INSTRUCTOR'S SOLUTIONS MANUAL
TO ACCOMPANY

A FIRST COURSE IN THE
FINITE ELEMENT METHOD

FIFTH EDITION

DARYL L. LOGAN

Contents

Chapter 1

1.1. A finite element is a small body or unit interconnected to other units to model a larger structure or system.

1.2. Discretization means dividing the body (system) into an equivalent system of finite elements with associated nodes and elements.

1.3. The modern development of the finite element method began in 1941 with the work of Hrennikoff in the field of structural engineering.

1.4. The direct stiffness method was introduced in 1941 by Hrennikoff. However, it was not commonly known as the direct stiffness method until 1956.

1.5. A matrix is a rectangular array of quantities arranged in rows and columns that is often used to aid in expressing and solving a system of algebraic equations.

1.6. As computer developed it made possible to solve thousands of equations in a matter of minutes.

1.7. The following are the general steps of the finite element method.

Step 1

Divide the body into an equivalent system of finite elements with associated nodes and choose the most appropriate element type.

Step 2

Choose a displacement function within each element.

Step 3

Relate the stresses to the strains through the stress/strain law—generally called the constitutive law.

Step 4

Derive the element stiffness matrix and equations. Use the direct equilibrium method, a work or energy method, or a method of weighted residuals to relate the nodal forces to nodal displacements.

Step 5

Assemble the element equations to obtain the global or total equations and introduce boundary conditions.

Step 6

Solve for the unknown degrees of freedom (or generalized displacements).

Step 7

Solve for the element strains and stresses.

Step 8

Interpret and analyze the results for use in the design/analysis process.

1.8. The displacement method assumes displacements of the nodes as the unknowns of the problem. The problem is formulated such that a set of simultaneous equations is solved for nodal displacements.

1.9. Four common types of elements are: simple line elements, simple two-dimensional elements, simple three-dimensional elements, and simple axisymmetric elements.

1.10 Three common methods used to derive the element stiffness matrix and equations are

(1) direct equilibrium method

(2) work or energy methods

(3) methods of weighted residuals

1.11. The term ‘degrees of freedom’ refers to rotations and displacements that are associated with each node.

1.12. Five typical areas where the finite element is applied are as follows.

(1) Structural/stress analysis

(2) Heat transfer analysis

(3) Fluid flow analysis

(4) Electric or magnetic potential distribution analysis

(5) Biomechanical engineering

1.13. Five advantages of the finite element method are the ability to

(1) Model irregularly shaped bodies quite easily

(2) Handle general load conditions without difficulty

(3) Model bodies composed of several different materials because element equations are evaluated individually

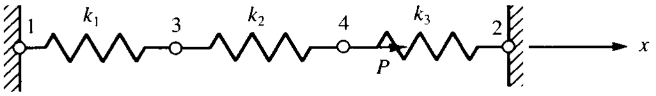
(4) Handle unlimited numbers and kinds of boundary conditions

(5) Vary the size of the elements to make it possible to use small elements where necessary

Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k_3{}^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\} \Rightarrow [K^{-1}]\,\{F\} = [K^{-1}]\,[K]\,\{d\}$$

$$\Rightarrow [K^{-1}]\,\{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$C_{11} = k_2 + k_3$ $\qquad$ $C_{21} = (-1)^3\,(-k_2)$

$C_{12} = (-1)^{1+2}\,(-k_2) = k_2$ $\qquad$ $C_{22} = k_1 + k_2$

$$[C] = \begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}$$

det $[K] = |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - (-k_2)\,(-k_2)$

$\Rightarrow \quad |\,[K]\,| = (k_1 + k_2)\,(k_2 + k_3) - k_2^2$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}}{(k_1+k_2)(k_2+k_3)-k_2^{\,2}} = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2+k_3 & k_2 \\ k_2 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow \quad u_3 = \frac{k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow \quad u_4 = \frac{(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K]\,\{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

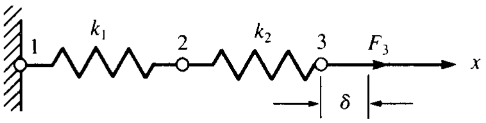
$$F_{1x} = -k_1\,u_3 = -k_1\,\frac{k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1\,k_2\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$F_{2x} = -k_3\,u_4 = -k_3\,\frac{(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3\,(k_1+k_2)\,P}{k_1\,k_2+k_1\,k_3+k_2\,k_3}$$

2.2



$$k_1 = k_2 = k_3 = 1000\ \frac{\text{lb}}{\text{in.}}$$

$$[k^{(1)}] = \begin{array}{cc} \begin{array}{cc} (1) & (2) \end{array} & \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{array}{c} (1) \\ (2) \end{array} \end{array} ; \quad [k^{(2)}] = \begin{array}{cc} \begin{array}{cc} (2) & (3) \end{array} & \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{array}{c} (2) \\ (3) \end{array} \end{array}$$

By the method of superposition the global stiffness matrix is constructed.

$$\begin{array}{c} \qquad\quad (1) \qquad (2) \qquad\quad (3) \\ [K] = \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \end{array}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} k & k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = \delta \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 = & 2k\,u_2 - k\delta \\ F_{3x} = & -k\,u_2 + k\delta \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{k\delta}{2\text{k}} = \frac{\delta}{2} = \frac{1\text{ in.}}{2} \Rightarrow u_2 = 0.5''$$

$$F_{3x} = -\,k\,(0.5'') + k\,(1'')$$

$$F_{3x} = (-\,1000\ \frac{\text{lb}}{\text{in.}})\ (0.5'') + (1000\ \frac{\text{lb}}{\text{in.}})\ (1'')$$

$$F_{3x} = 500 \text{ lbs}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0.5'' \end{Bmatrix}$$

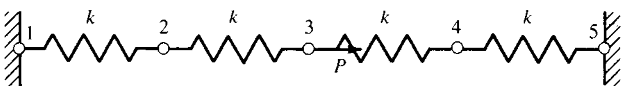
$$\Rightarrow \quad f_{1x}^{(1)} = (-\,1000\ \frac{\text{lb}}{\text{in.}})\ (0.5'') \Rightarrow f_{1x}^{(1)} = -\,500 \text{ lb}$$

$$f_{2x}^{(1)} = (1000\ \frac{\text{lb}}{\text{in.}})\ (0.5'') \Rightarrow f_{2x}^{(1)} = 500 \text{ lb}$$

Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 = 0.5'' \\ u_3 = 1'' \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2x}^{(2)} = -\,500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{array}$$

2.3



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K]\,\{d\}$ we have

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix}$$

(b) $\begin{Bmatrix} 0 \\ P \\ 0 \end{Bmatrix} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \Rightarrow \begin{matrix} 0 & = & 2ku_2 - ku_3 \\ P & = & -ku_2 + 2ku_3 - ku_4 \\ 0 & = & -ku_3 + 2ku_4 \end{matrix}$

$$\Rightarrow \quad u_2 = \frac{u_3}{2} \;;\; u_4 = \frac{u_3}{2}$$

Substituting in the equation in the middle

$$P = -\,k\,u_2 + 2k\,u_3 - k\,u_4$$

$$\Rightarrow \quad P = -\,k\left(\frac{u_3}{2}\right) + 2k\,u_3 - k\left(\frac{u_3}{2}\right)$$

$$\Rightarrow \quad P = k\,u_3$$

$$\Rightarrow \quad u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k} \;;\; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K]\,\{d\}$

$$F_{1x} = -\,k\,u_2 = -k\,\frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -\,k\,u_4 = -k\,\frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

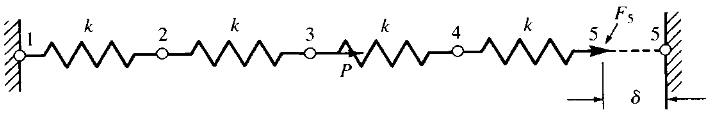
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2}\right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



(a) $[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K]\,\{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{Bmatrix}$$

(b) $0 = 2k\, u_2 - k\, u_3$ (1)

$0 = -\,ku_2 + 2k\, u_3 - k\, u_4$ (2)

$0 = -\,k\, u_3 + 2k\, u_4 - k\, \delta$ (3)

From (2)

$$u_3 = 2\, u_2$$

From (3)

$$u_4 = \frac{\delta + 2\, u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -\,k\,(u_2) + 2k\,(2\, u_2) - k\left(\frac{\delta + 2\,\delta_{2x}}{2}\right)$$

$$\Rightarrow -\,u_2 + 4\, u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2\,\frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\left(\frac{\delta}{4}\right)}{2} \Rightarrow u_4 = \frac{3\,\delta}{4}$$

(c) Going back to the global equation

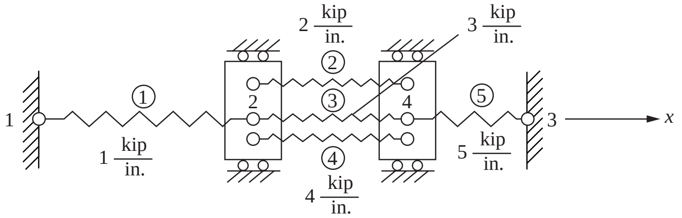
$$\{F\} = [K]\,\{d\}$$

$$F_{1x} = -\,k\;\, u_2 = -k\,\frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k\,\delta}{4}$$

$$F_{5x} = -\,k\, u_4 + k\,\delta = -\,k\left(\frac{3\,\delta}{4}\right) + k\,\delta$$

$$\Rightarrow \quad F_{5x} = \frac{k\,\delta}{4}$$

2.5



$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}; \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} d_2 & d_4 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} d_2 & d_4 \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array}; \quad [k^{(4)}] = \begin{array}{c} \begin{array}{cc} d_2 & d_4 \end{array} \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \begin{array}{c} \begin{array}{cc} d_4 & d_3 \end{array} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}$$

Assembling global $[K]$ using direct stiffness method

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2+3+4 & 0 & -2-3-4 \\ 0 & 0 & 5 & -5 \\ 0 & -2-3-4 & -5 & 2+3+4+5 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

2.6 Now apply + 2 kip at node 2 in spring assemblage of P 2.5.

$$\therefore \quad F_{2x} = 2 \text{ kip}$$

$$[K]\{d\} = \{F\}$$

$[K]$ from P 2.5

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 2 \\ F_3 \\ 0 \end{Bmatrix} \qquad \text{(A)}$$

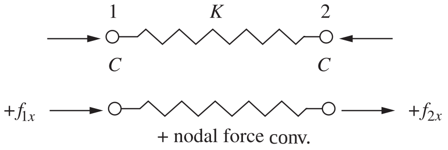
where $u_1 = 0$, $u_3 = 0$ as nodes 1 and 3 are fixed.
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 10 & -9 \\ -9 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 0.475 \text{ in.}, \quad u_4 = 0.305 \text{ in.}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

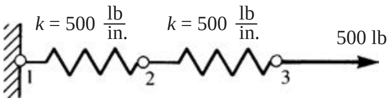
$$\therefore \quad f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{matrix} \text{same as for} \\ \text{tensile element} \end{matrix}$$

2.8



$$k_1 = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};\ k_2 = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

$$[K] = 500\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\Rightarrow \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 1000 \end{bmatrix} = 500\begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 1000\,u_2 - 500\,u_3 \qquad (1)$$

$$500 = -500\,u_2 + 500\,u_3 \qquad (2)$$

From (1)

$$u_2 = \frac{500}{1000}\,u_3 \Rightarrow u_2 = 0.5\,u_3 \qquad (3)$$

Substituting (3) into (2)

$$\Rightarrow \quad 500 = -500\,(0.5\,u_3) + 500\,u_3$$

$$\Rightarrow \quad 500 = 250\,u_3$$

$$\Rightarrow \quad u_3 = 2 \text{ in.}$$

$$\Rightarrow \quad u_2 = (0.5)\,(2 \text{ in.}) \Rightarrow u_2 = 1 \text{ in.}$$

Element 1–2

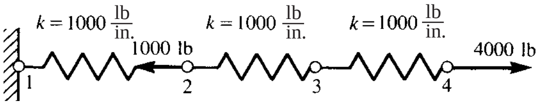
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \text{ in.} \\ 1 \text{ in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -500 \text{ lb} \\ f_{2x}^{(1)} = 500 \text{ lb} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \text{ in.} \\ 2 \text{ in.} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{matrix}$$

$$F_{1x} = 500\,[1\ -1\ \ 0]\begin{bmatrix} 0 \\ 1 \text{ in.} \\ 2 \text{ in.} \end{bmatrix} \Rightarrow F_{1x} = -500 \text{ lb}$$

2.9



$$\begin{array}{cc} & \begin{array}{cc}(1) & (2)\end{array} \\ [k^{(1)}] = & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cc}(2) & (3)\end{array} \\ [k^{(2)}] = & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cc}(3) & (4)\end{array} \\ [k^{(3)}] = & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc}(1) & (2) & (3) & (4)\end{array} \\ [K] = & \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -1000 & 0 \\ 0 & -1000 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix}\end{array}$$

$$\begin{Bmatrix} \cancel{F_{1x} = ?} \\ F_{2x} = -1000 \\ F_{3x} = 0 \\ F_{4x} = 4000 \end{Bmatrix} = \begin{bmatrix} \cancel{1000} & \cancel{-1000} & \cancel{0} & \cancel{0} \\ -1000 & 2000 & -1000 & 0 \\ 0 & -1000 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_1 &= 0 \text{ in.} \\ u_2 &= 3 \text{ in.} \\ u_3 &= 7 \text{ in.} \\ u_4 &= 11 \text{ in.} \end{aligned}$$

Reactions

$$F_{1x} = [1000 \quad -1000 \quad 0 \quad 0] \begin{Bmatrix} u_1 = & 0 \\ u_2 = & 3 \\ u_3 = & 7 \\ u_4 = & 11 \end{Bmatrix} \Rightarrow F_{1x} = -3000 \text{ lb}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x}^{(1)} &= -3000 \text{ lb} \\ f_{2x}^{(1)} &= 3000 \text{ lb} \end{aligned}$$

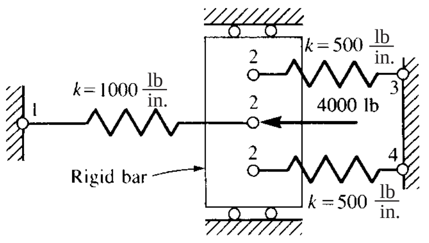
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 3 \\ 7 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2x}^{(2)} &= -4000 \text{ lb} \\ f_{3x}^{(2)} &= 4000 \text{ lb} \end{aligned}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 7 \\ 11 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= -4000 \text{ lb} \\ f_{4x}^{(3)} &= 4000 \text{ lb} \end{aligned}$$

2.10



$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -4000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = \frac{-4000}{2000} = -2 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -4000 \\ 1000 \\ 1000 \end{Bmatrix} \text{lb}$$

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -2000 \end{Bmatrix} \text{lb}$$

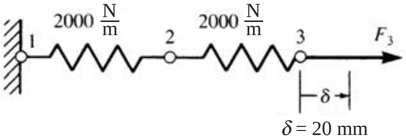
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{lb}$$

Element (3)

$$\begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{lb}$$

2.11



$$[k^{(1)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = ? \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 & 0 \\ -2000 & 4000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.02 \text{ m} \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0.01 \text{ m}$$

Reactions

$$F_{1x} = (-2000)\,(0.01) \Rightarrow F_{1x} = -20 \text{ N}$$

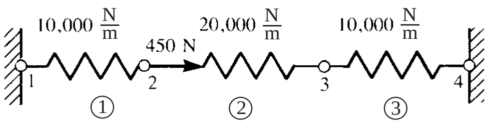
Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix} \text{ N}$$

2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 2 & -2 \\ -2 & 2 \end{Bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 4500 \text{ N} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$0 = -2\,u_2 + 3\,u_3 \Rightarrow u_2 = \frac{3}{2}\,u_3 \Rightarrow u_2 = 1.5\,u_3$$

$$450 \text{ N} = 30000\,(1.5\,u_3) - 20000\,u_3$$

$$\Rightarrow \quad 450 \text{ N} = (25000\,\frac{\text{N}}{\text{m}})\,u_3 \Rightarrow u_3 = 1.8 \times 10^{-2} \text{ m}$$

$$\Rightarrow \quad u_2 = 1.5\,(1.8 \times 10^{-2}) \Rightarrow u_2 = 2.7 \times 10^{-2} \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.7 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{array}{l} \hat{f}_{1x}{}^{(1)} = -270 \text{ N} \\ \hat{f}_{2x}{}^{(1)} = 270 \text{ N} \end{array}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.7\times10^{-2} \\ 1.8\times10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}^{(2)} = \ \ 180\text{ N} \\ \hat{f}_{3x}^{(2)} = -180\text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.8\times10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{3x}^{(3)} = \ \ 180\text{ N} \\ \hat{f}_{4x}^{(3)} = -180\text{ N} \end{matrix}$$

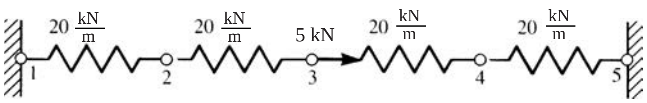
Reactions

$$\{F_{1x}\} = \left(10000\ \frac{\text{N}}{\text{m}}\right) [1 - 1] \begin{Bmatrix} 0 \\ 2.7\times10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -\,270\text{ N}$$

$$\{F_{4x}\} = \left(10000\ \frac{\text{N}}{\text{m}}\right) [-1 \quad 1] \begin{Bmatrix} 1.8\times10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad F_{4x} = -\,180\text{ N}$$

2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 10\text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 20 \begin{Bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

$$\left.\begin{matrix} 0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5\,u_3 \\ 0 = -\,u_3 + 2u_4 \Rightarrow u_4 = 0.5\,u_3 \end{matrix}\right\} \Rightarrow u_2 = u_4$$

$$\Rightarrow \quad 5\text{ kN} = -\,20\,u_2 + 40\,(2\,u_2) - 20\,u_2$$

$$\Rightarrow \quad 5 = 40\,u_2 \Rightarrow u_2 = 0.125\text{ m}$$

$$\Rightarrow \quad u_4 = 0.125\text{ m}$$

$$\Rightarrow \quad u_3 = 2(0.125) \Rightarrow u_3 = 0.25\text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}^{(1)} = -2.5\text{ kN} \\ \hat{f}_{2x}^{(1)} = \ \ 2.5\text{ kN} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0.25 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}^{(2)} = -2.5\text{ kN} \\ \hat{f}_{3x}^{(2)} = \ \ 2.5\text{ kN} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.25 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 2.5 \text{ kN} \\ f_{4x}^{(3)} = -2.5 \text{ kN} \end{matrix}$$

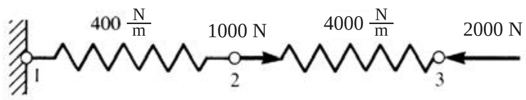
Element (4)

$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{5x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{4x}^{(4)} = 2.5 \text{ kN} \\ \hat{f}_{5x}^{(4)} = -2.5 \text{ kN} \end{matrix}$$

$$F_{1x} = 20 \, [1 \quad -1] \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow F_{1x} = -2.5 \text{ kN}$$

$$F_{5x} = 20 \, [-1 \quad 1] \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5 \text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 100 \\ F_{3x} = -200 \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$100 = 800\, u_2 - 400\, u_3$$

$$-200 = -400\, u_2 + 400\, u_3$$

$$-100 = 400\, u_2 \Rightarrow u_2 = -0.25 \text{ m}$$

$$100 = 800(-0.25) - 400\, u_3 \Rightarrow u_3 = -0.75 \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}^{(1)} = 100 \text{ N} \\ \hat{f}_{2x}^{(1)} = -100 \text{ N} \end{matrix}$$

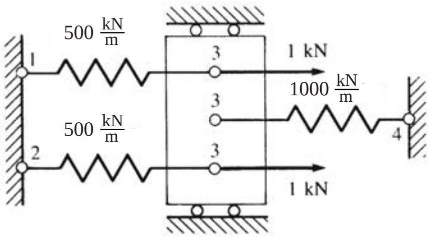
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.25 \\ -0.75 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}^{(2)} = 200 \text{ N} \\ \hat{f}_{3x}^{(2)} = -200 \text{ N} \end{matrix}$$

Reaction

$$\{F_{1x}\} = 400 \, [1 \quad -1] \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow F_{1x} = 100 \text{ N}$$

2.15



$$[k^{(1)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; \; [k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; \; [k^{(3)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = ? \\ F_{3x} = 2 \text{ kN} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & 0 & -500 & 0 \\ 0 & 500 & -500 & 0 \\ -500 & -500 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_3 = 0.001 \text{ m}$$

Reactions

$$F_{1x} = (-500)\,(0.001) \Rightarrow F_{1x} = -0.5 \text{ kN}$$

$$F_{2x} = (-500)\,(0.001) \Rightarrow F_{2x} = -0.5 \text{ kN}$$

$$F_{4x} = (-1000)\,(0.001) \Rightarrow F_{4x} = -1.0 \text{ kN}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

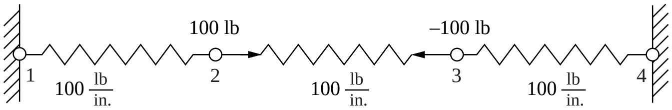
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0.001 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{Bmatrix} 1 \text{ kN} \\ -1 \text{ kN} \end{Bmatrix}$$

2.16



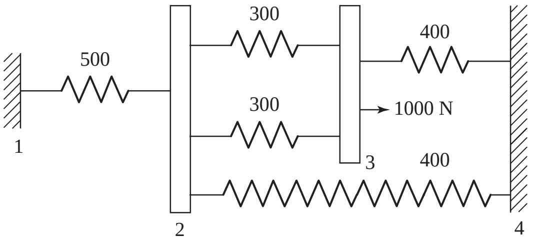
$$\begin{Bmatrix} F_{1x} \\ 100 \\ -100 \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100+100 & -100 & 0 \\ 0 & -100 & 100+100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 100 \\ -100 \end{Bmatrix} = \begin{Bmatrix} 200 & -100 \\ -100 & 200 \end{Bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_2 = \frac{1}{3} \text{ in.}$$

$$u_3 = -\frac{1}{3} \text{ in.}$$

2.17



$$\begin{Bmatrix} F_{1x} = ? \\ 0 \\ 1000 \text{ N} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & -500 & 0 & 0 \\ -500 & \begin{pmatrix} 400+300 \\ 500+300 \end{pmatrix} & -300-300 & -400 \\ 0 & -300-300 & (300+300+400) & -400 \\ 0 & -400 & -400 & 400+400 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = 0 \end{Bmatrix}$$

$$0 = 1500\, u_2 - 600\, u_3$$

$$1000 = -600\, u_2 + 1000\, u_3$$

$$u_3 = \frac{15\cancel{0}\,\cancel{0}}{6\cancel{0}\,\cancel{0}}\; u_2 = 2.5\, u_2$$

$$1000 = -600\, u_2 + 1000\,(2.5\, u_2)$$

$$1000 = 1900\, u_2$$

$$u_2 = \frac{1000}{1900} = \frac{1}{1.9} \text{ mm} = 0.526 \text{ mm}$$

$$u_3 = 2.5 \left(\frac{1}{1.9}\right) \text{ mm} = 1.316 \text{ mm}$$

$$F_{1x} = -500 \left(\frac{1}{1.9}\right) = -263.16 \text{ N}$$

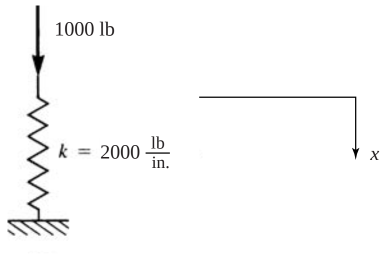
$$F_{4x} = -400 \left(\frac{1}{1.9}\right) - 400 \left(2.5\left(\frac{1}{1.9}\right)\right)$$

$$= -400 \left(\frac{1}{1.9} + \frac{2.5}{1.9}\right) = -736.84 \text{ N}$$

$$\Sigma F_x = -263.16 + 1000 - 736.84 = 0$$

2.18

(a)



As in Example 2.4

$$\pi_p = U + \Omega$$

$$U = \frac{1}{2}\, k\, x^2,\ \Omega = -Fx$$

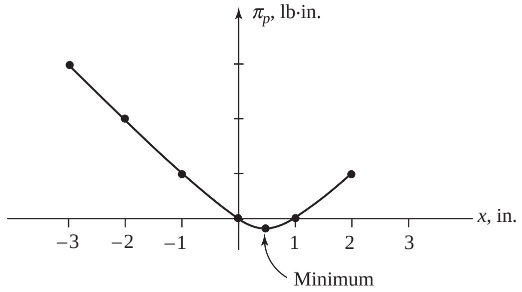
Set up table

$$\pi_p = \frac{1}{2}\,(2000)\, x^2 - 1000\, x = 1000\, x^2 - 1000\, x$$

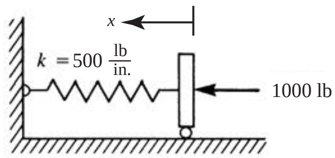
Deformation x, in.	π_p, lb·in.
– 3.0	6000
– 2.0	3000

– 1.0	1000
0.0	0
0.5	– 125
1.0	0
2.0	1000

$\dfrac{\partial \pi_p}{\partial x}$ = 2000 x – 1000 = 0 $\Rightarrow$ x = 0.5 in. yields minimum π_p as table verifies.



(b)



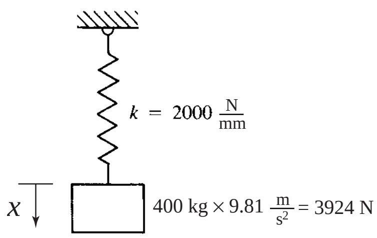
$$\pi_p = \frac{1}{2}\,kx^2 - F_x = 250\,x^2 - 1000\,x$$

x, in.	π_p, lb·in.
– 3.0	11250
– 2.0	3000
– 1.0	1250
0	0
1.0	– 750
2.0	– 1000
3.0	– 750

$$\frac{\partial \pi_p}{\partial x} = 500\,x - 1000 = 0$$

$\Rightarrow$ x = 2.0 in. yields π_p minimum

(c)



$$\pi_p = \frac{1}{2}(2000)\,x^2 - 3924\,x = 1000\,x^2 - 3924\,x$$

$$\frac{\partial \pi_p}{\partial x} = 2000\,x - 3924 = 0$$

$$\Rightarrow \quad x = 1.962 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}(2000)\,(1.962)^2 - 3924\,(1.962)$$

$$\Rightarrow \quad \pi_{p\,\min} = -\,3849.45 \text{ N·mm}$$

(d)

$$\pi_p = \frac{1}{2}(400)\,x^2 - 981\,x$$

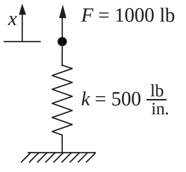
$$\frac{\partial \pi_p}{\partial x} = 400\,x - 981 = 0$$

$$\Rightarrow \quad x = 2.4525 \text{ mm yields } \pi_p \text{ minimum}$$

$$\pi_{p\,\min} = \frac{1}{2}(400)\,(2.4525)^2 - 981\,(2.4525)$$

$$\Rightarrow \quad \pi_{p\,\min} = -\,1202.95 \text{ N·mm}$$

2.19



Now let positive x be upward

$$\pi_p = \frac{1}{2}\,kx^2 - Fx$$

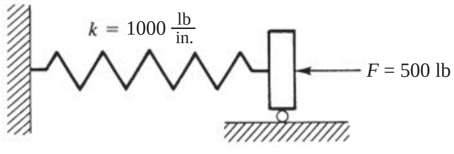
$$\pi_p = \frac{1}{2}(500)\,x^2 - 1000\,x$$

$$\pi_p = 250\,x^2 - 1000\,x$$

$$\frac{\partial \pi_p}{\partial x} = 500\,x - 1000 = 0$$

$$\Rightarrow \; x = 2.0 \text{ in.} \uparrow$$

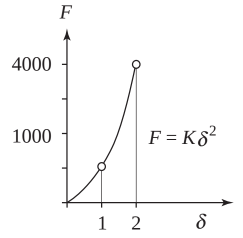
2.20



$$F = k\delta^2 \qquad (x = \delta)$$

$$dU = F\,dx$$

$$U = \int_0^x (kx^2)\, dx$$



$$U = \frac{k x^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3}\ kx^3 - 500\ x$$

$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 500$$

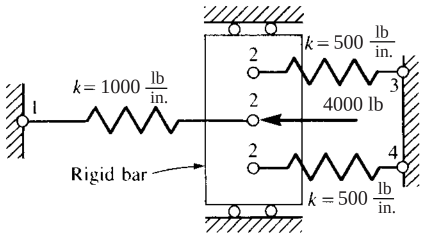
$$0 = 1000\ x^2 - 500$$

$$\Rightarrow \quad x = 0.707 \text{ in.} \quad \text{(equilibrium value of displacement)}$$

$$\pi_{p\ \min} = \frac{1}{3}\ (1000)\ (0.707)^3 - 500\ (0.707)$$

$$\pi_{p\ \min} = -235.7 \text{ lb}\cdot\text{in.}$$

2.21 Solve Problem 2.10 using P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2}\ k_1 (u_2 - u_1)^2 + \frac{1}{2}\ k_2 (u_3 - u_2)^2 + \frac{1}{2}\ k_3 (u_4 - u_2)^2$$

$$- f_{1x}^{(1)} u_1 - f_{2x}^{(1)} u_2 - f_{2x}^{(2)} u_2$$

$$- f_{3x}^{(2)} u_3 - f_{2x}^{(3)} u_2 - f_{4x}^{(3)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = -k_1 u_2 + k_1 u_1 - f_{1x}^{(1)} = 0 \qquad (1)$$

$$\frac{\partial \pi_p}{\partial u_2} = k_1 u_2 - k_1 u_1 - k_2 u_3 + k_2 u_2 - k_3 u_4$$

$$+ k_3 u_2 - f_{2x}^{(1)} - f_{2x}^{(2)} - f_{2x}^{(3)} = 0 \qquad (2)$$

$$\frac{\partial \pi_p}{\partial u_3} = k_2 u_3 - k_2 u_2 - f_{3x}^{(2)} = 0 \qquad (3)$$

$$\frac{\partial \pi_p}{\partial u_4} = k_3 u_4 - k_3 u_2 - f_{4x}^{(3)} = 0 \qquad (4)$$

In matrix form (1) through (4) become

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \\ f_{3x}^{(2)} \\ f_{4x}^{(3)} \end{Bmatrix} \qquad (5)$$

or using numerical values

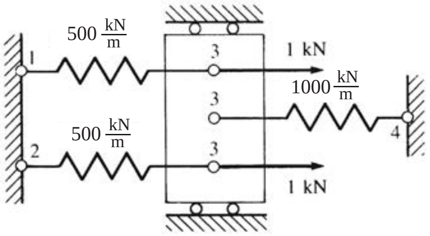
$$\begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ -4000 \\ F_{3x} \\ F_{4x} \end{Bmatrix} \qquad (6)$$

Solution now follows as in Problem 2.10

Solve 2nd of Equations (6) for $u_2 = -2$ in.

For reactions and element forces, see solution to Problem 2.10

2.22 Solve Problem 2.15 by P.E. approach



$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2} k_1 (u_3 - u_1)^2 + \frac{1}{2} k_2 (u_3 - u_2)^2$$

$$+ \frac{1}{2} k_3 (u_4 - u_3)^2 - f_{1x}^{(1)} u_1$$

$$- f_{3x}^{(1)} u_3 - f_{2x}^{(2)} u_2 - f_{3x}^{(2)} u_3$$

$$- f_{3x}^{(3)} u_3 - f_{3x}^{(4)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = -k_2 u_3 + k_2 u_2 - f_{2x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 + k_2 u_3 - k_2 u_2 - k_3 u_4 + k_3 u_3 - f_{3x}^{(2)} - f_{3x}^{(3)} - f_{3x}^{(1)} - k_1 u_1$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_3 u_4 - k_3 u_3 - f_{3x}^{(4)}$$

In matrix form

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} = 2\text{ kN} \\ F_{4x} \end{Bmatrix}$$

For rest of solution, see solutions of Problem 2.15.

2.23

$$I = a_1 + a_2 x$$

$$I(0) = a_1 = I_1$$

$$I(L) = a_1 + a_2 L = I_2$$

$$a_2 = \frac{I_2 - I_1}{L}$$

$$\therefore \qquad I = I_1 + \frac{I_2 - I_1}{L} x$$

Now $V = IR$

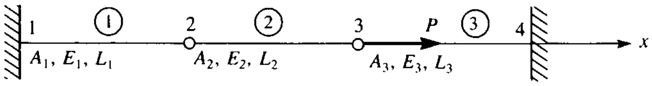
$$V = -V_1 = R(I_2 - I_1)$$

$$V = V_2 = R(I_2 - I_1)$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = R \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$

Chapter 3

3.1



(a) $\left[\hat{k}^{(1)}\right] = \dfrac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\left[\hat{k}^{(2)}\right] = \dfrac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\left[\hat{k}^{(3)}\right] = \dfrac{A_3E_3}{L_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[K] = \begin{bmatrix} \frac{A_1E_1}{L_1} & \frac{-A_1E_1}{L_1} & 0 & 0 \\ \frac{-A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & \frac{-A_2E_2}{L_2} & 0 \\ 0 & \frac{-A_2E_2}{L_2} & \frac{A_2E_2}{L_2} + \frac{A_3E_3}{L_3} & \frac{-A_3E_3}{L_3} \\ 0 & 0 & \frac{-A_3E_3}{L_3} & \frac{A_3E_3}{L_3} \end{bmatrix}$$

(b) $\dfrac{A_1E_1}{L_1} = \dfrac{A_2E_2}{L_2} = \dfrac{A_3E_3}{L_3} = \dfrac{AE}{L}$

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

It is known that $\{F\} = [K]\,\{d\}$

$$\Rightarrow \begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = P \\ F_{4x} = ? \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = \frac{2AE}{L}\,u_2 - \frac{AE}{L}\,u_3 \quad \Rightarrow u_3 = 2\,u_2$$

$$P = \frac{-AE}{L}\,u_2 + \frac{2AE}{L}\,u_3$$

$$\Rightarrow \quad P = \frac{-AE}{L}\,u_2 + \frac{2AE}{L}\,(2\,u_2)$$

$$\Rightarrow \quad u_2 = \frac{1}{3}\,\frac{PL}{AE}$$

$$\Rightarrow \quad u_3 = 2 \cdot \frac{1}{3}\frac{PL}{AE} \quad \Rightarrow \quad u_3 = \frac{2}{3}\frac{PL}{AE}$$

(c) $A = 1$ in.^{2}; $E = 10 \times 10^6$ psi; $L = 10$ in.

$P = 1000$ lbs

(i) $$u_2 = \frac{1}{3}\frac{PL}{AE} = \frac{1}{3}\frac{(1000)\,(10)}{(1)(10 \times 10^6)}$$

$$\Rightarrow \quad u_2 = 3.33 \times 10^{-4} \text{ in.}$$

$$u_3 = \frac{2}{3}\frac{PL}{AE} = 2\,u_2$$

$$\Rightarrow \quad u_3 = 6.67 \times 10^{-4} \text{ in.}$$

(ii) Going back to $\{F\} = [K]\,\{d\}$

$$F_{1x} = \frac{-AE}{L}\,u_2 = \frac{-AE}{L}\left(\frac{1}{3}\frac{PL}{AE}\right) = \frac{-1}{3}P$$

$$\Rightarrow \quad F_{1x} = \frac{-1}{3}\,(1000) \quad \Rightarrow \quad F_{1x} = -\,333.3 \text{ lbs}$$

$$F_{4x} = \frac{-AE}{L}\,u_3 = \frac{-AE}{L}\left(\frac{2}{3}\frac{PL}{AE}\right) = -\frac{2}{3}P$$

$$\Rightarrow \quad F_{4x} = \frac{-2}{3}\,(1000) \quad \Rightarrow \quad F_{4x} = -\,666.7 \text{ lbs}$$

(iii) $f = \sigma A$, where f = force, σ = stress and A = area.

Going back to the local system and substituting

Element (1)

$$\begin{Bmatrix} \sigma_{1x} = \frac{f_{1x}}{A} \\ \sigma_{2x} = \frac{f_{2x}}{A} \end{Bmatrix} = \frac{AE}{AL}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = 3.33 \times 10^{-4} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1x} = -\frac{E}{L}\,u_2 = -\,\frac{10 \times 10^6}{10}\,(3.33 \times 10^{-4})$$

$$\Rightarrow \quad \sigma_{1x}^{(1)} = -\,333.33 \text{ psi (C)}$$

$$\Rightarrow \quad \sigma_{2x} = \frac{E}{L}u_2 = \frac{10 \times 10^6}{10}\,(3.33 \times 10^{-4})$$

$$\Rightarrow \quad \sigma_{2x}^{(2)} = 333.33 \text{ psi (T)}$$

Element (2)

$$\begin{Bmatrix} \sigma_{2x} = \frac{f_{2x}}{A} \\ \sigma_{3x} = \frac{f_{3x}}{A} \end{Bmatrix} = \frac{AE}{AL}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 = 3.33 \times 10^{-4} \\ u_3 = 6.67 \times 10^{-4} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{2x} = \frac{E}{L}\,(u_2 - u_3) = \frac{10 \times 10^6}{10} \times 10^{-4}\,(3.33 - 6.67)$$

$$\Rightarrow\ \sigma_{2x}^{(2)} = -333.33 \text{ psi (C)}$$

$$\Rightarrow\ \sigma_{3x}^{(2)} = \frac{E}{L}(u_3 - u_2) = \frac{10\times10^6}{10}\times10^{-4}\,(6.67 - 3.33)$$

$$\Rightarrow\ \sigma_{3x}^{(2)} = 333.33 \text{ psi (T)}$$

Element (3)

$$\begin{Bmatrix} \sigma_{3x} = \frac{f_{3x}}{A} \\ \sigma_{4x} = \frac{f_{4x}}{A} \end{Bmatrix} = \frac{AE}{AL}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 = 6.67\times10^{-4} \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma_{3x} = \frac{E}{L}(u_3 - u_4) = \frac{10\times10^6}{10}\times10^{-4}\,(6.67 - 0)$$

$$\Rightarrow\ \sigma_{3x}^{(3)} = 666.7 \text{ psi (T)}$$

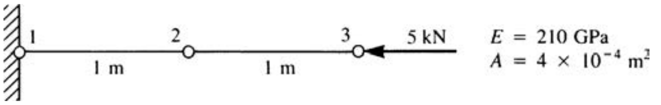
$$\Rightarrow\ \sigma_{4x}^{(3)} = \frac{E}{L}(u_4 - u_3) = \frac{10\times10^6}{10}\times10^{-4}\,(-6.67)$$

$$\Rightarrow\ \sigma_{4x}^{(3)} = -666.7 \text{ psi (C)}$$

So $\sigma^{(1)} = \sigma^{(2)} = -333.3$ psi (T) and

$\sigma^{(3)} = 666.7$ psi (C)

3.2



Element 1–2

$$[k_{1-2}] = 84\times10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element 2–3

$$[k_{2-3}] = 84\times10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{F\} = [K]\{d\}$ and $u_1 = 0$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = -5000 \end{Bmatrix} = 84\times10^6\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$\Rightarrow\ 2u_2 - u_3 = 0\ \Rightarrow\ u_3 = 2u_2 \qquad (1)$$

$$\Rightarrow\ -5000 = 84\times10^6\,[-u_2 + u_3] \qquad (2)$$

Substituting (1) in (2), we have

$$\frac{-5000}{84\times10^6} = -u_2 + 2u_2 \quad\Rightarrow\quad u_2 = -0.595\times10^{-4}\text{ m}$$

$$\Rightarrow\quad u_3 = -1.19\times10^{-4}\text{ m}$$

Element 1–2

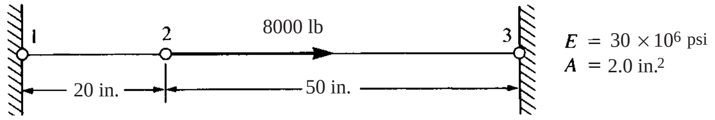
$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 84\times10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.595\times10^{-4} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = 5000\text{ N} \\ f_{2x}^{(1)} = -5000\text{ N} \end{matrix}$$

Element 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 84 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.595 \times 10^{-4} \\ -1.19 \times 10^{-4} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 5000 \text{ N} \\ f_{3x}^{(2)} = -5000 \text{ N} \end{matrix}$$

$$F_{1x} = 84 \times 10^6 \, [1 \; -1 \;\; 0] \begin{Bmatrix} 0 \\ -0.595 \times 10^{-4} \\ -1.19 \times 10^{-4} \end{Bmatrix} \Rightarrow F_{1x} = 5000\text{N}$$

3.3



$$[k_{1-2}] = 3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 1.2 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3+1.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} = 8000 \\ F_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix}$$

$$\Rightarrow \; u_2 = 1.905 \times 10^{-3} \text{ in.}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 4.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.905 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \; F_{1x} = -5715 \text{ lb}$$

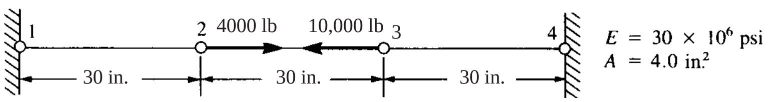
$$F_{2x} = 8000 \text{ lb}$$

$$F_{3x} = -2286 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.905 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -5715 \text{ lb} \\ f_{2x}^{(1)} = 5715 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 1.2 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.905 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 2286 \text{ lb} \\ f_{3x}^{(2)} = -2286 \text{ lb} \end{matrix}$$

3.4



$$[k_{1-2}] = 4 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 4 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{3-4}] = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix}F_{1x}=0\\ F_{2x}=4000\\ F_{3x}=-10000\\ F_{4x}=0\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3\\ u_4=0\end{Bmatrix}$$

$$\Rightarrow\; u_2 = -\,1.667\times10^{-4}\text{ in.}$$
$$u_3 = -\,1.333\times10^{-3}\text{ in.}$$

$$\begin{Bmatrix}F_{1x}\\ F_{2x}\\ F_{3x}\\ F_{4x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2=-1.667\times10^{-4}\\ u_3=-1.333\times10^{-3}\\ u_4=0\end{Bmatrix}$$

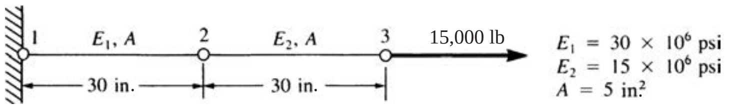
$$\Rightarrow\; F_{1x} = 666.7\text{ lb}$$
$$F_{2x} = 4000\text{ lb}$$
$$F_{3x} = -\,10000\text{ lb}$$
$$F_{4x} = 5333.3\text{ lb}$$

$$\begin{Bmatrix}f_{1x}\\ f_{2x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -1.667\times10^{-4}\end{Bmatrix} \Rightarrow \begin{matrix}f_{1x}^{(1)} = 666.7\text{ lb}\\ f_{2x}^{(1)} = -\,666.7\text{ lb}\end{matrix}$$

$$\begin{Bmatrix}f_{2x}\\ f_{3x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-1.667\times10^{-4}\\ -1.333\times10^{-3}\end{Bmatrix} \Rightarrow \begin{matrix}f_{2x}^{(2)} = 4666.67\text{ lb}\\ f_{3x}^{(2)} = -\,4666.7\text{ lb}\end{matrix}$$

$$\begin{Bmatrix}f_{3x}\\ f_{4x}\end{Bmatrix} = 4\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-1.333\times10^{-3}\\ 0\end{Bmatrix} \Rightarrow \begin{matrix}f_{3x}^{(3)} = -\,5333.3\text{ lb}\\ f_{4x}^{(3)} = 5333.3\text{ lb}\end{matrix}$$

3.5



Element 1–2

$$[k_{1-2}] = 5\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Element 2–3

$$[k_{2-3}] = 5\times10^6\begin{bmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\text{Global } [K] = 5\times10^6\begin{bmatrix}1 & -1 & 0\\ -1 & \frac{3}{2} & \frac{-1}{2}\\ 0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\{F\} = [K]\,\{d\} \text{ and } u_1 = 0$$

$$\begin{Bmatrix}F_{1x}=?\\ F_{2x}=0\\ F_{3x}=15000\end{Bmatrix} = 5\times10^6\begin{bmatrix}1 & -1 & 0\\ -1 & \frac{3}{2} & \frac{-1}{2}\\ 0 & \frac{-1}{2} & \frac{1}{2}\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3\end{Bmatrix}$$

$$\Rightarrow\ 0 = \frac{3}{2}\ u_2 - \frac{1}{2}\ u_3 \ \ \Rightarrow\ \ u_3 = 3\,u_2 \qquad (1)$$

$$\Rightarrow 15000 = 5 \times 10^6\ [\frac{1}{2}\ u_2 + \frac{1}{2}\ u_3] \qquad (2)$$

Substituting (1) in (2)

$$\frac{(2 \times 15000)}{5 \times 10^6} = -\,u_2 + 3\,u_2$$

$$\Rightarrow \qquad u_2 = 0.00075 \text{ in.}$$

$$\Rightarrow \qquad u_3 = 3\ (0.00075) \ \ \Rightarrow\ \ u_3 = 0.00225 \text{ in.}$$

Element 1–2

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.00075 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x} = -15000 \text{ lb} \\ f_{2x} = 15000 \text{ lb} \end{matrix}$$

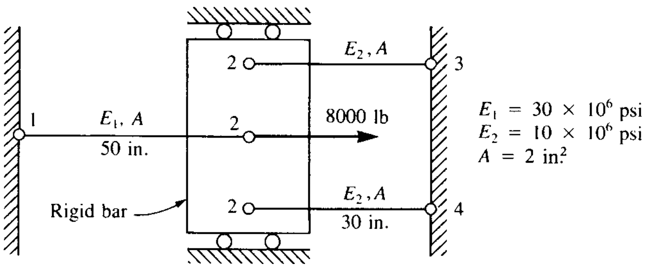
Element 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 5 \times 10^6 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0.00075 \\ 0.00225 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = -15000 \text{ lb} \\ f_{3x} = 15000 \text{ lb} \end{matrix}$$

$$F_{1x} = 5 \times 10^6\ [1 \quad -1 \quad 0] \begin{Bmatrix} 0 \\ 0.00075 \\ 0.00225 \end{Bmatrix}$$

$$\Rightarrow \qquad F_{1x} = -\,15000 \text{ lb}$$

3.6



$$[k_{1-2}] = 10^6 \begin{bmatrix} 1.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix}$$

$$[k_{2-3}] = [k_{2-4}] = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix}$$

$$\{F\} = [K]\ \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 8000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = 10^6 \begin{bmatrix} 1.2 & -1.2 & 0 & 0 \\ -1.2 & 2.533 & -0.667 & -0.667 \\ 0 & -0.667 & 0.667 & 0 \\ 0 & -0.667 & 0 & 0.667 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \qquad u_2 = 3.16 \times 10^{-3} \text{ in.}$$

Reactions

$$F_{1x} = (-\,1.2 \times 10^6)\ (u_2) \ \ \Rightarrow\ \ F_{1x} = -\,3789.5 \text{ lb}$$

$$F_{3x} = (-\,0.667 \times 10^6)\ (u_2) \ \ \Rightarrow\ \ F_{3x} = -\,2105.25 \text{ lb}$$

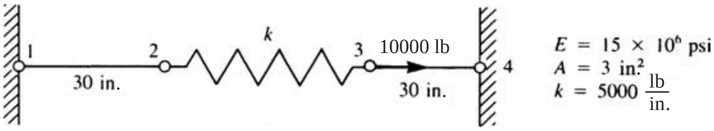
$$F_{4x} = (-0.667 \times 10^6)(u_2) \Rightarrow F_{4x} = -2105.25 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.16 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -3789.5 \text{ lb} \\ f_{2x}^{(1)} = 3789.5 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix} \begin{Bmatrix} 3.16 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 2105.25 \text{ lb} \\ f_{3x}^{(2)} = -2105.25 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 0.667 & -0.667 \\ -0.667 & 0.667 \end{bmatrix} \begin{Bmatrix} 3.16 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 2105.25 \text{ lb} \\ f_{4x}^{(2)} = -2105.25 \text{ lb} \end{matrix}$$

3.7



$$[k_{1-2}] = [k_{3-4}] = 1.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{2-3}] = 5000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = 0 \\ F_{3x} = 10000 \\ F_{4x} = 0 \end{Bmatrix} = 10^3 \begin{bmatrix} 1500 & -1500 & 0 & 0 \\ -1500 & 1505 & -5 & 0 \\ 0 & -5 & 1505 & -1500 \\ 0 & 0 & -1500 & 1500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 10000 \end{Bmatrix} = 10^3 \begin{bmatrix} 1505 & -5 \\ -5 & 1505 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \Rightarrow \begin{matrix} u_2 = 2.21 \times 10^{-5} \text{ in.} \\ u_3 = 6.65 \times 10^{-3} \text{ in.} \end{matrix}$$

Reactions

$$F_{1x} = (-1500 \times 10^3)(u_2) \Rightarrow F_{1x} = -33.15 \text{ lb}$$

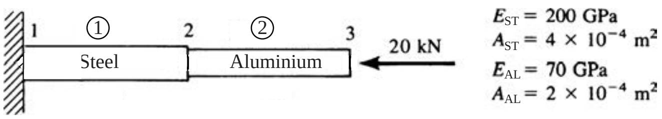
$$F_{4x} = (-1500 \times 10^3)(u_3) \Rightarrow F_{4x} = -9975 \text{ lb}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 1.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.21 \times 10^{-5} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -33.15 \text{ lb} \\ f_{2x}^{(1)} = 33.15 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 5000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.21 \times 10^{-5} \\ 6.65 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -33.15 \text{ lb} \\ f_{3x}^{(2)} = 33.15 \text{ lb} \end{matrix}$$

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 1.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 6.65 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x}^{(3)} = 9975 \text{ lb} \\ f_{4x}^{(3)} = -9975 \text{ lb} \end{matrix}$$

3.8



$$[k^{(1)}] = \frac{(4 \times 10^{-4}\,\text{m}^2)(200 \times 10^6\,\frac{\text{kN}}{\text{m}^2})}{1\,\text{m}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(1)}] = 800 \times 10^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

$$[k^{(2)}] = \frac{(2 \times 10^{-4}\,\text{m}^2)(70 \times 10^6\,\frac{\text{kN}}{\text{m}^2})}{1\,\text{m}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 140 \times 10^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} \\ F_{3x} = -20\text{ kN} \end{Bmatrix} = 10^2 \begin{bmatrix} 800 & -800 & 0 \\ -800 & 940 & -140 \\ 0 & -140 & 140 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 10^2\,(940\,u_2 - 140\,u_3) \;\Rightarrow\; u_3 = 6.741\,u_2 \qquad (1)$$

$$\Rightarrow -20000 = 10^2\,(-140\,u_2 + 140\,u_3) \qquad (2)$$

Substituting (1) into (2)

$$\Rightarrow \quad -20000 = 10^2\,(-140\,u_2 + 140\,(6.714)\,u_2)$$

$$\Rightarrow \quad u_2 = -0.25 \times 10^{-3}\text{ m}$$

$$\Rightarrow \quad u_3 = -1.678 \times 10^{-3}\text{ m}$$

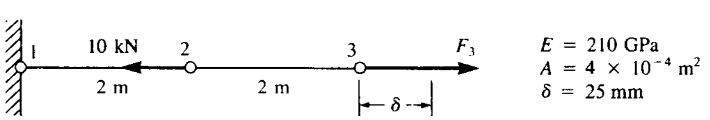
$$F_{1x} = 10^2\,(-800 \times (-0.25 \times 10^{-3}))$$

$$\Rightarrow \quad F_{1x} = 20\text{ kN}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 800 \times 10^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.25 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = 20\text{ kN} \\ f_{2x}^{(1)} = -20\text{ kN} \end{matrix}$$

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 140 \times 10^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.25 \times 10^{-3} \\ -1.678 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = 20\text{ kN} \\ f_{3x}^{(2)} = -20\text{ kN} \end{matrix}$$

3.9



$$[k_{1-2}] = [k_{2-3}] = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = -10\text{ kN} \\ F_{3x} = ? \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0.025\text{ m} \end{Bmatrix}$$

$$\Rightarrow \frac{-10\text{ kN}}{4.2 \times 10^4} = 2\,u_2 - 1(0.025)$$

$$\Rightarrow \quad u_2 = 0.01238\text{ m}$$

$$F_{3x} = 4.2 \times 10^4\,[-1\ 1] \begin{Bmatrix} 0.01238 \\ 0.025 \end{Bmatrix} \Rightarrow F_{3x} = 530\text{ kN}$$

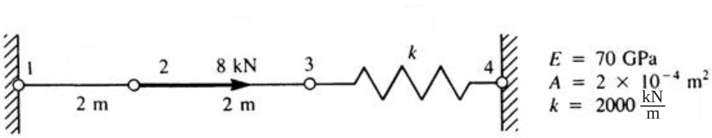
$$F_{1x} = 4.2 \times 10^4\,[1\ -1] \begin{Bmatrix} 0 \\ 0.01238 \end{Bmatrix} \Rightarrow F_{1x} = -520\text{ kN}$$

Element forces

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01238 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x}^{(1)} = -520\text{ kN} \\ f_{2x}^{(1)} = 520\text{ kN} \end{matrix}$$

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 4.2 \times 10^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.01238 \\ 0.025 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x}^{(2)} = -530\text{ kN} \\ f_{3x}^{(2)} = 530\text{ kN} \end{matrix}$$

3.10



$$[k^{(1)}] = [k^{(2)}] = 7000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 2000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 8\text{kN} \\ F_{3x} = 0 \\ F_{4x} = ? \end{Bmatrix} = 10^3 \begin{bmatrix} 7 & -7 & 0 & 0 \\ -7 & 14 & -7 & 0 \\ 0 & -7 & 9 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad 8 = 10^3\,[14\,u_2 - 7\,u_3] \tag{1}$$

$$0 = 10^3\,[-7\,u_2 + 9\,u_3]$$

$$\Rightarrow \quad u_3 = \frac{7}{9}\,u_2 \tag{2}$$

Substituting (2) into (1)

$$\Rightarrow \frac{8}{10^3} = 14\,u_2 - 7 \times \frac{7}{9}\,u_2$$

$$\Rightarrow \quad u_2 = 0.9351 \times 10^{-3}\text{ m}$$

$$\Rightarrow \quad u_3 = 0.7273 \times 10^{-3}\text{ m}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 7 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.9351 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} -6.546 \\ 6.546 \end{Bmatrix} \text{kN}$$

Element (2)

$$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = 7 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.9351 \times 10^{-3} \\ 0.7273 \times 10^{-3} \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{Bmatrix} 1.455 \\ -1.455 \end{Bmatrix} \text{kN}$$

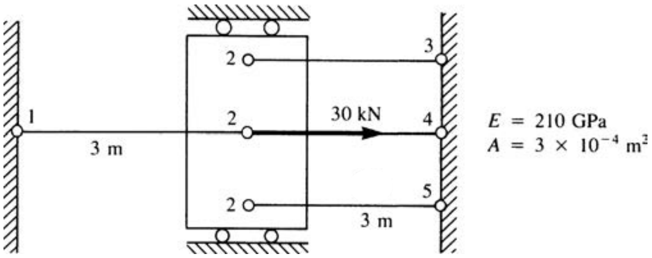
Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 2 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.7273 \times 10^{-3} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = \begin{Bmatrix} 1.455 \\ -1.455 \end{Bmatrix} \text{kN}$$

$$F_{1x} = 10^3 \, [7 \;\; -7] \begin{Bmatrix} 0 \\ 0.9351 \times 10^{-3} \end{Bmatrix} = F_{1x} = -\,6.546 \text{ kN}$$

$$F_{4x} = 10^3 \, [-2 \;\; 2] \begin{Bmatrix} 0.7273 \times 10^{-3} \\ 0 \end{Bmatrix} \;\Rightarrow\; F_{4x} = -1.455 \text{ kN}$$

3.11



$$[k_{1-2}] = [k_{2-3}] = [k_{2-4}] = [k_{2-5}] = 2.1 \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K] \, \{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{2x} = 30 \text{ kN} \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = 0 \end{Bmatrix} = 2.1 \times 10^7 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \\ u_5 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_2 = 3.572 \times 10^{-4} \text{ m}$$

Reactions

$$F_{1x} = (2.1 \times 10^7)\,(-\,1)\,(u_2) \;\Rightarrow\; F_{1x} = -\,7500 \text{ N}$$
$$F_{3x} = (2.1 \times 10^7)\,(-\,1)\,(u_2) \;\Rightarrow\; F_{3x} = -\,7500 \text{ N}$$
$$F_{4x} = (2.1 \times 10^7)\,(-\,1)\,(u_2) \;\Rightarrow\; F_{4x} = -\,7500 \text{ N}$$
$$F_{5x} = (2.1 \times 10^7)\,(-\,1)\,(u_2) \;\Rightarrow\; F_{5x} = -\,7500 \text{ N}$$

Element forces

$$f_{1-2} = -\,f_{2-3} = -\,f_{2-4} = -\,f_{2-5} = (2.1 \times 10^7)\,(u_2)$$

$$\Rightarrow \quad f_{1-2} = 7500 \text{ N}$$

$$f_{2-3} = -7500 \text{ N}$$
$$f_{2-4} = -7500 \text{ N}$$
$$f_{2-5} = -7500 \text{ N}$$

3.12

$$\frac{P}{A(x)} = E\frac{du}{dx}$$

$$u = \int \frac{P}{A(x)E}\, dx$$

$$u = \int \frac{P}{A_0\left(1+\frac{x}{L}\right)E}\, dx$$

$$= \int \frac{PL}{A_0 L\left(1+\frac{x}{L}\right)E}\, dx$$

$$= \int \frac{PL}{A_0\,(L+x)\,E}\, dx$$

$$= \int \frac{PL}{A_0 E\, u}\, du \quad \text{(Change variable } u = L + x \text{ and } du = dx\text{)}$$

$$= \frac{PL}{A_0 E}\int \frac{1}{u}\, du$$

$$= \frac{PL}{A_0 E}\; ln\; u$$

$$\Rightarrow \qquad u = \frac{PL}{A_0 E}\; ln\;(L + x)$$

$$u = \frac{(-1000)\,(20)}{2\times 10\times 10^6}\; ln\;(20 + x)$$

$$u = -10^{-3}\; ln\;(20 + x)$$

$$u(x = 0) = (-\,ln\;20)\times 10^{-3}$$
$$= -\,2.996\times 10^{-3} \text{ in.}$$

$$u(x = 10) = (-ln\;(20 + 10))\;(-10^{-3})$$
$$= -\,3.401\times 10^{-3} \text{ in.}$$

Two elements

$$A\left(\frac{L}{4}\right) = A_0\left(1+\frac{\frac{L}{4}}{L}\right) = A_0\left(1+\frac{1}{4}\right) = \frac{5}{4}A_0$$

$$A\left(\frac{3}{4}L\right) = A_0\left(1+\frac{\frac{3}{4}L}{L}\right) = A_0\left(1+\frac{3}{4}\right) = \frac{7}{4}A_0$$

$$[k^{(1)}] = \frac{5}{4}\frac{A_0 E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = \frac{7}{4}\frac{A_0 E}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} -P \\ 0 \\ F_{3x} \end{Bmatrix} = \frac{A_0E}{\frac{L}{2}} \begin{bmatrix} \frac{5}{4} & \frac{-5}{4} & 0 \\ \frac{-5}{4} & \frac{5}{4}+\frac{7}{4} & \frac{-7}{4} \\ 0 & \frac{-7}{4} & \frac{7}{4} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_{3=0} \end{Bmatrix}$$

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{5}{4} u_1 - \frac{5}{4} u_2\right) = -P \qquad (1)$$

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{-5}{4} u_1 + 3 u_2\right) = 0$$

$$\Rightarrow \quad u_2 = \frac{5}{4\times 3} u_1 = \frac{5}{12} u_1 \qquad (2)$$

Substituting (2) into (1)

$$\Rightarrow \quad \frac{A_0E}{\frac{L}{2}} \left(\frac{5}{4} u_1 - \frac{5}{4} \left(\frac{5}{12} u_1\right)\right) = -P$$

$$\Rightarrow \quad \left[\frac{12}{12}\left(\frac{5}{4}\right) - \frac{5}{4}\left(\frac{5}{12}\right)\right] u_1 = \frac{-PL}{2A_0E}$$

$$\Rightarrow \quad \left[\frac{60-25}{48}\right] u_1 = \frac{-PL}{2A_0E}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{\cancel{2} A_0E} \; \frac{\overset{24}{\cancel{48}}}{35}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{A_0E} \; \frac{24}{35}$$

$$\Rightarrow \quad u_2 = \frac{5}{12} \; \frac{24}{35}\left(\frac{-PL}{A_0E}\right)$$

$$\Rightarrow \quad u_2 = -\frac{2}{7} \; \frac{PL}{A_0E}$$

Now $\quad A_0 = 2$ in.^{2}, $L = 20$ in., $E = 10 \times 10^6$ psi

$P = 1000$ lb

$$u_1 = -\frac{(1000)\,(20)}{2(10\times 10^6)} \times \frac{24}{35}$$

$$\Rightarrow \quad u_1 = -0.6857 \times 10^{-3} \text{ in.}$$

$$\Rightarrow \quad u_2 = \frac{5}{12} \left(-0.6857 \times 10^{-3}\right)$$

$$\Rightarrow \quad u_2 = -0.2857 \times 10^{-3} \text{ in.}$$

One element

$$A = A_0 \left(1 + \frac{\frac{L}{2}}{L}\right) = A_0 \left(1 + \frac{1}{2}\right) = \frac{3}{2} A_0$$

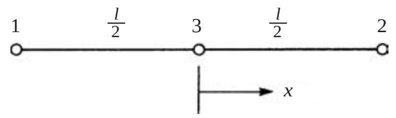
$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{\frac{3}{2} A_0 E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = \frac{-PL}{\frac{3}{2} A_0 E}$$

$$\Rightarrow \quad u_1 = -\frac{2}{3} \frac{(1000)(20)}{(2)(10 \times 10^6)}$$

$$\Rightarrow \quad u_1 = -0.667 \times 10^{-3} \text{ in.}$$

3.13



$$u = a_1 + a_2 x + a_3 x^2 \tag{A}$$

$$u(0) = u_2 = a_1 \tag{1}$$

$$u(-\frac{l}{2}) = u_1 = u_2 + a_2 (-\frac{l}{2}) + a_3 (-\frac{l}{2})^2 \tag{2}$$

$$u(\frac{l}{2}) = u_3 = u_2 + a_2 (\frac{l}{2}) + a_3 (\frac{l}{2})^2 \tag{3}$$

Solving for a_2 and a_3 from (2) and (3)

$$a_2 = \frac{u_3 - u_1}{l}, \; a_3 = \frac{2(u_1 + u_3 - 2u_2)}{l^2} \tag{4}$$

By (1) and (4) into (A)

$$u = u_2 + \left(\frac{u_3 - u_1}{l}\right) x + \frac{2(u_1 + u_3 - 2u_2)}{l^2} x^2 \tag{5}$$

$$u = [N] \{d\} \tag{6}$$

$$u = \begin{bmatrix} \frac{-x}{l} + \frac{2x^2}{l^2} & 1 - \frac{4x^2}{l^2} & \frac{x}{l} + \frac{2x^2}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{7}$$

$$\{\in\} = \frac{\partial u}{\partial x} = [B] \{d\} = \frac{\partial N}{\partial x} \{d\} \tag{8}$$

Using (7) in (8)

$$\{\in\} = \begin{bmatrix} -\frac{1}{l} + \frac{4x}{l^2} & \frac{-8x}{l^2} & \frac{1}{l} + \frac{4x}{l^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{9}$$

$$\therefore \quad [B] = \begin{bmatrix} -\frac{1}{l} + \frac{4x}{l^2} & \frac{-8x}{l^2} & \frac{1}{l} + \frac{4x}{l^2} \end{bmatrix} \tag{10}$$

$$[K] = A \int_{-l/2}^{l/2} [B^T] \, E \, [B] \, dx \tag{11}$$

A = cross sectional area of the bar

E = Young's Modulus of the bar

3.14 Given $u = a + bx^2$ for 2 noded bar

$$\in = \frac{du}{dx} = 2bx$$

$$u(0) = u_1 = a$$

$$u(L) = u_2 = u_1 + bL^2$$

$$\therefore \qquad b = \frac{u_2 - u_1}{L^2}$$

$$u = u_1 + \left[\frac{u_2 - u_1}{L^2}\right]x^2$$

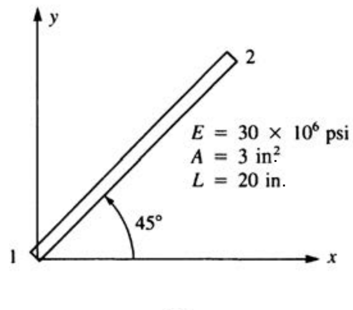
This displacement function allows for a rigid body displacement as the $a = u_1$ term does this. Also should allow for constant strain, but have $\in = 2bx$ or a linear strain. Therefore, not complete. Need to complete 2nd degree polynomial and 3rd node for compatible function.

Try $\qquad u = a_1 + a_2 x + a_3 x^2$

$$\frac{du}{dx} = a_2 + 2a_3 x$$

'a_2' allows for constant strain term.

3.15 (a)

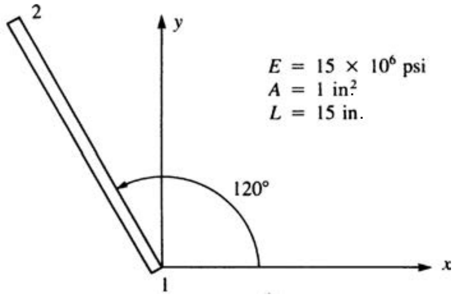


$$C = \frac{1}{\sqrt{2}}, \; S = \frac{1}{\sqrt{2}}$$

$$[K] = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ & & & S^2 \end{bmatrix}$$

$$[K] = 2.25 \times 10^6 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

(b)

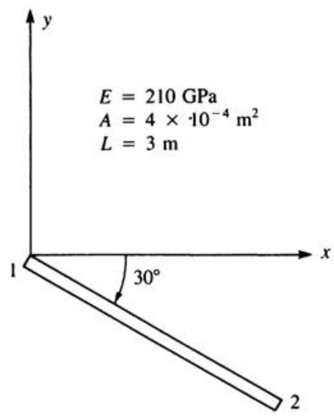


$$C = \frac{-1}{2},\ S = \frac{\sqrt{3}}{2}$$

$$[K] = \frac{15 \times 10^6 \times 1}{15}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

$$[K] = \frac{10^6}{4}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

(c)

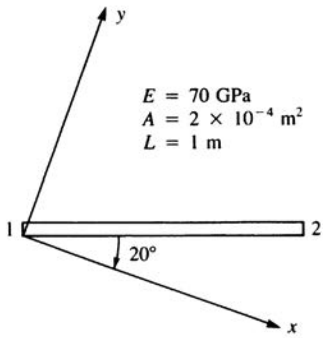


$$C = \frac{\sqrt{3}}{2}, \quad S = -\frac{1}{2}$$

$$[K] = \frac{(210 \times 10^6)(4 \times 10^{-4})}{3}\begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

$$\underline{K} = 7000\begin{bmatrix} 3 & \sqrt{3} & 3 & \sqrt{3} \\ \sqrt{3} & 1 & \sqrt{3} & 1 \\ 3 & \sqrt{3} & 3 & \sqrt{3} \\ \sqrt{3} & 1 & \sqrt{3} & 1 \end{bmatrix}\frac{\text{kN}}{\text{m}}$$

(d)



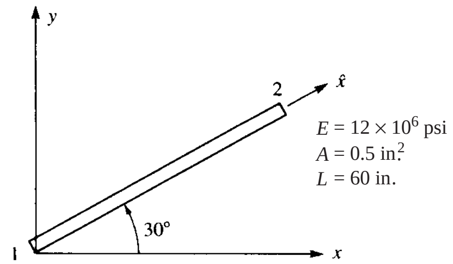
$C = 0.9397 \qquad C^2 = 0.883 \qquad CS = 0.321$

$S = 0.3420 \qquad S^2 = 0.117$

$$[K] = \frac{(70\times10^4)(2\times10^{-4})}{1}\begin{bmatrix} 0.883 & 0.321 & -0.883 & -0.321 \\ 0.321 & 0.883 & -0.321 & -0.883 \\ -0.883 & -0.321 & 0.883 & 0.321 \\ -0.321 & -0.883 & 0.321 & 0.883 \end{bmatrix}$$

$$[K] = 1.4\times10^7\begin{bmatrix} 0.883 & 0.321 & -0.883 & -0.321 \\ 0.321 & 0.883 & -0.321 & -0.883 \\ -0.883 & -0.321 & 0.883 & 0.321 \\ -0.321 & -0.883 & 0.321 & 0.883 \end{bmatrix}\frac{\text{N}}{\text{m}}$$

3.16 (a)



$C = 0.866 \qquad u_1 = 0.5 \text{ in.} \qquad v_1 = 0.0 \text{ in.}$

$S = 0.5 \qquad u_2 = 0.25 \text{ in.} \qquad v_2 = 0.75 \text{ in.}$

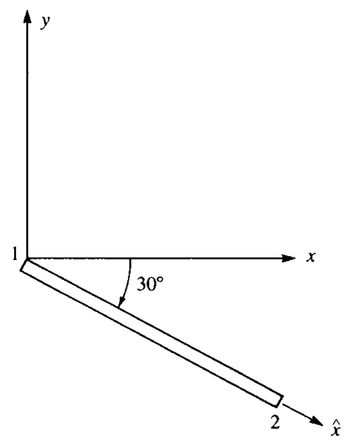
$$u_1' = u_1\,C + v_1\,S = 0.5\,(0.866) + (0.0)\,(0.5)$$

$$\Rightarrow \quad u_1' = 0.433 \text{ in.}$$

$$u_2' = u_2\,C + v_2\,S = (0.25)\,(0.866) + (0.75)\,(0.5)$$

$$u_2' = 0.592 \text{ in.}$$

(b)



$$C = \frac{\sqrt{3}}{2}, \quad S = -\frac{1}{2}$$

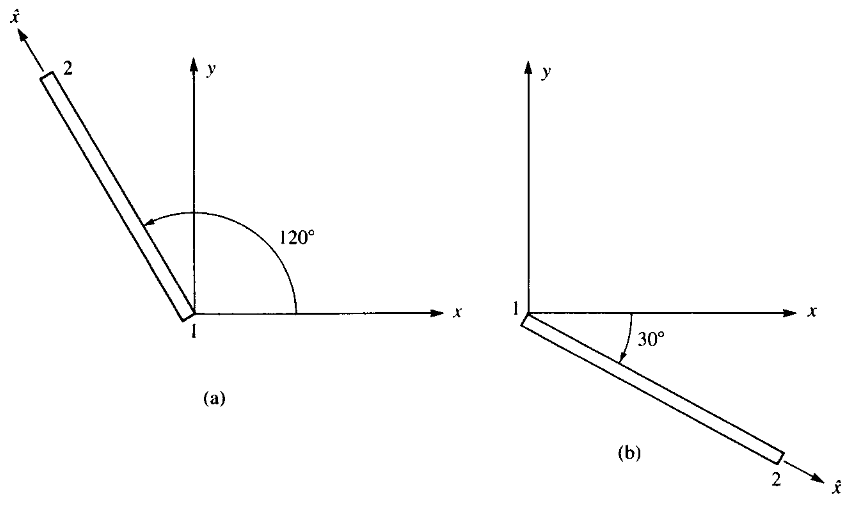
$$u_1' = u_1\,C + v_1\,S = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + (0)\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad u_1' = 0.433 \text{ in.}$$

$$u_2' = u_2\,C + v_2\,S = \left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad u_2' = -\,0.1585 \text{ in.}$$

3.17



$u_1 = 0.0$ $\quad u_2 = 5.0$ mm $\quad E = 210$ GPa

$v_1 = 2.5$ mm $\quad v_2 = 3.0$ mm $\quad A = 10 \times 10^{-4}$ m^2

$L = 3$ m

(a) We know that $\{d'\} = [T]\,\{d\}$

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$C =$ cos 120° = – 0.5, S = sin 120° = 0.866

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} -0.5 & 0.866 & 0 & 0 \\ -0.866 & -0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.866 \\ 0 & 0 & -0.866 & -0.5 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025 \\ 0.005 \\ 0.003 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{Bmatrix} 0.002165 \\ -0.00125 \\ 0.000098 \\ -0.00583 \end{Bmatrix} \text{ m} = \begin{Bmatrix} 2.165 \\ -1.25 \\ 0.098 \\ -5.830 \end{Bmatrix} \text{ mm}$$

(b) C = cos (– 30°) = 0.866, S = – 0.5

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0.866 & -0.5 \\ 0 & 0 & 0.5 & 0.866 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025 \\ 0.005 \\ 0.003 \end{Bmatrix}$$

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ 2.165 \\ 3.03 \\ 5.098 \end{Bmatrix} \text{ mm}$$

3.18

(a) $$\sigma = \frac{E}{L}\,[-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$C = \frac{\sqrt{2}}{2}, \quad S = \frac{\sqrt{2}}{2}, \quad E = 30 \times 10^6 \text{ psi}, \quad L = 60 \text{ in.}$$

$$\sigma = \frac{30 \times 10^6}{60} \left[-\frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \right] \begin{Bmatrix} 0 \\ 0 \\ 0.01 \\ 0.02 \end{Bmatrix}$$

$$\Rightarrow \sigma = 10600 \text{ psi}$$

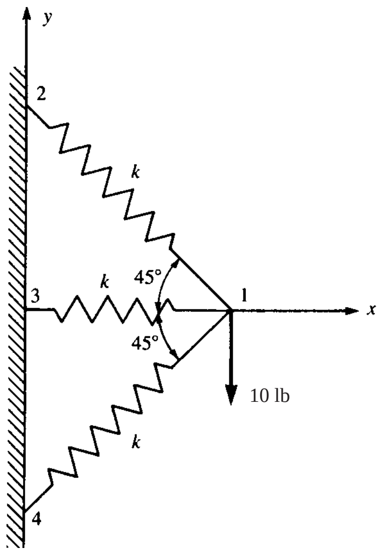
(b) $C = \dfrac{\sqrt{3}}{2}$, $S = \dfrac{1}{2}$, E = 210 GPa, L = 3 m

$$\sigma = \left[-\tfrac{\sqrt{3}}{2} \quad -\tfrac{1}{2} \quad \tfrac{\sqrt{3}}{2} \quad \tfrac{1}{2}\right]\begin{Bmatrix} 0.25 \\ 0 \\ 1.00 \\ 0 \end{Bmatrix} \times 10^{-3} \times \frac{210 \times 10^{6}}{3}$$

$$\Rightarrow \sigma = 45470\ \frac{\text{kN}}{\text{m}^2}$$

$$\Rightarrow \sigma = 45.47\ \text{MPa}$$

3.19



$$\{f\} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} \text{ and } \quad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

(a) For element 1–3; $\theta = 180°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{3x} \\ f_{3y} \end{Bmatrix} = K \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

For element 1–4; $\theta = 225°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{K}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

For element 1–2; $\theta = 135°$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{K}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \end{Bmatrix}$$

Total K

$$[K] = K \begin{bmatrix} 2 & 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) Applying boundary conditions

$u_4 = v_4 = u_2 = v_2 = u_3 = v_3 = 0$

$[K]$ is reduced to

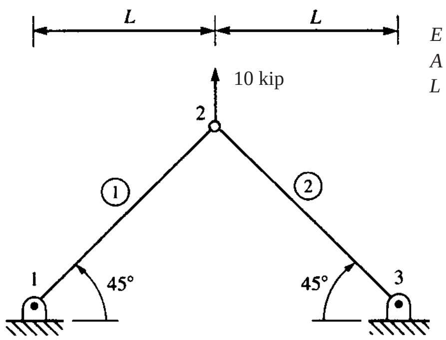
$$[K] = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 0 \\ -10 \end{Bmatrix} = K \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$\Rightarrow u_1 = 0$

$$v_1 = \frac{-10}{K}$$

3.20



$E = 1 \times 10^6$ psi
$A = 5$ in.^{2}
$L = 100$ in.

