Chapter 12

position, velocity, acceleration.

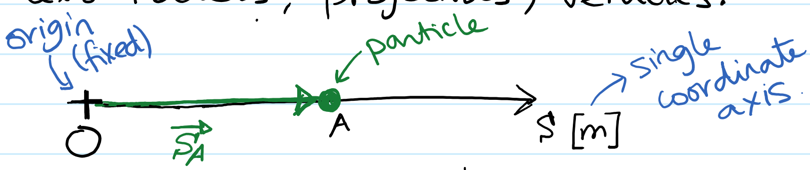
Kinematics of A Particle

straight line motion

$\boxed{12.2}$ Rectilinear Kinematics : Continuous Motion

* Particle : an object that has a mass but negligible size and shape. Rotation is neglected
ex: rockets, projectiles, vehicles.

1) Position



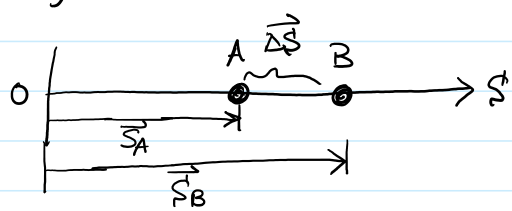
A vector quantity representing the location of a particle at any instant with respect to the point of origin O.

rate

→ magnitude of $\vec{S}_A$ is the distance between point O and the particle

→ direction (arbitrary) usually $\overset{-}{\longleftarrow}$ $\overset{+}{\longrightarrow}$

2) Displacement:



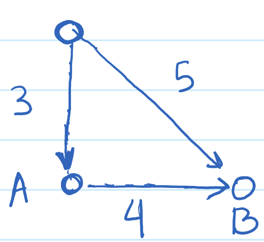
vector quantity representing the change in the particle's position

$\Delta \vec{S} = \vec{S}_B - \vec{S}_A$

$\Delta \vec{S}$ +ve → motion is to the right

$\Delta \vec{S}$ -ve → motion is to the left

Note: Difference between Displacement & Distance



7 = Distance is length of OA + AB

5 = Displacement is length of OB

3) Velocity

3) Velocity

$\vec{v}_{avg} = \dfrac{\Delta \vec{s}}{\Delta t}$, $\Delta \vec{s}$: Displacement (m) (ft)
Δt : time interval (s)

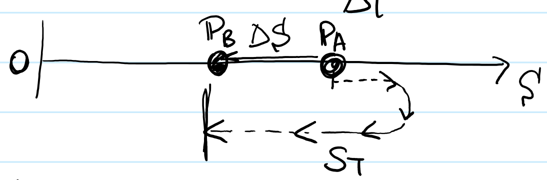
* take smaller values of $\Delta t \longrightarrow \Delta s$ becomes smaller *

Instantaneous velocity $\vec{v} = \lim_{\Delta t \to 0} \dfrac{\Delta \vec{s}}{\Delta t} = \dfrac{d\vec{s}}{dt}$ $\overset{+}{\longmapsto}$

Speed is the magnitude of velocity $= |\vec{v}|$ (m/s) (ft/s).

(Average speed) $(v_{sp})_{avg} = \dfrac{S_T}{\Delta t}$, S_T is the distance travelled.

ex:



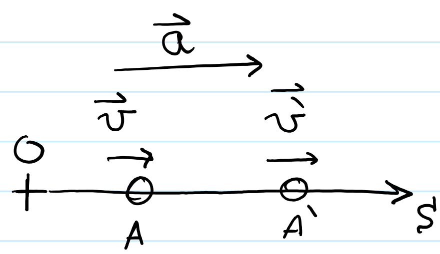
* Speed is always +ve (scalar, no direction)

$v_{avg} = \dfrac{-\Delta s}{\Delta t}$

$(v_{sp})_{avg} = \dfrac{S_T}{\Delta t} \cdot \left(\dfrac{\text{total distance}}{\text{total time}}\right)$

4) Acceleration:

Average acceleration $\vec{a}_{avg} = \dfrac{\Delta \vec{v}}{\Delta t}$



here $\Delta \vec{v} = \vec{v}' - \vec{v}$

Instantaneous Acceleration $\vec{a} = \lim_{\Delta t \to 0} \dfrac{\Delta \vec{v}}{\Delta t} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2 \vec{s}}{dt^2}$ (m/s²) (ft/s²)

Particle is accelerating $\rightarrow v' > v \longrightarrow a$ +ve
Particle is decelerating $\rightarrow v' < v \longrightarrow a$ -ve.
Constant velocity $\rightarrow v' = v \longrightarrow a$ is zero

$a = \dfrac{dv}{dt}$ $\qquad v = \dfrac{ds}{dt}$

$dt = \dfrac{dv}{a}$

$dt = \frac{dv}{a}$

$v = \frac{ds}{dv}\, a \Longrightarrow \boxed{a\, ds = v\, dv}$

* <u>Constant Acceleration</u> $\quad a = a_c$

1) Velocity as a function of time $(@ t = 0 \rightarrow \vec{v} = v_0)$

$a_c = \frac{dv}{dt} \Rightarrow dv = a_c\, dt$ integrate from $v_0 \rightarrow v$, $0 \rightarrow t$

$$\int_{v_0}^{v} dv = \int_0^t a_c\, dt \qquad v - v_0 = a_c(t - 0)$$

$$\boxed{v(t) = v_0 + a_c t}$$

2) Position as a function of time $(@ t = 0 , s = s_0)$

$v = \frac{ds}{dt} \qquad v_0 + a_c t = \frac{ds}{dt}$

$ds = (v_0 + a_c t)\, dt$ integrate from $s_0 \rightarrow s$, $0 \rightarrow t$

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\, dt$$

$$s - s_0 = \left(v_0 t + \frac{a_c}{2} t^2\right)\Big|_0^t$$

$$\boxed{s = v_0 t + \frac{a_c}{2} t^2 + s_0}$$

3) Velocity as a function of position $(@ t = 0 , s = s_0$, $v = v_0)$

$v\, dv = a_c\, ds$ integrate $s_0 \rightarrow s$, $v_0 \rightarrow v$

$$\int^{v} v\, dv = \int^{s} a_c\, ds$$

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a_c\,ds$$

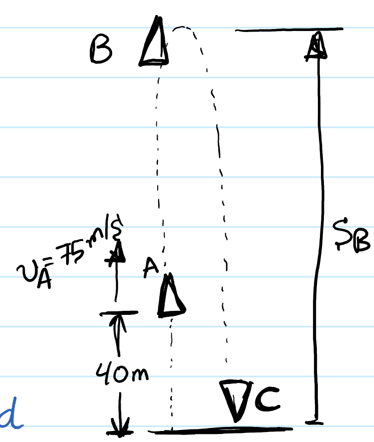
$$\frac{v^2}{2} = a_c s \Big|_{s_0}^{s} \qquad \frac{v^2}{2} - \frac{v_0^2}{2} = a_c(s - s_0)$$

