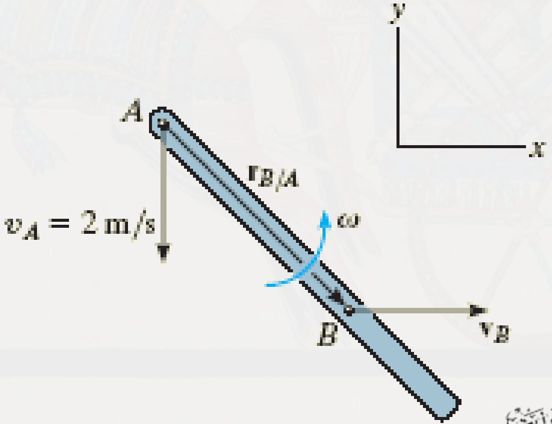
$$v_B\, \boldsymbol{i} = -2\, \boldsymbol{j} + (\omega\, \boldsymbol{k} \times (0.2 \sin 45\, \boldsymbol{i} - 0.2 \cos 45\, \boldsymbol{j}))$$

$$v_B\, \boldsymbol{i} = -2\, \boldsymbol{j} + 0.2\, \omega \sin 45\, \boldsymbol{j} + 0.2\, \omega \cos 45\, \boldsymbol{i}$$

Equating the ***i*** and ***j*** components gives:

$$v_B = 0.2\, \omega \cos 45$$

$$0 = -2 + 0.2\, \omega \sin 45$$




Solving:

$$\omega = 14.1 \text{ rad/s} \quad \text{or} \quad \boldsymbol{\omega}_{AB} = 14.1 \text{ rad/s}\, \boldsymbol{k}$$

$$v_B = 2 \text{ m/s} \quad \text{or} \quad \boldsymbol{v}_B = 2 \text{ m/s}\, \boldsymbol{i}$$

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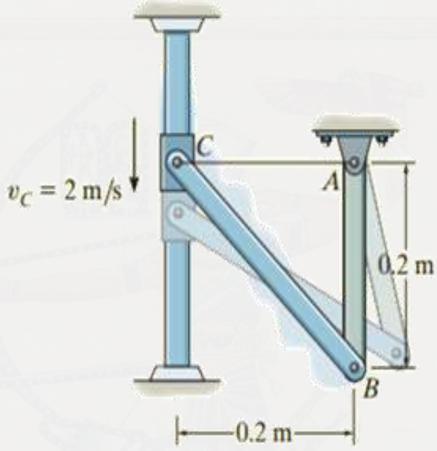
# Relative motion analysis: Velocity (16.5) Problem 2

The collar $C$ in Fig. 16–15$a$ is moving downward with a velocity of 2 m/s. Determine the angular velocity of $CB$ at this instant.



(a)

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# Relative motion analysis: Velocity (16.5) Problem 2

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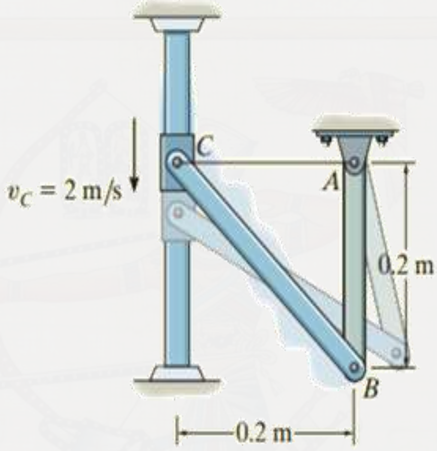



(a)

## SOLUTION I (VECTOR ANALYSIS)

**Kinematic Diagram.** The downward motion of $C$ causes $B$ to move to the right along a curved path. Also, $CB$ and $AB$ rotate counterclockwise.

**Velocity Equation.** Link $CB$ (general plane motion): See Fig. 16–15*b*.

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{CB} \times \mathbf{r}_{B/C}$$

$$v_B\mathbf{i} = -2\mathbf{j} + \omega_{CB}\mathbf{k} \times (0.2\mathbf{i} - 0.2\mathbf{j})$$

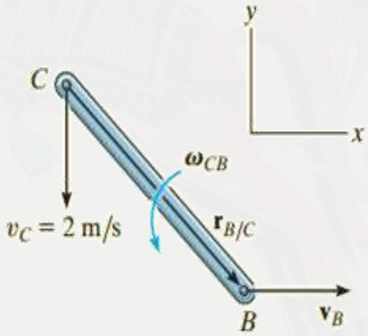
$$v_B\mathbf{i} = -2\mathbf{j} + 0.2\omega_{CB}\mathbf{j} + 0.2\omega_{CB}\mathbf{i}$$

$$v_B = 0.2\omega_{CB} \qquad (1)$$

$$0 = -2 + 0.2\omega_{CB} \qquad (2)$$

$$\omega_{CB} = 10 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

$$v_B = 2 \text{ m/s} \rightarrow$$




(b)

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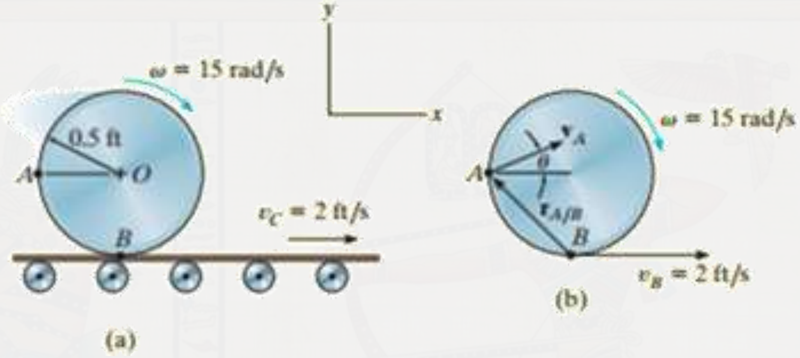
# Relative motion analysis: Velocity (16.5) Problem 3

The cylinder shown in Fig. 16–14*a* rolls without slipping on the surface of a conveyor belt which is moving at 2 ft/s. Determine the velocity of point *A*. The cylinder has a clockwise angular velocity $\omega = 15$ rad/s at the instant shown.



# Relative motion analysis: Velocity (16.5) Problem 3

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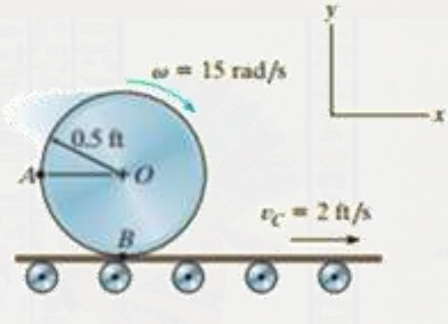
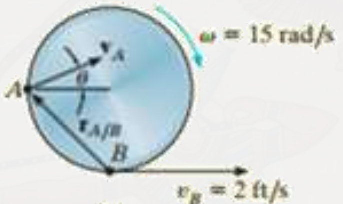


(a)



(b)

## SOLUTION I (VECTOR ANALYSIS)

**Kinematic Diagram.** Since no slipping occurs, point *B* on the cylinder has the same velocity as the conveyor, Fig. 16–14*b*. Also, the angular velocity of the cylinder is known, so we can apply the velocity equation to *B*, the base point, and *A* to determine $\mathbf{v}_A$.

**Velocity Equation.**

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + (-15\mathbf{k}) \times (-0.5\mathbf{i} + 0.5\mathbf{j})$$

$$(v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} = 2\mathbf{i} + 7.50\mathbf{j} + 7.50\mathbf{i}$$

so that

$$(v_A)_x = 2 + 7.50 = 9.50 \text{ ft/s} \qquad (1)$$

$$(v_A)_y = 7.50 \text{ ft/s} \qquad (2)$$

Thus,

$$v_A = \sqrt{(9.50)^2 + (7.50)^2} = 12.1 \text{ ft/s} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{7.50}{9.50} = 38.3^\circ \ \measuredangle \qquad \textit{Ans.}$$

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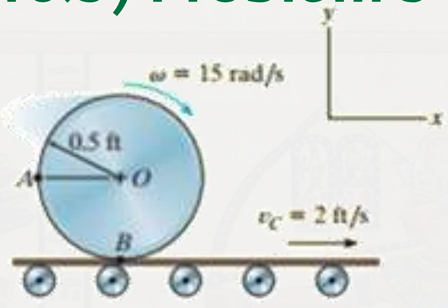
# Relative motion analysis: Velocity (16.5) Problem 3

The cylinder shown in Fig. 16–14*a* rolls without slipping on the surface of a conveyor belt which is moving at 2 ft/s. Determine the velocity of point *A*. The cylinder has a clockwise angular velocity $\omega = 15$ rad/s at the instant shown.



(a)

## SOLUTION II (SCALAR ANALYSIS)

As an alternative procedure, the scalar components of $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ can be obtained directly. From the kinematic diagram showing the relative "circular" motion which produces $\mathbf{v}_{A/B}$, Fig. 16–14*c*, we have

$$v_{A/B} = \omega r_{A/B} = (15\text{ rad/s})\left(\frac{0.5\text{ ft}}{\cos 45^\circ}\right) = 10.6\text{ ft/s}$$

Thus,

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\begin{bmatrix}(v_A)_x\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(v_A)_y\\ \uparrow\end{bmatrix} = \begin{bmatrix}2\text{ ft/s}\\ \rightarrow\end{bmatrix} + \begin{bmatrix}10.6\text{ ft/s}\\ \measuredangle 45^\circ\end{bmatrix}$$

Equating the $x$ and $y$ components gives the same results as before, namely,

$$(\overset{+}{\rightarrow}) \qquad (v_A)_x = 2 + 10.6\cos 45^\circ = 9.50\text{ ft/s}$$

$$(+\uparrow) \qquad (v_A)_y = 0 + 10.6\sin 45^\circ = 7.50\text{ ft/s}$$

Relative motion
(c)

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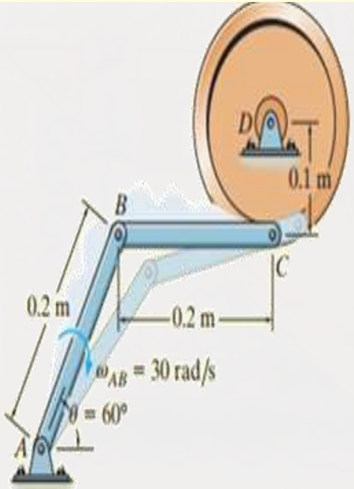
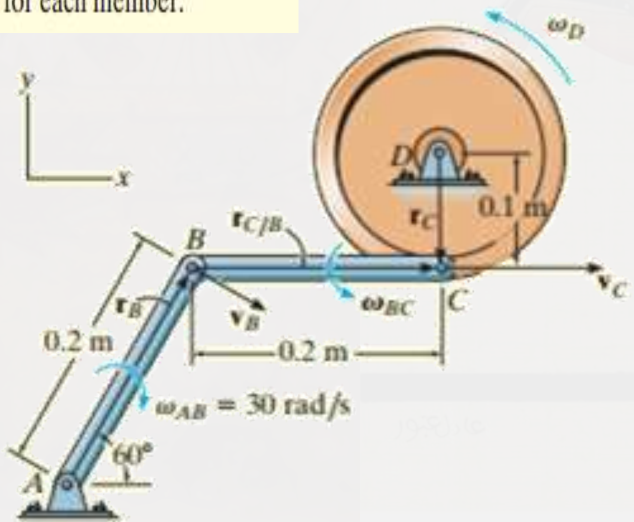
# Relative motion analysis: Velocity (16.5) Problem 4

The bar $AB$ of the linkage shown in Fig. 16–16*a* has a clockwise angular velocity of 30 rad/s when $\theta = 60^\circ$. Determine the angular velocities of member $BC$ and the wheel at this instant.

## SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** By inspection, the velocities of points $B$ and $C$ are defined by the rotation of link $AB$ and the wheel about their fixed axes. The position vectors and the angular velocity of each member are shown on the kinematic diagram in Fig. 16–16*b*. To solve, we will write the appropriate kinematic equation for each member.

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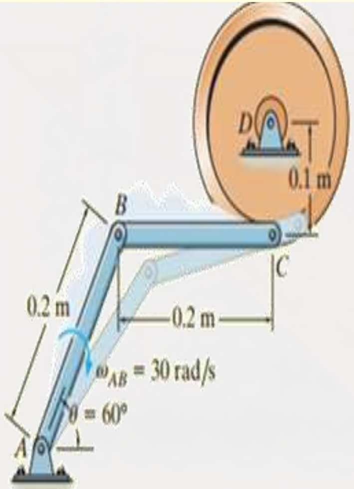
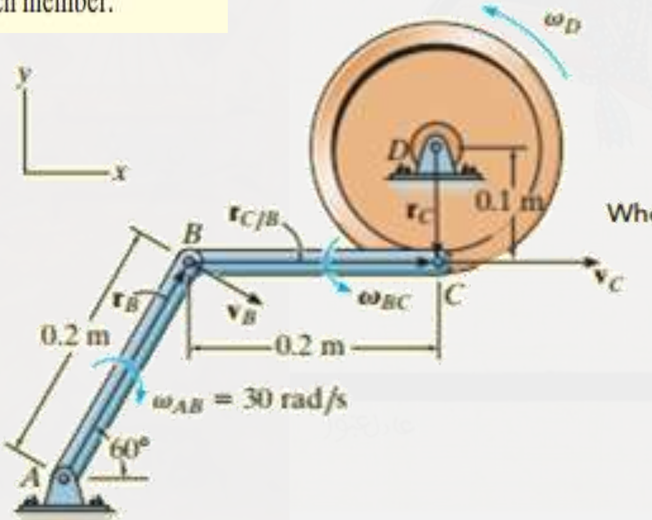
# Relative motion analysis: Velocity (16.5) Problem 4

The bar *AB* of the linkage shown in Fig. 16–16*a* has a clockwise angular velocity of 30 rad/s when $\theta = 60°$. Determine the angular velocities of member *BC* and the wheel at this instant.

SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** By inspection, the velocities of points *B* and *C* are defined by the rotation of link *AB* and the wheel about their fixed axes. The position vectors and the angular velocity of each member are shown on the kinematic diagram in Fig. 16–16*b*. To solve, we will write the appropriate kinematic equation for each member.




$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_B$$
$$= (-30\mathbf{k}) \times (0.2 \cos 60°\mathbf{i} + 0.2 \sin 60°\mathbf{j})$$
$$= \{5.20\mathbf{i} - 3.0\mathbf{j}\} \text{ m/s}$$

Link *BC* (general plane motion):

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$
$$v_C\mathbf{i} = 5.20\mathbf{i} - 3.0\mathbf{j} + (\omega_{BC}\mathbf{k}) \times (0.2\mathbf{i})$$
$$v_C\mathbf{i} = 5.20\mathbf{i} + (0.2\omega_{BC} - 3.0)\mathbf{j}$$
$$v_C = 5.20 \text{ m/s}$$
$$0 = 0.2\omega_{BC} - 3.0$$
$$\omega_{BC} = 15 \text{ rad/s} \circlearrowleft$$

*Ans.*

Wheel (rotation about a fixed axis):

$$\mathbf{v}_C = \boldsymbol{\omega}_D \times \mathbf{r}_C$$
$$5.20\mathbf{i} = (\omega_D\mathbf{k}) \times (-0.1\mathbf{j})$$
$$5.20 = 0.1\omega_D$$
$$\omega_D = 52.0 \text{ rad/s} \circlearrowleft$$

*Ans.*

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# Part 7

# Instantaneous center of zero velocity (16.6)

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# Instantaneous center of zero velocity (16.6)

## Applications :

The instantaneous center (IC) of zero velocity for this bicycle wheel is at the point in contact with ground. The velocity direction at any point on the rim is perpendicular to the line connecting the point to the IC.

# Instantaneous center of zero velocity (16.6)

For any body undergoing planar motion, there always exists a point in the plane of motion at which the velocity is instantaneously zero (if it were rigidly connected to the body).

This point is called the instantaneous center of zero velocity, or IC. It may or may not lie on the body!

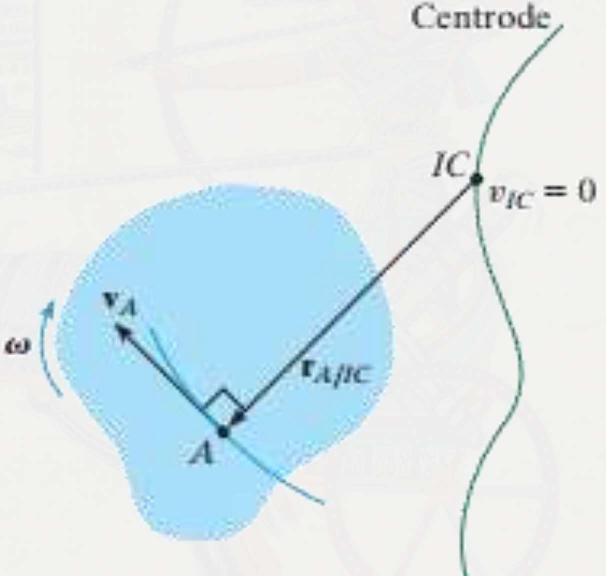
If the location of this point can be determined, the velocity analysis can be simplified because the body appears to rotate about this point at that instant.

# Instantaneous center of zero velocity (16.6)

To locate the IC, we can use the fact that the velocity of a point on a body is always perpendicular to the relative position vector from the IC to the point. Several possibilities exist.

First, consider the case when velocity $\boldsymbol{v}_A$ of a point A on the body and the angular velocity $\boldsymbol{\omega}$ of the body are known.

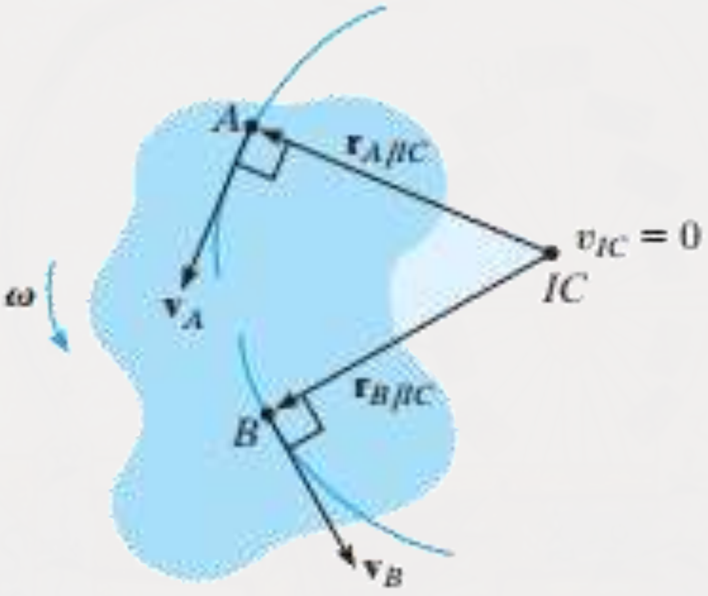
In this case, the IC is located along the line drawn perpendicular to $\boldsymbol{v}_A$ at A, a distance $r_{A/IC} = v_A/\omega$ from A. Note that the IC lies up and to the right of A since $\boldsymbol{v}_A$ must cause a clockwise angular velocity $\boldsymbol{\omega}$ about the IC.




Location of IC knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

(a)

# Instantaneous center of zero velocity (16.6)




Location of $IC$ knowing the directions of $\mathbf{v}_A$ and $\mathbf{v}_B$

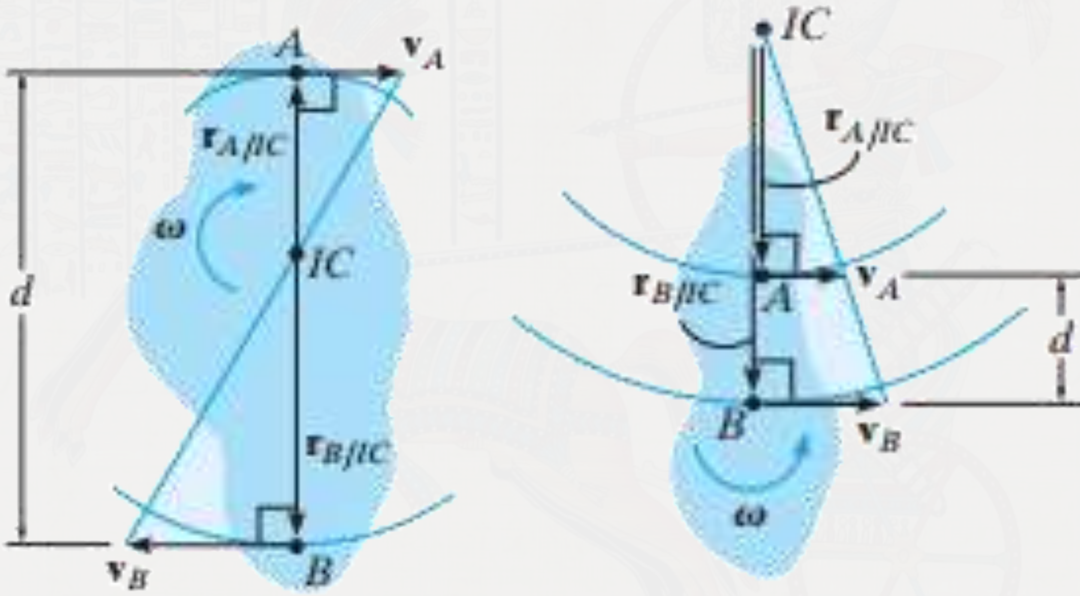
The second case is when the lines of action of two non-parallel velocities, $v_A$ and $v_B$, are known.

First, construct line segments from A and B perpendicular to $v_A$ and $v_B$. The point of intersection of these two line segments locates the IC of the body.

# Instantaneous center of zero velocity (16.6)

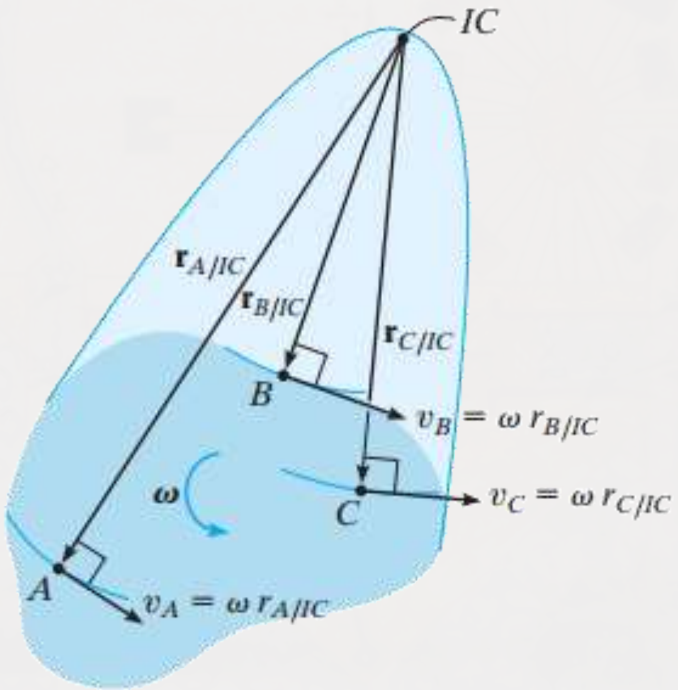
The third case is when the magnitude and direction of two parallel velocities at A and B are known.