Element 1–2

$$C = \frac{\sqrt{2}}{2}; \quad S = \frac{\sqrt{3}}{2}; \quad L_{1-2} = \sqrt{2}\ L$$

$$[k_{1-2}] = \frac{A_1E_1}{L_{1-2}}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Element 2–3

$$C = \frac{\sqrt{2}}{2}; \quad S = -\frac{\sqrt{2}}{2}; \quad L_{2-3} = \sqrt{2}\ L$$

$$[k_{2-3}] = \frac{A_2E_2}{L_{2-3}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Applying the boundary conditions

$$\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \frac{1 \times 5 \times 10^3}{\sqrt{2} \times 100}\begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow u_2 = 0$$

$$v_2 = \frac{10 \times \sqrt{2} \times 100}{5 \times 10^3}$$

$$\Rightarrow v_2 = 0.283 \text{ in.}$$

$$\{f'\} = [k']\{d'\} = [k'][T^*]\{d\}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{5 \times 10^3}{\sqrt{2} \times 100}\begin{bmatrix} C & S & -C & -S \\ -C & -S & C & S \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.283 \end{Bmatrix}$$

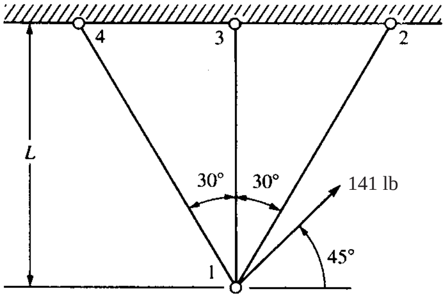
$$f'_{1x} = \frac{5 \times 10^3}{\sqrt{2} \times 100}\left[-\frac{\sqrt{2}}{2}(0.283)\right] = -7.07 \text{ kips}$$

$$f'_{2x} = \frac{5 \times 10^3}{\sqrt{2} \times 100}\left[\frac{\sqrt{2}}{2}(0.283)\right] = 7.07 \text{ kips}$$

$$\sigma_{1-2} = \frac{f'_{2x}}{A} = \frac{7.07 \text{ kips}}{5 \text{ in.}^2} \Rightarrow \sigma_{1-2} = 1414 \text{ psi (T)}$$

$$\sigma_{2-3} = \frac{f'_{3x}}{A} = \frac{7.07 \text{ kips}}{5 \text{ in.}^2} \Rightarrow \sigma_{2-3} = 1414 \text{ psi (T)}$$

3.21



Element 1–2

$$L_{1-2} = \frac{2}{\sqrt{3}}\,L;\ \theta = 60°$$

$$[k_{1-2}] = \frac{\sqrt{3}AE}{2L}\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

Element 1–3

$$L_{1-3} = L;\ \theta = 90°$$

$$[k_{1-2}] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Element 1–4

$$L_{1-4} = \frac{2}{\sqrt{3}}\,L;\ \theta = 120°$$

$$[k_{1-4}] = \frac{\sqrt{3}AE}{2L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

Applying the boundary conditions

$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$

$$[K] = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{2}(\frac{1}{4}) + 0 + \frac{\sqrt{3}}{2}(\frac{1}{4}) & \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{4}\right) + 0 + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{4}\right) \\ \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{4}\right) + 0 + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{4}\right) & \frac{\sqrt{3}}{2}(\frac{3}{4}) + 1 + \frac{\sqrt{3}}{2}(\frac{3}{4}) \end{bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}$$

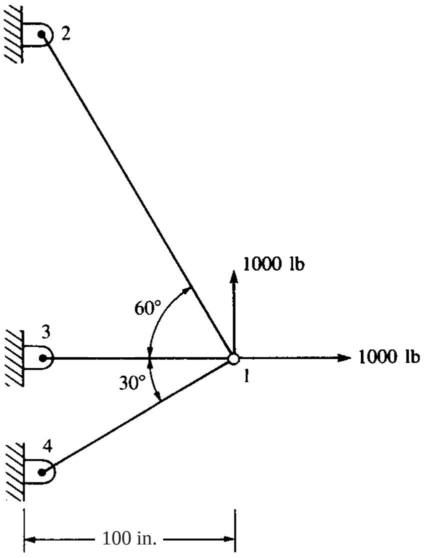
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow u_1 = \frac{400L}{\sqrt{3}AE} \Rightarrow u_1 = \frac{231\,L}{AE}$$

$$v_1 = \frac{400\,L}{(3\sqrt{3}+4)AE} \Rightarrow v_1 = \frac{43.5\,L}{AE}$$

3.22



$$[K] = [T^T]\,[k']\,[T] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

For element 1; $\theta = 120°$

$$[k^{(1)}] = \frac{AE}{2L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

For element 2; $\theta = 180°$

$$[k^{(2)}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For element 3; $\theta = 210°$

$$[k^{(3)}] = \frac{\sqrt{3}AE}{2\,L}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

Applying the boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{8}+1+\frac{3\sqrt{3}}{8} & -\frac{\sqrt{3}}{8}+0+\frac{3}{8} \\ -\frac{\sqrt{3}}{8}+0+\frac{3}{8} & \frac{3}{8}+0+\frac{\sqrt{3}}{8} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1.77 & 0.16 \\ 0.16 & 0.59 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow u_1 = \frac{422\,(100)}{1\times 10\times 10^6} \Rightarrow u_1 = 0.00422 \text{ in.}$$

$$\Rightarrow v_1 = \frac{1570\,(100)}{1\times 10\times 10^6} \Rightarrow v_1 = 0.0157 \text{ in.}$$

Element (1)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\begin{Bmatrix} 422\,\frac{L}{AE} \\ 1570\,\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow f_{2x} = 287 \text{ lb}$$

$$f_{2y} = -\,497 \text{ lb}$$

$$f^{(1)} = \sqrt{f_{2x}{}^2 + f_{2y}{}^2} \Rightarrow f^{(1)} = 5741 \text{ lb (C)}$$

$$\sigma^{(1)} = \frac{f^{(1)}}{A} = \frac{-\,5741}{\text{A}} \text{ psi}$$

$$\Rightarrow \sigma^{(1)} = -\,574 \text{ psi (C)}$$

Element (2)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 422\,\frac{L}{AE} \\ 1570\,\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$f_{3x} = -\,422 \text{ lb}$$

$$f_{3y} = 0 \text{ lb}$$

$$f^{(2)} = \sqrt{f_{3x}{}^2 + f_{3y}{}^2} \Rightarrow f^{(2)} = 422 \text{ lb (T)}$$

$$\sigma^{(2)} = \frac{f^{(2)}}{A} = \frac{422}{A} \text{ psi}$$

$$\Rightarrow \sigma^{(2)} = 422 \text{ psi (T)}$$

Element (3)

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{\sqrt{3}AE}{2\,L}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\begin{Bmatrix} 422\,\frac{L}{AE} \\ 1570\,\frac{L}{AE} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow f_{4x} = -\,862.8 \text{ lb}$$

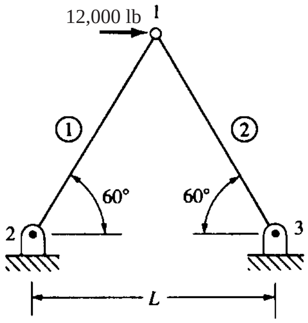
$$f_{4y} = -\,496 \text{ lb}$$

$$f^{(3)} = \sqrt{f_{4x}^{\ 2} + f_{4y}^{\ 2}} \Rightarrow f^{(3)} = 996 \text{ lb (T)}$$

$$\sigma^{(3)} = \frac{f^{(3)}}{A} = \frac{996}{A} \text{ psi (T)}$$

$$\sigma^{(3)} = 996 \text{ psi (T)}$$

3.23



$L = 100$ in.
$E = 10 \times 10^6$ psi
$A = 1$ in.^{2}

Element (1)

$$C = \frac{1}{2}; \quad S = \frac{\sqrt{3}}{2}$$

$$[k^{(1)}] = \frac{AE}{L}\left[\begin{array}{cc|cc} \frac{1}{4} & \frac{\sqrt{3}}{4} & & \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & \multicolumn{2}{c}{-\lambda} \\ \hline & & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \multicolumn{2}{c|}{-\lambda} & \frac{\sqrt{3}}{4} & \frac{3}{4} \end{array}\right]$$

Element (2)

$$[k^{(2)}] = \frac{AE}{L}\left[\begin{array}{cc|cc} \frac{1}{4} & -\frac{\sqrt{3}}{4} & & \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & \multicolumn{2}{c}{-\lambda} \\ \hline & & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \multicolumn{2}{c|}{-\lambda} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \end{array}\right]$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} 12000 \\ 0 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow 12000 = \frac{AE}{L}\,\frac{u_1}{2}$$

$$\Rightarrow u_1 = \frac{12000 \times 100 \times 2}{1 \times 10 \times 10^6}$$

$$\Rightarrow u_1 = 0.24 \text{ in.}$$

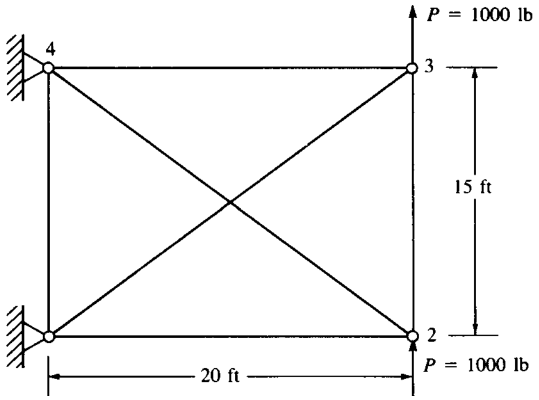
$$v_1 = 0$$

$$\sigma^{(1)} = [c']\,\{d\} = \frac{E}{L}\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_1 = 0.24 \\ v_1 = 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = \frac{10 \times 10^6}{10^2}\left[\frac{1}{2}(0.24)\right]$$

$$\Rightarrow \sigma^{(1)} = 12000 \text{ psi}$$

3.24



$L_{2-1} = 20'$	$L_{2-3} = 15'$	$L_{2-4} = 25'$
$\theta_{2-1} = 180°$	$\theta_{2-3} = 90°$	$\theta_{2-4} = 143.13°$
$\sin\theta_{2-1} = 0$	$\sin\theta_{2-3} = 1$	$\sin\theta_{2-4} = 1$
$\cos\theta_{2-1} = -1$	$\cos\theta_{2-3} = 0$	$\cos\theta_{2-4} = -0.8$

$L_{1-4} = 15'$	$L_{1-3} = 25'$	$L_{3-4} = 20'$
$\theta_{1-4} = 90°$	$\theta_{1-3} = 36.87°$	$\theta_{3-4} = 180°$
$\sin\theta_{1-4} = 1$	$\sin\theta_{1-3} = 0.6$	$\sin\theta_{3-4} = 0$
$\cos\theta_{1-4} = 0$	$\cos\theta_{1-3} = 0$	$\cos\theta_{3-4} = -1$

Boundary conditions $u_1 = v_1 = u_4 = v_4 = 0$

$$[k_{2-1}] = \frac{AE}{20}\begin{array}{c} \begin{array}{cccc} (2) & & (1) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} (2) \\ \\ (1) \\ \end{array} ; \quad [k_{2-3}] = \frac{AE}{15}\begin{array}{c} \begin{array}{cccc} (2) & & (3) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} (2) \\ \\ (3) \\ \end{array}$$

$$[k_{3-4}] = \frac{AE}{20}\begin{array}{c} \begin{array}{cccc} (3) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} (3) \\ \\ (4) \\ \end{array} ; \quad [k_{1-4}] = \frac{AE}{15}\begin{array}{c} \begin{array}{cccc} (1) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} (1) \\ \\ (4) \\ \end{array}$$

$$[k_{2-4}] = \frac{AE}{25}\begin{array}{c} \begin{array}{cccc} (2) & & (4) & \\ x & y & x & y \end{array} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array}\begin{array}{c} (2) \\ \\ (4) \\ \end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} & (1) & & (3) \\ x & y & x & y \end{array} \\
[k_{1-3}] = \dfrac{AE}{25} & \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}
\end{array}
\begin{array}{c} (1) \\ \\ (3) \end{array}
$$

$$\{F\} = [K]\,\{d\}$$

$$
\begin{Bmatrix} F_{1x} = ? \\ F_{1y} = ? \\ F_{2x} = 0 \\ F_{2y} = 1000 \\ F_{3x} = 0 \\ F_{3y} = 1000 \\ F_{4x} = ? \\ F_{4y} = ? \end{Bmatrix}
= AE
\begin{bmatrix}
0.0756 & 0.0192 & -0.05 & 0 & 0.0256 & 0.0192 & 0 & 0 \\
0.0192 & 0.0811 & 0 & 0 & -0.0192 & -0.0144 & 0 & -0.0667 \\
-0.05 & 0 & 0.0756 & -0.0192 & 0 & 0 & -0.0256 & 0.0192 \\
0 & 0 & -0.0192 & 0.0811 & 0 & -0.0667 & 0.0192 & -0.0144 \\
-0.0256 & -0.0192 & 0 & 0 & 0.0756 & 0.0192 & -0.05 & 0 \\
-0.0192 & -0.0144 & 0 & -0.0667 & 0.0192 & 0.0811 & 0 & 0 \\
0 & 0 & -0.0256 & 0.0192 & -0.05 & 0 & 0.0756 & -0.0192 \\
0 & 0.0667 & 0.0172 & 0.0144 & 0 & 0 & 0.0192 & 0.0811
\end{bmatrix}
\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = ? \\ v_2 = ? \\ u_3 = ? \\ v_3 = ? \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}
$$

$$\Rightarrow 0 = [0.0756\, u_2 - 0.0192\, v_2 + 0\, u_3 + 0\, v_3]\, AE$$

$$\Rightarrow \quad u_2 = 0.254\, v_2 \tag{1}$$

$$1000 = [-\,0.0192\, u_2 + 0.0811\, v_2 + 0\, u_3 - 0.0667\, v_3]\, AE$$

$$0 = [0\, u_2 + 0\, v_2 + 0.0756\, u_3 + 0.0192\, v_3]\, AE$$

$$\Rightarrow \quad u_3 = -\,0.254\, v_3 \tag{2}$$

$$1000 = [0\, u_2 - 0.0667\, v_2 + 0.0192\, u_3 + 0.0811\, v_3]\, AE$$

$$1000 = [-0.0192\,(0.254\, v_2) + 0.0811\, v_2 - 0.0667\, v_3]\, AE$$

$$\Rightarrow \quad 1000 = [0.0762\, v_2 - 0.0667\, v_3]\, AE \tag{3}$$

$$1000 = [\,-0.0667\, v_2 + 0.0192\,(-\,0.254\, v_3) + 0.0811\, v_3]\, AE$$

$$\Rightarrow \quad 1000 = [-\,0.0667\, v_2 + 0.0762\, v_3]\, AE \tag{4}$$

Multiplying (4) by $\left(\frac{0.0762}{0.0667}\right)$

$$\Rightarrow 1142.4 = [-0.0762\, v_2 + 0.0870\, v_3]\, AE \tag{5}$$

Adding (3) and (5)

$$2142.4 = [0\, v_2 + 0.204\, v_3]\, AE$$

$$\Rightarrow \quad v_3 = \frac{105021}{AE} \tag{6}$$

Substituting (6) into (3)

$$1000 = \left[0.0762\,v_2 - 0.0667\left(\frac{105021}{AE}\right)\right]AE$$

$$\Rightarrow \quad v_2 = \frac{105021}{AE} \Rightarrow v_2 = v_3$$

Substituting in (1) and (2)

$$u_2 = 0.254\,v_2 = 0.254\left(\frac{105021}{AE}\right)$$

$$\Rightarrow \quad u_2 = \frac{26675}{AE}$$

$$u_3 = -\,0.254\,v_3 = -\,0.254\left(\frac{105021}{AE}\right)$$

$$\Rightarrow \quad u_3 = -\frac{26675}{AE}$$

Going back to the local stiffness matrices

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_2 \\ u_1 = 0 \\ v_1 = 0 \end{Bmatrix} \begin{matrix} \Rightarrow \; f_{2x} = 1\,u_2\,\dfrac{AE}{20} \\ \Rightarrow \; f_{2x} = \dfrac{26675}{\cancel{AE}}\,\dfrac{\cancel{AE}}{20} \end{matrix}$$

$$\Rightarrow \; f_{2x} = 1333 \text{ lb} \;; \quad f_{2y} = 0 \text{ lb}$$

$$f_{1-2} = \sqrt{(f_{2x})^2 + (f_{2y})^2} \Rightarrow f_{1-2} = 1333 \text{ lb (T)}$$

Member 1–3

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{25}\begin{bmatrix} -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f_{3x} = \left[-0.64\left(\frac{-26675}{AE}\right) - 0.48\left(\frac{105021}{AE}\right)\right]\frac{AE}{25}$$

$$\Rightarrow \quad f_{3x} = -\,1333 \text{ lb}$$

$$f_{3y} = \left[-0.48\left(\frac{-26675}{AE}\right) - 0.36\left(\frac{105021}{AE}\right)\right]\frac{AE}{25}$$

$$\Rightarrow \quad f_{3y} = -\,1000 \text{ lb}$$

$$f_{1-3} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{1-3} = 1667 \text{ lb (T)}$$

Member 2–4

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{25}\begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f_{2x} = 1333 \text{ lb (C)}$$

$$f_{2y} = 1000 \text{ lb (C)}$$

$$f_{2\text{–}4} = \sqrt{(f_{2x})^2 + (f_{2y})^2}$$

$$\Rightarrow \quad f_{2\text{–}4} = 1667 \text{ lb (C)}$$

Member 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = 0 \\ f_{2y} = 0 \end{matrix}$$

$$\Rightarrow \quad f_{2\text{–}3} = 0 \text{ lb}$$

Member 3–4

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{20}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{3x} = -1333 \text{ lb} \\ f_{3y} = 0 \text{ lb} \end{matrix}$$

$$f_{3\text{–}4} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{3\text{–}4} = 1333 \text{ lb (C)}$$

Member 1-4

$$\begin{Bmatrix} f_{4x} \\ f_{4y} \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f_{1\text{–}4} = 0 \text{ lb}$$

3.25 The global stiffness matrix is changed since matrix $[k_{2\text{–}4}]$ is not incorporated in

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = 0 \\ F_{2x} = 0 \\ F_{2y} = 1000 \\ F_{3x} = 0 \\ F_{3y} = 1000 \\ F_{4x} = 0 \\ F_{4y} = 0 \end{Bmatrix} = AE \begin{array}{c} \begin{array}{cccccccc} \quad (1) & & \quad (2) & & \quad (3) & & \quad (4) & \end{array} \\ \begin{bmatrix} 0.0756 & 0.0192 & 0.05 & 0 & -0.0256 & -0.0192 & 0 & 0 \\ 0.0192 & 0.0811 & 0 & 0 & -0.0192 & -0.0144 & 0 & -0.0667 \\ -0.05 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0667 & 0 & -0.0667 & 0 & 0 \\ -0.0256 & -0.0192 & 0 & 0 & 0.0756 & 0.0192 & -0.05 & 0 \\ -0.0192 & -0.0144 & 0 & -0.0667 & 0.0192 & 0.0811 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 0 & 0.05 & 0 \\ 0 & -0.0667 & 0 & 0 & 0 & 0 & 0 & 0.0667 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = [0.05\, u_2 + 0\, v_2 + 0\, u_3 + 0\, v_3]\, AE$$

$$\Rightarrow \quad u_2 = 0 \qquad (1)$$

$$1000 = [0\, u_2 + 0.0667\, v_2 + 0\, u_3 - 0.0667\, v_3]\, AE \qquad (2)$$

$$0 = [0\, u_3 + 0\, v_2 + 0.0756\, u_3 + 0.0192\, v_3]\, AE \qquad (3)$$

$$\Rightarrow \quad u_3 = -\,0.254\, v_3 \qquad (3)$$

$$1000 = [0\, u_2 - 0.0667\, v_2 + 0.0192\, u_3 + 0.0811\, v_3]\, AE \qquad (4)$$

Adding (2) and (4)

$$2000 = [0.0192\, u_3 + 0.0144\, v_3]\, AE \qquad (5)$$

Substituting (3) in (5)

$$2000\, [0.0192\, (-\,0.254\, v_3) + 0.0144\, v_3]\, AE$$

$$\Rightarrow \quad v_3 = \frac{210000}{AE}$$

$$\Rightarrow \quad u_3 = (-\,0{\cdot}254)\, \frac{(210000)}{AE} \Rightarrow u_3 = \frac{-\,53340}{AE}$$

Substituting in (2)

$$\Rightarrow \quad 1000 = \left[0.0667\, v_2 - 0.0667\left(\frac{210000}{AE}\right)\right] AE$$

$$\Rightarrow \quad v_2 = \frac{224993}{AE}$$

Forces on members

Member 1–2

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{20} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_2 = 0 \\ v_2 = \\ u_1 = 0 \\ v_1 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2x} = 0 \\ f_{2y} = 0 \end{matrix}$$

$$\Rightarrow \quad f_{1-2} = 0$$

Member 1–3

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \frac{AE}{25} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f_{1x} = \frac{AE}{25}\left[-\,0.64\left(\frac{-\,53380}{AE}\right) - 0.48\left(\frac{210000}{AE}\right)\right]$$

$$\Rightarrow \quad f_{1x} = -\,2666.5 \text{ lb}$$

$$f_{1y} = \frac{AE}{25}\left[-\,0.48\left(\frac{-\,53380}{AE}\right) - 0.36\left(\frac{210000}{AE}\right)\right]$$

$$\Rightarrow \quad f_{1y} = -\,2000 \text{ lb}$$

$$f_{1-3} = \sqrt{(f_{1x})^2 + (f_{1y})^2} \qquad \Rightarrow f_{1-3} = 3333 \text{ lb (T)}$$

Member 2–3

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix} = \frac{AE}{15} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \quad f_{2x} = 0$$

$$\Rightarrow \quad f_{2y} = \frac{AE}{15} \left[1 \frac{(224993)}{AE} - \frac{(210000)}{AE} \right]$$

$$\Rightarrow \quad f_{2y} = -1000 \text{ lb (C)}$$

$$f_{2\text{–}3} = \sqrt{(f_{2x})^2 + (f_{2y})^2}$$

$$\Rightarrow \quad f_{2\text{–}3} = 1000 \text{ lb (C)}$$

Member 3–4

$$\begin{Bmatrix} f_{3x} \\ f_{3y} \end{Bmatrix} = \frac{AE}{20} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f_{3x} = 1 \ \frac{(-53340)}{AE} \ \frac{AE}{20} \Rightarrow f_{3x} = -2666.5 \text{ lb } (C)$$

$$\Rightarrow \quad f_{3y} = 0$$

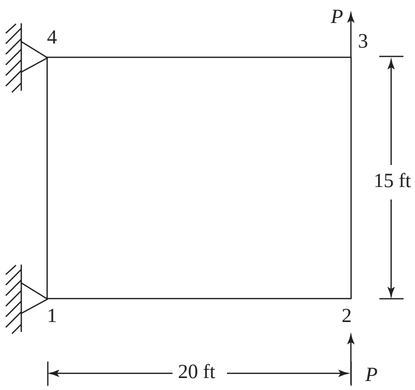
$$f_{3\text{–}4} = \sqrt{(f_{3x})^2 + (f_{3y})^2} \Rightarrow f_{3\text{–}4} = 2667 \text{ lb } (C)$$

Member 1–4

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \frac{AE}{15} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1x} = 0 \\ f_{1y} = 0 \end{matrix}$$

$$\Rightarrow \quad f_{1\text{–}4} = 0 \text{ lb}$$

3.26



Since both elements 2–4 and 1–3 are removed, the global stiffness matrix will change

$$\begin{Bmatrix} F_{1x}=? \\ F_{1y}=? \\ F_{2x}=0 \\ F_{2y}=1000 \\ F_{3x}=0 \\ F_{3y}=1000 \\ F_{4x}=? \\ F_{4y}=? \end{Bmatrix} = AE \begin{array}{c} \quad\;(1)\qquad\qquad\qquad\qquad(2)\qquad\qquad\qquad\qquad(3)\qquad\qquad\qquad\qquad(4) \\ \begin{bmatrix} 0.05 & 0 & -0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0667 & 0 & 0 & 0 & 0 & 0 & -0.0667 \\ -0.05 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0667 & 0 & -0.0667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & -0.0667 & 0 & 0.0667 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 0 & 0.05 & 0 \\ 0 & 0.0667 & 0 & 0 & 0 & 0 & 0 & 0.0667 \end{bmatrix} \end{array} \begin{Bmatrix} u_1=0 \\ v_1=0 \\ u_2=? \\ v_2=? \\ u_3=? \\ v_3=? \\ u_4=0 \\ v_4=0 \end{Bmatrix}$$

$\Rightarrow\;\; 0 = 0.05\,u_2 \qquad \Rightarrow u_2 = 0$

$1000 = 0.0667\,v_2 - 0.0667\,v_3$ (1)

$0 = 0.05\,u_3 \qquad \Rightarrow u_3 = 0$

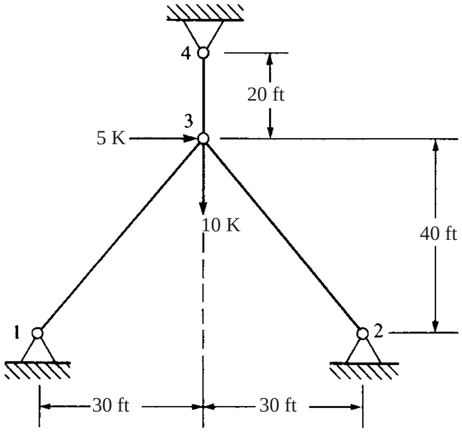
$1000 = -\,0.0067\,v_2 + 0.0667\,v_3$ (2)

Adding (1) to (2)

$2000 = 0\,v_2 + 0\,v_3$

The matrix, therefore, is singular and we get an inconsistent equation.
Of course this should have been expected since the truss is unstable.

3.27



	(1)	(2)	(3)
	1–3	2–3	3–4
L	50 ft	50 ft	20 ft
θ	53.13°	126.87°	90°
$\cos\theta$	0.6	– 0.6	0
$\sin\theta$	0.8	0.8	1

$$[k^{(1)}] = \frac{AE}{50} \begin{array}{c} \quad(1)\qquad\qquad\quad(3) \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & 0.48 \\ 0.48 & 0.64 & 0.48 & -0.64 \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{array}{c} (1) \\ \\ (3) \end{array}$$

$$
[k^{(2)}] = \frac{AE}{50}
\begin{array}{c}
\begin{array}{cc} \qquad (2) \qquad & \qquad (3) \qquad \end{array} \\
\left[\begin{array}{cc|cc}
0.36 & -0.48 & -0.36 & 0.48 \\
-0.48 & 0.64 & 0.48 & -0.64 \\
\hline
 & & & \\
 & & &
\end{array}\right]
\end{array}
\begin{array}{c} (2) \\ \\ (3) \end{array}
$$

$$
[k^{(3)}] = \frac{AE}{20}
\begin{array}{c}
\begin{array}{cc} (3) & (4) \end{array} \\
\left[\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
\hline
 & & & \\
 & & &
\end{array}\right]
\end{array}
\begin{array}{c} (3) \\ \\ (4) \end{array}
$$

Invoking boundary conditions. Therefore, need only 3–3

$$
[K] = \begin{bmatrix}
\frac{AE}{50}(0.36+0.36) + \frac{AE}{20}(0) & \frac{AE}{50}(0.48-0.48) + \frac{AE}{20}(0) \\
\frac{AE}{50}(0.48-0.48) + \frac{AE}{20}(0) & \frac{AE}{50}(0.64+0.64) + \frac{AE}{20}(1)
\end{bmatrix}
$$

$$
\Rightarrow \quad K = \begin{bmatrix}
(0.72)\frac{AE}{50} & 0 \\
0 & (1.28)\frac{AE}{50} + \frac{AE}{20}
\end{bmatrix}
$$

$$
\{F\} = [K]\{d\}
$$

$$
\begin{Bmatrix} F_{3x} = 5\text{ K} \\ F_{3y} = -10\text{ K} \end{Bmatrix} = AE \begin{bmatrix} \frac{0.72}{50} & 0 \\ 0 & \frac{1.28}{50} + \frac{1}{20} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}
$$

$$
\Rightarrow \qquad 5\text{ K} = \frac{(0.72)(AE)}{50}\, u_3
$$

$$
\Rightarrow \quad u_3 = \frac{(5000)\times(50)\times(12)}{(0.72)(3)(30\times 10^6)}
$$

$$
\Rightarrow \quad u_3 = 0.0463\text{ in.}
$$

$$
\Rightarrow \qquad -10\text{ K} = \left[\frac{1.28}{50} + \frac{1}{20}\right] AE\, v_3
$$

$$
\Rightarrow \quad v_3 = -0.0176\text{ in.}
$$

Forces on the members

Member 1–3 (1)

$$
\begin{Bmatrix} f'_{1x} \\ f'_{3x} \end{Bmatrix} = \frac{AE}{50} \begin{bmatrix} 0.6 & 0.8 & -0.6 & -0.8 \\ -0.6 & -0.8 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 \\ v_3 \end{Bmatrix}
$$

$$
\Rightarrow \quad f'^{(1)}_{1x} = -2.055\text{ kips}
$$

Member 2–3 (2)

$$
\begin{Bmatrix} f'_{2x} \\ f'_{3x} \end{Bmatrix} = \frac{AE}{50} \begin{bmatrix} -0.6 & 0.8 & 0.6 & -0.8 \\ 0.6 & -0.8 & -0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_2 = 0 \\ v_2 = 0 \\ u_3 \\ v_3 \end{Bmatrix}
$$

$$\Rightarrow \quad f'^{(2)}_{2x} = 6.279 \text{ kips}$$

Member 3–4 (3)

$$\begin{Bmatrix} f'_{3x} \\ f'_{4x} \end{Bmatrix} = \frac{AE}{20} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix}$$

$$\Rightarrow \quad f'_{3x} = \frac{AE}{20}\,(-\,0.0176)$$

$$\Rightarrow \quad f'_{3x} = -\,6.6 \text{ kips}$$

3.28

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

$$[T^T] = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

$$[T]\,[T^T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

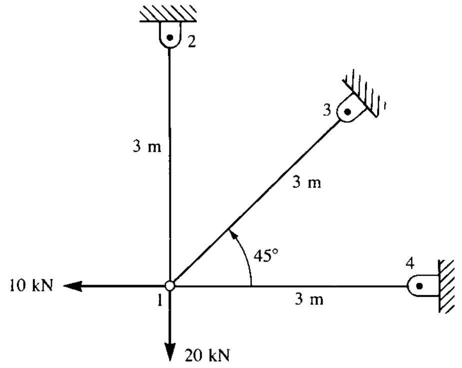
$$\Rightarrow T\,[T^T] = \begin{bmatrix} C^2+S^2 & -SC+SC & 0 & 0 \\ -CS+CS & S^2+C^2 & 0 & 0 \\ 0 & 0 & C^2+S^2 & -SC+SC \\ 0 & 0 & -CS+CS & S^2+C^2 \end{bmatrix}$$

$$[T]\,[T^T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{matrix}\text{Identity} \\ \text{matrix}\end{matrix} = [I]$$

But $$[T]\,[T^T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \Rightarrow$$

$$\Rightarrow \qquad [T^T] = [T^{-1}] = \begin{bmatrix} C & -S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & C & -S \\ 0 & 0 & S & C \end{bmatrix}$$

3.29



$E = 210$ GPa
$A = 4.0 \times 10^{-4}$ m^2

$$\frac{AE}{L} = \frac{(4 \times 10^{-4})\,(210 \times 10^{6})}{3} = 280 \times 10^{2}\ \frac{\text{kN}}{\text{m}}$$

Element (1)

$C = 0, S = 1$

$$\begin{array}{cc} & \begin{array}{cc}(1) & (2)\end{array} \\ [k^{(1)}] = 280 \times 10^{2} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$$C = \frac{\sqrt{2}}{2}, S = \frac{\sqrt{2}}{2}$$

$$\begin{array}{cc} & \begin{array}{cc}(1) & (3)\end{array} \\ [k^{(2)}] = 280 \times 10^{2} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Element (3)

$C = 1, S = 0$

$$[k^{(3)}] = 280 \times 10^{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = -10\text{ K} \\ F_{1y} = -20\text{ K} \end{Bmatrix} = 280 \times 10^{2} \begin{bmatrix} 1\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1\frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow\ u_1 = -\,0.893 \times 10^{-4}\,\text{m}$$

$$v_1 = -\,4.46 \times 10^{-4}\,\text{m}$$

Element stresses

$$\sigma^{(1)} = \frac{210\times10^6}{3}\ [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(1)} = 31.2 \text{ MPa (T)}$$

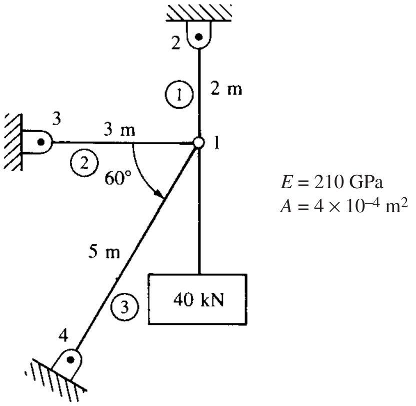
$$\sigma^{(2)} = 70\times10^6 \left[-\tfrac{\sqrt{2}}{2} \quad -\tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2} \quad \tfrac{\sqrt{2}}{2}\right] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(2)} = 26.5 \text{ MPa (T)}$$

$$\sigma^{(3)} = 70\times10^6 [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} -0.893\times10^{-4} \\ -4.46\times10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ \sigma^{(3)} = 6.25 \text{ MPa (T)}$$

3.30



Element 1–2

$$C = 0,\ S = 1$$

$$[k_{1-2}] = (4\times10^{-4})(210\times10^9) \begin{array}{c} \begin{array}{cc} (1) & \quad (2) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Element 1–3

$$C = -1,\ S = 0$$

$$[k_{1-3}] = 84\times10^6 \begin{array}{c} \begin{array}{cc} (1) & \quad (3) \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 1–4

$$C = -0.5,\ S = -\frac{\sqrt{3}}{2}$$

$$[k_{1\text{–}4}] = 84 \times 10^6 \begin{array}{c} \begin{array}{cccc} \quad (1) & & \quad (4) & \end{array} \\ \begin{bmatrix} 0.05 & 0.0866 & -0.05 & -0.0866 \\ 0.0866 & 0.15 & -0.0866 & -0.15 \\ -0.05 & -0.0866 & 0.05 & 0.0866 \\ -0.0866 & -0.15 & 0.0866 & 0.15 \end{bmatrix} \end{array}$$

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -40 \end{Bmatrix} = 84 \times 10^6 \begin{bmatrix} 0.3883 & 0.0866 \\ 0.0866 & 0.65 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 1.71 \times 10^{-4}\ \text{m}$$

$$v_1 = -7.55 \times 10^{-4}\ \text{m}$$

Element stresses

$$\sigma^{(1)} = \frac{210 \times 10^9}{2}\ [0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.65 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 79.28\ \text{MPa (T)}$$

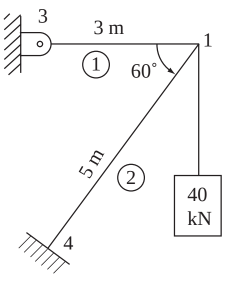
$$\sigma^{(2)} = \frac{210 \times 10^9}{3}\ [1 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.55 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = 11.97\ \text{MPa (T)}$$

$$\sigma^{(3)} = \frac{210 \times 10^9}{5} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} 1.71 \times 10^{-4} \\ -7.55 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(3)} = -23.87\ \text{MPa (C)}$$

3.31



$E = 210$ GPa
$A = 4 \times 10^{-4}\ \text{m}^2$

$[k_{1-3}]$ and $[k_{1-4}]$ are the same as in Problem 3.30

$$u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -40 \end{Bmatrix} = 8.4 \times 10^7 \begin{bmatrix} \frac{23}{60} & \frac{3\sqrt{3}}{60} \\ \frac{3\sqrt{3}}{60} & \frac{9}{60} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 8.248 \times 10^{-4}\,\text{m}$$

$$v_1 = -3.651 \times 10^{-3}\,\text{m}$$

Element stresses

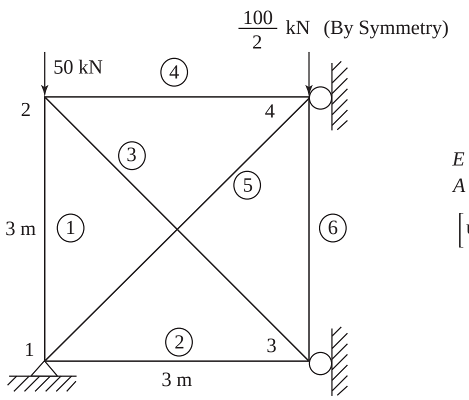
$$\sigma^{(1)} = \frac{210 \times 10^9}{3}\ [1 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} 8.248 \times 10^{-4} \\ -3.651 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 57.74\ \text{MPa (T)}$$

$$\sigma^{(2)} = \frac{210 \times 10^9}{5} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} 8.248 \times 10^{-4} \\ -3.657 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = -115.48\ \text{MPa (C)}$$

3.32



$E = 70$ GPa $= 70 \times 10^6\ \frac{\text{kN}}{\text{m}^2}$
$A = 3.0 \times 10^{-4}\,\text{m}^2$
$\left[\text{use } A_{3-4} = \frac{3.0}{2} \times 10^{-4}\,\text{m}^2\right]$
(due to symmetry)

$$[k_{1-2}] = 7000 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}; \quad [k_{1-3}] = 7000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_{2-3}] = 2475 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}; \quad [k_{2-4}] = 7000 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_{1-4}] = 2475\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix};\quad [k_{3-4}] = 3500\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$u_1 = v_1 = u_3 = v_4 = 0$$

$\therefore$ Global equations are

$$\begin{bmatrix} 9475 & -2475 & 2475 & 0 \\ & 9475 & -2475 & 0 \\ & & 5975 & -3500 \\ & & & 5975 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50 \\ 0 \\ -50 \end{Bmatrix}$$

Solving simultaneously

$$u_3 = 0.135 \times 10^{-2}\,\text{m}$$

$$v_2 = -0.850 \times 10^{-2}\,\text{m}$$

$$v_3 = -0.137 \times 10^{-1}\,\text{m}$$

$$v_4 = -0.164 \times 10^{-1}\,\text{m}$$

$$\sigma_{1-2} = \sigma_{5-6} = \frac{70\times10^6}{3}\,[0\ \ -1\ \ 0\ \ 1]\begin{Bmatrix} 0 \\ 0 \\ 0.135\times10^{-2} \\ -0.850\times10^{-2} \end{Bmatrix}$$

$$\Rightarrow\quad \sigma_{1-2} = \sigma_{5-6} = -198\ \text{MPa (C)}$$

$$f'_{x_{1-2}} = f'_{x_{5-6}} = \sigma_{1-2}\times A_{1-2} = -198000\times3\times10^{-4}$$

$$\Rightarrow\ f'_{x_{1-2}} = f'_{x_{5-6}} = -59.5\ \text{kN}$$

$$\sigma_{1-3} = \sigma_{5-3} = \frac{70\times10^6}{3}\,[-1\ \ 0\ \ 1\ \ 0]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.137\times10^{-1} \end{Bmatrix}$$

$$\Rightarrow\quad \sigma_{1-3} = \sigma_{5-3} = 0$$

$$\Rightarrow\quad f'_{x_{1-3}} = f'_{x_{5-3}} = 0$$

Similarly

$$\sigma_{2-3} = \sigma_{6-3} = \frac{70\times10^6}{3\sqrt{2}}\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} 0.135\times10^{-2} \\ -0.850\times10^{-2} \\ 0 \\ -0.137\times10^{-1} \end{Bmatrix}$$

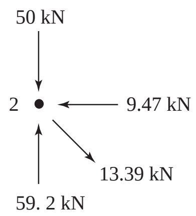
$$\Rightarrow\quad \sigma_{2-3} = \sigma_{6-3} = 44.6\ \text{MPa (T)}$$

$$f'_{x_{2-3}} = f'_{x_{6-3}} = 13.39\ \text{kN}$$

$$\sigma_{2-4} = \sigma_{6-4} = -31.6 \text{ MPa (C)}$$
$$f'_{x_{2-4}} = f'_{x_{6-4}} = -9.47 \text{ kN}$$
$$\sigma_{1-4} = \sigma_{5-4} = -191 \text{ MPa (C)}$$
$$f'_{x_{1-4}} = f'_{x_{5-4}} = -57.32 \text{ kN}$$
$$\sigma_{3-4} = -63.1 \text{ MPa (C)}$$
$$f'_{x_{3-4}} = -18.93 \text{ kN}$$

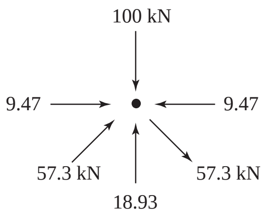
Force equilibrium

Node 2



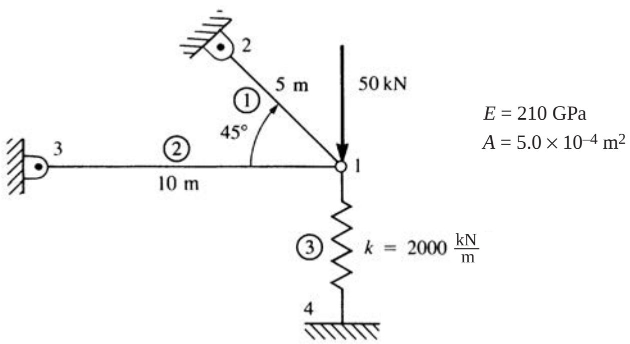
$$\Sigma F_y = -50 - 13.39 \sin 45° + 54.5$$
$$\Sigma F_y = 0$$
$$\Sigma F_x = -9.47 + 13.39 \cos 45° = -0.001 \text{ kN}$$

Node 4



$$\Sigma F_y = -100 + 57.3 \sin 45° \times 2 + 18.93$$
$$\Sigma F_y = -0.003 \text{ kN}$$
$$\Sigma F_x = 9.47 + 57.3 \cos 45° - 9.47 - 57.3 \cos 45°$$
$$\Sigma F_x = 0$$

3.33 (a)



Element 1–2 ; $\theta = 135°$

$$C^2 = 0.5,\ CS = -0.5,\ S^2 = 0.5$$

$$[k_{1-2}] = \frac{(210\times10^9)(5.0\times10^{-4})}{5}\begin{array}{c} \begin{array}{cc} (1) & (2) \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array}$$

$$\Rightarrow\ [k_{1-2}] = 105\times10^5\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Element 1–3 ; $\theta = 180°$

$$C^2 = 1.0,\ CS = 0,\ S^2 = 0$$

$$[k_{1-3}] = \frac{(210\times10^9)(5\times10^{-4})}{10}\begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow\ [k_{1-3}] = 105\times10^5\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 1–4 ; $\theta = 270°$

$$C^2 = 0,\ CS = 0,\ S^2 = 1.0$$

$$[k_{1-4}] = 20\times10^5\begin{array}{c} \begin{array}{cc} (1) & (4) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\{F\} = [K]\{d\}$$

Boundary conditions are

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

The final matrix (assembled)

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -50 \end{Bmatrix} = 10^5\begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$\Rightarrow\quad 0 = 210\,u_1 - 105\,v_1 \Rightarrow v_1 = 2\,u_1$$

$$-50000 = 10^5[-105\,u_1 + 125\,(2\,u_1)]$$

$$\Rightarrow\quad u_1 = -3.448\times10^{-3}\ \text{m}$$

$$\Rightarrow\quad v_1 = -6.896\times10^{-3}\ \text{m}$$

$$\sigma_{1-2} = \frac{210 \times 10^9}{5\text{ m}}\ [0.707 \quad -0.707 \quad -0.707 \quad 0.707] \begin{Bmatrix} -3.448 \times 10^{-3} \\ -6.896 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-2} = 102.4\text{ MPa (T)}$$

$$\sigma_{1-3} = \frac{210 \times 10^9}{10}\ [1.0 \quad 0 \quad -1.0 \quad 0] \begin{Bmatrix} -3.448 \times 10^{-3} \\ -6.896 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-3} = -72.4\text{ MPa (C)}$$

Note: Can show equilibrium at node 1

$$F_s = \left(2000\,\frac{\text{kN}}{\text{m}}\right)(6.896 \times 10^{-3}\text{ m})$$

$$= 13.792\text{ kN}$$

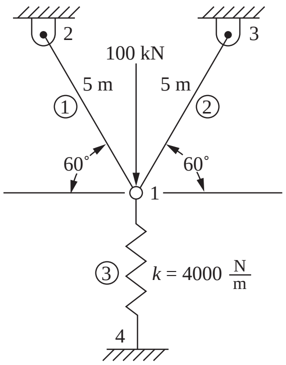
$$f_{1-3} = 35.6\text{ kN}$$

$$f_{1-2} = 51.2\text{ kN}$$

$$\Sigma F_y = 0$$

$$-50 + 13.79 + 36.198 = 0$$

(b)



$$A = 5 \times 10^{-4}\text{ m}^2,\ E = 210 \times 10^9\ \frac{\text{N}}{\text{m}^2}$$

$$L_1 = L_2 = 5\text{ m}$$

$$[k^{(1)}] = 2.1 \times 10^7 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & & \end{array} \\ \left[\begin{array}{cc|cc} \frac{1}{4} & -\frac{\sqrt{3}}{4} & & \\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & & \\ \hline & & & \\ & & & \end{array}\right] \end{array} \frac{\text{N}}{\text{m}}$$

$$[k^{(2)}] = 2.1 \times 10^7 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & & \end{array} \\ \left[\begin{array}{cc|cc} \frac{1}{4} & \frac{\sqrt{3}}{4} & & \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & & \\ \hline & & & \\ & & & \end{array}\right] \end{array} \frac{\text{N}}{\text{m}}$$

$$[k^{(3)}] = 4\times10^3 \begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 1 & \\ \hline & & \\ & & \end{array}\right] \end{array} \frac{\text{N}}{\text{m}}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = 0$$

$$u_4 = v_4 = 0$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -1\times10^5 \end{Bmatrix} = \begin{bmatrix} 5.25\times10^6 + 5.25\times10^6 & 0 \\ 0 & 1.58\times10^7 + 1.58\times10^7 + 4000 \\ & \end{bmatrix} \begin{Bmatrix} u_1 \\ \\ v_1 \end{Bmatrix}$$

Solving

$$0 = 1.05\times10^7\, u_1$$

$$\therefore \quad u_1 = 0$$

$$-1\times10^5 = 3.15\times10^7\, v_1$$

$$v_1 = -0.00317 \text{ m}$$

$$\sigma^{(1)} = \frac{E}{L}\,[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_1 \\ v_1 \\ v_2 \\ v_2 \end{Bmatrix}$$

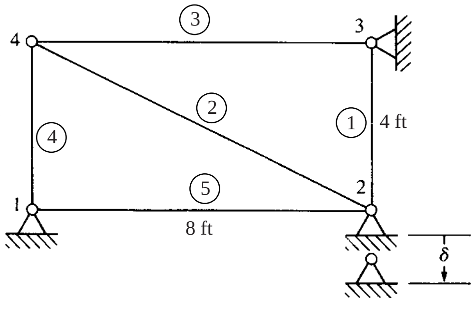
$$\sigma^{(1)} = \frac{210\times10^9}{5}\left[\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]\begin{Bmatrix} 0 \\ -0.00317 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 1.155\times10^8 \frac{\text{N}}{\text{m}^2} = 115 \text{ MPa}$$

$$\sigma^{(2)} = \frac{210\times10^9}{5}\left[-\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2}\right]\begin{Bmatrix} 0 \\ -0.00317 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(2)} = 115 \text{ MPa}$$

3.34



$$[k_{1-2}] = \frac{AE}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad [k_{1-4}] = \frac{AE}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k_{2-4}] = \frac{AE}{8.94}\begin{bmatrix} 0.80 & -0.40 & -0.80 & 0.40 \\ -0.40 & 0.20 & 0.40 & -0.20 \\ -0.80 & 0.40 & 0.80 & -0.40 \\ 0.40 & -0.20 & -0.40 & 0.20 \end{bmatrix}$$

$$[k_{2-3}] = \frac{AE}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}; \quad [k_{4-3}] = \frac{AE}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Boundary conditions

$u_1 = v_1 = u_3 = v_3 = u_2 = 0$, $v_2 = -0.05$ in.

Applying the boundary conditions and superimposing the [k]s

$$AE\begin{bmatrix} 0.522 & 0.0447 & -0.0223 \\ 0.0447 & 0.214 & -0.0447 \\ -0.0223 & -0.0447 & 0.272 \end{bmatrix}\begin{Bmatrix} v_2 = -0.05 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving

$$u_4 = 9.93 \times 10^{-3} \text{ in.}$$
$$v_4 = -2.46 \times 10^{-3} \text{ in.}$$

Element stresses

$$\sigma^{(1)} = \frac{30 \times 10^3}{4 \times 12}\,[0 \;\; -1 \;\; 0 \;\; 1]\begin{Bmatrix} u_2 = 0 \\ v_2 = -0.05 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 31.25 \text{ ksi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^3}{8.94 \times 12}\,[\,0.894 \;\; -0.447 \;\; -0.894 \;\; 0.447]\begin{Bmatrix} 0 \\ -0.05 \\ 9.93 \times 10^{-3} \\ -2.46 \times 10^{-3} \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = 3.459 \text{ ksi (T)}$$

$$\sigma^{(3)} = \frac{30 \times 10^3}{4 \times 12} \, [0 \quad -1 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 0 \\ 9.93 \times 10^{-3} \\ -2.46 \times 10^{-3} \end{bmatrix}$$

$$\Rightarrow \quad \sigma^{(3)} = -1.538 \text{ ksi (C)}$$

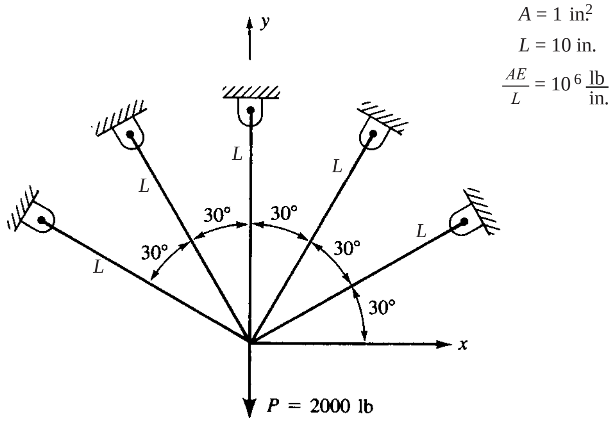
$$\sigma^{(4)} = \frac{30 \times 10^3}{8 \times 12} \, [-1 \quad 0 \quad 1 \quad 0] \begin{bmatrix} 9.93 \times 10^{-3} \\ -2.46 \times 10^{-3} \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \sigma^{(4)} = -3.103 \text{ ksi (C)}$$

$$\sigma^{(5)} = \sigma_{1-2} = 0$$

Note: This solution was also verified by a computer program.

3.35



Using symmetry

Element (5)

$$\theta = 30°$$

$$[k^{(5)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cc} (1) & (5) \end{array} \\ \left[\begin{array}{cc|c} 0.75 & 0.433 & \\ 0.433 & 0.25 & \\ \hline & & \end{array}\right] \end{array}$$

Element (4)

$$\theta = 60°$$

$$[k^{(4)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cc} (1) & (5) \end{array} \\ \left[\begin{array}{cc|c} 0.25 & 0.433 & \\ & 0.75 & \\ \hline & & \end{array}\right] \end{array}$$

Element (5)

$$\theta = 90°$$

$$[k^{(3)}] = \left(\frac{2AE}{L}\right)\frac{1}{2}\left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 1 & \\ \hline & & \end{array}\right]$$

Applying the boundary conditions and superimposing the K's

$$\frac{AE}{L}\begin{bmatrix} 1 & 0.866 \\ 0.866 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \end{Bmatrix}$$

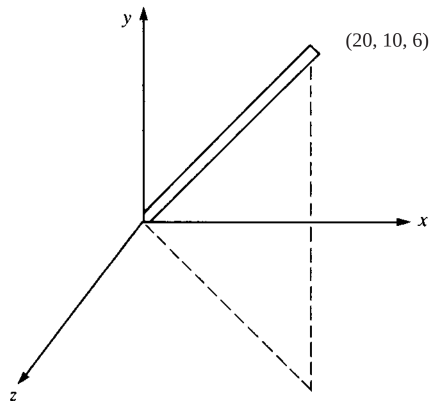
$$\Rightarrow \quad v_1 = \frac{-1000}{2 \times 10^6} \Rightarrow v_1 = -0.5 \times 10^{-3} \text{ in.}$$

$$u_1 = 0$$

$$\sigma^{(1)} = \sigma^{(5)} = \frac{E}{L}\ [0.866 \quad -0.5 \quad \ldots]\begin{Bmatrix} 0 \\ -0.0005 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 250 \text{ psi (T)}$$

3.36



$L = \sqrt{20^2 + 10^2 + 6^2}$

$L = 23.15$ in.

$$\{d'\} = [T^*]\{d\} \text{ and } [T^*] = [C_x \quad C_y \quad C_z]$$

$$C_x = \frac{20-0}{23.15} = 0.864 \qquad u_1 = 0.1 \text{ in.}$$

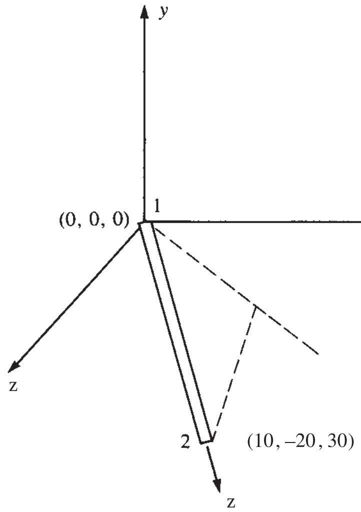
$$C_y = \frac{10-0}{23.15} = 0.432 \qquad v_1 = 0.2 \text{ in.}$$

$$C_z = \frac{6-0}{23.15} = 0.259 \qquad w_1 = 0.15 \text{ in.}$$

$$u_1' = 0.864\,(0.1) + 0.432\,(0.2) + 0.259\,(0.15)$$

$$\Rightarrow u_1' = 0.212 \text{ in.}$$

3.37



$L_{1-2} = 37.42$ in.

$$C_x = \frac{10}{37.42} = 0.267$$

$$C_y = \frac{-20}{37.42} = -0.534$$

$$C_z = \frac{30}{37.42} = 0.802$$

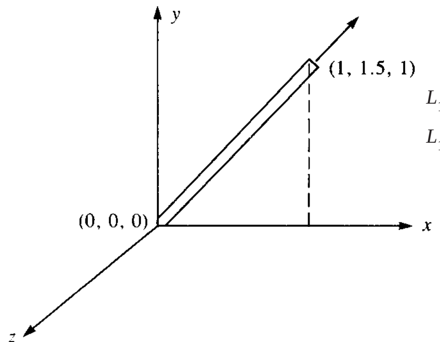
$$\{d'\} = [T]\{d\}$$

$$\{d'_{1x}\} = [C_x\ C_y\ C_z]\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$u'_1 = (0.267)\,(0.1) - (0.534)\,(0.2) + (0.802)\,(0.15)$$

$$\Rightarrow \quad u'_1 = 0.0397 \text{ in.}$$

3.38



$L_{1-2} = \sqrt{1^2 + 1.5^2 + 1^2}$

$L_{1-2} = 2.0616$ m

$$C_x = \frac{1}{2.0616} = 0.485$$

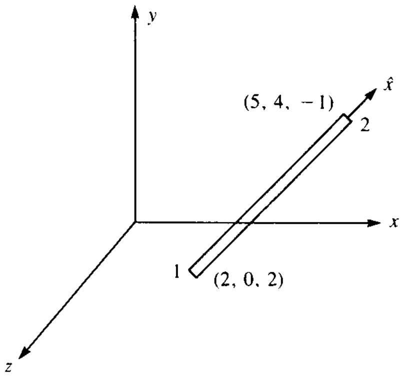
$$C_y = \frac{1.5}{2.0616} = 0.728$$

$$C_z = \frac{1}{2.0616} = 0.485$$

$$\{d'_{2x}\} = [0.485 \quad 0.728 \quad 0.485] \begin{Bmatrix} 5 \\ 10 \\ 15 \end{Bmatrix}$$

$$\Rightarrow \quad \hat{u}_2 = 16.98 \text{ mm}$$

3.39



$$L_{1-2} = \sqrt{(5-2)^2 + (4-0)^2 + (-1-2)^2}$$

$$L_{1-2} = 5.831 \text{ m}$$

$$C_x = \frac{3}{5.831} = 0.515$$

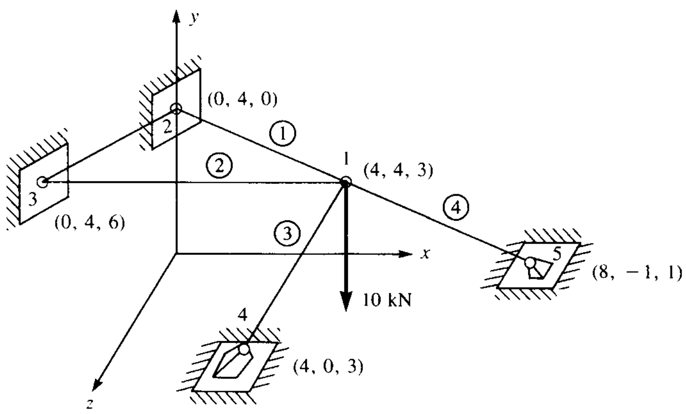
$$C_y = \frac{4}{5.831} = 0.686$$

$$C_z = -\frac{3}{5.831} = -0.515$$

$$\{d'_{2x}\} = [0.515 \quad 0.686 \quad -0.515] \begin{Bmatrix} 5 \\ 10 \\ 15 \end{Bmatrix}$$

$$\Rightarrow \quad \hat{u}_2 = 1.71 \text{ mm}$$

3.40 From Figure P. 3.40



$$L_{1-2} = \sqrt{(-4)^2 + 0 + (-3)^2} = 5 \text{ m}$$

$$C_x = \frac{0-4}{5} = -0.8;\ C_y = \frac{4-4}{5} = 0;\ C_2 = \frac{0-3}{5} = -0.6$$

$$L_{1-3} = \sqrt{(-4)^2 + 0 + (3)^2} = 5 \text{ m}$$

$$C_x = \frac{0-4}{5} = -0.8;\ C_y = \frac{4-4}{0} = 0;\ C_2 = \frac{6-3}{5} = 0.6$$

$$L_{1-4} = \sqrt{0 + (-4)^2 + 0} = 4 \text{ m}$$

$$C_x = 0;\ C_y = \frac{0-4}{4} = -1;\ C_z = 0$$

$$L_{1-5} = \sqrt{4^2 + (-5)^2 + (-2)^2} = 3\sqrt{5} \text{ m}$$

$$C_x = \frac{4}{3\sqrt{5}} = 0.596;\ C_y = \frac{-5}{3\sqrt{5}} = -0.745$$

$$C_z = \frac{-2}{3\sqrt{5}} = -0.298$$

Element 1–2

$$[\lambda] = \begin{bmatrix} 0.64 & 0 & 0.48 \\ 0 & 0 & 0 \\ 0.48 & 0 & 0.64 \end{bmatrix} \quad [k] = 42000 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–3

$$[\lambda] = \begin{bmatrix} 0.64 & 0 & -0.48 \\ 0 & 0 & 0 \\ -0.48 & 0 & 0.64 \end{bmatrix} \quad [k] = 42000 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–4

$$[\lambda] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [k] = 52500 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Element 1–5

$$[\lambda] = \begin{bmatrix} 0.356 & -0.444 & -0.178 \\ -0.444 & 0.556 & 0.222 \\ -0.178 & 0.222 & 0.0889 \end{bmatrix}$$

$$[k] = 31305 \begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix}$$

Applying the boundary conditions where all deflections at node 2, 3, 4 and 5 are zero. The global equations are

$$\begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix} = \begin{bmatrix} 64905 & -13899 & -5572 \\ -13899 & 69906 & 6950 \\ -5572 & 6950 & 33023 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$0 = 64905\, u_1 - 13899\, v_1 - 5572\, w_1 \qquad (1)$$

$$-10 = -13899\, u_1 + 69906\, v_1 + 6950\, w_1 \qquad (2)$$

$$0 = -5572\, u_1 + 6950\, v_1 + 33023\, w_1 \qquad (3)$$

From (1) and (3)

$$0 = 67058\, v_1 + 379094\, w_1 \qquad (4)$$

From (2) and (3)

$$-10 = 52570\, v_1 - 75424\, w_1 \qquad (5)$$

From (4) and (5), we get

$$w_1 = 2.68374 \times 10^{-5} \text{ m}$$

$$v_1 = -1.5171 \times 10^{-4} \text{ m}$$

Substituting in (1)

$$u_1 = -3.0183 \times 10^{-5} \text{ m}$$

Element stresses

$$\sigma_{1-2} = 42 \times 10^6 \, [0.8 \quad 0 \quad -0.6 \quad -0.8 \quad 0 \quad 0.6] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.6837 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-2} = -337.846 \ \frac{\text{kN}}{\text{m}^2} \ \text{(C)}$$

$$\text{Force}_{\ 1-2} = \frac{-337.846}{1000} = -0.337 \text{ kN (C)}$$

$$\sigma_{1-3} = 42 \times 10^6 \, [0.8 \quad 0 \quad -0.6 \quad -0.8 \quad 0 \quad 0.6] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.6837 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-3} = -1690 \ \frac{\text{kN}}{\text{m}^2} \ \text{(C)}$$

$$\text{Force}_{\ 1-3} = \frac{-1690}{1000} = -1.69 \text{ kN (C)}$$

$$\sigma_{1-4} = 52500 \times 10^3 \, [0 \quad 1 \quad 0 \quad 0 \quad -1 \quad 0] \begin{Bmatrix} -3.0183 \times 10^{-5} \\ -1.5171 \times 10^{-4} \\ 2.68374 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma_{1-4} = -7965 \ \frac{\text{kN}}{\text{m}^2} \ \text{(C)}$$

$$\text{Force}_{\ 1-4} = -\frac{7965}{1000} = -7.695 \text{ kN (C)}$$

$$\sigma_{1-5} = -2726 \ \frac{\text{kN}}{\text{m}^2}$$

$$\text{Force}_{\ 1-5} = -2.726 \text{ kN}$$

Force equilibrium at node 1

x direction

$$0 = 0.388 \times 0.8 + 1.69 \times 0.8 - 2.726 \times 0.596$$

$$0 = -0.00307$$

y direction

$$-10 = 0 + 0 + 7.965 \times 1 + 2.726 \times \frac{\sqrt{5}}{15}$$

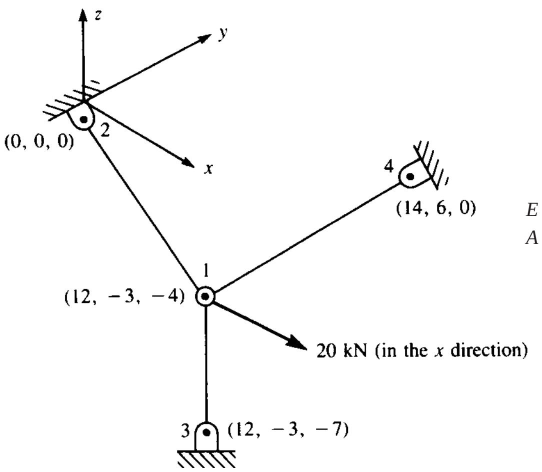
$$-10 = 9.9968$$

z direction

$$0 = 0.338 \times 0.6 - 1.69 \times 0.6 + 0 + 2.726 \times 0.298$$

$$0 = 0.0015$$

3.41



$E = 210$ GPa

$A = 10 \times 10^{-4}$ m^2

$$L_{1-2} = \sqrt{(12-0)^2 + (-3-0)^2 + (-4-0)^2} \Rightarrow L_{1-2} = 13 \text{ m}$$

$$L_{1-3} = \sqrt{(12-12)^2 + (-3+3)^2 + (-7+4)^2} \Rightarrow L_{1-3} = 3 \text{ m}$$

$$L_{1-4} = \sqrt{(14-12)^2 + (6+3)^2 + (0+4)^2} \Rightarrow L_{1-4} = 10.05 \text{ m}$$

Element	$C_x = \dfrac{x_j - x_i}{L_{i-j}}$	$C_y = \dfrac{y_j - y_i}{L_{i-j}}$	$C_z = \dfrac{z_j - z_i}{L_{i-j}}$
1 − 2	$\frac{-12}{13}$	$\frac{3}{13}$	$\frac{4}{13}$
1 − 3	0	0	−1
1 − 4	$\frac{2}{10.05}$	$\frac{9}{10.05}$	$\frac{4}{10.05}$

Element	$C_x^{\ 2}$	C_xC_y	C_xC_z	$C_y^{\ 2}$	C_yC_z	$C_2^{\ 2}$
1 − 2	0.852	−0.213	−0.284	0.053	0.071	0.095
1 − 3	0	0	0	0	0	1
1 − 4	0.040	0.178	0.079	0.802	0.356	0.158

$$[\lambda] = \begin{bmatrix} C_x^{\ 2} & C_xC_y & C_xC_z \\ & C_y^2 & C_yC_z \\ & & C_z^2 \end{bmatrix}$$

$$[\lambda_{1-2}] = \begin{bmatrix} 0.852 & -0.213 & -0.284 \\ -0.213 & 0.053 & 0.071 \\ -0.284 & 0.071 & 0.095 \end{bmatrix}$$

$$[\lambda_{1-3}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\lambda_{1-4}] = \begin{bmatrix} 0.040 & 0.178 & 0.079 \\ 0.178 & 0.802 & 0.356 \\ 0.079 & 0.356 & 0.158 \end{bmatrix}$$

$$u_2 = v_2 = w_2 = u_3 = v_3 = w_3 = 0$$
$$u_4 = v_4 = w_4 = 0$$

$$\left[k_{1-2}^{(1)}\right] = \frac{AE}{L_{1-2}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{13} & -\frac{[\lambda]}{13} \\ -\frac{[\lambda]}{13} & \frac{[\lambda]}{13} \end{bmatrix}$$

$$\left[k_{1-3}^{(2)}\right] = \frac{AE}{L_{1-3}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{3} & -\frac{[\lambda]}{3} \\ -\frac{[\lambda]}{3} & \frac{[\lambda]}{3} \end{bmatrix}$$

$$\left[k_{1-4}^{(3)}\right] = \frac{AE}{L_{1-4}}\begin{bmatrix} [\lambda] & [-\lambda] \\ [-\lambda] & [\lambda] \end{bmatrix} = AE\begin{bmatrix} \frac{[\lambda]}{10.05} & -\frac{[\lambda]}{10.05} \\ -\frac{[\lambda]}{10.05} & \frac{[\lambda]}{10.05} \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} = 20\,\text{kN} \\ F_{1y} = 0 \\ F_{1z} = 0 \end{Bmatrix} = 210 \times 10^3 \begin{bmatrix} 69.519 & 1.327 & -13.985 \\ 1.327 & 83.879 & 40.885 \\ -13.985 & 40.885 & 356.363 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = 1.383 \times 10^{-3}\text{ m}$$

$$v_1 = -5.119 \times 10^{-5}\text{ m}$$

$$w_1 = 6.015 \times 10^{-5}\text{ m}$$

$$\sigma^{(1)} = \frac{E}{L^{(1)}}\begin{bmatrix} \frac{12}{13} & -\frac{3}{13} & -\frac{4}{13} & -\frac{12}{13} & \frac{3}{13} & \frac{4}{13} \end{bmatrix} \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(1)} = 20.51\text{ MPa (T)}$$

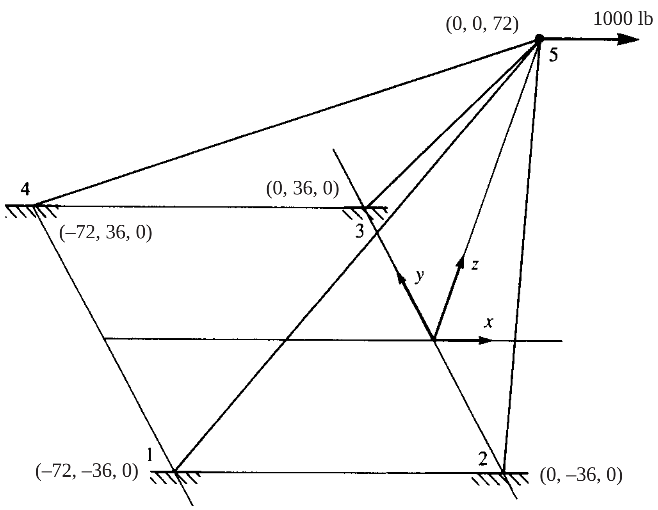
$$\sigma^{(2)} = \frac{E}{L^{(2)}}\,[0 \quad 0 \quad 1 \quad 0 \quad 0 \quad -1] \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(2)} = 4.21\text{ MPa (T)}$$

$$\sigma^{(3)} = \frac{E}{L^{(3)}}\begin{bmatrix} \frac{-2}{10.05} & -\frac{9}{10.05} & -\frac{4}{10.05} & \frac{2}{10.05} & \frac{9}{10.05} & \frac{4}{10.05} \end{bmatrix} \begin{Bmatrix} 1.383 \times 10^{-3} \\ -5.119 \times 10^{-5} \\ 6.015 \times 10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \sigma^{(3)} = -5.29\text{ MPa (C)}$$

3.42



Element 1–5

$$L_{1\text{–}5} = 108 \text{ in.}$$

$$C_x = \frac{x_5 - x_1}{L_{1\text{–}5}} = \frac{0-(-72)}{108} \Rightarrow C_x = 0.667$$

$$C_y = \frac{y_5 - y_1}{L_{1\text{–}5}} = \frac{0-(-36)}{108} \Rightarrow C_y = 0.333$$

$$C_z = \frac{z_5 - z_1}{L_{1\text{–}5}} = \frac{72-0}{108} \Rightarrow C_z = 0.667$$

$$[K] = \frac{4 \times 30 \times 10^6}{108}\begin{bmatrix} 0.444 & 0.222 & 0.444 & -0.444 & -0.222 & -0.444 \\ 0.222 & 0.111 & 0.222 & -0.222 & -0.111 & -0.222 \\ 0.444 & 0.222 & 0.444 & -0.444 & -0.222 & -0.444 \\ -0.444 & -0.222 & -0.444 & 0.444 & 0.222 & 0.444 \\ -0.222 & -0.111 & -0.222 & 0.222 & 0.111 & 0.222 \\ -0.444 & -0.222 & -0.444 & 0.444 & 0.222 & 0.444 \end{bmatrix}$$

Element 2–5

$$L_{2\text{–}5} = \sqrt{(0-0)^2 + (0-(-36)^2) + (72-0)^2}$$

$$\Rightarrow L_{2\text{–}5} = 80.5 \text{ in.}$$

$$C_x = \frac{0-0}{80.5} = 0$$

$$C_y = \frac{0-(-36)}{80.5} \Rightarrow C_y = 0.447$$

$$C_z = \frac{72-0}{80.5} \Rightarrow C_z = 0.894$$

$$[K] = \frac{4 \times 30 \times 10^6}{80.5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0 & -0.2 & -0.4 \\ 0 & 0.4 & 0.8 & 0 & -0.4 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & -0.4 & 0 & 0.2 & 0.4 \\ 0 & -0.4 & -0.8 & 0 & 0.4 & 0.8 \end{bmatrix}$$

Since the structure is symmetric to the x-z plane then we can assume $v_5 = 0$ and a load of 500 lbs.