$$\boxed{v^2 = v_0^2 + 2a_c(s - s_0)}$$

* These equations work only if $a = a_c$ (constant)
@ $t = 0 \rightarrow s = s_0$, $v = v_0$
* Signs of s_0, v_0, a_c depend on the positive direction specified by you.

Example: Free Fall (air resistance is neglected & Fall Distance is short)
Downward gravitational Acceleration is $9.81\ m/s^2$ or $32.2\ ft/s^2$.

Example 12.3 (textbook)



A rocket travels upward @ $75\ m/s$ when it is at $40\ m$ the engine fails. Rocket is subject to a downward acceleration of $9.81\ m/s^2$

Coordinate system: point O is on the ground
positive direction ↑

Find s_B
v_C

From A → B

$s_A = 40\ m$
$v_A = 75\ m/s$
$v_B = 0\ m/s$
$a_c = -9.81\ m/s^2$

$$v_B^2 = v_A^2 + 2a_c(s_B - s_A)$$

$$\boxed{s_B = 327\ m}$$

From B → C

$s_B = 327\ m$
$s_C = 0\ m$

$$v_C^2 = v_B^2 + 2a_c(s_C - s_B)$$

$s_B = 327$ m
$s_C = 0$ m
$v_B = 0$
$a_c = -9.81$ m/s^2

$v_C^2 = v_B^2 + 2a_c(s_C - s_B)$

$v_C = -80.1$ m/s
$\downarrow$

From A $\rightarrow$ C (another approach)

Note: $a_c = -9.81$ m/s all the time, it always acts in the opposite direction of positive v

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12.3 Rectilinear Kinematics: Erratic Motion.

* erratic motion: position, velocity and acceleration cannot be described by one function throughout the motion
→ series of functions for a series of functions
→ use graphs

The s-t, v-t, a-t graphs.

1) Given the s-t graph → v-t graph

$s(t)$, $v = \frac{ds}{dt}$ ⇒ velocity = slope of s-t graph

2) Given the v-t graph → a-t graph

$v(t)$, $a = \frac{dv}{dt}$ ⇒ acceleration = slope of v-t graph

3) Given the a-t graph → v-t graph

$$\Delta v = \int_{t_0}^{t_1} a\,dt$$

change in velocity = area under a-t graph

4) Given the v-t graph $\rightarrow$ s-t graph.

$$\Delta s = \int_{t_0}^{t_1} v\,dt$$

displacement (change in position) = area under v-t graph

The v-s and a-s graphs

1) Given a-s graph $\rightarrow$ v-s graph

$v_0 \rightarrow v_1$

$s_0 \rightarrow s_1$

$$\int_{v_0}^{v_1} v\,dv = \int_{s_0}^{s_1} a\,ds$$

$$\frac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a\,ds$$

$\rightarrow$ area under a-s graph

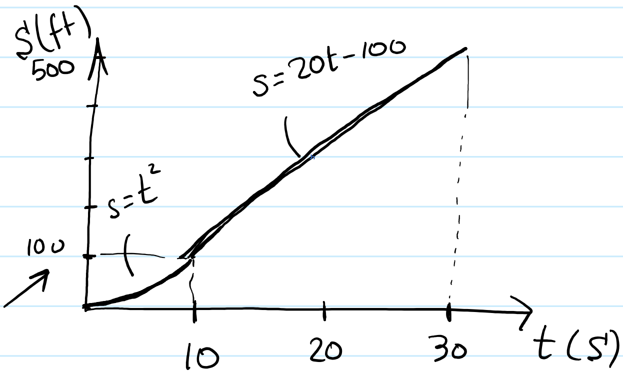
2) Given v-s graph $\rightarrow$ a-s graph

$$a = v\frac{dv}{ds}$$ $\rightarrow$ velocity times slope of v-s slope

Example 12.6 (textbook)

Bicycle moves along a straight road position is described by $\nearrow$

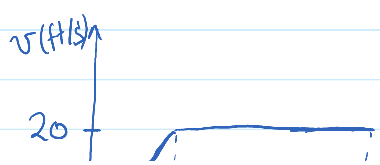
Construct v-t, a-t for $0 \leq t \leq 30$ s

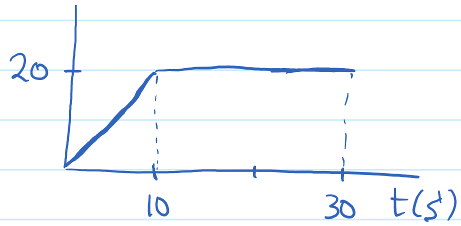


v-t graph

$0 \leq t \leq 10$ s $\quad s = t^2 \quad v = \frac{ds}{dt} = 2t$

$10 \leq t \leq 30$ s $\quad s = 20t - 100 \quad v = \frac{ds}{dt} = 20$

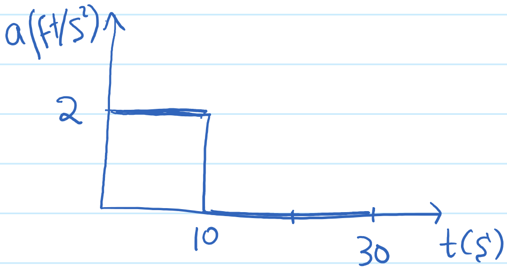




a-t graph

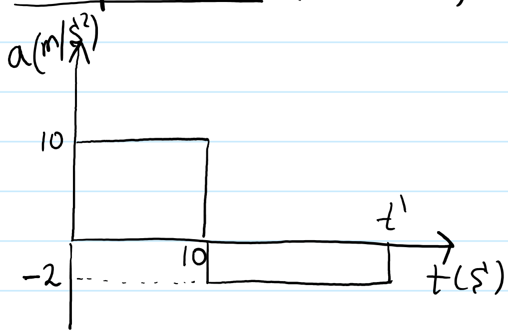
$0 \leqslant t \leqslant 10 \qquad v = 2t \qquad a = 2 \ ft/s^2$

$0 \leqslant t \leqslant 30 \qquad v = 20 \qquad a = 0$



Example 12.7 (textbook)

A car starts from rest and travels along a straight track. Draw v-t, s-t. Determine the time needed to stop the car t'. How far has the car travelled?