Here the location of the IC is determined by proportional triangles. As a special case, note that if the body is translating only ($v_A = v_B$), then the IC would be located at infinity. Then ω equals zero, as expected.



# Instantaneous center of zero velocity (16.6)

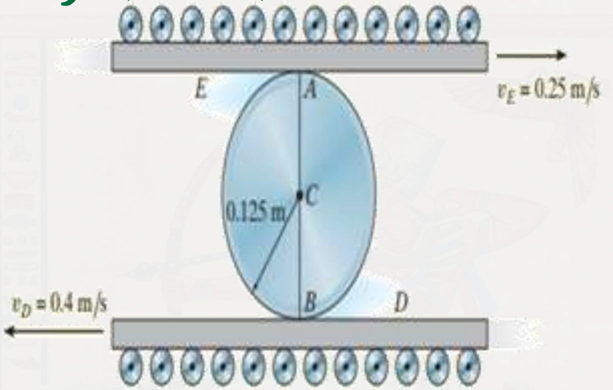
The velocity of any point on a body undergoing general plane motion can be determined easily once the instantaneous center of zero velocity of the body is located.




Since the body seems to rotate about the IC at any instant, as shown in this kinematic diagram, the magnitude of velocity of any arbitrary point is $v = \omega\, r$, where r is the radial distance from the IC to the point. The velocity's line of action is perpendicular to its associated radial line. Note the velocity has a sense of direction which tends to move the point in a manner consistent with the angular rotation direction.

# Instantaneous center of zero velocity (16.6)

The cylinder shown in Fig. 16–22*a* rolls without slipping between the two moving plates *E* and *D*. Determine the angular velocity of the cylinder and the velocity of its center *C*.




(a)

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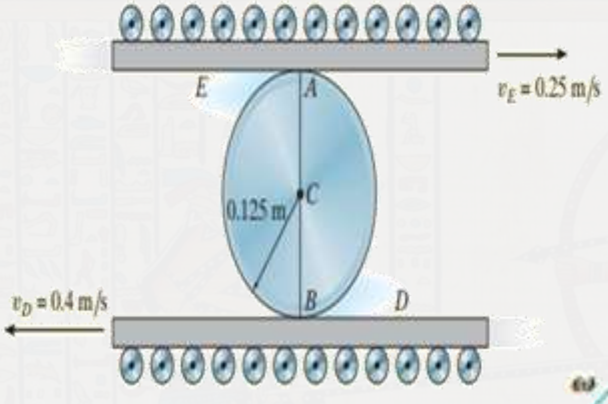
# Instantaneous center of zero velocity (16.6)

The cylinder shown in Fig. 16–22*a* rolls without slipping between the two moving plates *E* and *D*. Determine the angular velocity of the cylinder and the velocity of its center *C*.



(a)

SOLUTION

Since no slipping occurs, the contact points *A* and *B* on the cylinder have the same velocities as the plates *E* and *D*, respectively. Furthermore, the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are *parallel*, so that by the proportionality of right triangles the *IC* is located at a point on line *AB*, Fig. 16–22*b*. Assuming this point to be a distance *x* from *B*, we have

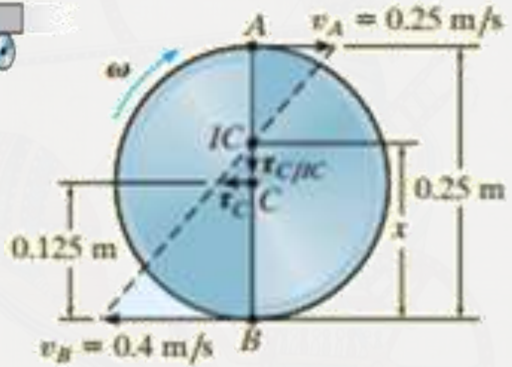
$$v_B = \omega x; \qquad 0.4 \text{ m/s} = \omega x$$

$$v_A = \omega(0.25 \text{ m} - x); \qquad 0.25 \text{ m/s} = \omega(0.25 \text{ m} - x)$$

Dividing one equation into the other eliminates $\omega$ and yields

$$0.4(0.25 - x) = 0.25x$$

$$x = \frac{0.1}{0.65} = 0.1538 \text{ m}$$



(b)

Hence, the angular velocity of the cylinder is

$$\omega = \frac{v_B}{x} = \frac{0.4 \text{ m/s}}{0.1538 \text{ m}} = 2.60 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

The velocity of point *C* is therefore

$$v_C = \omega r_{C/IC} = 2.60 \text{ rad/s}\,(0.1538 \text{ m} - 0.125 \text{ m})$$

$$= 0.0750 \text{ m/s} \leftarrow \qquad \textit{Ans.}$$

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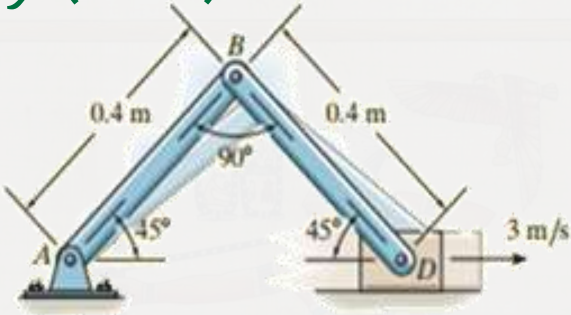
# Instantaneous center of zero velocity (16.6)

Block *D* shown in Fig. 16–21*a* moves with a speed of 3 m/s. Determine the angular velocities of links *BD* and *AB*, at the instant shown.

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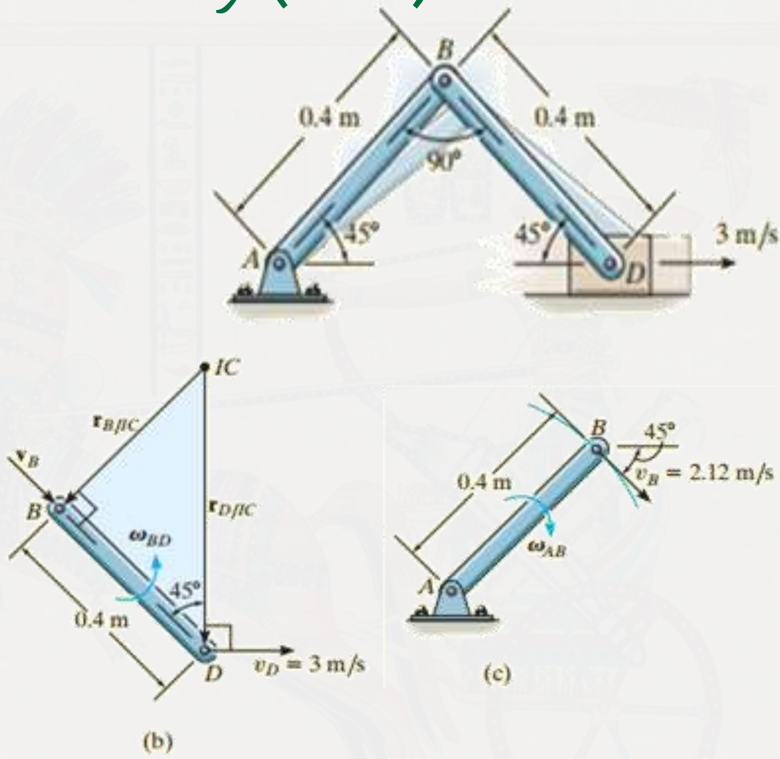
# Instantaneous center of zero velocity (16.6)

Block $D$ shown in Fig. 16–21$a$ moves with a speed of 3 m/s. Determine the angular velocities of links $BD$ and $AB$, at the instant shown.




**SOLUTION**

As $D$ moves to the right, it causes $AB$ to rotate clockwise about point $A$. Hence, $\mathbf{v}_B$ is directed perpendicular to $AB$. The instantaneous center of zero velocity for $BD$ is located at the intersection of the line segments drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_D$, Fig. 16–21$b$. From the geometry,

$$r_{B/IC} = 0.4 \tan 45° \text{ m} = 0.4 \text{ m}$$

$$r_{D/IC} = \frac{0.4 \text{ m}}{\cos 45°} = 0.5657 \text{ m}$$

Since the magnitude of $\mathbf{v}_D$ is known, the angular velocity of link $BD$ is

$$\omega_{BD} = \frac{v_D}{r_{D/IC}} = \frac{3 \text{ m/s}}{0.5657 \text{ m}} = 5.30 \text{ rad/s} \circlearrowleft \qquad \textit{Ans.}$$

The velocity of $B$ is therefore

$$v_B = \omega_{BD}(r_{B/IC}) = 5.30 \text{ rad/s } (0.4 \text{ m}) = 2.12 \text{ m/s} \quad \measuredangle 45°$$

From Fig. 16–21$c$, the angular velocity of $AB$ is

$$\omega_{AB} = \frac{v_B}{r_{B/A}} = \frac{2.12 \text{ m/s}}{0.4 \text{ m}} = 5.30 \text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

**NOTE:** Try and solve this problem by applying $\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$ to member $BD$.

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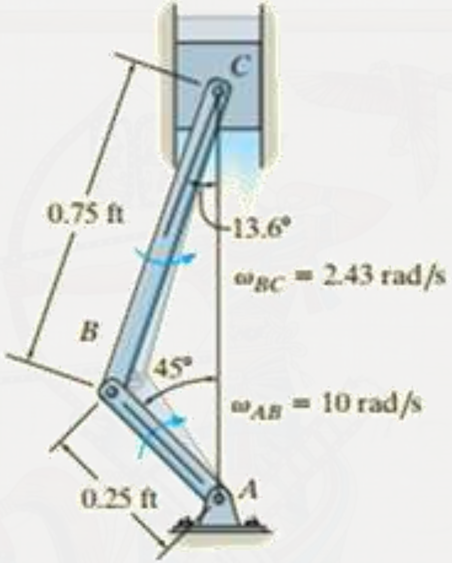
# Instantaneous center of zero velocity (16.6)

The crankshaft $AB$ turns with a clockwise angular velocity of 10 rad/s, Fig. 16–23$a$. Determine the velocity of the piston at the instant shown.



(a)

# Instantaneous center of zero velocity (16.6)

The crankshaft $AB$ turns with a clockwise angular velocity of 10 rad/s, Fig. 16–23*a*. Determine the velocity of the piston at the instant shown.