Disregarding all rows and columns of zero displacement we form the new global stiffness matrix comprised only of the non-zero displacements.

$$[K] = (4) \times 30 \times 10^6 \begin{bmatrix} 0.0041 & 0.0041 \\ 0.0041 & 0.014 \end{bmatrix}$$

$$\{F\} = [K]\,\{d\}$$

$$F_{5x} = 500 = 492000\, u_5 + 492000\, w_5$$

$$F_{5z} = 0 = 492000\, u_5 + 1685880\, w_5$$

$$\Rightarrow \quad u_5 = -\frac{1685880}{492000}\, w_5$$

$$\Rightarrow \quad 500 = 492000 \left[-\frac{1685880}{492000}\right] w_5 + 492000\, w_5$$

$$\Rightarrow \; w_5 = -0.00042''$$

$$\Rightarrow \; u_5 = 0.0014''$$

Element stresses

Element 1–5

$$\sigma_{1-5} = \frac{E}{L_{1-5}} \; [-C_x \quad -C_y \quad -C_z \quad C_x \quad C_y \quad C_z] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_5 \\ v_5 \\ w_5 \end{Bmatrix}$$

$$\sigma_{1-5} = \frac{30 \times 10^6}{108} [-0.667 \;\; -0.333 \;\; -0.667 \;\; 0.667 \;\; 0.333 \;\; 0.666] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0014 \\ 0 \\ -0.00042 \end{Bmatrix}$$

$$\Rightarrow \quad \sigma_{1-5} = 180 \text{ psi (T)}$$

Element 2–5

$$\sigma_{2-5} = \frac{30 \times 10^6}{80.5} \; [0 \quad -0.447 \quad -0.894 \quad 0 \quad 0.447 \quad 0.894] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0014 \\ 0 \\ -0.00042 \end{Bmatrix}$$

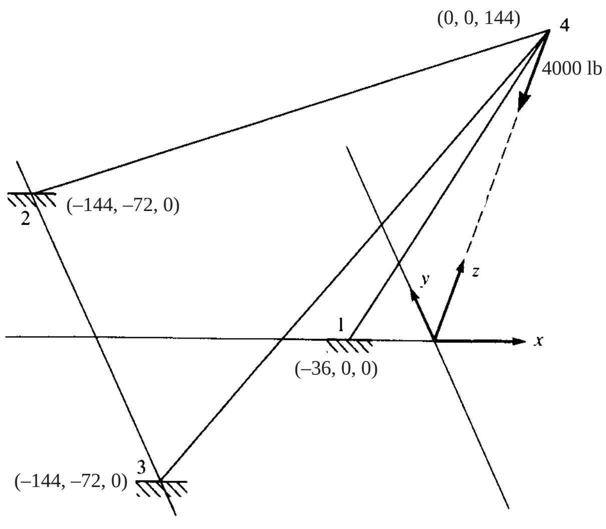
$$\Rightarrow \quad \sigma_{2-5} = -140 \text{ psi (C)}$$

From symmetry

$$\sigma_{3-5} = 140 \text{ psi (C)}$$

$$\sigma_{4-1} = 180 \text{ psi (T)}$$

3.43



Element 1–4

$$L_{1-4} = \sqrt{(0-(-36)^2)+0+(144-0)^2} \Rightarrow L_{1-4} = 148.4 \text{ in.}$$

$$C_x = \frac{x_4 - x_1}{L_{1-4}} = \frac{36}{148.6}$$

$$\Rightarrow C_x = 0.2426$$

$$C_y = 0$$

$$C_z = 0.9704$$

$$[k_{1-4}] = \frac{AE}{L_{1-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hdashline & & & 0.05885 & 0 & 0\cdot 2354 \\ & & & 0 & 0 & 0 \\ & & & 0\cdot 2354 & 0 & 0\cdot 9417 \end{array}\right]$$

Element 2–4

$$L_{2-4} = \sqrt{(144)^2 + (-72)^2 + 144^2} = 216 \text{ in.}$$

$$C_x = \frac{144}{216} = 0.667,\ C_y = \frac{-72}{216} = -0.3333,\ C_z = \frac{144}{216} = 0.6667$$

$$[k_{2-4}] = \frac{AE}{L_{2-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & 0.4425 & -0.2222 & 0.4445 \\ & & & -0.2222 & 0.1111 & -0.2222 \\ & & & 0.4444 & -0.2222 & 0.4445 \end{array}\right]$$

Element 3–4

$$L_{3-4} = 216 \text{ in.},\ C_x = 0.6667,\ C_y = 0.3333,\ C_z = 0.6667$$

$$[k_{3-4}] = \frac{AE}{L_{3-4}}\left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & 0.4425 & 0.2222 & 0.4425 \\ & & & 0.2222 & 0.1111 & 0.2222 \\ & & & 0.4445 & 0.2222 & 0.4445 \end{array}\right]$$

$$[K] = \frac{AE}{L_{1-4}}\begin{bmatrix} 0.05885+1.294 & 0 & 0.2354+1.294 \\ 0 & 0.3234 & 0 \\ 0.2354+1.294 & 0 & 0.9417+1.294 \end{bmatrix}$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ -4000 \end{Bmatrix} = \frac{AE}{148.4}\begin{bmatrix} 1.3529 & 0 & 1.5294 \\ 0 & 0.3234 & 0 \\ 1.5294 & 0 & 2.2357 \end{bmatrix}\begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$0 = \frac{AE}{148.4}\ [0\ u_4 + 0.3234\ v_4 + 0\ w_4]$$

$$\Rightarrow\ v_4 = 0$$

$$0 = \frac{AE}{148.4}\ [1.3529\ u_4 + 1.5294\ w_4]$$

$$\Rightarrow\ u_4 = -\ 1.1305\ w_4$$

$$-\ 4000 = \frac{AE}{148.4}\ [1.529\ u_4 + 2.2357\ w_4]$$

$$\Rightarrow\ -\ 4000 = \frac{AE}{148.4}\ [1.529\ (-\ 1.1305\ w_4) + 2.2357\ w_4]$$

$$\Rightarrow\ w_4 = \frac{1171501.87}{6 \times 30 \times 10^6}$$

$$\Rightarrow\ w_4 = -\ 0.00683 \text{ in.}$$

$$\Rightarrow\ u_4 = 0.00863 \text{ in.}$$

Stresses

$$\sigma_{1-4} = \frac{E}{L_{1-4}}\ [-\ C_x \quad -\ C_y \quad -\ C_z \quad C_x \quad C_y \quad C_z] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$\sigma_{1-4} = \frac{30 \times 10^6}{148.4}\ [-\ 0.2426 \quad 0 \quad -\ 0.9704 \quad 0.2426 \quad 0 \quad 0.9704] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.00863 \\ 0 \\ -\ 0.00683 \end{bmatrix}$$

$$\Rightarrow\ \sigma_{1-4} = -\ 916 \text{ psi (C)}$$

3.44 Derive Equation (3.7.21)

$$\sigma = \frac{f'_{2x}}{A}$$

$$f'_{2x} = \frac{AE}{L}\ [1\ \ -1] \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

$$\sigma = \frac{E}{L}\ [-1 \quad 1] \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

Now in 3–D

$$\{d'\} = [T^*]\ \{d\}$$

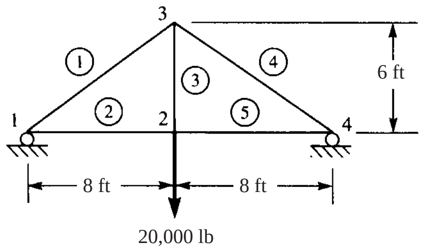
where by Equation (3.7.7)

$$[T^*] = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

$$\sigma = \frac{E}{L}\,[-1 \quad 1]\begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}\{d\}$$

$$\sigma = \frac{E}{L}\,[-C_x \quad -C_y \quad -C_z \quad C_x \quad C_y \quad C_z]\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

3.46

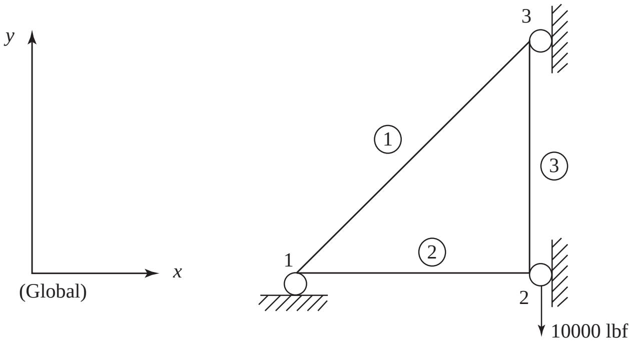


$$E = 30 \times 10^6 \text{ psi}$$

$$A^{(1)} = A^{(2)} = A^{(4)} = A^{(5)} = 10 \text{ in.}^2$$

$$A^{(2)} = 20 \text{ in.}^2$$

Reduce the given figure by symmetry.



$$A^{(1)} = A^{(2)} = A^{(3)} = 10 \text{ in.}^2$$

(Reducing given $A^{(3)}$ by half)

$$v_1 = 0,\ u_2 = u_3 = 0$$

find u_1, v_2, v_3

Data for reduced truss

Element	$\theta°$	C	S	C^2	S^2	CS
1	36.9°	0.8	0.6	0.64	0.36	0.48
2	0°	1.0	0	1.0	0	0
3	90°	0	1.0	0	1.0	0

$$[k^{(\alpha)}] = \left\{\frac{A_\alpha E_\alpha}{L_\alpha}\right\} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

$$\therefore \quad [k^{(1)}] = (10 \text{ in.}^2)\left(\frac{30\times 10^6}{\text{in.}^2}\text{ lbf}\right)\left(\frac{1}{10\text{ft}}\right)\left(\frac{\text{ft}}{12\text{in.}}\right) \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

So

$$[k^{(1)}] = 2.5 \times 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.36 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

Finally

$$[k^{(1)}] = 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1.6 & 1.2 & -1.6 & -1.2 \\ 1.2 & 0.9 & -1.2 & -0.9 \\ -1.6 & -1.2 & 1.6 & 1.2 \\ -1.2 & 0.9 & 1.2 & 0.9 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

$$[k^{(2)}] = (10 \text{ in.}^2)\left(\frac{30\times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{8\text{ft}}\right)\left(\frac{1\text{ ft}}{12\text{ in.}}\right) \begin{bmatrix} 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[k^{(2)}] = 10^6 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 3.125 & 0 & -3.125 & 0 \\ 0 & 0 & 0 & 0 \\ -3.125 & 0 & 3.125 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

and

$$[k^{(3)}] = (10 \text{ in.}^2)\left(\frac{30\times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{6\text{ft}}\right)\left(\frac{1\text{ ft}}{12\text{in.}}\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & -1.0 \\ 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 1.0 \end{bmatrix}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\
[k^{(3)}] = 10^6 & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4.167 & 0 & -4.167 \\ 0 & 0 & 0 & 0 \\ 0 & -4.167 & 0 & 4.167 \end{bmatrix} \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_1 & v_1 & u_3 & v_2 & u_3 & v_3 \end{array} \\
[K] = 10^6 & \left|\begin{array}{cccccc} 1.6+3.125 & 1.2 & -3.125 & & -1.6 & -1.2 \\ 1.2 & 0.9 & & & -1.2 & -0.9 \\ -3.125 & . & 3.125 & & & \\ & & & 4.167 & & -4.167 \\ -1.6 & -1.2 & & & 1.6 & 1.2 \\ -1.2 & -0.9 & & -4.167 & 1.2 & 4.1+4.167 \end{array}\right| \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
\end{array}
$$

$[K]\,\{d\} = [F]$ gives

$$
\begin{bmatrix} 4.725 & 1.2 & -3.125 & 0 & -1.6 & -1.2 \\ 1.2 & 0.9 & 0 & 0 & -1.2 & -0.9 \\ -3.125 & 0 & 3.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.167 & 0 & -4.167 \\ -1.6 & -1.2 & 0 & 0 & 1.6 & 1.2 \\ -1.2 & -0.9 & 0 & -4.167 & 1.2 & 5.067 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \\ 0 \\ v_2 \\ 0 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.01 \\ 0 \\ 0 \end{Bmatrix}
$$

and

$$
\begin{bmatrix} 4.725 & 0 & -1.200 \\ 0 & 4.167 & -4.167 \\ -1.2 & -4.167 & 5.067 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.01 \\ 0 \end{Bmatrix}
$$

$$
u_1 = \frac{\begin{vmatrix} 0 & 0 & -1.200 \\ -0.1 & 4.167 & -4.167 \\ 0 & -4.167 & 5.067 \end{vmatrix}}{\begin{vmatrix} 4.725 & 0 & -1.200 \\ 0 & 4.167 & -4.167 \\ -1.2 & -4.167 & 5.067 \end{vmatrix}}
$$

$$
u_1 = \frac{-1.2\begin{vmatrix} -0.01 & 4.167 \\ 0 & -4.167 \end{vmatrix}}{4.725\begin{vmatrix} 4.167 & -4.167 \\ -4.167 & 5.067 \end{vmatrix} - 1.2\begin{vmatrix} 0 & 4.167 \\ -1.2 & -4.167 \end{vmatrix}}
$$

$$
u_1 = \frac{-0.025}{17.72017 - 6.00048} = -0.00426 \text{ in.}
$$

$$v_2 = \frac{\begin{vmatrix} 4.725 & 0 & -1.2 \\ 0 & -0.01 & -4.167 \\ -1.2 & 0 & 5.067 \end{vmatrix}}{11.71969}$$

$$= \frac{4.725\{-0.02534\} - 1.2\{-0.006\}}{11.71969}$$

$$v_2 = -0.0192 \text{ in.}$$

$$v_3 = \frac{\begin{vmatrix} 4.725 & 0 & 0 \\ 0 & 4.167 & -0.01 \\ -1.2 & -4.167 & 0 \end{vmatrix}}{11.71969}$$

$$v_3 = \frac{4.725\{-(0.0)(4.167)\}}{11.71969} = -0.0168 \text{ in.}$$

now $\{\sigma\} = [C']\{d\}$ where $[C'] = \frac{E}{L}[-C \quad -S \quad C \quad S]$

So, for (1)

$$[C'] = \left(\frac{30 \times 10^6 \text{ lbf}}{120 \text{ in.}^2}\right)(-0.8 \;\; -0.6 \;\; 0.8 \;\; 0.6)\begin{Bmatrix} -0.00426 \\ 0 \\ 0 \\ -0.0168 \end{Bmatrix}$$

$$= \left(\frac{2.5 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(-0.0068 \text{ in.})$$

$$\sigma^{(1)} = -1668 \text{ psi (C)}$$

$$\sigma^{(2)} = \left(\frac{30 \times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{96 \text{ in.}}\right)[-1.0 \quad 0 \quad 1.0 \quad 0]\begin{Bmatrix} -0.00426 \\ 0 \\ 0 \\ -0.0192 \end{Bmatrix}$$

$$\sigma^{(2)} = \left(\frac{3.125 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(0.00426 \text{ in.})$$

$$\sigma^{(2)} = 1332 \text{ psi (T)}$$

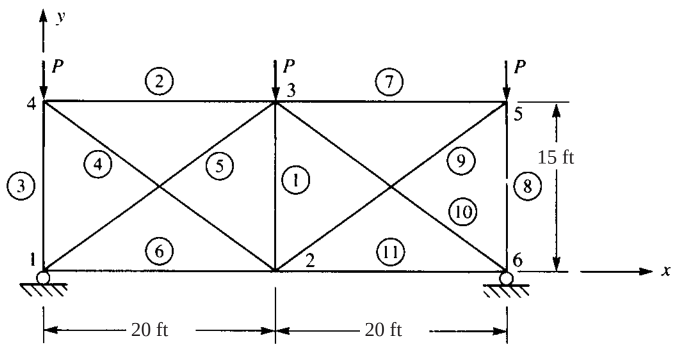
$$\sigma^{(3)} = \left(\frac{30 \times 10^6 \text{ lbf}}{\text{in.}^2}\right)\left(\frac{1}{72 \text{ in.}}\right)[0 \quad -1.0 \quad 0 \quad 1.0]\begin{Bmatrix} 0 \\ -0.0192 \\ 0 \\ -0.0168 \end{Bmatrix}$$

$$\sigma^{(3)} = \left(\frac{4.167 \times 10^5 \text{ lbf}}{\text{in.}^3}\right)(0.0096 - 0.0084)$$

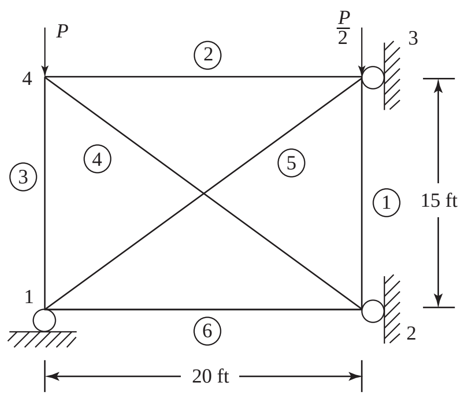
$$= 1000 \text{ psi}$$

$$\sigma^{(3)} = 1000 \text{ psi (T)}$$

3.47



Using symmetry



Boundary conditions

$$v_1 = u_2 = u_3 = 0$$

(see solution to Problem 3.24 for individual [k]'s for each element)
Global [K]

$$[K] = AE\begin{bmatrix} 0.0756 & 0.0192 & -0.05 & 0 & -0.0256 & -0.0192 & 0 & 0 \\ 0.0192 & 0.0811 & 0 & 0 & -0.0192 & -0.0144 & 0 & -0.0667 \\ -0.05 & 0.0 & 0.0756 & -0.0192 & 0 & 0 & -0.0256 & 0.0192 \\ 0 & 0 & -0.0192 & 0.0811 & 0 & -0.0667 & 0.0192 & -0.0144 \\ -0.0256 & -0.0192 & 0 & 0 & 0.0756 & 0.0192 & -0.05 & 0 \\ -0.0192 & -0.0144 & 0 & -0.0667 & 0.0192 & 0.0811 & 0 & 0 \\ 0 & 0 & -0.0256 & 0.0192 & -0.05 & 0 & 0.0756 & -0.0192 \\ 0 & -0.0667 & 0.0192 & -0.0144 & 0 & 0 & -0.0192 & 0.0811 \end{bmatrix}$$

Applying the boundary conditions, we obtain

$$AE\begin{bmatrix} 0.0756 & 0 & -0.0192 & 0 & 0 \\ 0 & 0.0811 & -0.0667 & 0.0192 & -0.0144 \\ -0.0192 & -0.0667 & 0.0811 & 0 & 0 \\ 0 & 0.0192 & 0 & 0.0756 & -0.0192 \\ 0 & -0.0144 & 0 & -0.0192 & 0.0811 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{-P}{Z} \\ 0 \\ -P \end{Bmatrix}$$

Using the simultaneous equation solver, we obtain

$$u_1 = -110\ \frac{P}{AE},\ v_2 = -405\ \frac{P}{AE}$$

$$v_3 = -433\ \frac{P}{AE},\ u_4 = 50\ \frac{P}{AE}$$

$$v_4 = -208\ \frac{P}{AE}$$

where all displacements are now in units of inches.
(Computer program TRUSS was also used to verify the above displacements (setting $P = 1$, $A = 1$, $E = 1$))

Stresses

$$\sigma^{(2)} = \frac{E}{L}\ [-1\quad 0\quad 1\quad 0] \begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{E}{L}\ [-1\quad 0\quad 1\quad 0] \begin{Bmatrix} 50 \\ -208 \\ 0 \\ -433 \end{Bmatrix} \frac{P}{AE}$$

$$= \frac{E}{240\text{ in.}}\left(-50\frac{P}{AE}\right)$$

$$\sigma^{(2)} = -0.208\ \frac{P}{A}$$

$$\sigma^{(3)} = \frac{E}{L}\ [0\ \ -1\ \ 0\ \ 1] \begin{Bmatrix} u_1 = -110 \\ v_1 = 0 \\ u_4 = 50 \\ v_4 = -208 \end{Bmatrix} \frac{P}{AE}$$

$$= \frac{-208}{15 \times 12}\frac{P}{A}$$

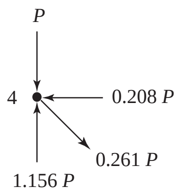
$$= -1.156\ \frac{P}{A}$$

$$\sigma^{(4)} = \frac{E}{L}\ [0.80\quad -0.60\quad -0.80\quad 0.60]\ \frac{P}{AE} \begin{Bmatrix} u_2 = 0 \\ v_2 = -405 \\ u_4 = 50 \\ v_4 = -208 \end{Bmatrix}$$

$$\sigma^{(4)} = \frac{(243 - 40 - 124.8)}{25' \times 12}\frac{P}{A}$$

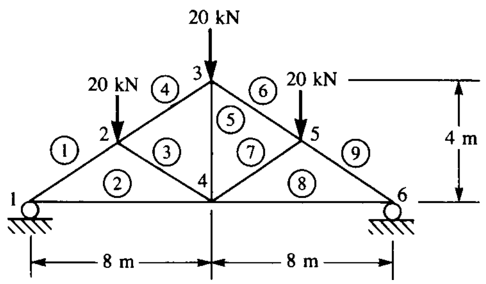
$$\sigma^{(4)} = 0.261\frac{P}{A}$$

Verify equilibrium at node 4



$$\Sigma F_x = -0.208\,P + 0.261\,P\,(0.8)$$
$$= (-0.208 + 0.208)P$$
$$= 0$$
$$\Sigma F_y = 1.156\,P - P + (0.261\,P)\,(0.6)$$
$$= (1.156 - 1 - 0.156)P$$
$$\cong 0$$

3.48



Element	L	θ	C	S	C^2	S^2	CS
(1)	4.47	26.565°	0.89443	0.44721	0.8	0.2	0.4
(2)	8	0	1	0	1	0	0
(3)	4.47	153.435°	– 0.89443	0.44721	0.8	0.2	– 0.4
(4)	4.47	26.565°	0.89443	0.44721	0.8	0.2	0.4
(5)	4	90°	0	1	0	1	0

$$[k^{(1)}] = \frac{AE}{4.47}\begin{array}{c}\begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ & 0.2 & -0.4 & -0.2 \\ & & 0.8 & 0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\end{array}\begin{array}{l} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$[k^{(2)}] = \frac{AE}{8}\begin{array}{c}\begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symmetry} & & & 0 \end{bmatrix}\end{array}\begin{array}{l} u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}$$

$$[k^{(3)}] = \frac{AE}{4.47}\begin{bmatrix} 0.8 & -0.4 & -0.8 & 0.4 \\ & 0.2 & 0.4 & -0.2 \\ & & 0.8 & -0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\begin{matrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{matrix}$$

$$[k^{(4)}] = \frac{AE}{4.47}\begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ & 0.2 & -0.4 & -0.2 \\ & & 0.8 & 0.4 \\ \text{Symmetry} & & & 0.2 \end{bmatrix}\begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

$$[k^{(5)}] = \frac{AE}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & 0 & 0 \\ \text{Symmetry} & & & \frac{1}{2} \end{bmatrix}\begin{matrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{matrix}$$

Boundary conditions

$$\because \quad v_1 = u_4 = u_3 = 0$$

$$[K] = 210 \times 10^6 \begin{array}{c} \begin{matrix} u_1 & \qquad u_2 & \qquad v_2 & \qquad v_3 & \qquad v_4 \end{matrix} \\ \begin{bmatrix} \frac{0.8}{4.47}+\frac{1}{8} & \frac{-0.8}{4.47} & \frac{-0.4}{4.47} & 0 & 0 \\ \frac{-0.8}{4.47} & \frac{2.4}{4.47} & \frac{0.4}{4.47} & \frac{-0.4}{4.47} & \frac{0.4}{4.47} \\ \frac{-0.4}{4.47} & \frac{0.4}{4.47} & \frac{0.6}{4.47} & \frac{-0.2}{4.47} & \frac{-0.2}{4.47} \\ 0 & \frac{-0.4}{4.47} & \frac{-0.2}{4.47} & \frac{0.2}{4.47}-\frac{0.5}{4} & \frac{-0.5}{4} \\ 0 & \frac{0.4}{4.47} & \frac{-0.2}{4.47} & \frac{-0.5}{4} & \frac{0.2}{4.47}+\frac{0.5}{4} \end{bmatrix} \end{array} \begin{matrix} \\ u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_4 \end{matrix}$$

$$\because \quad F_{2y} = -20 \text{ kN} \qquad F_{3y} = -10 \text{ kN}$$

$$F_{1x} = F_{1y} = F_{2x} = F_{3x} = F_{4x} = F_{4y} = ?$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ -20\times 10^3 \\ -10\times 10^3 \\ F_{4y} \end{Bmatrix} = \begin{bmatrix} \left(\frac{2}{5\sqrt{5}}+\frac{1}{8}\right) & \frac{-2}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} & 0 & 0 \\ \frac{-2}{5\sqrt{5}} & \frac{6}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} \\ \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{3}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} \\ 0 & \frac{-1}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) & \frac{-1}{8} \\ 0 & \frac{1}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{-1}{8} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \times 210 \times 10^6$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{4y} \\ -20\times 10^3 \\ -10\times 10^3 \end{Bmatrix} = \left[\begin{array}{ccc:cc} \left(\frac{2}{5\sqrt{5}}+\frac{1}{8}\right) & \frac{-2}{5\sqrt{5}} & 0 & \frac{-1}{5\sqrt{5}} & 0 \\ \frac{-2}{5\sqrt{5}} & \frac{6}{5\sqrt{5}} & \frac{-1}{10\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{-1}{5\sqrt{5}} \\ 0 & \frac{1}{5\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{8}\right) & \frac{-1}{10\sqrt{5}} & \frac{-1}{8} \\ \hdashline \frac{-1}{5\sqrt{5}} & \frac{1}{5\sqrt{5}} & \frac{1}{10\sqrt{5}} & \frac{3}{10\sqrt{5}} & \frac{-1}{10\sqrt{5}} \\ 0 & \frac{-1}{5\sqrt{5}} & \frac{-1}{8} & \frac{-1}{10\sqrt{5}} & \left(\frac{1}{10\sqrt{5}}+\frac{1}{5}\right) \end{array}\right]\begin{Bmatrix} u_1 \\ u_2 \\ v_4 \\ v_2 \\ v_3 \end{Bmatrix} \times 210 \times 10^6$$

$$\begin{Bmatrix} ? \\ P \end{Bmatrix} = 210 \times 10^6 \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\left[k_{11}^{-1}\right] = \begin{bmatrix} 4.192 & 1.532 & -0.8074 \\ 1.532 & 2.602 & -1.3716 \\ 0.807 & -1.3716 & 6.6148 \end{bmatrix}$$

$$[k_c] = [k_{21}]\left[k_{11}^{-1}\right][k_{12}] = \begin{bmatrix} 0.04759 & 0.02923 \\ 0.02923 & 0.0935 \end{bmatrix}$$

$$[k_c] = [k_{22}] - [k_{21}]\left[k_{11}^{-1}\right][k_{12}] = \begin{bmatrix} 0.08657 & -0.073956 \\ -0.073956 & 0.076213 \end{bmatrix}$$

$$[k_c] = [N^1]^{-1} = \begin{bmatrix} 67.539 & 65.539 \\ 65.539 & 76.719 \end{bmatrix}$$

$$\begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 67.539 & 65.539 \\ 65.539 & 76.719 \end{bmatrix} \begin{Bmatrix} \frac{-2}{21} \times 10^{-3} \\ \frac{-1}{21} \times 10^{-3} \end{Bmatrix}$$

$$= \begin{Bmatrix} -9.5532489 \times 10^{-3} \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_4 \end{Bmatrix} = -\left[k_{11}^{-1}\right][k_{12}]\{d_2\} = -\begin{bmatrix} -0.201853644 & -0.03610867 \\ 0.157096846 & -0.06134 \\ -0.346288346 & -0.704175 \end{bmatrix} \begin{Bmatrix} -9.5532489 \times 10^{-3} \\ -9.8951503 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} -2.285658 \times 10^{-3} \\ 8.93812942 \times 10^{-4} \\ -1.027609703 \times 10^{-2} \end{Bmatrix}$$

The stresses in each element

$$\sigma^{(1)} = \frac{E}{L}\,[-C \;\; -S \;\; C \;\; S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\sigma^{(1)} = \frac{210 \times 10^9}{2\sqrt{5}} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{Bmatrix} -2.285658 \times 10^{-3} \\ 0 \\ 8.93812942 \times 10^{-4} \\ -9.5532489 \times 10^{-3} \end{Bmatrix}$$

$$\sigma^{(1)} = -67.08044733 \times 10^6 \; \frac{\text{N}}{\text{mm}^2} = -67.08 \text{ MPa (C)}$$

$$\sigma^{(2)} = \frac{210\times10^9}{8}\ [-1\ 0\ 1\ 0]\ \begin{Bmatrix} -2.285658\times10^{-3} \\ 0 \\ 0 \\ -1.027609703\times10^{-2} \end{Bmatrix}$$

$$\sigma^{(2)} = 59.9985525\times10^6\ \frac{\text{N}}{\text{mm}^2} = 60.0 \text{ MPa (T)}$$

$$\sigma^{(3)} = \frac{210\times10^9}{2\sqrt{5}}\ \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ -1.027609703\times10^{-2} \\ 8.93812942\times10^{-4} \\ -9.5532489\times10^{-3} \end{Bmatrix}$$

$$= -22.36033275\times10^6\ \frac{\text{N}}{\text{mm}^2} = -22.36 \text{ MPa (C)}$$

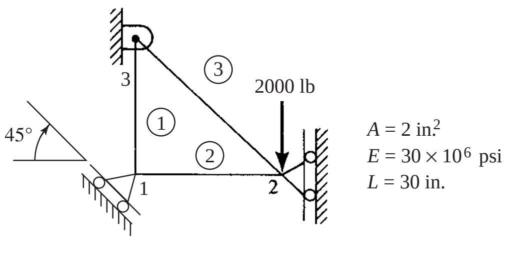
$$\sigma^{(4)} = \frac{210\times10^9}{2\sqrt{5}}\ \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\ \begin{Bmatrix} 8.93812942\times10^{-4} \\ 9.5532489\times10^{-3} \\ 0 \\ -9.8951503\times10^{-3} \end{Bmatrix}$$

$$= -44.72007296\times10^6\ \frac{\text{N}}{\text{mm}^2} = -44.72 \text{ MPa (C)}$$

$$\sigma^{(5)} = \frac{210\times10^9}{4}\ [0\ \ -1\ \ 0\ \ 1]\ \begin{Bmatrix} 0 \\ -1.027609703\times10^{-2} \\ 0 \\ -9.8951503\times10^{-3} \end{Bmatrix}$$

$$= 19.99970332\times10^6\ \frac{\text{N}}{\text{mm}^2} = 20.00 \text{ MPa (T)}$$

3.49



Element (1)

$$\theta = 90°$$

$$[k^{(1)}] = \frac{2\times30\times10^6}{30} \begin{array}{c} \begin{array}{cc} (1) & (3) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$\theta = 0°$

$$[k^{(2)}] = \frac{2 \times 30 \times 10^6}{30} \begin{array}{c} \begin{matrix} \quad (1) & & \quad (2) & \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (3)

$\theta = 135°$

$$[k^{(3)}] = \frac{2 \times 30 \times 10^6}{30\sqrt{2}} \begin{array}{c} \begin{matrix} \qquad (2) & & \qquad (3) & \end{matrix} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array}$$

Assembling the stiffness matrices

$$[K] = 2 \times 10^6 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1+\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} \\ 0 & 0 & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\ 0 & 0 & -\frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} \\ 0 & -1 & \frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & -\frac{0.5}{\sqrt{2}} & 1+\frac{0.5}{\sqrt{2}} \end{bmatrix}$$

$$[K] = 10^6 \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & -2 & 0.707 & -0.707 & -0.707 & 2.707 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T^T] = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T]\,[K]\,[T^T] = 10^6 \begin{bmatrix} 2.0 & 0 & -1.4 & 0 & 0 & 1.4 \\ 0 & 2.0 & -1.4 & 0 & 0 & -1.4 \\ -1.4 & -1.4 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 1.4 & -1.4 & 0.707 & -0.707 & -0.7 & 2.707 \end{bmatrix}$$

Applying the boundary conditions

$$v_1 = u_2 = u_3 = v_3 = 0$$

$$\begin{Bmatrix} F'_{1x} = 0 \\ F_{2y} = -2000 \end{Bmatrix} = 10^6 \begin{bmatrix} 2.0 & 0 \\ 0 & 0.707 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \quad u'_1 = 0$$

$$v_2 = -\,0.00283 \text{ in.}$$

$$\begin{Bmatrix} F'_{1x} = 0 \\ F'_{1y} = 0 \\ F_{2x} = 0 \\ F_{2y} = -2000 \\ F_{3x} = 0 \\ F_{3y} = 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 2.707 & -0.707 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 & 0.707 & -0.707 \\ 0 & -2 & 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00283 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow F_{1x} = 0$$

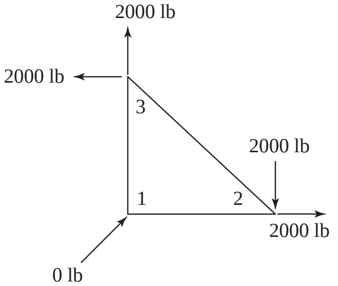
$$F_{1y} = 0$$

$$F_{2x} = 2000 \text{ lb}$$

$$F_{2y} = -\,2000 \text{ lb}$$

$$F_{3x} = -\,2000 \text{ lb}$$

$$F_{3y} = 2000 \text{ lb}$$



Element stresses

$$\{\sigma\} = [C']\,\{d\}$$

$$[C'] = \frac{E}{L}\,[-C \quad -S \quad C \quad S]$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{30}\,[0 \quad -1 \quad 0 \quad 1] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(1)} = 0$$

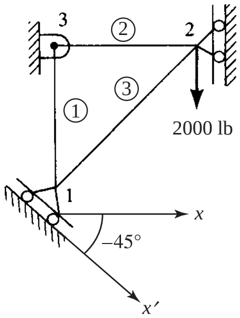
$$\sigma^{(2)} = \frac{30 \times 10^6}{30}\ [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 = -0.00283 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(2)} = 0$$

$$\sigma^{(3)} = \frac{30 \times 10^6}{30\sqrt{2}}\ [0.707 \quad -0.707 \quad -0.707 \quad 0.707] \begin{Bmatrix} 0 \\ -0.00283 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \ \sigma^{(3)} = 1414 \text{ psi (T)}$$

3.50



$$[k^{(1)}] = 2 \times 10^6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 2 \times 10^6 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(3)}] = 1.414 \times 10^6 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

Assembling the stiffness matrices

$$[K] = 10^6 \begin{bmatrix} 0.707 & 0.707 & -0.707 & -0.707 & 0 & 0 \\ 0.707 & 2.707 & -0.707 & -0.707 & 0 & -2 \\ -0.707 & -0.707 & 2.707 & 0.707 & -2.0 & 0 \\ -0.707 & -0.707 & 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & -2.0 & 0 & -2.0 & 0 \\ 0 & -2.0 & 0 & 0 & 0 & 2.0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T]\,[K]\,[T^T] = 10^6 \begin{bmatrix} 1.0 & -1.0 & 0 & 0 & 0 & 1.4 \\ -1.0 & 2.4 & -1.0 & -1.0 & 0 & -1.4 \\ 0 & -1.0 & 2.707 & 0.707 & -2 & 0 \\ 0 & -1.0 & 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 2.0 & 0 & -2.0 & 0 \\ 1.4 & -1.4 & 0 & 0 & 0 & 2.0 \end{bmatrix}$$

Applying the boundary conditions

$$v_1 = u_2 = u_3 = v_3 = 0$$

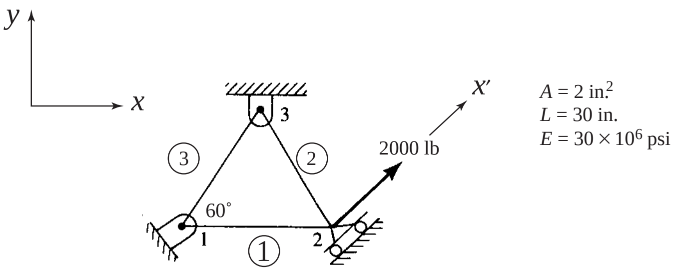
$$\begin{Bmatrix} F'_{1x} = 0 \\ F_{2y} = -2000 \end{Bmatrix} = \begin{bmatrix} 1000000 & 0 \\ 0 & 707106.75 \end{bmatrix} \begin{Bmatrix} u'_1 = 0 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \; u'_1 = 0 \qquad \sigma_{1-2} = -1414 \text{ psi}$$

$$v_2 = -\,0.00283 \text{ in.} \qquad \sigma_{1-3} = 0$$

$$\sigma_{3-2} = 0$$

3.51



Element (1)

$$C = 1,\, S = 0,\; \theta = 0°$$

$$[k^{(1)}] = 2 \times 10^6 \begin{array}{c} \begin{array}{cc} \quad(1) & \qquad(2) \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (2)

$$C = -\frac{1}{2},\ S = \frac{\sqrt{3}}{2},\ \ \theta = 120°$$

$$\begin{array}{cc} & \begin{array}{cc}\quad (2) & \qquad\qquad (3)\end{array} \\ [k^{(2)}] = 2\times 10^{6} & \begin{bmatrix} 0.25 & \frac{-\sqrt{3}}{4} & -0.25 & \frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{-3}{4} \\ -0.25 & \frac{\sqrt{3}}{4} & 0.25 & \frac{-\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{-3}{4} & \frac{-\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}\end{array}$$

Element (3)

$$C = \frac{1}{2},\quad S = \frac{\sqrt{3}}{2}$$

$$\begin{array}{cc} & \begin{array}{cc}\quad (1) & \qquad\qquad (3)\end{array} \\ [k^{(3)}] = 2\times 10^{6} & \begin{bmatrix} 0.25 & \frac{\sqrt{3}}{4} & -0.25 & \frac{-\sqrt{3}}{4} \\ & 0.75 & \frac{-\sqrt{3}}{4} & -0.75 \\ & & 0.25 & \frac{\sqrt{3}}{4} \\ \text{Symmetry} & & & 0.75 \end{bmatrix}\end{array}$$

Global $[K]$

$$[K] = 2\times 10^{6}\begin{bmatrix} 1.25 & \frac{\sqrt{3}}{4} & -1 & 0 & -0.25 & \frac{-\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0.75 & 0 & 0 & \frac{-\sqrt{3}}{4} & -0.75 \\ -1 & 0 & 1.25 & \frac{-\sqrt{3}}{4} & -0.25 & \frac{\sqrt{3}}{4} \\ 0 & 0 & \frac{-\sqrt{3}}{4} & 0.75 & \frac{\sqrt{3}}{4} & -0.75 \\ -0.25 & \frac{-\sqrt{3}}{4} & -0.75 & \frac{\sqrt{3}}{4} & 0.5 & 0 \\ \frac{-\sqrt{3}}{4} & -0.75 & \frac{\sqrt{3}}{4} & \frac{-3}{4} & 0 & 1.5 \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K^*] = [T_1]\,[K]\,\left[T_1^T\right]$$

with boundary conditions

$$u_1 = v_1 = v_2 = u_3 = v_3 = 0$$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F'_{2x} = 2000 \\ F'_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = [K^*] \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u'_2 = 0 \\ v'_2 = 0 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

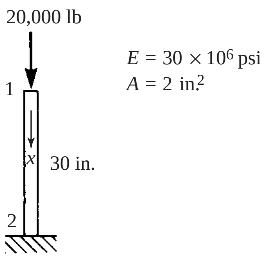
$$[T_1]\,[K] = \begin{bmatrix} 2.5\times10^{6} & 8.66\times10^{5} & -2\times10^{6} & 0 & -5\times10^{-5} & -8.66\times10^{5} \\ 8.66\times10^{5} & 1.5\times10^{6} & 0 & 0 & -8.66\times10^{5} & -1.5\times10^{6} \\ -1.414\times10^{6} & 0 & 1.155\times10^{6} & 4.483\times10^{5} & 2.588\times10^{5} & -4.483\times10^{5} \\ 1.414\times10^{6} & 0 & -2.38\times10^{6} & 1.673\times10^{6} & 9.659\times10^{5} & -1.673\times10^{6} \\ -5\times10^{5} & -8.66\times10^{5} & -1.5\times10^{6} & 8.66\times10^{5} & 1\times10^{6} & 0 \\ -8.66\times10^{5} & -1.5\times10^{6} & 8.66\times10^{5} & -1.5\times10^{6} & 0 & 3\times10^{6} \end{bmatrix}$$

$$[K^*] = [T_1]\,[K]\,\left[T_1^T\right] = \begin{bmatrix} 2.5\times10^{6} & 8.66\times10^{5} & -1.414\times10^{6} & 1.414\times10^{6} & -5\times10^{5} & -8.66\times10^{5} \\ 8.66\times10^{5} & 1.5\times10^{6} & 0 & 0 & -8.66\times10^{5} & -1.5\times10^{6} \\ -1.414\times10^{6} & 0 & 1.134\times10^{6} & -5\times10^{5} & 2.588\times10^{5} & -4.483\times10^{5} \\ 1.414\times10^{6} & 0 & -5\times10^{5} & 2.866\times10^{6} & 9.659\times10^{5} & -1.673\times10^{6} \\ -5\times10^{-5} & -8.66\times10^{5} & -4.483\times10^{5} & 1.673\times10^{6} & 1\times10^{6} & 0 \\ -8.66\times10^{5} & -1.5\times10^{6} & -4.483\times10^{5} & -1.673\times10^{6} & 0 & 3\times10^{6} \end{bmatrix}$$

Solving the third equation of $[K^*]\,\{d'\} = \{F'\}$ yields

$$u'_2 = \frac{2000}{1.134\times10^{6}} = 1.764\times10^{-3}\text{ in.}$$

3.52 (a)



Using Equation (3.9.19), we get

$$\pi_p = \frac{AL}{2}\ \{d^T\}\ [B^T]\,[D]\,[B]\ \{d\} - \{d^T\}\{f\}$$

For the bar above, Equation 3.9.26 yields

$$\pi_p = \frac{AL}{2} u_1^2\ \frac{E}{L^2} - u_1 f$$

$$\pi_p = \frac{AE}{2L} u_1^2 - u_1 f$$

Putting in the numerical values

$$\pi_p = \frac{(2)(30 \times 10^6)}{(2)(30)}\ u_1^2\ - u_1\,(20000)$$

$$\pi_p = 10^6\ u_1^2\ - 2 \times 10^4\ u_1$$

u_1, in.	π_p lb·in.
–0.004	96
–0.002	44
0	0
0.002	–36
0.004	–54
0.006	–84
0.008	–96
0.010	–100 ←
0.012	96

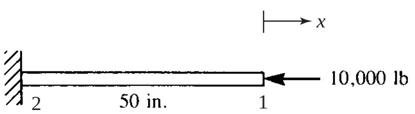
$\pi_{p\,\min} = -\,100$ lb·in.

Also by calculus

$$\frac{\partial \pi_p}{\partial u_1} = 2 \times 10^6\ u_1 - 2 \times 10^4 = 0$$

$\Rightarrow\ u_1 = 0.01$ in. for π_p minimum

(b)



$$\pi_p = \frac{AE}{2L} u_1^2 - u_1 f \text{ (see Problem 3.52 (a))}$$

$$\pi_p = \frac{2(30 \times 10^6)}{2(50)} u_1^2 - u_1 (10000)$$

$$\pi_p = 6 \times 10^5 u_1^2 - 0.1 \times 10^5 u_1$$

u_1 in.	π_p lb·in.
–0.004	49.6
–0.002	44.0
–0.001	10.6
0	0
0.002	4.0
0.004	–30.4
0.006	–38.4
0.008	–41.6 ←
0.010	–40.6

$$\pi_{p\,\min} = -41.67 \text{ lb·in.}$$

By calculus

$$\frac{\partial \pi_p}{\partial u_1} = 12 \times 10^5 u_1 - 0.1 \times 10^5$$

$$\Rightarrow \quad u_1 = 0.00833 \text{ in. yields } \pi_{p\,\min}.$$

3.53

$$du = \sigma_x \, d\,\varepsilon_x \, du$$

$$U = \iiint_v \left\{ \int_0^{\varepsilon_x} \sigma_x \, d\,\varepsilon_x \right\} dv$$

$$dv = A(x)\, dx \qquad A(x) = A_0 \left(1 + \frac{x}{L}\right)$$

$$\therefore \qquad U = \frac{1}{2}\int_0^L \sigma_x \, \varepsilon_x A_0 \left(1 + \frac{x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \{\hat{d}\}^T \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \{d\} \; A_0 \left(1 + \frac{x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \{\hat{u}_1 \quad \hat{u}_2\} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} A_0 \left(1 + \frac{x}{L}\right) dx$$

$$U = \frac{1}{2}\int_0^L \frac{EA_0}{L^2}(u_1^2 - 2u_1 u_2 + u_2^2) \left(1 + \frac{x}{L}\right) dx$$

$$\frac{\partial U}{\partial u_1} = \frac{A_0 L}{2}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right] + \frac{A_0}{2}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right]\frac{1}{2}$$

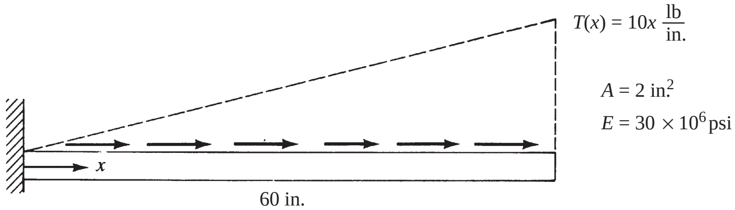
$$\frac{\partial U}{\partial u_1} = \frac{3A_0\, L}{4}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right] = f_1$$

Similarly

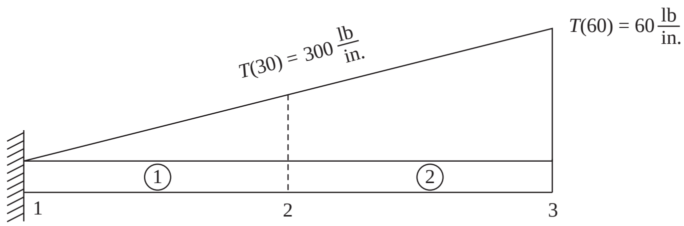
$$\frac{\partial U}{\partial u_2} = \frac{3AL}{4}\left[\frac{E}{L^2}(2u_2 - 2u_1)\right] = f_2$$

$$\therefore \quad K = \frac{3AE}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

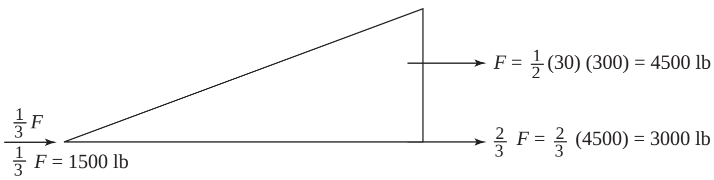
3.54



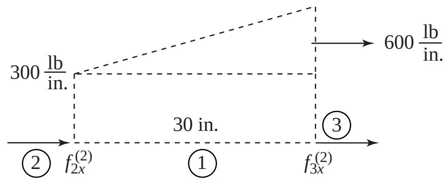
(a) Two element solution



For element (1) force matrix is from Example 3.9



$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1500 \\ 3000 \end{Bmatrix}$$



$$f_{2x}^{(2)} = \frac{1}{3}\,(4500) + \frac{1}{2}\,(30 \times 300)$$

$$f_{3x}^{(2)} = \frac{2}{3}\,(4500) + 30 \times 300 \times \frac{1}{2}$$

$$f_{2x}^{(2)} = 6000 \text{ lb} \qquad f_{3x}^{(2)} = 7500 \text{ lb}$$

$$\therefore \quad \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 6000 \\ 7500 \end{Bmatrix}$$

$$\therefore \quad \{F\} = \begin{Bmatrix} F_{1x} + 1500 \\ 3000 + 6000 \\ 7500 \end{Bmatrix}$$

Stiffness matrices

$$[k^{(1)}] = \frac{AE}{30} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(2)}] = \frac{AE}{30} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{AE}{30} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Global equations

$$\{F\} = [K]\,\{d\}$$

$$\begin{Bmatrix} F_{1x} + 1500 \\ 9000 \\ 7500 \end{Bmatrix} = \frac{(2)(30 \times 10^6)}{30} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving equations 2 and 3 for u_2 and u_3, we obtain

$$u_2 = 0.00825 \text{ in.}$$

$$u_3 = 0.012 \text{ in.}$$

Element stresses

$$\sigma^{(1)} = [C']\,\{d\} = \frac{E}{L}\,[-C \;\; -S \;\; C \;\; S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

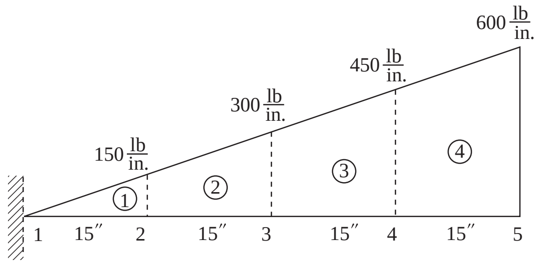
$$= \frac{30 \times 10^6}{30}\,[-1 \;\; 0 \;\; 1 \;\; 0] \begin{Bmatrix} 0 \\ 0 \\ 0.00825 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 10^6\,(0.00825) = 8250 \text{ psi (T)}$$

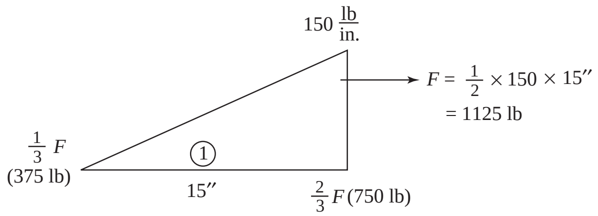
$$\sigma^{(2)} = \frac{30\times10^6}{30}\,[-1\ \ 0\ \ 1\ \ 0]\begin{Bmatrix} u_2 = 0.00825 \\ v_2 = 0 \\ u_3 = 0.012 \\ v_3 = 0 \end{Bmatrix}$$

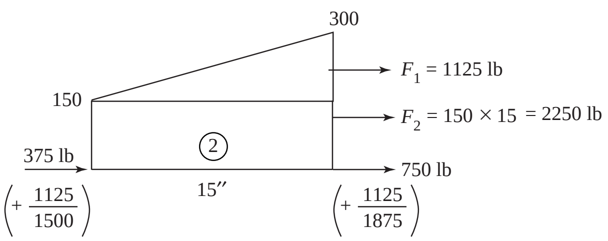
$$\sigma^{(2)} = 3750 \text{ psi (T)}$$

Four element solution

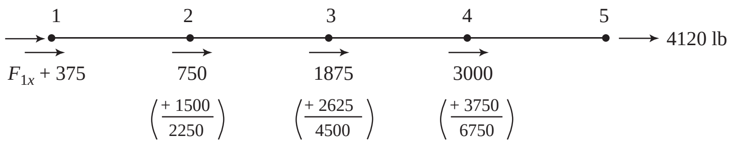


Basic triangular load





Total global forces at nodes



Global equations

$$\begin{Bmatrix} \cancel{F_{1x}+375} \\ 2250 \\ 4500 \\ 6750 \\ 4120 \end{Bmatrix} = \frac{AE}{15}\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{-1} & 2 & -1 & 0 & 0 \\ \cancel{0} & -1 & 2 & -1 & 0 \\ \cancel{0} & 0 & -1 & 2 & -1 \\ \cancel{0} & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \cancel{u_1 = 0} \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

Solve the last four equations for u_2 through u_5

$$2250 = \frac{AE}{15}\,(2\,u_2 - u_3) \qquad (1)$$

$$4500 = \frac{AE}{15}\ (-u_2 + 2u_3 - u_4) \tag{2}$$

$$6750 = \frac{AE}{15}\ (-u_3 + 2u_4 - u_5) \tag{3}$$

$$4120 = \frac{AE}{15}\ (-u_4 + u_5) \tag{4}$$

Using Gaussian Elimination, divide (1) by (2)

$$1125 = \left(u_2 - \frac{1}{2}u_5\right)\frac{AE}{15} \tag{5}$$

Add (5) to (2)

$$5625 = (1\tfrac{1}{2}\ u_3 - u_4)\ \frac{AE}{15} \tag{6}$$

$$6750 = (-u_3 + 2u_4 - u_5)\ \frac{AE}{15} \tag{7}$$

$$4120 = (-u_4 + u_5)\ \frac{AE}{15} \tag{8}$$

$$1125 = (u_2 - \tfrac{1}{2}\ u_3)\ \frac{AE}{15} \tag{9}$$

Divide (6) by 1.5

$$3750 = \left(0 + u_3 - \frac{1}{1.5}u_4\right)\frac{AE}{15} \tag{10}$$

Add (10) to (7)

$$10500 = [(2 - 0.067)\ u_4 - u_5)]\ \frac{AE}{15} \tag{11}$$

$$4120 = (-u_4 + u_5)\ \frac{AE}{15} \tag{12}$$

$$1125 = \left(u_2 - \frac{1}{2}u_3\right)\frac{AE}{15} \tag{13}$$

$$3750 = \left(-\frac{1}{1.5}u_4\right)\frac{AE}{15} \tag{14}$$

$$\frac{3}{4} \times (11)$$

$$\Rightarrow 7875 = \left(u_4 - \frac{3}{4}u_5\right)\frac{AE}{15} \tag{15}$$

Add (15) to (12)

$$11995 = \frac{1}{4}u_5\ \frac{AE}{15} \tag{16}$$

Solve (16) for u_5

$$u_5 = \frac{11995 \times 4 \times 15}{2 \times 30 \times 10^6} = 0.012 \text{ in.}$$

By (15)

$$u_4 = 7875 \times \frac{15}{2 \times 30 \times 10^6} + \frac{3}{4} \ (0.012 \text{ in.})$$

$$\Rightarrow \quad u_4 = 0.01097 \text{ in.}$$

By (10)

$$u_3 = \frac{3750\,(15'')}{2 \times 30 \times 10^6} + \frac{1}{1.5} \ (0.01097)$$

$$\Rightarrow \quad u_3 = 0.00825 \text{ in.}$$

By (9)

$$u_2 = \frac{1125 \times 15}{2 \times 30 \times 10^6} + \frac{1}{2} \ (0.00825)$$

$$\Rightarrow \quad u_2 = 0.00441 \text{ in.}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{15} \ [-1 \ \ 1] \begin{Bmatrix} 0 \\ 0.00441 \end{Bmatrix}$$

$$\sigma^{(1)} = 8812 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{15} \ [-1 \ \ 1] \begin{Bmatrix} 0.00441 \\ 0.00825 \end{Bmatrix}$$

$$\sigma^{(2)} = 7688 \text{ psi (T)}$$

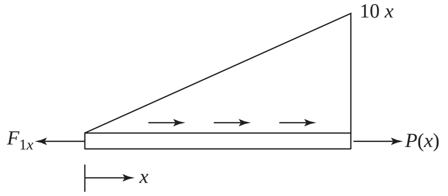
$$\sigma^{(3)} = \frac{30 \times 10^6}{15} \ [-1 \ \ 1] \begin{Bmatrix} 0.00825 \\ 0.01097 \end{Bmatrix}$$

$$\sigma^{(3)} = 5440 \text{ psi (T)}$$

$$\sigma^{(4)} = \frac{30 \times 10^6}{15} \ [-1 \ \ 1] \begin{Bmatrix} 0.01097 \\ 0.012 \end{Bmatrix}$$

$$\sigma^{(4)} = 2060 \text{ psi (T)}$$

Exact solution



$$F_{1x} = \frac{1}{2}\ (60'')\ (600\ \frac{\text{lb}}{\text{in.}})$$

$$F_{1x} = 18000\ \text{lb}$$

$$P(x) = 18000 - \frac{1}{2}\ (10\ x)\ x$$

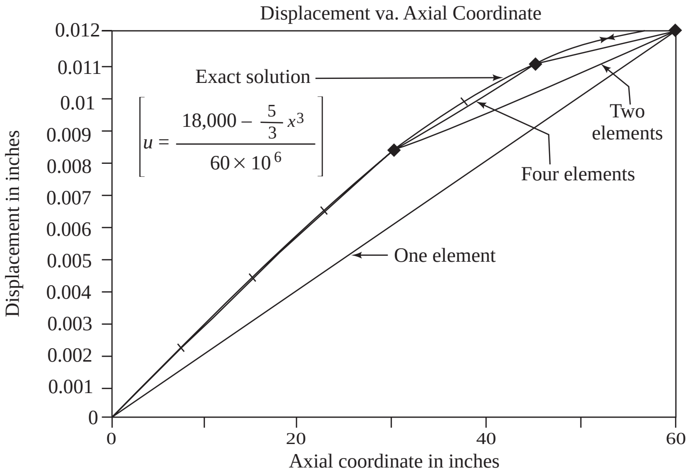
$$P(x) = 18000 - 5\ x^2$$

$$u(x) = \int_0^x \frac{(18000 - 5\ x^2)}{AE}\ dx$$

$$= \frac{1}{AE}\left[18000\ x - \frac{5}{3}\ x^3\right] + C$$

$$u(0) = 0 = C$$

$$u = \frac{18000 - \frac{5}{3}\ x^3}{60 \times 10^6}$$

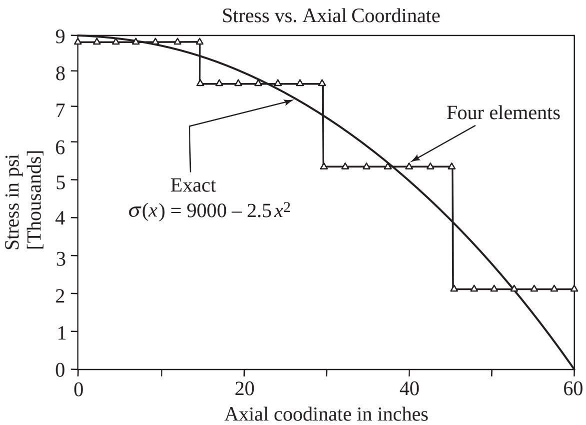


Analytical comparison with FEM

Element stress

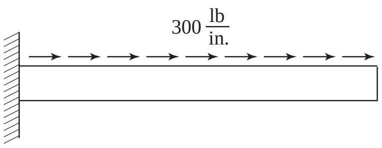
Exact $\sigma(x)$

$$\sigma(x) = \frac{P(x)}{A} = 9000 - 2.5\ x^2$$



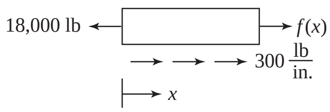
Analytical comparison with FEM

3.55



Analytical solution

$F(x)$ at wall = – 300 (60) = – 18000 lb



$$\therefore \quad f(x) = 18000 - 300\,x$$

$$\delta x = \text{displacement}$$

$$\sigma(x) = \frac{f(x)}{A} = \frac{18000 - 300\,x}{2}$$

$$\Rightarrow \quad \sigma(x) = 9000 - 150\,x \qquad (1)$$

$$\delta(x) = \int_0^x \frac{\sigma(x)}{E}\,dx$$

$$\Rightarrow \quad \delta(x) = \frac{1}{E}\left[\frac{-150\,x^2}{2} + 9000\,x\right] + C$$

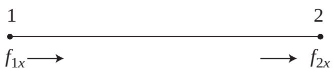
Applying the boundary conditions

$$\delta(x = 0) = 0 = C$$

$$\delta(x) = -2.5 \times 10^{-6}\,x^2 + 3 \times 10^{-4}\,x \qquad (2)$$

Finite element solutions

(i) One element



Replace the distributed force with a concentrated force.

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \end{Bmatrix}$$

$$F_{1x} = F_{2x} = \frac{1}{2}\,F = \frac{1}{2} \times 300 \times 60 = 9000 \text{ lb}$$

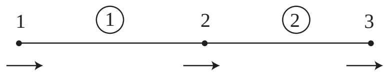
Solving for d_{2x}

$$u_2 = \frac{F_{2x}\,L}{A\,E} = \frac{9000 \times 60}{2 \times 30 \times 10^6}$$

$$u_2 = 0.009 \text{ in.}$$

$$\sigma = \frac{f_{2x}}{A} = \frac{9000}{2} = 4500 \text{ psi}$$

(2) Two elements



$$f_{1x} = \frac{300 \times 30}{2}$$

$$f_{2x} = \frac{300 \times 30}{2}(1+1)$$

$$f_{3x} = \frac{300 \times 30}{2}$$

$$f_{1x} = 4500 \text{ lb}$$

$$f_{2x} = 9000 \text{ lb}$$

$$f_{3x} = 4500 \text{ lb}$$

Global equation

$$\begin{Bmatrix} F_{1x} = 4500 \\ F_{2x} = 9000 \\ F_{3x} = 4500 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving the two equations

$$u_2 = 6.75 \times 10^{-3} \text{ in.}$$

$$u_3 = 0.009 \text{ in.}$$

$$\sigma^{(1)} = \frac{E}{L} [-1 \ \ 1] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{30} [-1 \ \ 1] \begin{Bmatrix} 0 \\ 6.75 \times 10^{-3} \end{Bmatrix}$$

$$\Rightarrow \quad \sigma^{(1)} = 6750 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{30} [-1 \ \ 1] \begin{Bmatrix} 6.75 \times 10^{-2} \\ 0.009 \end{Bmatrix}$$

$$\Rightarrow \quad \sigma^{(2)} = 2250 \text{ psi (T)}$$

Computer solutions

One element

NUMBER OF ELEMENTS (NELE) = 1

NUMBER OF NODES (KNODE) = 2

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
9.000000E+03	0.000000E+00	0.000000E+00

ELEMENTS

K	NODE (I, K)		E(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.45000E+04

Two elements

NUMBER OF ELEMENTS (NELE) = 2
NUMBER OF MODES (KNODE) = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	3.000000E+01	0.000000E+00	0.000000E+00
3	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
9.000000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00

ELEMENTS

K	MODE (I, K)		K(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00
2	2	3	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.6750E–02	0.0000E+00	0.0000E+00
NODE NUMBER 3	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENT (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.67500E+04
2 =	0.22500E+04

Four elements

NUMBER OF ELEMENTS (NELE) = 4
NUMBER OF MODES (KNODE) = 5

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00
2	0 1 1	1.500000E+01	0.000000E+00	0.000000E+00
3	0 1 1	3.000000E+01	0.000000E+00	0.000000E+00
4	0 1 1	4.500000E+01	0.000000E+00	0.000000E+00
5	0 1 1	6.000000E+01	0.000000E+00	0.000000E+00

FORCE (1, K)	FORCE (2, K)	FORCE (3, K)
0.000000E+00	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
4.500000E+03	0.000000E+00	0.000000E+00
2.250000E+03	0.000000E+00	0.000000E+00

ELEMENTS

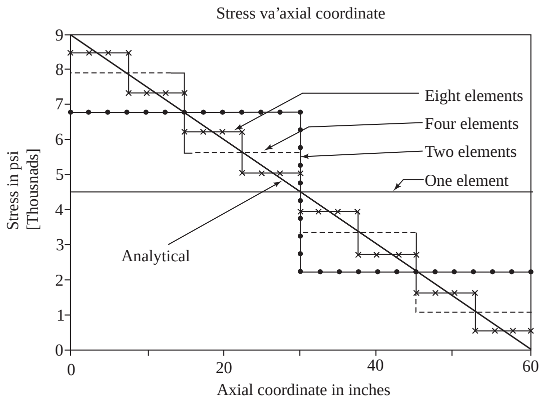
K	MODE (I, K)		K(K)	A(K)
1	1	2	3.0000E+07	2.0000E+00
2	2	3	3.0000E+07	2.0000E+00
3	3	4	3.0000E+07	2.0000E+00
4	4	5	3.0000E+07	2.0000E+00

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD) = 5

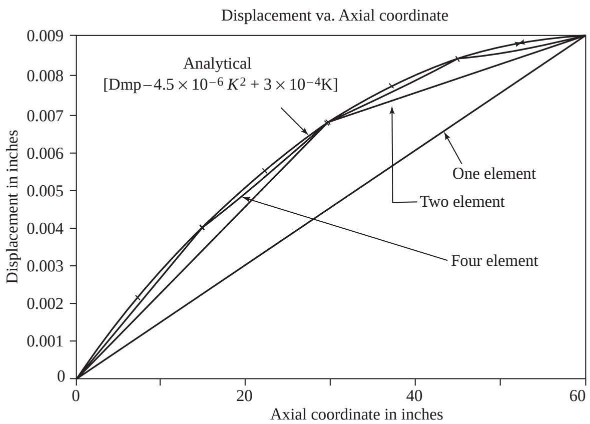
DISPLACEMENTS	X	Y	Z
NODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
NODE NUMBER 2	0.3937E–02	0.0000E+00	0.0000E+00
NODE NUMBER 3	0.6750E–02	0.0000E+00	0.0000E+00
NODE NUMBER 4	0.8437E–02	0.0000E+00	0.0000E+00
NODE NUMBER 5	0.9000E–02	0.0000E+00	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.78750E+04
2 =	0.56250E+04
3 =	0.33750E+04
4 =	0.11250E+04

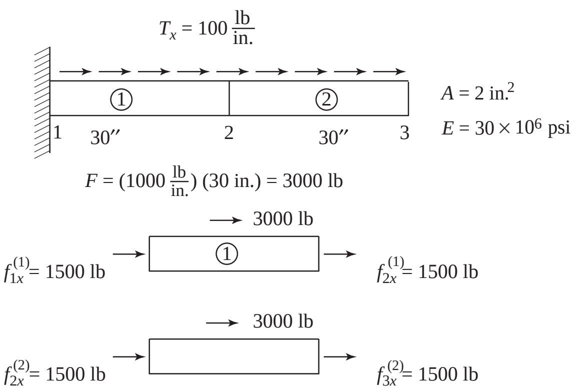


Analytical comparison with FEM



Analytical comparison with FEM

3.56



$$\{F\} = \begin{Bmatrix} F_{1x} + 1500 \text{ lb} \\ 1500 + 1500 \\ F_{3x} + 1500 \end{Bmatrix}$$

$$[k^{(1)}] = \frac{AE}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = \frac{AE}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global equations

$$\frac{(2)(30\times10^6)}{30}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1=0\\ u_2\\ u_3=0\end{Bmatrix}=\begin{Bmatrix}F_{1x}+1500\\ 3000\\ F_{3x}+1500\end{Bmatrix}$$

Solving Equation (2)

$2\times10^6\,(2\,u_2) = 3000$

$$u_2 = 0.75\times10^{-3}\text{ in.}$$

Element stresses

$$\sigma^{(1)} = [C']\,\{d\} = \frac{E}{L}\,[-C \quad -S \quad C \quad S]\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\end{Bmatrix}$$

$$= \frac{30\times10^6}{30}\,[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix}0\\ 0\\ 0.75\times10^{-3}\\ 0\end{Bmatrix}$$

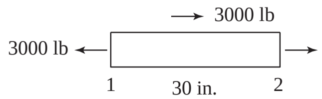
$$\Rightarrow \sigma^{(1)} = 750\text{ psi (T)}$$

$$F_{1x} + 1500 = 2\times10^6\,(-1)\,(0.75\times10^{-3})$$

$$\Rightarrow F_{1x} = -3000\text{ lb} \qquad (\leftarrow)$$

and $\qquad F_{3x} = -3000\text{ lb} \qquad (\leftarrow)$

Total applied force = $60\times100 = 6000$ lb ($\rightarrow$)

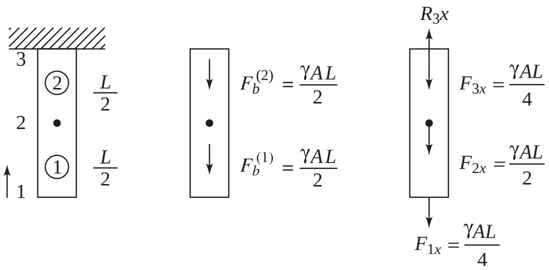


$$\sigma = 0\text{ (at node 2)}$$

$$\sigma\,(x = 15'') = \frac{3000-1500}{2} = 750\text{ psi}$$

3.57 Bar hanging under own weight

Two element solution



$$\frac{AE}{\frac{L}{2}}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3=0\end{Bmatrix}=\begin{Bmatrix}\frac{\gamma AL}{4}\\ \frac{\gamma AL}{2}\\ -R_{3x}+\frac{\gamma AL}{4}\end{Bmatrix}$$

$$u_1 - u_2 = \frac{\gamma AL^2}{2\,AE}\left(\frac{1}{4}\right) \qquad (1)$$

$$-u_1 + 2u_2 = \frac{\gamma AL^2}{2\,AE}\left(\frac{1}{2}\right) \qquad (2)$$

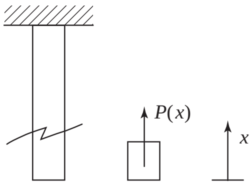
Adding (1) and (2)

$$u_2 = \frac{\gamma L^2}{2\,E}\left(\frac{3}{4}\right)$$

$$\Rightarrow \qquad u_2 = \frac{3\,\gamma L^2}{8\,E}(\downarrow)$$

$$u_1 = \frac{\gamma L^2}{8\,E} + \frac{3\,\gamma L^2}{8E} = \frac{\gamma L^2}{2\,E} \ (\downarrow)$$

Analytical solution



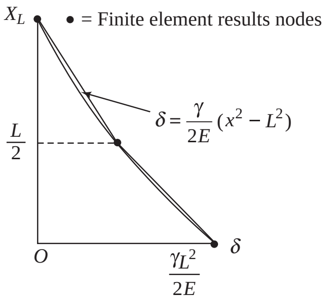
$$W_x = \gamma V(x) = \gamma Ax = P(x)$$

$$\delta = \int_0^x \frac{P(x)}{AE}\,dx = \int_0^x \frac{\gamma\,A\,x}{AE}\,dx$$

$$\delta = \frac{\gamma\,x^2}{2\,E} + C$$

$$\delta(L) = 0 = \frac{\gamma L^2}{2\,E} + C \Rightarrow C = -\frac{\gamma L^2}{2\,E}$$

$$\therefore \quad \delta = \frac{\gamma x^2}{2\,E} - \frac{\gamma L^2}{2\,E} = \frac{\gamma}{2\,E}\,(x^2 - L^2)$$



$$\sigma^{(1)} = \frac{E}{L}\,[1 \ \ -1]\begin{Bmatrix} u_1 = \frac{\gamma L^2}{2\,E} \\ u_2 = \frac{3\,\gamma L^2}{8\,E} \end{Bmatrix}$$

$$= \gamma L = \left[\frac{1}{2} - \frac{3}{8}\right]$$

$$\sigma^{(1)} = \frac{\gamma L}{8} \text{ (T)}$$

$$\sigma^{(2)} = \frac{E}{L}\,[1 \;\; -1] \begin{Bmatrix} u_2 = \frac{3\gamma L^2}{8E} \\ u_3 = 0 \end{Bmatrix}$$

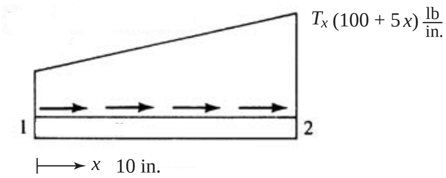
$$\sigma^{(2)} = \frac{3\gamma L}{8} \text{ (T)}$$

$$-R_{3x} = \frac{\gamma A L}{4} + \frac{2A\cancel{E}}{\cancel{L}}\left(-\frac{3\gamma L^{\cancel{2}}}{8\cancel{E}}\right)$$

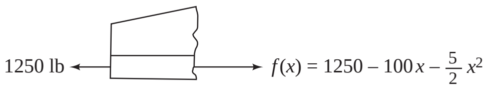
$$R_{3x} = \frac{\gamma A L}{4} + \frac{3}{4}\,\gamma A L$$

$$R_{3x} = \gamma A L$$

3.58



$$\text{Total } T_x = 100 \times 10 + \frac{1}{2} \times 5(10)^2 \text{ lb} = 1250 \text{ lb}$$



$$f_{1x}u_1 + f_{2x}u_2 = \int_0^{10} (100 + 5x)\left[\left(\frac{u_1 - u_2}{10}\right)x + u_1\right]dx$$

$$= \int_0^{10}\left[10\,(u_2 - u_1)\,x + 100\,u_1 + \frac{(u_2 - u_1)}{2}x^2 + 5u_1 x\right]dx$$

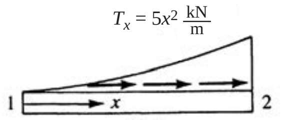
$$= 500\,(u_2 - u_1) + 1000\,u_1 + \frac{(u_2 - u_1)\,1000}{6} + 250\,u_1$$

Let $u_1 = 1$; $u_2 = 0$

$$\therefore \quad f_{1x} = -500 + 1000 - \frac{1000}{6} + 250 = 583.3$$

$$\therefore \quad f_{1x} = 583.33 \text{ lb}$$

(b)



$$f_{1x}\,u_1 + f_{2x}\,u_2 = \int_0^4 (5x^2)\left[\frac{u_2 - u_1}{4}\,x + u_1\right]$$

$$= \frac{5}{16}\,(u_2 - u_1)\,(4^4) + \frac{5\,u_1\,(4)^3}{3}$$

Let $u_1 = 0$; $u_2 = 1$

$$f_{2x} = (16)\,5 = 80 \text{ kN} = f_{2x}$$

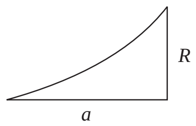
$$u_2 = 0;\ u_1 = 1$$

$$f_{1x} = -(16)\,5 + \frac{5(4)^3}{3} + 26.67 \text{ kN} = f_{1x}$$

$$f_{1x} + f_{2x} = 106.67 \text{ kN (Total force)}$$

Check parabolic yields

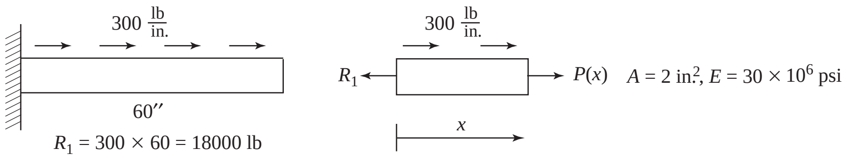
$$A = \frac{a\,R}{3}$$



$$= \frac{(4)\,\left[5\,(4^2)\right]}{3}$$

$$= 106.67 \text{ kN total force}$$

3.59



Exact solution

$$P(x) = 18000 - 300\,x$$

$$u = \frac{1}{AE}\int_0^x P(x)\,dx$$

$$= \frac{1}{AE}\int_0^x (18000 - 300\,x)\,dx$$

$$u = \frac{1}{AE}\left(18000\,x - 300\frac{x^2}{2}\right) + C_1$$

$$u\,(0) = 0 \Rightarrow C_1 = 0$$

$$u = \frac{1}{AE}\ (18000\,x - 150\,x^2)$$

$$u = \frac{1}{2 \times 30 \times 10^6}\ (18000\,x - 150\,x^2)$$

Now choose $u = C_0 + C_1 x + C_2 x^2$ (1)

$$\frac{du}{dx} = C_1 + 2C_2 x$$

Differential equation $AE\,\frac{du}{dx} - P(x) = 0$ (2)

$$u(0) = 0 \Rightarrow C_0 = 0$$

Substituting (1) into (2) to yield R (error function)

$AE\,[C_1 + 2C_2\,x] - (18000 - 300\,x) = R$ (3)

By collocation:

2 unknowns so evaluate R at 2 points

a + $x = \frac{L}{2}$ and $x = L$

$$R\left(C, x = \frac{L}{2} = 0\right) = AE\left[C_1 + 2C_2\,\frac{L}{2}\right] - 18000 + 300\,\frac{L}{2} = 0$$

$R(C, x = L) = 0 = AE\,[C_1 + 2C_2\,L] - 18000 + 300\,L = 0$ (4)

Simplifying (4)

$-AE\,[C_1 + C_2\,L] + 18000 - 150\,L = 0$ (5.1)

$AE\,[C_1 + 2C_2\,L] - 18000 + 300\,L = 0$ (5.2)

Substituting (5.1) from (5.2)

$$AE\,C_2\,\cancel{L} + 0 + 150\,\cancel{L} = 0$$

$C_2 = \frac{-150}{AE}$ (6)

Substituting (6) into (5.1)

$$AE\left[C_1 + \frac{-150}{AE}\,L\right] - 18000 + 150\,L = 0$$

$$AEC_1 = 18000 - \cancel{150L} + \cancel{150L} = 0$$

$C_1 = \frac{18000}{AE}$ (7)

$u = \frac{18000}{AE}\,x + \frac{150}{AE}\,x^2 = \frac{1}{AE}(18000\,x - 150\,x^2)$ This gives the result which is same as the exact solution of differential equation.

Subdomain method:

Use 2 subintervals as 2 C's

$$\int_0^{L/2} R\,dx = 0 = \int_0^{L/2} \{AE\,[C_1 + 2C_2\,x] - 18000 + 300x\}dx = 0$$

$$\int_{L/2}^{L} R dx = 0 = \int_{L/2}^{L} \{AE\,[C_1 + 2C_2x] - 18000 + 300x\}dx = 0 \quad (8)$$

$$\left\{AE\left[C_1x + 2C_2\frac{x^2}{2}\right] - 18000x + 300\frac{x^2}{2}\right\}\Bigg|_0^{L/2} = 0$$

$$\left\{AE\left[C_1x + 2C_2\frac{x^2}{2}\right] - 18000x + 150\,x^2\right\}\Bigg|_{L/2}^{L} = 0 \quad (9)$$

$$AE\left[C_1\frac{L}{2} + C_2\left(\frac{L}{2}\right)^2\right] - 18000\left(\frac{L}{2}\right) + 150\left(\frac{L}{2}\right)^2 = 0$$

and $$AE\left[C_1\left(L - \frac{L}{2}\right) + C_2\left(L^2 - \left(\frac{L}{2}\right)^2\right)\right] - 18000\left(L - \frac{L}{2}\right) + 150\left[L^2 - \left(\frac{L}{2}\right)^2\right] = 0 \quad (10)$$

Simplifying (10)

$$-AE\left[C_1\frac{L}{2} + C_2\left(\frac{L}{2}\right)^2\right] - 9000\,L + 150\left(\frac{L}{2}\right)^2 = 0 \quad (11.1)$$

$$AE\left[C_1\frac{L}{2} + C_2\frac{3L^2}{4}\right] - 9000\,L + 150\left(\frac{3L^2}{4}\right) = 0 \quad (11.2)$$

Substituting (11.1) from (11.2)

$$AE\,C_2\,\frac{L^{\cancel{2}}}{\cancel{2}} + 0 + 150\,\frac{L^{\cancel{2}}}{\cancel{2}} = 0$$

$$C_2 = \frac{-150}{AE} \quad (12)$$

Substituting (12) into (11.1)

$$AE\left[C_1\,\frac{L}{2} + \cancel{\left(\frac{-150}{AE}\right)\left(\frac{L}{2}\right)^2}\right] - 9000\,\cancel{L} + 150\,\cancel{\left(\frac{L}{2}\right)^2} = 0$$

$$C_1 = \frac{18000}{AE} \quad (13)$$

Same values for C_1 and C_2 as previous solution. Same u as exact solution

Least square method:

2 C's need 2 integrals

$$\int_0^L R\,\frac{\partial R}{\partial C_1}\,dx = 0 \quad \text{by (3.13.10)} \quad (14)$$

$$\frac{\partial R}{\partial C_1}\ \underline{\underline{(3)}}\ AE\,(1) \text{ and } \frac{\partial R}{\partial C_2} = AE\,2x \quad (15)$$

$$\int_0^L \{AE\,[C_1 + 2C_2x] - 18000 + 300\,x\}\,\cancel{AE}\,dx = 0$$

$$\int_0^L [AE\,(C_1 + 2C_2x] - 18000 + 300\,x\}\,AE\,2x\,dx = 0 \quad (16)$$

Integrating and simplifying

$$-\frac{x}{2}\,AE\,[C_1 x + \cancel{2}\,C_2\,\frac{x^2}{\cancel{2}}] - 18000\,x + \frac{300\,x^2}{2} = 0$$

$$AE\left[C_1\frac{x^2}{2} + 2C_2\frac{x^3}{3}\right] - 18000\,\frac{x^2}{2} + 300\,\frac{x^3}{3} = 0 \tag{17}$$

$$AE\left[\frac{2}{3}x^3 - \frac{x^3}{2}\right]C_2 + 0 + 300\left(\frac{x^3}{3} - \frac{x^3}{4}\right) = 0$$

$$AE\;\cancel{x^3}\,\frac{1}{6}\;C_2 + 300\;\cancel{x^3}\left(\frac{1}{12}\right) = 0$$

$$C_2 = \frac{-300}{12}\,\frac{6}{AE} = \frac{-150}{AE} \tag{18}$$

Substituting (18) into (17)

$$AE\left[C_1\cancel{x} + \cancel{\left(\frac{-150}{AE}\right)}x^2\right] - 18{,}000\,\cancel{x} + 150\,\cancel{x}^2 = 0$$

$$C_1 = \frac{18000}{AE} \tag{19}$$

Galerkin’s method:

$$\int R\,W_i\,dx = 0 \quad \text{by Equation (3.13.13)} \tag{20}$$

Need 2 equations

$$\text{Let } W_1 = x \text{ and } W_2 = x^2 \tag{21}$$

$$\int_0^L \{AE\,[C_1 + 2C_2 x] - 18000 + 300\,x]\,x\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2 x] - 18000 + 300\,x\}\,x^2\,dx = 0 \tag{22}$$

$$\left\{AE\left[C_1\frac{x^2}{2} + 2C_2\frac{x^3}{3}\right] - 18000\frac{x^2}{2} + 300\frac{x^3}{3}\right\}_0^L = 0$$

$$\left\{AE\left[C_1\frac{x^3}{3} + 2C_2\frac{x^4}{4}\right] - 18000\frac{x^3}{3} + 300\frac{x^4}{4}\right\}\Bigg|_0^L = 0$$

$$\frac{-2L}{3}\;AE\left[C_1\frac{L^2}{2} + \frac{2}{3}C_2 L^3\right] - 9000\,L^2 + 100\,L^3 = 0$$

$$AE\left[C_1\frac{L^3}{3} + C_2\frac{L^4}{2}\right] - 6000\,L^3 + \frac{300}{4}\;L^4 = 0 \tag{23}$$

$$AE\left[\frac{-4}{9}C_2\;\cancel{L^4} + C_2\,\frac{\cancel{L^4}}{2}\right] + 0 + \left[\frac{300}{4} - \frac{200}{3}\right]\cancel{L^4} = 0$$

$$AE\left[\left(\frac{-8+9}{18}\right)C_2\right] + \frac{900-800}{12} = 0$$

$$C_2 = -\left(\frac{100}{\cancel{12}}\right)\frac{\overset{1.5}{\cancel{18}}}{AE}$$

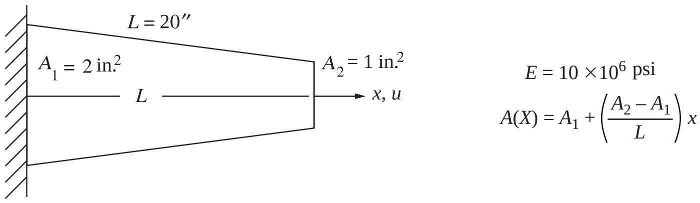
$$C_2 = \frac{-150}{AE} \tag{24}$$

(24) into (23)

$$AE\left[C_1\frac{L^2}{2}+\frac{2}{3}\left(\cancel{\frac{-150}{AE}}L^3\right)\right]-9000\,L^3+\cancel{100\,L^3}=0$$

$$C_1 = \frac{18000}{AE} \tag{25}$$

3.60



Exact solution

$$P - EA(x)\,\frac{du}{dx} = 0$$

$$du = \frac{P\,dx}{EA\,(x)}$$

$$\int_0^x du = \int_0^x \frac{P\,dx}{EA\,(x)}\,dx$$

$$u(x) = \int_0^x \frac{P\,dx}{E\left[A_1+\left(\frac{A_2-A_1}{L}\right)x\right]}$$

$$\therefore\quad u(x) = \frac{PL}{E(A_2-A_1)}\left\{ln\left[A_1+\left(\frac{A_2-A_1}{L}\right)x\right]-ln\,A_1\right\}$$

x, in.	$u(x)$, in.
0	0
6.66	3.642×10^{-4}
13.32	8.099×10^{-4}
19.98	1.384×10^{-3}

Collocation method:

Let $u(x) = C_1x + C_2x^2 + C_3x^3$

$u(0) = 0$ satisfied

3 C_i*'s*. So need error to go to zero at 3 points.

$$x = \frac{L}{3},\ \frac{2L}{3},\ L$$

$$A(x)\,E\,\frac{du}{dx} - P = 0$$

$$\therefore \qquad \left[A_1 + \left(\frac{A_2 - A_1}{L}\right)x\right]E(C_1 + 2C_2 x + 3C_3 x^2) - P = R \text{ } (R = \text{the error})$$

$$\therefore \qquad R(C, x)\Big|_{x=\frac{L}{3}} = 0,\ R(C, x)\Big|_{x=\frac{2L}{3}} = 0,\ R(C, x)\Big|_{x=L} = 0$$

3 equations follow using Mathcad

Given

$$\left(2 - \frac{1\left(\frac{20}{3}\right)}{20}\right)E\left[C_1 + 2\,C_2\left(\frac{20}{3}\right) + 3\,C_3\left(\frac{20}{3}\right)^2\right] - 1000 = 0$$

$$\left[2 - \left(\frac{1}{20}\right)\left(\frac{40}{3}\right)\right]E\left[C_1 + 2\,C_2\left(\frac{40}{3}\right) + 3\,C_3\left(\frac{40}{3}\right)^2\right] - 1000 = 0$$

$$[2 - (1)]\,E\,(C_1 + 2\,C_2\ 20 + 3\,C_3\,20^2) - 1000 = 0$$

$$\text{Find }(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{11}{200000} \\ 0 \\ \frac{3}{80000000} \end{pmatrix}$$

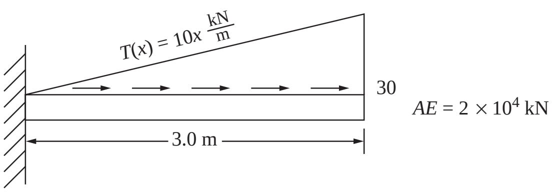
$$C_1 = \frac{11}{200000};\qquad C_2 = 0;\qquad C_3 = \frac{3}{80000000}$$

$$uc(x) = C_1\,x + C_2\,x^2 + C_3\,x^3$$

$x =$	Collocation solution $uc(x) =$	Exact solution $u(x) =$
0	0	0
6.66	3.774×10^{-4}	3.642×10^{-4}
13.32	8.212×10^{-4}	8.099×10^{-4}
19.98	1.398×10^{-3}	1.384×10^{-3}

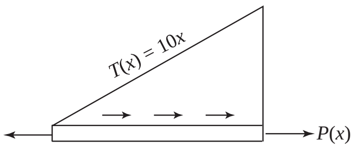
(For use of other methods see P. 3.59 and P. 3.61)

3.61



Exact solution

$$u = \frac{1}{AE}\int_0^x P(x)\,dx$$



$$R_1 = \frac{1}{2}\ (30 \times 3) = 45 \text{ kN}$$

$$P(x) = 45 - 10x\ \frac{x}{2} = 45 - 5x^2$$

$$u = \frac{1}{AE} \int_0^x (45 - 5x^2)\, dx$$

$$u = \frac{1}{AE}\left(45\,x - \frac{5x^3}{3}\right) + C$$

$$u(0) = 0 \Rightarrow C = 0$$

$$\therefore \quad u = \frac{1}{AE}\left(45\,x - \frac{5x^3}{3}\right) \qquad \text{(A)}$$

Collocation method:

$$\text{Let } u = C_1x + C_2x^2 + C_3x^3 \qquad \text{(B)}$$

$$AE\ \frac{du}{dx} - P(x) = 0 \qquad (1)$$

$$\frac{du}{dx} = C_1 + 2C_2x + 3C_3x^2 \qquad (2)$$

$$R = AE\ [C_1 + 2C_2\,x + 3C_3x^2] - (45 - 5x^2) = 0 \qquad (3)$$

3 *C's*, therefore, need 3 equations

$$R\left(C,\, x = \frac{L}{3}\right) = 0$$

$$R\left(C,\, x = \frac{2L}{3}\right) = 0 \qquad (4)$$

$$R\ (C,\ x = L) = 0$$

Substituting for *R* using Equation (3) into (4)

$$AE\left[C_1 + 2C_2\frac{L}{3} + 3C_3\left(\frac{L}{3}\right)^2\right] - 45 + 5\left(\frac{L}{3}\right)^2 = 0$$

$$AE\left[C_1 + 2C_2\left(\frac{2L}{3}\right) + 3C_3\left(\frac{2L}{3}\right)^2\right] - 45 + 5\left(\frac{2L}{3}\right)^2 = 0$$

$$AE\ [C_1 + 2\ C_2L + 3\ C_3\,L^2\,] - 45 + 5\ L^2 = 0 \qquad (5)$$

Solving for $C_1 - C_3$ in Mathcad

Given

$$AE\left(3C_3\,\frac{L^2}{9} + C_1 + 2\,C_2\,\frac{L}{3}\right) = 45 - 5\ \frac{L^2}{9}$$

$$AE\left(3\,C_3\,4\,\frac{L^2}{9}+C_1+2\,C_2\,2\,\frac{L}{3}\right)=45-5\times 4\times\frac{L^2}{9}$$

$$AE\;(C_1+2\,C_2\;L+3\,C_3\;L^2)=45-5\,L^2$$

$$\text{Find }(C_1, C_2, C_3)\rightarrow\begin{pmatrix}\frac{45}{AE}\\ 0\\ \frac{-5}{3\,AE}\end{pmatrix}\qquad(6)$$

Equation (6) into (B)

$$u=\frac{45\,x}{AE}+0+\frac{-5\,x^3}{3\,AE}\qquad(7)$$

Equation (7) is identical to exact solution given by Equation (A).