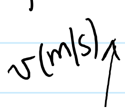
$0 \leqslant t \leqslant 10 \qquad v_i = 0 \qquad a = 10$

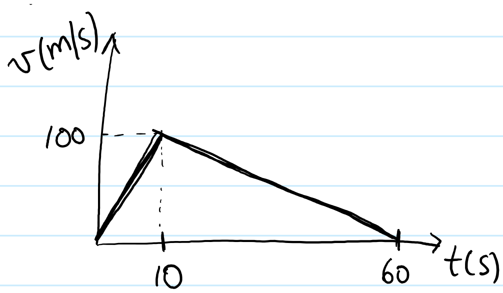
$$\int_0^v dv = \int_0^t 10\, dt \Rightarrow v = 10t$$

$10 \leqslant t \leqslant t' \qquad v_i = v(10) = 100 \ m/s \qquad a = -2$

$$\int_{100}^{v} dv = \int_{10}^{t} -2\, dt \Rightarrow v = (-2t + 120)$$

to find $t' \rightarrow v(t') = 0 = -2t' + 120 \Rightarrow \boxed{t' = 60 \ s}$





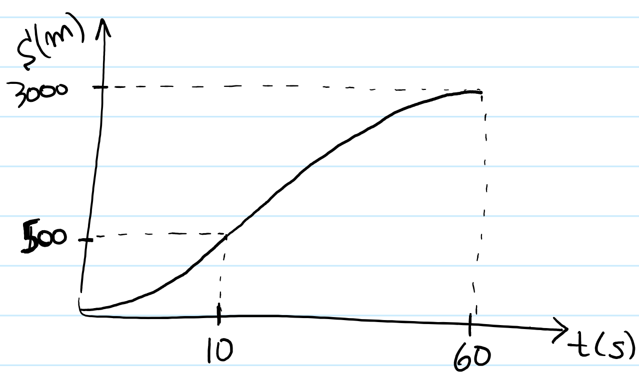
$0 \leq t \leq 10 \qquad s_i = 0 \qquad v = 10t$

$$\int_0^{s} ds = \int_0^{10} 10t \, dt \Rightarrow s = 5t^2$$

$10 \leq t \leq 60 \qquad s_i = s(10) = 500 \text{ m} \qquad v(t) = -2t + 120$

$$\int_{500}^{s} ds = \int_{10}^{t} -2t + 120 \, dt \Rightarrow s(t) = -t^2 + 120t - 600$$

for $t = 60$ s $\qquad \boxed{s(60) = 3000 \text{ m}}$



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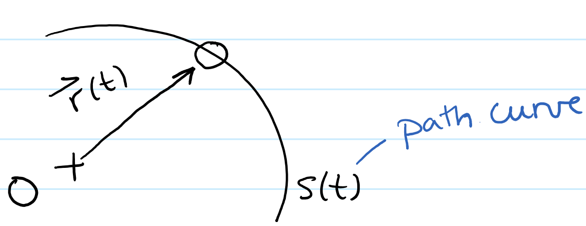
12.4 General curvilinear motion:

Curvilinear motion → particle moves along a curved path

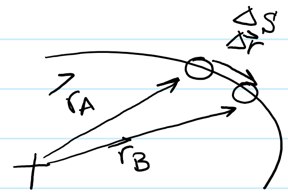
↳ 3D ⇒ vector analysis

1) Position:



$\vec{r}(t) \rightarrow$ position of the particle measured from a fixed point O
$\vec{r}(t)$ changes in magnitude and direction as the particle moves along the path $S(t)$.

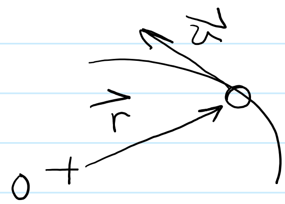
2) Displacement:



Δr is very very small

$$\vec{\Delta r} = \vec{r}_B - \vec{r}_A$$
(vector subtraction)

3) velocity:



Average velocity: $\vec{v}_{avg} = \dfrac{\Delta \vec{r}}{\Delta t}$

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

dt

$d\vec{r}$ is tangent to the curve $\Rightarrow \vec{v}$ is tangent to the curve

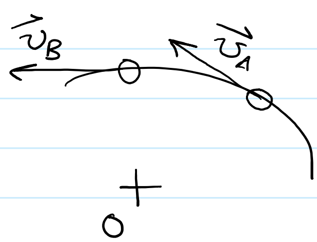
Speed: $\Delta \vec{r} \rightarrow \Delta s$ as $\Delta t \rightarrow 0$

$$v_{sp} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Remark: Differentiating the displacement $\rightarrow \vec{v}_{inst}$
Differentiating the path function $\rightarrow v_{sp}$.

* The magnitude of v is calculated by realizing that $\Delta\vec{r} \rightarrow \Delta s$ as $\Delta t \rightarrow 0$.

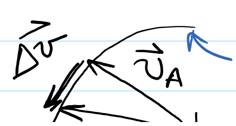
4) Acceleration:



$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A$$

$$a_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

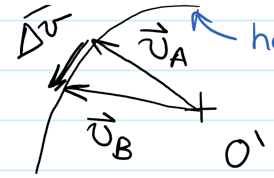
Plot both velocity vectors such that their tails are at a new origin point O` and their arrows touch points on a curve.



hodograph

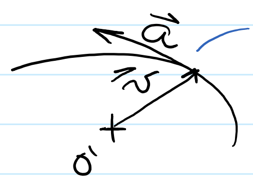
on a curve.

Hodograph: a curve describing the locus of points for the arrowheads of velocity vectors



Instantaneous Acceleration $\vec{a}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

$a_{inst} = \frac{d^2\vec{r}}{dt^2}$

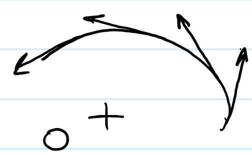


Question: is $\vec{a}$ tangent to the motion path?