## SOLUTION

The crankshaft rotates about a fixed axis, and so the velocity of point $B$ is

$$v_B = 10\text{ rad/s}\,(0.25\text{ ft}) = 2.50\text{ ft/s} \measuredangle 45^\circ$$

Since the directions of the velocities of $B$ and $C$ are known, then the location of the $IC$ for the connecting rod $BC$ is at the intersection of the lines extended from these points, perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$, Fig. 16–23*b*. The magnitudes of $\mathbf{r}_{B/IC}$ and $\mathbf{r}_{C/IC}$ can be obtained from the geometry of the triangle and the law of sines, i.e.,

$$\frac{0.75\text{ ft}}{\sin 45^\circ} = \frac{r_{B/IC}}{\sin 76.4^\circ}$$

$$r_{B/IC} = 1.031\text{ ft}$$

$$\frac{0.75\text{ ft}}{\sin 45^\circ} = \frac{r_{C/IC}}{\sin 58.6^\circ}$$

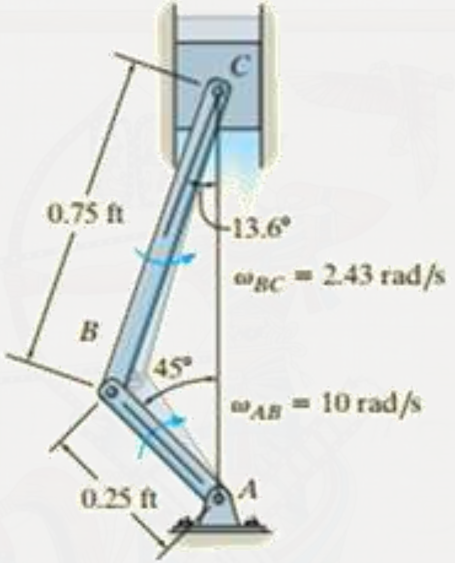
$$r_{C/IC} = 0.9056\text{ ft}$$

The rotational sense of $\boldsymbol{\omega}_{BC}$ must be the same as the rotation caused by $\mathbf{v}_B$ about the $IC$, which is counterclockwise. Therefore,

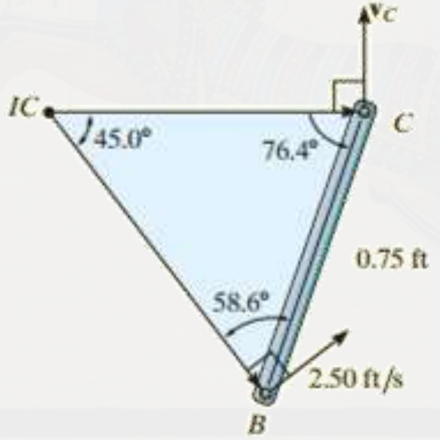
$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{2.5\text{ ft/s}}{1.031\text{ ft}} = 2.425\text{ rad/s}$$

Using this result, the velocity of the piston is

$$v_C = \omega_{BC} r_{C/IC} = (2.425\text{ rad/s})(0.9056\text{ ft}) = 2.20\text{ ft/s} \qquad \textit{Ans.}$$



(a)



(b)

Fig. 16–23

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# Part 8

# Relative motion analysis: Acceleration

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# Relative motion analysis: Acceleration (16.7) Applications

In the mechanism for a window, link AC rotates about a fixed axis through C, while point B slides in a straight track. The components of acceleration of these points can be inferred since their motions are known.

To prevent damage to the window, the accelerations of the links must be limited.

# Relative motion analysis: Acceleration (16.7)

The equation relating the accelerations of two points on the body is determined by differentiating the velocity equation with respect to time.

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

These are absolute accelerations of points A and B. They are measured from a set of fixed x, y axes.

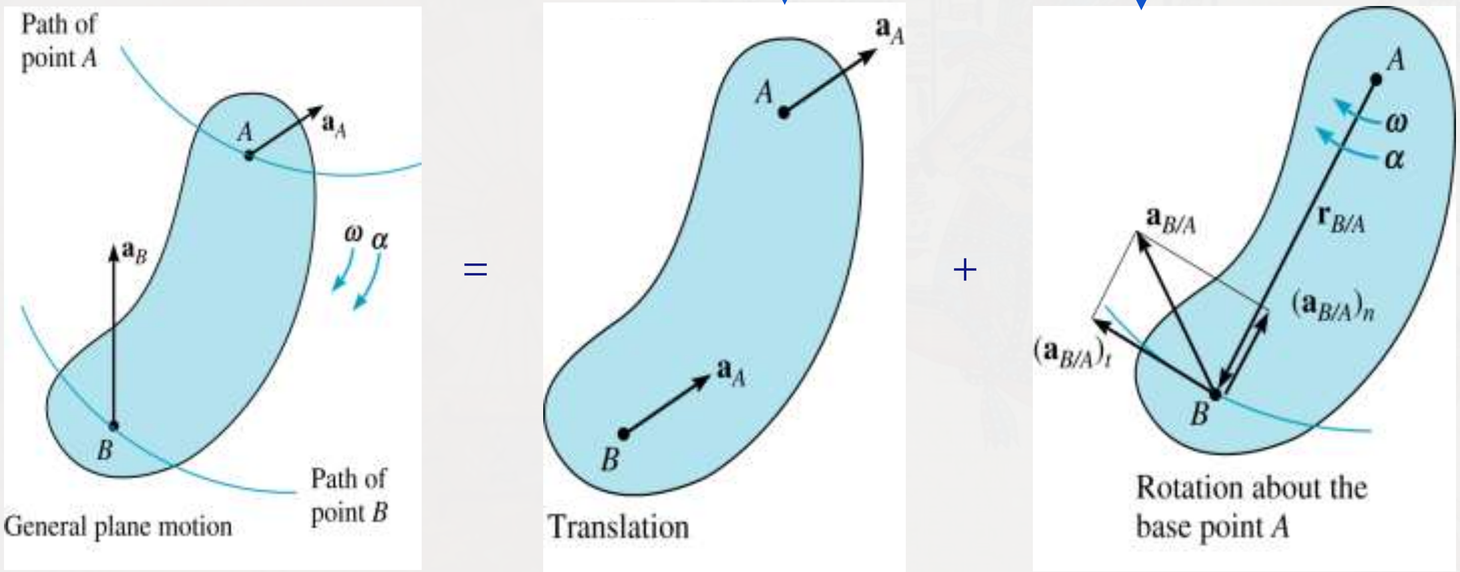
This term is the acceleration of B with respect to A.
It will develop tangential and normal components.

The result is $\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{a}_{B/A})_t + (\boldsymbol{a}_{B/A})_n$

# Relative motion analysis: Acceleration (16.7)

Graphically:

$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{a}_{B/A})_t + (\boldsymbol{a}_{B/A})_n$$




The relative tangential acceleration component $(\boldsymbol{a}_{B/A})_t$ is ($\alpha$ x $\boldsymbol{r}_{B/A}$) and perpendicular to $\boldsymbol{r}_{B/A}$.

The relative normal acceleration component $(\boldsymbol{a}_{B/A})_n$ is ($-\omega^2$ $\boldsymbol{r}_{B/A}$) and the direction is always from B towards A.

# Relative motion analysis: Acceleration (16.7)

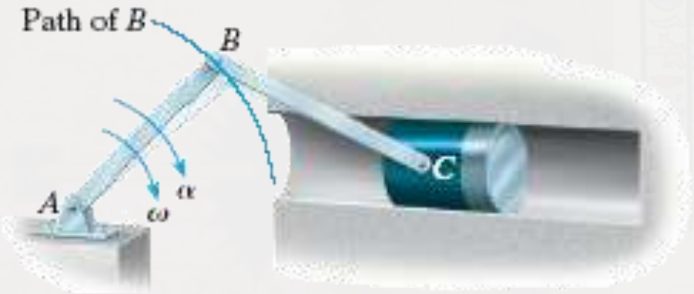
Since the relative acceleration components can be expressed as $(\boldsymbol{a}_{B/A})_t = \alpha \times \boldsymbol{r}_{B/A}$ and $(\boldsymbol{a}_{B/A})_n = -\,\omega^2\, \boldsymbol{r}_{B/A}$ the relative acceleration equation becomes

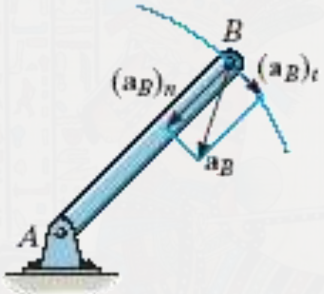
$$\boldsymbol{a}_B = \boldsymbol{a}_A + \alpha \times \boldsymbol{r}_{B/A} - \omega^2\, \boldsymbol{r}_{B/A}$$

Note that the last term in the relative acceleration equation is not a cross product. It is the product of a scalar (square of the magnitude of angular velocity, $\omega^2$) and the relative position vector, $\boldsymbol{r}_{B/A}$.

# Application of relative acceleration equation

In applying the relative acceleration equation, the two points used in the analysis (A and B) should generally be selected as points which have a known motion, such as pin connections with other bodies.





In this mechanism, point B is known to travel along a circular path, so $\boldsymbol{a}_B$ can be expressed in terms of its normal and tangential components. Note that point B on link BC will have the same acceleration as point B on link AB.

Point C, connecting link BC and the piston, moves along a straight-line path. Hence, $\boldsymbol{a}_C$ is directed horizontally.

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# Procedure of analysis (16.7)

1. Establish a fixed coordinate system.

2. Draw the kinematic diagram of the body.

3. Indicate on it $\boldsymbol{a}_A$, $\boldsymbol{a}_B$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\boldsymbol{r}_{B/A}$. If the points A and B move along curved paths, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\boldsymbol{a}_A = (\boldsymbol{a}_A)_t + (\boldsymbol{a}_A)_n$ and $\boldsymbol{a}_B = (\boldsymbol{a}_B)_t + (\boldsymbol{a}_B)_n$.

4. Apply the relative acceleration equation:

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \alpha \times \boldsymbol{r}_{B/A} - \omega^2\, \boldsymbol{r}_{B/A}$$

5. If the solution yields a negative answer for an unknown magnitude, it indicates the sense of direction of the vector is opposite to that shown on the diagram.

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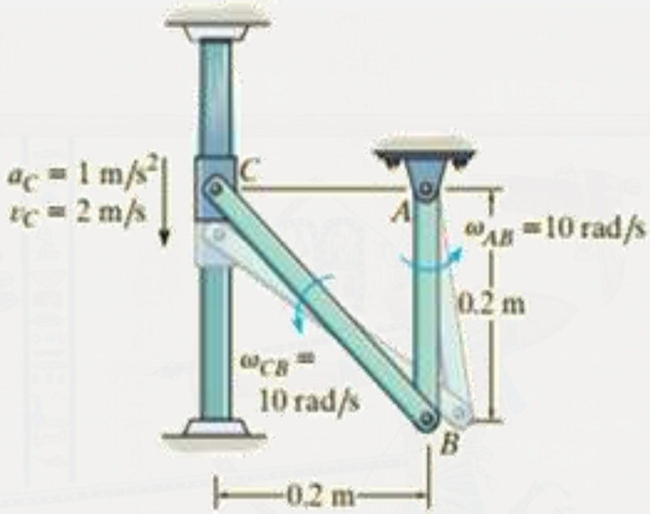
# Example 1

The collar $C$ in Fig. 16–30$a$ moves downward with an acceleration of $1\ \text{m/s}^2$. At the instant shown, it has a speed of 2 m/s which gives links $CB$ and $AB$ an angular velocity $\omega_{AB} = \omega_{CB} = 10$ rad/s. (See Example 16.8.) Determine the angular accelerations of $CB$ and $AB$ at this instant.

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# Example 1

The collar $C$ in Fig. 16–30$a$ moves downward with an acceleration of and 1 $\text{m/s}^2$. At the instant shown, it has a speed of 2 m/s which gives links riate $CB$ and $AB$ an angular velocity $\omega_{AB} = \omega_{CB} = 10$ rad/s. (See Example 16.8.) Determine the angular accelerations of $CB$ and $AB$ at this instant.