Subdomain method:

3*C's*, 3 intervals needed

$$\int_0^{L/3} Rd_x=0$$

$$0=\int_0^{L/3}\{AE\,[C_1+2C_2x+3C_3x^2]-(45-5x^2)\}dx$$

$$0=\int_{L/3}^{2L/3}\{AE\,[C_1+2C_2x+3C_3x^2]-(45-5x^2)\}dx$$

$$0=\int_{2L/3}^{L}\{AE\,[C_1+2C_2\,x+3C_3x^2]-(45-5x^2)\}dx\qquad(8)$$

Integrate and simplify (8)

$$AE\left[C_1\,\frac{L}{3}+C_2\left(\frac{L}{3}\right)^2+C_3\left(\frac{L}{3}\right)^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{L}{3}\right)^3=0$$

$$AE\left[C_1\,\frac{L}{3}+C_2\left(\frac{1}{3}L^2\right)+C_3\,\frac{7}{27}\,L^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{7}{27}\right)L^3=0$$

$$AE\left[C_1\,\frac{L}{3}+C_2\,\frac{5}{9}L^2+C_3\left(\frac{19}{27}\right)L^3\right]-45\,\frac{L}{3}+\frac{5}{3}\left(\frac{19}{27}\right)L^3=0\qquad(9)$$

Solve using Mathcad for C_1-C_3

Given

$$AE\left[C_1\left(\frac{L}{3}\right)+C_2\left(\frac{L}{3}\right)^2+C_3\left(\frac{L}{3}\right)^3\right]-45\left(\frac{L}{3}\right)+\frac{5}{3}\left(\frac{L}{3}\right)^3=0$$

$$\frac{1}{3}\,AE\;C_1\,L+\frac{1}{3}\,AE\,C_2\,L^2+\frac{7}{27}\,AE\,C_3\,L^3-15\,L+\frac{35}{81}\;L^3=0$$

$$\frac{1}{3}\,AE\;C_1\,L+\frac{5}{9}\,AE\;C_2\,L^2+\frac{19}{27}\,AE\,C_3\,L^3-15\,L+\frac{95}{81}\,L^3=0$$

$$\text{Find }(C_1, C_2, C_3)\rightarrow\begin{pmatrix}\frac{45}{AE}\\ 0\\ \frac{-5}{3\,AE}\end{pmatrix}\quad\text{Same C's as in collocation method}\qquad(10)$$

Least squares method:

3 C's need 3 integrals

$$\int_0^L R\,\frac{\partial R}{\partial C_i}\,dx = 0 \quad \text{by (3.13.10)} \tag{11}$$

$$\frac{\partial R}{\partial C_1} = AE, \qquad \frac{\partial R}{\partial C_2} = AE2x$$

$$\frac{\partial R}{\partial C_3} = AE3x^2 \tag{12}$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2]\,AE\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\,AE\,2x\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\,AE\,3x^2\,dx = 0 \tag{13}$$

Simplifying (13)

$$AE\,[C_1L + C_2L^2 + C_3L^3 - 45\,L + \frac{5}{3}\,L^3 = 0$$

$$AE\left[C_1\frac{L^2}{2} + \frac{2C_2L^3}{3} + \frac{3C_3L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5\,L^4}{4} = 0$$

$$AE\left[C_1\frac{L^3}{3} + C_2\frac{L^4}{2} + \frac{3}{5}C_3\,L^5\right] - \frac{45}{3}L^3 + L^5 = 0 \tag{14}$$

Solving (14) for $C_1 - C_3$ using Mathcad.

Least squares method:

Given

$$AE\,(C_1\,L + C_2\,L^2 + C_3\,L^3) - 45\,L + \left(\frac{5}{3}\right)L^3 = 0$$

$$AE\left[C_1\left(\frac{L^2}{2}\right) + 2\,C_2\left(\frac{L^3}{3}\right) + 3\,C_3\left(\frac{L^4}{4}\right)\right] - 45\left(\frac{L^2}{2}\right) + 5\left(\frac{L^4}{4}\right) = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + C_2\left(\frac{L^4}{2}\right) + \left(\frac{3}{5}\right)C_3\,L^5\right] - \frac{45}{3}L^3 + L^5 = 0$$

Find (C_1, C_2, C_3) $\begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$ Same C's as in other methods

Galerkin's method:

$$\int_0^L R\,W_i\,dx = 0 \qquad (3.13.13)$$

Need 3 equations

Let $W_1 = x$, $W_2 = x^2$, $W_3 = x^3$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\; x\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\; x^2\,dx = 0$$

$$\int_0^L \{AE\,[C_1 + 2C_2x + 3C_3x^2] - 45 + 5x^2\}\; x^3\,dx = 0$$

$$AE\left[C_1\frac{x^2}{2} + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4}\right] - 45\frac{x^2}{2} + \frac{5x^4}{4}\Bigg|_0^L = 0$$

$$AE\left[C_1\frac{x^3}{3} + \frac{2C_2x^4}{4} + \frac{3C_3x^5}{5}\right] - 45\frac{x^3}{4} + \frac{5x^5}{5}\Bigg|_0^L = 0$$

$$AE\left[C_1\frac{x^4}{4} + \frac{2C_2x^5}{5} + \frac{3C_3x^6}{6}\right] - 45\frac{x^4}{4} + \frac{5x^6}{6}\Bigg|_0^L = 0$$

Simplifying

$$AE\left[C_1\frac{L^2}{2} + \frac{2C_2L^3}{3} + \frac{3C_3L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5}{4}L^4 = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + \frac{C_2L^4}{2} + \frac{3C_3L^5}{5}\right] - 15L^3 + L^5 = 0$$

$$AE\left[C_1\frac{L^4}{4} + \frac{2C_2L^5}{5} + \frac{C_3L^6}{2}\right] - \frac{45}{4}L^4 + \frac{5L^6}{6} = 0$$

Solve for $C_1 - C_3$ using Mathcad

Given

$$AE\left[C_1\left(\frac{L^2}{2}\right) + 2\,C_2\,\frac{L^3}{3} + 3\,C_3\,\frac{L^4}{4}\right] - 45\frac{L^2}{2} + \frac{5}{4}L^4 = 0$$

$$AE\left[C_1\left(\frac{L^3}{3}\right) + C_2\,\frac{L^4}{2} + 3\,C_3\,\frac{L^5}{5}\right] - 15\,L^3 + L^5 = 0$$

$$AE\left[C_1\frac{L^4}{4} + 2\,C_2\,\frac{L^5}{5} + C_3\,\frac{L^6}{2}\right] - 45\frac{L^4}{4} + 5\frac{L^6}{6} = 0$$

$$\text{Find }(C_1, C_2, C_3) \rightarrow \begin{pmatrix} \frac{45}{AE} \\ 0 \\ \frac{-5}{3\,AE} \end{pmatrix}$$

3.62

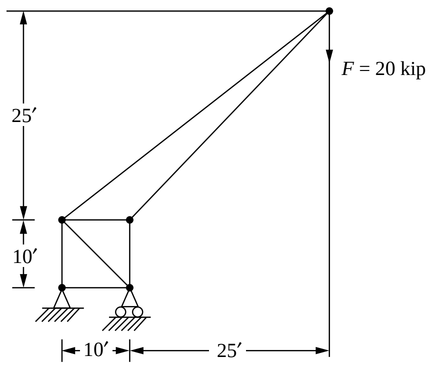


Figure P3–62 Derrick truss (FS = 4.0)

PRINT ELEMENT TABLE ITEMS PER ELEMENT
***** POST 1 ELEMENT TABLE LISTING ***
STAT CURRENT CURRENT

ELEM	SAXL	MFORX
1	$-3.3538E-12$	
2	50000	50000
3	$4.74E-12$	$4.74E-12$
4	-70000	-70000
5	-70000	-70000
6	86023	86023
7	-98995	-98995

MINIMUM	VALUES	
ELEM	7	7
VALUE	–98995	–98995
MAXIMUM	VALUES	
ELEM	6	6
VALUES	86023	86023

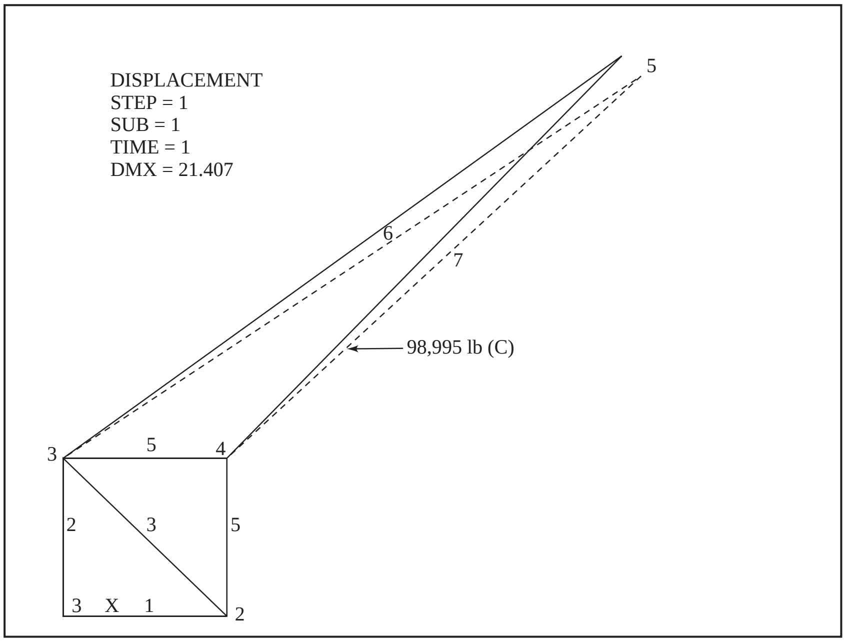


Figure 4 Free and deformed states

Try $W\,21 \times 122$ $A36$ steel based on critical buckling member 7. Assumed $L_e = 2.1\,L$ (conservative).

3.63

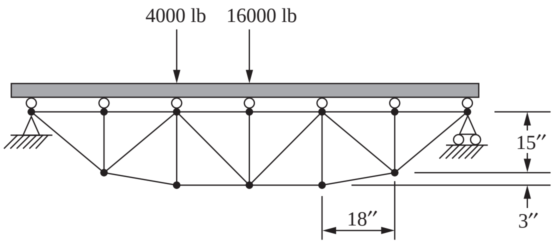
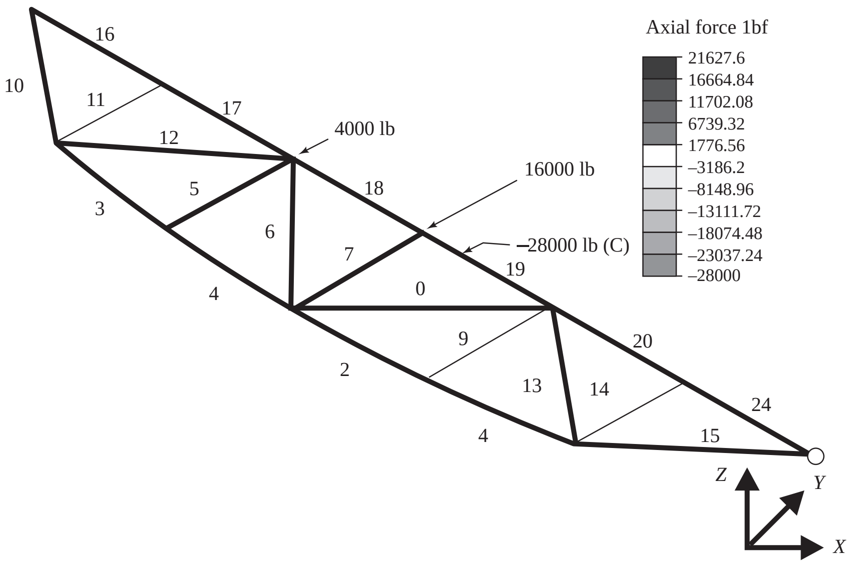


Figure P3–63 Truss bridge (FS = 3.0)



Load case: 1 of 1
Maximum value: 21627.6 lbf
Minimum value: –28000 lbf elements numbered
Try S 8×18.4 *A*36 steel
or $4.5'' \times 4.5'' \times \frac{5}{16}''$ square tube
(assume pin-pin ends of members)

3.64

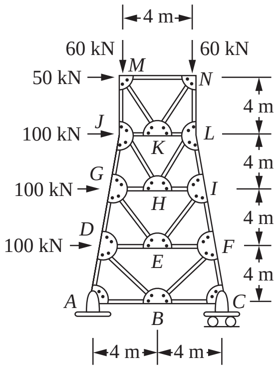


Figure P3–64 Tower (FS = 2.5)

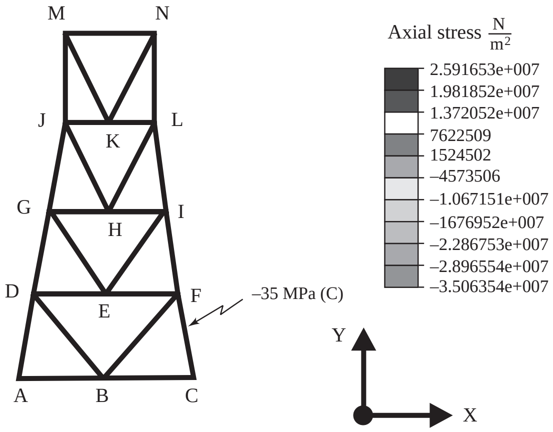


Figure 4 Stress analysis model

Try $S\ 460 \times 140$ (mm $\times \dfrac{\text{kg}}{\text{m}}$) *A*36 steel

$P_{\max}$ comp = – 466.35 kN in member CF

3.65

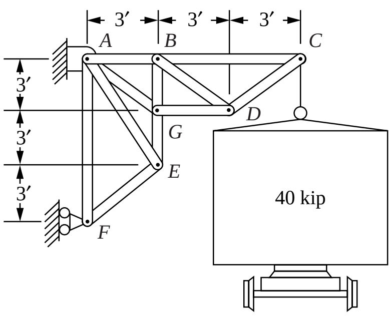


Figure P3–65 Boxcar lift (FS = 3.0)

Try one of these cross sections – 1) a square solid bar, 3.25 in. × 3.25 in., 2) a 6 × 6 × $\frac{1}{2}$ in. structural square tube, 3) a W 8 × 35 wide flange section. Any of these made of A 36 steel. The critical force of – 120,000 lb is in element *EG*. So Johnson buckling formula dictates the section selected.

3.66

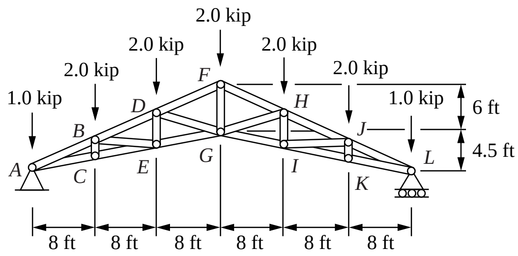


Figure P3–66 Howe scissors roof truss (FS = 2.0)

Try an *S* 6 × 12.5 or a 3 in. × 3 in. × $\frac{3}{6}$ in. structural square tube made of A 36 steel. The critical elements are *AB* and *JL* with force of –21,830 lb. Buckling dictates the cross section sizes recommended here.

3.67

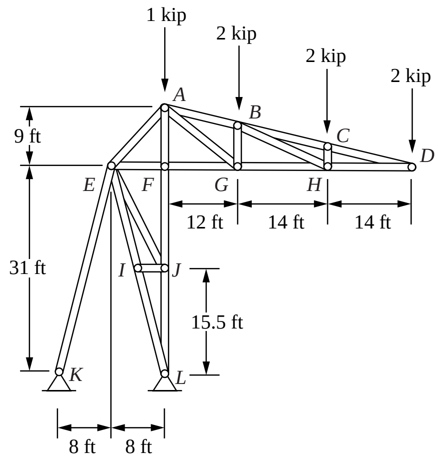
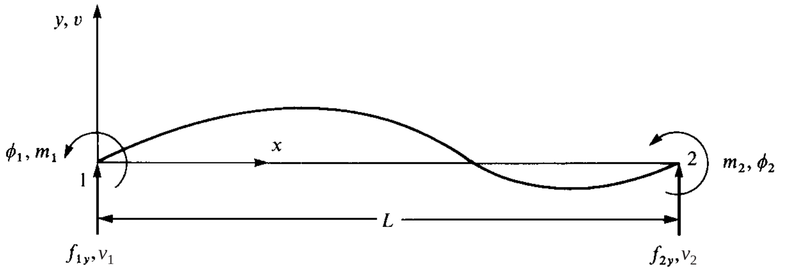


Figure P3–67 Stadium roof truss (FS = 3.0)

Try a W 6 × 25 of A 36 steel dictated by compressive force of –20,500 lb in elements *AF*, *FJ*, *JL*.

Chapter 4

4.1



Using Equation (4.1.7) plot N_1, $\frac{dN_2}{dx}$, N_3, $\frac{dN_4}{dx}$

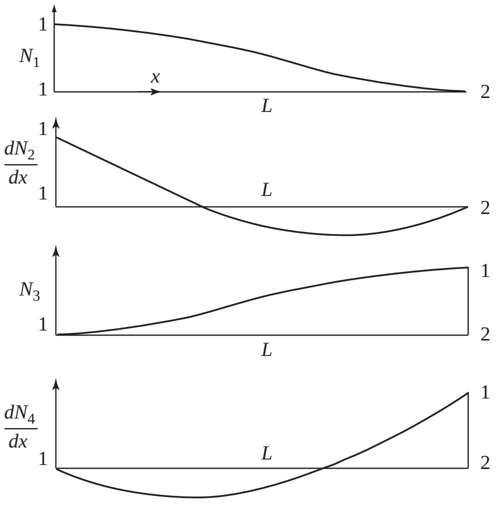
x	N_1	$\frac{dN_2}{dx}$	N_3	$\frac{dN_4}{dx}$
0	1	1	0	0
0.2L	0.896	0.32	0.104	–0.28
0.4L	0.648	–0.12	0.352	–0.32
0.6L	0.352	–0.32	0.648	–0.12
0.8L	0.104	–0.28	0.896	0.32
1.0L	0	0	1.00	1.00

where by Equation (4.1.7)

$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3)$$

$$N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \text{ etc.}$$

Plots



4.2

Assumed nodal sign conventions

Note: moments opposite those in text.

$$v(0) = v_1 = a_4$$

$$\frac{dv(0)}{dx} = -\phi_1 = a_3$$

$$v(L) = v_2 = a_1 L^3 + a_2 L^2 + \underset{a_3 L}{\overset{-\phi_1}{\Uparrow}} \quad \underset{a_4}{\overset{-v_1}{\Uparrow}}$$

$$\frac{dv(L)}{dx} = -\phi_2 = 3a_1 L^2 + 2a_2 L + a_3$$

$$v_2 + \frac{1}{2}\phi_2(L) = a_1\left(L^3 - \frac{3}{2}L^3\right) + a_2(0) + a_3 L - \frac{1}{2}a_3 L + a_4$$

$$v_2 + \frac{\phi_2 L}{2} - \frac{1}{2}(-\phi_1)L - v_1 = a_1\left(\frac{-1}{2}L^3\right)$$

$$a_1 = \frac{2}{L^3}\left(-v_2 - \frac{\phi_2 L}{2} - \frac{\phi_1 L}{2} + v_1\right)$$

$$v_2 = \frac{2}{\cancel{L^3}}\left(-v_2 - \frac{\phi_2 L}{2} - \frac{\phi_1 L}{2} + v_1\right)\cancel{L^3} + a_2 L^2 - \phi_1 L + v_1$$

$$a_2 L^2 = v_2 + 2v_2 + \phi_2 L + \phi_1 L - 2v_1 + \phi_1 L - v_1$$

$$\therefore a_2 = \frac{3}{L^2}(v_2 - v_1) + \frac{(2\phi_1 + \phi_2)}{L^{\cancel{2}}}\cancel{L}$$

$$v = \left[\frac{2}{L^3}(v_1 - v_2) - \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2$$

$$- \phi_1 x + v_1$$

Note: Terms with $\phi_i s$ are opposite signs from v in Equation (4.1.4)

$$f_{1y} = V = \frac{EI}{L^3}\frac{d^3 v(0)}{dx^3} = \frac{EI}{L^3}(12v_1 - 12v_2 - 6L\phi_1 - 6L\phi_2)$$

$$m_1 = m = EI\,\frac{d^2v(0)}{dx^2} = \frac{EI}{L^3}(-6Lv_1 + 6Lv_2 + 4L^2\phi_1 + 2L^2\phi_2)$$

$$f_{2y} = V = -EI\,\frac{d^3v(L)}{dx^3} = \frac{EI}{L^3}(-12v_1 + 12v_2 + 6\phi_1 L + 6\phi_2 L)$$

$$m_2 = -m = -EI\,\frac{d^2v(L)}{dx^2} = \frac{EI}{L^3}(-12Lv_1 + 12Lv_2 + 6\phi_1 L^2 +$$

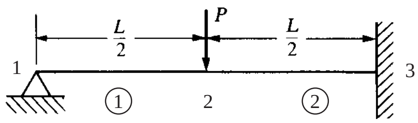
$$6\phi_2 L^2 + 6Lv_1 - 6Lv_2 - 4\phi_1 L^2 - 2\phi L^2)$$

$$m_2 = \frac{EI}{L^3}\;(-6Lv_1 + 6Lv_2 + 2L^2\phi_1 + 4L^2\phi_2)$$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Note: All 6L terms have opposite signs from Equation (4.1.13), m_1 and m_2 are now negative of previous results.

4.3



Let $\qquad \frac{L}{2} = l$

Element 1–2 $\qquad\qquad$ Element 2–3

$$[k_{1-2}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}; \quad [k_{2-3}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1y} = ? \\ M_1 = 0 \\ F_{2y} = -P \\ M_2 = 0 \\ F_{3y} = ? \\ M_3 = ? \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = ? \\ v_2 = ? \\ \phi_2 = ? \\ v_3 = 0 \\ \phi_3 = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 24 & 0 \\ 2l^2 & 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Rearrange

$$\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix} = \frac{EI}{l^3}\begin{array}{c} \begin{array}{cc} \alpha\alpha & \alpha\beta \end{array} \\ \left[\begin{array}{cc|c} 4l^2 & 2l^2 & -6l \\ 2l^2 & 8l^2 & 0 \\ \hline -6l & 0 & 24 \end{array}\right] \\ \begin{array}{cc} \beta\alpha & \beta\beta \end{array} \end{array} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ v_2 \end{Bmatrix}$$

Apply partition method

$$N = k_{\beta\beta} - k_{\beta\alpha}\ k_{\alpha\alpha}^{-1}\ k_{\alpha\beta}$$

$$= \frac{EI}{l^3}[24 - [-6l\ \ 0]\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}^{-1}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}] = 13.7148\frac{EI}{l^3}$$

$$d_\beta = N^{-1}\ F \Rightarrow v_2 = \frac{l^3}{13.7148\ EI}\ (-P)$$

$$= \frac{-7Pl^3}{96EI} \Rightarrow \boxed{v_2 = \frac{-7PL^3}{768EI}}$$

$$\{d_\alpha\} = -[k_{\alpha\alpha}^{-1}]\ [k_{\alpha\beta}]\ \{d_\beta\}$$

$$\Rightarrow \{d_\alpha\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = -\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}\left[\frac{-7Pl^3}{96EI}\right]$$

$$= \frac{1}{l^2}\begin{bmatrix} 0.2857 & -0.0714 \\ -0.0714 & 0.1429 \end{bmatrix}\begin{Bmatrix} -6l \\ 0 \end{Bmatrix}\left[\frac{-7Pl^3}{96EI}\right]$$

$$\Rightarrow \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} \frac{-Pl^2}{8EI} \\ \frac{Pl^2}{32\ EI} \end{bmatrix} = \begin{bmatrix} \frac{-PL^2}{32\ EI} \\ \frac{PL^2}{128\ EI} \end{bmatrix}$$

Substituting back in the global matrix equation we have

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-Pl^2}{8EI} \\ \frac{-7\ Pl^3}{96\ EI} \\ \frac{Pl^2}{32\ EI} \\ 0 \\ 0 \end{Bmatrix}$$

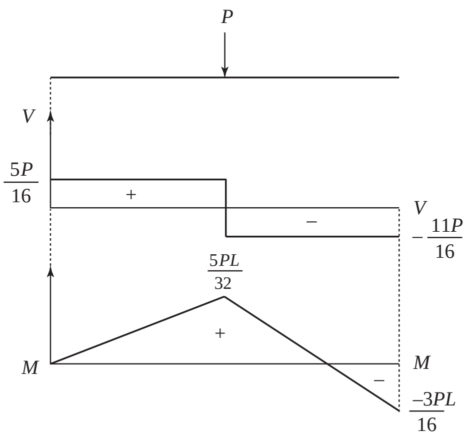
$$F_{1y} = \frac{EI}{l^3}\left[\frac{-6Pl^3}{8EI} + \frac{7Pl^3}{8EI} + \frac{6Pl^3}{32EI}\right] = \frac{EI}{l^3}\frac{10Pl^3}{32EI} \Rightarrow \boxed{F_{1y} = \frac{5P}{16}}$$

Similarly

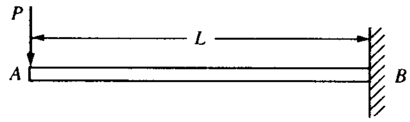
$$\boxed{M_1 = 0} \qquad \boxed{F_{3y} = \frac{11P}{16}}$$

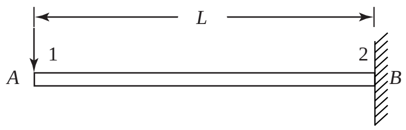
$$\boxed{F_{2y} = -P} \qquad \boxed{M_3 = \frac{-3PL}{16}}$$

$$\boxed{M_2 = 0}$$



4.4





$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & 6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Boundary conditions

$$v_2 = \phi_2 = 0$$

$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \end{Bmatrix} \Rightarrow \boxed{\phi_1 = \frac{PL^2}{2EI}}$$

$$\boxed{v_1 = \frac{-PL^3}{3EI}}$$

Matrix forces

$$\begin{Bmatrix} F_{1x} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}\begin{Bmatrix} \frac{-PL^3}{3EI} \\ \frac{PL^2}{2EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow\ F_{1y} = \frac{EI}{L^3}\left[-12\left(\frac{-PL^3}{3\,EI}\right) + 6L\left(\frac{-PL^2}{2\,EI}\right)\right]$$

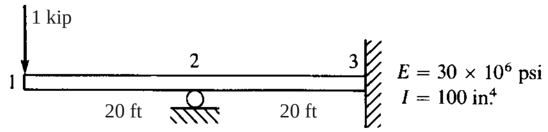
$$\Rightarrow\ F_{1y} = -P$$

Similarly $\quad M_1 = 0$

$$F_{2y} = P$$

$$M_2 = -PL$$

4.5



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$

$E = 30 \times 10^6$, $I = 100$ in.^{4}, $L = 20$ ft $= 240$ in.

$$\{F\} = [K]\{d\} \Rightarrow \begin{Bmatrix} F_{1y} = -10 \\ M_1 = 0 \\ F_{2y} = ? \\ M_2 = 0 \\ F_{3y} = ? \\ M_3 = ? \end{Bmatrix} = [K]\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \text{ where } v_2 = v_3 = \phi_3 = 0$$

$$\begin{Bmatrix} -1000 \\ 0 \\ 0 \end{Bmatrix} = \frac{30 \times 10^6 (100)}{(240)^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix} \qquad (1)$$

Solving for the displacements we have

$\phi_1 = 0.0144$ rad, $\phi_2 = 0.0048$ rad, $v_1 = -2.688$ in.

Substituting in the equation $\{F\} = [K]\{d\}$ we have

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{30 \times 10^6 (100)}{(240)}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -2.688 \text{ in.} \\ 0.0144 \\ 0 \\ 0.0048 \\ 0 \\ 0 \end{Bmatrix}$$

$F_{1y} = -1000$, $M_1 = 0$,

$F_{2y} = 2500$ lb, $M_2 = 0$, $F_{3y} = -1500$ lb, $M_3 = 10$ ft $\cdot$ kip

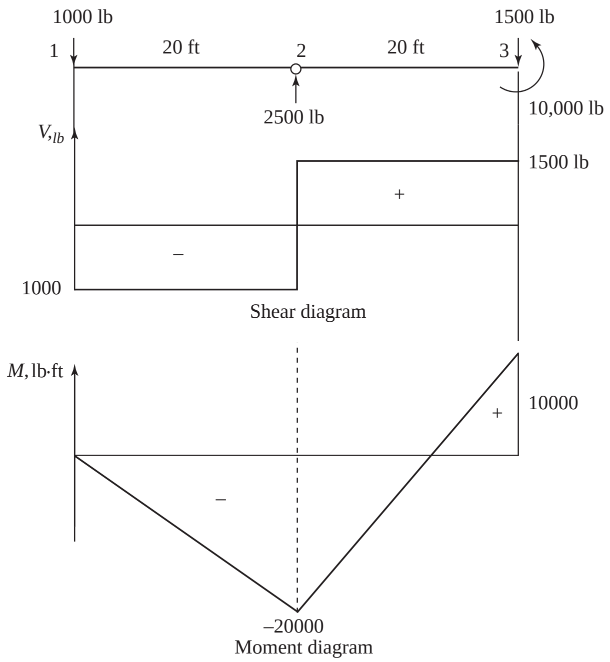
Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -2.688 \\ 0.0144 \\ 0 \\ 0.0048 \end{Bmatrix}$$

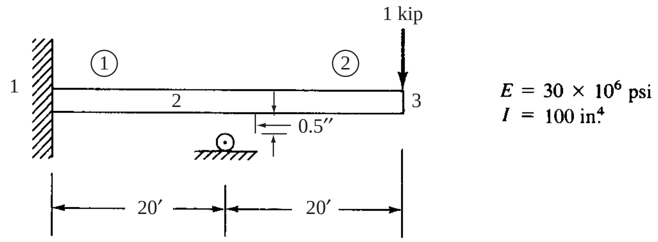
$$f_{1y} = -1000 \text{ lb}$$
$$m_1 = 0$$
$$f_{2y} = 1000 \text{ lb}$$
$$m_2 = -20000 \text{ ft}\cdot\text{lb}$$

Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.0048 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{array}{l} f_{2y} = 1500 \text{ lb} \\ m_2 = 20000 \text{ ft}\cdot\text{lb} \\ f_{3y} = -1500 \text{ lb} \\ m_3 = 10000 \text{ ft}\cdot\text{lb} \end{array}$$



4.6



$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad (1)$$

Boundary conditions $v_1 = 0$, $\phi_1 = 0$, $v_2 = -0.5$ in.

$$\begin{Bmatrix} F_{2y} \\ 0 \\ -1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^3 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \tag{2}$$

$$\frac{EI}{L^3} = \frac{(30\times 10^6\text{psi})(100\text{ in.}^4)}{(240\text{ in.})^3} = 217\ \frac{\text{lb}}{\text{in.}}$$

$$\begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix} = 217\begin{bmatrix} 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \tag{3}$$

Solving (3)

$$v_3 = -\ 3.938\text{ in.}$$

$$\phi_2 = -\ 0.007925\text{ rad}$$

$$\phi_3 = -\ 0.01753\text{ rad}$$

Back substituting into (1)

$$F_{1y} = -\ 1174\text{ lb}$$

$$M_1 = -\ 41{,}875\text{ lb}\cdot\text{in.}$$

$$F_{2y} = 2174\text{ lb}$$

Element 1

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = -\ 0.5\text{ in.} \\ \phi_2 = -\ 0.00793 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1174\text{ lb} \\ -41{,}875\text{ lb}\cdot\text{in.} \\ 1174\text{ lb} \\ -240{,}000\text{ lb}\cdot\text{in.} \end{Bmatrix}$$

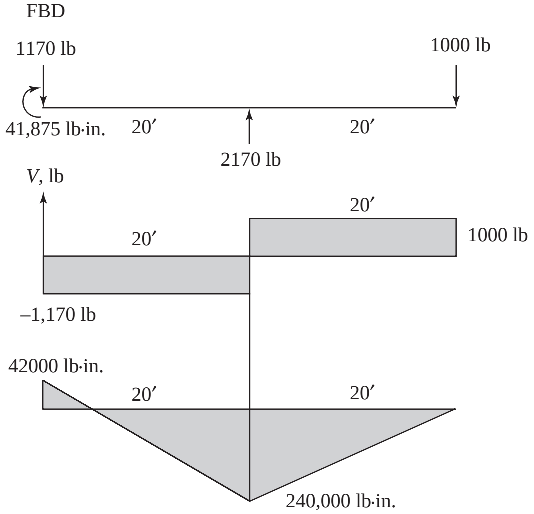
Element 2

$$\begin{Bmatrix} f_{2y}^{(2)} \\ m_2^{(2)} \\ f_{3y}^{(2)} \\ m_3^{(2)} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.5\text{ in.} \\ -0.0079 \\ -3.938\text{ in.} \\ -0.01753 \end{Bmatrix}$$

$$f_{2y}^{(2)} = 1000\text{ lb} = -\ f_{3y}^{(2)}$$

$$m_2^{(2)} = 240{,}000\text{ lb}\cdot\text{in.}$$

$$m_3^{(2)} = 0$$



4.7

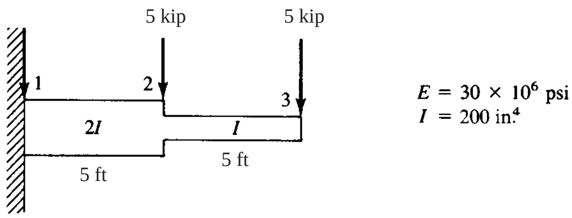
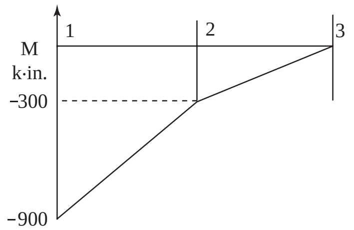


Figure P4-7



$$\begin{Bmatrix} F_{1y} \\ M_1 \\ -5000 \\ 0 \\ -5000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 12L & -24 & 12L & 0 & 0 \\ 12L & 8L^2 & -12L & 4L^2 & 0 & 0 \\ -24 & -12L & 36 & -6L & -12 & 6L \\ 12L & 4L^2 & -6L & 12L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad \text{(A)}$$

Solving the last four equations of (A)

$$\begin{aligned} v_2 &= -\,0.105 \text{ in.} \\ \phi_2 &= -\,0.003 \text{ rad} \\ v_3 &= -\,0.345 \text{ in.} \\ \phi_3 &= -\,0.0045 \text{ rad} \end{aligned} \qquad \text{(B)}$$

(B) in (A)

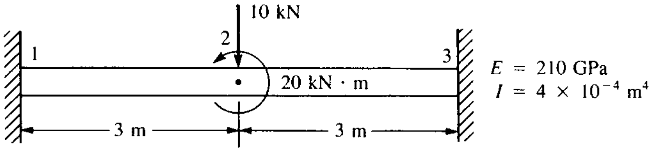
$$\Rightarrow F_{1y} = \frac{(30\times10^6)(200)}{60^3(1000)}\,[-\,24\,(-\,0.105) + 720\,(-\,0.003)]$$

$$= 10 \text{ kip}$$

$$M_1 = \frac{(30\times10^6)(200)}{60^3(1000\times12)}\ [-720\ (-0.105) + 14400\ (-0.003)]$$

$$= 75\ \text{kip}\cdot\text{ft}$$

4.8



$$[k_{1-2}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

$$[k_{2-3}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Boundary conditions

$$v_1 = \phi_1 = v_3 = \phi_3 = 0$$

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{2y} = -1000\ \text{N} \\ M_2 = 20000\ \text{N}\cdot\text{m} \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$-0.0032142 = 24\ v_2 \Rightarrow v_2 = -1.34\times10^{-4}\ \text{m}$$

$$0.0064285 = 72\ \phi_2 \Rightarrow \phi_2 = 8.93\times10^{-5}\ \text{rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -1.34\times10^{-4} \\ 8.93\times10^{-5} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow F_{1y} = 3.1\times10^6\ (-12\ (1.34\times10^{-4}) + 6(3)\ (8.93\times10^{-5})) \Rightarrow F_{1y} = 10000\ \text{N}$$

$$M_1 = 3.1\times10^6\ (-6(3)\ (1.34\times10^{-4}) + 2(3)^2\ (8.93\times10^{-5})) \Rightarrow M_1 = 12500\ \text{N}\cdot\text{m}$$

Similarly

$$F_{2y} = -10000\ \text{N}$$

$$M_2 = 20000\ \text{N}\cdot\text{m}$$

$$F_{3y} = 1.87\ \text{N}$$

$$M_3 = -2500\ \text{N}\cdot\text{m}$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -1.34\times10^{-4} \\ 8.93\times10^{-5} \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 10000 \text{ N}$$

$$m_1 = 12500 \text{ N}\cdot\text{m}$$

$$f_{2y} = -10000 \text{ N}$$

$$m_2 = 17500 \text{ N}\cdot\text{m}$$

Element 2–3

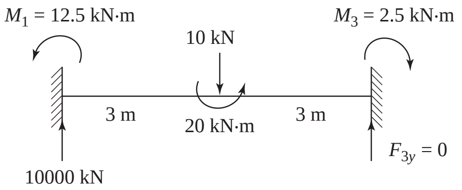
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{(210\times10^9)(4\times10^{-4})}{(3)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -1.34\times10^{-4} \\ 8.93\times10^{-5} \\ 0 \\ 0 \end{Bmatrix}$$

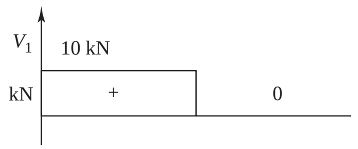
$$\Rightarrow f_{2y} = -1.87 \text{ N}$$

$$m_2 = 2500 \text{ N}\cdot\text{m}$$

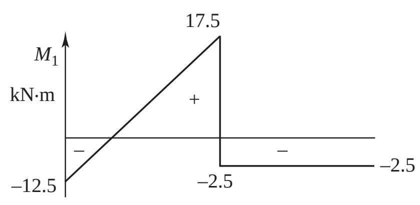
$$f_{3y} = 1.87 \text{ N}$$

$$m_3 = -2500 \text{ N}\cdot\text{m}$$



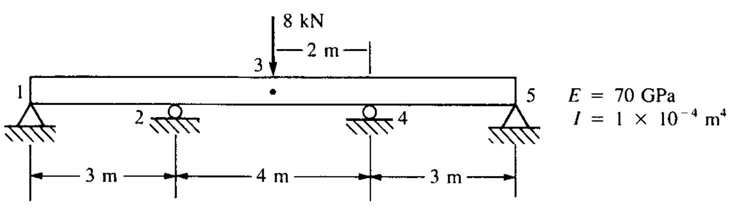


Shear diagram

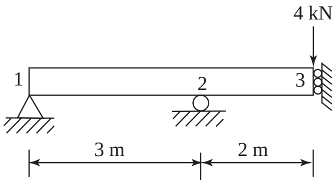


Moment diagram

4.9



Using symmetry



$$[k_{1-2}] = EI \begin{bmatrix} \frac{12}{27} & \frac{6}{9} & \frac{-12}{27} & \frac{6}{9} \\ & \frac{4}{3} & \frac{-6}{9} & \frac{2}{3} \\ & & \frac{12}{27} & \frac{-6}{9} \\ \text{Symmetry} & & & \frac{4}{3} \end{bmatrix}$$

$$[k_{2-3}] = EI \begin{bmatrix} \frac{12}{8} & \frac{6}{4} & \frac{-12}{8} & \frac{6}{4} \\ & \frac{4}{2} & \frac{-6}{4} & \frac{3}{2} \\ & & \frac{12}{8} & \frac{-6}{4} \\ & & & \frac{4}{2} \end{bmatrix}$$

Applying the boundary conditions

$v_1 = v_2 = \phi_3 = 0$ we have

$$\begin{Bmatrix} M_1 = 0 \\ M_2 = 0 \\ F_{3y} = -4000 \text{ N} \end{Bmatrix} = EI \begin{bmatrix} \frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{10}{3} & \frac{-3}{2} \\ 0 & \frac{-3}{2} & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ v_3 \end{Bmatrix}$$

which implies

$$0 = \frac{4}{3}\ \phi_1 + \frac{2}{3}\ \phi_2 \Rightarrow \phi_1 = -\frac{1}{2}\ \phi_2$$

$$0 = \frac{2}{3}\ \left(-\frac{1}{2}\phi_2\right) + \frac{10}{3}\ (\phi_2) - \frac{3}{2}\ v_3 \Rightarrow \phi_2 = \frac{1}{2}\ v_3$$

$$-4000 \text{ N} = (70 \times 10^9)\ (1 \times 10^{-4})\ \left[-\frac{3}{2}\left(\frac{1}{2}v_3\right) + \frac{3}{2}v_3\right]$$

$$v_3 = -7.619 \times 10^{-4} \text{ m}$$

$$\Rightarrow \quad \phi_2 = \frac{-1}{2}\ (-7.619 \times 10^{-4}) \text{ m} \Rightarrow \phi_2 = -3.809 \times 10^{-4} \text{ rad}$$

$$\Rightarrow \quad \phi_1 = \frac{-1}{2}\ \phi_2$$

$$\Rightarrow \quad \phi_1 = 1.904 \times 10^{-4} \text{ rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = (70 \times 10^9)\ (1 \times 10^{-4}) \begin{bmatrix} \frac{4}{9} & \frac{2}{3} & \frac{-4}{9} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{-4}{9} & \frac{-2}{3} & \frac{35}{18} & \frac{5}{6} & \frac{-3}{2} & \frac{3}{2} \\ \frac{2}{3} & \frac{2}{3} & \frac{5}{6} & \frac{4}{3} & \frac{-3}{2} & 1 \\ 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ 0 & 0 & \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1.904 \times 10^{-4} \\ 0 \\ -3.809 \times 10^{-4} \\ -7.619 \times 10^{-4} \\ 0 \end{Bmatrix}$$

$$F_{1y} = (7 \times 10^6)\left[\frac{2}{3}(1.904 \times 10^{-4}) + \frac{2}{3}\left(-3.809 \times 10^{-4}\right)\right]$$

$$\Rightarrow \quad F_{1y} = -889 \text{ N}$$

Similarly

$$F_{2y} = 4889 \text{ N}, \; M_3 = 5333 \text{ N}\cdot\text{m}$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = (70 \times 10^9)\,(1 \times 10^{-4}) \begin{bmatrix} \frac{4}{9} & \frac{2}{3} & \frac{-4}{9} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{-4}{9} & \frac{-2}{3} & \frac{4}{9} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-2}{3} & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} 0 \\ 1.904 \times 10^{-4} \\ 0 \\ -3.809 \times 10^{-4} \end{Bmatrix}$$

$$\begin{aligned} \Rightarrow \quad f_{1y} &= -889 \text{ N} \\ m_1 &= 0 \\ f_{2y} &= 889 \text{ N} \\ m_2 &= -2667 \text{ N}\cdot\text{m} \end{aligned}$$

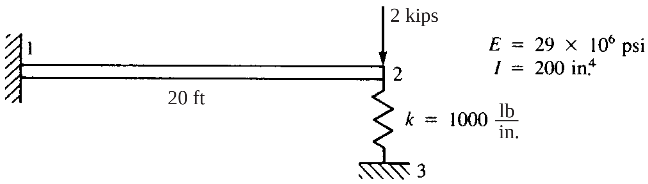
Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = (70 \times 10^9)\,(1 \times 10^{-4}) \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ -3.809 \times 10^{-4} \\ -7.619 \times 10^{-4} \\ 0 \end{Bmatrix}$$

$$\begin{aligned} \Rightarrow \quad f_{2y} &= 4000 \text{ N} = -f_{3y} \\ m_2 &= 2667 \text{ N}\cdot\text{m}, \; m_3 = 5333 \text{ N}\cdot\text{m} \end{aligned}$$

Elements 3–4 and 4–5 have same forces due to symmetry. Moments though will have opposite signs.

4.10



$$[K] = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 + \frac{KL^3}{EI} & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{array}$$

Boundary conditions

$$v_1 = \phi_1 = 0$$

Applying the boundary conditions on equation $\{F\} = [K]\,\{d\}$

$$\begin{Bmatrix} F_2 = -2000 \text{ lb} \\ M_2 = 0 \end{Bmatrix} = \frac{(29 \times 10^6)(200)}{(20 \times 12)^3} \begin{bmatrix} 12 + \frac{KL^3}{EI} & -6L \\ -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$0 = -6(240)v_2 + 4(240)^2\,\phi_2$$

$$\Rightarrow 0 = -6v_2 + 4(240)\phi_2 \Rightarrow v_2 = 160\phi_2$$

$$-2000 = 419.56\,[(12 + (2.38)\,160\phi_2 - 6(240)\phi_2]$$

$$\Rightarrow \quad \phi_2 = -0.005538 \text{ rad}$$

$$\Rightarrow \quad v_2 = 160(-0.005538) \Rightarrow v_2 = -0.886 \text{ in.}$$

Beam element

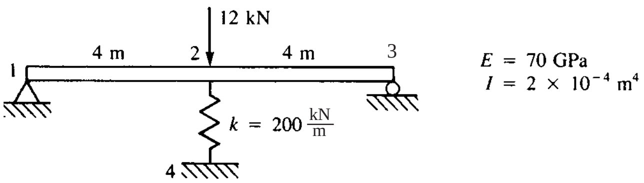
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{(29\times10^6)(200)}{(20\times12)^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.886 \\ -0.005538 \end{Bmatrix}$$

$$F_{1y} = 1115 \text{ lbs} \uparrow,\ M_1 = -267 \text{ kip}\cdot\text{in.}$$

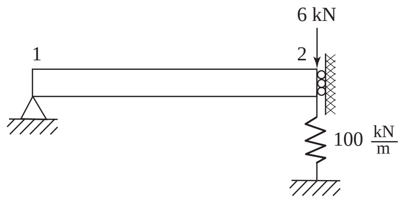
$$F_{2y} = -1115 \text{ lbs} \downarrow,\ M_2 = 0$$

The extra force at node 2 is resisted by the spring.

4.11



Applying symmetry



$$[K] = \frac{EI}{L_3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & \frac{12+KL^3}{EI} & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}$$

Applying the boundary conditions $v_1 = 0$, $\phi_2 = 0$ we have

$$\begin{Bmatrix} M_1 = 0 \\ F_{2y} = -6000 \text{ N} \end{Bmatrix} = \frac{(70\times10^9)(2\times10^{-4})}{4^3}\begin{bmatrix} 4L^2 & -6L \\ -6L & \frac{12+KL^3}{EI} \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \end{Bmatrix}$$

$$\Rightarrow \quad 0 = 4L^2\phi_1 - 6Lv_2 \Rightarrow \phi_1 = \frac{6}{4L}\,v_2 \Rightarrow \phi_1 = \frac{6}{16}\,v_2$$

$$-6000 = 218750\left[-24\left(\frac{6}{16}\right)v_2 + 12.457\,v_2\right]$$

$$\Rightarrow \quad v_2 = -7.9338\times10^{-3} \text{ m}$$

$$\phi_1 = \frac{6}{16}(-7.9338 \times 10^{-3}) \Rightarrow \phi_1 = -2.9752 \times 10^{-3} \text{ rad}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{(70 \times 10^9)(2 \times 10^{-4})}{4^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12.457 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 2 \\ -2.9752 \times 10^{-3} \\ -7.9336 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$F_{1y} = 5.208 \text{ kN} \uparrow, \; M_2 = 20.83 \text{ kN} \cdot \text{m} \circlearrowright$$

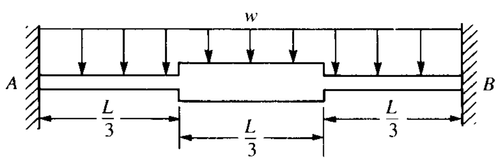
$$F_{2y} = 0 \text{ kN} \downarrow$$

$$F_{\text{spring}} = \left(200 \; \frac{\text{kN}}{\text{m}}\right)(7.9338 \times 10^{-3} \text{ m})$$

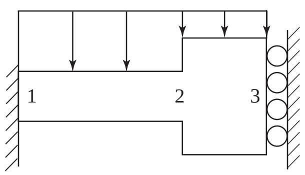
$$F_{\text{spring}} = 1.587 \text{ kN}$$

From symmetry $F_{3y} = 5.208 \text{ kN} \uparrow$

4.12

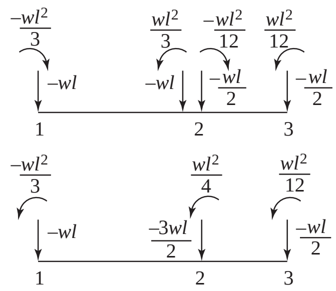


From symmetry



Let $\frac{L}{6} = l$

$\Rightarrow \frac{L}{3} = 2l$



Final nodal forces and moments

$$[k_{1-2}] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} \frac{3}{2} & \frac{3}{2}l & \frac{-3}{2} & \frac{3}{2}l \\ \frac{3}{2}l & 2l^2 & \frac{-3}{2}l & l^2 \\ \frac{-3}{2} & \frac{-3}{2}l & \frac{3}{2} & \frac{-3}{2}l \\ \frac{3}{2}l & l^2 & \frac{-3}{2}l & 2l^2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{array}$$

$$
\begin{array}{cccccc}
 & v_2 & \phi_2 & v_3 & \phi_3 & \\
[k_{2-3}] = & \left[\begin{array}{cccc} 24 & 12l & -24 & 12l \\ 12l & 8l^2 & -12l & 4l^2 \\ -24 & -12l & 24 & -12l \\ 12l & 4l^2 & -12l & 8l^2 \end{array}\right. & & & \left.\right] & \begin{array}{c} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{array}
\end{array}
$$

$$
\begin{Bmatrix} -wl \\ \frac{-wl^2}{3} \\ \frac{-3wl}{2} \\ \frac{wl^2}{4} \\ \frac{-wl}{2} \\ \frac{wl^2}{12} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 1.5 & 1.5l & 1.5 & 1.5l & 0 & 0 \\ 1.5l & 2l^2 & -1.5l & l^2 & 0 & 0 \\ -1.5 & -1.5l & 25.5 & 10.5l & -24 & 12l \\ 1.5l & l^2 & 10.5l & 10l^2 & -12l & 4l^2 \\ 0 & 0 & -24 & -12l & 24 & -12l \\ 0 & 0 & 12l & 4l^2 & 12l & 8l^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = ? \\ \phi_2 = ? \\ v_3 = ? \\ \phi_3 = 0 \end{Bmatrix}
$$

Adding third row equation to fifth row equation we have

$$\frac{l^3}{EI}\left(\frac{-3wl}{2} - \frac{wl}{2}\right) = 25.5v_2 + 10.5l\phi_2 - 24v_3 - 24v_2 - 12l\phi_2 + 24v_3$$

$$\Rightarrow \frac{-2wl^4}{EI} = 1.5v_2 - 1.5l\phi_2$$

$$\Rightarrow \quad v_2 = \frac{-4wl^4}{3EI} + l\phi_2 \tag{A}$$

Multiplying fourth row equation by –2 and third row equation by l and adding we have

$$\frac{l^3}{EI}\left(\frac{-3wl}{2} - \frac{wl^2}{2}\right) = 25.5lv_2 + 10.5l^2\phi_2 - 24lv_3 - 21lv_2 - 20l^2\phi_2 + 24lv_3$$

$$\Rightarrow \frac{-2wl^5}{EI} = 4.5lv_2 - 9.5l^2\phi_2 \tag{B}$$

Substituting (A) into (B)

$$\frac{-2wl^5}{EI} = 4.5l\left[\frac{-4wl^4}{3EI} + l\phi_2\right] - 9.5l^2\phi_2$$

$$\Rightarrow \frac{-2wl^5}{EI} = \frac{-18wl^5}{3EI} + 4.5l^2\phi_2 - 9.5l^2\phi_2$$

$$\Rightarrow \frac{4wl^5}{EI} = -5l^2\phi_2 \Rightarrow \boxed{\phi_2 = \frac{-4wl^3}{5EI}}$$

$$v_2 = \frac{-4wl^4}{3EI} + l\left(\frac{-4wl^3}{5EI}\right) \quad \Rightarrow \quad \boxed{v_2 = \frac{-32wl^4}{15EI}}$$

Substituting in third row equation.

$$\frac{-3wl^4}{2EI} = 25.5\left(\frac{-32}{15}\frac{wl^4}{EI}\right) + 10.5l\left(\frac{-4}{5}\frac{wl^3}{EI}\right) - 24v_3$$

$$\frac{-3wl^4}{2EI}+\frac{816wl^4}{15EI}+\frac{42wl^4}{5EI} = -24v_3 \;\Rightarrow\; \boxed{v_3=\frac{-1839\,wl^4}{720\,EI}}$$

(Remember that $l = \frac{L}{6}$)

Now from symmetry

$$\boxed{v_4=v_2} \text{ and } \boxed{\phi_4=-\phi_2}$$

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \\ F_{3y}^{(e)} \\ M_3^{(e)} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 1.5 & 1.5l & -1.5 & 1.5l & 0 & 0 \\ 1.5l & 2l^2 & -1.5l & l^2 & 0 & 0 \\ -1.5 & -1.5l & 25.5 & 10.5l & -24 & 12l \\ 1.5l & l^2 & 10.5l & 10l^2 & -12l & 4l^2 \\ 0 & 0 & -24 & -12l & 24 & -12l \\ 0 & 0 & 12l & 4l^2 & -12l & 8l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-32wl^4}{15EI} \\ \frac{-4wl^3}{5EI} \\ \frac{-1839wl^4}{720\,EI} \\ 0 \end{Bmatrix}$$

$$F_{1y}^{(e)} = 2\,wl,\; M_1^{(e)} = 2.4\,wl^2,\; F_{2y}^{(e)} = -1.5\,wl$$

$$M_2^{(e)} = 0.25\,wl^2,\; F_{3y}^{(e)} = -0.5\,wl,\; M_3^{(e)} = 1.85\,wl^2$$

The equation

$\{F\} = [K]\{d\} - \{F_0\}$ is now used to find the global nodal concentrated forces.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 2wl \\ 2.4wl^2 \\ -1.5wl \\ 0.25wl \\ -0.5wl \\ 1.85wl^2 \end{Bmatrix} - \begin{Bmatrix} -wl \\ \frac{-wl^2}{3} \\ \frac{-3}{2wl} \\ \frac{wl^2}{4} \\ \frac{-wl}{2} \\ \frac{wl^2}{12} \end{Bmatrix} \Rightarrow$$

$$F_{1y}=3wl=3w\left(\frac{L}{6}\right)$$

$$\Rightarrow \boxed{F_{1y}=\frac{wL}{2}}$$

$$M_1 = 2.73\,wl^2 = 3w\left(\frac{L}{6}\right)^2$$

$$\Rightarrow \boxed{M_1=\frac{wL^2}{12}}$$

$$F_{2y}=0,\; M_2=0,\; F_{3y}=0$$

$$M_3 = \frac{21.5wl^2}{12}=\frac{21.5}{12}w\left(\frac{L}{6}\right)^2 \Rightarrow M_3=\frac{wL^2}{24}$$

In our case M_3 is the maximum at midspan. From symmetry from elements 3–4 and 4–5

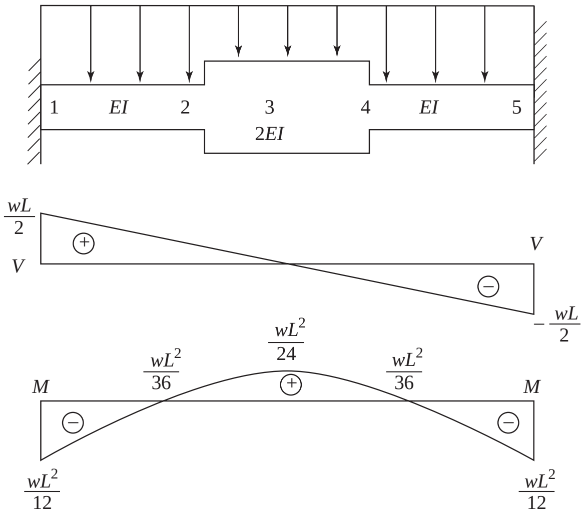
$$\Rightarrow M_3 = \frac{-wL^2}{24}$$

So $M_3 = 0$, $F_{4y} = 0$, $M_4 = 0$

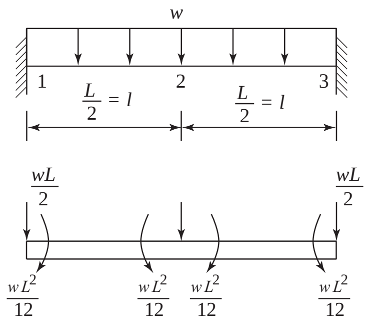
$$\boxed{F_{5y}=\frac{wL}{2}} \quad \boxed{M_5=\frac{-wL^2}{12}}$$

Going back to the deflections we have

$$v_3 = \frac{-1839\,wl^4}{720\,EI} = \frac{-1839}{720}\;\frac{w\left(\frac{L}{6}\right)^4}{EI} \;\Rightarrow\; \boxed{v_3=\frac{-wL^4}{507\,EI}}$$



4.13



Applying the boundary conditions

$$v_1 = \phi_1 = v_3 = \phi_3 = 0$$

$$\begin{Bmatrix} -wl \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \Rightarrow 0 = \frac{EI}{l^3}\ (8l^2\phi_2) \Rightarrow \boxed{\phi_2 = 0}$$

$$-\,wl = \frac{EI}{l^3}\ (24v_2) \ \Rightarrow\ \boxed{v_2 = \frac{-wl^4}{24EI}}$$

$$f_{1y}^{(e)} = \frac{wl}{2}, m_1^{(e)} = \frac{wl^2}{4},\ f_{2y}^{(e)} = -\,wl,\ m_2^{(e)} = 0,$$

$f_{3y}^{(e)} = \dfrac{wl}{2}, m_3 = \dfrac{-wl^2}{4}$. These are obtained from the following matrix equation.

$$\begin{Bmatrix} f_{1y}^{(e)} \\ m_1^{(e)} \\ f_{2y}^{(e)} \\ m_2^{(e)} \\ f_{3y}^{(e)} \\ m_3^{(e)} \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-wl^4}{24EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{F\} = [K]\{d\} - \{F_0\}$$

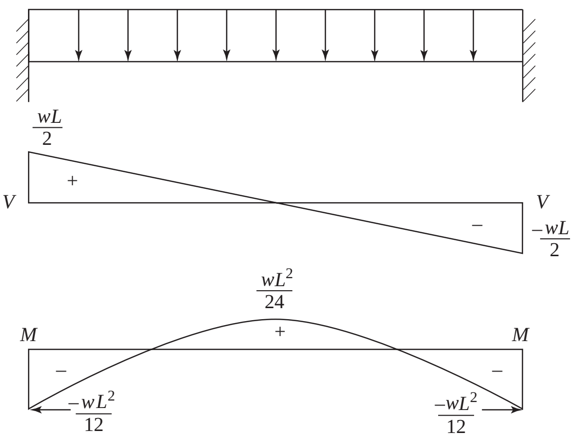
$$F_{1y} = \frac{wl}{2} - \left(\frac{-wl}{2}\right) = wl = \frac{wL}{2}$$

$$M_1 = \frac{wl^2}{4} - \left(\frac{-wl^2}{12}\right) = \frac{wl^2}{3} = \frac{wL^2}{12}$$

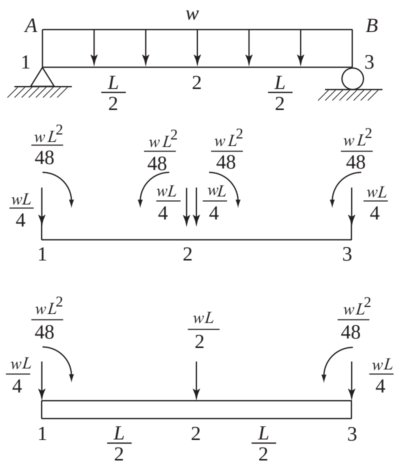
$$F_{2y} = -wl - (-wl) = 0, M_2 = 0 - 0 = 0$$

$$F_{3y} = \frac{wl}{2} - \left(\frac{-wl}{2}\right) = \frac{wL}{2}, M_3 = \frac{wl^2}{4} - \frac{-wl^2}{12} = \frac{wL^2}{12}$$

$$v_2 = \frac{wl^4}{24EI} = \frac{w(\frac{l}{2})^4}{24EI} \Rightarrow \boxed{v_2 = \frac{-wL^4}{384EI}}$$



4.14



After applying the boundary conditions

$v_1 = v_3 = \phi_2 = 0$ we have

$$\begin{Bmatrix} \frac{-wL^2}{48} \\ \frac{wL}{2} \\ \frac{-wL^2}{48} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} L^2 & -3L & 0 \\ -3L & 24 & 3L \\ 0 & 3L & L^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \Rightarrow \boxed{\phi_1 = \frac{-wL^3}{24EI}} \quad \boxed{v_2 = \frac{-5wL^4}{384EI}} \quad \boxed{\phi_3 = \frac{wL^3}{24EI}}$$

Now $\{F^{(e)}\} = [K]\,\{d\}$

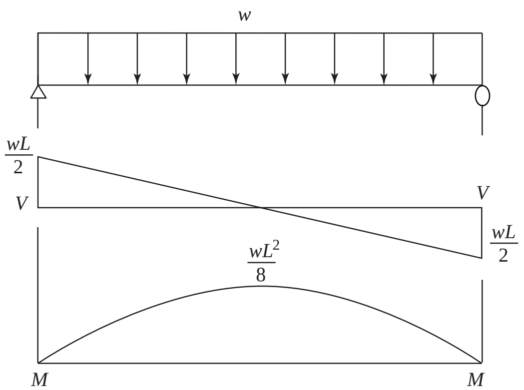
$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \\ F_{3y}^{(e)} \\ M_3^{(e)} \end{Bmatrix} = \frac{EI}{\left(\frac{L}{2}\right)^3}\begin{bmatrix} 12 & 3L & -12 & 3L & 0 & 0 \\ 3L & L^2 & -3L & \frac{L^2}{2} & 0 & 0 \\ -12 & -3L & 24 & 0 & -12 & 3L \\ 3L & \frac{L^2}{2} & 0 & 2L^2 & -3L & \frac{L^2}{2} \\ 0 & 0 & -12 & -3L & 12 & -3L \\ 0 & 0 & 3L & \frac{L^2}{2} & -3L & L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{24EI} \\ \frac{-5wL^4}{384EI} \\ 0 \\ 0 \\ \frac{wL^3}{24EI} \end{Bmatrix}$$

$$F_{1y}^{(e)} = \frac{wL}{4},\ M_1^{(e)} = \frac{-wL^2}{48},\ F_{2y}^{(e)} = \frac{wL}{2},\ M_2^{(e)} = 0,$$

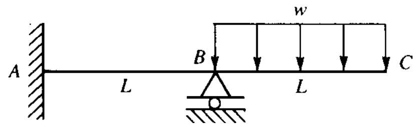
$$F_{3y}^{(e)} = \frac{wL}{4},\ M_3 = \frac{wL^2}{48}$$

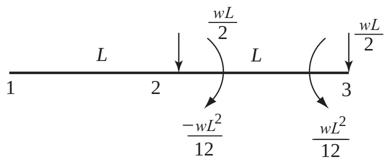
$$\{F\} = \{F^{(e)}\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{3y} \\ M_3 \end{Bmatrix} = \begin{Bmatrix} \frac{wL}{4} \\ \frac{-wL^2}{48} \\ \frac{wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{4} \\ \frac{-wL^2}{48} \\ \frac{-wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix} \Rightarrow \boxed{F_{1y} = \frac{wL}{2},\ M_1 = 0} \quad \boxed{F_{3y} = \frac{wL}{2},\ M_3 = 0}$$



4.15





Total $[K]$ for the whole beam

$$[K] = \frac{EI}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 & 0 & 0 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 & 0 & 0 \\ -12L & -6L^2 & 24L & 0 & -12L & 6L^2 \\ 6L^2 & 2L^3 & 0 & 8L^3 & -6L^2 & 2L^3 \\ 0 & 0 & -12L & -6L^2 & 12L & -6L^2 \\ 0 & 0 & 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}$$

$\{F\} = [K]\,\{d\}$

After applying boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_2 = 0$$

$$\begin{Bmatrix} M_2 = \frac{-wL^2}{12} \\ F_{3y} = \frac{-wL}{2} \\ M_3 = \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^4}\begin{bmatrix} 8L^3 & -6L^2 & 2L^3 \\ -6L^2 & 12L & -6L^2 \\ 2L^3 & -6L^2 & 4L^3 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_2 \\ \phi_3 \end{Bmatrix}$$

Solving the 3 equations we get

$$\boxed{\phi_2 = \frac{-wL^3}{8EI} \text{ or } \circlearrowright} \quad \boxed{v_3 = \frac{-wL^4}{4EI}\downarrow} \quad \boxed{\phi_3 = \frac{-7wL^3}{24EI} \text{ or } \circlearrowright}$$

$$f_{1y}^{(1)} = \frac{EI}{L^4}\,(6L^2\phi_2) = \frac{3wL^5 EI}{4EIL^4} = \frac{-3wL}{4} \text{ or } \downarrow$$

$$m_1^{(1)} = \frac{EI}{L^4}\,(2L^3\phi_2) = \frac{-2wL^6}{8EI}\,\frac{EI}{L^4} = \frac{-wL^2}{4} \text{ or } \circlearrowright$$

$$f_{2y}^{(1)} = \frac{EI}{L^4}\,(-6L^2\phi_2) = \frac{3wL}{4}\uparrow$$

Reactions

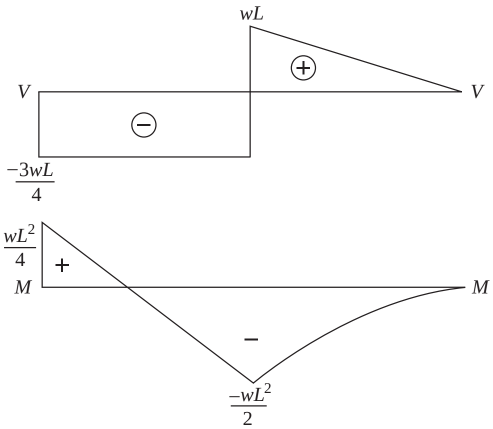
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \end{Bmatrix} = \begin{Bmatrix} \frac{-3wL}{4} \\ \frac{-wL^2}{4} \\ \frac{3wL}{4} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ \frac{-wL}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{-3wL}{4} \\ \frac{-wL^2}{4} \\ \frac{7wL}{4} \end{Bmatrix}$$

using

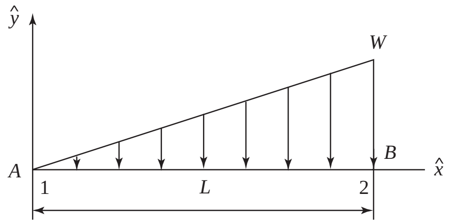
$$\{f\} = [k]\,\{d\} - \{f^{(0)}\}$$

$$\begin{bmatrix} f_{2y}{}^{(2)} \\ m_2{}^{(2)} \\ f_{3y}{}^{(2)} \\ m_3{}^{(3)} \end{bmatrix} = \frac{EI}{L^4}\begin{bmatrix} 12L & 6L^2 & -12L & 6L^2 \\ 6L^2 & 4L^3 & -6L^2 & 2L^3 \\ -12L & -6L^2 & 12L & -6L^2 \\ 6L^2 & 2L^3 & -6L^2 & 4L^3 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{8EI} \\ \frac{-wL^4}{4EI} \\ \frac{-7wL^3}{24EI} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\Rightarrow \quad m_2^{(2)} = \frac{-wL^2}{2}$$



4.16



$$W_{\text{distributed}} = \int_0^L w(x)v(x)dx \quad \text{and}$$

$$W_{\text{discrete}} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

and $W_{\text{distributed}} = W_{\text{discrete}}$

$$\int_0^L w(x)\,v(x)\,dx = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

Now evaluting the left side of the equation by substituting where $w(x) = \dfrac{-wx}{L}$ (since the load is linear and increasing to the right), and in $V(x)$ we substitute with a's already evaluated, we get

$$\int_0^L w(x)v(x)dx = \int_0^L \frac{-wx}{L}\,[a_1x^3 + a_2x^2 + a_3x + a_4]$$

$$= \int_0^L \frac{-wx}{L}\left\{\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 - \right.$$

$$\left.\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + v_1\right\} dx$$

$$= \int_0^L \frac{-w}{L}\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^4 -$$

$$\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]x^3 + d_1 x^2 + v_1\, x\Big]dx$$

$$= \frac{-w}{L}\left[\frac{2}{3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + \frac{w}{L}\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right]$$

$$\frac{x^4}{4}\bigg|_0^L - \frac{w}{L}\phi_1 \frac{x^3}{3}\bigg|_0^L - \frac{w}{L} v_1 \frac{x^2}{2}\bigg|_0^L$$

$$= \frac{-wL^4}{5}\left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] +$$

$$\frac{wL^3}{4}\left[\frac{3}{L^2}(v_1 - v_2) + \frac{1}{L}(2\phi_1 + \phi_2)\right] - \frac{wL^2}{3\phi_1} - \frac{wL}{2} v_1$$

$$= \frac{-2wL}{5}(v_1 - v_2) + \frac{wL^2}{5}(\phi_1 + \phi_2) + \frac{3wL}{4}(v_1 - v_2)$$

$$+ \frac{wL^2}{4}(2\phi_1 + \phi_2) - \frac{wL^2}{3}\phi_1 - \frac{wL}{2} v_1$$

$$= m_1\phi_1 + m_2\phi_2 + f_{1y} v_1 + f_{2y} v_2$$

Now if we take the last equation and set $\phi_1 = \phi_2 = v_1 = 0$ and $v_1 = 1$, we have

$$\frac{-2wL}{5} + \frac{3wL}{4} - \frac{wL}{2} = f_{1y}\, 1$$

$$\Rightarrow \quad \frac{-8wL + 15wL - 10wL}{20} = f_{1y} \quad \Rightarrow \quad \boxed{f_{1y} = \frac{-3wL}{20}}$$

If $\quad \phi_2 = v_2 = v_1 = 0 \quad$ and $\quad \phi_1 = 1$

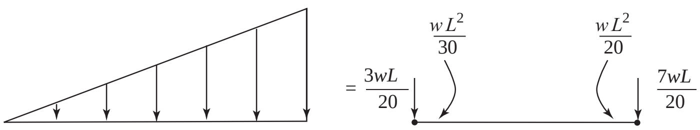
$$\Rightarrow \frac{-wL}{5} + \frac{2wL^2}{4} - \frac{wL^2}{3} = m_1 \quad \Rightarrow \quad \boxed{m_1 = \frac{-wL^2}{30}}$$

If $\quad \phi_2 = \phi_1 = v_1 = 0$ and $v_2 = 1$

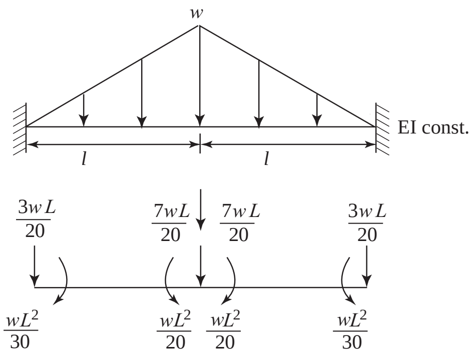
$$\Rightarrow \quad \frac{2wL}{5} - \frac{3wL}{4} = f_{2y} \quad \Rightarrow \quad \boxed{f_{2y} = \frac{-7wL}{20}}$$

If $\phi_1 = v_1 = v_2 = 0$ and $\phi_2 = 1$

$$\Rightarrow \quad \frac{-wL^5}{5} + \frac{wL^2}{4} = m_2\, 1 \quad \Rightarrow \quad \boxed{m_2 = \frac{wL^2}{20}}$$



4.17



Work equivalent load system

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 12+12 & -6l+6l & -12 & 6l \\ & & & 4l^2+4l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1=0 \\ \phi_1=0 \\ v_2 \\ \phi_2 \\ v_3=0 \\ \phi_3=0 \end{Bmatrix} = \begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y}=-\frac{14wl}{20} \\ M_2=0 \\ F_{3y} \\ M_3 \end{Bmatrix} \begin{matrix} \\ \\ (3) \\ (4) \\ \\ \\ \end{matrix}$$

Boundary Conditions $v_1 = \phi_1 = v_3 = \phi_3 = 0$

Use Equations (3) and (4)

$$\begin{Bmatrix} \frac{-14wl}{20} \\ 0 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix}$$

$$v_2 = \frac{-7wl^4}{240EI} = \frac{-7L^4w}{3840EI} \quad (L = 2l)$$

$$\phi_2 = 0$$

Reactions

$$\{F\} = [K]\,\{d\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ & 4l^2 & -6l & 2l^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6l \\ & & & 8l^2 & -6l & 2l^2 \\ & & & & 12 & -6l \\ \text{Symmetry} & & & & & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-7wl^4}{240\,EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -\frac{3wl}{20} \\ -\frac{wl^2}{30} \\ \frac{-7wl}{10} \\ 0 \\ -\frac{3wl}{20} \\ \frac{wl^2}{30} \end{Bmatrix}$$

$$F_{1y} = \frac{EI}{l^3}\,(-12)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-3wl}{20}\right)$$

$$F_{1y} = \frac{7}{20}\,wl + \left(\frac{3wl}{20}\right) = \frac{10}{20}\,wl = \frac{wl}{2} = \frac{wl}{4} \quad \left(\frac{L}{2} = l\right)$$

$$M_1 = \frac{EI}{l^3}\ (-\,6l)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-wl^2}{30}\right) = \left(\frac{7}{40} + \frac{1}{30}\right) wl^2$$

$$M_1 = \frac{5}{24}\,wl^2 = \frac{5wL^2}{96}$$

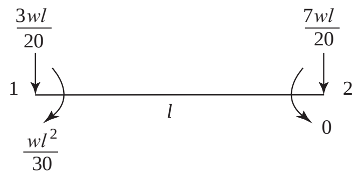
$$F_{2y} = \frac{EI}{l^3}\ (24)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-7wl}{10}\right) = 0$$

$$M_2 = \frac{EI}{l^3}\ (0) - 0 = 0$$

$$F_{3y} = \frac{EI}{l^3}\ (-\,12)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{-3wl}{20}\right) = \frac{wl}{2} = \frac{wL}{4}$$

$$M_3 = \frac{EI}{l^3}\ (6l)\left(\frac{-7wl^4}{240EI}\right) - \left(\frac{wl^2}{30}\right) = \frac{-5}{24}\ wl^2 = \frac{-5}{96}\ wL^4$$

Note: Could use symmetry



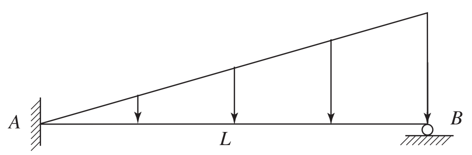
With $\quad v_1 = \phi_1 = \phi_2 = 0$

Get

$$\frac{12EI}{l^3}\ v_2 = \frac{-7wl}{20}$$

$$v_2 = \frac{-7wl^4}{240EI}\ \text{ as before}$$

4.18



$$\{F_0\} = [K]\ \{d\}$$

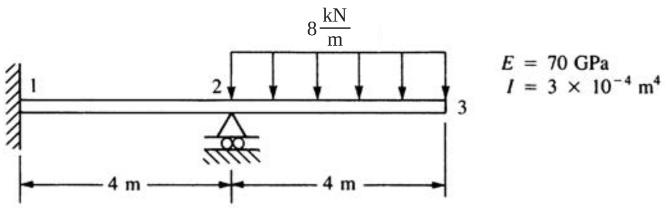
$$\begin{Bmatrix} F_{01y} = -\frac{3wL}{20} \\ M_{01} = -\frac{wL^2}{30} \\ F_{02y} = -\frac{7wL}{20} \\ M_{02} = \frac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 = 0 \\ \phi_1 = 0 \\ v_2 = 0 \\ \phi_2 \end{Bmatrix}$$

$$\Rightarrow \frac{wL^2}{20} = \frac{EI}{L^3}\, 4L^2\phi_2 \Rightarrow \boxed{\phi_2 = \frac{wL^3}{80EI}}$$

$$\begin{Bmatrix} F_{1y}^{(e)} \\ M_1^{(e)} \\ F_{2y}^{(e)} \\ M_2^{(e)} \end{Bmatrix} = \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{wL^3}{80\,EI} \end{Bmatrix} \Rightarrow \begin{aligned} F_{1y}^{(e)} &= \frac{3wL}{40} \\ M_1^{(e)} &= \frac{wL^2}{40} \\ F_{2y}^{(e)} &= -\frac{3wL}{40} \\ M_2^{(e)} &= \frac{wL^2}{20} \end{aligned}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \frac{3wL}{40} \\ \frac{wL^2}{40} \\ \frac{-3wL}{40} \\ \frac{wL^2}{20} \end{Bmatrix} - \begin{Bmatrix} \frac{-3wL}{20} \\ \frac{-wL^2}{30} \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix} \Rightarrow \boxed{F_{1y} = \frac{9wL}{40}} \quad \boxed{M_1 = \frac{7wL^2}{120}} \quad \boxed{F_{2y} = \frac{11wL}{40}} \quad \boxed{M_2 = 0}$$

4.19



$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

After imposing the boundary conditions and using work equivalence

$v_1 = \phi_1 = v_2 = 0$, we have in $\{F\} = [K]\{d\}$

$$\begin{Bmatrix} \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

From (1) and (3)

$$\frac{wL^2}{12} = \frac{EI}{L^3}\,[-8L^2\phi_2 + 6Lv_3 - 2L^2\phi_3]$$

$$\frac{wL^2}{12} = \frac{EI}{L^3}\,[2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3]$$

$$\frac{wL^3}{6} = \frac{EI}{L^3}\,[-6L^2\phi_2 + 2L^2\phi_3] \qquad (4)$$

From (2) and (3)

$$\frac{-wL^2}{4} = \frac{EI}{L^3}\ [-3L^2\phi_2 + 6Lv_3 - 3L^2\phi_3]$$

$$\frac{wL^2}{12} = \frac{EI}{L^3}\ [2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3]$$

$$\frac{-wL^2}{6} = \frac{EI}{L^3}\ [-L^2\phi_2 + L^2\phi_3] \tag{5}$$

Adding (4) and (5) we have

$$\frac{wL^2}{6} = \frac{EI}{L^3}\ [-6L^2\phi_2 + 2L^2\phi_3]$$

$$\frac{2wL^2}{6} = \frac{EI}{L^3}\ [2L^2\phi_2 - 2L^2\phi_3]$$

$$\frac{wL^2}{6} = \frac{EI}{L^3}\ [-4L^2\phi_2] \ \Rightarrow\ \boxed{\phi_2 = \frac{-wL^3}{8EI} = -3.046\times10^{-3}\,\text{rad}}$$