No, in order for the particle to stay along the desired path, the directional change always swings the velocity inwards

$\vec{v}$ is tangent to the path

$\vec{a}$ is tangent to the hodograph



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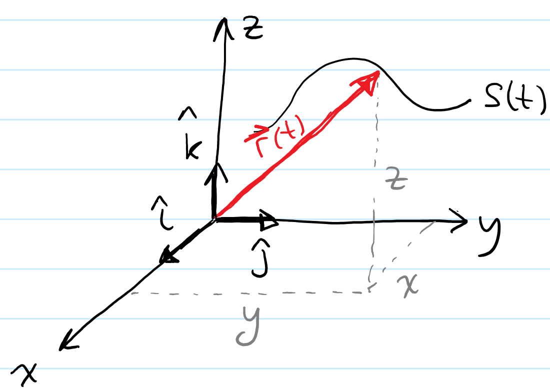
$\boxed{12.5}$ Curvilinear Motion: Rectangular Components.



Position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

magnitude of $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

direction of $\vec{r}$: $\vec{u}_r = \frac{\vec{r}}{r}$

unit vector in the direction of $\vec{r}$

Remember!

r is $r(t)$
x is $x(t)$
y is $y(t)$
z is $z(t)$

Functions of time

take into account the change in mag and direction

Velocity $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i}) + \frac{d}{dt}(y\hat{j}) + \frac{d}{dt}(z\hat{k})$

$$\frac{d}{dt}(x\hat{i}) = \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt}$$

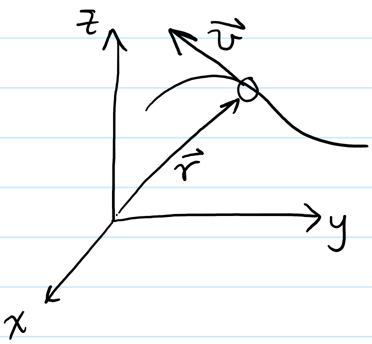
$\frac{d\hat{i}}{dt} = 0$ since the xyz frame of reference is fixed.

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

where $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$ (dot → first time derivative)

magnitude $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

direction $\vec{u}_v = \frac{\vec{v}}{v}$ → unit vector in the direction of $\vec{v}$ (tangent to the path)



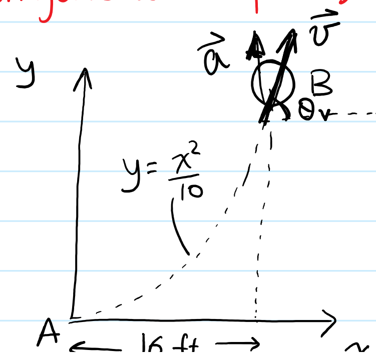
Acceleration $\vec{a} = \frac{dv}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

where
$$a_x = \dot{v}_x = \ddot{x}$$
$$a_y = \dot{v}_y = \ddot{y}$$
$$a_z = \dot{v}_z = \ddot{z}$$
(double dot → second time derivative).

magnitude $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

direction $\vec{u}_a = \frac{\vec{a}}{a}$ — unit vector in the direction of a (not tangent to the path).

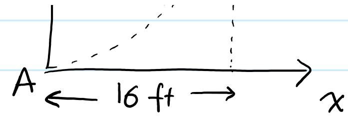
Example 12.9 (textbook)



$x = 8t$ ft (horizontal position of the balloon)

$y = \frac{x^2}{10}$ ft (equation of the path) of the balloon)

$y = \frac{x^2}{10}$ ft (equation of the path)



determine the magnitude and direction of $\vec{v}$, $\vec{a}$ @ $t = 2$ s.

Note $\quad x = x(t)$, $y = y(x)$ } 1) use implicit differentiation 2) write $y = y(t)$

$$v_x = \dot{x} = \frac{d}{dt} 8t = 8 \text{ ft/s} \rightarrow$$

$$v_y = \dot{y} = \frac{d}{dt}\left(\frac{x^2}{10}\right) = \frac{2x}{10}\dot{x} = \frac{2(\overset{x(2)}{16})(8)}{10} = 25.6 \text{ ft/s} \uparrow$$

or $\quad y = \frac{64t^2}{10} \rightarrow \dot{y} = \frac{2 \times 64\, t}{10} \rightarrow v_y = \frac{2 \times 64 \times 2}{10} = 25.6 \text{ ft}\uparrow$

$$|v| = \sqrt{v_x^2 + v_y^2} = \sqrt{8^2 + 25.6^2} = 26.8 \text{ ft/s}$$

$$\vec{u}_v = \frac{\vec{v}}{|v|} = \frac{8}{26.8}\hat{i} + \frac{25.6}{26.8}\hat{j} \quad \text{(tangent to the path)}$$

$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{25.6}{8} = 72.6^\circ$$

$$a_x = \dot{v}_x = \frac{d}{dt}8 = 0$$

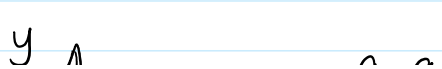
$$a_y = \dot{v}_y = \frac{d}{dt}\left(\frac{2 \times 64\, t}{10}\right) = \frac{2 \times 64}{10} = 12.8 \text{ ft/s}^2 \uparrow$$

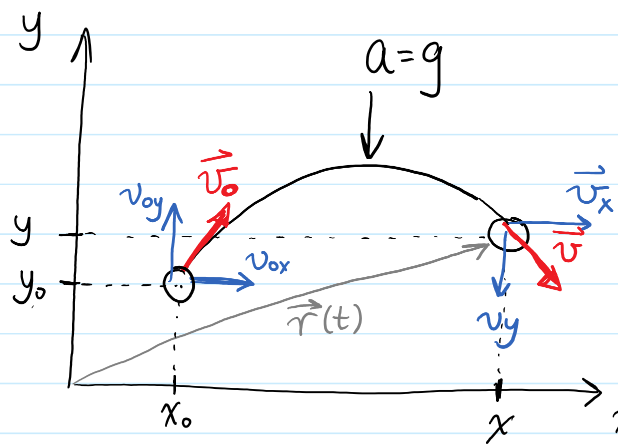
$$|a| = \sqrt{0^2 + 12.8^2} = 12.8 \text{ ft/s}^2$$

$$\vec{u}_a = \frac{\vec{a}}{|a|} = \frac{0}{12.8}\hat{i} + \frac{12.8}{12.8}\hat{j} = \hat{j}$$

$$\theta_a = 90^\circ$$

$\boxed{12.6}$ Motion of a Projectile





Air Resistance is neglected $\Rightarrow$ Constant downward acc.