(a)

## SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams of *both* links $AB$ and $CB$ are shown in Fig. 16–30$b$. To solve, we will apply the appropriate kinematic equation to each link.

**Acceleration Equation.**

Link $AB$ (rotation about a fixed axis):

$a_C = 1$

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$

$$\mathbf{a}_B = (\alpha_{AB}\mathbf{k}) \times (-0.2\mathbf{j}) - (10)^2(-0.2\mathbf{j})$$

$$\mathbf{a}_B = 0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j}$$

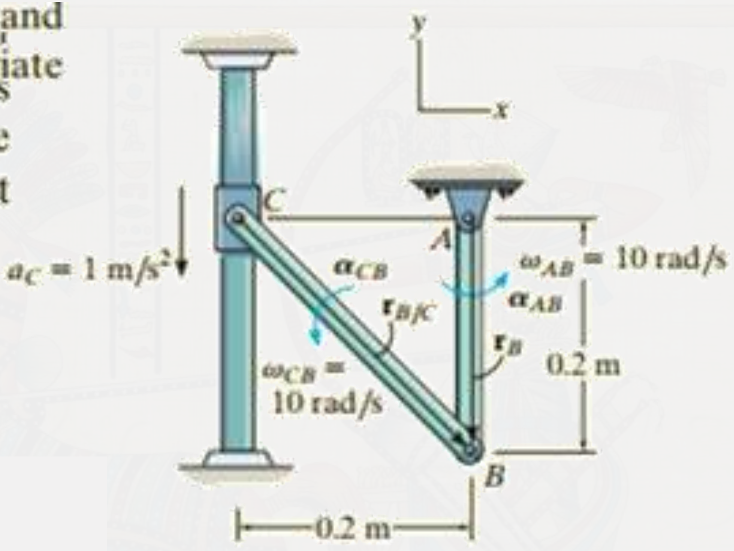
Note that $\mathbf{a}_B$ has $n$ and $t$ components since it moves along a *circular path*.

Link $BC$ (general plane motion): Using the result for $\mathbf{a}_B$ and applying Eq. 16–18, we have

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \omega_{CB}^2 \mathbf{r}_{B/C}$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^2(0.2\mathbf{i} - 0.2\mathbf{j})$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\alpha_{CB}\mathbf{j} + 0.2\alpha_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$




(b)

**Fig. 16–30**

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$

$$20 = -1 + 0.2\alpha_{CB} + 20$$

Solving,

$$\alpha_{CB} = 5 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

$$\alpha_{AB} = -95 \text{ rad/s}^2 = 95 \text{ rad/s}^2 \circlearrowright \qquad \textit{Ans.}$$

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# Example 2

The crankshaft $AB$ turns with a clockwise angular acceleration of 20 rad/s$^2$, Fig. 16–31*a*. Determine the acceleration of the piston at the instant $AB$ is in the position shown. At this instant $\omega_{AB}$ = 10 rad/s and $\omega_{BC}$ = 2.43 rad/s (See Example 16.13.)

SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams for both $AB$ and $BC$ are shown in Fig. 16–31*b*. Here $\mathbf{a}_C$ is vertical since $C$ moves along a straight-line path.

# Example 2

The crankshaft $AB$ turns with a clockwise angular acceleration of 20 rad/s$^2$, Fig. 16–31*a*. Determine the acceleration of the piston at the instant $AB$ is in the position shown. At this instant $\omega_{AB}$ = 10 rad/s and $\omega_{BC}$ = 2.43 rad/s (See Example 16.13.)

SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams for both $AB$ and $BC$ are shown in Fig. 16–31*b*. Here $\mathbf{a}_C$ is vertical since $C$ moves along a straight-line path.

**Acceleration Equation.** Expressing each of the position vectors in Cartesian vector form

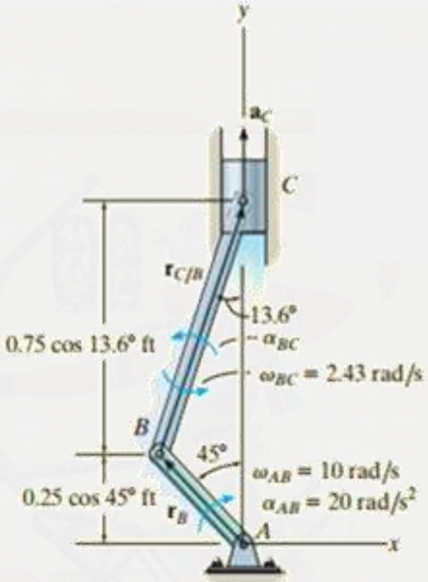
$$\mathbf{r}_B = \{-0.25 \sin 45°\mathbf{i} + 0.25 \cos 45°\mathbf{j}\}\text{ ft} = \{-0.177\mathbf{i} + 0.177\mathbf{j}\}\text{ ft}$$

$$\mathbf{r}_{C/B} = \{0.75 \sin 13.6°\mathbf{i} + 0.75 \cos 13.6°\mathbf{j}\}\text{ ft} = \{0.177\mathbf{i} + 0.729\mathbf{j}\}\text{ ft}$$

Crankshaft *AB* (rotation about a fixed axis):

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B \\ &= (-20\mathbf{k}) \times (-0.177\mathbf{i} + 0.177\mathbf{j}) - (10)^2(-0.177\mathbf{i} + 0.177\mathbf{j}) \\ &= \{21.21\mathbf{i} - 14.14\mathbf{j}\}\text{ ft/s}^2\end{aligned}$$

Connecting Rod *BC* (general plane motion): Using the result for $\mathbf{a}_B$ and noting that $\mathbf{a}_C$ is in the vertical direction, we have




$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.177\mathbf{i} + 0.729\mathbf{j}) - (2.43)^2(0.177\mathbf{i} + 0.729\mathbf{j})$$

$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + 0.177\alpha_{BC}\mathbf{j} - 0.729\alpha_{BC}\mathbf{i} - 1.04\mathbf{i} - 4.30\mathbf{j}$$

$$0 = 20.17 - 0.729\alpha_{BC}$$

$$a_C = 0.177\alpha_{BC} - 18.45$$

Solving yields

$$\alpha_{BC} = 27.7\text{ rad/s}^2 ↺$$

$$a_C = -13.5\text{ ft/s}^2 \qquad \textit{Ans.}$$

NOTE: Since the piston is moving upward, the negative sign for $a_C$ indicates that the piston is decelerating, i.e., $\mathbf{a}_C = \{-13.5\mathbf{j}\}$ ft/s$^2$. This causes the speed of the piston to decrease until $AB$ becomes vertical, at which time the piston is momentarily at rest.

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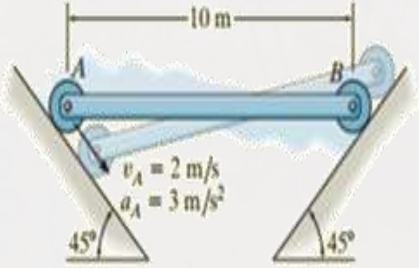
# Example 3

The rod $AB$ shown in Fig. 16–27$a$ is confined to move along the inclined planes at $A$ and $B$. If point $A$ has an acceleration of 3 m/s$^2$ and a velocity of 2 m/s, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

## SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points $A$ and $B$ on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283$ rad/s ↺ using either the velocity equation or the method of instantaneous centers.




(a)

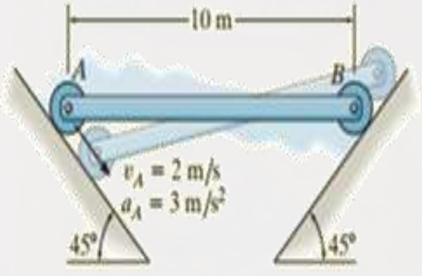
# Example 3

The rod $AB$ shown in Fig. 16–27*a* is confined to move along the inclined planes at $A$ and $B$. If point $A$ has an acceleration of 3 m/s$^2$ and a velocity of 2 m/s, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

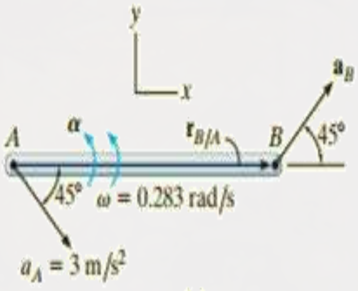
## SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points $A$ and $B$ on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283$ rad/s ↻ using either the velocity equation or the method of instantaneous centers.

**Kinematic Diagram.** Since points $A$ and $B$ both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns in Fig. 16–27*b*, namely, $a_B$ and $\alpha$.




(a)




(b)

**Acceleration Equation.**

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

$$a_B \cos 45°\mathbf{i} + a_B \sin 45°\mathbf{j} = 3\cos 45°\mathbf{i} - 3\sin 45°\mathbf{j} + (\alpha\mathbf{k}) \times (10\mathbf{i}) - (0.283)^2(10\mathbf{i})$$

Carrying out the cross product and equating the **i** and **j** components yields

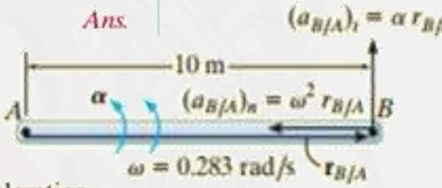
$$a_B \cos 45° = 3\cos 45° - (0.283)^2(10) \quad (1)$$

$$a_B \sin 45° = -3\sin 45° + \alpha(10) \quad (2)$$

Solving, we have

$$a_B = 1.87 \text{ m/s}^2 \measuredangle 45°$$

$$\alpha = 0.344 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$




(c)

## SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$, Fig. 16–27*c*, we have

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$\begin{bmatrix} a_B \\ \measuredangle 45° \end{bmatrix} = \begin{bmatrix} 3 \text{ m/s}^2 \\ \sphericalangle 45° \end{bmatrix} + \begin{bmatrix} \alpha(10 \text{ m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283 \text{ rad/s})^2(10 \text{ m}) \\ \leftarrow \end{bmatrix}$$

Equating the $x$ and $y$ components yields Eqs. 1 and 2, and the solution proceeds as before.

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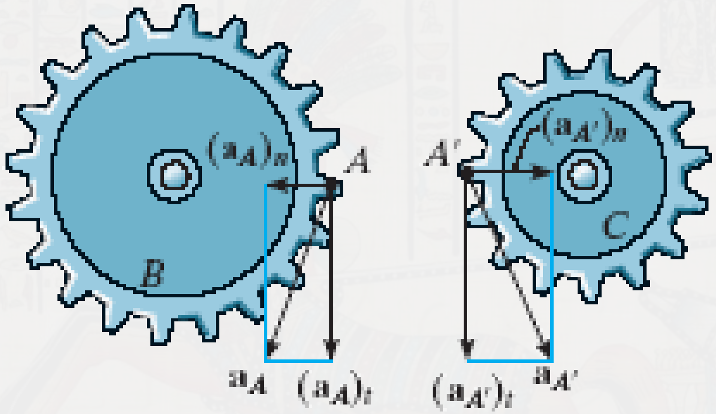
# Bodies in contact (16.7)

Consider two bodies in contact with one another without slipping, where the points in contact move along different paths.