Substituting in (5) $\Rightarrow \dfrac{-wL^2}{6} = \dfrac{EI}{L^3}\left[-L^2\left(\dfrac{-wL^3}{8EI}\right) + L^2\phi_3\right]$

$$\Rightarrow \boxed{\phi_3 = \frac{-7wL^3}{24EI} = -0.00711\,\text{rad}}$$

Finally substituting in (1)

$$\Rightarrow \boxed{v_3 = \frac{-wL^4}{4EI} = -0.0244\ \text{m}}$$

Reactions can be found from the global equation

$$\{F\} = [K]\{d\} - \{F_0\}$$

$$F_{1y} = \frac{EI}{L^3}[6L\phi_2] - 0 = \frac{EI}{L^3}6L\left(\frac{-wL^3}{8EI}\right) = \frac{-3wL}{4} = -24\ \text{kN}$$

$$M_1 = \frac{EI}{L^3}\ [2L^2\phi_2] - 0 = \frac{EI}{L^3}2L^2\left(\frac{-wL^3}{8EI}\right) = \frac{-wL^2}{4} = -32\ \text{kN}\cdot\text{m}$$

$$F_{2y} = \frac{EI}{L^3}\ [-12v_3 + 6L\phi_3] - \left[-\frac{wL}{2}\right] = \frac{7wL}{4} = 56\ \text{kN}$$

and $M_2 = 0$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{-wL^3}{8EI} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$f_{1y} = \frac{-3}{4}wL = -24 \text{ kN},\; m_1 = \frac{-wL^2}{4} = -32 \text{ kN}\cdot\text{m}$$

$$f_{2y} = \frac{3}{4}\, wL = 24 \text{ kN},\; m_2 = -64 \text{ kN}\cdot\text{m}$$

Element 2–3

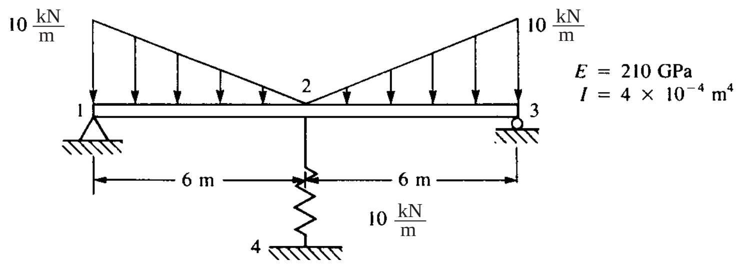
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{-wL^3}{8EI} \\ \frac{-wL^4}{4EI} \\ \frac{-7wL^3}{24EI} \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{2} \\ \frac{-wL^2}{12} \\ \frac{-wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\Rightarrow f_{2y} = 32 \text{ kN}$$

$$m_2 = 64 \text{ kN}\cdot\text{m}$$

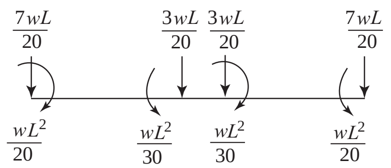
$$f_{3y} = m_3 = 0$$

4.20



Global stiffness matrix

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L & 2L^2 & 0 & 0 \\ & & \frac{24+KL^3}{EI} & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix} \end{array}\begin{array}{c} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{array}$$



Boundary conditions

$$v_1 = v_3 = \phi_2 = 0 \text{ and } \phi_1 = -\phi_3$$

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} \frac{-wL^2}{20} \\ \frac{-6wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & -6L & 0 \\ -6L & \frac{24+KL^3}{EI} & 6L \\ 0 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ v_2 \\ \phi_3 \end{Bmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

since $\phi_1 = -\phi_3$ we can ignore Equation (3)

$\Rightarrow$ Multiplying (1) by 3 and (2) by L and adding we have

$$\frac{-3\,wL^2}{20} = \frac{EI}{L^3}\,[12L^2\phi_1 - 18Lv_2]$$

$$\frac{-6\,wL^2}{20} = \frac{EI}{L^3}\left[-12L^2\phi_1 + \left(24 + \frac{KL^4}{EI}\right)v_2\right]$$

$$-9\,\frac{wL^2}{20} = \left(\frac{6EI}{L^2} + KL\right)v_2$$

$$\Rightarrow\ -162000 = 140{,}060{,}000\ v_2 \Rightarrow \boxed{v_2 = -0.011522\text{ m}}$$

Substituting in (1) we have

$$-18000 = 5.6 \times 10^7\ \phi_1 + 16.1309 \times 10^4$$

$$\Rightarrow\quad \boxed{\phi_1 = -0.0032019\text{ rad}}$$

$$\text{since } \phi_1 = -\phi_3 \Rightarrow \boxed{\phi_3 = 0.0032019\text{ rad}}$$

The reactions can be found by the global matrix $\{F\} = [K]\{d\} - \{F_0\}$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & \frac{24+KL^3}{EI} & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.0032019 \\ -0.0011522 \\ 0 \\ 0 \\ 0.0032019 \end{Bmatrix} - \begin{Bmatrix} \frac{-7wL}{20} \\ \frac{-wL^2}{20} \\ \frac{-6wL}{20} \\ 0 \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix}$$

$$\Rightarrow\quad F_{1y} = 29.94\text{ kN},\ M_1 = 0$$

$$F_{2y} = 0.11522\text{ kN},\ M_2 = 0$$

$$F_{3y} = 29.94\text{ kN},\ M_3 = 0$$

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetry} & & & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.00332019 \\ -0.011522 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{-7wL}{20} \\ \frac{-wL^2}{20} \\ \frac{-3wL}{20} \\ \frac{wL^2}{30} \end{Bmatrix}$$

$$\boxed{\Rightarrow\ f_{1y} = 29.44\text{kN},\ m_1 = 0,\ f_{2y} = 0.058\ \text{kN},\ m_2 = 59.65\ \text{kN}\cdot\text{m}}$$

Element 2–3

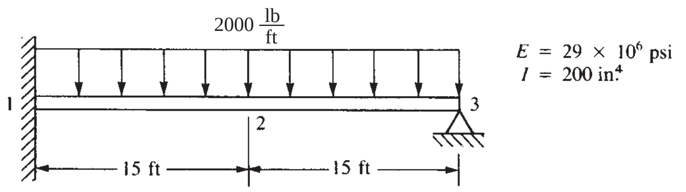
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.011522 \\ 0 \\ 0 \\ -0.0032019 \end{Bmatrix} - \begin{Bmatrix} \frac{-3wL}{20} \\ \frac{-wL^2}{30} \\ \frac{-7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix}$$

$$\boxed{\Rightarrow\ f_{2y} = 0,\ m_3 = -59.65\ \text{kN}\cdot\text{m},\ f_{3y} = 29.94\ \text{kN},\ m_3 = 0}$$

Force in spring

$$F_S = \frac{10\,\text{kN}}{\cancel{m}} \times (0.011522)\cancel{m} = -0.1152\ \text{kN}$$

4.21



Global stiffness matrix of the beam

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

After imposing the boundary conditions $v_1 = \phi_1 = v_3 = 0$ in $\{F\} = [K]\ \{d\}$

$$\begin{Bmatrix} -wL \\ 0 \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Multiplying (1) by L and (3) by – 4 and adding them

$$-wL^2 = \frac{EI}{L^3}\,[24Lv_2 + 0\phi_2 + 6L^2\phi_3]$$

$$\frac{-wL^2}{3} = \frac{EI}{L^3}\,[-24Lv_2 - 8L^2\phi_2 - 16L^2\phi_3]$$

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_2 - 10L^2\phi_3] \qquad (4)$$

Adding (2) and (4) we get

$$\frac{-4wL^2}{3} = \frac{EI}{L^3}\,[-8L^2\phi_3] \Rightarrow \boxed{\phi_3 = \frac{wL^3}{6EI}}$$

Substituting in (2)

$$0 = \frac{EI}{L^3}\left[8L^2\phi_2 + \frac{2L^2wL^3}{6EI}\right] \Rightarrow \boxed{\phi_2 = \frac{-wL^3}{24EI}}$$

Substituting in (1)

$$-wL = \frac{EI}{L^3}\left[24v_2 + \frac{6LwL^3}{6EI}\right] \Rightarrow -wL - wL = 24\,\frac{EI}{L^3}\,v_2$$

$$\Rightarrow \boxed{v_2 = \frac{-wL^4}{12EI}}$$

$$\Rightarrow \phi_3 = \frac{(\frac{2000}{12})(15\times 12)^3}{6\left(29\times 10^6\right)200} = 0.02793 \text{ rad}$$

$$\phi_2 = \frac{-(\frac{2000}{12})(15\times 12)^3}{24\left(29\times 10^6\right)200} = -0.0069827 \text{ rad}$$

$$v_2 = \frac{-1}{12}\,\frac{(\frac{2000}{12})(15\times 12)^4}{29\times 10^6\times 200} = -2.5138 \text{ in.}\downarrow$$

Substituting back in the global equation

$\{F\} = [K]\,\{d\} - \{F_0\}$ we can find the reactions

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -2.5138 \\ -0.0069827 \\ 0 \\ 0.027931 \end{Bmatrix}$$

$$-\begin{Bmatrix} \frac{-wL}{2} = -15000 \text{ lb} \\ \frac{-wL^2}{12} = -450000 \text{ lb}\cdot\text{in.} \\ -wL = -30000 \text{ lb} \\ 0 \\ \frac{-wL}{2} = -15000 \text{ lb} \\ \frac{-wL^2}{12} = 450000 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$

$$F_{1y} = 37500 \text{ lbs}, M_1 = 225000 \text{ lb}\cdot\text{in.}$$

$$F_{2y} = 0, M_2 = 0$$

$$F_{1y} = 22500 \text{ lb}, M_3 = 0$$

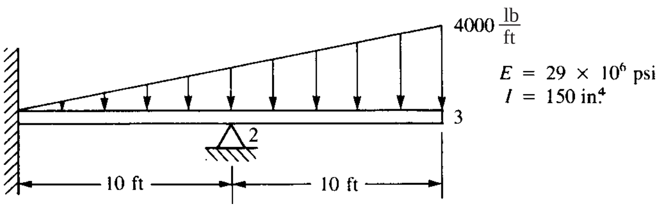
Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.5138 \\ -0.0069827 \end{Bmatrix} - \begin{Bmatrix} -15000 \\ -450000 \\ -15000 \\ 450000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{1y} = 37500 \text{ lb} \\ m_1 = 225000 \text{ lb}\cdot\text{in.} \\ f_{2y} = -37500 \text{ lb} \\ m_2 = 112500 \text{ lb}\cdot\text{in.} \end{matrix}$$

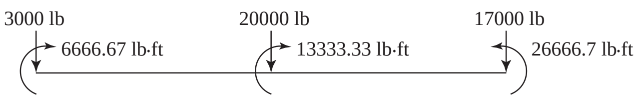
Element 2–3

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -2.5138 \\ -0.0069827 \\ 0 \\ 0.027931 \end{Bmatrix} - \begin{Bmatrix} -15000 \\ -450000 \\ -15000 \\ 450000 \end{Bmatrix} \Rightarrow \begin{matrix} f_{2y} = 7500 \text{ lb} \\ m_2 = -112500 \text{ lb}\cdot\text{in.} \\ f_{3y} = 22500 \text{ lb} \\ m_3 = 0 \end{matrix}$$

4.22



$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6 & 2L^2 & 0 & 0 \\ & & 24 & 0 & -12 & 6L \\ & & & 8L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ \text{Symmetry} & & & & & 4L^2 \end{bmatrix}$$



After applying the boundary conditions

$$v_1 = \phi_1 = v_2 = 0 \text{ in } \{F_0\} = [k]\{d\}$$

$$\begin{Bmatrix} -13333.33 \text{ ft}\cdot\text{lb} \\ -17000 \text{ lb} \\ 26666.67 \text{ ft}\cdot\text{lb} \end{Bmatrix} = 30208.33 \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

Rewriting equations (1) (2) and (3) we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3 \tag{1}$$

$$-0.562759 = -6L\phi_2 + 12v_3 - 6L\phi_3 \tag{2}$$

$$0.882759 = 2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3 \tag{3}$$

Adding (1) to $-4 \times (3)$ we get

$$-0.441379 = 8L^2\phi_2 - 6Lv_3 + 2L^2\phi_3$$
$$-3.53103 = -8L^2\phi_2 + 24Lv_3 - 16L^2\phi_3$$

$$-3.9724 = 18Lv_3 - 14L^2\phi_3 \qquad (4)$$

Adding $L \times (2)$ to $3 \times (3)$ we get (where L = 10 ft)

$$-5.62759 = -6L^2\phi_2 + 12Lv_3 - 6L^2\phi_3$$
$$2.64827 = 6L^2\phi_2 - 18Lv_3 + 12L^2\phi_3$$

$$-297931 = -6Lv_3 + 6L^2\phi_3 \qquad (5)$$

Adding Equation (4) to $3 \times (5)$ we have

$$-12.91034 = 4L^2\phi_3 \;\Rightarrow\; \boxed{\phi_3 = -3.22758 \times 10^{-2} \text{ rad}}$$

Substituting in (4)

$$\Rightarrow\; -3.9724 = 180v_3 - 1400\,(-3.22758 \times 10^{-2})$$
$$\Rightarrow\; \boxed{v_3 = -2.73103 \times 10^{-1} \text{ ft} = -3.27724 \text{ in.}}$$

Substituting in (1)

$$\Rightarrow -0.441379 = 8L^2\phi_2 - 6L\,(-2.73103 \times 10^{-1}) + 2L^2\,(-3.22758)$$
$$\Rightarrow \quad \boxed{\phi_2 = -1.29655 \times 10^{-2} \text{ rad}}$$

$$F_{1y}^{(e)} = \frac{6EI}{L^2}\phi_2 = \frac{6(29 \times 10^6)(150 \text{ in.}^4)}{(120)\text{ in.}^2}\;(-1.29655 \times 10^{-2}) = -23500 \text{ lb}$$

$$M_1^{(e)} = \frac{2EI}{L}\phi_2 = \frac{2(29 \times 10^6)(150 \text{ in.}^4)}{120 \times 12''} = (-1.29655 \times 10^{-2}) = 78333 \text{ lb}\cdot\text{ft}$$

$$F_{2y}^{(e)} = \frac{-12EI}{L^3}v_3 + \frac{6EI}{L^2}\phi_3$$
$$= \frac{-12(29 \times 10^6)(150)}{120 \times 120 \times 120}\;(-3.27724) + \frac{6(29 \times 10^6)}{(120)^2} \times 150$$
$$\times (-3.22758 \times 10^{-1}) = 40500 \text{ lb}$$

$$M_2^{(e)} = \frac{8L^2EI}{L^3}\phi_3 - \frac{6LEI}{L^3}v_2 + \frac{2L^2EI}{L^3}\phi_3 = -13333.33 \text{ ft}\cdot\text{lb}$$

$$F_{3y}^{(e)} = \frac{-6LEI}{L^3}\phi_2 + \frac{12LEI}{L^3}v_3 - \frac{6LEI}{L^3}\phi_3 = -17000 \text{ lb}$$

$$M_3^{(e)} = \frac{2L^2EI}{L^3}\phi_2 - \frac{6LEI}{L^3}v_3 + \frac{4L^2EI}{L^3}\phi_3 = 26{,}666.67 \text{ ft}\cdot\text{lb}$$

Global forces

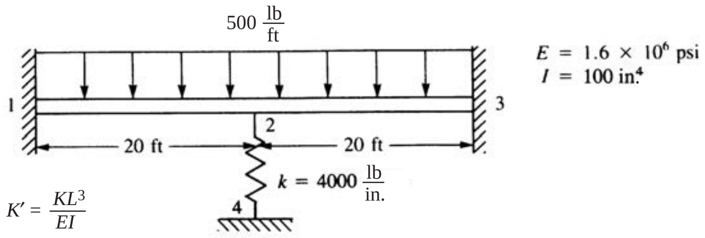
$$F_{1y} = -23500 + 3000 = -20500 \text{ lb}$$
$$M_1 = -78333.33 + 6666.67 = -71{,}666.67 \text{ ft}\cdot\text{lb}$$
$$F_{2y} = 40{,}500 + 20{,}000 = 60500 \text{ lb}$$
$$M_2 = -13333.33 + 13333.33 = 0$$

$F_{3y} = -17000 + 17000 = 0$

$M_3 = 26666.67 - 26666.67 = 0$

Element 1–2	Element 2–3
$f_{1y} = -20500$ lb	$f_{2y} = -30{,}000$ lb
$m_1 = -71666.67$ ft·lb	$m_2 = 2000$ kip·in.
$f_{2y} = 30{,}500$ lb	$f_{3y} = 0$
$m_2 = -2000$ kip·in.	$m_3 = 0$

4.23



After applying the boundary conditions on $\{F\} = [K]\{d\}$ we have

$v_1 = \phi_1 = v_3 = \phi_3 = \phi_2 = 0$

So

$$-wL = \frac{EI}{L^3}\left[24 + \frac{KL^3}{EI}\right] v_2$$

$$\Rightarrow -500 \times 20 = \frac{1.6 \times 10^6 \times 100}{(20 \times 12)^3}\left[24 + \frac{4000(20 \times 12)^3}{1.6 \times 10^6 (100)}\right] v_2$$

$$\Rightarrow \quad -10000 = 277.78\, v_2 \quad \Rightarrow \quad v_2 = -2.338 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = 11.574 \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24+k^1 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.338 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -5000\text{ lb} \\ -200000\text{ in}\cdot\text{lb} \\ -10000 \\ 0 \\ -5000\text{ lb} \\ 200000\text{ in}\cdot\text{lb} \end{Bmatrix}$$

$\Rightarrow \quad F_{1y} = 5325$ lb

$M_1 = 19{,}914$ lb·ft

$F_{2y} = 0$ lb

$M_2 = 0$

$F_{3y} = 5325$ lb

$M_3 = -19{,}914$ lb·ft

Element 1–2

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = 11.574 \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.338 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -5000 \\ -200000 \\ -5000 \\ 200000 \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 5325 \text{ lb}, \; m_1 = 19914 \text{ lb}\cdot\text{ft}$$

$$f_{2y} = 4675 \text{ lb}, \; m_2 = -13419 \text{ lb}\cdot\text{ft}$$

Element 2–3

$$f_{2y} = 4675 \text{ lb}$$

$$m_2 = 13419 \text{ lb}\cdot\text{ft}$$

from symmetry

$$f_{3y} = 5325 \text{ lb}$$

$$m_3 = -19914 \text{ lb}\cdot\text{ft}$$

Note: Spring force is

$$F_s = (4000 \frac{\text{lb}}{\text{in.}})(2.338 \text{ in.}) = 9352 \text{ lb}$$

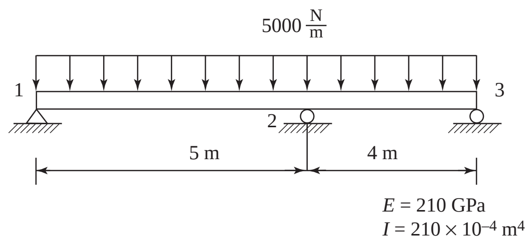
Equilibrium at node 2

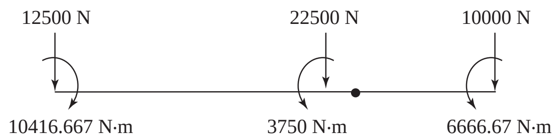
$\downarrow$ 4675 lb from element 1 $\qquad \Sigma F_y = 0$

$\downarrow$ 4675 lb from element 2

$\uparrow F_s =$ 9352 lb

4.24





$$v_1 = 0 = v_2 = v_3$$

$$[k_{1-2}] = EI \begin{array}{c} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \end{array} \begin{array}{l} \phi_1 \\ \phi_2 \end{array}$$

$$[k_{2-3}] = EI \begin{array}{c} \phi_2 \quad \phi_3 \\ \begin{bmatrix} \frac{4}{4} & \frac{2}{4} \\ \frac{2}{4} & \frac{4}{4} \end{bmatrix} \end{array}$$

$$\{F_0\} = [K]\{d\}$$

$$\begin{Bmatrix} -10416.667 \\ 3750 \\ 6666.67 \end{Bmatrix} = (210 \times 10^9)\,(2 \times 10^{-4}) \begin{bmatrix} 0.8 & 0.4 & 0 \\ 0.4 & 1.8 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$-10416.667 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.8\phi_1 + 0.4\phi_2) \quad (1)$$

$$3750 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.4\phi_1 + 1.8\phi_2 + 0.5\phi_3] \quad (2)$$

$$6666.67 = (210 \times 10^9)\,(2 \times 10^{-4})\,[0.5\phi_2 + \phi_3] \quad (3)$$

Multiplying – 2 *x* (2) and adding it to (1) we have

$$-10416.667 = 4.2 \times 10^7\,[0.8\phi_1 + 0.4\phi_2]$$

$$-7500 = 4.2 \times 10^7\,[-0.8\phi_1 - 3.6\phi_2 - \phi_3]$$

$$-17916.667 = 4.2 \times 10^7\,[-3.2\phi_2 - \phi_3] \quad (4)$$

Adding (3) to (4) we have

$$-17916.667 = 4.2 \times 10^7\,[-3.2\phi_2 - \phi_3]$$

$$6666.667 = 4.2 \times 10^7\,[0.5\phi_2 + \phi_3]$$

$$-11250 = 4.2 \times 10^7\,(-2.7\phi_2)$$

$$\Rightarrow \boxed{\phi_2 = 9.92 \times 10^{-5} \text{ rad}}$$

Substituting into (4) we have

$$-17916.667 = 4.2 \times 10^7\,[-3.2\,(9.92 \times 10^{-5}) - \phi_3]$$

$$\Rightarrow \boxed{\phi_3 = 1.091 \times 10^{-4} \text{ rad}}$$

Substituting in (1)

$$\Rightarrow -10416.667 = 4.2 \times 10^7\,[0.8 + (u_0 + (-8)) + 0.4(9.92 \times 10^{-5})]$$

$$\Rightarrow \boxed{\phi_1 = -3.596 \times 10^{-4} \text{ rad}}$$

Element 1–2

$$\begin{Bmatrix} f_{1y}^{(e)} \\ m_1^{(e)} \\ f_{2y}^{(e)} \\ m_2^{(2)} \end{Bmatrix} = \frac{(210 \times 10^9)(2 \times 10^{-4})}{5^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ -3.596 \times 10^{-4} \\ 0 \\ 9.92 \times 10^{-5} \end{Bmatrix}$$

$$\Rightarrow \hat{f}_{1y}^{(e)} = -2625 \text{ N}$$

$$m_1^{(e)} = -10416.67 \text{ N} \cdot \text{m}$$

$$f_{2y}^{(e)} = 2625 \text{ N}$$

$$m_2^{(e)} = -2708.33 \text{ N} \cdot \text{m}$$

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \begin{Bmatrix} -2625 \\ -10416.667 \\ 2625 \\ -2708.33 \end{Bmatrix} - \begin{Bmatrix} -12500 \\ -10416.667 \\ -12500 \\ -10416.667 \end{Bmatrix}$$

$$\Rightarrow f_{1y} = 9875 \text{ N},\ m_1 = 0$$

$$f_{2y} = 15125 \text{ N},\ m_2 = -13125 \text{ N} \cdot \text{m}$$

Element 2–3

$$\begin{Bmatrix} f_{2y}^{(e)} \\ m_2^{(e)} \\ f_{3y}^{(e)} \\ m_3^{(e)} \end{Bmatrix} = \frac{(210\times10^9)(2\times10^{-4})}{4^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 9.92\times10^{-5} \\ 0 \\ -1.091\times10^{-4} \end{Bmatrix}$$

$$\Rightarrow f_{2y} = 13281.25 \text{ N}$$

$$m_2 = 13125 \text{ N}\cdot\text{m}$$

$$f_3 = 6718.5 \text{ N}$$

$$m_3 = 0$$

Global

$$F_{1y} = f_{1y} = 9875 \text{ N}$$

$$M_1 = m_1 = 0$$

$$F_{2y} = 1525 + 13281.25 = 28406.25 \text{ N}$$

$$M_2 = -13125 + 13125 = 0$$

$$F_{3y} = f_{3y} = 6718.75 \text{ N}$$

$$M_3 = m_3 = 0$$

4.25

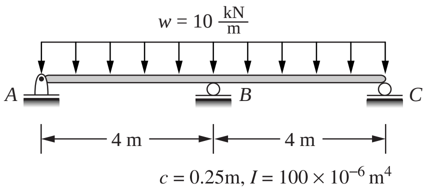
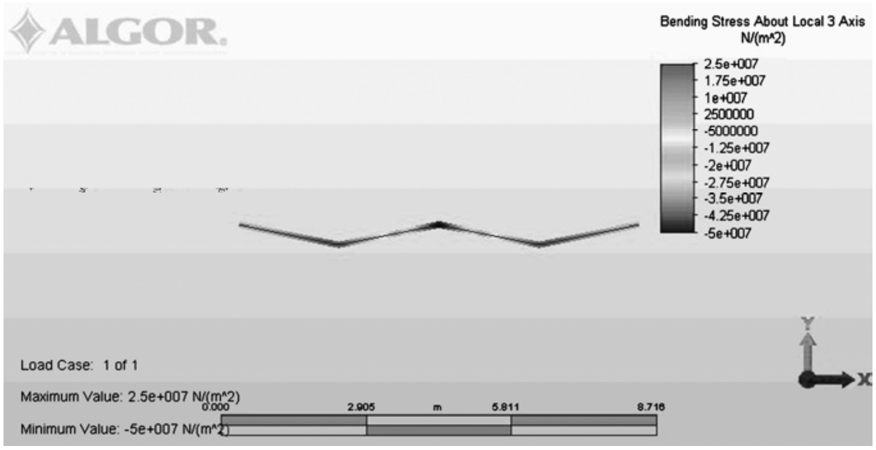
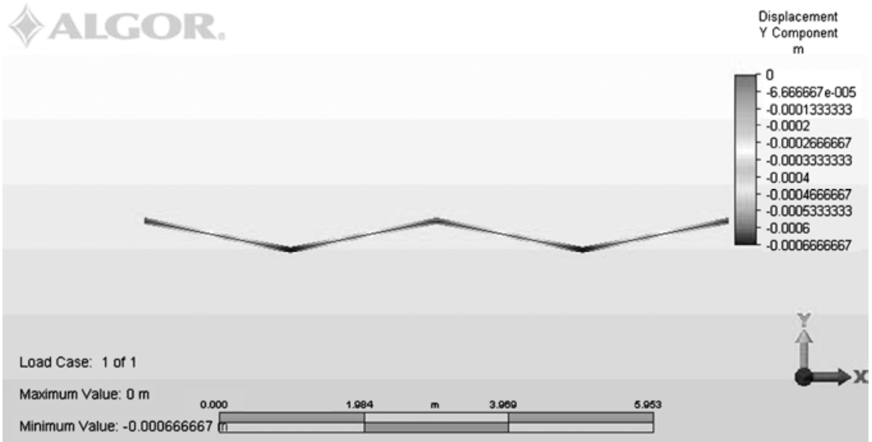


Figure P4–25





4.26

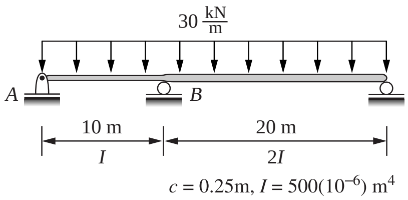
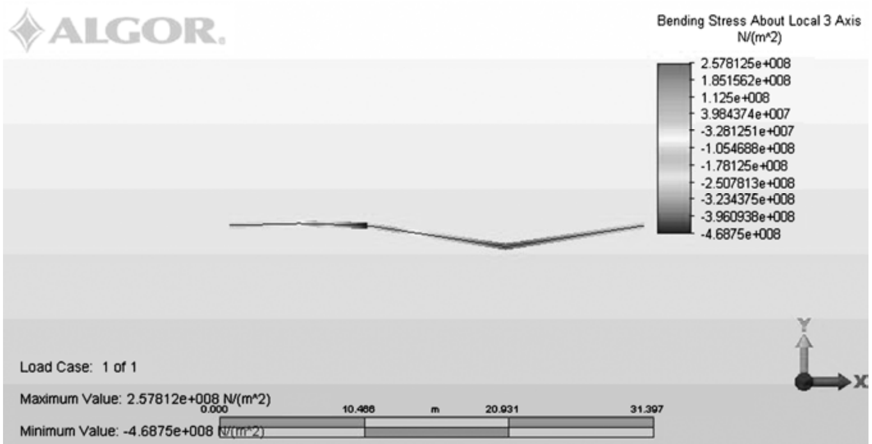
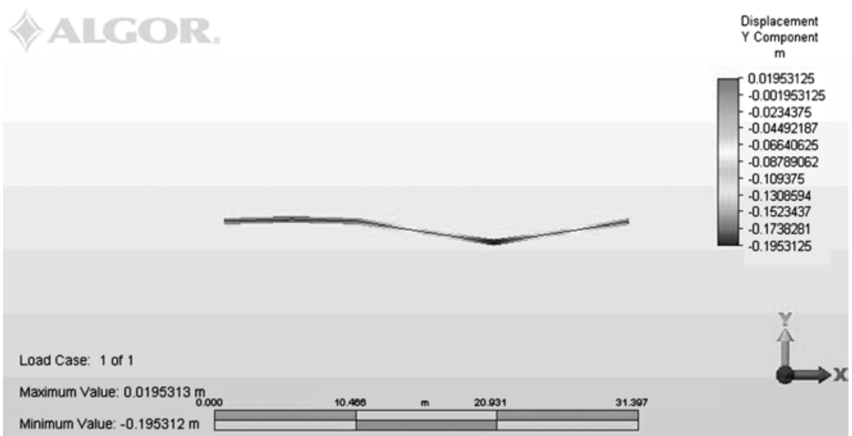


Figure P4–26





4.27

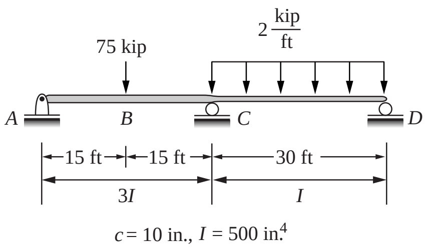
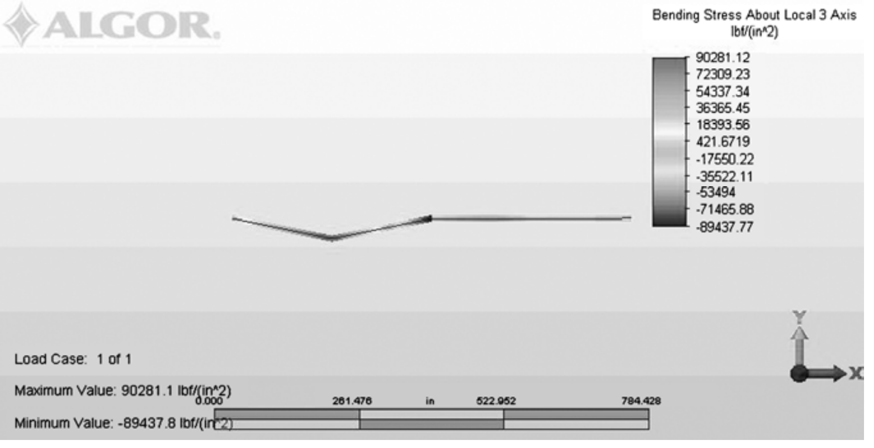
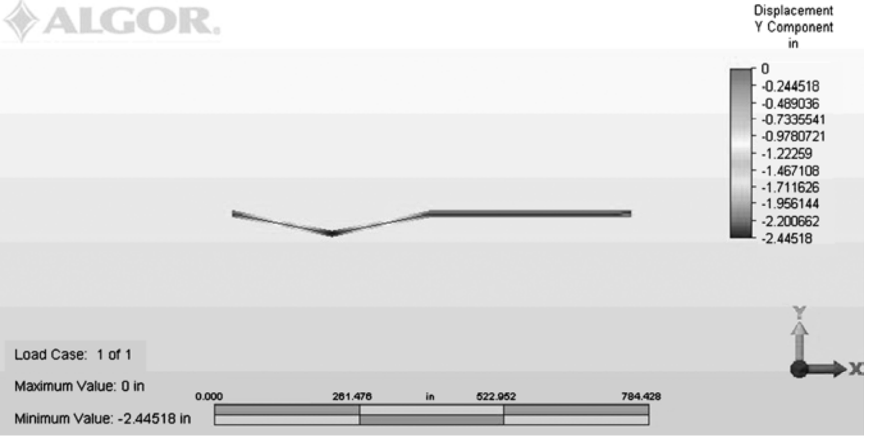


Figure P4–27





4.28

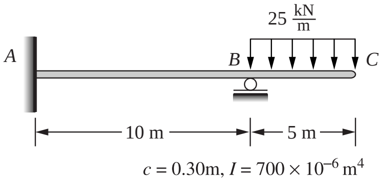


Figure P4–28

1 **** BEAM ELEMENTS

number of beam elements	= 2
number of area property sets	= 1
number of fixed end force sets	= 4

number of materials = 1
number of intermediate load sets = 4

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	4.688E+04	0.000E+00	0.000E+00	0.000E+00	1.562E+05
		0.000E+00	4.688E+04	0.000E+00	0.000E+00	0.000E+00	–3.125E+05
2	1	0.000E+00	–1.250E+05	0.000E+00	0.000E+00	0.000E+00	–3.125E+05
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A-M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	6.6973+07	–6.697E+07	6.697E+07
		0.000E+00	0.000E+00	0.000E+00	–1.339E+08	1.339E+08	1.339E+08
2	1	0.000E+00	0.000E+00	0.000E+00	–1.339E+08	1.339E+08	1.339E+08
		0.000E+00	0.000E+00	0.000E+00	–1.192E–07	1.192E–07	1.192E–07

4.29

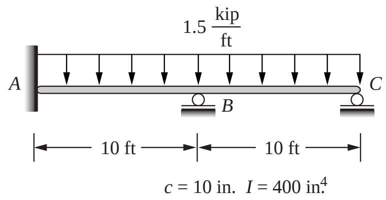
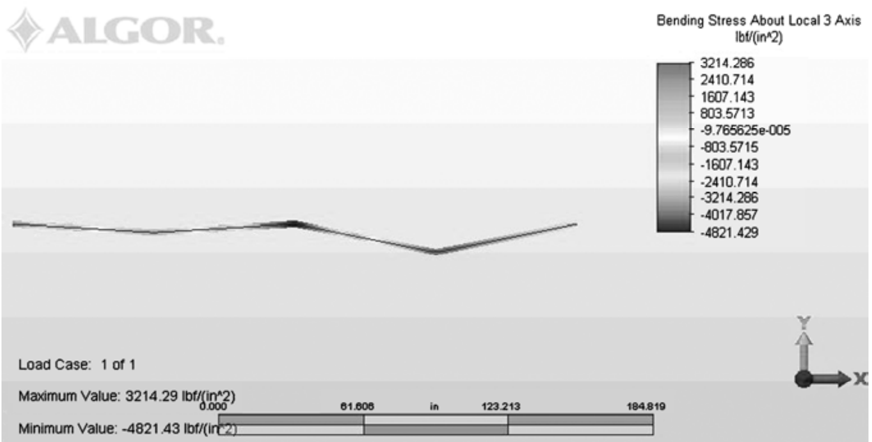
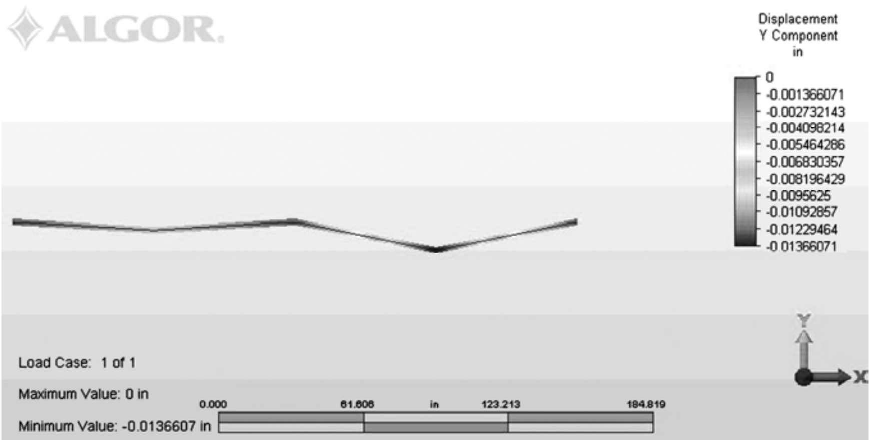
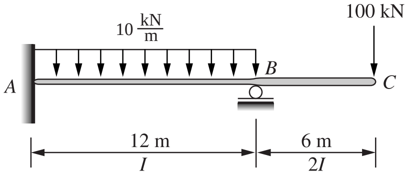


Figure P4–29



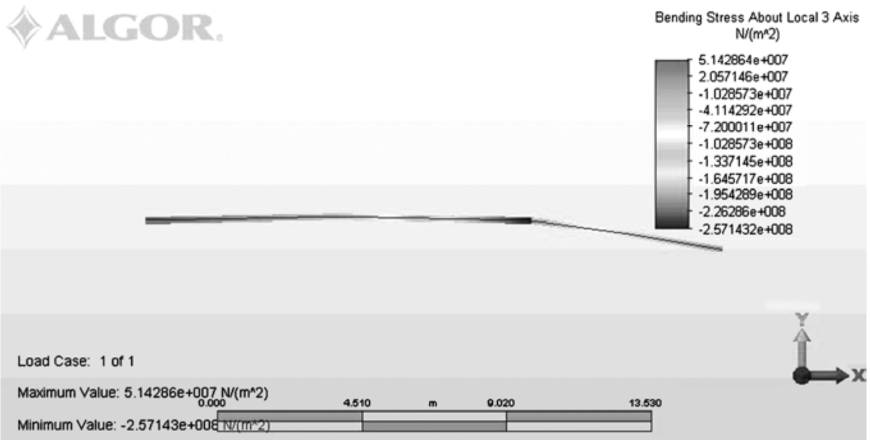


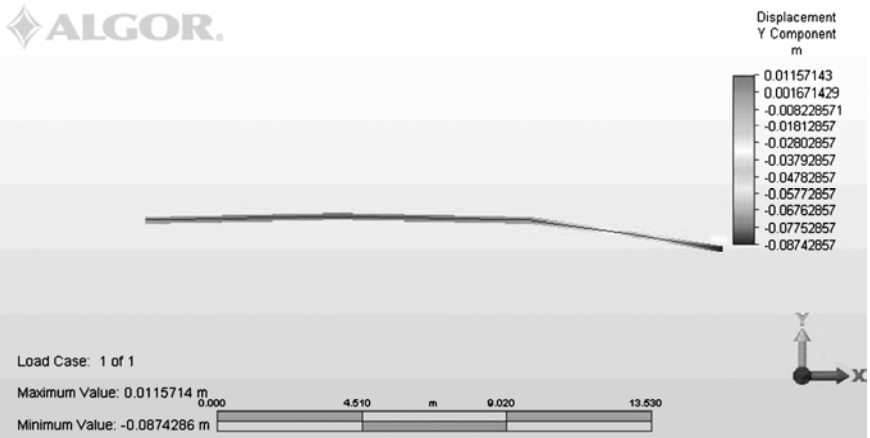
4.30



$c = 0.30\text{m}$, $I = 700 \times 10^{-6}\ \text{m}^4$

Figure P4–30





4.31 Design a beam of ASTM A36 steel with allowable bending stress of 160 MPa to support the load shown in Figure P4–31. Assume a standard wide flange beam from Appendix F or some other source can be used.

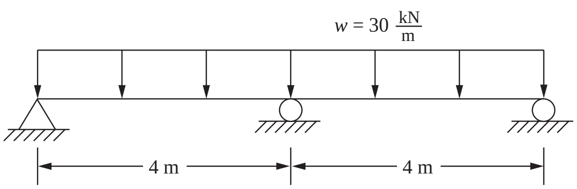


Figure P4–31

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–4.500E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
2	1	0.000E+00	1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
		0.000E+00	7.500E+04	0.000E+00	0.000E+00	0.000E+00	–6.000E+04
3	1	0.000E+00	–7.500E+04	0.000E+00	0.000E+00	0.000E+00	–6.000E+04
		0.000E+00	–1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
4	1	0.000E+00	–1.500E+04	0.000E+00	0.000E+00	0.000E+00	3.000E+04
		0.000E+00	4.500E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A-M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
2	1	0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
		0.000E+00	0.000E+00	0.000E+00	–9.317E+06	9.317E+06	9.317E+06
3	1	0.000E+00	0.000E+00	0.000E+00	–9.317E+06	9.317E+06	9.317E+06
		0.000E+00	0.000E+00	0.000E+00	4.658E+06	–4.658E+06	4.658E+06
4	1	0.000E+00	0.000E+00	0.000E+00	4.658E+06	-4.658E+06	4.658E+06
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

S75 × 11.2	1430	76	64	6.6	8.9	1.20	31.6	29.0	0.254	7.72	13.1
8.5	1070	76	59	6.6	4.3	1.03	27.1	31.0	0.190	6.44	13.3

† It may be noted that an American Standard Beam is designated by the letter S followed by the nominal depth in millimeters and the mass in kilograms per meter. S75 × 8.5 acceptable for σ_{max} ≤ 160 MPa. But not for deflection. Try larger section. W 10 × 112 works.

4.32 Select a standard steel pipe from Appendix F to support the load shown. The allowable bending stress must not exceed 24 ksi, and the allowable deflection must note exceed $\frac{L}{360}$ of any span.

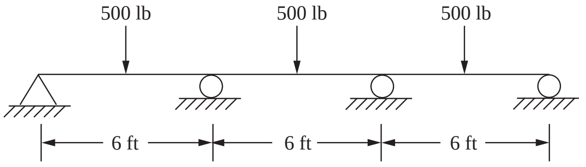
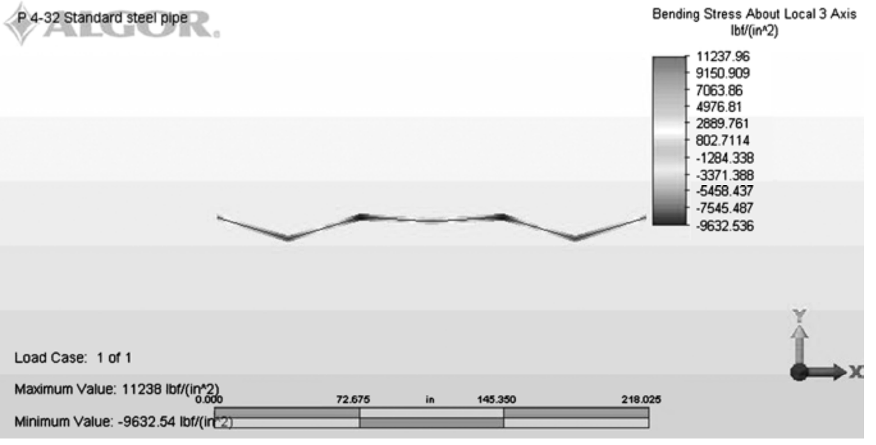
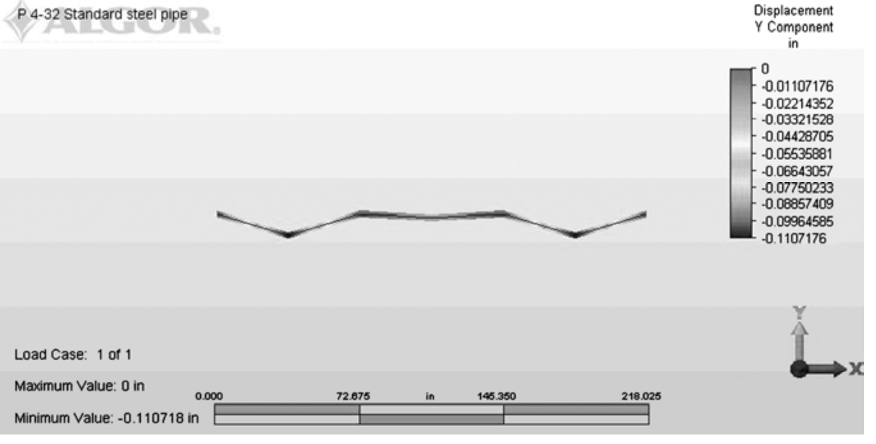


Figure P4–32





1 **** BEAM ELEMENTS

number of beam elements = 6
number of area property sets = 1
number of fixed end force sets = 4
number of materials = 1
number of intermediate load sets = 4

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT No.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–1.751E+02	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	–1.751E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
2	1	0.000E+00	3.249E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
		0.000E+00	3.249E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
3	1	0.000E+00	–2.500E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
		0.000E+00	–2.500E+02	0.000E+00	0.000E+00	0.000E+00	3.606E+03
4	1	0.000E+00	2.500E+02	0.000E+00	0.000E+00	0.000E+00	3.606E+03
		0.000E+00	2.500E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
5	1	0.000E+00	–3.249E+02	0.000E+00	0.000E+00	0.000E+00	–5.394E+03
		0.000E+00	–3.249E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
6	1	0.000E+00	1.751E+02	0.000E+00	0.000E+00	0.000E+00	6.303E+03
		0.000E+00	1.751E+02	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1 **** BEAM ELEMENT STRESSES

ELEMENT No.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
2	1	0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
		0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
3	1	0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
		0.000E+00	0.000E+00	0.000E+00	6.432E+03	–6.432E+03	6.432E+03
4	1	0.000E+00	0.000E+00	0.000E+00	6.432E+03	–6.432E+03	6.432E+03
		0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
5	1	0.000E+00	0.000E+00	0.000E+00	–9.621E+03	9.621E+03	9.621E+03
		0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
6	1	0.000E+00	0.000E+00	0.000E+00	1.124E+04	–1.124E+04	1.124E+04
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

σ_{max} = 11.24 ksi < σ_{allow} = 24 ksi (For 2 in. schedule 40 steel pipe, I = 0.666 in.^{4})

4.33 Select a rectangular structural tube from Appendix F to support the loads shown for the beam in Figure P4–33. The allowable bending stress should not exceed 24 ksi.

Rectangular tube $4'' \times 2\frac{1}{2}'' \times \frac{5}{16}''$
I_2 = 2.89 in.^{4}
I_3 = 6.13 in.^{4}

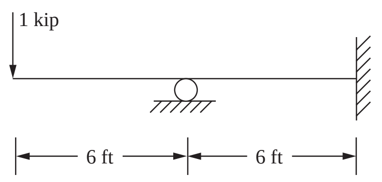


Figure P4–33

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT No.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–3.600E+04
2	1	0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–3.600E+04
		0.000E+00	1.000E+03	0.000E+00	0.000E+00	0.000E+00	–7.200E+04
3	1	0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–7.200E+04
		0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–1.800E+04
4	1	0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	–1.800E+04
		0.000E+00	–1.500E+03	0.000E+00	0.000E+00	0.000E+00	3.600E+04

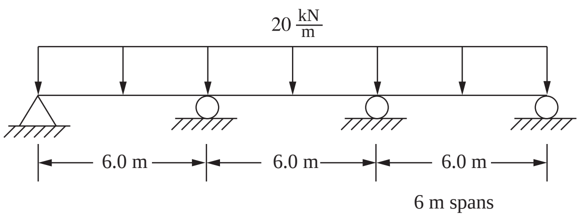
1 **** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		0.000E+00	0.000E+00	0.000E+00	–1.176E+04	1.176E+04	1.176E+04

2	1	0.000E+00	0.000E+00	0.000E+00	–1.176E+04	1.176E+04	1.176E+04
		0.000E+00	0.000E+00	0.000E+00	–2.353E+04	2.353E+04	2.353E+04
3	1	0.000E+00	0.000E+00	0.000E+00	–2.353E+04	2.353E+04	2.353E+04
		0.000E+00	0.000E+00	0.000E+00	–5.882E+03	5.882E+03	5.882E+03
4	1	0.000E+00	0.000E+00	0.000E+00	–5.882E+03	5.882E+03	5.882E+03
		0.000E+00	0.000E+00	0.000E+00	1.176E+04	–1.176E+04	1.176E+04

σ_{max} = 23,530 psi < σ_{allow} = 24,000 psi

4.34 Select a standard W section from Appendix F or some other source to support the loads shown for the beam in Figure P4–34. The bending stress must not exceed 160 MPa.



$W310 \times 39$ (mm $\times \frac{kg}{m}$)

σ_{max} = 139 MPa < 160 MPa

Figure P4 –34

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	–4.2861E–03	0.0000E+00	0.0000E+00	0.0000E+00	–5.8470E–03
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	2.3388E–02
4	0.0000E+00	–2.4492E–03	0.0000E+00	0.0000E+00	0.0000E+00	1.7541E–02
5	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–9.3553E–02
6	0.0000E+00	–9.7968E–03	0.0000E+00	0.0000E+00	0.0000E+00	–6.4317E–02

1 **** BEAM ELEMENT FORCE AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R1	SHEAR FORCE R2	SHEAR FORCE R3	TORSION MOMENT M1	BENDING MOMENT M2	BENDING MOMENT M3
1	1	0.000E+00	–6.115E+04	0.000E+00	0.000E+00	0.000E+00	–6.231E+04
		0.000E+00	–1.154E+03	0.000E+00	0.000E+00	0.000E+00	3.115E+04
2	1	0.000E+00	–1.154E+03	0.000E+00	0.000E+00	0.000E+00	3.115E+04
		0.000E+00	5.885E+04	0.000E+00	0.000E+00	0.000E+00	–5.538E+04
3	1	0.000E+00	–5.654E+04	0.000E+00	0.000E+00	0.000E+00	–5.538E+04
		0.000E+00	3.462E+03	0.000E+00	0.000E+00	0.000E+00	2.423E+04
4	1	0.000E+00	3.462E+03	0.000E+00	0.000E+00	0.000E+00	2.423E+04
		0.000E+00	6.346E+04	0.000E+00	0.000E+00	0.000E+00	–7.615E+04
5	1	0.000E+00	–7.269E+04	0.000E+00	0.000E+00	0.000E+00	–7.615E+04
		0.000E+00	–1.269E+04	0.000E+00	0.000E+00	0.000E+00	5.192E+04
6	1	0.000E+00	–1.269E+04	0.000E+00	0.000E+00	0.000E+00	5.192E+04
		0.000E+00	4.731E+04	0.000E+00	0.000E+00	0.000E+00	0.000E+00

1**** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A + M2/S2	P/A–M2/S2	P/A+M3/S3	P/A-M3/S3	WORST SUM
1	1	0.000E+00	0.000E+00	0.000E+00	–1.139E+08	1.139E+08	1.139E+08
		0.000E+00	0.000E+00	0.000E+00	5.695E+07	– 5.695E+07	5.695E+07
2	1	0.000E+00	0.000E+00	0.000E+00	5.695E+07	– 5.569E+07	5.695E+07
		0.000E+00	0.000E+00	0.000E+00	–1.013E+08	1.013E+08	1.013E+08
3	1	0.000E+00	0.000E+00	0.000E+00	–1.013E+08	1.013E+08	1.013E+08
		0.000E+00	0.000E+00	0.000E+00	4.430E+07	– 4.430E+07	4.430E+07
4	1	0.000E+00	0.000E+00	0.000E+00	4.430E+07	–4.430E+07	4.430E+07
		0.000E+00	0.000E+00	0.000E+00	–1.392E+08	1.392E+08	1.392E+08
5	1	0.000E+00	0.000E+00	0.000E+00	–1.392E+08	1.392E+08	1.392E+08
		0.000E+00	0.000E+00	0.000E+00	9.492E+07	–9.492E+07	9.492E+07
6	1	0.000E+00	0.000E+00	0.000E+00	9.492E+07	–9.492E+07	9.492E+07
		0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

4.35 For the beam shown in Figure P4–35, determine a suitable sized W section from Appendix F or from another suitable source such that the bending stress does not exceed 150 MPa and the maximum deflection does not exceed $\frac{L}{360}$ of any span.

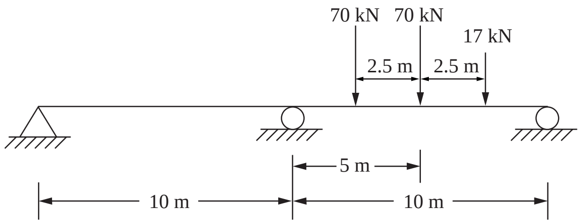


Figure P4–35

ASTM A36 steel

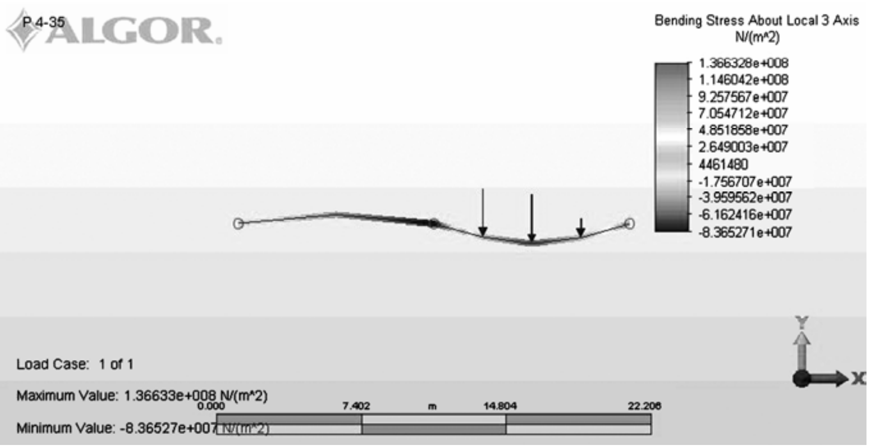
$$\frac{L}{360} = \frac{10\text{ m}}{360} = 0.0278\text{ m}$$

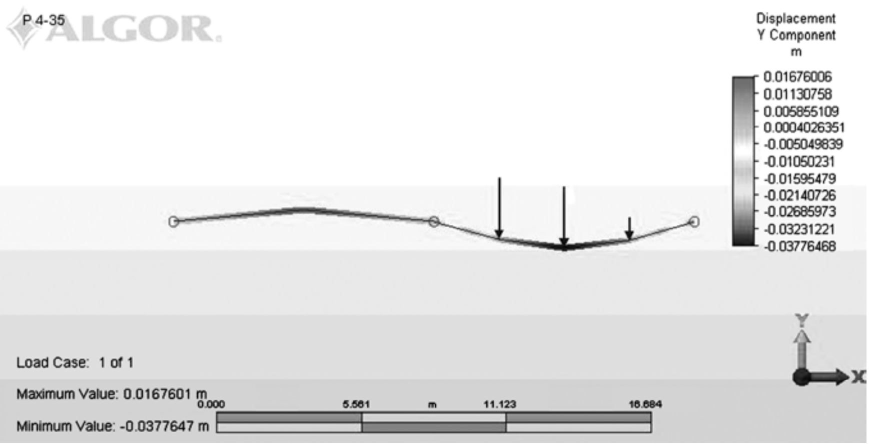
$$\Delta Y_{\text{max}} = 0.0556\text{ m}$$

Bending stress max = 150 MPa = $1.50 \times 10^8 \ \frac{\text{N}}{\text{m}^2}$

Beam	I_3 (m^4)	S_3(m^3)	A(m^2)	Bending stress ($\frac{\text{N}}{\text{m}^2}$)	ΔY_{max} (m)
W310 × 143	0.000348	0.002150	0.018200	1.010×10^8	– 0.0269
W460 × 158	0.000796	0.00340	0.0201	6.389×10^7	– 0.0118
W760 × 257	0.003420	0.008850	0.0326	2.455×10^7	– 0.00275
W310 × 44.5	0.0000992	0.000634	0.00569	3.426×10^8	– 0.0944
W310 × 74	0.000165	0.001060	0.009480	2.049×10^8	– 0.0568
W310 × 107	0.000248	0.001590	0.013600	1.366×10^8	– 0.03776

For the problem given a W310 × 107 beam was chosen made of ASTM A36 steel. This made for a maximum deflection of 0.0378 m (see above table) which is less than the 0.0556 m maximum restraint.





4.36 For the stepped shaft shown in Figure P4–36, determine a solid circular cross section for each section shown such that the bending stress does not exceed 160 MPa and the maximum deflection does not exceed $\frac{L}{360}$ of the span.

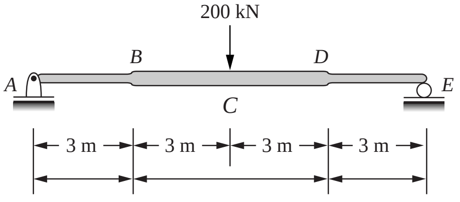


Figure P4–36

Try small radius of 140 mm

Large radius of 166 mm

Yields $\sigma_{\max} = $ 166 MPa close to

$\sigma_{\max}$ allow = 160 MPa.

Need to increase smaller diameter

$$\frac{L}{360} = \frac{12 \text{ m}}{360} = \frac{1}{30} = 0.0333 \text{ m}$$

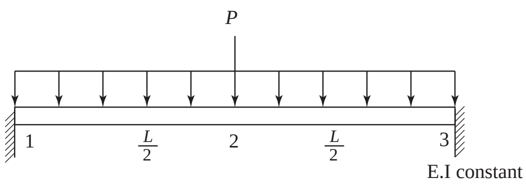
Other trials shown below

d_{AB} and d_{DE}	d_{BD}, mm	σ (MPa)	δ (m)	
279.6	332.5	166.3 MPa > 160	0.064 > 0.033	
290	340	155.5	0.058	δ Too Large
310	360	131	0.046	δ Too Large
340	390	103	0.0330	δ Finally at Limit of $\frac{L}{360}$

$\therefore$ Final $d_{AB} =$ 340 mm $= d_{DE}$

$d_{BD} =$ 390 mm

4.37



Applying the boundary conditions

$v_1 = \phi_1 = \phi_2 = v_3 = \phi_3 = 0$ in the global equation $\{F\} = [K]\,\{d\}$ we have

$$-\left(P + \frac{wL}{2}\right) = \frac{EI}{(\frac{L}{2})^3}\;[24 v_2]$$

$$\Rightarrow\; -P - \frac{wL}{2} = \frac{8EI}{L^3}\;24 v_2 \;\Rightarrow\; \boxed{v_2 = \frac{-PL^3}{192EI} - \frac{wL^4}{384EI}}$$

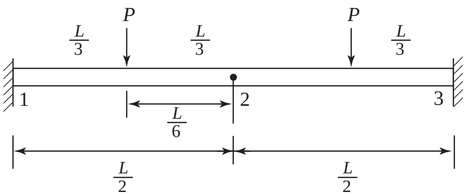
Reactions

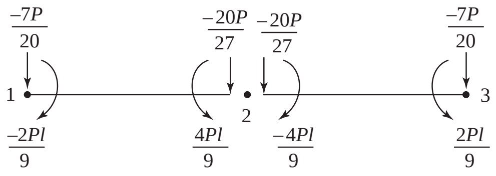
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{8EI}{L^3}\begin{bmatrix} 12 & \frac{6L}{2} & -12 & \frac{6L}{2} & 0 & 0 \\ \frac{6L}{2} & 4\left(\frac{L}{2}\right)^2 & \frac{-6L}{2} & \frac{2L^2}{4} & 0 & 0 \\ -12 & \frac{-6L}{2} & 24 & 0 & -12 & \frac{6L}{2} \\ 6L & \frac{2L^2}{4} & 0 & \frac{8L^2}{4} & \frac{-6L}{2} & \frac{2L^2}{4} \\ 0 & 0 & -12 & \frac{-6L}{2} & 12 & \frac{-6L}{2} \\ 0 & 0 & \frac{6L}{2} & \frac{2L^2}{2} & \frac{-6L}{2} & \frac{4L^2}{4} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-PL^3}{192\,EI} - \frac{wL^4}{384\,EI} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \frac{-wL}{4} \\ \frac{-wL^2}{48} \\ \frac{-wL}{2} \\ 0 \\ \frac{-wL}{4} \\ \frac{wL^2}{48} \end{Bmatrix}$$

$$\Rightarrow\; F_{1y} = \frac{P + wL}{2},\; M_1 = \frac{PL}{8} + \frac{wL^2}{12},\; F_{2y} = 0$$

$$M_2 = 0,\; F_{3y} = \frac{P + wL}{2},\; M_3 = \frac{-PL}{8} - \frac{wL^2}{12}$$

4.38





After applying the boundary conditions

$$v_1 = \phi_1 = \phi_2 = v_2 = \phi_3 = 0$$

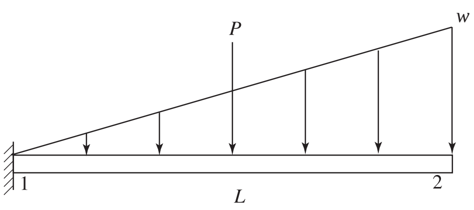
We have in the equation $\{F\} = [K]\ \{d\}$ the following

$$\frac{-40P}{27} = \frac{EI}{(3l)^3}\ [24v_2] \Rightarrow v_2 = \frac{-40Pl^3}{24EI}$$

$$\Rightarrow\ v_2 = \frac{-5Pl^3}{3EI}\ \text{since } l = \frac{L}{6}$$

$$\Rightarrow\ v_2 = \frac{-5P(\frac{L}{6})^3}{3EI} \quad \Rightarrow \quad \boxed{v_2 = \frac{-5PL^3}{648\,EI}}$$

4.39



Applying the boundary conditions $v_1 = \phi_1 = 0$

$$\Rightarrow \begin{Bmatrix} -\dfrac{P}{2} - \dfrac{7wL}{20} \\ \dfrac{PL}{8} + \dfrac{wL^2}{20} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Multiplying (1) by L and (2) by 2 and then adding

$$\frac{-PL}{2}-\frac{7wL^2}{20} = \frac{EI}{L^3}\ [12Lv_2 - 6L^2\phi_2]$$

$$\frac{PL}{4}+\frac{2wL^2}{20} = \frac{EI}{L^3}\ [-12Lv_2 + 8L^2\phi_2]$$

$$\frac{-PL}{4}-\frac{wL^2}{4} = \frac{EI}{L^3}\ (2L^2\phi_2) \Rightarrow \boxed{\phi_2 = \frac{-(PL^2 + wL^3)}{8EI}}$$

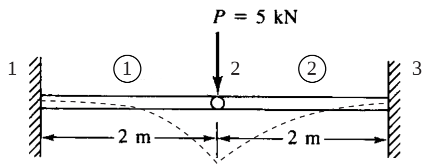
Substituting in (1) $\Rightarrow \dfrac{-P}{2}-\dfrac{7wL}{20} = \dfrac{EI}{L^3}\left[12v_2 - 6L\left[\dfrac{-PL^2 - wL^3}{8EI}\right]\right]$

$$\Rightarrow\ \frac{-5P}{4}-\frac{22wL}{20} = \frac{12EI}{L^3}\ v_2 \Rightarrow \boxed{v_2 = \frac{-(25P+22wL)L^3}{240EI}}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{-(25P+22wL)L^3}{240EI} \\ \frac{-PL^2-wL^3}{EI} \end{Bmatrix} - \begin{Bmatrix} -\frac{P}{2}-\frac{3wL}{20} \\ -\frac{PL}{8}-\frac{wL^2}{30} \\ -\frac{P}{2}-\frac{7wL}{20} \\ \frac{PL}{8}+\frac{wL^2}{20} \end{Bmatrix}$$

$$\Rightarrow \begin{array}{l} F_{1y} = P + \dfrac{wL}{2}, M_1 = \dfrac{PL}{2} + \dfrac{1}{3}wL^2 \\ F_{2y} = 0, M_2 = 0 \end{array}$$

4.40



Assume the hinge as a part of the first element. Therefore, stiffness matrix for element 1 is

$$[k^{(1)}] = \frac{3EI}{8}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Stiffness matrix of element 2 is

$$[k^{(2)}] = \begin{array}{cc} \begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix} & \begin{array}{c} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{array} \end{array}$$

Adding the matrices by superposition

$$[K] = \begin{array}{c} \begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 3 & 6 & -3 & 0 & 0 & 0 \\ 6 & 12 & 6 & 0 & 0 & 0 \\ -3 & -6 & 15 & 12 & -12 & 12 \\ 0 & 0 & 12 & 16 & -12 & 8 \\ 0 & 0 & 12 & -12 & 12 & -12 \\ 0 & 0 & 12 & 8 & 12 & 16 \end{bmatrix} \end{array}$$

Applying the boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_3 = 0,\ \phi_3 = 0$$

$$\therefore\ \frac{EI}{8}\begin{bmatrix} 15 & 12 \\ 12 & 16 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -5\ \text{kN} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \frac{EI}{8}[15v_2 + 12\phi_2] = -5000 \qquad (1)$$

$$12v_2 + 16\phi_2 = 0 \qquad (2)$$

$$\text{From (2), } v_2 = -\frac{4}{3}\phi_2 \qquad (3)$$

Substituting (3) in (1)

$$\overset{5}{\cancel{15}} \times \left(-\frac{4}{\cancel{3}}\right)\phi_2 + 12\phi_2 = \frac{-5 \times 10^3 \times 8}{210 \times 10^9 \times 2 \times 10^{-4}}$$

$$\Rightarrow -8\phi_2 = \frac{-5 \times 10^3 \times 8}{420 \times 10^5}$$

$$\Rightarrow \quad \phi_2 = \frac{5000}{420 \times 10^5}$$

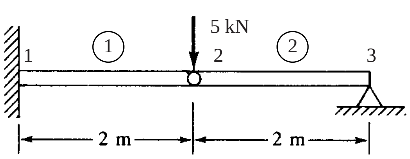
$$\Rightarrow \quad \boxed{\phi_2^{(2)} = 1.19 \times 10^{-4}\ \text{rad}}$$

$$\therefore \quad v_2 = -\frac{4}{3}\phi_2$$

$$= -\frac{4}{3} \times 1.19 \times 10^{-4}$$

Hence $\boxed{v_2 = -1.57 \times 10^{-4}\ \text{m}}$

4.41



$$[k^{(1)}] = \frac{3EI}{8}\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \frac{EI}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix}$$

By superposition

$$[K] = \frac{EI}{8}\begin{array}{c} \begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 3 & 6 & -3 & 0 & 0 & 0 \\ 6 & 12 & -6 & 0 & 0 & 0 \\ -3 & -6 & 15 & 12 & -12 & 12 \\ 0 & 0 & 12 & 16 & -12 & 8 \\ 0 & 0 & -12 & -12 & 12 & -12 \\ 0 & 0 & 12 & 8 & -12 & 16 \end{bmatrix} \end{array}$$

Boundary conditions

$$v_1 = 0,\ \phi_1 = 0,\ v_3 = 0$$

$$\frac{EI}{8}\begin{bmatrix} 15 & 12 & 12 \\ 12 & 16 & 8 \\ 12 & 8 & 16 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -5000 \\ 0 \\ 0 \end{Bmatrix}$$

Solving by Gaussian Elimination we have

$$\left[\begin{array}{ccc:c} 15 & 12 & 12 & -9.5\times10^{-4} \\ 12 & 16 & 8 & 0 \\ 12 & 8 & 16 & 0 \end{array}\right]$$

Select $a_{11} = 15$ as the pivot

(a) Add the multiple $\frac{-a_{21}}{a_{11}} = \frac{-12}{15} = \frac{-4}{5}$ of the first row to the second row

(b) Add the multiple $\frac{-a_{31}}{a_{11}} = \frac{-4}{5}$ of the first row to the third row.

$$\left[\begin{array}{ccc:c} 15 & 12 & 12 & -9.5\times10^{-4} \\ 0 & 6.4 & -1.6 & 7.6\times10^{-4} \\ 0 & -1.6 & 6.4 & 7.6\times10^{-4} \end{array}\right]$$

Select $a_{22} = 6.4$ as the pivot

Repeating the same procedure

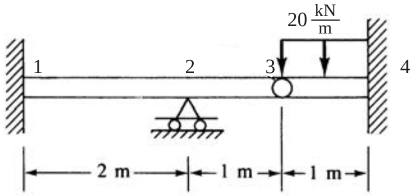
$$\begin{bmatrix} 15 & 12 & 12 & -9.5\times10^{-4} \\ 0 & 6.4 & -1.6 & 7.6\times10^{-4} \\ 0 & 0 & 6 & 9.5\times10^{-4} \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{\phi_3 = 1.583 \times 10^{-4} \text{ rad}}}$$

$$\underline{\underline{\phi_2 = 1.583 \times 10^{-4} \text{ rad}}}$$

$$\underline{\underline{v_2 = -\,3.175 \times 10^{-4} \text{ m}}}$$

4.42



$$[K^{(1)}] = \frac{EI}{8}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix} \end{array} = EI\begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{3}{2} & \frac{-3}{2} \\ \frac{3}{2} & 1 & \frac{-3}{2} & 2 \end{bmatrix}$$

Assume the hinge as *a* + right end part of element (2)

$$[K^{(2)}] = \frac{3EI}{(1)^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[K^{(3)}] = \frac{EI}{(1)^3}\begin{array}{c} \begin{array}{cccc} v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array}$$

By superposition

$$[K] = \begin{array}{c} \begin{array}{cccccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 & v_4 & \phi_4 \end{array} \\ \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{-3}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 2 & \frac{-3}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{-3}{2} & \frac{-3}{2} & \frac{9}{2} & \frac{-3}{2} & -3 & 0 & 0 & 0 \\ \frac{3}{2} & 1 & \frac{-3}{2} & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & -3 & 15 & 6 & -12 & 6 \\ 0 & 0 & 0 & 0 & 6 & 4 & -6 & 2 \\ 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \end{array}$$

Applying the boundary conditions

$$v_1 = \phi_1 = v_2 = v_4 = \phi_4 = 0$$

$$\Rightarrow \; EI\begin{bmatrix} 5 & -3 & 0 \\ -3 & 15 & 6 \\ 0 & 6 & 4 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10\text{ kN} \\ -1.67\text{ kN}\cdot\text{m} \end{Bmatrix}$$

$$\Rightarrow \; \phi_2 = \frac{3}{5}v_3 \tag{1}$$

$$6v_3 + 4\phi_3 = \frac{1667}{EI}$$

$$\Rightarrow \; \phi_3 = \left[\frac{-416.75}{EI} - 1.5v_3\right] \tag{2}$$

$$EI(-3\phi_2 + 15v_3 + 6\phi_3) = -10000 \tag{3}$$

Substituting (2) and (1) in (3)

$$-\frac{9}{5}v_3 + 15v_3 - \frac{2500}{EI} - 9v_3 = \frac{-10000}{EI}$$

$$\Rightarrow \quad \frac{21}{5}v_3 = -\frac{12500}{EI}$$

$$\underline{\underline{v_3 = -4.252\times 10^{-5}\text{ m}}}$$

$$\underline{\underline{\phi_2 = -2.551\times 10^{-5}\text{ rad}}}$$

$$\underline{\underline{\phi_3 = 5.386\times 10^{-5}\text{ rad}}}$$

4.43 From Equation (4.7.15)

$$\pi_p = \int_0^L \frac{EI}{2}\{d\}[B]^T[B]\{d\}\,dx - \int_0^L w\{d\}^T[N]^T\,dx - \{d\}^T\{P\}$$

$$\{d\} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

EI $\rightarrow$ constant

$$\frac{\partial\pi_p}{\partial v_2} = \left(\frac{\cancel{2}EI}{\cancel{2}}\int_0^L [B]^T[B]dx\right)d_{1y} - \left(\int_0^L N_1 w dx\right) - f_{1y} = 0 \tag{1}$$

$$\frac{\partial\pi_p}{\partial\phi_1} = \left(\frac{2EI}{2}\int_0^L [B]^T[B]dx\right)\phi_1 - \left(\int_0^L N_2 w dx\right)1_1 - m_1 = 0 \tag{2}$$

$$\frac{\partial\pi_p}{\partial v_2} = \left(\frac{\cancel{2}EI}{\cancel{2}}\int_0^L [B]^T[B]dx\right)d_{2y} - \left(\int_0^L N_3 w dx\right)1_1 - f_{2y} = 0 \tag{3}$$

$$\frac{\partial\pi_p}{\partial\phi_2} = \left(\frac{2EI}{2}\int_0^L [B]^T[B]dx\right)\phi_2 - \left(\int_0^L N_4 w dx\right)1_1 - m_2 = 0 \tag{4}$$

Equations (1) – (4) in matrix form are

$$EI\int_0^L [B]^T[B]dx\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} - \int_0^L \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} wdx - \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = 0$$

Simplifying

$$EI\int_0^L [B]^T[B]dx\{d\} - \int_0^L [N]^T wdx - \{P\} = 0$$

4.44

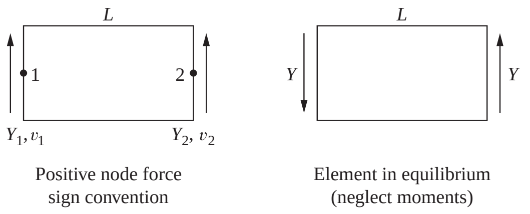


Figure P4–44

$$Y = A_W\,\tau = A_W\,G\gamma = A_w\,G\left(\frac{v_2 - v_1}{L}\right)$$

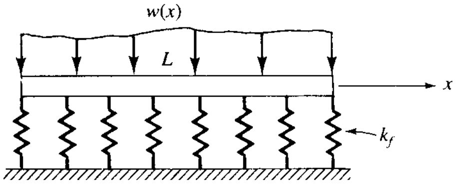
$$Y_1 = -Y = \frac{-A_W\,G}{L}\;(v_2 - v_1)$$

$$Y_2 = Y = \frac{A_W\,G}{L}\;(v_2 - v_1)$$

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \frac{A_W\,G}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

$$[k] = \frac{A_W\,G}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.47



$$\pi_p = \int_0^L \frac{1}{2}EI(v'')^2\,dx + \int_0^L \frac{k_f v^2}{2}dx - \int_0^L wvdx$$

$$v = [N]\,\{d\} \qquad \varepsilon_x = -y\frac{d^2v}{dx^2} = -yv''$$

$$\varepsilon_x = -\;\overline{y}\,[B]\,\{d\}$$

$[B]$ from Equation (4.7.10)

$$\pi_p = \int_0^L\int_A \frac{1}{2}\{\sigma_x^T\}\{\varepsilon_x\}dAdx - \int_0^L bT_y vdx\;\int_0^L \frac{1}{2}k_f\,\{d^T\}[N^T][N]\{d\}\,dx \qquad \underline{bT_y = w}$$

$$= \frac{EI}{2}\int_0^L \{d^T\}B^T[B]\{d\}\,dx - \int_0^L w\,\{d^T\}[N^T]dx + \int_0^L \frac{k_f}{2}\{d^T\}[N^T][N]\{d\}\,dx$$

$$\frac{\partial \pi_p}{\partial \underline{d}} = EI\int_0^L [B^T][B]dx\,\{d\} - \int_0^L [N^T]w\,dx + \int_0^L K_f[N^T][N]dx\,\{d\}$$

$$\therefore\ [k] = \underbrace{EI\int_0^L [B^T][B]dx}_{\text{Equation.(4.7.19)}} + \underbrace{k_f\int_0^L [N^T][N]dx}_{\text{New part similar to convection part of heat transfer stiffness matrix.}}$$

4.77 Find the deflection at the mid-span using four beam elements, making the shear area zero and then making the shear area equal to $\frac{5}{6}$ times the cross-sectional area (b times h). Then make the beam have decreasing spans of 200 mm, 100 mm, and 50 mm with zero shear area and then $\frac{5}{6}$ times the cross-sectional area. Compare the answers. Based on your program answers, can you conclude whether your program includes the effects of transverse shear deformation?

Beam Span (m)	Shear Area	Displacement at center (m)	% difference
0.400	0	1.28E-03	4.61.E+00
0.400	0.001042	1.34E-03	
0.200	0	1.60E-04	16.21
0.200	0.001042	1.91E-04	
0.100	0	2.00E-05	43.62
0.100	0.001042	3.55E-05	
0.0500	0	2.50E-06	75.58
0.0500	0.001042	1.02E-05	

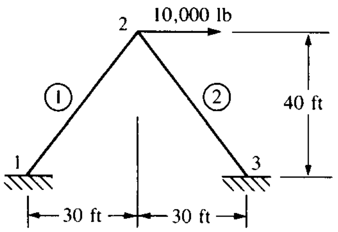
It would appear that the program **DOES** include the effects of transverse shear area which can be seen in the increasing per cent differences as the width of the beam approaches the span of the beam. As these width and span get closer and closer together the shear area becomes a larger factor, this would be the expected outcome if the program includes the effect of transverse shear area in the calculations.

Note: For all of the beams the element definitions remained the same except the beam spans were changed. The figure below shows one example of when the 400mm beam was run with shear area included.