$$a_c = g = 9.81\ m/s^2$$
$$= 32.2\ ft/s^2$$

Horizontal motion $\quad a_c = 0$

$\overset{+}{\rightarrow} \quad v = v_0 + a_c t \quad \longrightarrow \quad v_x = v_{ox}$

$\overset{+}{\rightarrow} \quad x = x_0 + v_0 t + \frac{1}{2} a_c t^2 \quad \longrightarrow \quad x = x_0 + v_{ox} t$

$\overset{+}{\rightarrow} \quad v^2 = v_0^2 + 2 a_c (x - x_0) \quad \longrightarrow \quad v_x = v_{ox}$

Vertical Motion $\quad a_c = -g \quad$ (positive direction is $\uparrow +$)

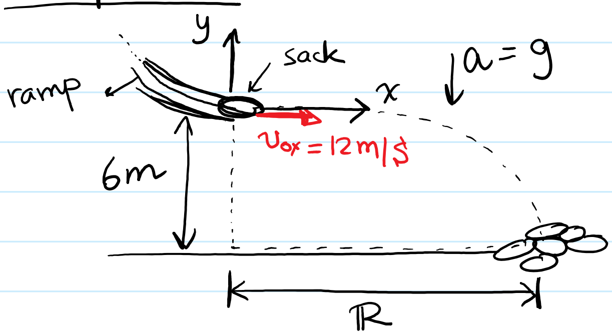
$\uparrow + \quad v = v_0 + a_c t \quad \longrightarrow \quad v_y = v_{oy} - gt$

$\uparrow + \quad y = y_0 + v_0 t + \frac{1}{2} a_c t^2 \quad \longrightarrow \quad y = y_0 + v_{oy} t - \frac{1}{2} g t^2$

$\uparrow + \quad v^2 = v_0^2 + 2 a_c (y - y_0) \quad \longrightarrow \quad v_y^2 = v_{oy}^2 - 2g(y - y_0)$

Last equation results from eliminating time from first two, only 2 of the 3 equations are independent.

Example 12.11 (text book)



Calculate the time needed to hit the floor and the Range R

$v_{ox} = 12\ m/s$

$v_{oy} = 0 \ m/s$
$a_c = -g = -9.81 \ m/s^2$
$x_o = 0$
$x = R$ (unknown)
$y_o = 0 \ m$
$y = -6 \ m$
t (unknown)
$v_x = v_{ox} = 12 \ m/s$
$v_y =$ unknown

<u>vertical motion</u>.

$$y = y_o + v_{oy} t - \frac{1}{2} g t^2$$

$$-6 = 0 + 0 - \frac{1}{2}(9.81) t^2 \Rightarrow t = 1.11 \ s$$

<u>Horizontal motion</u>

$$x = x_o + v_{ox} t$$

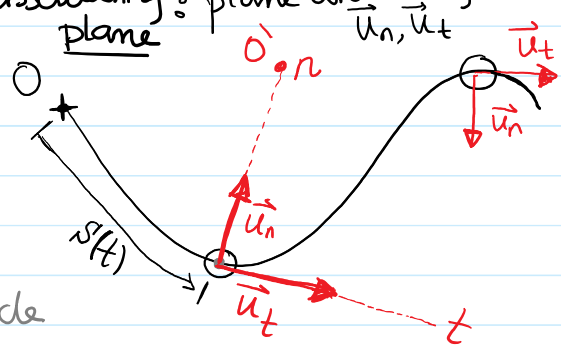
$$= 0 + 12 \times 1.11 \Rightarrow \boxed{x = R = 13.3 m}$$

* * * *

12.7 Curvilinear Motion: Normal & Tangential Components.

<u>Planar Position</u>

osculating plane: plane containing $\vec{u}_n, \vec{u}_t$



- Particle is at position s from origin point O
- Coordinate system with an origin on the curve, which at this instant coincides with the particle

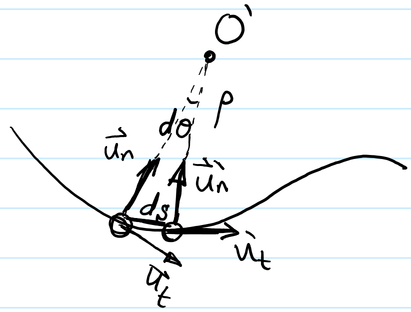
t-axis: tangent to the curve, +ve → increasing s
n-axis: normal to the curve, +ve → center of curvature

<u>Velocity</u> $\vec{v} = v \vec{u}_t$. (tangential only)

Magnitude $|\vec{v}| = \dfrac{ds}{dt} = \dot{s}$

Direction $\vec{u}_t$ (always tangent to the curve)

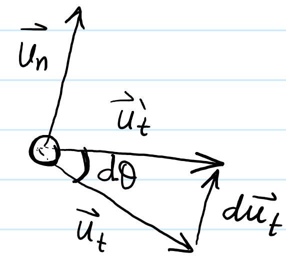
Acceleration



$$\vec{a} = \dot{\vec{v}} = \dot{v}\,\vec{u}_t + v\,\dot{\vec{u}}_t$$

what is $\dot{\vec{u}}_t$

$\vec{u}_t$ does not change in magnitude (only direction)



$$\vec{u}_t' = \vec{u}_t + d\vec{u}_t$$

$d\vec{u}_t$ has a magnitude of $1\,d\theta$
$d\vec{u}_t$ has a direction // $\vec{u}_n$

$d\vec{u}_t = d\theta\,\vec{u}_n$ divide by dt

$$\dot{\vec{u}}_t = \dot{\theta}\,\vec{u}_n$$

$$ds = \rho\, d\theta \Rightarrow \dot{s} = \rho\,\dot{\theta} \Rightarrow \dot{\theta} = \frac{\dot{s}}{\rho} = \frac{v}{\rho}$$

$$\dot{\vec{u}}_t = \frac{\dot{s}}{\rho}\vec{u}_n$$

Thus: $\vec{a} = \underbrace{\dot{v}}_{a_t}\,\vec{u}_t + \underbrace{\frac{v^2}{\rho}}_{a_n}\,\vec{u}_n$

tangential component

normal component.

$$|a| = \sqrt{a_t^2 + a_n^2} = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Remark 1: 1) Straight line $\rho = \infty$

$a_t = \dot{v}$ (time rate of change in the mag. of the velocity)

$a_n = \dfrac{v^2}{\rho} = 0$

2) Curve, constant speed.