In this case, the tangential components of acceleration will be the same, i. e.,

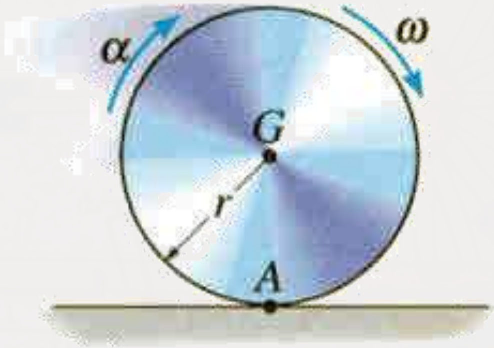
$$(\boldsymbol{a}_A)_t = (\boldsymbol{a}_{A'})_t \text{ (which implies } \alpha_B r_B = \alpha_C r_C \text{).}$$

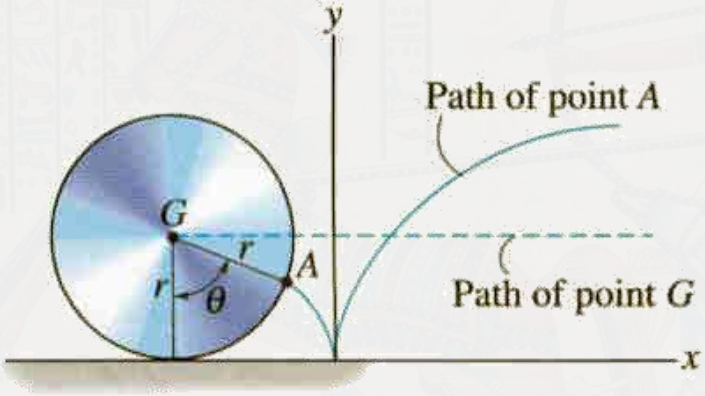
The normal components of acceleration will not be the same.

$$(\boldsymbol{a}_A)_n \neq (\boldsymbol{a}_{A'})_n \text{ so } \boldsymbol{a}_A \neq \boldsymbol{a}_{A'}$$

# Rolling motion (16.7)

Another common type of problem encountered in dynamics involves rolling motion without slip; e.g., a ball or disk rolling along a flat surface without slipping. This problem can be analyzed using relative velocity and acceleration equations.

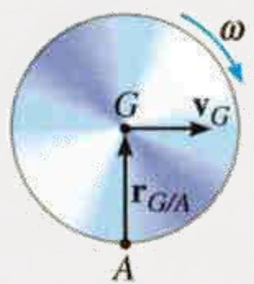



As the cylinder rolls, point G (center) moves along a straight line, while point A, on the rim of the cylinder, moves along a curved path called a cycloid. If $\omega$ and $\alpha$ are known, the relative velocity and acceleration equations can be applied to these points, at the instant A is in contact with the ground.

# Rolling motion (16.7)

- Velocity:



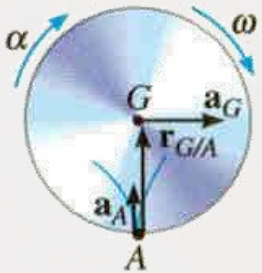

Since no slip occurs, $\boldsymbol{v}_A = \boldsymbol{0}$ when A is in contact with ground. From the kinematic diagram:

$$\boldsymbol{v}_G = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{G/A}$$

$$v_G \boldsymbol{i} = \boldsymbol{0} + (-\omega \boldsymbol{k}) \times (r \boldsymbol{j})$$

$$v_G = \omega r \quad \text{or} \quad \boldsymbol{v}_G = \omega r \boldsymbol{i}$$

- Acceleration:



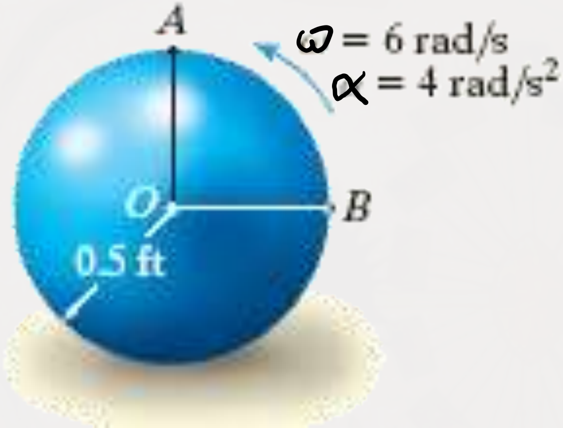

Since G moves along a straight-line path, $\boldsymbol{a}_G$ is horizontal. Just before A touches ground, its velocity is directed downward, and just after contact, its velocity is directed upward. Thus, point A accelerates upward as it leaves the ground.

$$\boldsymbol{a}_G = \boldsymbol{a}_A + \boldsymbol{\alpha} \times \boldsymbol{r}_{G/A} - \omega^2 \boldsymbol{r}_{G/A} \Rightarrow a_G \boldsymbol{i} = a_A \boldsymbol{j} + (-\alpha \boldsymbol{k}) \times (r \boldsymbol{j}) - \omega^2 (r \boldsymbol{j})$$

Evaluating and equating ***i*** and ***j*** components:

$$a_G = \alpha r \quad \text{and} \quad a_A = \omega^2 r \quad \text{or} \quad \boldsymbol{a}_G = \alpha r \boldsymbol{i} \quad \text{and} \quad \boldsymbol{a}_A = \omega^2 r \boldsymbol{j}$$

# Example 4




**Given:** The ball rolls without slipping.

**Find:** The accelerations of points A and B at this instant.

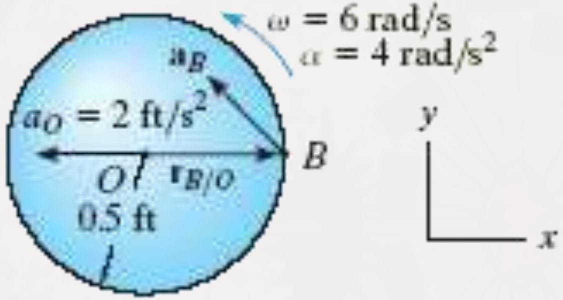
**Plan:** Follow the solution procedure.

**Solution:** Since the ball is rolling without slip, $\boldsymbol{a}_O$ is directed to the left with a magnitude of:

$$a_O = \alpha r = (4\ \text{rad/s}^2)(0.5\ \text{ft}) = 2\ \text{ft/s}^2$$

# Example 4

Now, apply the relative acceleration equation between points O and B.

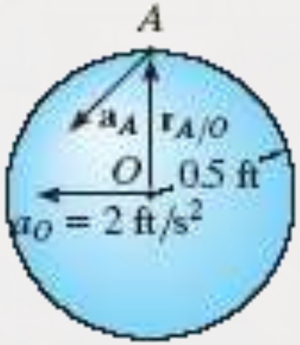



$$\boldsymbol{a}_B = \boldsymbol{a}_O + \boldsymbol{\alpha} \times \boldsymbol{r}_{B/O} - \omega^2 \boldsymbol{r}_{B/O}$$

$$\boldsymbol{a}_B = -2\boldsymbol{i} + (4\boldsymbol{k}) \times (0.5\boldsymbol{i}) - (6)^2(0.5\boldsymbol{i})$$

$$= (-20\boldsymbol{i} + 2\boldsymbol{j}) \text{ ft/s}^2$$

Now do the same for point A.




$$\boldsymbol{a}_A = \boldsymbol{a}_O + \boldsymbol{\alpha} \times \boldsymbol{r}_{A/O} - \omega^2 \boldsymbol{r}_{A/O}$$

$$\boldsymbol{a}_A = -2\boldsymbol{i} + (4\boldsymbol{k}) \times (0.5\boldsymbol{j}) - (6)^2(0.5\boldsymbol{j})$$

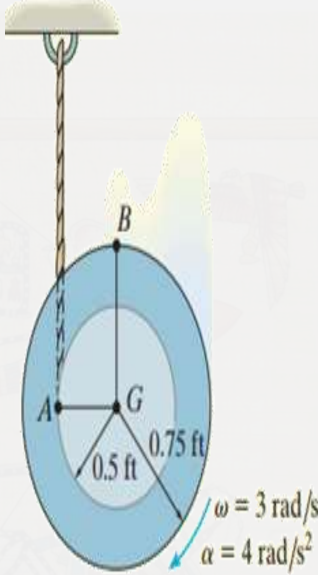
$$= (-4\boldsymbol{i} - 18\boldsymbol{j}) \text{ ft/s}^2$$

# Example 5

The spool shown in Fig. 16–29*a* unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s$^2$. Determine the acceleration of point *B*.

## SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point *A*. Therefore, we can use the results of Example 16.15 to determine the acceleration of point *G*, i.e.,



(a)

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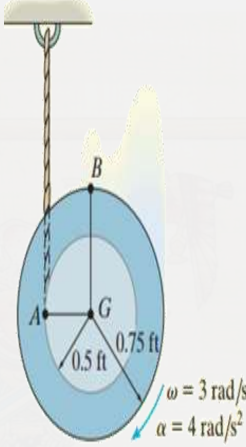
# Example 5

The spool shown in Fig. 16–29*a* unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s$^2$. Determine the acceleration of point *B*.




(a)

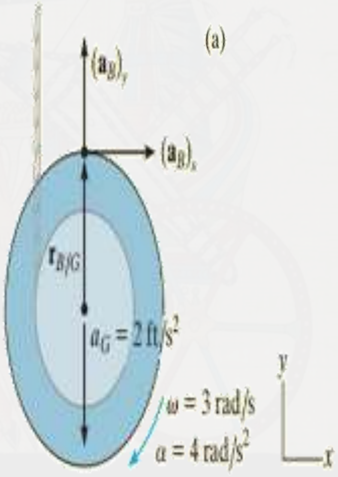
## SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point *A*. Therefore, we can use the results of Example 16.15 to determine the acceleration of point *G*, i.e.,

$$a_G = \alpha r = (4\ \text{rad/s}^2)(0.5\ \text{ft}) = 2\ \text{ft/s}^2$$

We will apply the acceleration equation to points *G* and *B*.

**Kinematic Diagram.** Point *B* moves along a *curved path* having an *unknown* radius of curvature.* Its acceleration will be represented by its unknown *x* and *y* components as shown in Fig. 16–29*b*.




(b)

**Acceleration Equation.**

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \omega^2 \mathbf{r}_{B/G}$$

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -2\mathbf{j} + (-4\mathbf{k}) \times (0.75\mathbf{j}) - (3)^2(0.75\mathbf{j})$$

Equating the **i** and **j** terms, the component equations are

$$(a_B)_x = 4(0.75) = 3\ \text{ft/s}^2 \rightarrow \qquad (1)$$

$$(a_B)_y = -2 - 6.75 = -8.75\ \text{ft/s}^2 = 8.75\ \text{ft/s}^2 \downarrow \qquad (2)$$

The magnitude and direction of $\mathbf{a}_B$ are therefore

$$a_B = \sqrt{(3)^2 + (8.75)^2} = 9.25\ \text{ft/s}^2 \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{8.75}{3} = 71.1° \ \measuredangle \qquad \textit{Ans.}$$

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Part 9

# Relative Motion Analysis Using Rotating Axes

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# Relative Motion Analysis Using Rotating Axes (16.8)

- In the previous sections the relative-motion analysis for velocity and acceleration was described using a translating coordinate system. This type of analysis is useful for determining the motion of points on the same rigid body, or the motion of points located on several pin-connected bodies.

- In some problems the rigid bodies (mechanisms) are constructed such that sliding will occur at their connections. The kinematic analysis for such cases is best performed if the motion is analyzed using a coordinate system which both translates and rotates. Furthermore, this frame of reference is useful for analyzing the motions of two points on a mechanism which are not located in the same body and for specifying the kinematics of particle motion when the particle moves along a rotating path.