Figure 1: 400mm beam deflection with shear area included

Chapter 5

5.1



$E = 30 \times 10^6$ psi
$A = 10$ in.^{2}
$I = 500$ in.^{4}

Element (1)

$$L^{(1)} = \sqrt{40^2 + 30^2} = 50 \text{ ft} = 600 \text{ in.}$$

$$\cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{30 - 0}{50} = 0.6$$

$$\sin\theta = \frac{y_2 - y_1}{L^{(1)}} = \frac{40 - 0}{50} = 0.8$$

$$\frac{E}{L} = 50000, \qquad \frac{12\,I}{L^2} = 0.0167, \qquad \frac{6\,I}{L} = 5.0$$

$$[k^{(1)}] = 50000\begin{bmatrix} 3.61 & 4.79 & -4 & -3.61 & -4.79 & -4 \\ 4.79 & 6.41 & 3 & -4.79 & -6.41 & 3 \\ -4 & 3 & 2000 & 4 & -3 & 1000 \\ -3.61 & -4.79 & 4 & 3.61 & 4.79 & 4 \\ -4.79 & -6.41 & -3 & 4.79 & 6.41 & -3 \\ -4 & 3 & 1000 & 4 & -3 & 2000 \end{bmatrix}$$

Element (2)

$$L^{(2)} = 50 \text{ ft} = 600 \text{ in.}$$

$$\cos\theta = \frac{-30 - 0}{50} = -0.6 \quad \sin\theta = \frac{40 - 0}{50} = 0.8$$

$$[k^{(2)}] = 50000\begin{bmatrix} 3.61 & -4.79 & -4 & -3.61 & 4.79 & -4 \\ -4.79 & 6.41 & -3 & 4.79 & -6.41 & -3 \\ -4 & -3 & 2000 & 4 & 3 & 1000 \\ -3.61 & 4.79 & 4 & 3.61 & -4.79 & 4 \\ 4.79 & -6.41 & 3 & -4.79 & 6.41 & 3 \\ -4 & -3 & 1000 & 4 & 3 & 2000 \end{bmatrix}$$

After imposing the boundary conditions on each element stiffness matrix and assembling, we have

$$\begin{Bmatrix} F_{2x} = 10000 \\ F_{2y} = 0 \\ M_2 = 0 \end{Bmatrix} = 50000 \begin{bmatrix} 7.22 & 0 & 8 \\ 0 & 12.82 & 0 \\ 0 & 0 & 4000 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously, we obtain

$$u_2 = 0.0278 \text{ in.}, \; v_2 = 0, \; \phi_2 = -0.555 \times 10^{-4} \text{ rad}$$

The element forces are obtained using

$$\{f'\} = [k'][T]\{d\}$$

Element (1)

$$\{f'\} = [k'][T]\{d\} = 50000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0167 & 5 & 0 & -0.0167 & 5 \\ 0 & 5 & 2000 & 0 & -5 & 1000 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0167 & -5 & 0 & 0.0167 & -5 \\ 0 & 5 & 0 & 0 & -5 & 2000 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ 0.0278 \\ 0 \\ -0.555 \times 10^{-4} \end{Bmatrix} = \begin{Bmatrix} f'_{1x} = -8300 \text{ lb} \\ f'_{1y} = 4.6 \text{ lb} \\ m_1 = 2775 \text{ lb} \cdot \text{in.} \\ f'_{2x} = 8300 \text{ lb} \\ f'_{2y} = -4.6 \text{ lb} \\ m_2 = 0 \end{Bmatrix}$$

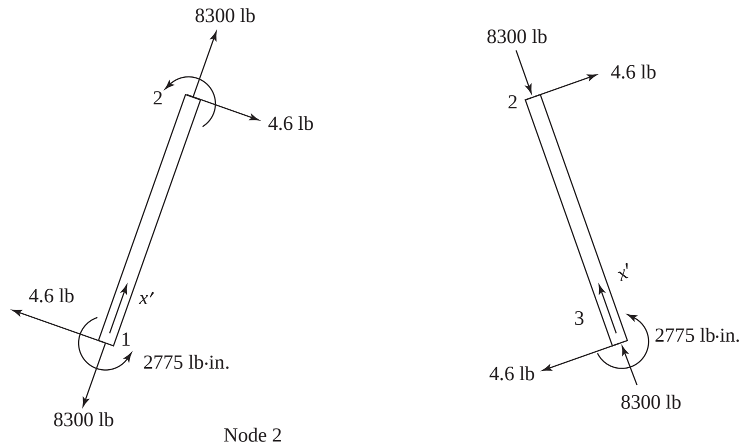
Element (2)

$$\{f'\} = \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0167 & 5 & 0 & -0.0167 & 5 \\ 0 & 5 & 2000 & 0 & -5 & 1000 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0167 & -5 & 0 & 0.0167 & -0.5 \\ 0 & 5 & 1000 & 0 & -0.5 & 2000 \end{bmatrix}$$

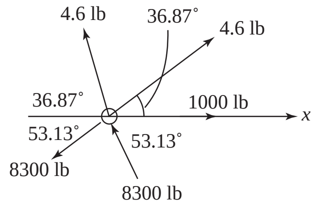
$$\times \begin{bmatrix} -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0278 \\ 0 \\ -0.555 \times 10^{-4} \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{3x} = 8300 \text{ lb} \\ f'_{3y} = 4.6 \text{ lb} \\ m_3 = 2775 \text{ lb} \cdot \text{in.} \\ f'_{2x} = -8300 \text{ lb} \\ f'_{2y} = -4.6 \text{ lb} \\ m_2 = 0 \end{Bmatrix}$$

Equilibrium check



Node 2



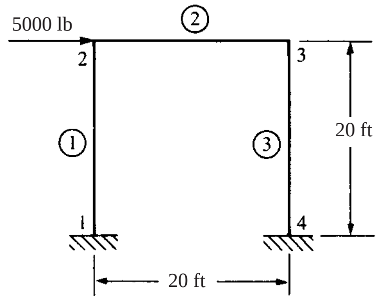
$\Sigma F_x = 0$

$1000 - 2(8300) \cos 53.13° - 2\,(4.6) \cos 36.87° = 32.6 \text{ lb} \cong 0$

$\Sigma F_y = 0$

$8300\,(\sin 53.13° - \sin 53.13°) + 4.6\,(\sin 36.87° - \sin 36.87°) = 0$

5.2



$$\frac{12\,I}{L^2} = 0.04167,\ \frac{6\,I}{L} = 5.0,\ \frac{E}{L} = 125{,}000$$

Element (1)

$C^{(1)} = 0, \quad S^{(1)} = 1$

$$[k^{(1)}] = 125{,}000 \begin{bmatrix} 0.04167 & 0 & -5 & -0.04167 & 0 & -5 \\ & 10 & 0 & 0 & -10 & 0 \\ & & 800 & 5 & 0 & 400 \\ & & & 0.04167 & 0 & 5 \\ & & & & 10 & 0 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Element (2)

$$C^{(2)} = 1, \quad S^{(2)} = 0$$

$$[k^{(2)}] = 125{,}000 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ & 0.04167 & 5 & 0 & 0.04167 & 5 \\ & & 800 & 0 & -5 & 400 \\ & & & 10 & 0 & 0 \\ & & & & 0.04169 & -5 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Element (3)

$$C^{(3)} = 0, \quad S^{(3)} = -1$$

$$[k^{(3)}] = 125{,}000 \begin{bmatrix} 0.04167 & 0 & 10 & -0.04167 & 0 & 10 \\ & 10 & 0 & 0 & -10 & 0 \\ & & 800 & -10 & 0 & 400 \\ & & & 0.167 & 0 & -10 \\ & & & & 10 & 0 \\ \text{Symmetry} & & & & & 800 \end{bmatrix}$$

Boundary conditions are

$$u_1 = v_1 = \phi_1 = 0, u_4 = v_4 = \phi_4 = 0$$

Global equations $\{F\} = [K]\ \{d\}$ are

$$125000 \begin{bmatrix} 9.958 & 0 & -5 & -9.958 & 0 & 0 \\ & -9.958 & 5 & 0 & 10.0417 & 5 \\ & & 1200 & 10 & -5 & 1200 \\ & & & 0 & 0 & -10 \\ & & & & -9.958 & -5 \\ \text{Symmetry} & & & & & 1200 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 5000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = 0.688 \text{ in.}, v_2 = 0.00171 \text{ in.}, \phi_2 = -0.00173 \text{ rad}$$

$$u_3 = 0.686 \text{ in.}, v_3 = -0.00171 \text{ in.}, \phi_3 = -0.00172 \text{ rad}$$

Local element forces

$$\{f'\} = [k']\{d'\} = [k']\ [T]\ \{d\} \text{ for each element}$$

Element (1)

$$[T]^{(1)}\ \{d\}^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.688 \\ 0.171 \times 10^{-2} \\ -0.173 \times 10^{-2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.171 \times 10^{-2} \\ -0.688 \\ -0.173 \times 10^{-2} \end{Bmatrix}$$

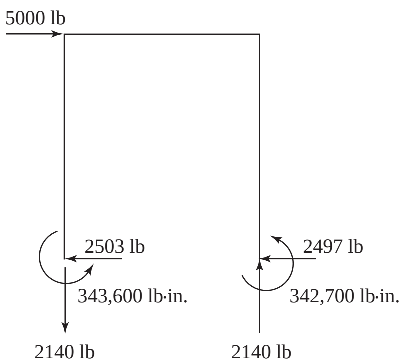
$$\{f'\}^{(1)} = [k']^{(1)}[T]^{(1)}\{d\}^{(1)} = \begin{Bmatrix} f'^{(1)}_{1x} = -2140 \text{ lb} \\ f'^{(1)}_{1y} = 2503 \text{ lb} \\ m'^{(1)}_1 = 343{,}600 \text{ lb} \cdot \text{in.} \\ f'^{(1)}_{2x} = 2140 \text{ lb} \\ f'^{(1)}_{2y} = -2503 \text{ lb} \\ m'^{(1)}_2 = 257{,}000 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$

Similarly

$$\{f'\}^{(2)} = [k']^{(2)}[T]^{(2)}\{d\}^{(2)} = \begin{Bmatrix} f_{2x} = 2497 \text{ lb} \\ f_{2y} = -2140 \text{ lb} \\ m_2 = -257{,}000 \text{ lb} \cdot \text{in.} \\ f_{3x} = -2497 \text{ lb} \\ f_{3y} = 2140 \text{ lb} \\ m_3 = -256{,}600 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$

$$\{f'\}^{(3)} = [k']^{(3)}[T]^{(3)}\{d\}^{(3)} = \begin{Bmatrix} f_{3x} = 2140 \text{ lb} \\ f_{3y} = 2497 \text{ lb} \\ m_3 = 256{,}600 \text{ lb} \cdot \text{in.} \\ f_{4x} = -2140 \text{ lb} \\ f_{4y} = -2497 \text{ lb} \\ m_4 = 342.700 \text{ lb} \cdot \text{in.} \end{Bmatrix}$$

Free body diagram of frame
(using local force results)



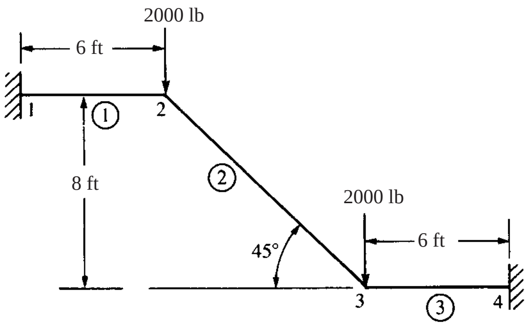
Check equation

$$\Sigma F_x = 0: 5000 - 2503 - 2497 \cong 0$$

$$\Sigma F_y = 0: -2140 + 2140 = 0$$

$$\Sigma M_1 = 343{,}600 + 342{,}700 + 2140\,(20')\left(12\ \frac{\text{in.}}{\text{ft}}\right) - 5000\,(20')\left(12\ \frac{\text{in.}}{\text{ft}}\right) \cong 0$$

5.3



$$\sigma_{\text{bend}} < \frac{2}{3}\,\sigma_y < 24000 \text{ psi}$$

$$\sigma_b = \frac{Mc}{I}$$

Assume channel section, $C6 \times 8.2$

$$I_x = 13.1 \text{ in.}^4,\ A = 2.40 \text{ in.}^2$$

Element (1)

$$C = 1 \qquad S = 0$$

$$\frac{12\,I}{L^2} = 2.3148 \times 10^{-3}\,I = 0.03032$$

$$\frac{6\,I}{L} = 8.33 \times 10^{-2}\,I = 1.092$$

$$\frac{E}{L} = \frac{29 \times 10^6}{(12)\,(6)} = 4.028 \times 10^5$$

$$[k^{(1)}] = (4.028 \times 10^5)\ \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ 0 & 0.0303 & 1.0917 & 0 & 0.303 & 1.0917 \\ 0 & 1.0917 & 52.4 & 0 & -1.0917 & 26.2 \\ -2.4 & 0 & 0 & 2.4 & 0 & 0 \\ 0 & -0.0303 & -1.0917 & 0 & 0.0303 & -1.0917 \\ 0 & 1.0917 & 26.2 & 0 & -1.0917 & 52.4 \end{bmatrix} \end{array}$$

Element (2)

$$C = \cos 45° = 0.707,\ S = \sin 45° = 0.707$$

$$\frac{12\,I}{L^2} = \frac{12\,I}{(8^2 + 8^2)^{\frac{1}{2}}\,12} = 8.528 \times 10^{-3}$$

$$\frac{6\,I}{L} = 0.5789,\ \frac{E}{L} = 2.136 \times 10^6$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & (2) & & & (3) & \end{array} \\
[k^{(2)}] = 2.136 \times 10^5 & \begin{bmatrix} 1.204 & 1.196 & -0.409 & -1.204 & -1.196 & -0.4094 \\ 1.196 & 1.204 & 0.409 & -1.196 & -1.204 & 0.4094 \\ & & 52.4 & 0.4094 & -0.4094 & 26.2 \\ & & & 1.204 & 1.196 & 0.4094 \\ & & & & 1.204 & -0.4094 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix}
\end{array}
$$

Element (3)

$$C = \cos 0° = 1, \quad S = \sin 0° = 0$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & (3) & & & (4) & \end{array} \\
[k^{(3)}] = 4.028 \times 10^5 & \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & -1.092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix}
\end{array}
$$

Boundary conditions

$$u_1 = v_1 = \phi_1 = 0 \qquad u_4 = v_4 = \phi_4 = 0$$

Applying boundary conditions the reduced global equations become

$$
10^5 \begin{bmatrix} 12.24 & 2.554 & -0.8745 & -2.573 & -2.554 & -0.8745 \\ & 2.695 & -3.523 & -2.554 & -2.573 & 0.8745 \\ & & 323 & 0.8745 & -0.8745 & 55.97 \\ & & & 12.24 & 2.55 & 0.8745 \\ & & & & 2.69 & 3.523 \\ \text{Symmetry} & & & & & 323 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} F_{2x} = 0 \\ F_{2y} = -2000 \\ M_2 = 0 \\ F_{3x} = 0 \\ F_{3y} = -2000 \\ M_3 = 0 \end{Bmatrix}
$$

Solving simultaneously, we obtain

$$u_2 = -3.008 \times 10^{-9} \text{ in.}$$

$$v_2 = -0.402 \text{ in.}$$

$$\phi_2 = -6.663 \times 10^{-3} \text{ rad}$$

$$u_3 = 3.30 \times 10^{-9} \text{ in.}$$

$$v_3 = -0.402 \text{ in.}$$

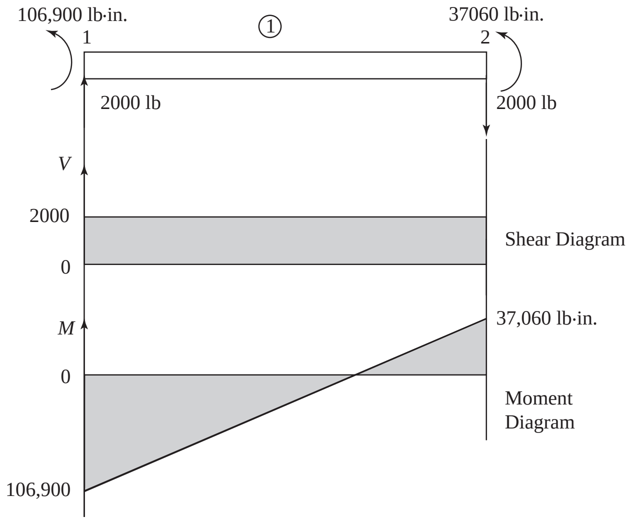
$$\phi_3 = 6.663 \times 10^{-3} \text{ rad}$$

Element forces

Element (1)

$$\begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1^{(1)} \\ f'^{(1)}_{2x} \\ f'^{(1)}_{2y} \\ m_2^{(1)} \end{Bmatrix} = 4.028\times10^5 \begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & 0.1092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.008\times10^{-9} \\ -0.402 \\ -6.663\times10^{-3} \end{Bmatrix}$$

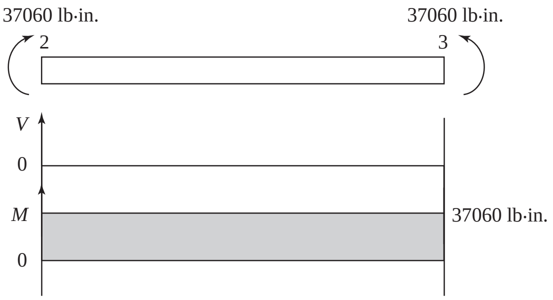
$$= \begin{Bmatrix} f'^{(1)}_{1x} = 0 \\ f'^{(1)}_{1y} = 2000 \text{ lb} \\ m_1^{(1)} = 106{,}900 \text{ lb}\cdot\text{in.} \\ f'^{(1)}_{2x} = 0 \\ f'^{(1)}_{2y} = -2000 \text{ lb} \\ m_2^{(1)} = 37060 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Element (2)

$$2.316\times10^5 \begin{bmatrix} 1.204 & 1.196 & -0.409 & -1.204 & -1.196 & -0.409 \\ & 1.204 & 0.409 & -1.196 & -1.204 & 0.409 \\ & & 52.4 & 0.409 & -0.409 & 26.2 \\ & & & 1.204 & 1.196 & 0.409 \\ & & & & 1.204 & -0.409 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix} \begin{Bmatrix} 3.008\times10^{-9} \\ -0.402 \\ -6.66\times10^{-3} \\ 3.3\times10^{-9} \\ -0.402 \\ 6.66\times10^{-3} \end{Bmatrix}$$

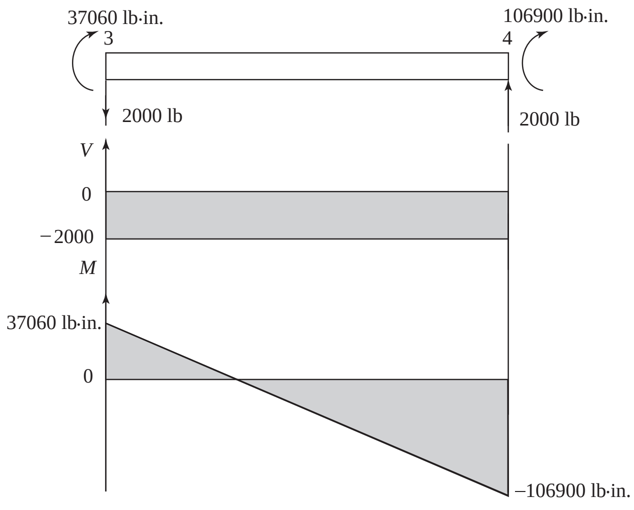
$$= \begin{Bmatrix} f'^{(2)}_{2x} \\ f'^{(2)}_{2y} \\ m_2^{(2)} \\ f'^{(2)}_{3x} \\ f'^{(2)}_{3y} \\ m_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -37060 \text{ lb}\cdot\text{in.} \\ 0 \\ 0 \\ 37060 \text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Element (3)

$$4.028\times 10^{5}\begin{bmatrix} 2.4 & 0 & 0 & -2.4 & 0 & 0 \\ & 0.0303 & 1.092 & 0 & -0.0303 & 1.092 \\ & & 52.4 & 0 & -1.092 & 26.2 \\ & & & 2.4 & 0 & 0 \\ & & & & 0.0303 & -1.092 \\ \text{Symmetry} & & & & & 52.4 \end{bmatrix}\begin{Bmatrix} 3.3\times 10^{-9} \\ -0.402 \\ 6.66\times 10^{-3} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} f'^{(3)}_{3x} \\ f'^{(3)}_{3y} \\ m_3^{(3)} \\ f'^{(3)}_{4x} \\ f'^{(3)}_{4y} \\ m_4^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2000\text{ lb} \\ -37060\text{ lb}\cdot\text{in.} \\ 0 \\ 2000\text{ lb} \\ -106900\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Reactions

$$F_{1x} = f'^{(1)}_{1x} = 0$$

$$F_{1y} = f'^{(1)}_{1y} = 2000\text{ lb}$$

$$M_1 = m_1^{(1)} = 106900\text{ lb}\cdot\text{in.}$$

$$F_{4x} = f'^{(3)}_{4x} = 0$$

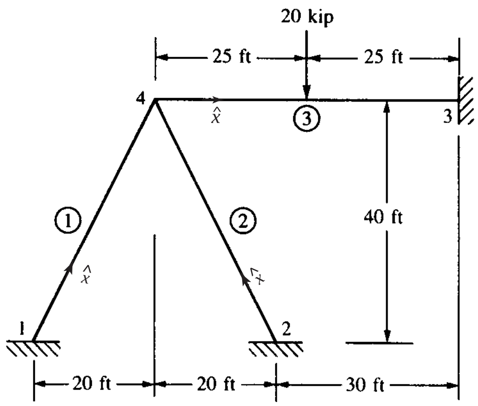
$$F_{4y} = f'^{(3)}_{4y} = 2000 \text{ lb}$$

$$M_4 = m_4^{(3)} = -\,106900 \text{ lb}\cdot\text{in.}$$

$$\sigma_b = \frac{Mc}{I} = \frac{106900\,(3\text{ in.})}{13.1\text{ in.}^4} = 24{,}480 \text{ psi}$$

Close to σ_{allow} = 24,000 psi

5.4



$E = 30 \times 10^3$ ksi
$A = 8$ in^2
$I = 800$ in.^{4}

Element (1)

$$C = \frac{x_4 - x_1}{L} = \frac{20}{44.7} = 0.447,\ S = \frac{40}{44.7} = 0.895$$

$$\frac{12I}{L^2} = 0.0334,\ \frac{6I}{L} = 8.949,\ \frac{E}{L} = 55.93$$

Imposing boundary conditions $u_1 = v_1 = \phi_1 = 0$, $u_2 = v_2 = \phi_2 = 0$, $u_3 = v_3 = \phi_3 = 0$

$$\begin{array}{cc} & \begin{array}{ccc} u_4 & v_4 & \phi_4 \end{array} \\ [k^{(1)}] = & \begin{bmatrix} 90.87 & 178.2 & 447.9 \\ & 358.8 & -223.7 \\ \text{Symmetry} & & 179000 \end{bmatrix} \end{array}$$

Element (2)

$$[k^{(2)}] = \begin{bmatrix} 90.87 & -178.2 & 447.9 \\ & 358.8 & 223.7 \\ \text{Symmetry} & & 179000 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 400 & 0 & 0 \\ & 1.334 & 400 \\ \text{Symmetry} & & 160000 \end{bmatrix}$$

Equivalent forces (element 3)

$$m_{04} = -\frac{PL}{8} = \frac{-(20\text{ kip})\,(50\text{ ft})\,(12\,\frac{\text{in.}}{\text{ft}})}{8} = -\,1500\text{ kip}\cdot\text{in.}$$

$$f_{04y} = -\,10\text{ kip}$$

Global equations

$$\begin{Bmatrix} 0 \\ -10\text{ kip} \\ -1500\text{ kip}\cdot\text{in.} \end{Bmatrix} = \begin{bmatrix} 582 & 0 & 896 \\ & 719 & 400 \\ \text{Symmetry} & & 517900 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ \phi_4 \end{Bmatrix}$$

Solving

$$u_4 = 0.445 \times 10^{-2}\text{ in.},\ v_4 = -\,0.123 \times 10^{-1}\text{ in.}$$

$$\phi_4 = -\,0.290 \times 10^{-2}\text{ rad}$$

Element forces

$$\{f'\} = [k']\,[T]\,\{d\} - \{f'_0\}$$

Element (1)

$$\begin{bmatrix} 447 & 0 & 0 & -477 & 0 & 0 \\ 0 & 1.868 & 500.5 & 0 & -1.868 & 500.5 \\ 0 & 500.5 & 179000 & 0 & -500.5 & 89490 \\ -447 & 0 & 0 & 447 & 0 & 0 \\ 0 & -1.868 & -500.5 & 0 & 1.868 & -500.5 \\ 0 & 500.5 & 89490 & 0 & -500.5 & 179000 \end{bmatrix} \overset{[T]\,\{d\}}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.90193\times 10^{-2} \\ -0.94808\times 10^{-2} \\ -0.2895\times 10^{-2} \end{Bmatrix}} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{4x} \\ f'_{4y} \\ m_4 \end{Bmatrix} = \begin{Bmatrix} 4.04\text{ kip} \\ -1.43\text{ kip} \\ -254\text{ kip}\cdot\text{in.} \\ -4.04\text{ kip} \\ 1.43\text{ kip} \\ -513\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Element (2)

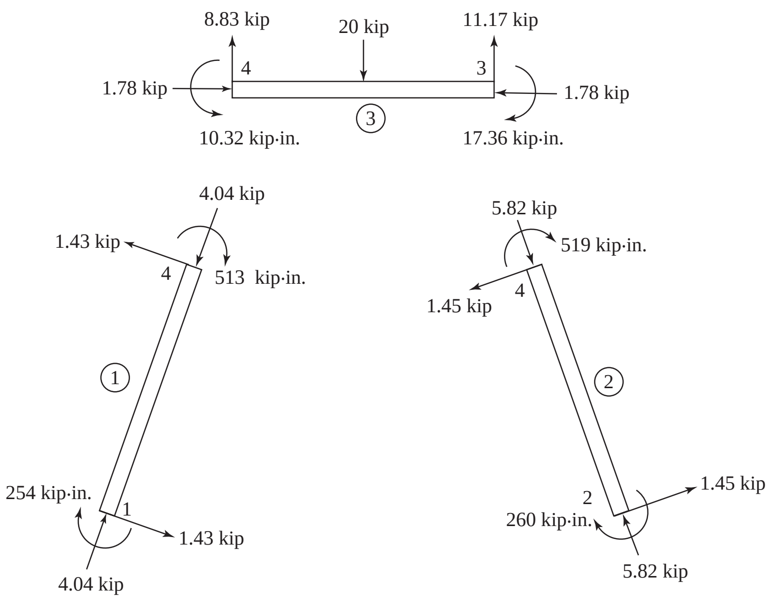
Similarly

$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{4x} \\ f'_{4y} \\ m_4 \end{Bmatrix} = \begin{Bmatrix} 5.82\text{ kip} \\ -1.45\text{ kip} \\ -260\text{ kip}\cdot\text{in.} \\ -5.82\text{ kip} \\ 1.45\text{ kip} \\ -519\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} f'_{4x} \\ f'_{4y} \\ m_4 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} -1.78\text{ kip} \\ 1.17\text{ kip} \\ -468\text{ kip}\cdot\text{in.} \\ 1.78\text{ kip} \\ -1.17\text{ kip} \\ -236\text{ kip}\cdot\text{in.} \end{Bmatrix} - \overset{\{f'_0\}}{\begin{Bmatrix} 0 \\ 10 \\ -1500 \\ 0 \\ 10 \\ 1500 \end{Bmatrix}} = \begin{Bmatrix} -1.78\text{ kip} \\ -8.83\text{ kip} \\ 1032\text{ kip}\cdot\text{in.} \\ 1.78\text{ kip} \\ -11.17\text{ kip} \\ -1736\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

Free-body diagrams at each element

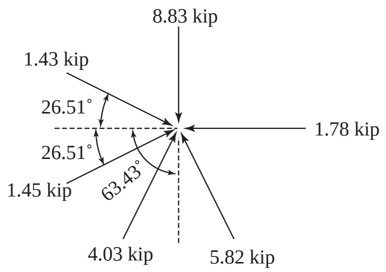


Equilibrium at node 4

$$M_4 = -513\ \text{kip}\cdot\text{in.} - 519\ \text{kip}\cdot\text{in.}\ + 1032\ \text{kip}\cdot\text{in.} \approx 0$$

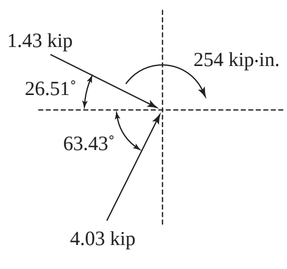
$$\Sigma F_x = 1.43 \cos 26.57° + 1.45 \cos 26.57° - 1.78 + 4.03 \cos 63.43° - 5.82 \cos 63.43° \approx 0$$

$$\Sigma F_y = \sin 26.57° \ (1.45 - 1.43) - 8.83 + \sin 63.43° \ (4.03 + 5.82) \approx 0$$



Reactions

Support node 1

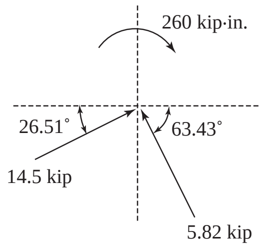


$$\Sigma F_y = 4.03 \sin 63.43° - 1.43 \sin 26.57° = 2.96\ \text{kip} \uparrow$$

$$\Sigma F_x = 4.03 \cos 63.43° + 1.43 \cos 26.57° = 31\ \text{kip} \rightarrow$$

$$M = 254\ \text{kip}\cdot\text{in.}$$

Reactions support node 2

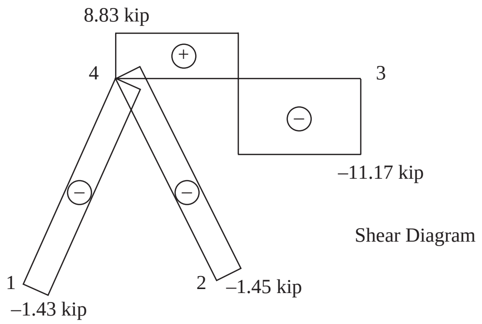


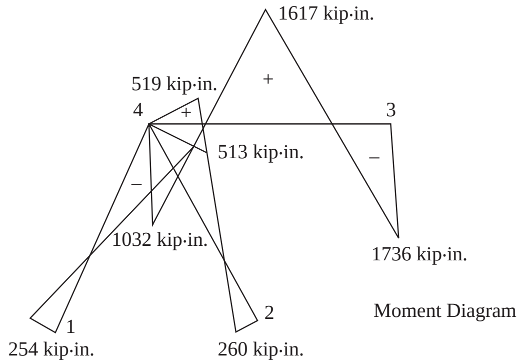
$$\Sigma F_x = 1.45 \cos 26.57° - 5.82 \cos 63.43° = -1.31 \text{ kip or} \leftarrow$$
$$\Sigma F_y = 5.82 \sin 63.43° + 1.45 \sin 26.57° = 58.86 \text{ kip} \uparrow$$
$$M_z = 260 \text{ kip} \cdot \text{in.}$$

Reactions support node 3

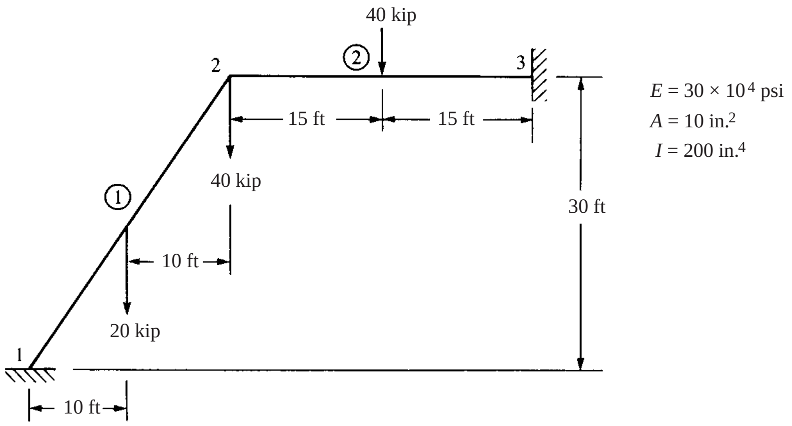
Already in global x-y directions

$$\Sigma F_y = 11.17 \text{ kip}, \Sigma F_x = 1.78 \text{ kip} \leftarrow, M = 1736 \text{ kip} \cdot \text{in.}$$





5.5



Element 1–2 (1)

$$\frac{12I}{L^2} = 0.0178,\ \frac{E}{L} = 69.338,\ \frac{6I}{L} = 2.7735,\ C = 0.555,\ S = 0.832$$

After imposing the boundary conditions $u_1 = v_1 = \phi_1$ and $u_3 = v_3 = \phi_3 =$ we have

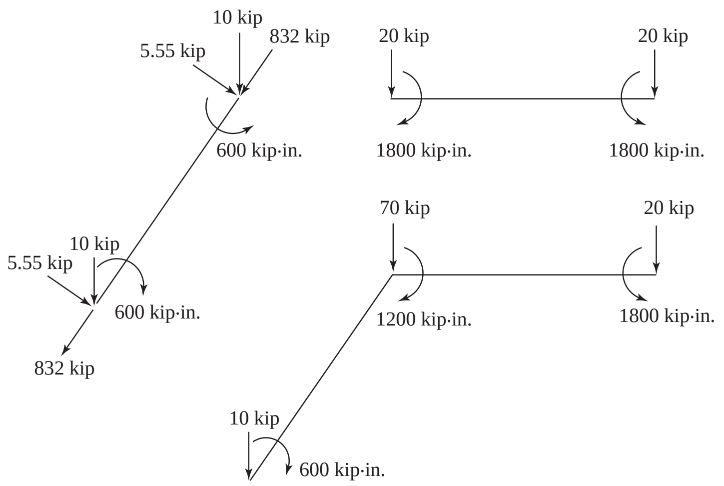
$$[k_{1-2}] = \begin{bmatrix} 214.2 & 319.8 & 160 \\ 319.8 & 480.2 & -106.7 \\ 160 & -106.7 & 55470 \end{bmatrix}$$

Element 2–3 (2)

$$\frac{12I}{L^2} = 0.0185,\ \frac{6I}{L} = 3.33,\ \frac{E}{L} = 85.53,\ C = 1;\ S = 0$$

$$[k_{2-3}] = \begin{bmatrix} -853.3 & 0 & 0 \\ 0 & 1.54 & 277.49 \\ 0 & 277.49 & 66664 \end{bmatrix}$$

Equivalent forces



Assembled global equations.

$$\begin{Bmatrix} 0 \\ -70 \\ -1200 \end{Bmatrix} = \begin{bmatrix} 1047.5 & 319.8 & 160 \\ & 481.74 & 170.8 \\ & & 122134.4 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$u_2 = 0.0562 \text{ in.}$$

$$v_2 = -0.1792 \text{ in.}$$

$$\phi_2 = -0.00965 \text{ rad}$$

Element forces

$$\{f'\} = [k'] \{d'\} - \{f_0\} = [k'] [T] \{d\} - \{f'_0\}$$

Element (1)

$$[T]\,\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.1179\text{ in.} \\ -0.1462\text{ in.} \\ -0.00965\text{ rad} \end{Bmatrix}$$

$$[k']\,[T]\,\{d\} - \{f'_0\} =$$

$$\begin{bmatrix} 693.38 & 0 & 0 & -693.38 & 0 & 0 \\ & 0.8875 & 192 & 0 & -0.8875 & 192 \\ & & 55380 & 0 & -192 & 27690 \\ & & & 693.33 & 0 & 0 \\ & & & & 0.8875 & -192 \\ \text{Symmetry} & & & & & 55380 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.1179 \\ -0.1462 \\ -0.00965 \end{Bmatrix} - \begin{Bmatrix} -8.32 \\ -5.55 \\ -600 \\ -8.32 \\ -5.55 \\ 600 \end{Bmatrix} = \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = \begin{matrix} f'_{1x} = 90.07\text{ kip} \\ f'_{1y} = 3.83\text{ kip} \\ m_1 = 360.86\text{ kip}\cdot\text{in.} \\ f'_{2x} = -73.43\text{ kip} \\ f'_{2y} = 7.27 \\ m_2 = -110.635\text{ kip}\cdot\text{in.} \end{matrix}$$

Reactions

From the free body diagram of equivalent forces gives

$$F_{1x} = f'_{1x}\,(0.555) - f'_{1y}\,(0.832) \Rightarrow F_{1x} = 46.8\text{ kip}$$

$$F_{1y} = f'_{1x}\,(0.832) - f'_{1y}\,(0.555) \Rightarrow F_{1y} = 77.06\text{ kip}$$

$$M_1 = 360.86\text{ kip}\cdot\text{in.}$$

Element (2)

$$\begin{bmatrix} 833.3 & 0 & 0 & -833.3 & 0 & 0 \\ & 1.5416 & 277.486 & 0 & -1.5416 & 277.488 \\ & & 66591 & 0 & -277.488 & 33299 \\ & & & 833.3 & 0 & 0 \\ & & & & 1.5416 & -277.488 \\ \text{Symmetry} & & & & & 66597 \end{bmatrix} \begin{Bmatrix} 0.05618 \\ -0.1792 \\ -0.00965 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$- \begin{Bmatrix} 0 \\ -160 \\ -1800 \\ 0 \\ -20 \\ 1800 \end{Bmatrix} = \begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} 46.8\text{ kip} \\ 17.1\text{ kip} \\ 1108\text{ kip}\cdot\text{in.} \\ -46.8\text{ kip} \\ 22.95\text{ kip} \\ -2171\text{ kip}\cdot\text{in.} \end{Bmatrix}$$

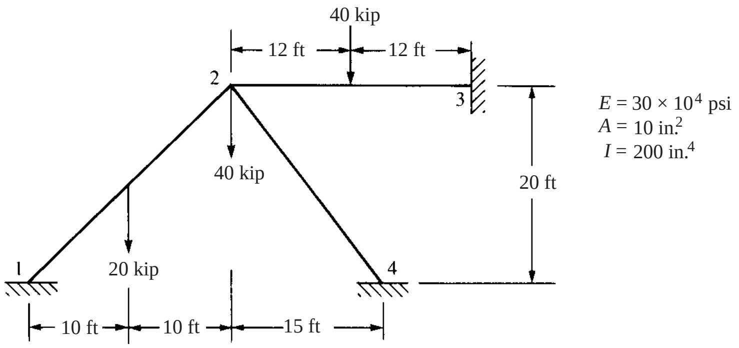
Reactions

$$F_{3x} = f'_{3x} = -46.8\text{ kip} \leftarrow$$

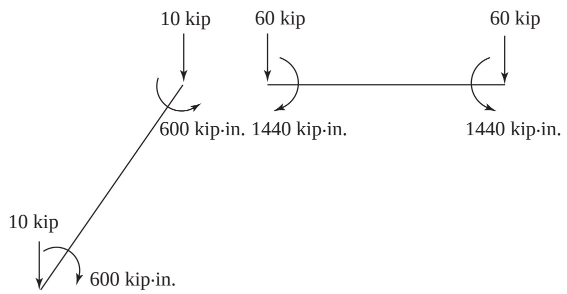
$$F_{3y} = f'_{3y} = 22.95\text{ kip} \uparrow$$

$$M_3 = m_3 = -2171\text{ kip}\cdot\text{in.}$$

5.6



Replacement (Equivalent) force system



Element 1–2 (1)

$$C = S = 0.707,\ \frac{12\,I}{L^2} = 0.0208, \frac{E}{L} = 88.3881\ \frac{\text{kip}}{\text{m}^3}, \frac{6I}{L} = 3.536$$

Since $u_1 = v_1 = \phi_1 = 0$

$$\begin{array}{c c} & \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ [k_{1-2}] = & \begin{bmatrix} 442.82 & 441.06 & 220.97 \\ 441.06 & 442.82 & -220.97 \\ 220.97 & -220.97 & 70710 \end{bmatrix} \end{array}$$

Element 2–3 (2)

$$C = 1, S = 0,\ \frac{12\,I}{L^2} = 0.0289,\ \frac{6\,I}{L} = 4.166,\ \frac{E}{L} = 104.167$$

$$u_3 = v_3 = \phi_3 = 0$$

$$\begin{array}{c c} & \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ [k_{2-3}] = & \begin{bmatrix} 1041.67 & 0 & 0 \\ 0 & 3.01 & 434.03 \\ 0 & 434.03 & 83333.28 \end{bmatrix} \end{array}$$

Element 4–2 (3)

$$C = -0.60, S = 0.80,\ \frac{12\,I}{L^2} = 0.0267, \frac{6I}{L} = 4, \frac{E}{L} = 100$$

$$[k_{4-2}] = \begin{bmatrix} 361.71 & -478.72 & 320 \\ -478.72 & 641 & 240 \\ 320 & 240 & 80000 \end{bmatrix}$$

By direct stiffness method

$$\begin{Bmatrix} 0 \\ -70 \\ -840 \end{Bmatrix} = \begin{bmatrix} 1846.2 & -37.66 & 540.97 \\ -37.66 & 1086.83 & 453.06 \\ 540.97 & 453.06 & 234043.28 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -0.000269 \text{ in.} \\ -0.0363 \text{ in.} \\ -0.00347 \text{ rad} \end{Bmatrix}$$

Element 1–2 (1)

$$[T]\,\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0447 \text{ in.} \\ -0.0426 \text{ in.} \\ -0.00347 \text{ rad} \end{Bmatrix}$$

$$\begin{array}{ccc} \{f'\} & [k'] & [T]\{d\} \\ \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = (10^6) & \begin{bmatrix} 883.88 & 0 & 0 & -883.88 & 0 & 0 \\ & 1.838 & 311.92 & 0 & -1.838 & 311.92 \\ & & 70579.2 & 0 & -311.92 & 352.90 \\ & & & 883.88 & 0 & 0 \\ & & & & 1.838 & -311.92 \\ \text{Symmetry} & & & & & 70579.2 \end{bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0447 \\ -0.0426 \\ -0.00347 \end{Bmatrix} \end{array}$$

$$\begin{array}{c} -\{f'_0\} \\ -\begin{Bmatrix} -7.07 \\ -7.07 \\ -600 \\ -7.07 \\ -7.07 \\ 600 \end{Bmatrix} \end{array} = \begin{array}{l} f'_{1x} = 46.6 \text{ kip} \\ f'_{1y} = 6.07 \text{ kip} \\ m_1 = 491.3 \text{ kip}\cdot\text{in.} \\ f'_{2x} = -32.4 \text{ kip} \\ f'_{2y} = 8.07 \text{ kip} \\ m_2 = -831.3 \text{ kip}\cdot\text{in.} \end{array}$$

From FBD of element (1)

$$F_{1x} = \frac{1}{2}\sqrt{2}\;(46.6 - 6.07) = 28.65 \text{ kip}$$

$$F_{1y} = \frac{1}{2}\sqrt{2}\;(46.6 + 6.07) = 37.24 \text{ kip}$$

$$M_1 = m_1 = 491.3 \text{ kip}\cdot\text{in.}$$

Similarly

Element 2–3 (2)

$$f'_{2x} = -0.28 \text{ kip}$$

$$f'_{2y} = 58.31 \text{ kip}$$

$$m_2 = 1123.9 \text{ kip}\cdot\text{in.}$$

$$f'_{3x} = 0.28 \text{ kip}$$

$$f'_{3y} = 21.69 \text{ kip}\cdot\text{in.}$$

$$m_3 = -1611 \text{ kN}\cdot\text{in.}$$

Element 4–2 (3)

$$f'_{4x} = 50.2 \text{ kip}$$

$$f'_{4y} = -1.49 \text{ kip}$$

$$m_4 = -154.2 \text{ kip}\cdot\text{in.}$$

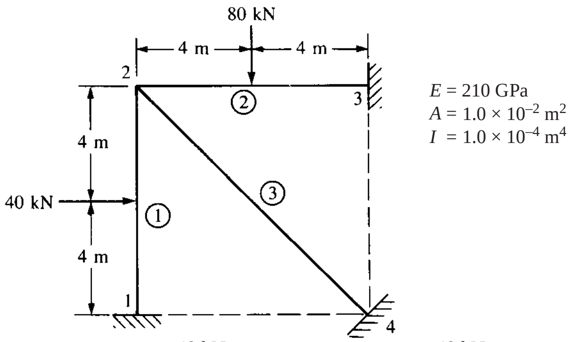
$$f'_{2x} = -50.2 \text{ kip}$$

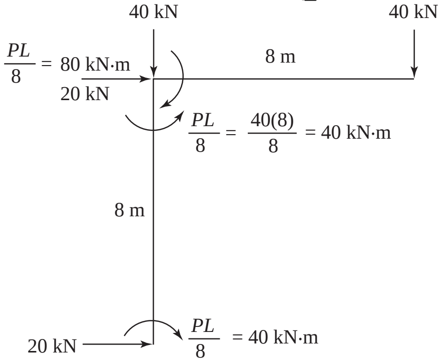
$$f'_{2y} = 1.49$$

$$m_2 = -293.2 \text{ kip}\cdot\text{in.}$$

$$\left.\begin{array}{l} F_{2x} = 0.28\ \text{kip} \\ F_{3y} = 21.69\ \text{kip} \\ M_3 = m_3 = -1611.8\ \text{kip}\cdot\text{in.} \end{array}\right\} \leftarrow \text{Reaction} \rightarrow \left\{\begin{array}{l} F_{4x} = -28.93\ \text{kip} \\ F_{4y} = 41.05\ \text{kip} \\ M_4 = -154.2\ \text{kip}\cdot\text{in.} \end{array}\right.$$

5.7





Equivalent force system

Boundary conditions $\quad u_1 = v_1 = \phi_1 = 0$

$u_3 = v_3 = \phi_3 = 0$

$u_4 = v_4 = \phi_4 = 0$

Element (1)

$$C = 0, \quad S = 1$$

$$\frac{12I}{L^3} = 2.344\times10^{-6},\ \frac{4I}{L} = 5\times10^{-5}$$

$$\frac{6I}{L^2} = 9.375\times10^{-6},\ \frac{A}{L} = 1.25\times10^{-3}$$

$$[k^{(1)}] = \frac{210\times10^6}{8}\begin{bmatrix} 1.875\times10^{-5} & 0 & 9.375\times10^{-6} \\ & 1.0\times10^{-2} & 0 \\ \text{Symmetry} & & 4.0\times10^{-4} \end{bmatrix}$$

Element (2)

$$C = 1, \quad S = 0$$

$$[k^{(2)}] = \frac{210\times10^6}{8}\begin{bmatrix} 1.0\times10^2 & 0 & 0 \\ & 1.875\times10^{-5} & 9.375\times10^{-6} \\ \text{Symmetry} & & 4.0\times10^{-4} \end{bmatrix}$$

Element (3)

$$C = 0.707, \quad S = -0.707$$

$$\frac{12I}{L^3} = 8.286\times10^{-7}, \quad \frac{4I}{L} = 3.536\times10^{-5}$$

$$\frac{6I}{L^2} = 4.6875\times10^{-6}, \quad \frac{A}{L} = 8.839\times10^{-4}$$

$$[k^{(3)}] = \frac{210\times10^6}{8\sqrt{2}}\begin{bmatrix} 5.005\times10^{-3} & -4.995\times10^{-3} & 3.75\times10^{-4} \\ & 5.005\times10^{-3} & 3.75\times10^{-5} \\ \text{Symmetry} & & 4.0\times10^{-4} \end{bmatrix}$$

Global equations

$$\begin{Bmatrix} 20 \\ -40 \\ -40 \end{Bmatrix} = \frac{210\times10^6}{8\sqrt{2}}\begin{bmatrix} 1.917\times10^{-2} & -4.995\times10^{-3} & 1.436\times10^{-4} \\ & 1.917\times10^{-2} & 1.436\times10^{-4} \\ \text{Symmetry} & & 1.531\times10^{-3} \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = 0.4308\times10^{-4}\text{ m},\ v_2 = -0.9067\times10^{-4}\text{ m}$$

$$\phi_2 = -0.1403\times10^{-2}\text{ rad}$$

Local element forces

Element (1)

Effective forces = $[k']\{d'\}$

$$f_{1x}^{(e)} = 23.8\text{ kN},\ f_{1y}^{(e)} = -2.74\text{ kN},\ m_1^{(e)} = -7.28\text{ kN}\cdot\text{m}$$

$$f_{2x}^{(e)} = -23.8\text{ kN},\ f_{2y}^{(e)} = 2.74\text{ kN},\ m_2^{(e)} = -14.65\text{ kN}\cdot\text{m}$$

Actual forces

$$\{f'\} = [k']\{d\} - \{f_0'\}$$

$$f'_{1x} = 23.8 - 0 = 23.8\text{ kN} \uparrow$$

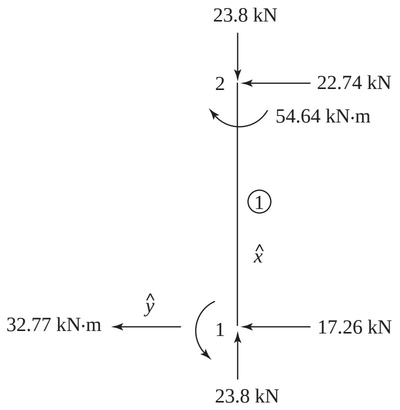
$$f'_{1y} = -2.74 + 20.0 = 17.26\text{ kN} \leftarrow$$

$$m_1 = -7.28 + 40 = 32.77\text{ kN}\cdot\text{m} \circlearrowleft$$

$$f'_{2x} = -23.8\text{ kN} \downarrow$$

$$f'_{2y} = 2.74 + 20 = 22.74\text{ kN} \leftarrow$$

$$m_2 = -14.65 - 40 = -54.64\text{ kN}\cdot\text{m} \circlearrowright$$



FBD element (1)

Element (2)
Effective forces

$$f_{2x}^{(e)} = 11.31\text{ kN},\ f_{2y}^{(e)} = -2.81\text{ kN},\ m_2^{(e)} = -14.9\text{ kN}\cdot\text{m}$$

$$f_{3x}^{(e)} = -11.31\text{ kN},\ f_{3y}^{(e)} = 2.81\text{ kN},\ m_3^{(e)} = -7.54\text{ kN}\cdot\text{m}$$

Actual forces

$$\{f'\} = [k']\{d\} - \{f_0'\}$$

$$f'_{2x} = 11.31 - 0 = 11.31\text{ kN} \rightarrow$$

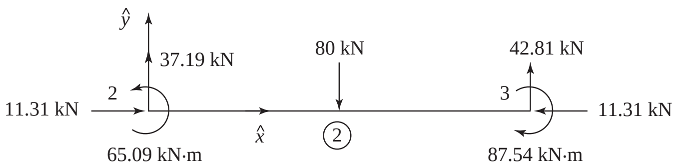
$$f'_{2y} = -2.81 - (-40) = 37.19\text{ kN} \uparrow$$

$$m_2 = -14.91 - (-80) = 65.09\text{ kN}\cdot\text{m}$$

$$f'_{3x} = -11.31 - 0 = -11.31\text{ kN} \leftarrow$$

$$f'_{3y} = 2.81 - (-40) = 42.81\text{ kN} \uparrow$$

$$m_3 = -7.54 - 80 = -87.54\text{ kN}\cdot\text{m}$$



FBD of element 2

Element (3)

$$\{f_0'\} = 0$$

$$f'_{2x} = 17.55\text{ kN}$$

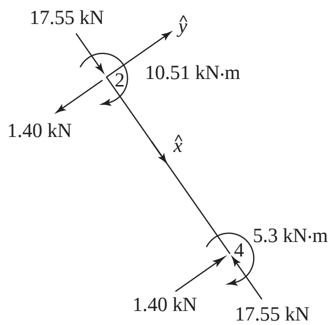
$$f'_{2y} = -1.40\text{ kN}$$

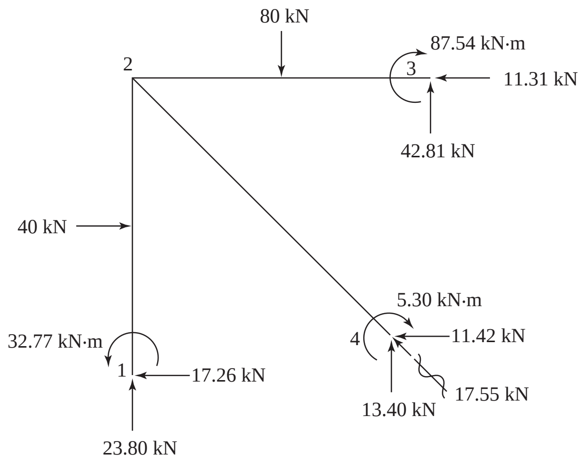
$$m_2 = -10.51\text{ N}\cdot\text{m}$$

$$f'_{4x} = -17.55\text{ kN}$$

$$f'_{4y} = 1.40\text{ kN}$$

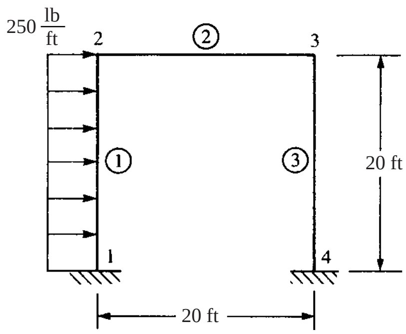
$$m_4 = -5.30\text{ kN}\cdot\text{m}$$



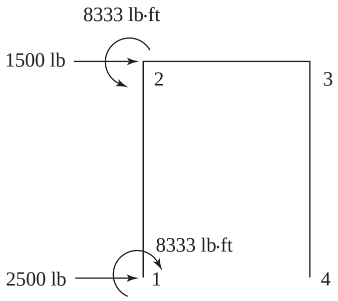


FBD of frame

5.8



$E = 30 \times 10^6$ psi
$A = 15$ in.^{2}
$I = 250$ in.^{4}



Equivalent force system

$$F = \frac{wL}{2} = \frac{250 \times 20}{2} = 2500 \text{ lb}$$

$$M = \frac{wL^2}{12} = \frac{250\,(20)^2}{12} = 8333 \text{ lb·ft} = 100{,}000 \text{ lb·in.}$$

Calculate $[k]$'*s* based on node 2 and 3 contributions as

$$u_1 = v_1 = \phi_1 = 0,\ u_4 = v_4 = \phi_4 = 0$$

Element (1)

$$C = 0, \quad S = 1$$

$$(3)$$

$$[k^{(1)}] = \frac{E}{L}\begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ \text{Symmetry} & & 4I \end{bmatrix}$$

Element (2)

$C = 1, \quad S = 0$

$$\begin{array}{cc} & \begin{array}{cccccc} & (2) & & & (3) & \end{array} \\ [k^{(2)}] = \dfrac{E}{L} & \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ & \frac{12I}{L^2} & \frac{16I}{L} & 0 & \frac{-12I}{L^2} & \frac{16I}{L} \\ & & 4I & 0 & \frac{-6I}{L} & 2I \\ & & & A & 0 & 0 \\ & & & & \frac{12I}{L^2} & \frac{-6I}{L} \\ \text{Symmetry} & & & & & 4I \end{bmatrix} \end{array}$$

Element (3)

$C = 0, \quad S = -1$

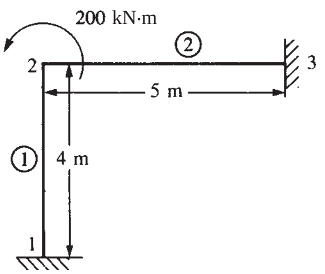
$$\begin{array}{cc} & \begin{array}{c} (3) \end{array} \\ [k^{(3)}] = \dfrac{E}{L} & \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ \text{Symmetry} & & 4I \end{bmatrix} \end{array}$$

Assemble global $[K]$ and Equations $\{F\} = [K]\,\{d\}$ use numerical values for E, I, A, L

$$\begin{Bmatrix} 2500 \\ 0 \\ 100{,}000 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{L} \begin{bmatrix} 15.05 & 0 & 6.25 & -15 & 0 & 0 \\ & 15.05 & 6.25 & 0 & 0.0521 & 6.25 \\ & & 2000 & 0 & -6.25 & 500 \\ & & & 15.05 & 0 & 6.25 \\ & & & & 15.05 & -6.25 \\ \text{Symmetry} & & & & & 2000 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Solve simultaneously using an equation solver

5.9



$E = 210$ GPa
$A = 2 \times 10^{-2}$ m^2
$I = 2 \times 10^{-4}$ m^4

Element $[k]$'s

Element (1)

$C = 0, \quad S = 1$

$$\frac{12I}{L^2} = \frac{12(2 \times 10^{-4})}{4^2} = 1.5 \times 10^{-4} \text{ m}^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{4} = 3.0\times10^{-4}\ \text{m}^3$$

$$\frac{E}{L} = \frac{210\times10^6}{4} = 5.25\times10^7\ \frac{\text{kN}}{\text{m}}$$

$$[k^{(1)}] = 5.25\times10^7 \begin{array}{c} \begin{array}{ccc} \quad u_2 \quad & \quad v_2 \quad & \quad \phi_2 \quad \end{array} \\ \begin{bmatrix} 1.5\times10^{-4} & 0 & 3\times10^{-4} \\ & 2\times10^{-2} & 0 \\ & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \quad S = 0$$

$$\frac{12I}{L^2} = \frac{12(2\times10^{-4})}{5^2} = 9.6\times10^{-5}\ \text{m}^2$$

$$\frac{6I}{L} = \frac{6(2\times10^{-4})}{5} = 2.4\times10^{-4}\ \text{m}^3$$

$$\frac{E}{L} = \frac{210\times10^6}{5} = 4.2\times10^7\ \frac{\text{kN}}{\text{m}}$$

$$[k^{(2)}] = 4.2\times10^7 \begin{array}{c} (2) \\ \begin{bmatrix} 2\times10^{-2} & 0 & 0 \\ & 9.6\times10^{-5} & 2.4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Assemble global equations for node 2

$$\begin{Bmatrix} 0 \\ 0 \\ 200 \end{Bmatrix} = 10^7 \begin{bmatrix} 8.48\times10^{-2} & 0 & 1.58\times10^{-3} \\ & 1.05\times10^{-1} & 1.01\times10^{-3} \\ \text{Symmetry} & & 7.56\times10^{-3} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving simultaneously

$$u_2 = -4.95\times10^{-5}\ \text{m},\ v_2 = -2.56\times10^{-5}\ \text{m}$$

$$\phi_2 = 2.66\times10^{-3}\ \text{rad}$$

Element forces $\{f'\} = [k']\ [T]\ \{d\}$

$$[T]\ \{d\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -4.95\times10^{-5} \\ -2.56\times10^{-5} \\ 2.66\times10^{-3} \end{Bmatrix}$$

$$\{f'\} = [k']^{(1)} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56\times10^{-5} \\ 4.95\times10^{-6} \\ 2.66\times10^{-3} \end{Bmatrix}$$

$$\{f'\} = 5.25\times10^{7} \begin{bmatrix} : & : & : & -2\times10^{-2} & 0 & 0 \\ . & . & . & 0 & -1.5\times10^{-4} & 3\times10^{-4} \\ . & . & . & 0 & -3\times10^{-4} & 4\times10^{-4} \\ . & . & . & 2\times10^{-2} & 0 & 0 \\ . & . & . & 0 & 1.5\times10^{-4} & -3\times10^{-4} \\ : & : & : & 0 & -3\times10^{-4} & 8\times10^{-4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.56\times10^{-5} \\ 4.95\times10^{-5} \\ 2.66\times10^{-3} \end{Bmatrix}$$

Multiplying matrices yields

$$f'^{(1)}_{1x} = 26.9 \text{ kN} = -f'^{(1)}_{2x},\ f'^{(1)}_{1y} = 42 \text{ kN} = -f'^{(1)}_{2y}$$

$$m_1^{(1)} = 55.9 \text{ kN}\cdot\text{m},\ m_2^{(1)} = 111.7 \text{ kN}\cdot\text{m}$$

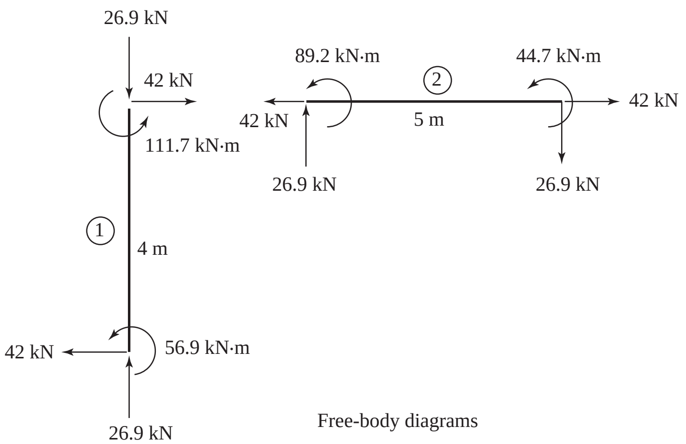
Similarly for element (2)

$$\{f'\}^{(2)} = [K']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)} \text{ yields}$$

$$f'^{(2)}_{2x} = 42.0 \text{ kN} = -f'^{(2)}_{3x}$$

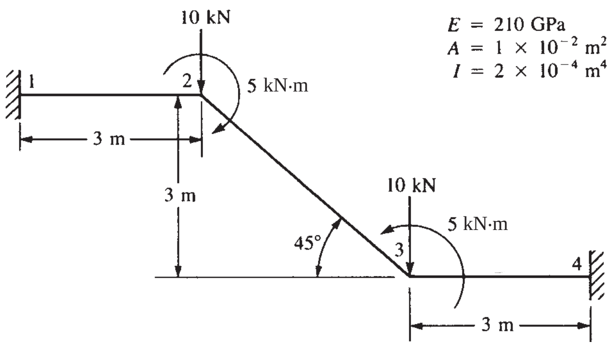
$$f'^{(2)}_{2y} = -26.9 \text{ kN} = -f'^{(2)}_{3y}$$

$$m_2^{(2)} = 89.2 \text{ kN}\cdot\text{m},\ m_3^{(2)} = 44.7 \text{ kN}\cdot\text{m}$$



Free-body diagrams

5.10



Element (1)

$$\frac{E}{L} = \frac{210\times10^9}{3\text{m}} = 70\times10^9,\ \frac{12I}{L^2} = 2.67\times10^{-4}$$

$$\frac{6I}{L} = 4\times10^{-4},\ C = 1,\ S = 0$$

$$u_1 = v_1 = \phi_1 = 0 \text{ and } u_4 = v_4 = \phi_4 = 0$$

$$[k^{(1)}] = 70\times10^9 \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1\times10^{-2} & 0 & 0 \\ & 2.67\times10^{-4} & -4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$\frac{E}{L} = \frac{210\times10^9}{3\sqrt{2}} = 49.5\times10^9,\quad \frac{12I}{L^2} = \frac{12(2\times10^{-4})}{18} = 1.33\times10^{-4}$$

$$\frac{6I}{L} = 2.83\times10^{-4},\ C = 0.707 = -S,\ 4I = 8\times10^{-4}$$

$$[k^{(2)}] = \begin{bmatrix} 3.58 & -3.48 & 0.1415 & 3.58 & 3.48 & 0.1415 \\ & 3.58 & 0.1415 & -3.48 & -3.58 & 0.1415 \\ & & 0.566 & -0.1415 & -0.1415 & 0.2828 \\ & & & 3.58 & -3.48 & -0.1415 \\ & & & & 3.58 & -0.1415 \\ \text{Symmetry} & & & & & 0.566 \end{bmatrix} 70\times10^6$$

Element (3)

$$\frac{E}{L} = 70\times10^9,\ \frac{12I}{L^2} = 2.67\times10^{-4},\ \frac{6I}{L} = 4\times10^{-4}$$

$$C = 1,\ S = 0$$

$$[k^{(3)}] = 70\times10^9 \begin{array}{c} \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 1\times10^{-2} & 0 & 0 \\ 0 & 2.67\times10^{-4} & 4\times10^{-4} \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Global equations

$$10\times10^{9}\begin{bmatrix} 0.0135 & -0.0035 & 0.00014 & 0.0036 & 0.0035 & 0.00014 \\ & 0.0038 & -0.0003 & -0.0035 & -0.0036 & 0.00014 \\ & & 0.0014 & -0.00014 & -0.00014 & 0.00028 \\ & & & 0.0135 & -0.0035 & -0.00014 \\ & & & & 0.0038 & 0.0003 \\ \text{Symmetry} & & & & & 0.0014 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10000 \\ -5000 \\ 0 \\ -10000 \\ 5000 \end{Bmatrix}$$

Using an equation solver

$$u_2 = 0.16\times10^{-10}\text{ m} \qquad u_3 = 0.85\times10^{-11}$$

$$v_2 = -0.1423\times10^{-2}\text{ m} \qquad v_3 = -0.1423\times10^{-2}\text{ m}$$

$$\phi_2 = -0.5917\times10^{-3}\text{ rad} \qquad \phi_3 = 0.5917\times10^{-3}\text{ rad}$$

Element forces

Element (1)

$$[T]\,\{d\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \sim 0 \\ 0 \\ -0.1423\times10^{-2} \\ -0.5917\times10^{-3} \end{Bmatrix}$$

$$\{f'\}^{(1)} = [k']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)}$$

$$= 70\times10^{9}\begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ & & 4C_2L^2 & 0 & -6C_2L & 2C_2L \\ & & & C_1 & 0 & 0 \\ & & & & 12C_2 & -6C_2L \\ \text{Symmetry} & & & & & 4C_2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \overset{0}{\cancel{0.16\times10^{-10}}} \\ -0.1923\times10^{-2} \\ -0.5917\times10^{-3} \end{Bmatrix}$$

$$f'^{(1)}_{1x} = 0$$

$$f'^{(1)}_{1y} = 10028\text{ N}$$

$$m_1^{(1)} = 23276\text{ N}\cdot\text{m}$$

$$f'^{(1)}_{2x} = 0$$

$$f'^{(1)}_{2y} = -10028\text{ N}$$

$$m_2^{(1)} = 6709\text{ N}\cdot\text{m}$$

Element (2)

$$[T] = \left[\begin{array}{ccc:ccc} 0.707 & -0.707 & 0 & 0 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline & & & & & \\ & & & & & \\ & & & & & \end{array}\right]$$

$$\{f'\}^{(2)} = [k']^{(2)}\,[T]^{(2)}\,\{d\}^{(2)} = \begin{Bmatrix} 0 \\ -0.387\times10^{-4} \cong 0 \\ -11,720 \\ 0 \\ 0.387\times10^{-4} \cong 0 \\ 11,720 \end{Bmatrix}$$

where $[k']^{(2)}$ from Equation (6.1.8) text

$$\{d\}^{(2)} = \begin{Bmatrix} 0.16\times10^{-10} = 0 \\ -0.1423\times10^{-2} \\ -0.5917\times10^{-3} \\ 0.85\times10^{-11} = 0 \\ -0.1423\times10^{-2} \\ 0.5917\times10^{-3} \end{Bmatrix}$$

Similarly for element (3)

$$f'^{(3)}_{3x} = 0$$

$$f'^{(3)}_{3y} = -10{,}028 \text{ N}$$

$$m_3^{(3)} = -6709 \text{ N}\cdot\text{m}$$

$$f'^{(3)}_{4x} = 0$$

$$f'^{(3)}_{4y} = 10{,}028 \text{ N}$$

$$m_4^{(3)} = -23276 \text{ N}\cdot\text{m}$$

5.11

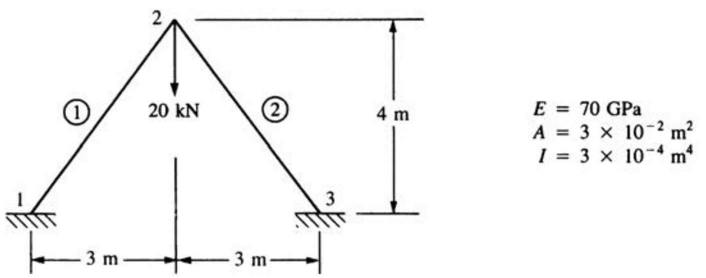


Figure P5–11

This problem is done using symmetry and Mathcad

$$E = 70 \times 10^9$$
$$A = 3 \times 10^{-2}$$
$$I = 3 \times 10^{-4}$$

Element 1

$$x_1 = 0 \qquad x_2 = 3$$
$$y_1 = 0 \qquad y_2 = 4$$
$$L_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$C = \frac{x_2 - x_1}{L_1} \qquad S = \frac{y_2 - y_1}{L_1}$$
$$C = 0.6 \qquad S = 0.8$$
$$N = \frac{12I}{L_1^{\,2}} \qquad M = \frac{6I}{L_1}$$

$$k_1 = \left(\frac{E}{L_1}\right)\begin{bmatrix} AC^2 + NS^2 & (A-N)CS & -MS & -(AC^2 + NS^2) & -(A-N)CS & -MS \\ (A-N)CS & AS^2 + NC^2 & MC & -(A-N)CS & -(AS^2 + NC^2) & MC \\ -MS & MC & 4I & MS & -MC & 2I \\ -(AC^2 + NS^2) & -(A-N)CS & MS & AC^2 + NS^2 & (A-N)CS & MS \\ -(A-N)CS & -(AS^2 + NC^2) & -MC & (A-N)CS & AS^2 + NC^2 & -MC \\ -MS & MC & 2I & MS & -MC & 4I \end{bmatrix}$$

Boundary conditions with symmetry

$$u_1 = 0 \qquad M_2 = 0 \qquad u_2 = 0$$
$$v_1 = 0 \qquad \phi_2 = 0$$
$$\phi_1 = 0$$

Reduced set of equations

Guess

$$v_2 = 1$$

Given

$$F_{2y} = -10 \times 10^3$$
$$(F_{2y}) = \left[\frac{E}{L_1}\left(AS^2 + NC^2\right)\right](v_2)$$
$$v_2 = \text{find}\,(v_2)$$

$v_2 = -3.7102 \times 10^{-5}$ Displacement of node 2

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \end{pmatrix} = k_1 \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} \quad \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3.71\times10^{-5} \\ 0 \end{pmatrix} \text{ Displacements and rotations}$$

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \end{pmatrix} = \begin{pmatrix} 7443.901 \\ 10000 \\ 112.197 \end{pmatrix} \text{ Reaction forces}$$

$$F_1 = \sqrt{F_{1x}{}^2 + F_{1y}{}^2}$$

$F_1 =$ 12466.422 Reaction magnitude at node 1 same as node 3 by symmetry.

Forces in elements

$$T = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix}$$

$$C_1 = \frac{AE}{L_1} \qquad C_2 = \frac{EI}{L_1^3}$$

$$[k'] = \begin{pmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2 L_1 & 0 & -12C_2 & 6C_2 L_1 \\ 0 & 6C_2 L_1 & 4C_2 L_1^2 & 0 & -6C_2 L_1 & 2C_2 L_1^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2 L_1 & 0 & 12C_2 & -6C_2 L_1 \\ 0 & 6C_2 L_1 & 2C_2 L_1^2 & 0 & -6C_2 L_1 & 4C_2 L_1^2 \end{pmatrix}$$

$$\begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix} = [k'] \, [T] \, \{d\} \qquad \begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix} = \begin{pmatrix} 12466.341 \\ 44.879 \\ 112.197 \\ -12466.341 \\ -44.879 \\ 112.197 \end{pmatrix}$$

Forces in element 1 and 2 by symmetry.

5.12 Determine displacements and rotations of the nodes, element forces, and reactions.

Material properties & geometry

$EE = 210 \times 10^9$ Pa Modulus of elasticity

$AA = 80 \times 10^{-3}$ m^2 Area of cross section of all elements

$\mathrm{II} = 1.2 \times 10^{-4}$ m^4 Area moment of inertia of all elements

$LL_1 =$ 3 m Length of element 1

$LL_2 =$ 6 m Length of element 2

$\theta_1 =$ 0 deg Angle between local x and global x for element 1.

$\theta_2 =$ – 90 deg Angle between local x and global x for element 2.

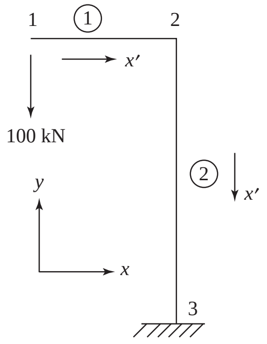


Figure P5.12

Applied loads

$F_{1y} =$ – 100000 N Applied load (down at node 1)

Boundary conditions

$d_{3x} =$ 0 x-displacement 3 is zero.

$d_{3y} =$ 0 y-displacement at node 3 is zero.

$\phi_3 =$ 0 Angular displacement. at node 3 is zero.

Defining element properties in unitless format. (Mathcad does not allow elements with dissimilar units within the same matrix.)

$E = \dfrac{EE}{Pa}$ Modulus of elasticity (Pa).

$A_1 = \dfrac{AA}{m^2}$ Cross sectional area of element 1 (m^2)

$A_2 = \dfrac{AA}{m^2}$ Cross sectional area of element 2 (m^2)

$I_1 = \dfrac{II}{m^4}$ Area moment of element 1 (m^4)

$I_2 = \dfrac{II}{m^4}$ Area moment of element 2 (m^4)

$L_1 = \dfrac{LL_1}{m}$ Length of element 1(m)

$L_2 = \dfrac{LL_2}{m}$ Length of element 2 (m)

$C_1 = \cos(\theta_1)$ Cosine of angle between local x and global x for element 1.

$S_1 = \sin(\theta_1)$ Sine of angle between local x and global x for element 1.

$C_2 = \cos(\theta_2)$	Cosine of angle between local x and global x for element 2.
$S_2 = \sin(\theta_2)$	Sine of angle between local x and global x for element 2.

Functional equations for the global stiffness matrix for a 2D beam/frame element with axial effects.

Refer to text Equation (5.1.11)

Functional expressions for large repeated terms in global stiffness matrix.

$$M_1(A, C, S, I, L) = AC^2 + \frac{12I}{L^2}\ S^2$$

$$M_2(A, C, S, I, L) = \left(A - \frac{12I}{L^2}\right) CS$$

$$M_3(A, C, S, I, L) = AS^2 + \frac{12I}{L^2}\ C^2$$

Functional equation for global stiffness matrix of beam/frame element.

$$k(A, C, S, E, I, L) = \frac{E}{L}\begin{bmatrix} M_1(A,C,S,I,L) & M_2(A,C,S,I,L) & \left(\frac{-6I}{L}S\right) & -M_1(A,C,S,I,L) & -M_2(A,C,S,I,L) & \frac{-6I}{L}S \\ M_2(A,C,S,I,L) & M_3(A,C,S,I,L) & \frac{6I}{L}C & -M_2(A,C,S,I,L) & -M_3(A,C,S,I,L) & \frac{-6I}{L}C \\ \frac{-6I}{L}S & \frac{6I}{L}C & 4I & \frac{6I}{L}S & \frac{-6I}{L}C & 2I \\ -M_1(A,C,S,I,L) & -M_2(A,C,S,I,L) & \frac{6I}{L}S & M_1(A,C,S,I,L) & M_2(A,C,S,I,L) & \frac{6I}{L}S \\ -M_2(A,C,S,I,L) & -M_3(A,C,S,I,L) & \frac{-6I}{L}C & M_2(A,C,S,I,L) & M_3(A,C,S,I,L) & -\left(\frac{6I}{L}C\right) \\ \frac{-6I}{L}S & \frac{6I}{L}C & 2I & \frac{6I}{L}S & -\left(\frac{6I}{L}C\right) & 4I \end{bmatrix}$$

Calculate global stiffness matrix for 1st element.

$i = 1$	Set $i = 1$ so that the properties of element 1 are used in the functional expression.

$$k_1 = k(A_i, C_i, S_i, E, I_i, L_i)$$

$$k_1 = \begin{pmatrix} 5.6\times10^9 & 0 & 0 & -5.6\times10^9 \\ 0 & 1.12\times10^7 & 1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 3.36\times10^7 & 0 \\ -5.6\times10^9 & 0 & 0 & 5.6\times10^9 \\ 0 & -1.12\times10^7 & -1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 1.68\times10^7 & 0 \end{pmatrix}$$

$$\left.\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ 0 & 0 \\ 1.12\times10^{7} & -1.68\times10^{7} \\ -1.68\times10^{7} & 3.36\times10^{7} \end{matrix}\right)$$

Augment element global k matrix with rows and columns of zeros to facilitate the assembly of the total global stiffness matrix. The global k matrix for each element needs to have the same number of rows and columns as there is degrees of freedom in the model. In this case, it is 9 (3 nodes, $\frac{3\text{ DOF}}{\text{node}}$).

$$\text{ZeroCol} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{ZeroRow} = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$kI_a = \text{augment}\ (k_1, \text{ZeroCol, ZeroCol, ZeroCol})$$

$$kI_b = \text{stack}(k_{1a}, \text{ZeroRow, ZeroRow, ZeroRow})$$

$k_{Ib} =$

0	1	2	3	4	5	6	7	8	9
1	5.6×10^{9}	0	0	-5.6×10^{9}	0	0	0	0	0
2	0	1.12×10^{7}	1.68×10^{7}	0	-1.12×10^{7}	1.68×10^{7}	0	0	0
3	0	1.68×10^{7}	3.36×10^{7}	0	-1.68×10^{7}	1.68×10^{7}	0	0	0
4	-5.6×10^{9}	0	0	5.6×10^{9}	0	0	0	0	0
5	0	-1.12×10^{7}	-1.68×10^{7}	0	1.12×10^{7}	-1.68×10^{7}	0	0	0
6	0	1.68×10^{7}	1.68×10^{7}	0	-1.68×10^{7}	3.36×10^{7}	0	0	0
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Calculate global stiffness matrix for 2nd element.

$i = 2$ Set $i = 2$ so that the properties of element 2 are used in the functional expression.

$$k_2 = k\ (A_i, C_i, S_i, E, I_i, L_i)$$

$$K_2 = \begin{pmatrix} 1.4\times10^{6} & \cancelto{0}{-1.714\times10^{-7}} & 4.2\times10^{6} & -1.4\times10^{6} & \cancelto{0}{1.714\times10^{-7}} & 4.2\times10^{6} \\ \cancelto{0}{-1.714\times10^{-7}} & 2.8\times10^{9} & \cancelto{0}{2.572\times10^{-10}} & \cancelto{0}{1.714\times10^{-7}} & -2.8\times10^{9} & \cancelto{0}{2.572\times10^{-10}} \\ 4.2\times10^{6} & \cancelto{0}{2.572\times10^{-10}} & 1.68\times10^{7} & -4.2\times10^{6} & -2.572\times10^{-10} & 8.4\times10^{6} \\ -1.4\times10^{6} & \cancelto{0}{1.714\times10^{-7}} & -4.2\times10^{6} & 1.4\times10^{6} & \cancelto{0}{-1.714\times10^{-7}} & -4.2\times10^{6} \\ \cancelto{0}{1.714\times10^{-7}} & -2.8\times10^{9} & \cancelto{0}{-2.572\times10^{-10}} & \cancelto{0}{-1.714\times10^{-7}} & 2.8\times10^{9} & \cancelto{0}{-2.572\times10^{-10}} \\ 4.2\times10^{6} & \cancelto{0}{2.572\times10^{-10}} & 8.4\times10^{6} & -4.2\times10^{6} & \cancelto{0}{-2.572\times10^{-10}} & 1.68\times10^{7} \end{pmatrix}$$

Augment element global k matrix with rows and columns of zeros to facilitate the assembly of the total global stiffness matrix. As before, we need 9 rows and columns.

$$k_{2a} = \text{augment (ZeroCol,ZeroCol, ZeroCol, } k_2)$$

$$k_{2b} = \text{stack(ZeroRow, ZeroRow, ZeroRow, } k_{2a})$$

$k_{2b} =$

0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	1.4×10^{6}	-1.714×10^{-7}	4.2×10^{6}	-1.4×10^{6}	1.714×10^{-7}	4.2×10^{6}
5	0	0	0	-1.714×10^{-7}	2.8×10^{9}	2.572×10^{-10}	1.714×10^{-7}	-2.8×10^{9}	2.572×10^{-10}
6	0	0	0	4.2×10^{6}	2.572×10^{-10}	1.68×10^{7}	-4.2×10^{6}	-2.572×10^{-10}	8.4×10^{6}
7	0	0	0	-1.4×10^{6}	1.714×10^{-7}	-4.2×10^{6}	1.4×10^{6}	-1.714×10^{-7}	-4.2×10^{6}
8	0	0	0	1.714×10^{-7}	-2.8×10^{9}	-2.572×10^{-10}	-1.714×10^{-7}	2.8×10^{9}	-2.572×10^{-10}
9	0	0	0	4.2×10^{6}	2.572×10^{-10}	8.4×10^{6}	-4.2×10^{6}	-2.572×10^{-10}	1.68×10^{7}

Calculate total global stiffness matrix by adding augmented matrices for each element.

$$K = k_{1b} + k_{2b}$$

$K =$

0	1	2	3	4	5	6	7	8	9
1	5.6×10^{9}	0	0	-5.6×10^{9}	0	0	0	0	0
2	0	1.12×10^{7}	1.68×10^{7}	0	-1.12×10^{7}	1.68×10^{7}	0	0	0
3	0	1.68×10^{7}	3.36×10^{7}	0	-1.68×10^{7}	1.68×10^{7}	0	0	0
4	-5.6×10^{9}	0	0	5.601×10^{9}	-1.714×10^{-7}	4.2×10^{6}	-1.4×10^{6}	1.714×10^{-7}	4.2×10^{6}
5	0	-1.12×10^{7}	-1.68×10^{7}	-1.714×10^{-7}	2.811×10^{9}	-1.68×10^{7}	1.714×10^{-7}	-2.8×10^{9}	2.572×10^{-10}
6	0	1.68×10^{7}	1.68×10^{7}	4.2×10^{6}	-1.68×10^{7}	5.04×10^{7}	-4.2×10^{6}	2.572×10^{-10}	8.4×10^{6}
7	0	0	0	-1.4×106	1.714×10^{-7}	-4.2×10^{6}	1.4×10^{6}	-1.714×10^{-7}	-4.2×10^{6}
8	0	0	0	1.714×10^{-7}	-2.8×10^{9}	2.572×10^{-10}	-1.714×10^{-7}	2.8×10^{9}	2.572×10^{-10}
9	0	0	0	4.2×10^{6}	2.572×10^{-10}	8.4×10^{6}	-4.2×10^{6}	2.572×10^{-10}	1.68×10^{7}

Solve for displacements and rotations at node 1 and 2.