$a_t = \dot{v} = 0$

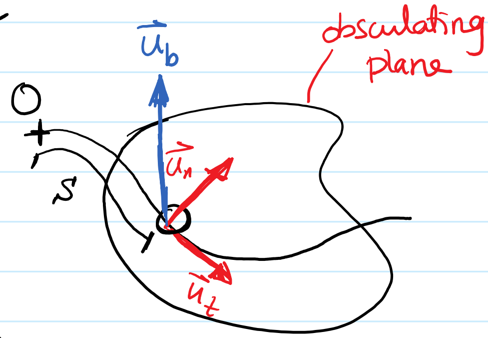
$$a_n = \frac{v^2}{\rho}$$ (time rate of change in the direction of the velocity)

Centripital acceleration

Remark 2: 3D motion

$\vec{u}_b$: binormal axis

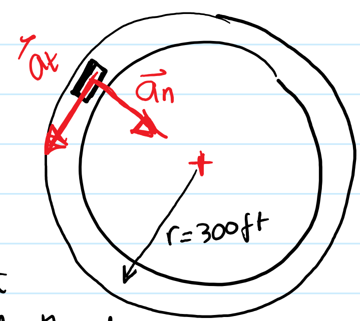
$$\vec{u}_b = \vec{u}_t \times \vec{u}_n$$



Example 12.15 (textbook)

starting from rest

$$a = 7 \ ft/s^2$$



Determine the time needed to get to $8 \ ft/s^2$ and the speed at that point

@ $t = 0$ $\quad |a| = \sqrt{a_t^2 + a_n^2}$

Note that $a_n = \dfrac{v^2}{\rho} = \dfrac{0}{\rho} = 0$

then $\vec{a}_t = 7 \ ft/s^2$

Velocity as a function of time

$$v = v_0 + at$$

$$v = 0 + 7t \Rightarrow v = 7t$$

$$a_n = \frac{v^2}{\rho} = \frac{49t^2}{300} = 0.163t^2 \ ft/s^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(7)^2 + (0.163t^2)^2} = 8 \ ft/s^2$$

$$\boxed{t = 4.87 \ s} \Rightarrow v(4.87) = 7(4.87) = 34.1 \ ft/s$$

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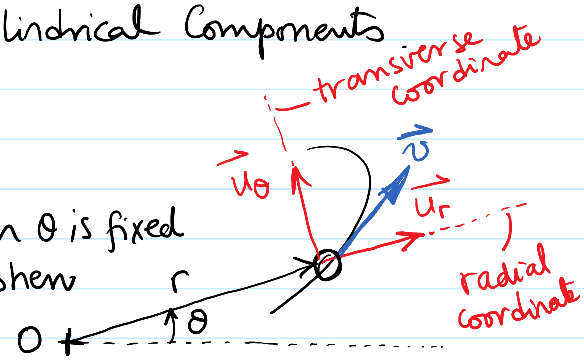
$\boxed{12.8}$ Curvilinear Motion: Cylindrical Components

Polar Coordinates (2D)



$\vec{u}_r$: direction of increasing r when θ is fixed

$\vec{u}_\theta$: direction of increasing θ when r is fixed.

Position: $\vec{r} = r\,\vec{u}_r$

Velocity:

$$\vec{v} = \dot{\vec{r}} = \dot{r}\,\vec{u}_r + r\,\dot{\vec{u}}_r \qquad \dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta$$

$$= \underbrace{\dot{r}}_{\text{radial vel } v_r}\vec{u}_r + \underbrace{r\dot{\theta}}_{v_\theta \text{ transverse vel}}\vec{u}_\theta$$

$\vec{v}_r$: rate of increase or decrease of the length of r

$\vec{v}_\theta$: rate of motion along the circumference of a circle with radius r

Note $\dot{\theta}$: angular velocity (rad/s)

$$|v| = \sqrt{v_r^2 + v_\theta^2} \qquad \vec{v} : \text{tangent to the curve.}$$

Acceleration

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\,\vec{u}_r + \dot{r}\,\dot{\vec{u}}_r + \dot{r}\dot{\theta}\,\vec{u}_\theta + r\ddot{\theta}\,\vec{u}_\theta + r\dot{\theta}\,\dot{\vec{u}}_\theta$$

$$\dot{\vec{u}}_\theta = -\dot{\theta}\,\vec{u}_r$$
$$\dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta$$

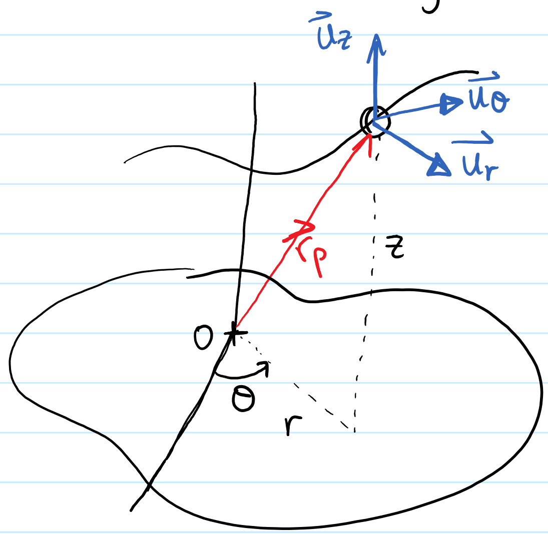
$$\vec{a} = \ddot{r}\,\vec{u}_r + \dot{r}\dot{\theta}\,\vec{u}_\theta + \dot{r}\dot{\theta}\,\vec{u}_\theta + r\ddot{\theta}\,\vec{u}_\theta - r\dot{\theta}^2\,\vec{u}_r$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\vec{u}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\vec{u}_\theta$$

$$|a| = \sqrt{a_r^2 + a_\theta^2} \qquad a \text{ is not tangent to the path}$$

Note: $\ddot{\theta}$: angular acceleration (rad/s^2)