# Application

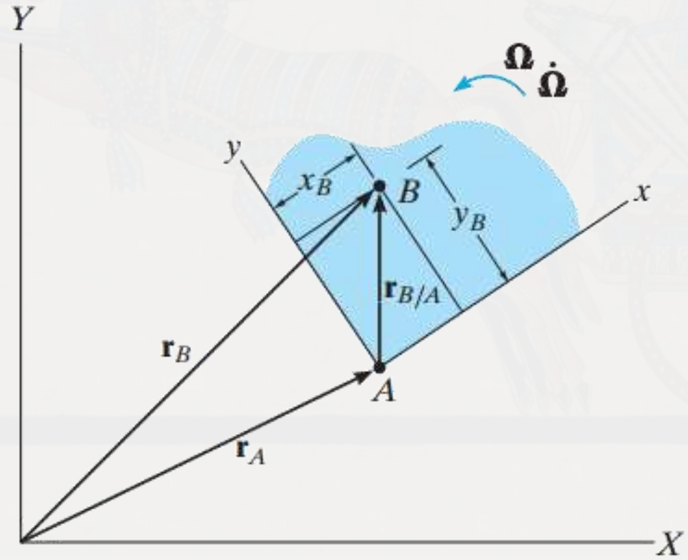
The rotation of the dumping bin of the truck about point C is operated by the extension of the hydraulic cylinder AB. To determine the rotation of the bin due to this extension, we can use the equations of relative motion and fix the x, y axes to the cylinder so that the relative motion of the cylinder's extension occurs along the y axis.

# Position

- Consider the two points A and B shown in Fig. below. Their location is specified by the position vectors $r_A$ and $r_B$ which are measured with respect to the fixed X,Y, Z coordinate system. As shown in the figure, the "base point" A represents the origin of the x, y, z coordinate system, which is assumed to be both translating and rotating with respect to the X,Y, Z system. The position of B with respect to A is specified by the relative-position vector $r_{B/A}$
- The components of the vector $r_{B/A}$ may be expressed either in terms of unit vectors along the X, Y axes, i.e., I and J, or by unit vectors along the x, y axes, i.e., i and j.

$$\mathbf{r}_{B/A} = x_B\mathbf{i} + y_B\mathbf{j}$$

$$\boxed{\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}}$$

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# Velocity

- At the instant considered, point A has a velocity $V_A$ and an acceleration $a_A$ while the angular velocity and angular acceleration of the x, y axes are Ω (omega) and $\dot{\Omega} = d\Omega/\mathrm{dt}$ respectively.

- The velocity of point B is determined by taking the time derivative of the previous equation, which yields

$$\mathbf{v}_B = \mathbf{v}_A + \frac{d\mathbf{r}_{B/A}}{dt}$$

# Velocity

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}}$$

where

$\mathbf{v}_B$ = velocity of $B$, measured from the $X, Y, Z$ reference

$\mathbf{v}_A$ = velocity of the origin $A$ of the $x, y, z$ reference, measured from the $X, Y, Z$ reference

$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "$B$ with respect to $A$," as measured by an observer attached to the rotating $x, y, z$ reference

$\boldsymbol{\Omega}$ = angular velocity of the $x, y, z$ reference, measured from the $X, Y, Z$ reference

$\mathbf{r}_{B/A}$ = position of $B$ with respect to $A$

# Velocity

$\mathbf{v}_B$ — absolute velocity of $B$ — motion of $B$ observed from the $X, Y, Z$ frame

(equals)

$\mathbf{v}_A$ — absolute velocity of the origin of $x, y, z$ frame

(plus)

$\boldsymbol{\Omega} \times \mathbf{r}_{B/A}$ — angular velocity effect caused by rotation of $x, y, z$ frame

($\mathbf{v}_A$ and $\boldsymbol{\Omega} \times \mathbf{r}_{B/A}$:) motion of $x, y, z$ frame observed from the $X, Y, Z$ frame

(plus)

$(\mathbf{v}_{B/A})_{xyz}$ — velocity of $B$ with respect to $A$ — motion of $B$ observed from the $x, y, z$ frame

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# Acceleration

- The acceleration of point B is determined by taking the time derivative of the previous equation, which yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times \frac{d\mathbf{r}_{B/A}}{dt} + \frac{d(\mathbf{v}_{B/A})_{xyz}}{dt}$$

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}}$$

$\mathbf{a}_B$ = acceleration of $B$, measured from the $X, Y, Z$ reference

$\mathbf{a}_A$ = acceleration of the origin $A$ of the $x, y, z$ reference, measured from the $X, Y, Z$ reference

$(\mathbf{a}_{B/A})_{xyz}, (\mathbf{v}_{B/A})_{xyz}$ = acceleration and velocity of $B$ with respect to $A$, as measured by an observer attached to the *rotating* $x, y, z$ reference

$\dot{\boldsymbol{\Omega}}, \boldsymbol{\Omega}$ = angular acceleration and angular velocity of the $x, y, z$ reference, measured from the $X, Y, Z$ reference

$\mathbf{r}_{B/A}$ = position of $B$ with respect to $A$

# Acceleration

| | | |
|---|---|---|
| $\mathbf{a}_A$ | absolute acceleration of the origin of $x$, $y$, $z$ frame | motion of $x$, $y$, $z$ frame observed from the $X$, $Y$, $Z$ frame |
| | (plus) | |
| $\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A}$ | angular acceleration effect caused by rotation of $x$, $y$, $z$ frame | |
| | (plus) | |
| $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A})$ | angular velocity effect caused by rotation of $x$, $y$, $z$ frame | |
| | (plus) | |
| $2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$ | combined effect of $B$ moving relative to $x$, $y$, $z$ coordinates and rotation of $x$, $y$, $z$ frame | interacting motion |
| | (plus) | |
| $(\mathbf{a}_{B/A})_{xyz}$ | acceleration of $B$ with respect to $A$ | motion of $B$ observed from the $x$, $y$, $z$ frame |

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# Procedure for Analysis

Equations 16–24 and 16–27 can be applied to the solution of problems involving the planar motion of particles or rigid bodies using the following procedure.

## Coordinate Axes.

- Choose an appropriate location for the origin and proper orientation of the axes for both fixed $X$, $Y$, $Z$ and moving $x$, $y$, $z$ reference frames.
- Most often solutions are easily obtained if at the instant considered:
  1. the origins are coincident
  2. the corresponding axes are collinear
  3. the corresponding axes are parallel
- The moving frame should be selected fixed to the body or device along which the relative motion occurs.

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# Procedure for Analysis

## Kinematic Equations.

- After defining the origin $A$ of the moving reference and specifying the moving point $B$, Eqs. 16–24 and 16–27 should be written in symbolic form

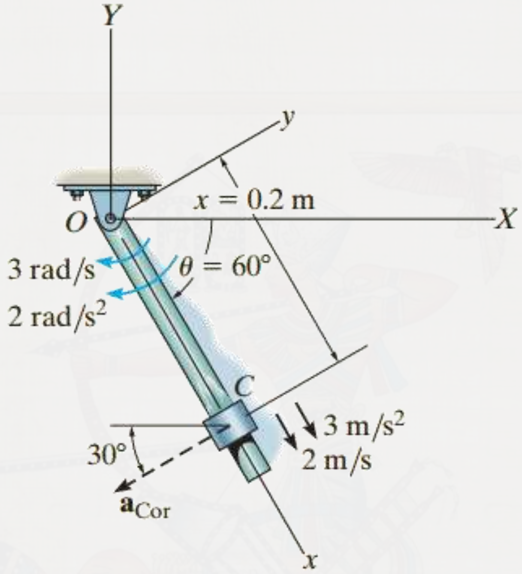
$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- The Cartesian components of all these vectors may be expressed along either the $X$, $Y$, $Z$ axes or the $x$, $y$, $z$ axes. The choice is arbitrary provided a consistent set of unit vectors is used.
- Motion of the moving reference is expressed by $\mathbf{v}_A$, $\mathbf{a}_A$, $\boldsymbol{\Omega}$, and $\dot{\boldsymbol{\Omega}}$; and motion of $B$ with respect to the moving reference is expressed by $\mathbf{r}_{B/A}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$.

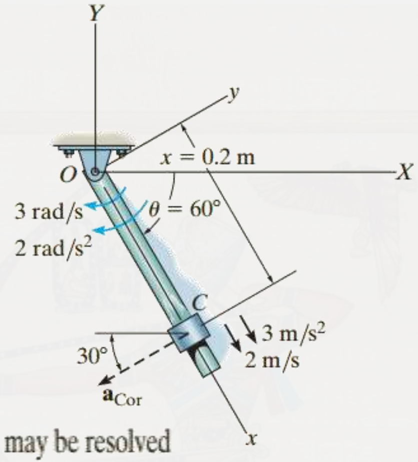
# Example 1

At the instant $\theta = 60°$, the rod in Fig. 16–33 has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s$^2$. At this same instant, collar $C$ travels outward along the rod such that when $x = 0.2$ m the velocity is 2 m/s and the acceleration is 3 m/s$^2$, both measured relative to the rod. Determine the Coriolis acceleration and the velocity and acceleration of the collar at this instant.



# Example 1

At the instant $\theta = 60°$, the rod in Fig. 16–33 has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s$^2$. At this same instant, collar $C$ travels outward along the rod such that when $x = 0.2$ m the velocity is 2 m/s and the acceleration is 3 m/s$^2$, both measured relative to the rod. Determine the Coriolis acceleration and the velocity and acceleration of the collar at this instant.




## SOLUTION

**Coordinate Axes.** The origin of both coordinate systems is located at point $O$, Fig. 16–33. Since motion of the collar is reported relative to the rod, the moving $x, y, z$ frame of reference is *attached* to the rod.

**Kinematic Equations.**

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz} \quad (1)$$

$$\mathbf{a}_C = \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/O} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/O}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz} \quad (2)$$

It will be simpler to express the data in terms of **i**, **j**, **k** component vectors rather than **I**, **J**, **K** components. Hence,

| Motion of moving reference | Motion of $C$ with respect to moving reference |
|---|---|
| $\mathbf{v}_O = \mathbf{0}$ | $\mathbf{r}_{C/O} = \{0.2\mathbf{i}\}$ m |
| $\mathbf{a}_O = \mathbf{0}$ | $(\mathbf{v}_{C/O})_{xyz} = \{2\mathbf{i}\}$ m/s |
| $\boldsymbol{\Omega} = \{-3\mathbf{k}\}$ rad/s | $(\mathbf{a}_{C/O})_{xyz} = \{3\mathbf{i}\}$ m/s$^2$ |
| $\dot{\boldsymbol{\Omega}} = \{-2\mathbf{k}\}$ rad/s$^2$ | |

The Coriolis acceleration is defined as

$$\mathbf{a}_{Cor} = 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} = 2(-3\mathbf{k}) \times (2\mathbf{i}) = \{-12\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

This vector is shown dashed in Fig. 16–33. If desired, it may be resolved into **I**, **J** components acting along the $X$ and $Y$ axes, respectively.