First partition out rows and columns associated with homogenous boundary conditions (rows/columns 7, 8 and 9)

$$K_{\text{part}} = \text{submatrix } (K, 1, 6, 1, 6)$$

$$K_{\text{part}} = \begin{pmatrix} 5.6\times10^{9} & 0 & 0 & -5.6\times10^{9} \\ 0 & 1.12\times10^{7} & 1.68\times10^{7} & 0 \\ 0 & 1.68\times10^{7} & 3.36\times10^{7} & 0 \\ -5.6\times10^{9} & 0 & 0 & 5.601\times10^{9} \\ 0 & -1.12\times10^{7} & -1.68\times10^{7} & -1.714\times10^{-7} \\ 0 & 1.68\times10^{7} & 1.68\times10^{7} & 4.2\times10^{6} \end{pmatrix}$$

$$\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ -1.714\times10^{-7} & 4.2\times10^{6} \\ 2.811\times10^{9} & -1.68\times10^{7} \\ -1.68\times10^{7} & 5.04\times10^{7} \end{matrix}\Bigg)$$

Define partitioned vector of applied loads.

$$F_{\text{part}} = \begin{pmatrix} 0 & \dfrac{F_{1y}}{N} & 0 & 0 & 0 & 0 \end{pmatrix}^{T}$$

Solve for displacements and rotations at node 1 and 2.

$$\begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = K_{\text{part}}{}^{-1} F_{\text{part}}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -0.214 \\ -0.25 \\ 0.089 \\ -0.214 \\ -3.571\times10^{-5} \\ 0.071 \end{pmatrix} \begin{matrix} \text{(m)} \\ \text{(m)} \\ \text{(rad)} \\ \text{(m)} \\ \text{(m)} \\ \text{(rad)} \end{matrix}$$ [Displacements are in meters, rotations in radians.]

$0.089 \text{ rad} = 5.099°$

$0.071 \text{ rad} = 4.068°$

Solving for reactions.

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{pmatrix} = K \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{pmatrix}$$ Multiplying global stiffness matrix by displacement vector.

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F_{3x} \\ F_{3y} \\ M_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -10\times10^4 \\ 2.328\times \overset{0}{\cancel{10^{-10}}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1\times10^5 \\ -3\times10^5 \end{pmatrix}$$

Forces are in newtons, moment in $\text{N}\cdot\text{m}$.

Values agree with what one would expect given the simple nature of the problem. Vertical reaction at node 3 should be equal and opposite the applied load F_{1y} and it is. Moment reaction at 3 (M_3) should equal the applied load (F_{1y}) times it's moment arm (L_1) and it does.

Solve for element forces.

Functional equations for local beam elements

Functional equation for local stiffness matrix for a 2D beam/frame element with axial affects. Refer to text Equation (5.1.8).

$$K_{\text{local}}\,(A, C, S, E, I, L) = \begin{pmatrix} \frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix}$$

Functional equation for transformation matrix between local and global coordinates.

$$T(C, S) = \begin{pmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solving for element forces in element 1

$$i = 1$$

$k_{\text{local}}1 = k_{\text{local}}(A_i, C_i, S_i, E, I_i, L_i)$ Local k matrix for element 1

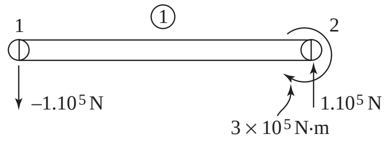
$$k_{\text{local}}1 = \left(\begin{matrix} 5.6\times10^9 & 0 & 0 & -5.6\times10^9 \\ 0 & 1.12\times10^7 & 1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 3.36\times10^7 & 0 \\ -5.6\times10^9 & 0 & 0 & 5.6\times10^9 \\ 0 & -1.12\times10^7 & -1.68\times10^7 & 0 \\ 0 & 1.68\times10^7 & 1.68\times10^7 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 \\ -1.12\times10^{7} & 1.68\times10^{7} \\ -1.68\times10^{7} & 1.68\times10^{7} \\ 0 & 0 \\ 1.12\times10^{7} & -1.68\times10^{7} \\ -1.68\times10^{7} & 3.36\times10^{7} \end{matrix}\right)$$

$T_1 = T(C_i, S_i)$ Transformation matrix for element 1.

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_1 = k_{\text{local}}\, T_1 \begin{pmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{pmatrix}$$ Calculate local forces/moments in element 1.



$$f_1 = \begin{pmatrix} 0 \\ -100000 \\ 0 \\ 0 \\ 100000 \\ -300,000 \end{pmatrix} \begin{pmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{pmatrix}$$

Forces are in Newtons, moments in $\text{N}\cdot\text{m}$.

f_{1y} comes back out of the equations as expected.

f_{2y} must be equal and opposite—and it is.

m_2 also checks (resists in CW direction the applied load of 1*10^5N over 3m moment arm)

Solving for element forces in element 2

$i = 2$

$k_{\text{local}}2 = k_{\text{local}}(A_\text{i}, \text{C}_\text{i}, S_\text{i}, E, I_\text{i}, L_\text{i})$ Local k matrix for a element 2

$$k_{\text{local}}2 = \begin{pmatrix} 2.8\times10^9 & 0 & 0 & -2.8\times10^9 \\ 0 & 1.4\times10^6 & 4.2\times10^6 & 0 \\ 0 & 4.2\times10^6 & 1.68\times10^7 & 0 \\ -2.8\times10^9 & 0 & 0 & 2.8\times10^9 \\ 0 & -1.4\times10^6 & -4.2\times10^6 & 0 \\ 0 & 4.2\times10^6 & 8.4\times10^6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ -1.4\times10^6 & 4.2\times10^6 \\ -4.2\times10^6 & 8.4\times10^6 \\ 0 & 0 \\ 1.4\times10^6 & -4.2\times10^6 \\ -4.2\times10^6 & 1.68\times10^7 \end{pmatrix}$$

$T_2 = T(C_i, S_i)$ Transformation matrix for element 2.

$$T_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

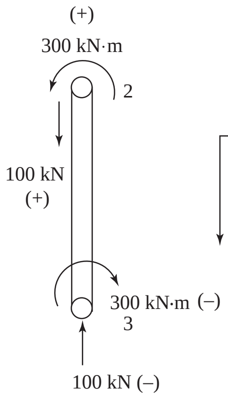
Calculate local forces/moments in element 1.

$$f_2 = k_{\text{local}}\, 2\, T_2 \begin{pmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{pmatrix}$$

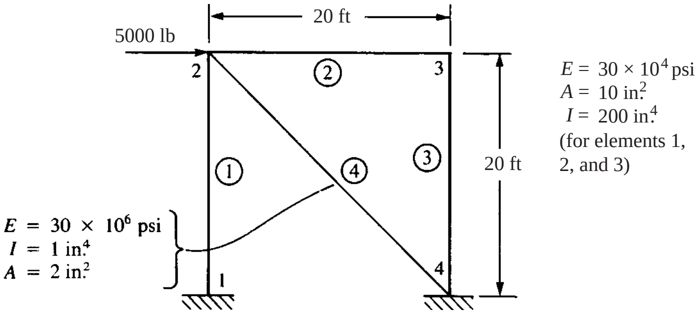
Forces are in Newtons, moments in $\text{N}\cdot\text{m}$.
f_{2y} and f_{3y} again are of same magnitude as applied load at node 1 as expected.

m_2 and m_3 also have correct magnitude. m_3 must be equal and opposite of applied moment. m_2 must be opposite of m_3 for equilibrium.

$$f_2 = \begin{pmatrix} 100{,}000 \\ 0 \\ 300000 \\ -100000 \\ 0 \\ -300000 \end{pmatrix} \quad \begin{pmatrix} f_{2x} \\ f_{2y} \\ m_2 \\ f_{3x} \\ f_{3y} \\ m_3 \end{pmatrix}$$



5.13



Boundary conditions $\quad u_1 = v_1 = \phi_1 = 0, \quad u_4 = v_4 = \phi_4 = 0$

Element (1) by Equation (6.1.11)

$$C = 0; \quad S = 1$$

$$[k^{(1)}] = \frac{E}{L} \begin{array}{c} \begin{matrix} u_2 & v_2 & \phi_2 \end{matrix} \\ \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ 0 & A & 0 \\ \frac{6I}{L} & 0 & 4I \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \quad S = 0$$

$$[k^{(2)}] = \frac{E}{L} \begin{array}{c} \begin{matrix} u_2 & v_2 & \phi_2 & u_3 & v_3 & \phi_3 \end{matrix} \\ \begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ & \frac{12I}{L^2} & \frac{6I}{L} & 0 & \frac{-12I}{L^2} & \frac{6I}{L} \\ & & 4I & 0 & \frac{-6I}{L} & 2I \\ & & & A & 0 & 0 \\ & & & & \frac{12I}{L^2} & \frac{-6I}{L} \\ \text{Symmetry} & & & & & 4I \end{bmatrix} \end{array}$$

Element (3)

$$C = 0, \quad S = -1$$

$$[k^{(3)}] = \frac{E}{L} \begin{array}{c} \begin{array}{ccc} u_3 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} \\ & A & 0 \\ & & 4I \end{bmatrix} \end{array}$$

Element (4)

$$C = \frac{-\sqrt{2}}{2}, \quad S = \frac{-\sqrt{2}}{2}, \quad L = 20, \quad L^{(4)} = \sqrt{2}\ L$$

$$[k^{(4)}] = \frac{E}{L^{(4)}} \begin{bmatrix} \frac{A}{2} + \frac{6L}{L^2} & \frac{-A}{2} + \frac{6I}{L^2} & \sqrt{2}\ \frac{3I}{L} \\ & \frac{A}{2} + \frac{6I}{L^2} & \sqrt{2}\ \frac{3I}{L} \\ \text{Symmetry} & & 4I \end{bmatrix}$$

Assembling

$$\begin{Bmatrix} 5000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{L} \begin{bmatrix} 10.75 & -0.707 & 5.0 & -10 & 0 & 0 \\ & 10.75 & 5.0 & 0 & -0.0417 & 5 \\ & & 1603 & 0 & -5 & 400 \\ & & & 10.04 & 0 & 5 \\ & & & & 10.04 & -5 \\ & & & & & 1600 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

$$\frac{E}{L} = \frac{30 \times 10^6}{20' \left(12 \frac{\text{in.}}{\text{ft}}\right)} = 0.125 \times 10^6 \ \frac{\text{lb}}{\text{in.}^3}$$

Solving simultaneously

$$u_2 = 0.055916 \text{ in.} \qquad u_3 = 0.05576 \text{ in.}$$
$$v_2 = 0.003817 \text{ in.} \qquad v_3 = -0.000133 \text{ in.}$$
$$\phi_2 = -0.00015 \text{ rad} \qquad \phi_3 = -0.000149 \text{ rad}$$

PLANE FRAME PROBLEM 6.13
NUMBER OF ELEMENTS = 4
NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	240.000000	0.000000	5000.000000
3	0 0 0	240.000000	240.000000	0.000000	0.000000
4	1 1 1	240.000000	0.000000	0.000000	0.000000

ELEMENTS

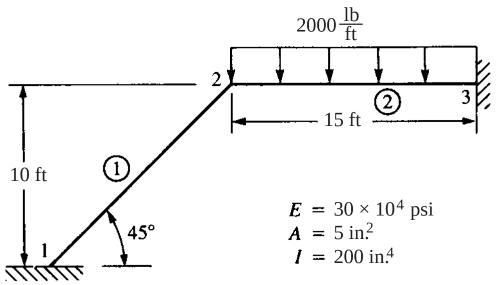
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
2	2	3	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
3	3	4	3.0000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
4	2	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00

NODE	DISPLACEMENTS		Z-ROTATION
	X	Y	THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.55918E–01	0.38170E–02	– 0.14987E–03
3	0.55760E–01	– 0.13304E–03	– 0.14913E–03
4	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z–MOMENT
1	1	2	– 0.4771E+04	0.1976E+03	0.2746E+05	– 0.4771E+04	– 0.1976E+03	0.1996E+05
2	2	3	0.1972E+03	– 0.1663E+03	– 0.1997E+05	– 0.1972E+03	0.1663E+03	– 0.1994E+05
3	3	4	0.1663E+03	0.1972E+03	0.1994E+05	– 0.1663E+03	– 0.1972E+03	0.2739E+05
4	2	4	0.6513E+04	0.1547E+00	0.1301E+02	– 0.6513E+04	– 0.1547E+00	0.3950E+02

5.14



Element (1)

$$C = \frac{\sqrt{2}}{2}\ ,\ S = \frac{\sqrt{2}}{2}$$

Use only node 2 part of $[k^{(1)}]$

$$[k^{(1)}] = \frac{E}{L_1}\begin{bmatrix} AC^2 + \frac{12I}{L_1^2}S^2 & \left(A - \frac{12I}{L_1^2}\right)CS & \frac{6I}{L_1}S \\ & AS^2 + \frac{12I}{L_1^2}C^2 & \frac{-6I}{L_1}C \\ \text{Symmetry} & & 4I \end{bmatrix}$$

$$
\begin{array}{cccc}
 & u_2 & v_2 & \phi_2 \\
= & \multicolumn{3}{c}{\begin{bmatrix} \frac{61}{24} & \frac{59}{24} & 5\sqrt{2} \\ & \frac{61}{24} & -5\sqrt{2} \\ \text{Symmetry} & & 800 \end{bmatrix} \times 125\sqrt{2} \times 10^3}
\end{array}
$$

Element (2)

$$C = 1, \; S = 0$$

$$[k^{(2)}] = \frac{E}{L_2} \begin{bmatrix} AC^2 + \frac{12I}{L_2^2} S^2 & \left(A - \frac{12I}{L_2^2}\right) CS & \frac{-6I}{L_2} S \\ & AS^2 + \frac{12I}{L_2^2} C^2 & \frac{6I}{L_2} C \\ \text{Symmetry} & & 4I \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{2}{27} & \frac{20}{3} \\ & & 800 \end{bmatrix} \times \frac{5}{3} \times 10^5$$

$$[K] = [k^{(1)}] + [k^{(2)}]$$

Then

$$\{F\} = [K]\,\{d\}$$

Equivalent nodal forces

$$f_{2x} = 0 \qquad f_{2y} = \frac{-wL_2}{2} = -15000 \text{ lb}$$

$$m_2 = \frac{-wL_2^2}{12} = -45{,}000 \text{ lb} \cdot \text{in.}$$

$$\begin{Bmatrix} 0 \\ -15000 \\ -45000 \end{Bmatrix} = \begin{Bmatrix} 1282600 & 434580 & 125000 \\ & 461650 & -138900 \\ & & 27{,}475{,}000 \end{Bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

Solving

$$u_2 = 0.0174 \text{ in.}$$

$$v_2 = -0.0481 \text{ in.}$$

$$\phi_2 = -0.00165 \text{ rad}$$

Element forces

$$\{f'\}^{(1)} = [K']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)}$$

Element one (1)

$$\begin{array}{c} \\ (1) \end{array}\quad \begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1^{(1)} \\ f'^{(1)}_{2x} \\ f'^{(1)}_{2y} \\ m_2^{(1)} \end{Bmatrix} = 10^6 \begin{Bmatrix} \frac{5}{4\sqrt{2}} & 0 & 0 & \frac{-5}{4\sqrt{2}} & 0 & 0 \\ & \frac{1}{48\sqrt{2}} & \frac{5}{4} & 0 & \frac{-1}{48\sqrt{2}} & \frac{5}{4} \\ & & 100\sqrt{2} & 0 & \frac{-5}{4} & 50\sqrt{2} \\ & & & \frac{5}{4\sqrt{2}} & 0 & 0 \\ & & & & \frac{1}{48\sqrt{2}} & \frac{-5}{4} \\ \text{Symmetry} & & & & & 100\sqrt{2} \end{Bmatrix} \begin{array}{c} [T]\{d\} \\ \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.02167 \\ 0.0463 \\ -0.00165 \end{Bmatrix} \end{array}$$

$$f'^{(1)}_{1x} = 19160 \text{ lb} = -f'^{(1)}_{2x}$$

$$f'^{(1)}_{2y} = -1385 \text{ lb} = -f'^{(1)}_{2y}$$

$$m_1^{(1)} = -59050 \text{ lb}\cdot\text{in.}$$

$$m_2^{(1)} = -176000 \text{ lb}\cdot\text{in.}$$

Element two (2)

$$\{f'\}^{(2)} = [K']^{(2)}[T]^{(2)}\{d\}^{(2)} - \{f'_0\}^{(2)}$$

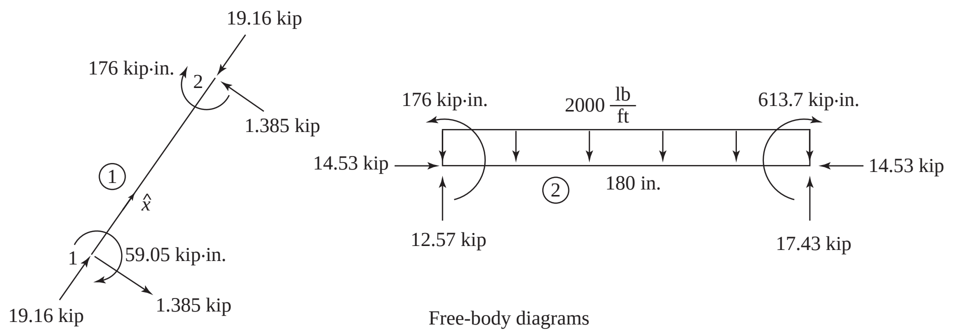
$$[T]^{(2)}\{d\}^{(2)} = \begin{bmatrix} 0.0174 \\ -0.0481 \\ -0.00165 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{Bmatrix} f'^{(2)}_{2xe} \\ f'^{(2)}_{2ye} \\ m_{2e}^{(2)} \\ f'^{(2)}_{3xe} \\ f'^{(2)}_{3ye} \\ m_{3e}^{(2)} \end{Bmatrix} = 10^6 \begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{-5}{6} & 0 & 0 \\ & \frac{1}{81} & \frac{10}{9} & 0 & \frac{-1}{81} & \frac{10}{9} \\ & & \frac{400}{3} & 0 & \frac{-10}{9} & \frac{200}{3} \\ & & & \frac{5}{6} & 0 & 0 \\ & & & & \frac{1}{81} & \frac{-10}{9} \\ & & & & & \frac{400}{3} \end{bmatrix} \begin{Bmatrix} 0.0174 \\ -0.0481 \\ -0.00165 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f'_{(2)}\}^{(2)} = \begin{Bmatrix} 14530 \\ -2432 \\ -273,980 \\ -14530 \\ 2432 \\ -163,700 \end{Bmatrix}$$

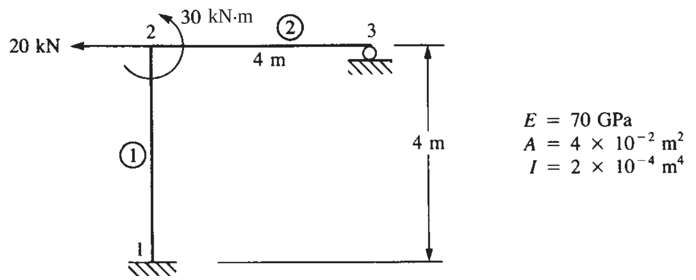
Finally $\{f'\} = \{f'_e\} - \{f'_0\}$

$$\begin{Bmatrix} f'^{(2)}_{2x} \\ f'^{(2)}_{2y} \\ m^{(2)}_2 \\ f'^{(2)}_{3x} \\ f'^{(2)}_{3y} \\ m^{(2)}_3 \end{Bmatrix} = \begin{Bmatrix} 14530 \\ -2432 \\ -273{,}980 \\ -14530 \\ 2432 \\ -163{,}700 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -15000 \\ -450{,}000 \\ 0 \\ -15000 \\ 450{,}000 \end{Bmatrix} = \begin{Bmatrix} 14530\text{ lb} \\ 12568\text{ lb} \\ 176020\text{ lb}\cdot\text{in.} \\ -14530\text{ lb} \\ 17432\text{ lb} \\ -613700\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Free-body diagrams

5.15



Element (1)

$$C = 0, \quad S = 1$$

$$\frac{12\,I}{L^2} = \frac{12\,(2\times10^{-4})}{(4)^2} = 1.5\times10^{-4}\,\text{m}^2$$

$$\frac{6\,I}{L} = 3.0\times10^{-4}\,\text{m}^3$$

$$\frac{E}{L} = \frac{70\times10^6}{4} = 1.75\times10^7\ \frac{\text{kN}}{\text{m}}$$

$$[k^{(1)}] = 1.75\times10^7 \begin{array}{c} \begin{array}{ccc} u_2 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 1.5\times10^{-4} & 0 & 3.0\times10^{-4} \\ & 4\times10^{-2} & 0 \\ \text{Symmetry} & & 8\times10^{-4} \end{bmatrix} \end{array}$$

Element (2)

$$S = 0, \quad C = 1$$

$$
\begin{array}{cccccc}
 & u_2 & v_2 & \phi_2 & u_3 & \phi_3 \\
[k^{(2)}] = 1.75 \times 10^7 & \multicolumn{5}{l}{\left[\begin{array}{ccccc}
4\times10^{-2} & 0 & 0 & -4\times10^{-2} & 0 \\
 & 1.5\times10^{-4} & 3\times10^{-4} & 0 & 3\times10^{-4} \\
 & & 8\times10^{-4} & 0 & 4\times10^{-4} \\
 & & & 4\times10^{-2} & 0 \\
\text{Symmetry} & & & & 8\times10^{-4}
\end{array}\right]}
\end{array}
$$

where boundary conditions

$$u_1 = v_1 = \phi_1 = 0 \text{ and } v_3 = 0$$

have been used in $[k^{(1)}]$ and $[k^{(2)}]$

The global equations are

$$
1.75 \times 10^7 \begin{bmatrix}
4.015\times10^{-2} & 0 & 3\times10^{-4} & -4\times10^{-2} & 0 \\
 & 4.015\times10^{-2} & 3\times10^{-4} & 0 & 3\times10^{-4} \\
 & & 1.6\times10^{-3} & 0 & 4\times10^{-4} \\
 & & & 4\times10^{-2} & 0 \\
\text{Symmetry} & & & & 8\times10^{-4}
\end{bmatrix}
\begin{Bmatrix} u_2 \\ v_2 \\ \phi_3 \\ u_3 \\ \phi_3 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} F_{2x} = -20 \\ F_{2y} = 0 \\ M_2 = 30 \\ F_{3x} = 0 \\ M_3 = 0 \end{Bmatrix}
$$

Solving simultaneously

$$u_2 = -1.76 \times 10^{-2}\text{ m},\ v_2 = -1.87 \times 10^{-5}\text{ m}$$

$$\phi_2 = 5 \times 10^{-3}\text{ rad}$$

$$u_3 = -1.76 \times 10^{-2}\text{ m},\quad \phi_3 = -2.49 \times 10^{-3}\text{ rad}$$

Element forces

Element (1)

$$\{f'\}^{(1)} = [k']^{(1)}\ [T]^{(1)}\ \{d\}^{(1)}$$

$$
[T]^{(1)}\{d\}^{(1)} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \phi_1 = 0 \\ u_2 = -1.76\times10^{-2} \\ v_2 = -1.87\times10^{-5} \\ \phi_2 = 5.0\times10^{-3} \end{Bmatrix}
$$

$$
[T]^{(1)}\{d\}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.87\times10^{-5} \\ 1.76\times10^{-2} \\ 5.00\times10^{-3} \end{Bmatrix}
$$

$$\{f'\}^{(1)} = [k']^{(1)}\,[T]^{(1)}\,\{d\}^{(1)} =$$

$$\begin{bmatrix} 7\times10^{-5} & 0 & 0 & -7\times10^{-5} & 0 & 0 \\ & 2.625\times10^{3} & 5.25\times10^{3} & 0 & -2.625\times10^{3} & 5.25\times10^{3} \\ & & 1.4\times10^{4} & 0 & -5.25\times10^{3} & 7\times10^{3} \\ & & & 7\times10^{5} & 0 & 0 \\ & & & & 2.625\times10^{3} & -5.25\times10^{3} \\ \text{Symmetry} & & & & & 1.4\times10^{4} \end{bmatrix}$$

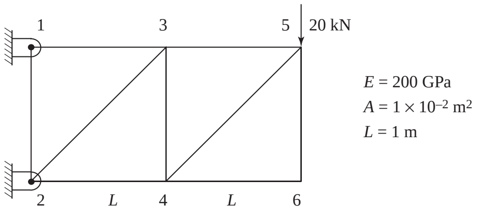
$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.87\times10^{-5} \\ 1.76\times10^{-2} \\ 5.0\times10^{-3} \end{Bmatrix}$$

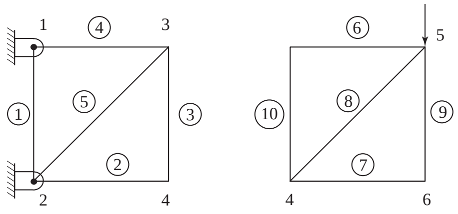
$$f'_{1x} = 13.1 \text{ kN}$$
$$f'_{1y} = -20.0 \text{ kN}$$
$$m_1 = -57.4 \text{ kN}\cdot\text{m}$$
$$f'_{2x} = -13.1 \text{ kN}$$
$$f'_{2y} = 20.0 \text{ kN}$$
$$m_2 = -22.4 \text{ kN}\cdot\text{m}$$

Similarly for element (2)

$$\{f'\}^{(2)} = [k']^{(2)}[T]^{(2)}\,\{d\}^{(2)}$$
$$f'_{2x} = f'_{3x} = 0$$
$$f'_{2y} = 13.1 \text{ kN},\ f'_{3y} = -13.1 \text{ kN}$$
$$m_2 = 52.47 \text{ kN}\cdot\text{m},\ m_3 = 0$$

5.16





Element (1)

$$C = 0, \quad S = 1$$

$$[k^{(1)}] = \frac{\overset{E}{200 \times 10^9}\ \overset{A}{(1 \times 10^{-2})}}{\underset{L}{1}} \begin{array}{c} \begin{array}{cccc} (2) & & (1) & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (3)

$$C = 0, \quad S = 1$$

$$[k^{(3)}] = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} (4) & & (3) & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Element (2)

$$C = 1, \quad S = 0$$

$$[k^{(2)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} (2) & & (4) & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (4)

$$C = 1, \quad S = 0$$

$$[k^{(4)}] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} (1) & & (3) & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element (5)

$$C = \frac{\sqrt{2}}{2}, \quad S = \frac{\sqrt{2}}{2}$$

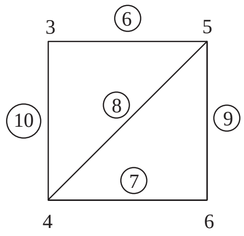
$$[k^{(5)}] = \frac{AE}{\sqrt{2}L} \begin{array}{c} \begin{array}{cccc} (2) & & (3) & \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Boundary conditions

$u_1 = v_1 = u_2 = v_2 = 0$

Assemble appropriate parts of $[k]$'s

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 2 \times 10^9 \begin{bmatrix} 1.3535 & 0.3535 & 0 & 0 \\ & 1.3535 & 0 & -1 \\ & & 1 & 0 \\ \text{Symmetry} & & & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad \text{(a)}$$



Similarly assembling $[k^{(6)}]$ through $[k^{(10)}]$ we obtain

$$\begin{array}{r} \\ [K_{ee}] = \\ \\ [K] = \\ \\ [K_{ie}] = \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccccc} u_3 & v_3 & u_4 & v_4 & u_5 & v_5 & u_6 & v_6 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3535 & 0.3535 & -0.3535 & -0.3535 & -1 & 0 \\ 0 & -1 & 0.3535 & 1.3535 & -0.3535 & -0.3535 & 0 & 0 \\ \hline -1 & 0 & -0.3535 & -0.3535 & 1.3535 & 0.3535 & 0 & 0 \\ 0 & 0 & -0.3535 & -0.3535 & 0.3535 & 1.3535 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ = [K_{ei}] \\ \\ \times 2 \times 10^9 \\ \\ = [K_{ii}] \\ \\ \\ \end{array}$$

Now $[K_{ii}] - [K_{ie}]\,[K_{ee}^{-1}]\,[K_{ei}]\,\{u_i\} = \{f_i\} - [K_{ie}]\,[K_{ee}^{-1}]\,\{f_e\}$

$$\left\{ \begin{bmatrix} 1.3535 & 0.3535 & 0 & 0 \\ 0.3535 & 1.3535 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1.3535 & 0.3535 \\ 0 & -1 & 0.3535 & 1.3535 \end{bmatrix}^{-1} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.3535 & -0.3535 & -1 & 0 \\ -0.3535 & -0.3535 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} -1 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -0.3535 & -0.3535 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1.3535 & 0.3535 \\ 0 & -1 & 0.3535 & 1.3535 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -20000 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore \quad 2\times10^{9}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_3\\ v_3\\ u_4\\ v_4\end{Bmatrix}=\begin{Bmatrix}-20000\\ -20000\\ 20000\\ 0\end{Bmatrix} \qquad \text{(b)}$$

Adding two sections (a) and (b)

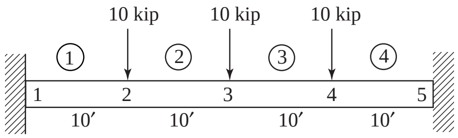
$$2\times10^{9}\begin{bmatrix}1.3535 & 0.3535 & 0 & 0\\ 0.3535 & 2.3535 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\end{bmatrix}\begin{Bmatrix}u_3\\ v_3\\ u_4\\ v_4\end{Bmatrix}=\begin{Bmatrix}-20000\\ -20000\\ 20000\\ 0\end{Bmatrix}$$

Solving

$$u_3 = 2.832\times10^{-11}\,\text{m},\ v_3 = -2.828\times10^{-5}\,\text{m}$$

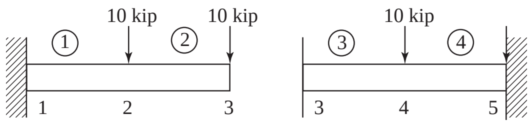
$$u_4 = 1.0\times10^{-5}\,\text{m},\ v_4 = -2.828\times10^{-5}\,\text{m}$$

5.17



Substructure 1 Substructure 2

Substructure 1 Substructure 2



$[k]$'s for each element are

$$[k^{(1)}]=[k^{(2)}]=[k^{(3)}]=[k^{(4)}]=\frac{29\times10^{3}\times10^{3}}{(120)^{3}}\begin{array}{c}\begin{array}{cccc} & (1) & & (2)\end{array}\\ \begin{bmatrix}12 & 720 & -12 & 720\\ 720 & 57600 & -720 & -28800\\ & & 12 & -720\\ \text{Symmetry} & & & 57600\end{bmatrix}\end{array}$$

Adding $[k]$'s of elements (1) and (2) for substructure (1) and apply boundary conditions

$$v_1 = \phi_1 = 0$$

$$16.78\begin{bmatrix}24 & 0 & -12 & 720\\ & 115200 & -720 & 28800\\ & & 12 & -720\\ \text{Symmetry} & & & 57600\end{bmatrix}\begin{Bmatrix}v_2\\ \phi_2\\ v_3\\ \phi_3\end{Bmatrix}=\begin{Bmatrix}-10\\ 0\\ -10\\ 0\end{Bmatrix}$$

Now rearrange the equations with interface displacement first

$$16.78\begin{bmatrix} 12 & -720 & -12 & -720 \\ -720 & 57600 & 720 & 28800 \\ -12 & 720 & 24 & 0 \\ -720 & 28800 & 0 & 115200 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

Using Equation (6.6.6) $[K_{ii}] - [K_{ie}]\ [K_{ee}^{-1}]\ [K_{ei}]\ \{u_i\} = \{f_i\} - [K_{ie}]\ [K_{ee}^{-1}]\ \{f_e\}$

$$\left\{16.78\begin{bmatrix} 12 & -720 \\ -720 & 57600 \end{bmatrix} - \begin{bmatrix} -12 & -720 \\ 720 & 28800 \end{bmatrix}\begin{bmatrix} 24 & 0 \\ 0 & 115200 \end{bmatrix}^{-1}\begin{bmatrix} -12 & 720 \\ -720 & 28800 \end{bmatrix}\right\}$$

$$\times \begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 0 \end{Bmatrix} - \begin{bmatrix} -12 & -720 \\ 720 & 28800 \end{bmatrix}\begin{bmatrix} -24 & 0 \\ 0 & 115200 \end{bmatrix}^{-1}\begin{Bmatrix} -10 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 25.17 & -3020 \\ -3020 & 483260 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -15 \\ 300 \end{Bmatrix} \tag{1}$$

Considering substructure (2) with boundary conditions $v_5 = \phi_5 = 0$

$$16.78\begin{bmatrix} 12 & 720 & -12 & 720 \\ & 57600 & -720 & 28800 \\ & & 24 & 0 \\ \text{Symmetry} & & & 115200 \end{bmatrix}\begin{Bmatrix} u_3 \\ \phi_3 \\ v_4 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -10 \\ 0 \end{Bmatrix}$$

Simplifying as per substructure (1)

$$\begin{bmatrix} 25.17 & 3020 \\ 3020 & 483264 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -5 \\ -300 \end{Bmatrix} \tag{2}$$

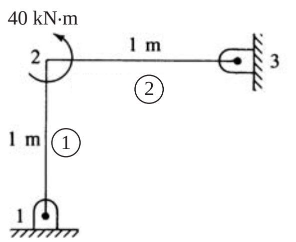
Adding (1) and (2)

$$\begin{bmatrix} 50.34 & 0 \\ 0 & 966528 \end{bmatrix}\begin{Bmatrix} v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \end{Bmatrix}$$

Solving

$$v_3 = -0.3973 \text{ in.} \quad \phi_3 = 0$$

5.18



$E = 200$ GPa
$I = 2 \times 10^{-4}$ m^4
$A = 1 \times 10^{-2}$ m^2

$$\frac{12\,I}{L^2} = \frac{12 \times 2 \times 10^{-4}}{1^2} = 2.4 \times 10^{-3}$$

$$\frac{6\,I}{L} = \frac{6 \times 2 \times 10^{-4}}{1} = 1.2 \times 10^{-3}$$

$$\frac{E}{L} = 200 \times 10^9 \ \frac{\text{N}}{\text{m}^3}$$

Considering substructure (1) and applying boundary conditions

$$u_1 = v_1 = 0$$

$$C = 0, S = 1$$

$$\begin{array}{cc} & \begin{array}{cc}[k^{(1)}] & \qquad\qquad \{d\}\end{array} \\ = 200 \times 10^9 & \begin{bmatrix} 0.0008 & 0.0012 & 0 & 0.0004 \\ & 0.0024 & 0 & 0.0012 \\ & & 0.01 & 0 \\ \text{Symmetry} & & & 0.0008 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ u_2 \\ v_2 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 40 \times 10^3 \end{Bmatrix} \end{array}$$

Rearranging equations (rows and columns) we place interface displacements first

$$200 \times 10^9 \left[\begin{array}{ccc:c} 0.0024 & 0 & 0.0012 & 0.0012 \\ & 0.01 & 0 & 0 \\ & & 0.0008 & 0.0004 \\ \hdashline \text{Symmetry} & & & 0.0008 \end{array}\right] \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 40 \times 10^3 \\ 0 \end{Bmatrix}$$

Using Equation (6.6.6)

$$200 \times 10^9 \left\{ \begin{bmatrix} 0.0024 & 0 & 0.0012 \\ & 0.01 & 0 \\ \text{Symmetry} & & 0.0008 \end{bmatrix} - \begin{Bmatrix} 0.0012 \\ 0 \\ 0.0004 \end{Bmatrix} [1250][0.0012 \quad 0 \quad 0.0004] \right\} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 20 \times 10^3 \end{Bmatrix} - \begin{Bmatrix} 0.0012 \\ 0 \\ 0.0004 \end{Bmatrix} [1250]\ [0]$$

Simplifying

$$1 \times 10^9 \begin{bmatrix} 0.12 & 0 & 0.12 \\ & 2.0 & 0 \\ & & 0.12 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 40 \times 10^3 \end{Bmatrix} \qquad (1)$$

Considering substructure (2) and applying boundary conditions

$$C = 1, S = 0$$

$$200 \times 10^9 \left[\begin{array}{ccc:c} 0.01 & 0 & 0 & -0.01 \\ 0 & 0.0024 & 0.0012 & 0 \\ 0 & 0.0012 & 0.0008 & 0 \\ \hdashline -0.01 & 0 & 0 & 0.01 \end{array}\right] \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying by applying Equation (6.6.6)

$$200 \times 10^9 \left\{ \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.0024 & 0.0012 \\ & & 0.0008 \end{bmatrix} - \begin{Bmatrix} -0.01 \\ 0 \\ 0 \end{Bmatrix} \left[\frac{1}{0.01} \right] [-0.01 \ 0 \ 0] \right\} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$= \{0\} - \{0\}$$

$$\therefore \quad 1 \times 10^9 \begin{bmatrix} 2.0 & 0 & 0 \\ & 0.12 & 0.12 \\ \text{Symmetry} & & 0.12 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2)$$

Adding (1) and (2)

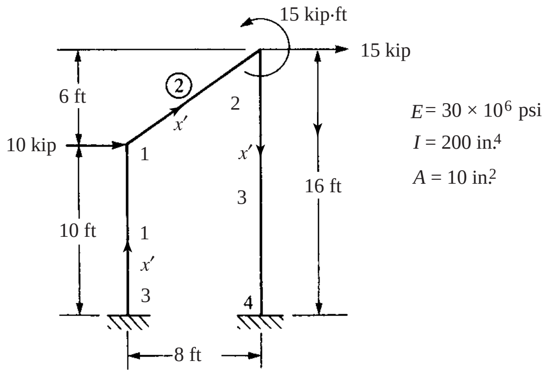
$$1 \times 10^9 \begin{bmatrix} 2.12 & 0 & 0.12 \\ & 2.12 & 0.12 \\ \text{Symmetry} & & 0.24 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 40 \times 10^3 \end{Bmatrix}$$

Solving

$$u_2 = v_2 = -0.010 \times 10^{-3} \text{ m}$$

$$\phi_2 = 17.67 \times 10^{-5} \text{ rad}$$

5.19



NUMBER OF ELEMENTS = 3
NUMBER OF NODES = 4
NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)	FORCE(2, K)	FORCE (3, K)
1	0 0 0	0.000000	120.000000	0.000000	10000.000000	0.000000	0.000000
2	0 0 0	96.000000	192.000000	0.000000	15000.000000	0.000000	180000.000000
3	1 1 1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	1 1 1	96.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

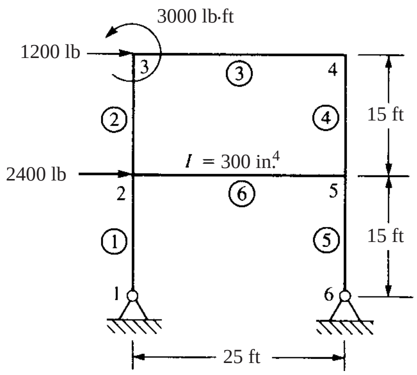
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	3	1	3.0000000E+07	1.0000000E+00	1. 0000000E+01	2.0000000E+02
2	1	2	3.0000000E+07	1.0000000E+00	1.0000000E+01	2.0000000E+02
3	2	4	3.0000000E+07	1.0000000E+00	1.0000000E+01	2.0000000E+02

DISPLACEMENT		Z-ROTATION
X	Y	THETA
0.70180E+00	0.79708E–02	– 0.44578E–02
0.72656E+00	– 0.12753E–01	– 0.49949E–03
0.00000E+00	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	3	1	– 0.1993E+05	0.1810E+05	0.1309E+07	0.1993E+05	– 0.1810E+05	0.8629E+06
2	1	2	– 0.1843E+05	– 0.1108E+05	– 0.8629E+06	0.1843E+05	0.1108E+05	– 0.4671E+06
3	2	4	0.1993E+05	0.6903E+04	0.6471E+06	– 0.1993E+05	– 0.6903E+04	0.6783E+06

5.20



$E = 30 \times 10^6$ psi
$I_{1\text{-}5} = 200$ in.^{4}
$A = 10$ in.^{2}
$I_6 = 300$ in.^{4}

PLANE FRAME PROBLEM 5.20
NUMBER OF ELEMENTS = 6
NUMBER OF NODES = 6

NODE POINTS

K	IFIX			XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1	1	0	0.000000	0.000000	0.000000	0.000000
2	0	0	0	0.000000	180.000000	0.000000	2400.000000
3	0	0	0	0.000000	360.000000	0.000000	1200.000000
4	0	0	0	300.000000	360.000000	0.000000	0.000000
5	0	0	0	300.000000	180.000000	0.000000	0.000000
6	1	1	0	300.000000	0.000000	0.000000	0.000000

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
0.000000	0.000000
0.000000	36000.000000
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
2	2	3	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
3	3	4	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
4	4	5	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
5	5	6	2.9000000E+07	0.0000000E+00	1.0000000E+01	2.0000000E+02
6	2	5	2.9000000E+07	0.0000000E+00	1.0000000E+01	3.0000000E+02

DISPLACEMENTS X	Y	Z-ROTATION THETA
0.00000E+00	0.00000E+00	– 0.69680E–02
0.95468E+00	0.17130E–02	– 0.19754E–02
0.12408E+01	0.20315E–02	– 0.55648E–03
0.12403E+01	0.20315E–02	– 0.79750E–03
0.95333E+00	– 0.17130E–02	– 0.19213E–02
0.00000E+00	0.00000E+00	– 0.69838E–02

ELEMENTS

K	NODE(I, K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE
1	1	2	– 0.2760E+04	0.1787E+04	0.7438E–01	0.2760E+04	– 0.1787E+04
2	2	3	– 0.5131E+03	0.6953E+03	0.1686E+05	0.5131E+03	– 0.6953E+03
3	3	4	0.5045E+03	– 0.5131E+03	– 0.7230E+05	– 0.5045E+03	0.5131E+03
4	4	5	0.5131E+03	0.5046E+03	0.8162E+05	– 0.5131E+03	– 0.5046E+03
5	5	6	0.2760E+04	0.1813E+04	0.3263E+06	– 0.2760E+04	– 0.1813E+04
6	2	5	0.1308E+04	– 0.2247E+04	– 0.3386E+06	– 0.1308E+04	0.2247E+04

		Z-MOMENT
1	2	0.3217E+06
2	3	0.1083E+06
3	4	– 0.8162E+05
4	5	0.9200E+04
5	6	0.3091E–01
2	5	– 0.3355E+06

5.21 For the slant-legged rigid frame shown in Figure P5–21, size the structure for minimum weight based on a maximum bending stress of 20 ksi in the horizontal beam elements and a maximum compressive stress (due to bending and direct axial load) of 15 ksi in the slant-legged elements. Use the same element size for the two slant-legged elements and the same element size for the two 10-foot sections of the horizontal element. Assume A36 steel is used.

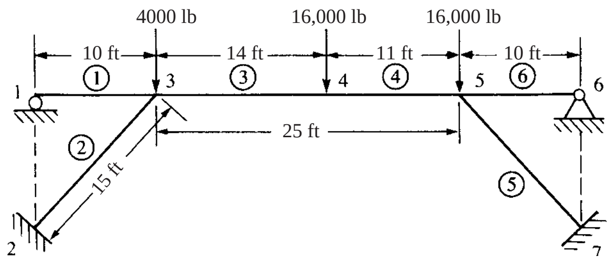


Figure P5–21

- I chose to use an 'I' beam for my cross sectional area because it is commonly used in bridges. My final design was chosen because the design meets the constraints of the problem while not being overly large.
- The center part of the cross member (25 foot section) was taken to be W14 $\times$ 26. This has a thickness of 0.420 inches, a depth of 13.91 inches and a width of 5.025 inches.
- The two angle members and the outside 10 foot members are also designed as 'I' beams. These members were taken to be W14 $\times$ 22. This size has a thickness of 0.335 inches, a depth of 13.74 inches and a width of 5.00 inches.
- The maximum bending stress (about the local axis 3) in this member is 19.51891 ksi. This is under to 20 ksi constraint put on the design in the problem. It is near the center of the cross member.
- The maximum stress in the angle member is – 11.51989 ksi and is well below the 15 ksi allowed in the problem.
- The sizes were determined by taking some commonly used sizes from my Mechanics of Materials book and using trial and error. When I got the lower members too small the bending stress went too high in the cross member. The final design was chosen because it minimized the size of the cross section while also minimizing the size of the angle members.

5.22 For the rigid building frame shown in Figure P5–22, determine the forces in each element and calculate the bending stresses. Assume all the vertical elements have $A = 10$ in.^{2} and $I = 100$ in^4 and all horizontal elements have $A = 15$ in.^{2} and $I = 150$ in.^{4}. Let $E = 29 \times 10^6$ psi for all elements. Let $c = 5$ in. for the vertical elements and $c = 6$ in. for the horizontal elements, where c denotes the distance from the neutral axis to the top or bottom of the beam cross section, as used in the bending stress formula $\sigma = \left(\frac{M_c}{I}\right)$.

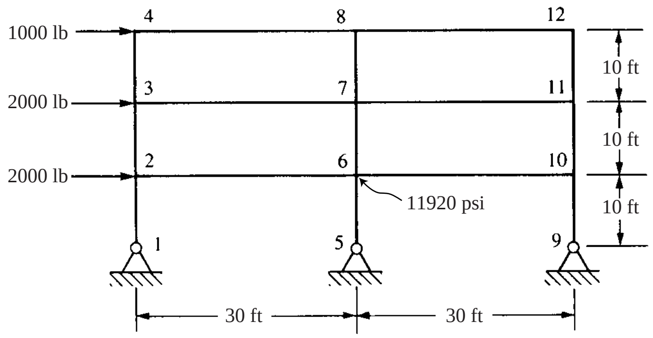
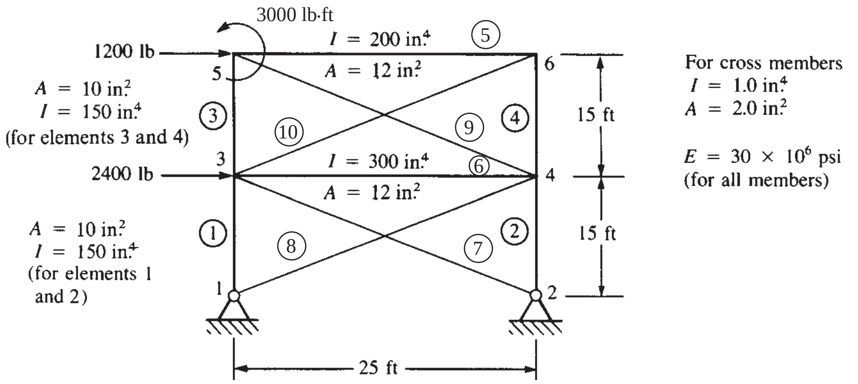


Figure P5–22

1**** BEAM ELEMENT STRESSES

ELEMENT NO.	CASE (MODE)	P/A	P/A+M2/S2	P/A–M2/S2	P/A+M3/S3	P/A–M3/S3	WORST SUM
1	1	1.500E+02	1.500E+02	1.500E+02	1.500E+02	1.500E+02	1.500E+02
		1.500E+02	1.500E+02	1.500E+02	4.676E+03	–4.376E+03	4.676E+03
2	1	1.500E+02	1.500E+02	1.500E+02	4.676E+03	–4.376E+03	4.676E+03
		1.500E+02	1.500E+02	1.500E+02	9.203E+03	–8.903E+03	9.203E+03
3	1	5.591E+01	5.591E+01	5.591E+01	1.605E+02	–4.866E+01	1.605E+02
		5.591E+01	5.591E+01	5.591E+01	2.054E+03	–1.942E+03	2.0543+03
4	1	5.591E+01	5.591E+01	5.591E+01	2.054E+03	–1.942E+03	2.054E+03
		5.591E+01	5.591E+01	5.591E+01	3.947E+03	–3.835E+03	3.947E+03
5	1	1.480E+01	1.480E+01	1.480E+01	1.329E+02	–1.033E+02	1.329E+02
		1.480E+01	1.480E+01	1.480E+01	7.855E+02	–7.559E+02	7.855E+02
6	1	1.480E+01	1.480E+01	1.480E+01	7.855E+02	–7.559E+02	7.855E+02
		1.480E+01	1.480E+01	1.480E+01	1.438E+03	–1.409E+03	1.438E+03
7	1	8.096E–02	8.096E–02	8.096E–02	8.096E–02	8.096E–02	8.096E–02
		8.096E–02	8.096E–02	8.096E–02	5.962E+03	–5.962E+03	5.962E+03
8	1	8.096E–02	8.096E–02	8.096E–02	5.962E+03	–5.962E+03	5.962E+03
		8.096E–02	8.096E–02	8.096E–02	1.192E+04	–1.192E+04	1.192E+04
9	1	1.920E–01	1.920E–01	1.920E–01	–4.031E+03	4.032E+03	4.032E+03
		1.920E–01	1.920E–01	1.920E–01	1.177E+03	–1.177E+03	1.177E+03
10	1	1.920E–01	1.920E–01	1.920E–01	1.177E+03	–1.177E+03	1.177E+03
		1.920E–01	1.920E–01	1.920E–01	6.386E+03	–6.386E+03	6.386E+03
11	1	1.098E–01	1.098E–01	1.098E–01	–8.724E+02	8.726E+02	8.726E+02
		1.098E–01	1.098E–01	1.098E–01	8.076E+02	–8.074E+02	8.076E+02
12	1	1.098E–01	1.098E–01	1.098E–01	8.076E+02	–8.074E+02	8.076E+02
		1.098E–01	1.098E–01	1.098E–01	2.488E+03	–2.487E+03	2.488E+03
13	1	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02	–1.500E+02
		–1.500E+02	–1.500E+02	–1.500E+02	4.361E+03	–4.661E+03	–4.661E+03
14	1	–1.500E+02	–1.500E+02	–1.500E+02	4.361E+03	–4.661E+03	–4.661E+03
		–1.500E+02	–1.500E+02	–1.500E+02	8.873E+03	–9.173E+03	–9.173E+03
15	1	–5.610E+01	–5.610E+01	–5.610E+01	3.192E+01	–1.441E+02	–1.441E+02
		–5.610E+01	–5.610E+01	–5.610E+01	1.930E+03	–2.042E+03	–2.042E+03
16	1	–5.610E+01	–5.610E+01	–5.610E+01	1.930E+03	–2.042E+03	–2.042E+03
		–5.610E+01	–5.610E+01	–5.610E+01	3.828E+03	–3.940E+03	–3.940E+03
17	1	–1.491E+01	–1.491E+01	–1.491E+01	8.670E+01	–1.165E+02	–1.165E+02
		–1.491E+01	–1.491E+01	–1.491E+01	7.542E+02	–7.840E+02	–7.840E+02
18	1	–1.491E+01	–1.491E+01	–1.491E+01	7.542E+02	–7.840E+02	–7.840E+02
		–1.491E+01	–1.491E+01	–1.491E+01	1.422E+03	–1.451E+03	–1.451E+03

5.23



Problem 5-23
NUMBER OF ELEMENTS = 10
NUMBER OF NODES = 6
NODE POINTS

K	IFII	XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000
2	1 1 0	300.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	180.000000	0.000000	2400.000000
4	0 0 0	300.000000	180.000000	0.000000	0.000000
5	0 0 0	0.000000	360.000000	0.000000	1200.000000
6	0 0 0	300.000000	360.000000	0.000000	0.000000

ELEMENTS

K	NODE (I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
2	2	4	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
3	3	5	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
4	4	6	3.0000000E+07	0.0000000E+00	1.0000000E+01	1.5000000E+02
5	5	6	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02
6	3	4	3.0000000E+07	0.0000000E+00	1.2000000E+01	3.0000000E+02
7	3	2	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
8	1	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
9	5	4	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00
10	3	6	3.0000000E+07	0.0000000E+00	2.0000000E+00	1.0000000E+00

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000
0.000000	0.000000
0.000000	36000.000000
0.000000	0.000000

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.90513E–04
2	0.00000E+00	0.00000E+00	– 0.11175E–03
3	0.15236E–01	0.10352E–02	– 0.72478E–04
4	0.14269E–01	– 0.99244E–03	– 0.13558E–04
5	0.20440E–01	0.12193E–02	0.20690E–03
6	0.20082E–01	– 0.11183E–02	– 0.74145E–04

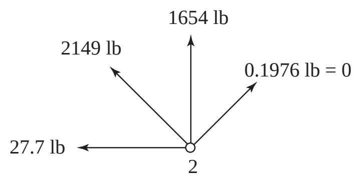
ELEMENTS

K	NODE(I,	K)	X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.1725E+04	0.5246E+01	0.2132E+02	0.1725E+04	– 0.5246E+01	0.9230E+03
2	2	4	0.1654E+04	0.2770E+02	0.3793E+02	0.1654E+04	– 0.2770E+02	0.4948E+04
3	3	5	– 0.3069E+03	0.1602E+03	0.7434E+04	0.3069E+03	– 0.1602E+03	0.2140E+05
4	4	6	0.2097E+03	– 0.1926E+02	– 0.2187E+03	– 0.2097E +03	0.1926E+02	– 0.3248E+04
5	5	6	0.4288E+03	0.5934E+02	0.1452E+05	– 0.4288E+03	– 0.5934E+02	0.3279E+04
6	3	4	0.1160E+04	– 0.4351E+02	– 0.8294E+04	– 0.1160E+04	0.4351E+02	– 0.4759E+04
7	3	2	0.2149E+04	– 0.1976E+00	– 0.3119E+02	– 0.2149E+04	0.1976E+00	– 0.3793E+02
8	1	4	– 0.2011E+04	– 0.8417E–01	– 0.2132E+02	0.2011E+04	0.8417E–01	– 0.8126E+01
10	3	6	– 0.5227E+03	0.1791E+00	– 0.3119E+02	0.5227+03	0.1791E+00	– 0.3148E+02

Reactions

Node 2

From x' and y' forces in elements (2) and (7)



$$F_{2x} = -27.7 - 1843 = 1871 \text{ lb} \leftarrow$$

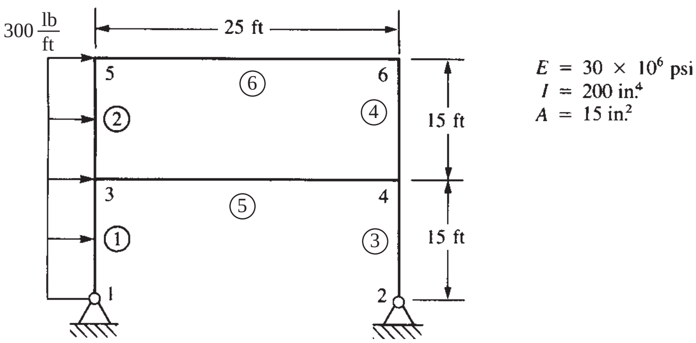
$$F_{2y} = 1654 + 1106 = 2760 \text{ lb} \uparrow$$

Similarly

$$F_{1x} = 1730 \text{ lb} \leftarrow$$

$$F_{1y} = 2760 \text{ lb} \downarrow$$

5.24



Problem 5.24

NUMBER OF ELEMENTS = 6

NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)	FORCE(2, K)	FORCE(3, K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	–67500.000000
2	1 1 0	300.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	180.000000	0.000000	4500.000000	0.000000	0.000000
4	0 0 0	300.000000	180.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	0.000000	360.000000	0.000000	2250.000000	0.000000	67500.000000
6	0 0 0	300.000000	360.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

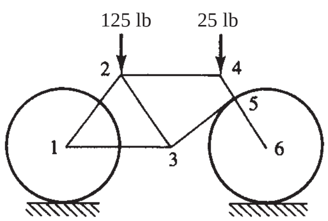
K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
2	3	5	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
3	2	4	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
4	4	6	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
5	3	4	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02
6	5	6	3.0000000E+07	0.0000000E+00	1.5000000E+01	2.0000000E+02

NODE	DISPLACEMENTS		Z-ROTATION
	X(in.)	Y(in.)	THETA
1	0.00000E+00	0.00000E+00	– 0.15591E–01
2	0.00000E+00	0.00000E+00	– 0.15054E–01
3	0.21212E+01	0.21601E–02	– 0.51828E–02
4	0.21193E+01	– 0.21601E–02	– 0.52135E–02
5	0.28221E+01	0.26614E–02	– 0.13916E–02
6	0.28215E+01	– 0.26614E–02	– 0.17770E–02

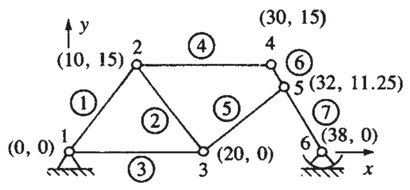
ELEMENTS

K	NODE (I, K)		X-FORCE (lb)	Y-FORCE (lb)	Z-MOMENT (lb · in.)	X-FORCE (lb)	Y-FORCE (lb)	Z-MOMENT (lb · in.)
1	1	3	– 0.5400E+04	0.3105E+04	– 0.6750E+05	0.5400E+04	– 0.3105E+04	0.6264E+06
2	3	5	– 0.1253E+04	0.1349E+04	– 0.4968E+04	0.1253E+04	– 0.1349E+04	0.2478E+06
3	2	4	0.5400E+04	0.3645E+04	0.8592E–01	– 0.5400E+04	– 0.3645E+04	0.6561E+06
4	4	6	0.1253E+04	0.9017E+03	– 0.3340E+05	– 0.1253E+04	– 0.9017E+03	0.1957E+06
5	3	4	0.2744E+04	– 0.4147E+04	– 0.6214E+06	– 0.2744E+04	0.4147E+04	– 0.6227E+06
6	5	6	0.9014E+03	– 0.1253E+04	– 0.1803E+06	– 0.9014E+03	0.1253E+04	– 0.1957E+06

5.25 (a)



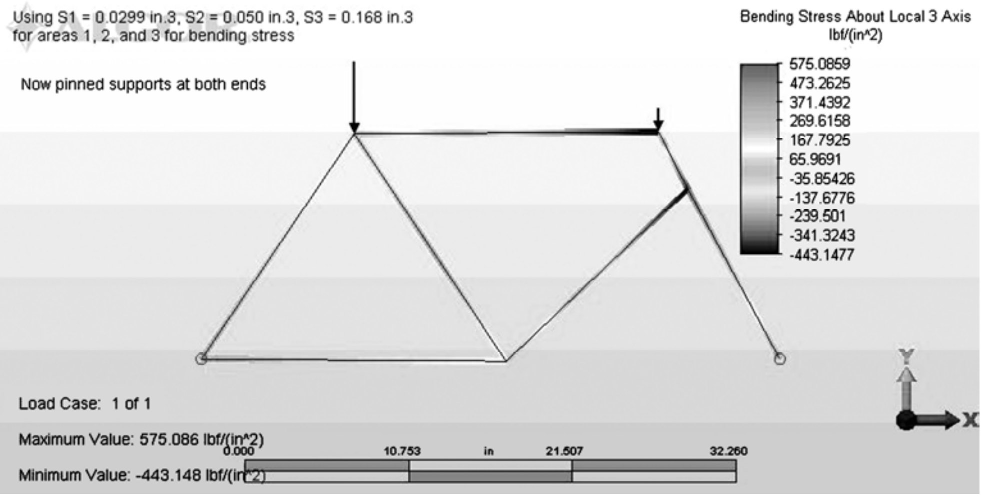
(a) Design 1

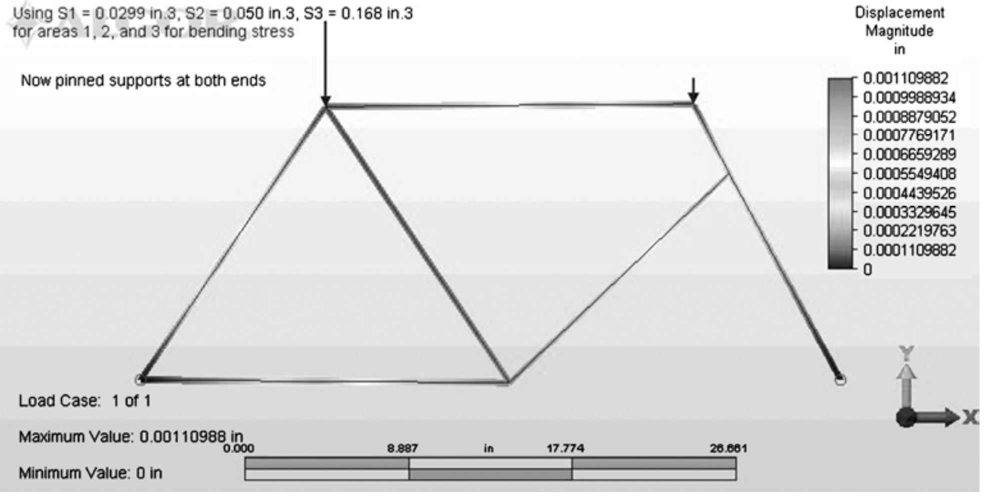


(a) Design 1

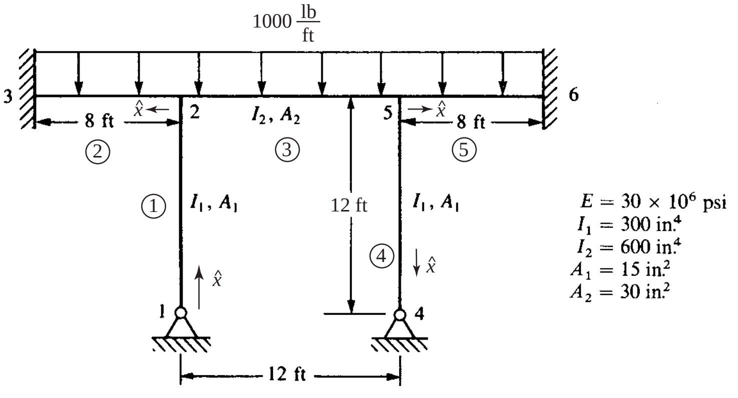
Case 1
$E = 30 \times 10^6$ psi
Case 2
$E = 10 \times 10^6$ psi

$A_1 = 0.1$ in.^{2}
$A_2 = A_3 = A_4 = A_5 = 0.15$ in.^{2}
$A_6 = A_7 = A_8 = 0.3$ in.^{2}
$I_1 = 0.01$ in.^{4}
$I_2 = I_3 = I_4 = I_5 = 0.02$ in.^{4}
$I_6 = I_7 = I_8 = 0.1$ in.^{4}

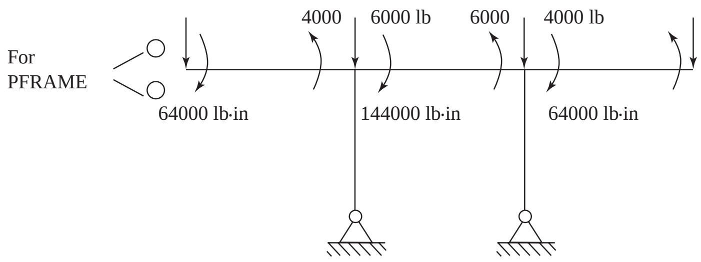




5.26



Solution: From appendix D for distributed load



NUMBER OF ELEMENTS = 5

NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)
1	1 1 0	–72.000000	0.000000	0.000000	0.000000
2	0 0 0	–72.000000	144.000000	0.000000	0.000000
3	1 1 1	–168.00000	144.000000	0.000000	0.000000
4	1 1 0	72.000000	0.000000	0.000000	0.000000
5	0 0 0	72.000000	144.000000	0.000000	0.000000
6	1 1 1	168.00000	144.000000	0.000000	0.000000

FORCE(2, K)	FORCE(3, K)
0.000000	0.000000
–10000.000000	–80000.000000
0.000000	0.000000
0.000000	0.000000
–10000.000000	80000.000000
0.0000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.50000000E+01	3.0000000E+02
2	2	3	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02
3	2	5	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02
4	5	4	3.0000000E+07	0.0000000E+00	1.50000000E+01	3.0000000E+02
5	5	6	3.0000000E+07	0.0000000E+00	3.00000000E+01	6.0000000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.49989E–04
2	0.59560E–05	– 0.33163E–02	– 0.10010E–03
3	0.00000E+00	0.00000E+00	0.00000E+00
4	0.00000E+00	0.00000E+00	– 0.49989E–04
5	– 0.59560E–05	– 0.33163E–02	0.10010E–03
6	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I, K)		X-FORCE	Y- FORCE	Z-MOMENT	X-FORCE	Y-FORCE
1	1	2	0.1036E+05	– 0.1303E+03	– 0.7533E-04	– 0.1036E+05	0.1303E+03
2	2	3	– 0.5584E+02	– 0.3634E+03	– 0.3621E+05	0.5584E+02	0.3634E+03
3	2	5	0.7445E+02	0.1703E–04	– 0.2503E+05	– 0.7445E+02	– 0.1703E–04
4	5	4	0.1036E+05	0.1303E+03	0.1876E+05	– 0.1036E+05	– 0.1303E+03
5	5	6	– 0.5584E+02	0.3634E+03	0.3621E+05	0.5584E+02	– 0.3634E+03

		K	Z-MOMENT
1	1	2	– 1876E+05
2	2	3	.1325E+04
3	2	5	.2503E+05

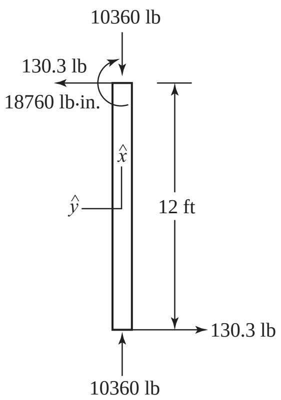
4	5	4	– 1137E+02
5	5	6	– 1325E+04

Refer to computer print out for displacement and rotations

Corrected elemental forces

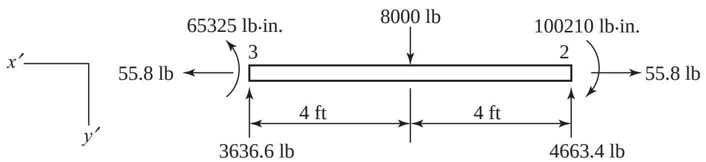
Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ m_1^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \\ m_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 10360\text{ lb} \\ -130.3\text{ lb} \\ 0\text{ lb}\cdot\text{in.} \\ -10360\text{ lb} \\ 130.3\text{ lb} \\ -18760\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



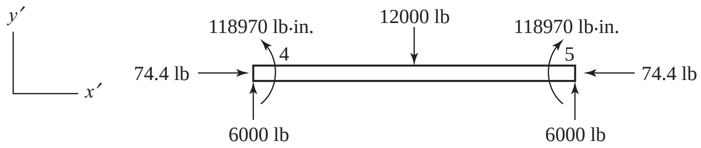
Element (2)

$$\begin{Bmatrix} f'_{2x} \\ f'_{2y} \\ m_2 \\ f'_{3x} \\ f'_{3y} \\ m_3 \end{Bmatrix} = \begin{Bmatrix} -55.8\text{ lb} \\ -363.4\text{ lb} \\ -36210\text{ lb}\cdot\text{in.} \\ 55.8\text{ lb} \\ 363.4\text{ lb} \\ 1325\text{ lb}\cdot\text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ +4000 \\ +64,000 \\ 0 \\ +4000 \\ -64,000 \end{Bmatrix} = \begin{Bmatrix} -55.8\text{ lb} \\ -4363.4\text{ lb} \\ -100,210\text{ lb}\cdot\text{in.} \\ 55.8\text{ lb} \\ -3636.6\text{ lb} \\ 65325\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



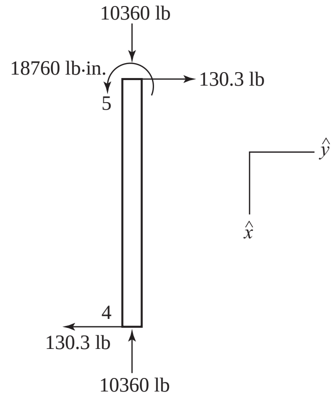
Element (3)

$$\begin{Bmatrix} f'^{(3)}_{2x} \\ f'^{(3)}_{2y} \\ m_2^{(3)} \\ f'^{(3)}_{5x} \\ f'^{(2)}_{5y} \\ m_5^{(2)} \end{Bmatrix} = \begin{Bmatrix} 74.4\text{ lb} \\ 0 \\ -25030\text{ lb} \\ -74.4\text{ lb} \\ 0 \\ 25030\text{ lb}\cdot\text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -6000 \\ -144,000 \\ 0 \\ -6000 \\ 144,000 \end{Bmatrix} = \begin{Bmatrix} 74.4\text{ lb} \\ 6000\text{ lb} \\ -118870\text{ lb}\cdot\text{in.} \\ -74.4\text{ lb} \\ 6000\text{ lb} \\ -118870\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



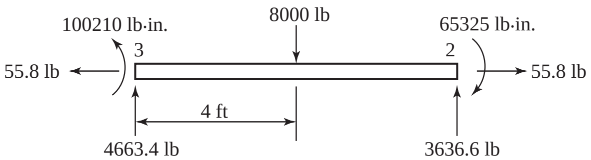
Element (4) No correction refer to computer print out

Element (4)



Element (5)

$$\begin{Bmatrix} f'^{(5)}_{5x} \\ f'^{(5)}_{5y} \\ m^{(5)}_5 \\ f'^{(5)}_{6x} \\ f'^{(5)}_{6y} \\ m^{(5)}_6 \end{Bmatrix} = \begin{Bmatrix} -55.81\text{ lb} \\ -363.4\text{ lb} \\ 36210\text{ lb}\cdot\text{in.} \\ 55.8\text{ lb} \\ -363.4\text{ lb} \\ -1325\text{ lb}\cdot\text{in.} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -4000 \\ -64{,}000 \\ 0 \\ -4000 \\ 64{,}000 \end{Bmatrix} = \begin{Bmatrix} -55.81\text{ lb} \\ 4363.4\text{ lb} \\ 100{,}210\text{ lb}\cdot\text{in.} \\ 55.8\text{ lb} \\ 3636.6\text{ lb} \\ -65325\text{ lb}\cdot\text{in.} \end{Bmatrix}$$



Reactions

NODE 1

$$F_{1x} = f'^{(1)}_{1y} = +\,130.3\text{ lb}$$

$$F_{1y} = f'^{(1)}_{1x} = +\,10360\text{ lb}$$

$$M_1 = -\,m_1^{(1)} = 0\text{ lb}\cdot\text{in.}$$

NODE 3

$$F_{3x} = f'^{(2)}_{3x} = 55.8\text{ lb}$$

$$F_{3y} = f'^{(2)}_{3y} = -\,3636.6\text{ lb}$$

$$M_3 = -m_3^{(2)} = -\,65235\text{ lb}\cdot\text{in.}$$

NODE 4

$$F_{4x} = -\ f'^{(4)}_{4y} = 130.3 \text{ lb}$$

$$F_{4y} = \quad f'^{(4)}_{4x} = -\ 10360 \text{ lb}$$

$$M_4 = -\ m_4^{(4)} = 0 \text{ lb}\cdot\text{in.}$$

NODE 6

$$F_{6x} = -\ f'^{(5)}_{6x} = -\ 55.8 \text{ lb}$$

$$F_{6y} = -\ f'^{(5)}_{6y} = -\ 3636.6 \text{ lb}$$

$$M_6 = -\ m_6^{(5)} = \quad 65325 \text{ lb}\cdot\text{in.}$$

5.28

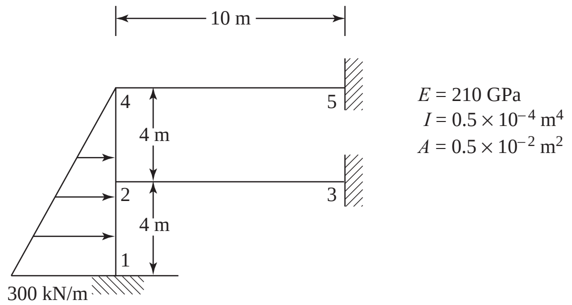
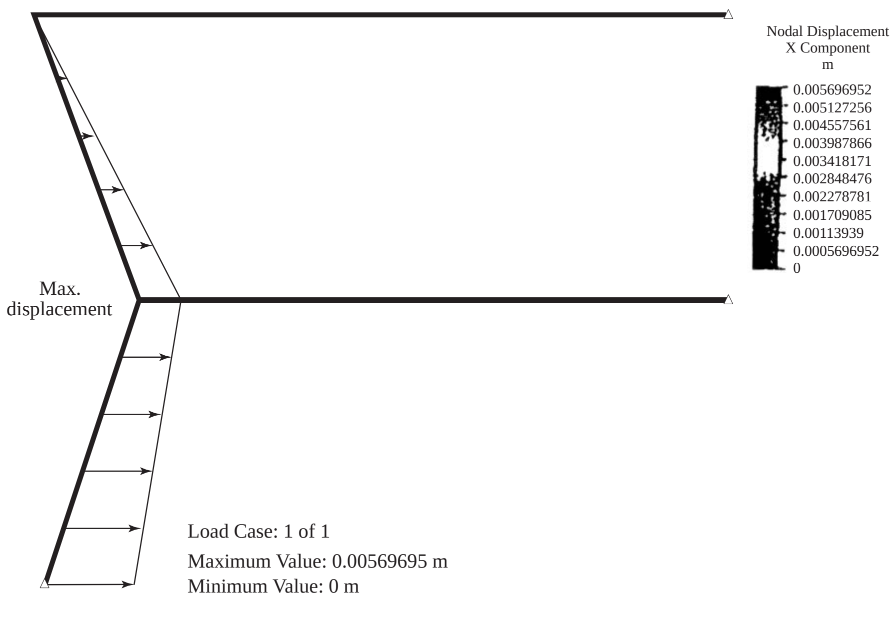


Figure P5-28



5.29

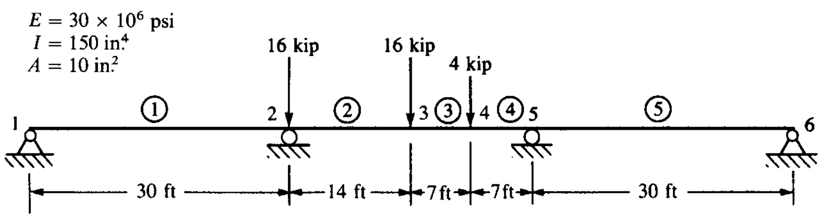
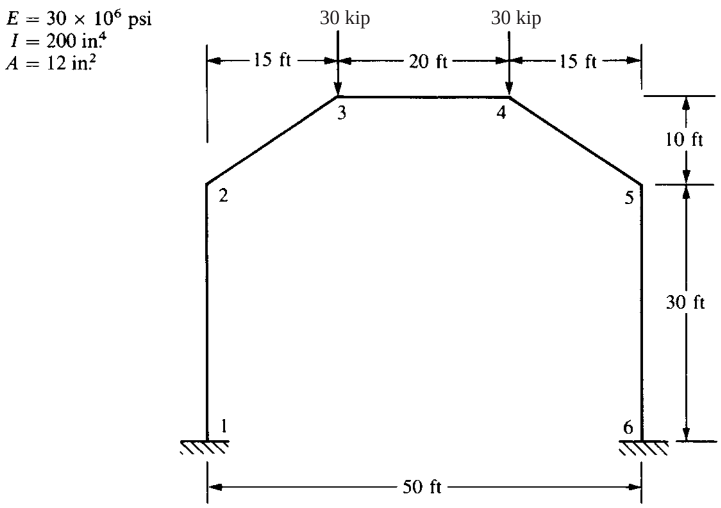


Figure P5–29

Displacements/Rotations (degrees) of nodes

Node number	X–translation	Y–translation	Z–translation	X–rotation	Y–rotation	Z–rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	3.4030E–01
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–6.8060E–01
3	0.0000E+00	–1.8330E+00	0.0000E+00	0.0000E+00	0.0000E+00	–3.7774E–02
4	0.0000E+00	–1.2242E+00	0.0000E+00	0.0000E+00	0.0000E+00	7.6168E–01
5	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	7.4186E–01
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–3.7093E–01
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.30



Plane Frame-Problem 5.30

NUMBER OF ELEMENTS = 5
NUMBER OF NODES = 6

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1, K)	FORCE(2, K)	FORCE(3, K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	360.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	180.000000	480.000000	0.000000	0.000000	–30000.000000	0.000000
4	0 0 0	420.000000	480.000000	0.000000	0.000000	–30000.000000	0.000000
5	0 0 0	600.000000	360.000000	0.000000	0.000000	0.000000	0.000000
6	1 1 1	600.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Elements

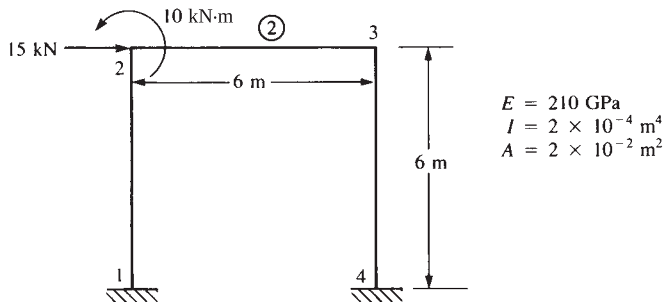
K	NODE (I, K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
2	2	3	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
3	3	4	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
4	4	5	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00
5	5	6	3.0000000E+07	0.0000000E+00	1.2000000E+01	2.0000000E+02	0.0000000E+00

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+0
2	– 0.44021E+01	– 0.30000E–01	– 0.17622E–0
3	0.33879E–02	– 0.66668E+01	– 0.32000E–0
4	– 0.44038E–02	– 0.66669E+01	0.32000E–0
5	0.44011E+01	– 0.30000E–01	0.17624E–0
6	0.00000E+00	0.00000E+00	0.00000E+0

ELEMENTS

K	NODE (I,K)		X–FORCE	Y–FORCE	Z– MOMENT	X–FORCE	Y–FORCE	Z–MOMENT
1	1	2	0.3000E+05	– 0.1169E+05	– 0.1810E+07	– 0.3000E+05	0.1169E+05	– 0.2398E+07
2	2	3	0.2637E+05	0.1848E+05	0.2398E+07	– 0.2637E+05	– 0.1848E+05	0.1600E+07
3	3	4	0.1169E+05	0.1738E+00	– 0.1600E+07	– 0.1169E+05	– 0.1738E+00	0.1600E+07
4	4	5	0.2637E+05	– 0.1848E+05	– 0.1600E+07	– 0.2637E+05	0.1848E+05	– 0.2397E+07
5	5	6	0.3000E+05	0.1169E+05	0.2397E+07	– 0.3000E+05	– 0.1169E+05	0.1810E+07