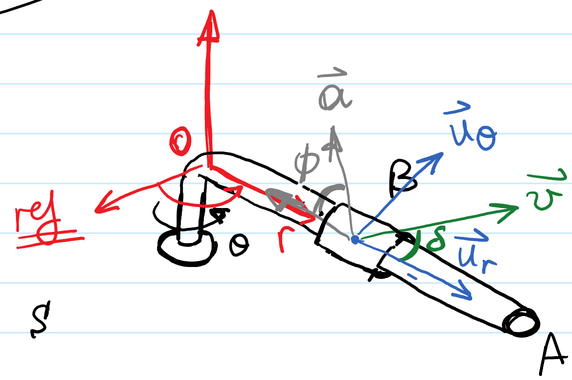
Remark: 3D Motion $\rightarrow$ Cylindrical Coordinates



$\vec{u}_z$ constant mag & direction

$$\vec{r}_p = r\vec{u}_r + z\vec{u}_z$$
$$\vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + \dot{z}\vec{u}_z$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\vec{u}_\theta + \ddot{z}\vec{u}_z.$$

Example 12.18 (textbook)



$\theta = t^3$ rad

$r = 100t^2$ mm

Find $\vec{v}_B$, $\vec{a}_B$ @ $t = 1$ s

$r(1) = 100$ mm

$\dot{r} = 200t \Rightarrow \dot{r}(1) = 200$ mm/s

$\ddot{r} = 200 \Rightarrow \ddot{r}(1) = 200$ mm/s^2

$\theta(1) = 1$ rad

$\dot{\theta} = 3t^2 \rightarrow \dot{\theta}(1) = 3$ rad/s

$\ddot{\theta} = 6t \rightarrow \ddot{\theta}(1) = 6$ rad/s^2

$$\vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta$$
$$= 200\vec{u}_r + 300\vec{u}_\theta \text{ mm/s}$$

$$|\vec{v}| = \sqrt{200^2 + 300^2} = 361 \text{ mm/s}$$

$$\delta = \tan^{-1}\frac{300}{200} = 56.3^\circ \Rightarrow \delta + 1\text{ rad} = 56.3 + 57.3 \simeq 114^\circ.$$

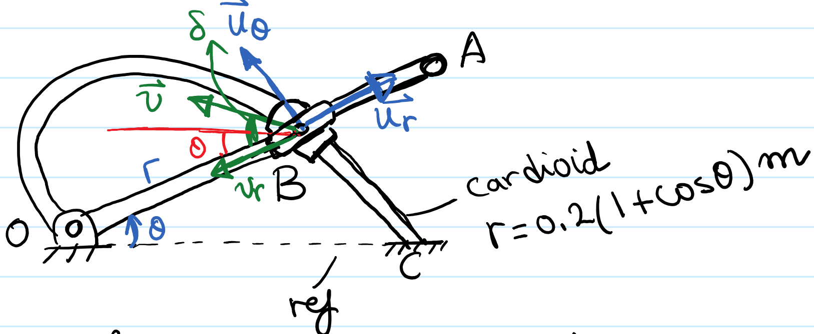
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta$$

$$= -700\,\vec{u}_r + 1800\,\vec{u}_\theta \quad mm/s^2$$

$$|\vec{a}| = \sqrt{700^2 + 1800^2} = 1930 \; mm/s^2$$

$$\phi = \tan^{-1}\frac{1800}{700} = 68.7^\circ \qquad (180 - \phi) + 1\text{ rad} = 169^\circ.$$

Example fundamental problem 12-37 (textbook)



@ $\theta = 30^\circ$ $\quad \dot{\theta} = 3$ rad/s $\quad$ Find $\vec{v}_B$ (mag + dir). at this point.

$$\vec{v}_B\big|_{30^\circ} = \dot{r}\big|_{30^\circ}\vec{u}_r + r\dot{\theta}\big|_{30^\circ}\vec{u}_\theta$$

$$r(30) = 0.2(1+\cos 30) = 0.2\left(1+\frac{\sqrt{3}}{2}\right) = 0.373\text{ m}$$

$$\dot{\theta}(30) = 3 \text{ rad/s}$$

$$\dot{r} = -0.2\sin\theta\,\dot{\theta} \Rightarrow \dot{r}(30) = -0.2 \times \frac{1}{2} \times 3 = -0.3\text{ m/s}$$

$$\vec{v} = -0.3\,\vec{u}_r + 1.119\,\vec{u}_\theta$$

$$|\vec{v}| = \sqrt{0.3^2 + 1.12^2} = 1.16 \text{ m/s}$$

$$\delta = \tan^{-1}\frac{1.12}{0.3} = 75^\circ$$

$$\delta - \theta = 75^\circ - 30 = 45^\circ$$

$$\text{angle of } \vec{v} = 180 - 45 = 135^\circ$$

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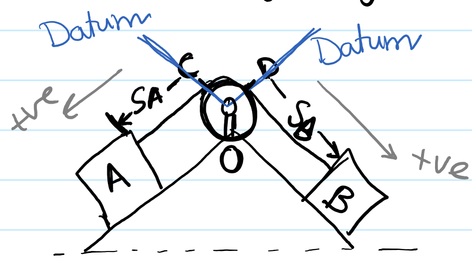
12.9 Absolute Dependent Motion – Analysis of 2 Particles.