The velocity and acceleration of the collar are determined by substituting the data into Eqs. 1 and 2 and evaluating the cross products, which yields

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_O + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} + (\mathbf{v}_{C/O})_{xyz} \\ &= \mathbf{0} + (-3\mathbf{k}) \times (0.2\mathbf{i}) + 2\mathbf{i} \\ &= \{2\mathbf{i} - 0.6\mathbf{j}\} \text{ m/s} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_O + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/O} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/O}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/O})_{xyz} + (\mathbf{a}_{C/O})_{xyz} \\ &= \mathbf{0} + (-2\mathbf{k}) \times (0.2\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.2\mathbf{i})] + 2(-3\mathbf{k}) \times (2\mathbf{i}) + 3\mathbf{i} \\ &= \mathbf{0} - 0.4\mathbf{j} - 1.80\mathbf{i} - 12\mathbf{j} + 3\mathbf{i} \\ &= \{1.20\mathbf{i} - 12.4\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

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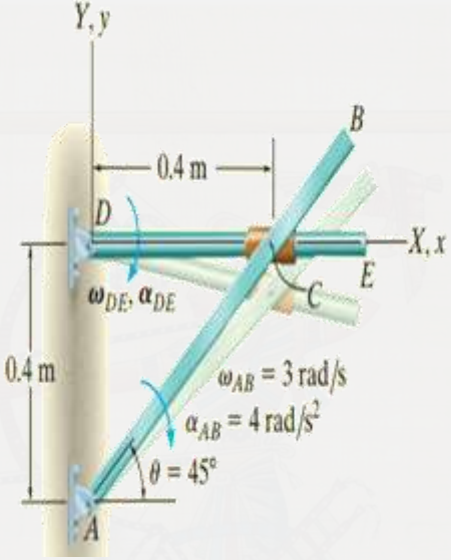
# Example 2

Rod *AB*, shown in Fig. 16–34, rotates clockwise such that it has an angular velocity $\omega_{AB} = 3$ rad/s and angular acceleration $\alpha_{AB} = 4$ rad/s$^2$ when $\theta = 45°$. Determine the angular motion of rod *DE* at this instant. The collar at *C* is pin connected to *AB* and slides over rod *DE*.

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SOLUTION

**Coordinate Axes.** The origin of both the fixed and moving frames of reference is located at $D$, Fig. 16–34. Furthermore, the $x$, $y$, $z$ reference is attached to and rotates with rod $DE$ so that the relative motion of the collar is easy to follow.

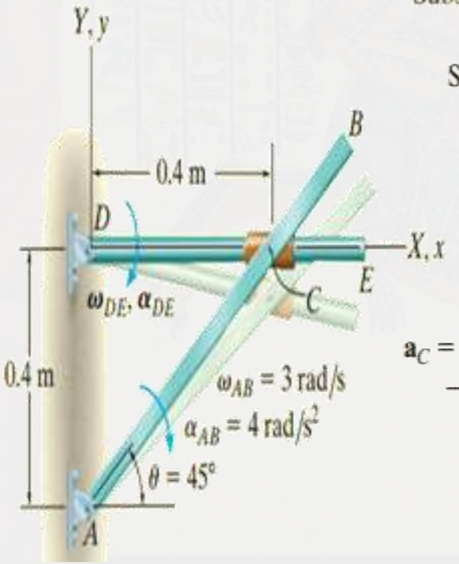
**Kinematic Equations.**

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz} \qquad (1)$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz} \qquad (2)$$

All vectors will be expressed in terms of **i**, **j**, **k** components.

| Motion of moving reference | Motion of C with respect to moving reference |
|---|---|
| $\mathbf{v}_D = \mathbf{0}$ | $\mathbf{r}_{C/D} = \{0.4\mathbf{i}\}$m |
| $\mathbf{a}_D = \mathbf{0}$ | $(\mathbf{v}_{C/D})_{xyz} = (v_{C/D})_{xyz}\mathbf{i}$ |
| $\boldsymbol{\Omega} = -\omega_{DE}\mathbf{k}$ | $(\mathbf{a}_{C/D})_{xyz} = (a_{C/D})_{xyz}\mathbf{i}$ |
| $\dot{\boldsymbol{\Omega}} = -\alpha_{DE}\mathbf{k}$ | |




**Motion of C:** Since the collar moves along a *circular path* of radius $AC$, its velocity and acceleration can be determined using Eqs. 16–9 and 16–14.

$$\mathbf{v}_C = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{C/A} = (-3\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) = \{1.2\mathbf{i} - 1.2\mathbf{j}\}\text{ m/s}$$

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{C/A} - \omega_{AB}^2\mathbf{r}_{C/A}$$

$$= (-4\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) - (3)^2(0.4\mathbf{i} + 0.4\mathbf{j}) = \{-2\mathbf{i} - 5.2\mathbf{j}\}\text{ m/s}^2$$

Substituting the data into Eqs. 1 and 2, we have

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$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\Omega} \times \mathbf{r}_{C/D} + (\mathbf{v}_{C/D})_{xyz}$$

$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} + (-\omega_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (v_{C/D})_{xyz}\mathbf{i}$$

$$1.2\mathbf{i} - 1.2\mathbf{j} = \mathbf{0} - 0.4\omega_{DE}\mathbf{j} + (v_{C/D})_{xyz}\mathbf{i}$$

$$(v_{C/D})_{xyz} = 1.2\text{ m/s}$$

$$\omega_{DE} = 3\text{ rad/s} \circlearrowright \qquad \textit{Ans.}$$

$$\mathbf{a}_C = \mathbf{a}_D + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/D} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/D}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/D})_{xyz} + (\mathbf{a}_{C/D})_{xyz}$$

$$-2\mathbf{i} - 5.2\mathbf{j} = \mathbf{0} + (-\alpha_{DE}\mathbf{k}) \times (0.4\mathbf{i}) + (-3\mathbf{k}) \times [(-3\mathbf{k}) \times (0.4\mathbf{i})]$$

$$+ 2(-3\mathbf{k}) \times (1.2\mathbf{i}) + (a_{C/D})_{xyz}\mathbf{i}$$

$$-2\mathbf{i} - 5.2\mathbf{j} = -0.4\alpha_{DE}\mathbf{j} - 3.6\mathbf{i} - 7.2\mathbf{j} + (a_{C/D})_{xyz}\mathbf{i}$$

$$(a_{C/D})_{xyz} = 1.6\text{ m/s}^2$$

$$\alpha_{DE} = -5\text{ rad/s}^2 = 5\text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

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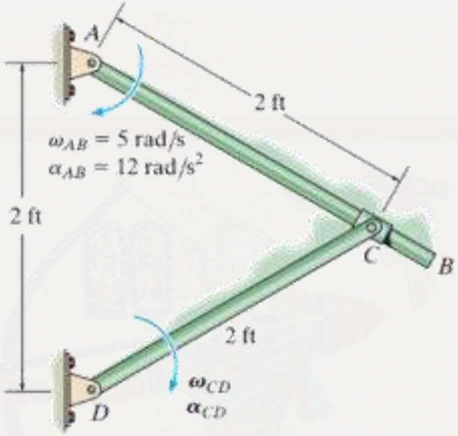
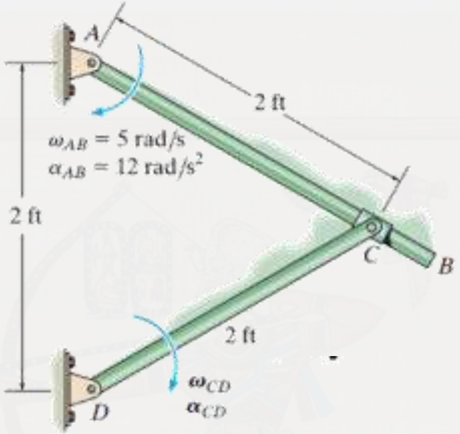
# Example 3

**16–143.** At a given instant, rod *AB* has the angular motions shown. Determine the angular velocity and angular acceleration of rod *CD* at this instant. There is a collar at *C*.



# Example 3

**16–143.** At a given instant, rod *AB* has the angular motions shown. Determine the angular velocity and angular acceleration of rod *CD* at this instant. There is a collar at *C*.




$\mathbf{v}_A = \mathbf{0}$

$\mathbf{a}_A = \mathbf{0}$

$\Omega = \{-5\mathbf{k}\}$ rad/s

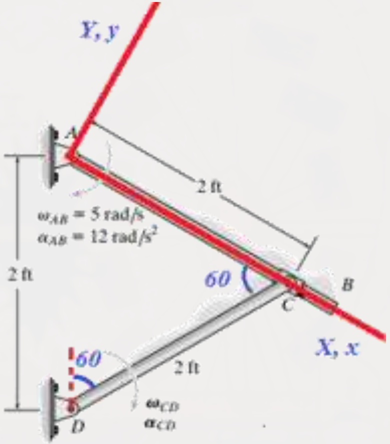
$\dot{\Omega} = \{-12\mathbf{k}\}$ rad/s$^2$

$\mathbf{r}_{C/A} = \{2\mathbf{i}\}$ ft

$(\mathbf{v}_{C/A})_{xyz} = (v_{C/A})_{xyz}\,\mathbf{i}$

$(\mathbf{a}_{C/A})_{xyz} = (a_{C/A})_{xyz}\,\mathbf{i}$

$\mathbf{v}_C = \mathbf{v}_A + \Omega \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz}$




$\mathbf{v}_C = \mathbf{0} + (-5\mathbf{k}) \times (2\mathbf{i}) + (v_{C/A})_{xyz}\mathbf{i}$

$= (v_{C/A})_{xyz}\,\mathbf{i} - 10\mathbf{j}$

$$\mathbf{v}_C = \omega_{CD} \times \mathbf{r}_{CD}$$

$(v_{C/A})_{xyz}\,\mathbf{i} - 10\mathbf{j} = (-\omega_{CD}\mathbf{k}) \times (2\cos 60°\mathbf{i} + 2\sin 60°\mathbf{j})$

$(v_{C/A})_{xyz}\,\mathbf{i} - 10\mathbf{j} = 1.732\omega_{CD}\,\mathbf{i} - \omega_{CD}\mathbf{j}$

Solving:

$$\omega_{CD} = 10 \text{ rad/s}$$

$(v_{C/A})_{xyz} = 1.732(10) = 17.32$ ft/s

$\mathbf{a}_C = \mathbf{a}_A + \dot{\Omega} \times \mathbf{r}_{C/A} + \Omega \times (\Omega \times \mathbf{r}_{C/A}) + 2\Omega \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz}$

$\mathbf{a}_C = \mathbf{0} + (-12\mathbf{k}) \times (2\mathbf{i}) + (-5\mathbf{k}) \times [(-5\mathbf{k}) \times (2\mathbf{i})] + 2(-5\mathbf{k}) \times [(v_{C/A})_{xyz}\mathbf{i}] + (a_{C/A})_{xyz}\,\mathbf{i}$

$= [(a_{C/A})_{xyz} - 50]\mathbf{i} - [10(v_{C/A})_{xyz} + 24]\mathbf{j}$

$\mathbf{a}_C = \alpha_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2\mathbf{r}_{CD}$

$[(a_{C/A})_{xyz} - 50]\mathbf{i} - [10(17.32) + 24]\mathbf{j} = (-\alpha_{CD}\mathbf{k}) \times (2\cos 60°\mathbf{i} + 2\sin 60°\mathbf{j}) - (10)^2(2\cos 60°\mathbf{i} + 2\sin 60°\mathbf{j})$

$[(a_{C/A})_{xyz} - 50]\mathbf{i} - (10(17.32) + 24]\mathbf{j} = (1.732\,\alpha_{CD} - 100)\mathbf{i} - (\alpha_{CD} + 173.2)\mathbf{j}$

Solving:

$-[10(17.32) + 24] = -(\alpha_{CD} + 173.2)$ $\quad \alpha_{CD} = 24$ rad/s$^2$ **Ans.**

$(a_{C/A})_{xyz} - 50 = 1.732(24) - 100$ $\quad (a_{C/A})_{xyz} = -8.43$ ft/s$^2$

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# References

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2. Dr. Balasie PowerPoints, Dynamic Course, Birzeit University.


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