* Cord is inextensible
* pulleys are frictionless

$$L_r = s_A + L_{CD} + s_B$$

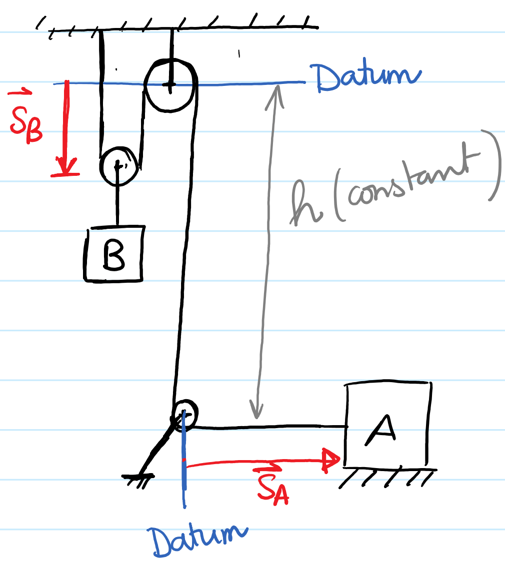
Datums & O are fixed.

$$\frac{d}{dt}\left(L_r = s_A + L_{CD} + s_B\right)$$

$$0 = \frac{ds_A}{dt} + \frac{ds_B}{dt} \Rightarrow \boxed{v_A = -v_B}$$

Similarly $\Rightarrow \boxed{a_A = -a_B}$

Example



$$2s_B + h + s_A = L$$

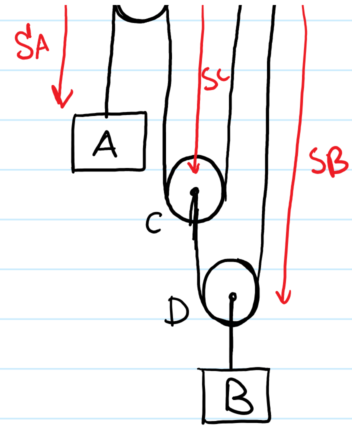
$$2\frac{ds_B}{dt} + \frac{ds_A}{dt} = 0 \Rightarrow v_A = -2v_B$$

Similarly $\Rightarrow a_A = -2a_B$

Example 12.22 (textbook)

$\vec{v}_B = 6$ ft/s ↑
Find $\vec{v}_A$

and v_A



$$L_1 = S_B + S_B - S_C$$

$$L_2 = S_A + 2 S_C$$

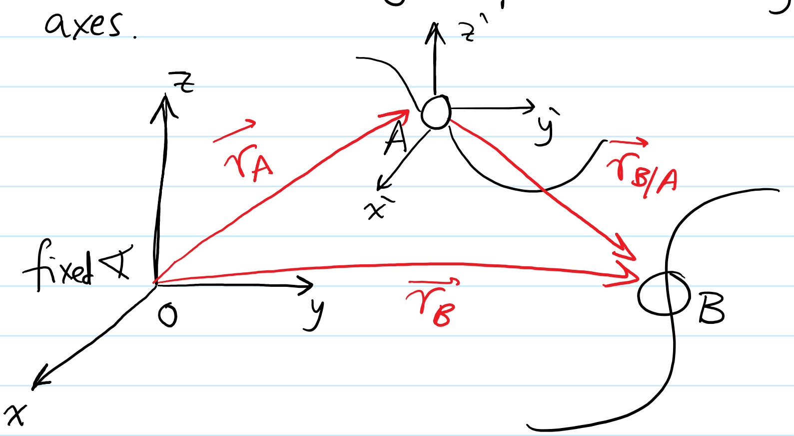
$$2 v_B = v_C$$

$$v_A = -2 v_C$$

$$v_A = -4 v_B = -4 \times 6 = -24 \text{ ft/s}$$

downwards.

$\boxed{12.10}$ Relative motion of 2 particles using translating axes.



1) Position $\vec{r}_A$, $\vec{r}_B$ Absolute positions wrt the fixed frame

$\vec{r}_{B/A}$ Relative position of B wrt the moving frame A.

$$\boxed{\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}}$$

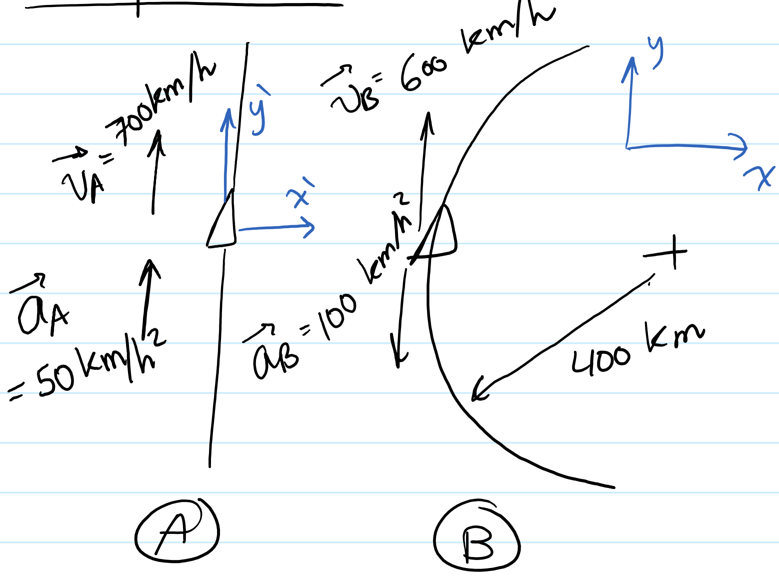
2) Velocity: $$\vec{v}_B = \frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt}$$

$$\boxed{\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}}$$

x', y', z' translate and do not rotate so they are direction is fixed.

3) <u>Acceleration:</u> $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$

<u>Example 12.26</u>



Determine $\vec{v}_{B/A}$, $\vec{a}_{B/A}$

WRT A $\Rightarrow$ translating frame $x'y'z'$ @ A

$\uparrow +$ $\quad \vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$

$$600 = 700 + \vec{v}_{B/A} \Rightarrow \vec{v}_{B/A} = -100 \text{ km/h}$$

100 km/h $\downarrow$

$\rightarrow$ Curved path $\rightarrow$ $\vec{a}_B$ has $\vec{a}_n$ & $\vec{a}_t$

$$(\vec{a}_B)_t = -100 \text{ km/h}^2$$

$$(\vec{a}_B)_n = \frac{v_B^2}{\rho} = \frac{600^2}{400} = 900 \text{ km/h}^2$$

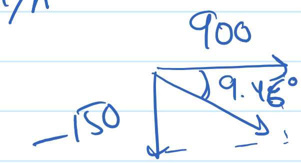
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$900\,\hat{i} - 100\,\hat{j} = 50\,\hat{j} + \vec{a}_{B/A}$

$\vec{a}_{B/A} = 900\,\hat{i} - 150\,\hat{j}$ km/h^2

$|\vec{a}_{B/A}| = \sqrt{900^2 + 150^2} = 912$ km/h^2

$\theta = \tan^{-1} \dfrac{-150}{900} = -9.46°$



* Example 12.27 from the book.