5.32



Problem 5.32
NUMBER OF ELEMENTS = 3
NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	0.000000	6.000000	0.000000	15000.000000	0.000000	10000.000000
3	0 0 0	6.000000	6.000000	0.000000	0.000000	0.000000	0.000000
4	1 1 1	6.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04
2	2	3	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04
3	3	4	2.1000000E+11	1.0000000E+00	2.0000000E-02	2.0000000E-04

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.42966E–02	0.71361E–05	– 0.24093E–03
3	0.42871E–02	– 0.71361E–05	– 0.47744E–03
4	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

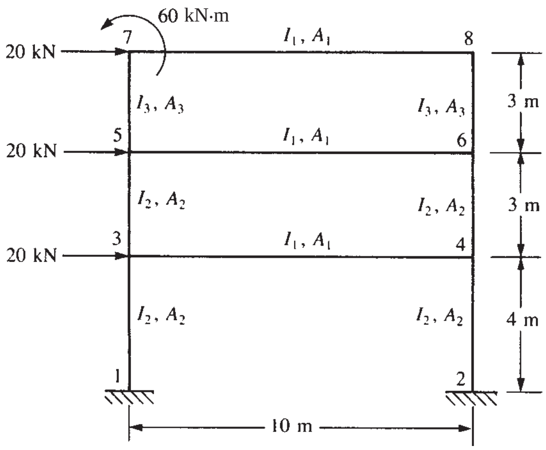
K	NODE (I,K)		X-FORCE	Y-FORCE	Z- MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	– 0.4995E+04	0.8339E+04	0.2620E+05	0.4995E+04	– 0.8339E+04	0.2333E+05
2	2	3	0.6661E+04	– 0.4995E+04	– 0.1333E+05	– 0.6661E+04	0.4995E+04	– 0.1664E+05
3	3	4	0.4995E+04	0.6661E+04	0.1664E+05	– 0.4999E+04	– 0.6661E+04	0.2333E+05

Reactions

$$F_{1x} = -8339 \text{ N},\ F_{1y} = -4995 \text{ N},\ M_1 = 26{,}700 \text{ N}\cdot\text{m}$$

$$F_{4x} = -6661 \text{ N},\ F_{4y} = 4995 \text{ N},\ M_4 = 23{,}330 \text{ N}\cdot\text{m}$$

5.33



E = 210 GPa
$I_1 = 2 \times 10^{-4}$ m^4
$A_1 = 2 \times 10^{-2}$ m^2
$I_2 = 1 \times 10^{-4}$ m^4
$A_2 = 1 \times 10^{-2}$ m^2
$I_3 = 0.5 \times 10^{-4}$ m^4
$A_3 = 0.5 \times 10^{-2}$ m^2

***** Frame Problem 5.33 ******
NUMBER OF ELEMENTS = 9
NUMBER OF NODES = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	–5.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	1 1 1	5.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	–5.000000	4.000000	0.000000	20.000000	0.000000	0.000000
4	0 0 0	5.000000	4.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	–5.000000	7.000000	0.000000	20.000000	0.000000	0.000000
6	0 0 0	5.000000	7.000000	0.000000	0.000000	0.000000	0.000000
7	0 0 0	–5.000000	10.000000	0.000000	20.000000	0.000000	60.000000
8	0 0 0	5.000000	10.000000	0.000000	0.0000000	0.000000	0.000000

ELEMENTS

K	NODE (I,K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	3	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
2	2	4	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
3	3	4	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00
4	3	5	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00

5	4	4	2.1000000E+06	0.0000000E+00	9.9999998E–03	9. 9999997E–03	0.0000000E+00
6	5	6	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00
7	5	7	2.1000000E+06	0.0000000E+00	4.9999999E–03	4. 9999999E–03	0.0000000E+00
8	6	8	2.1000000E+06	0.0000000E+00	4.9999999E–03	4. 9999999E–03	0.0000000E+00
9	7	8	2.1000000E+06	0.0000000E+00	2.0000000E–02	1. 9999999E–04	0.0000000E+00

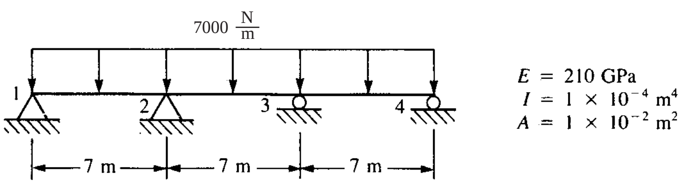
NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.00000E+00	0.00000E+00	0.00000E+00
3	0.13109E–01	0.40236E–04	–0.26944E–02
4	0.13092E–01	–0.40236E–04	–0.27869E–02
5	0.22043E–01	0.50737E–04	–0.19861E–02
6	0.21994E–01	–0.50737E–04	–0.15658E–02
7	0.26380E–01	0.46314E–04	0.17098E–02
8	0.26376E–01	–0.46314E–04	–0.11140E–02

ELEMENTS

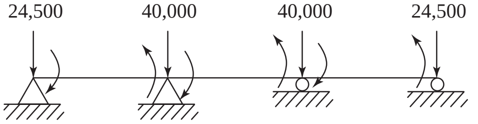
K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	–0.2112E+02	0.3040E+02	0.7493E+02
2	2	4	0.2112E+02	0.2960E+02	0.7383E+02
3	3	4	0.7451E+01	–0.1377E+02	–0.4847E+02
4	3	5	–0.7351E+01	0.1785E+02	0.2182E+02
5	4	6	0.7351E+01	0.2215E+02	0.2468E+02
6	5	6	0.2046E+02	–0.8899E+01	–0.4626E+02
7	5	7	0.1548E+01	0.1831E+02	0.1453E+02
8	6	8	–0.1548E+01	0.1693E+01	0.9578E+00
9	7	8	0.1691E+01	0.1548E+01	0.1960E+02

X-FORCE	Y-FORCE	Z-MOMENT
0.2112E+02	0.3040E+02	0.4665E+02
–0.2112E+02	–0.2960E+02	0.4457E+02
–0.7451E+01	0.1377E+02	–0.6925E+02
0.7351E+01	–0.1785E+02	0.3173E+02
–0.7351E+01	–0.2215E+02	0.4177E+02
–0.2046E+02	0.8899E+01	–0.4273E+02
–0.1548E+01	–0.1831E+02	0.4040E+02
0.1548E+01	–0.1693E+01	0.4120E+01
–0.1691E+01	–0.1548E+01	–0.4120E+01

5.34



Solution: From appendix D for distributed load



NUMBER OF ELEMENTS = 3
NUMBER OF NODES = 4

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	28583.000000
2	1 1 0	7.000000	0.000000	0.000000	0.000000	0.000000	0.000000

3	0 1 0	14.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0 1 0	21.000000	0.000000	0.000000	0.000000	0.000000	–28583.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04
2	2	3	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04
3	3	4	2.1000000E+11	1.0000000E+00	1.0000000E–02	1.0000000E–04

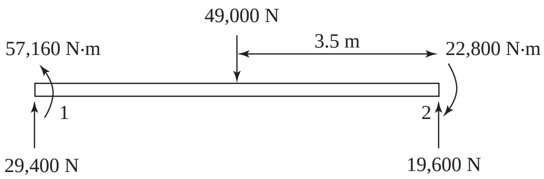
NODE	DISPLACEMENTS		Z-ROTATION
	X	Y	THETA
1	0.00000E+00	0.00000E+00	0.28583E–02
2	0.00000E+00	0.00000E+00	– 0.95277E–03
3	0.00000E+00	0.00000E+00	0.95277E–03
4	0.00000E+00	0.00000E+00	– 0.28583E–02

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.4900E+04	0.2858E+05	0.0000E+00	– 0.4900E+04	0.5717E+04
2	2	3	0.0000E+00	0.1201E–03	–0.5717E+04	0.0000E+00	– 0.1201E–03	0.5717E+04
3	3	4	0.0000E+00	– 0.4900E+04	–0.5717E+04	0.0000E+00	0.4900E+04	0.2858E+05

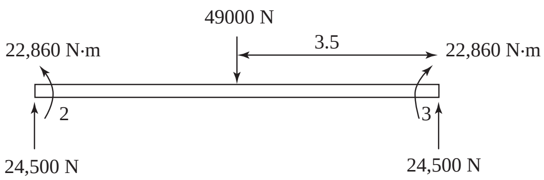
Element (1)

$$\begin{Bmatrix} f'^{(1)}_{1x} \\ f'^{(1)}_{1y} \\ m_1 \\ f'_{2x} \\ f'_{2y} \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4900 \\ 28580 \\ 0 \\ -4900 \\ 5717 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 29400\text{ N} \\ 57160\text{ N}\cdot\text{m} \\ 0\text{ N} \\ 19600\text{ N} \\ -22860\text{ N}\cdot\text{m} \end{Bmatrix}$$



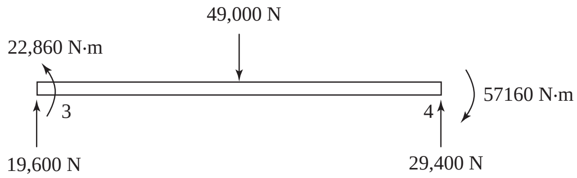
Element (2)

$$\begin{Bmatrix} f'^{(2)}_{2x} \\ f'^{(2)}_{2y} \\ m^{(2)}_2 \\ f'^{(2)}_{3x} \\ f'^{(2)}_{3y} \\ m^{(2)}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -5717 \\ 0 \\ 0 \\ 5717 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 24500\text{ N} \\ 22{,}860\text{ N}\cdot\text{m} \\ 0 \\ 24500\text{ N} \\ -22860\text{ N}\cdot\text{m} \end{Bmatrix}$$



Element (3)

$$\begin{Bmatrix} f'^{(3)}_{3x} \\ f'^{(3)}_{3y} \\ m^{(3)}_{3} \\ f'^{(3)}_{4x} \\ f'^{(3)}_{4y} \\ m^{(3)}_{4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4900 \\ -5717 \\ 0 \\ 4900 \\ -28580 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -24500 \\ -28580 \\ 0 \\ -24500 \\ 28580 \end{Bmatrix} = \begin{Bmatrix} 0\text{ N} \\ 19600\text{ N} \\ 22860\text{ N}\cdot\text{m} \\ 0\text{ N} \\ 29400\text{ N} \\ -57160\text{ N}\cdot\text{m} \end{Bmatrix}$$



Reactions

Node 1

$$F_{1x} = -f'^{(1)}_{1x} = 0\text{ N}$$

$$F_{1y} = -f'^{(1)}_{1y} = -29400\text{ N}$$

$$M_1 = -m^{(1)}_1 = 0\text{ N}\cdot\text{m}$$

Node 3

$$F_{3x} = -f'^{(2)}_{3x} = 0\text{ N}$$

$$F_{3y} = -\left(f'^{(2)}_{3y} + f'^{(1)}_{3y}\right) = -44100\text{ N}$$

$$M_3 = 0\text{ N}\cdot\text{m}$$

Node 2

$$F_{2x} = -f'^{(2)}_{2x} = 0\text{ N}$$

$$F_{2y} = -\left(f'^{(1)}_{2y} + f'^{(2)}_{2y}\right) = -44100\text{ N}$$

$$M_2 = 0\text{ N}\cdot\text{m}$$

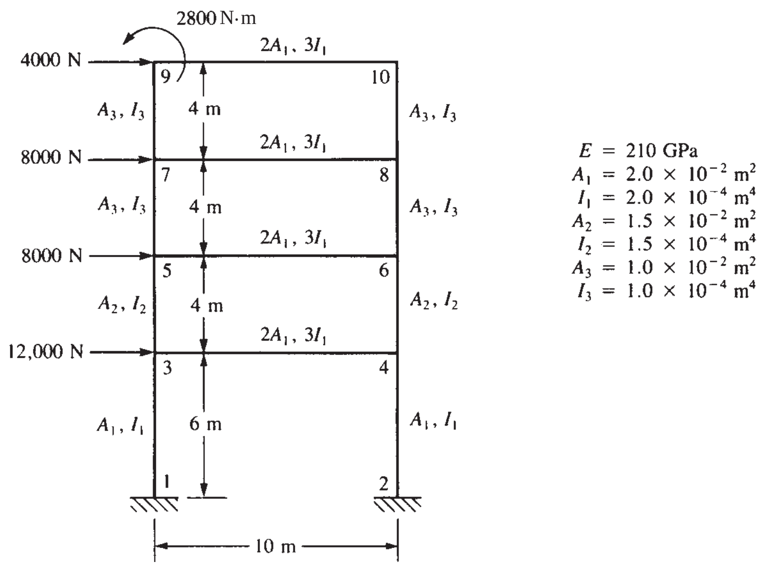
Node 4

$$F_{4x} = -f'^{(3)}_{3x} = 0\text{ N}$$

$$F_{4y} = -f'^{(3)}_{4y} = -29400\text{ N}$$

$$M_4 = -m^{(3)}_4 = 0\text{ N}\cdot\text{m}$$

5.35



NUMBER OF ELEMENTS = 12
NUMBER OF NODES = 10

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)
1	1 1 1	0.000000	0.000000	0.000000
2	1 1 1	10.000000	0.000000	0.000000
3	0 0 0	0.000000	6.000000	0.000000
4	0 0 0	10.000000	6.000000	0.000000
5	0 0 0	0.000000	10.000000	0.000000
6	0 0 0	10.000000	10.000000	0.000000
7	0 0 0	0.000000	14.000000	0.000000
8	0 0 0	10.000000	14.000000	0.000000
9	0 0 0	0.000000	18.000000	0.000000
10	0 0 0	10.000000	18.000000	0.000000

FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
0.000000	0.000000	0.000000
0.000000	0.000000	0.000000
12000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
8000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
8000.000000	0.000000	0.000000
0.000000	0.000000	0.000000
4000.000000	0.000000	2800.000000
0.000000	0.000000	0.000000

ELEMENTS

K	NODE(1,K)		E(K)	G(K)	A(K)	XI(K)
1	1	3	2.1000000E+11	0.00000000E+00	2.0000000E–02	2.0000000E–04
2	3	5	2.1000000E+11	0.00000000E+00	1.5000000E–02	1.5000000E–04
3	5	7	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
4	7	9	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
5	2	4	2.1000000E+11	0.00000000E+00	2.0000000E–02	2.0000000E–04
6	4	6	2.1000000E+11	0.00000000E+00	1.5000000E–02	1.5000000E–04
7	6	8	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
8	8	10	2.1000000E+11	0.00000000E+00	1.0000000E–02	1.0000000E–04
9	3	4	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
10	5	6	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
11	7	8	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04
12	9	10	2.1000000E+11	0.00000000E+00	4.0000000E–02	6.0000000E–04

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	0.00000E+00
2	0.00000E+00	0.00000E+00	0.00000E+00
3	0.92499E–02	0.32295E–04	– 0.79668E–03
4	0.92428E–02	– 0.32295E–04	– 0.79610E–03
5	0.13440E–01	0.45836E–04	– 0.45169E–03
6	0.13435E–01	– 0.45836E–04	– 0.45347E–03
7	0.16322E–01	0.53376E–04	– 0.23126E–03
8	0.16317E–01	– 0.53376E–04	– 0.22138E–03
9	0.17395E–01	0.54706E–04	– 0.25437E–04
10	0.17393E–01	– 0.54706E–04	– 0.88782E–04

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.2261E+05	0.1601E+05	0.5360E+05	0.2261E+05	– 0.1601E+05	0.4244E+05
2	3	5	– 0.1066E+05	0.1000E+05	0.1728E+05	0.1066E+05	– 0.1000E+05	0.2272E+05
3	5	7	– 0.3959E+04	0.5969E+04	0.1078E+05	0.3959E+04	– 0.5969E+04	0.1310E+05
4	7	9	– 0.6981E+03	0.2206E+04	0.3331E+04	0.6981E+03	– 0.2206E+04	0.5492E+04
5	2	4	0.2261E+05	0.1599E+05	0.5355E+05	– 0.2261E+05	– 0.1599E+05	0.4241E+05
6	4	6	0.1066E+05	0.1000E+05	0.1730E+05	– 0.1066E+05	– 0.1000E+05	0.2270E+05
7	6	8	0.3959E+04	0.6032E+04	0.1085E+05	– 0.3959E+04	– 0.6032E+04	0.1328E+05
8	8	10	0.6981E+03	0.1796E+04	0.2896E+04	– 0.6981E+03	– 0.1796E+04	0.4288E+04
9	3	4	0.5993E+04	– 0.1194E+05	– 0.5973E+05	– 0.5993E+04	0.1194E+05	– 0.5971E+05
10	5	6	0.3968E+04	– 0.6704E+04	– 0.3350E+05	– 0.3968E+04	0.6704E+04	– 0.3354E+05
11	7	8	0.4237E+04	– 0.3261E+04	– 0.1643E+05	– 0.4237E+04	0.3261E+04	– 0.1618E+05
12	9	10	0.1799E+04	– 0.6981E+03	– 0.2692E+04	–0.1795E+04	0.6981E+03	– 0.4288E+04

5.36

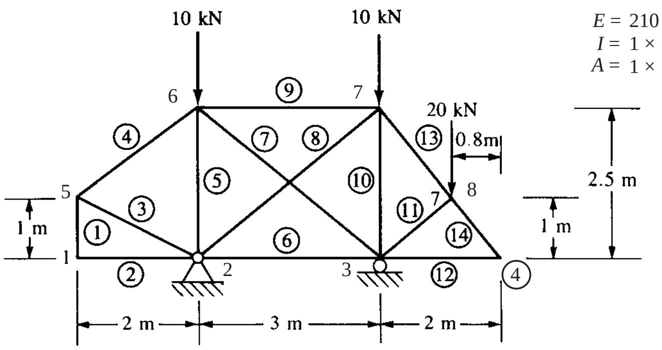
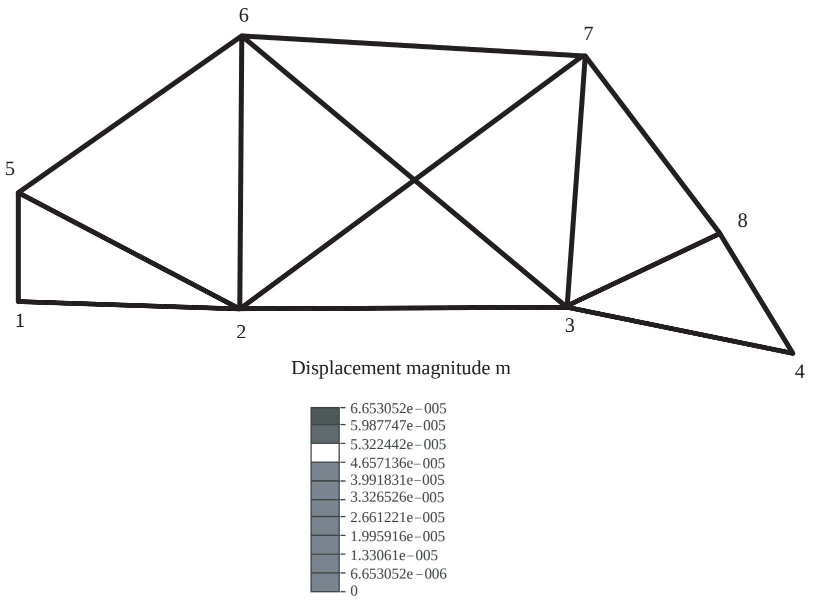


Figure P5–36

Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation
1	–1.5416E–07	9.7204E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.2761E–04
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	–2.9451E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	–1.2720E–03
4	–3.0149E–06	–6.6462E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.7954E–03
5	1.6304E–05	–4.5862E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.1306E–04
6	5.2760E–06	9.7805E–06	0.0000E+00	0.0000E+00	0.0000E+00	–3.6825E–04
7	2.7047E–05	–2.6136E–05	0.0000E+00	0.0000E+00	0.0000E+00	–3.4281E–04
8	2.2989E–05	–4.5872E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.4049E–03



5.37

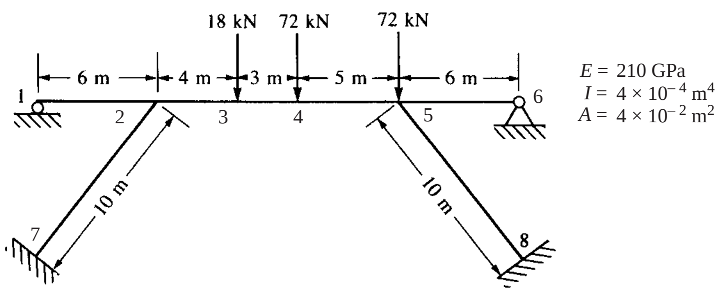
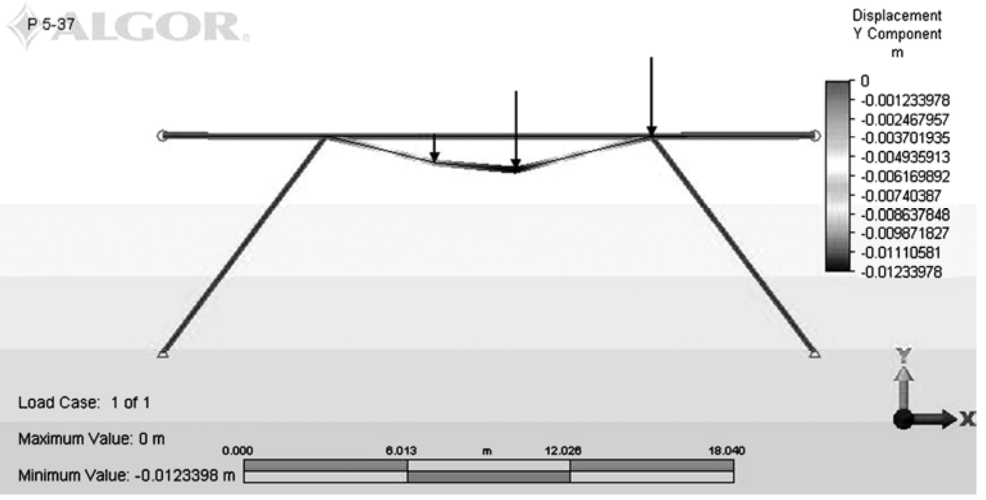


Figure P5–37

Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation (deg)
1	1.1203E–05	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	4.0957E–02
2	1.1203E–05	–1.1000E–04	0.0000E+00	0.0000E+00	0.0000E+00	–8.5066E–02
3	–3.5606E–06	–1.0371E–02	0.0000E+00	0.0000E+00	0.0000E+00	–1.3515E–01
4	–1.4633E–05	–1.2879E–02	0.0000E+00	0.0000E+00	0.0000E+00	6.2412E–02
5	–3.3087E–05	–2.9915E–04	0.0000E+00	0.0000E+00	0.0000E+00	8.5353E–02
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00



5.38

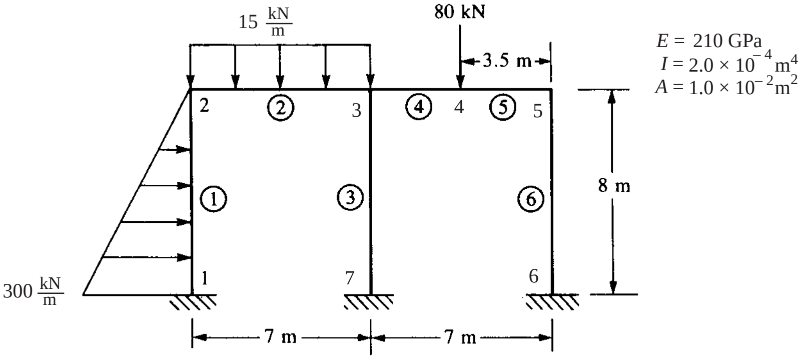


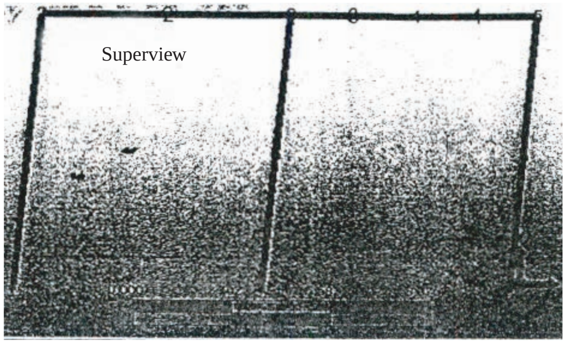
Figure P5–38

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation (deg)
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	1.4372E–01	–1.1344E-04	0.0000E+00	0.0000E+00	0.0000E+00	1.4397E–01
3	1.4300E–01	–1.3696E–04	0.0000E+00	0.0000E+00	0.0000E+00	–4.1178E–01
4	1.4282E–01	–2.1948E–03	0.0000E+00	0.0000E+00	0.0000E+00	2.2790E–01
5	1.4265E–01	–4.7155E–04	0.0000E+00	0.0000E+00	0.0000E+00	–5.1623E–01
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

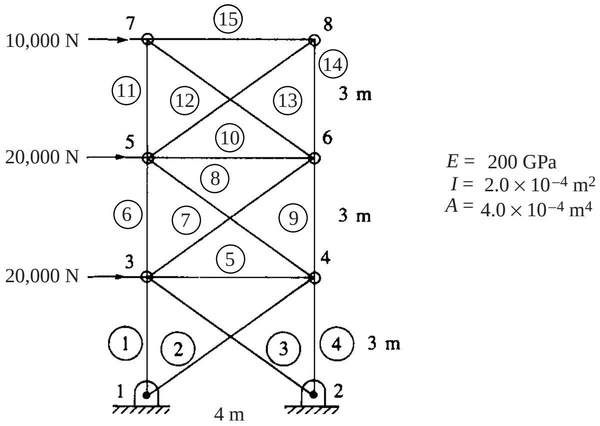
- BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	–2.907E+04	–9.878E+05	0.000E+00	0.000E+00	0.000E+00	–1.538E+06
		–2.907E+04	2.122E+05	0.000E+00	0.000E+00	0.000E+00	–3.605E+04
2	1	–2.122E+05	–2.907E+04	0.000E+00	0.000E+00	0.000E+00	–3.605E+04
		–2.122E+05	7.593E+04	0.000E+00	0.000E+00	0.000E+00	–2.001E+05
3	1	–1.024E+05	4.084E+04	0.000E+00	0.000E+00	0.000E+00	2.022E+05
		–1.024E+05	4.084E+04	0.000E+00	0.000E+00	0.000E+00	5.932E+04
4	1	–1.024E+05	1.208E+05	0.000E+00	0.000E+00	0.000E+00	5.932E+04
		–1.024E+05	1.208E+05	0.000E+00	0.000E+00	0.000E+00	–3.636E+05
5	1	–1.208E+05	–1.024E+05	0.000E+00	0.000E+00	0.000E+00	–4.560E+05
		–1.208E+05	–1.024E+05	0.000E+00	0.000E+00	0.000E+00	3.636E+05

6	1	–3.510E+04	–1.098E+05	0.000E+00	0.000E+00	0.000E+00	–4.760E+05
		–3.510E+04	–1.098E+05	0.000E+00	0.000E+00	0.000E+00	4.023E+05



5.39



(a) Truss model
NUMBER OF ELEMENTS (NELE) = 15
NUMBER OF NODES (KNODE) = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE (1,K)	FORCE (2,K)	FORCE (3,K)
1	1 1 1	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
2	1 1 1	4.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
3	0 0 1	0.000000E+00	3.000000E+00	0.000000E+00	2.000000E+04	0.000000E+00	0.000000E+00
4	0 0 1	4.000000E+00	3.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
5	0 0 1	0.000000E+00	6.000000E+00	0.000000E+00	2.000000E+04	0.000000E+00	0.000000E+00
6	0 0 1	4.000000E+00	6.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
7	0 0 1	0.000000E+00	9.000000E+00	0.000000E+00	1.000000E+04	0.000000E+00	0.000000E+00
8	0 0 1	4.000000E+00	9.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

ELEMENTS

K	NODE(I,K)		E(K)	A(K)
1	1	3	2.0000E+11	2.0000E–04
2	1	4	2.0000E+11	2.0000E–04
3	2	3	2.0000E+11	2.0000E–04
4	2	4	2.0000E+11	2.0000E–04
5	3	4	2.0000E+11	2.0000E–04
6	3	5	2.0000E+11	2.0000E–04
7	3	6	2.0000E+11	2.0000E–04
8	4	5	2.0000E+11	2.0000E–04
9	4	6	2.0000E+11	2.0000E–04
10	5	6	2.0000E+11	2.0000E–04

11	5	7	2.0000E+11	2.0000E–04
12	5	8	2.0000E+11	2.0000E–04
13	6	7	2.0000E+11	2.0000E–04
14	6	8	2.0000E+11	2.0000E–04
15	7	8	2.0000E+11	2.0000E–04

NUMBER OF NONZERO UPPER CO-DIAGONALS (MUD)-11

DISPLACEMENTS	X	Y	Z
MODE NUMBER 1	0.0000E+00	0.0000E+00	0.0000E+00
MODE NUMBER 2	0.0000E+00	0.0000E+00	0.0000E+00
MODE NUMBER 3	0.7935E–02	0.3730E–02	0.0000E+00
MODE NUMBER 4	0.7315E–02	– 0.3583E–02	0.0000E+00
MODE NUMBER 5	0.1738E–01	0.5276E–01	0.0000E+00
MODE NUMBER 6	0.1681E–01	– 0.4849E–02	0.0000E+00
MODE NUMBER 7	0.2603E–01	0.5662E–02	0.0000E+00
MODE NUMBER 8	0.2572E–01	– 0.5026E–02	0.0000E+00

STRESSES IN ELEMENTS (IN CURRENT UNITS)

ELEMENT NUMBER	STRESS
1 =	0.24864E+09
2 =	0.14809E+09
3 =	– 0.16441E+09
4 =	– 0.23886E+09
5 =	– 0.30998E+08
6 =	0.10311E+09
7 =	0.78155E+08
8 =	– 0.10934E+09
9 =	– 0.84393E+08
10 =	– 0.28261E+08
11 =	0.25697E+08
12 =	0.19671E+08
13 =	– 0.42828E+08
14 =	– 0.11803E+08
15 =	– 0.15737E+08

(b) Rigid frame model

NUMBER OF ELEMENTS = 15

NUMBER OF NODES = 8

NODE POINTS

K	IFIK	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE (3,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	1 1 0	4.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	3.000000	0.000000	20000.000000	0.000000	0.000000
4	0 0 0	4.000000	3.000000	0.000000	0.000000	0.000000	0.000000
5	0 0 0	0.000000	6.000000	0.000000	20000.000000	0.000000	0.000000
6	0 0 0	4.000000	6.000000	0.000000	0.000000	0.000000	0.000000
7	0 0 0	0.000000	9.000000	0.000000	10000.000000	0.000000	0.000000
8	0 0 0	4.000000	9.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I, K)		E(K)	C(K)	A(K)	XI(K)
1	1	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
2	1	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
3	2	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
4	2	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
5	3	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
6	3	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
7	3	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
8	4	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04

9	4	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
10	5	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
11	5	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
12	5	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
13	6	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
14	6	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04
15	7	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	4.0000000E–04

NODE	DISPLACEMENTS X	Y	K-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.14936E–02
2	0.00000E+00	0.00000E+00	– 0.14200E–02
3	0.54772E–02	0.32294E–02	– 0.17329E–02
4	0.51685E–02	– 0.32554E–02	– 0.16980E–02
5	0.11556E–01	0.40230E–02	– 0.20908E–02
6	0.11199E–01	– 0.40706E–02	– 0.20924E–02
7	0.18021E–01	0.42395E–02	– 0.21639E–02
8	0.17667E–01	– 0.42509E–02	– 0.21228E–02

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	3	– 0.4306E+05	0.2267E+05	0.4038E+05	0.4306E+05	– 0.2267E+05	0.2762E+05
2	1	4	– 0.1745E+05	– 0.1746E+05	– 0.4038E+05	0.1745E+05	0.1746E+05	– 0.4692E+05
3	2	3	0.1955E+05	– 0.1545E+05	– 0.3363E+05	– 0.1955E+05	0.1545E+05	– 0.4364E+05
4	2	4	0.4340E+05	0.1748E+05	0.3363E+05	– 0.4340E+05	– 0.1748E+05	0.1860E+05
5	3	4	0.3087E+04	– 0.5654E+04	– 0.1201E+05	– 0.3087E+04	0.5654E+04	– 0.1061E+05
6	3	5	– 0.1058E+05	0.1220E+05	0.2784E+05	0.1058E+05	– 0.1220E+05	0.8749E+04
7	3	6	– 0.1582E+04	– 0.2226E+04	0.1879E+03	0.1582E+04	0.2226E+04	– 0.1132E+05
8	4	5	0.5942E+04	0.1406E+04	0.9799E+04	– 0.5942E+04	– 0.1406E+04	– 0.2770E+04
9	4	6	0.1087E+05	0.1228E+05	0.2893E+05	– 0.1087E+05	– 0.1228E+05	0.7899E+04
10	5	6	0.3563E+04	– 0.4091E+04	– 0.8150E+04	– 0.3563E+04	0.4091E+04	– 0.8214E+04
11	5	7	– 0.2887E+04	0.2978E+04	0.6416E+04	0.2887E+04	– 0.2978E+04	0.2517E+04
12	5	8	0.6004E+03	– 0. 1903E+04	– 0.4245E+04	– 0.6004E+03	0.1903E+04	– 0.5270E+04
13	6	7	0.3772E+04	0.7733E+03	0.3077E+04	– 0.3772E+04	– 0.7733E+03	0.7894E+03
14	6	8	0.2405E+04	0.5163E+04	0.8555E+04	– 0.2405E+04	– 0.5163E+04	0.6933E+04
15	7	8	0.3541E+04	– 0.1243E+04	– 0.3307E+04	– 0.3541E+04	0.1243E+04	– 0.1664E+04

(c) Use program PFRAME to model a truss

(Use PFRAME to model a Truss, i.e., MAKE $I \cong 0$).

NUMBER OF ELEMENTS = 15

NUMBER OF NODES = 8

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)
1	1 1 0	0.000000	0.000000	0.000000	0.000000
2	1 1 0	4.000000	0.000000	0.000000	0.000000
3	0 0 0	0.000000	3.000000	0.000000	20000.000000
4	0 0 0	4.000000	3.000000	0.000000	0.000000
5	0 0 0	0.000000	6.000000	0.000000	20000.000000
6	0 0 0	4.000000	6.000000	0.000000	0.000000
7	0 0 0	0.000000	9.000000	0.000000	10000.000000
8	0 0 0	4.000000	9.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	C(K)	A(K)	XI(K)
1	1	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
2	1	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
3	2	3	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
4	2	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06

5	3	4	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
6	3	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
7	3	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
8	4	5	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
9	4	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
10	5	6	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
11	5	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
12	5	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
13	6	7	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
14	6	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06
15	7	8	2.0000000E+11	0.0000000E+00	2.0000000E–04	1.0000000E–06

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.00000E+00	– 0.21033E–02
2	0.00000E+00	0.00000E+00	– 0.19714E–02
3	0.79234E–02	0.37278E–02	– 0.23957E–02
4	0.73056E–02	– 0.35821E–02	– 0.23447E–02
5	0.17337E–01	0.52713E–02	– 0.28898E–02
6	0.16772E–01	– 0.48468E–02	– 0.29231E–02
7	0.25985E–01	0.56570E–02	– 0.27580E–02
8	0.25670E–01	–0.50248E–02	– 0.27544E–02

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE at 1st node	Z-MOMENT	X-FORCE	Y-FORCE at 2nd node	Z-MOMENT
1	1	3	– 0.4970E+05	0.1044E+03	0.1762E+03	0.4970E+05	– 0.1044X+03	0.1372E+03
2	1	4	– 0.2956E+05	– 0.7432E+02	– 0.1762E+03	0.2956E+05	– 0.7432E+02	– 0.1955E+03
3	2	3	0.3282E+05	– 0.6108E+02	– 0.1357E+03	– 0.3282E+05	0.6108E+02	– 0.1697E+03
4	2	4	0.4776E+05	0.7391E+02	0.1357E+03	– 0.4776E+05	– 0.7391E+02	0.8597E+02
5	3	4	0.6178E+04	– 0.8141E+02	– 0.1654E+03	– 0.6178E+04	0.8141E+02	– 0.1603E+03
6	3	5	– 0.2058E+05	0.1320E+03	0.2309E+03	0.2058E+05	– 0.1320E+03	0.1650E+03
7	3	6	– 0.1548E+05	– 0.2166E+02	– 0.3305E+02	0.1548E+05	0.2166E+02	– 0.7524E+02
8	4	5	0.2170E+05	0.2894E+00	0.2253E+02	– 0.2170E+05	– 0.2894E+00	– 0.2108E+02
9	4	6	0.1686E+05	0.1391E+03	0.2472E+03	– 0.1686E+05	– 0.1391E+03	0.1701E+03
10	5	6	0.5641E+04	– 0.5654E+02	– 0.1114E+03	– 0.5641E+04	0.5654E+02	– 0.1148E+03
11	5	7	– 0.5143E+04	0.1572E+02	0.1480E+02	0.5143E+04	– 0.1572E+02	0.3237E+02
12	5	8	– 0.3912E+04	– 0.1677E+02	– 0.4735E+02	0.3912E+04	0.1677E+02	– 0.3652E+02
13	6	7	0.8543E+04	– 0.5227E+01	– 0.1967E+02	– 0.8543E+04	0.5227E+01	– 0.6461E+01
14	6	8	0.2373E+04	0.3387E+02	0.3956E+02	– 0.2373E+04	– 0.3387E+02	0.6206E+02
15	7	8	0.3153E+04	– 0.1286E+02	– 0.2591E+02	– 0.3153E+04	0.1286E+02	– 0.2554E+02

Comparison of TRUSS, PFRAME and modeling a truss using PFRAME

DISPLACEMENTS

	u_5	v_5	u_7	v_7
TRUSS	0.01738	0.005276	0.02603	0.005662
PFRAME	0.011556	0.004023	0.018021	0.0042395
Truss using PFRAME	0.017337	0.0052713	0.025985	0.005657

Note: Global displacements in meters

FORCES

	ELEMENT 1 f_{1x}	ELEMENT 1 f_{1y}	ELEMENT 2 f_{1x}	ELEMENT 2 f_{1y}	ELEMENT 3 f_{2x}	ELEMENT 3 f_{2y}
Truss	–49728	0	–29618	0	32882	0
PFRAME	–43060	22670	–17450	–17460	19550	–15450
Truss using PFRAME	–49700	104.4	–29560	–74.32	32820	–61.00

Note 1: From equilibrium, only forces for one node of an element are shown
Note 2: All forces are in local element coordinates and in Newtons.

5.40

For the two-story, two-bay rigid frame shown, determine (1) the nodal displacement components and (2) the shear force and bending moments in each member. Let E = 200 GPa, $I = 2 \times 10^{-4}$ m^4 for each horizontal member and $I = 1.5 \times 10^{-4}$ m^4 for each vertical member.

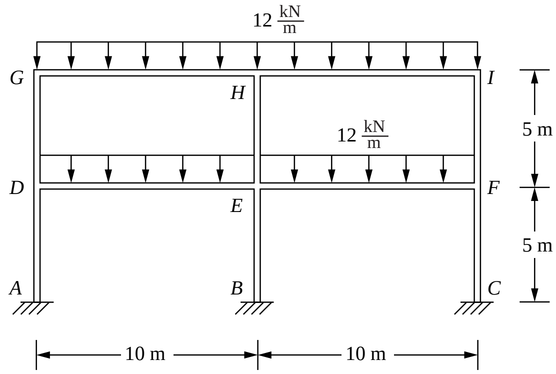


Figure P5–40

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	–7.2482E-04	–1.4007E–06	0.0000E+00	0.0000E+00	0.0000E+00	1.6616E–02
5	0.0000E+00	–3.1986E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
6	7.2482E-04	–1.4007E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.6616E–02
7	–7.7467E–07	–2.8014E–06	0.0000E+00	0.0000E+00	0.0000E+00	–6.6411E–02
8	–3.8733E–07	–9.2660E–03	0.0000E+00	0.0000E+00	0.0000E+00	1.6572E–02
9	0.0000E+00	–6.3972E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
10	3.8733E–07	–9.2660E–03	0.0000E+00	0.0000E+00	0.0000E+00	–1.6572E–02
11	7.7467E–07	–2.8014E–06	0.0000E+00	0.0000E+00	0.0000E+00	6.6411E–02
12	–6.2096E–04	–3.4868E–06	0.0000E+00	0.0000E+00	0.0000E+00	4.7408E–02
13	0.0000E+00	–8.0263E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
14	6.2096E–04	–3.4868E–06	0.0000E+00	0.0000E+00	0.0000E+00	–4.7408E–02
15	1.1921E–06	–4.1723E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.2336E–01
16	5.9603E–07	–1.0511E–02	0.0000E+00	0.0000E+00	0.0000E+00	3.0792E–02
17	0.0000E+00	–9.6555E–06	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
18	–5.9603E–07	–1.0511E–02	0.0000E+00	0.0000E+00	0.0000E+00	–3.0792E–02
19	–1.1921E–06	–4.1723E–06	0.0000E+00	0.0000E+00	0.0000E+00	1.2336E–01

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	1.391E+04
		–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–6.955E+03
2	1	–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
3	1	–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	–1.391E+04
		–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	6.955E+03
4	1	–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–6.955E+03

		–1.121E+05	8.348E+03	0.000E+00	0.000E+00	0.000E+00	–2.782E+04
5	1	–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–2.559E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
6	1	–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	6.955E+03
		–1.121E+05	–8.348E+03	0.000E+00	0.000E+00	0.000E+00	2.782E+04
7.	1	–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.364E+04
		–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.963E+03
8	1	–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
9	1	–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.364E+04
		–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.963E+04
10	1	–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–5.963E+03
		–5.484E+04	2.384E+04	0.000E+00	0.000E+00	0.000E+00	–6.557E+04
11	1	–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
		–1.303E+05	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
12	1	–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	5.963E+03
		–5.484E+04	–2.384E+04	0.000E+00	0.000E+00	0.000E+00	6.557E+04

5.41 For the two-story, three-bay rigid frame shown, determine (1) the nodal displacements and (2) the member end shear forces and bending moments. (3) Draw the shear force and bending moment diagrams for each member. Let E = 200 GPa, $I = 1.29 \times 10^{-4}\ \text{m}^4$ for the beams and $I = 0.462 \times 10^{-4}\ \text{m}^4$ for the columns. The properties for I correspond to a W 610×155 and a W 410×114 wide-flange section, respectively, in metric units.

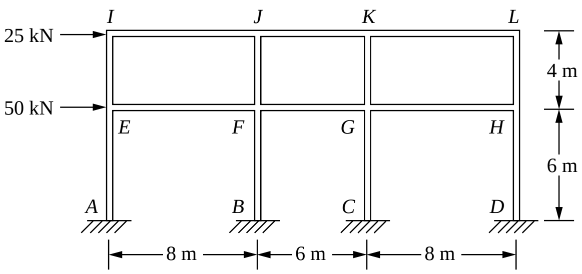


Figure P5–41

1**** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
1	1	1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–5.719E+04
		1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
2	1	2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–6.219E+04
		2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
3	1	–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–6.218E+04
		–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
4	1	–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–5.718E+04
		–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
5	1	1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
		1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	4.782E+04
6	1	2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
		2.471E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	5.782E+04
7	1	–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	–2.184E+03
		–2.470E+03	–2.000E+04	0.000E+00	0.000E+00	0.000E+00	5.782E+04
8	1	–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	–4.685E+03
		–1.347E+04	–1.750E+04	0.000E+00	0.000E+00	0.000E+00	4.781E+04
9	1	2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	–5.747E+02
		2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
10	1	1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	–1.668E+04
		1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03

11	1	–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	–1.669E+04
		–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
12	1	–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	–5.786E+02
		–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
13	1	2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
		2.686E+03	–3.022E+03	0.000E+00	0.000E+00	0.000E+00	1.151E+04
14	1	1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
		1.063E+03	–9.477E+03	0.000E+00	0.000E+00	0.000E+00	2.122E+04
15	1	–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.269E+03
		–1.063E+03	–9.478E+03	0.000E+00	0.000E+00	0.000E+00	2.122E+04
16	1	–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	5.469E+03
		–2.686E+03	–3.024E+03	0.000E+00	0.000E+00	0.000E+00	1.152E+04

5.42 For the rigid frame shown, determine (1) the nodal displacements and rotations and (2) the member shear forces and bending moments. Let E = 200 GPa, $I = 0.795 \times 10^{-4}\ \text{m}^4$ for the horizontal members and $I = 0.316 \times 10^{-4}\ \text{m}^4$ for the vertical members. These I values correspond to a W 460 × 158 and a W 410 × 85 wide-flange section, respectively.

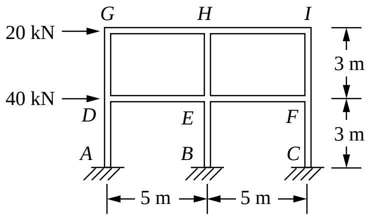


Figure P5–42

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	7.8454E–04	8.2939E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.5312E–02
3	1.5075E–03	1.1468E–05	0.0000E+00	0.0000E+00	0.0000E+00	–9.6245E–03
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	7.3108E–04	2.5665E–08	0.0000E+00	0.0000E+00	0.0000E+00	–1.2692E–02
6	1.4698E–03	1.8869E–07	0.0000E+00	0.0000E+00	0.0000E+00	–7.8815E–03
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	7.0278E–04	–8.3195E–06	0.0000E+00	0.0000E+00	0.0000E+00	–1.4567E–02
9	1.4574E–03	–1.1657E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.0523E–02
10	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
11	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
12	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
13	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
14	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
15	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
			Columns				
1	1	1.106E+04	–2.085E+04	0.000E+00	0.000E+00	0.000E+00	–4.540E+04
		1.106E+04	–2.085E+04	0.000E+00	0.000E+00	0.000E+00	1.716E+04
2	1	4.232E+03	–3.811E+03	0.000E+00	0.000E+00	0.000E+00	–4.719E+02
		4.232E+03	–3.811E+03	0.000E+00	0.000E+00	0.000E+00	1.096E+04
3	1	3.422E+01	–2.168E+04	0.000E+00	0.000E+00	0.000E+00	–4.422E+04
		3.422E+01	–2.168E+04	0.000E+00	0.000E+00	0.000E+00	2.081E+04

4	1	2.174E+02	–1.088E+04	0.000E+00	0.000E+00	0.000E+00	–1.188E+04
		2.174E+02	–1.088E+04	0.000E+00	0.000E+00	0.000E+00	2.075E+04
5	1	–1.109E+04	–1.747E+04	0.000E+00	0.000E+00	0.000E+00	–3.964E+04
		–1.109E+04	–1.747E+04	0.000E+00	0.000E+00	0.000E+00	1.277E+04
6	1	–4.450E+03	–5.314E+03	0.000E+00	0.000E+00	0.000E+00	–4.242E+03
		–4.450E+03	–5.314E+03	0.000E+00	0.000E+00	0.000E+00	1.170E+04

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
				Beams			
1	1	–1.619E+04	4.232E+03	0.000E+00	0.000E+00	0.000E+00	1.096E+04
		–1.619E+04	4.232E+03	0.000E+00	0.000E+00	0.000E+00	–1.020E+04
2	1	–5.314E+03	4.450E+03	0.000E+00	0.000E+00	0.000E+00	1.055E+04
		–5.314E+03	4.450E+03	0.000E+00	0.000E+00	0.000E+00	–1.170E+04
3	1	–2.296E+04	6.825E+03	0.000E+00	0.000E+00	0.000E+00	1.763E+04
		–2.296E+04	6.825E+03	0.000E+00	0.000E+00	0.000E+00	–1.649E+04
4	1	–1.216E+04	6.642E+03	0.000E+00	0.000E+00	0.000E+00	1.620E+04
		–1.216E+04	6.642E+03	0.000E+00	0.000E+00	0.000E+00	–1.701E+04

5.43 For the rigid frame shown, determine (1) the nodal displacements and rotations and (2) the shear force and bending moments in each member. Let $E = 29 \times 106$ psi, $I = 3100$ in.^{4} for the horizontal members and $I = 1110$ in.^{4} for the vertical members. The I values correspond to a W 24 × 104 and a W 16 × 77.

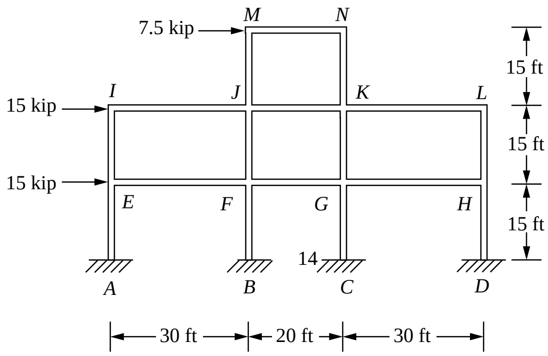


Figure P5–43

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	1.8762E–03	1.4645E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5869E–03
3	3.0993E-03	2.1717E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.4409E–03
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	1.5614E–03	3.5423E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.3154E–03
6	2.7811E–03	7.7093E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.4087E–03
7	3.6829E–03	1.2865E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5467E–03
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
9	1.4331E–03	–4.4935E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.2573E–03
10	2.6228E–03	–8.8701E–05	0.0000E+00	0.0000E+00	0.0000E+00	–1.3909E–03
11	3.5926E-03	–1.4025E-04	0.0000E+00	0.0000E+00	0.0000E+00	–1.5614E–03
12	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
13	1.3394E–03	–1.3694E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.3297E–03
14	2.4458E–03	–2.0556E–04	0.0000E+00	0.0000E+00	0.0000E+00	–1.3014E–03
15	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
16	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
17	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
18	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

19	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
20	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

1 **** BEAM ELEMENT FORCES AND MOMENTS

ELEMENT NO.	CASE (MODE)	AXIAL FORCE R_1	SHEAR FORCE R_2	SHEAR FORCE R_3	TORSION MOMENT M_1	BENDING MOMENT M_2	BENDING MOMENT M_3
			Columns				
1	1	6.363E+03	–1.139E+04	0.000E+00	0.000E+00	0.000E+00	–1.444E+05
		6.363E+03	–1.139E+04	0.000E+00	0.000E+00	0.000E+00	2.653E+04
2	1	3.073E+03	–5.646E+03	0.000E+00	0.000E+00	0.000E+00	–3.692E+04
		3.073E+03	–5.646E+03	0.000E+00	0.000E+00	0.000E+00	4.777E+04
3	1	1.539E+03	–9.487E+03	0.000E+00	0.000E+00	0.000E+00	–1.200E+05
		1.539E+03	–9.487E+03	0.000E+00	0.000E+00	0.000E+00	2.231E+04
4	1	1.810E+03	–5.894E+03	0.000E+00	0.000E+00	0.000E+00	–4.766E+04
		1.810E+03	–5.894E+03	0.000E+00	0.000E+00	0.000E+00	4.074E+04
5	1	2.240E+03	–3.517E+03	0.000E+00	0.000E+00	0.000E+00	–3.150E+04
		2.240E+03	–3.517E+03	0.000E+00	0.000E+00	0.000E+00	2.125E+04
6	1	–1.952E+03	–8.662E+03	0.000E+00	0.000E+00	0.000E+00	–1.117E+05
		–1.952E+03	–8.662E+03	0.000E+00	0.000E+00	0.000E+00	1.828E+04
7	1	–1.902E+03	–5.757E+03	0.000E+00	0.000E+00	0.000E+00	– 4.814E+04
		–1.902E+03	–5.757E+03	0.000E+00	0.000E+00	0.000E+00	3.822E+04
8	1	–2.240E+03	–3.983E+03	0.000E+00	0.000E+00	0.000E+00	–3.620E+04
		–2.240E+03	–3.983E+03	0.000E+00	0.000E+00	0.000E+00	2.355E+04
9	1	–5.950E+03	–7.958E+03	0.000E+00	0.000E+00	0.000E+00	–1.091E+05
		–5.950E+03	–7.958E+03	0.000E+00	0.000E+00	0.000E+00	1.031E+04
10	1	–2.982E+03	–5.203E+03	0.000E+00	0.000E+00	0.000E+00	–3.797E+04
		–2.982E+03	–5.203E+03	0.000E+00	0.000E+00	0.000E+00	4.008E+04

5.44 A structure is fabricated by welding together three lengths of I-shaped members as shown in Figure P5-44. The yield strength of the members is 36 ksi, E = 29e6 psi, and Poisson's ratio is 0.3. The members all have cross-section properties corresponding to a W 18 × 76. That is, A = 22.3 in.^{2}, depth of section is d = 18.21 in., I_x = 1330 in.^{4}, S_x = 146 in.^{3}, I_y = 152 in.^{4}, and S_y = 27.6 in.^{3}. Determine whether a load of Q = 10,000 lb downward is safe against general yielding of the material. The factor of safety against general yielding is to be 2.0. Also, determine the maximum vertical and horizontal deflections of the structure.

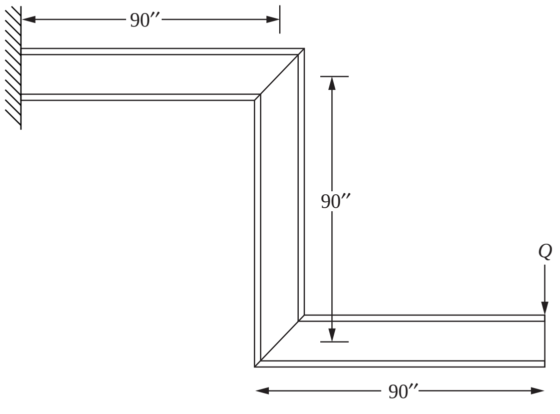
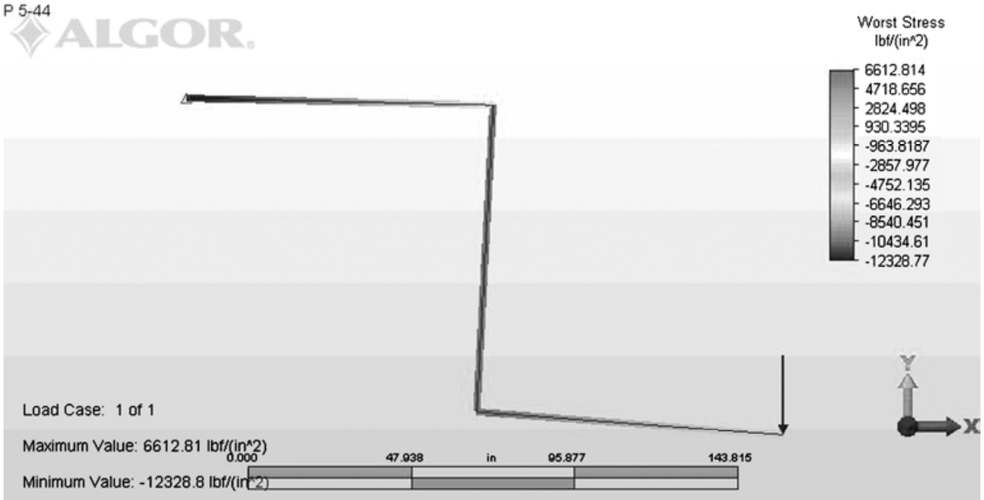


Figure P5-44

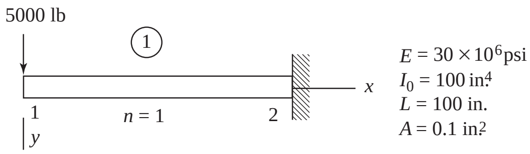


$$\sigma_{max} = 12{,}329 \text{ psi} < \frac{36000}{2} = 18{,}000 \text{ psi}$$

$\therefore$ Safe against yielding

5.45

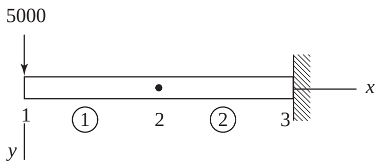
Tapered beam using 1 element



$$I(x) = I_0\left(1 + n\frac{x}{L}\right) = 100\left(1 + \frac{x}{100}\right) = 100 + x$$

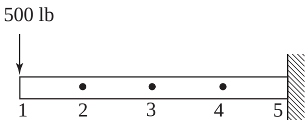
$$I\left(\frac{L}{2}\right) = 150 \text{ in.}^4$$

Tapered beam using 2 elements



$$I_{1-2}\left(\frac{L}{4}\right) = 125 \text{ in.}^4;\ I_{2-3} = \left(\frac{3L}{4}\right) = 175 \text{ in.}^4$$

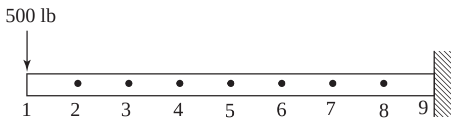
Tapered beam using 4 elements



$$I_{1-2}\left(\frac{L}{8}\right) = 112.5 \text{ in.}^4;\ I_{2-3}\left(\frac{3L}{8}\right) = 137.5 \text{ in.}^4;\ I_{3-4}\left(\frac{5L}{8}\right) = 162.5 \text{ in.}^4$$

$$I_{4-5} = \left(\frac{7L}{8}\right) = 187.5 \text{ in.}^4$$

Tapered beam using 8 elements



$$I_{1-2}\left(\frac{L}{16}\right) = 106.25 \qquad I_{2-3}\left(\frac{3L}{16}\right) = 118.75$$

$$I_{3-4}\left(\frac{5L}{16}\right) = 131.25 \qquad I_{4-5}\left(\frac{7L}{16}\right) = 143.75$$

$$I_{5-6}\left(\frac{9L}{16}\right) = 156.25 \qquad I_{6-7}\left(\frac{11L}{16}\right) = 168.75$$

$$I_{7-8}\left(\frac{13L}{16}\right) = 181.25 \qquad I_{8-9}\left(\frac{15L}{16}\right) = 193.75$$

Analytical solution

$$v = \frac{Pl^2}{n^2\, EI_0}\left[\left(\frac{n}{2l}x^2 + 2x + \frac{l}{n}\right) - \frac{1}{n}\left(1+\frac{n}{l}x\right) ln\left(1+\frac{n}{l}x\right)\right] + Ax + B$$

$$A = \frac{Pl^2}{n^2\, EI_0}\ [ln\,(1+n) - (1+n)]$$

$$B = \frac{Pl^3}{n^2\, EI_0}\left[\frac{1}{n} ln\,(1+n) + \frac{n}{2} - 1 - \frac{1}{n}\right]$$

$$x = 0,\ n = 1,\ P = 500,\ L = 100,\ E = 30\times 10^6,\ I_0 = 100$$

$$A = \frac{(500)\,(100 \text{ in.})^2}{(1)^2\,(30\times 10^6)\,(100 \text{ in.}^4)}\ [ln(2) - 2]$$

$$= -2.1781\times 10^{-3}$$

$$B = \frac{500\,(100 \text{ in.})^3}{(1)^2\,(30\times 10^6)\,(100 \text{ in.}^4)}\left[ln\,(2) + \frac{1}{2} - 1 - 1\right]$$

$$= -1.3448\times 10^{-1}$$

$$v = \frac{500\,(100 \text{ in.})^2}{(1)^2\,(30\times 10^6)\,(100 \text{ in.})}\left[\left(\frac{100 \text{ in.}}{(1)}\right)\right] - 1.3448\times 10^{-1}$$

$$= 0.032187 \text{ in.}$$

PROBLEM 2 USING 1 ELEMENT--
NUMBER OF ELEMENTS = 1
NUMBER OF NODES = 2

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3,0000000E+07	1.0000000E+00	1.0000000E–01	1.5000000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.37037E–01	– 0.55556E–03
2	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.2434E–03	0.0000E+00	– 0.5000E+03	0.5000E+05

PROBLEM 2 USING 2 ELEMENTS--
NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	0 0 0	50.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

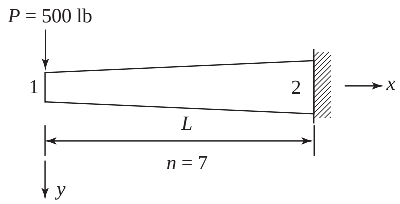
ELEMENTS

K	NODE(1,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	1.0000000E+00	1.0000000E–01	1.2500000E+02
2	2	3	3.0000000E+07	1.0000000E+00	1.0000000E–01	1.7500000E+02

NODE	DISPLACEMENTS X	Y	Z-ROTATION THETA
1	0.00000E+00	0.33333E–01	– 0.52381E–03
2	0.00000E+00	0.99206E–02	– 0.35714E–03
3	0.00000E+00	0.00000E+00	0.000000E+00

ELEMENTS

K	NODE (I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.6515E–02	0.0000E+00	– 0.5000E+03	0.2500E+05
2	2	3	0.0000E+00	0.5000E+03	– 0.2500E+05	0.0000E+00	– 0.5000E+03	0.5000E+05



For 1 element

$$I_1 = 100\left(1+\frac{7}{2}\right) = 450 \text{ in.}^4$$

For 2 elements

$$I_1 = 100\left(\frac{11}{4}\right) = 275 \text{ in.}^4$$

$$I_2 = 100\left(\frac{25}{4}\right) = 625 \text{ in.}^4$$

For 4 elements

$$I_1 = 100\left(\frac{15}{8}\right) = 187.5 \text{ in.}^4$$

$$I_2 = 100\left(\frac{29}{8}\right) = 362.5 \text{ in.}^4$$

$$I_3 = 100\left(\frac{43}{8}\right) = 537.5 \text{ in.}^4$$

$$I_4 = 100\left(\frac{57}{8}\right) = 712.5 \text{ in.}^4$$

For 8 elements

$$I_1 = 100\left(\frac{23}{16}\right) = 143.75 \text{ in.}^4$$

$$I_2 = 100\left(\frac{37}{16}\right) = 231.25 \text{ in.}^4$$

$$I_3 = 100\left(\frac{51}{16}\right) = 318.75 \text{ in}^4$$

$$I_4 = 100\left(\frac{65}{16}\right) = 406.25 \text{ in.}^4$$

$$I_5 = 100\left(\frac{79}{16}\right) = 493.75 \text{ in.}^4$$

$$I_6 = 100\left(\frac{93}{16}\right) = 581.25 \text{ in.}^4$$

$$I_7 = 100\left(\frac{107}{16}\right) = 668.75 \text{ in.}^4$$

$$I_8 = 100\left(\frac{121}{16}\right) = 756.25 \text{ in.}^4$$

The analytical solution is

$$v_{\max} = y_{\max} = \frac{1}{17.55}\,\frac{(500)\,(100)^3}{(30\times 10^6)\,(100)} = 0.0095 \text{ in.}$$

		$y_{\max}$
	Analytical	0.0095 in.
FEM	1 element	0.0123 in.
FEM	2 elements	0.0103 in.
FEM	4 elements	0.0097 in.
FEM	8 elements	0.0096 in.

NUMBER OF ELEMENTS = 1
NUMBER OF NODES = 2

NODE POINTS

K	IFIK	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E–01	4.5000000E+02

NODE	DISPLACEMENTS		Z-ROTATION
	X	Y	THETA
1	0.00000E+00	0.12346E-01	–0.18519E-03
2	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE(I,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	–0.7706E–02	0.0000E+00	–0.5000E+03	0.5000E+05

NUMBER OF ELEMENTS = 4
NUMBER OF NODES = 5

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	0.000000	0.000000	0.000000	0.000000	500.000000	0.000000
2	0 0 0	25.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0 0 0	50.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0 0 0	75.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	1 1 1	100.000000	0.000000	0.000000	0.000000	0.000000	0.000000

ELEMENTS

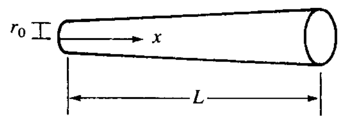
K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)
1	1	2	3.0000000E+07	0.0000000E+00	1.0000000E–01	1.8750000E+02
2	2	3	3.0000000E+07	0.0000000E+00	1.0000000E–01	3.6250000E+02
3	3	4	3.0000000E+07	0.0000000E+00	1.0000000E–01	5.3750000E+02
4	4	5	3.0000000E+07	0.0000000E+00	1.0000000E–01	7.1250000E+02

NODE	DISPLACEMENTS		Z-ROTATION
	X	Y	THETA
1	0.00000E+00	0.97156E–02	– 0.17050E–03
2	0.00000E+00	0.56845E–02	– 0.14272E–03
3	0.00000E+00	0.25953E–02	– 0.99620E–04
4	0.00000E+00	0.67008E–03	– 0.51170E–04
5	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (1,K)		X-FORCE	Y-FORCE	Z-MOMENT	X-FORCE	Y-FORCE	Z-MOMENT
1	1	2	0.0000E+00	0.5000E+03	0.3812E –01	0.0000E+00	– 0.5000E+03	0.1250E+05
2	2	3	0.0000E+00	0.5000E+03	– 0.1250E+05	0.0000E+00	– 0.5000E+03	0.2500E+05
3	3	4	0.0000E+00	0.5000E+03	– 0.2500E+05	0.0000E+00	– 0.5000E+03	0.3750E+05
4	4	5	0.0000E+00	0.5000E+03	– 0.3750E+05	0.0000E+00	– 0.5000E+03	0.5000E+05

5.46



$$U = \frac{1}{2}\int_V \tau\ \gamma dV \qquad \tau = \frac{Tr}{J} = G\gamma$$

$$U = \frac{1}{2}\int_V \frac{T^2 r^2}{G\,J^2}\; dV \qquad\qquad \gamma = \frac{\gamma}{L}\;\phi$$

$$= \frac{1}{2}\int \frac{T^2}{GJ^2}\left(\int r^2\, dA\right)\, dx$$

$$= \int \frac{T^2}{2GJ(x)}\; dx$$

Now

$$\gamma = \frac{\gamma}{L}\;(\phi_{2x} - \phi_{1x})$$

$$= \gamma \underbrace{\begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}}_{[B]} \underbrace{\begin{Bmatrix} \phi_{1x} \\ \phi_{2x} \end{Bmatrix}}_{\{\phi\}}$$

$$\tau_{\max} = G\gamma_{\max} = G\,[B]\;\{\phi\}$$

$$U = \frac{1}{2}\int_A \int \gamma\,\{\phi^T\}\,B^T\,\{G\}\,[B]\,\{\phi\}\,\gamma\; dx\;\; dA$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^R r^2\, r\, dr\, d\theta \int_0^L \{\phi^T\}\,[B^T]\,G\,[B]\,\{\phi\}\,dx \quad \text{(If J constant)}$$

or

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^R r^3\;\; dr\, d\theta \int_0^L \{\phi^T\}\,[B^T]\,G\,[B]\,\{\phi\}\,dx$$

$$r = r_0\left(1 + \frac{x}{L}\right) \qquad dr = \frac{r_0}{L}\; dx$$

$$\frac{\partial U}{\partial \phi_{1x}} = \int_0^{2\pi}\int_0^L \left(r_0 + r_0\,\frac{x}{L}\right)^3 \quad \frac{r_0}{L}\,dx\,\frac{GL}{2}\,(2\,\phi_{1x} - 2\,\phi_{2x})$$

$$\frac{\partial U}{\partial \phi_{2x}} = 2\pi\,\frac{r_0^{\,4}}{L}\int_0^L \left(1 + \frac{x}{L}\right)^3 dx\,\frac{GL}{2L^2}\,(-2\phi_{1x} + 2\phi_{2x})$$

Let $$u = 1 + \frac{x}{L},\ du = \frac{dx}{L}$$

$$\int_1^2 u^3 L\, du = \left.\frac{u^4}{4}\,L\right|_1^2$$

$$\therefore\;\; \frac{\partial U}{\partial \phi_{1x}} = \frac{2\,\pi\,r_0^{\,4}}{4}\,\frac{L}{4}\;\left. u^4\right|_1^2\;\; \frac{G}{2L}\,(2\phi_{1x} - 2\phi_{2x})$$

$$= J_0(16 - 1)\;\frac{G}{L}\,(\phi_{2x} - \phi_{1x})$$

$$\frac{\partial U}{\partial \phi_{2x}} = J_0(16 - 1)\;\frac{G}{L}\,(\phi_{2x} - \phi_{1x})$$

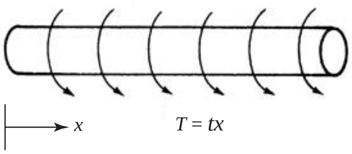
$$\therefore \quad [K] = \frac{15\,G\,J_0}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

5.47

$$U = \frac{1}{2}\int \frac{T^2}{2\,GJ^2}\left(\int r^2\,dA\right)\,dx$$

$$= \frac{1}{2}\int \frac{T^2}{2\,GJ}\,dx$$

$$T = t\,x \qquad t\left(\frac{\text{lb}\cdot\text{in.}}{\text{in.}}\right)$$

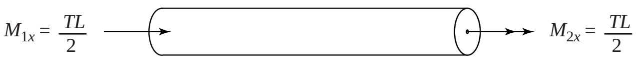


$$U = \frac{1}{2}\int_0^L \frac{(-\,tx)^2}{GJ}\,dx$$

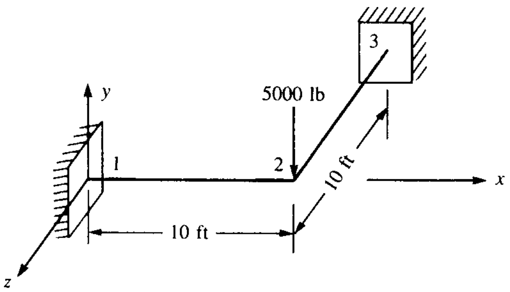
$$= \frac{1}{2\,GJ}\int_0^L (-\,tx)^2\,dx$$

$$= \frac{t^2}{2\,GJ}\,\frac{x^3}{3}\bigg|_0^L$$

$$U = \frac{t^2\,L^3}{6\,GJ} \text{ Total strain energy}$$



5.48



NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	1 1 1	0.0000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0 0 0	120.000000	0.000000	0.000000	–5.000000	0.000000	120.000000
3	1 1 1	120.000000	0.000000	–120.000000	0.000000	0.000000	120.000000

ELEMENTS

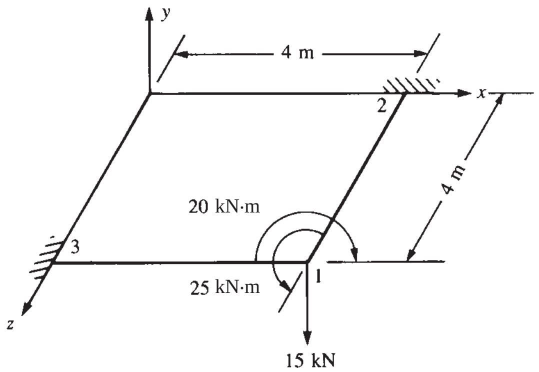
K	NODE(I,K)		E(K)	XI(K)	XJ(K)	G(K)
1	1	2	3.0000000E+04	2.0000000E+02	1.0000000E+02	1.0000000E+04
2	2	3	3.0000000E+04	2.0000000E+02	1.0000000E+02	1.0000000E+04

NODE	DISPLACEMENT	THETA-X	THETA-Z
1	0.00000E+00	0.00000E+00	0.00000E+00
2	–0.21429E+00	0.25714E–02	–0.25714E–02
3	0.00000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (I,K)		Y-FORCE	X-MOMENT	Z-MOMENT	Z-MOMENT	Y-FORCE	X-MOMENT
1	1	2	0.2500E+01	–0.2143E+02	0.2786E+03	0.2143E+02	–0.2500E+01	0.2143E+02
2	2	3	–0.2500E+01	0.2143E+02	–0.2143E+02	–0.2786E+03	0.2500E+01	–0.2143E+02

5.51



NUMBER OF ELEMENTS = 2
NUMBER OF NODES = 3

NODE POINTS

K	IFIX	XC(K)	YC(K)	ZC(K)	FORCE(1,K)	FORCE(2,K)	FORCE(3,K)
1	0 0 0	4.000000	0.000000	4.000000	–15.000000	25.000000	–20.000000
2	1 1 1	4.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	1 1 1	0.000000	0.000000	4.000000	0.000000	0.000000	0.000000

ELEMENTS

K	NODE(I,K)		E(K)	G(K)	A(K)	XI(K)	XJ(K)
1	1	2	2.1000000E+08	8.4000000E+07	9.9999998E–03	1.9999999E–04	9.9999997E–05
2	1	3	2.1000000E+08	8.4000000E+07	9.9999998E–03	1.9999999E–04	9.9999997E–05

NODE	DISPLACEMENT	THETA-X	THETA-Z
1	–0.69048E–02	0.30329E–02	–0.29195E–02
2	0.000000E+00	0.00000E+00	0.00000E+00
3	0.000000E+00	0.00000E+00	0.00000E+00

ELEMENTS

K	NODE (I,K)		Y-FORCE	X-MOMENT	Y-MOMENT	Y-FORCE	X-MOMENT	Z-MOMENT
1	1	2	–0.6607E+01	0.6131E+01	0.1863E+02	0.6607E+01	–0.6131E+01	–0.4506E+02
2	1	3	–0.8393E+01	–0.6369E+01	0.1387E+02	0.8393E+01	0.6369E+01	–0.4744E+02

5.52

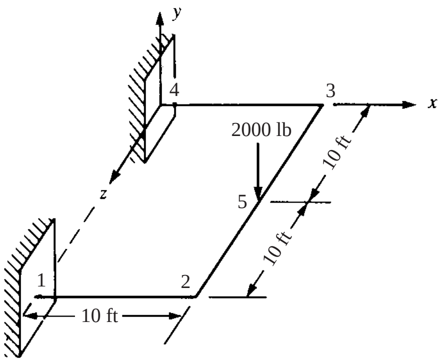
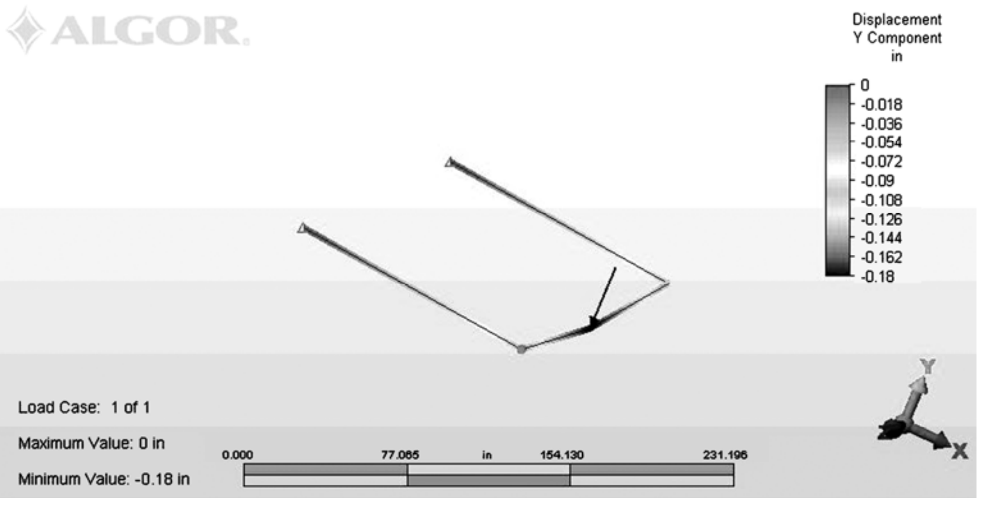


Figure P5–52



5.58-5.59 Determine the displacements and reactions for the space frames shown in Figures P5–58 and P5–59. Let I_x = 100 in.^{4}, I_y = 200 in.^{4}, I_z = 1000 in.^{4}, E = 30,000 ksi, G = 10,000 ksi, and A = 100 in.^{2} for both frames.

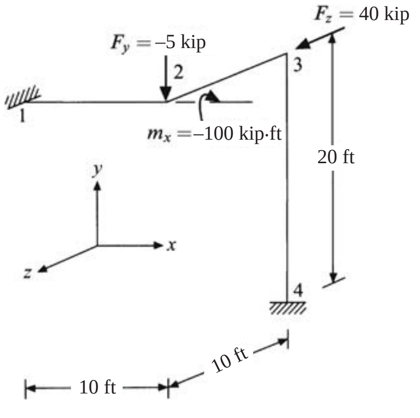


Figure P5–58

Displacements/Rotations (degrees) of nodes

NODE number	X– translation	Y– translation	Z– translation	X– rotation	Y– rotation	Z– rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	6.8927E–01	–2.6320E–03	4.4137E–01	3.1601E–01	–7.5875E–01	–4.8913E–01

3	8.3682E–01	1.0091E+00	–5.8406E–02	–3.6119E–01	1.1418E–01	–4.8006E–01
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.59

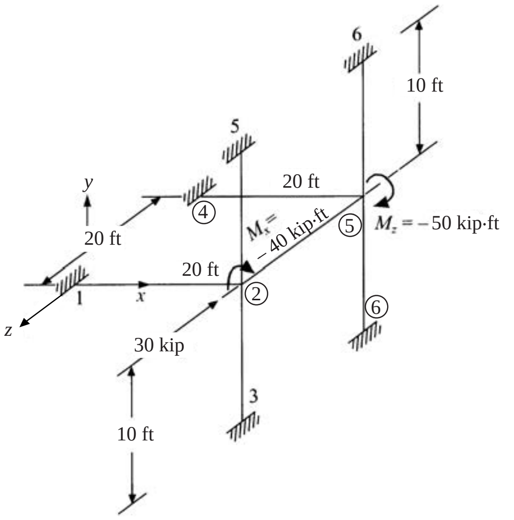
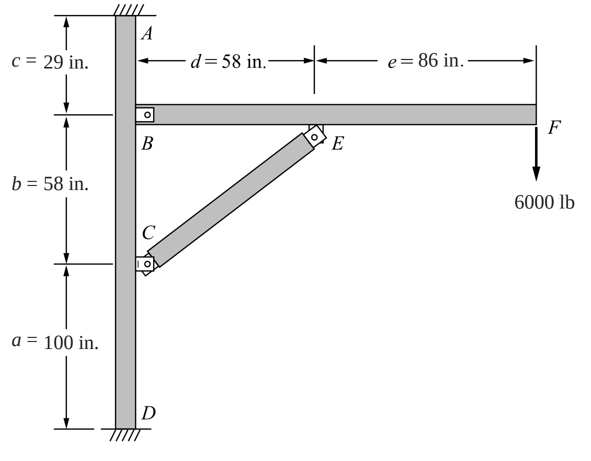


Figure P5–59

Displacements/Rotations (degrees) of nodes

NODE number	X–translation	Y–translation	Z–translation	X–rotation	Y–rotation	Z–rotation
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	3.1055E–04	2.5111E–01	1.0906E–05	–1.1112E–01	2.1045E–08	3.6115E–02
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5	–3.1055E–04	2.4992E–01	–2.5905E–05	1.1109E–02	6.8756E–02	3.5829E–02
6	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
7	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
8	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

5.60 Design a jib crane as shown in Figure P5–60 that will support a downward load of 6000 lb. Choose a common structural steel shape for all members. Use allowable stresses of 0.66 S_y (S_y is the yield strength of the material) in bending, and 0.60 S_y in. tension



FigureP5–60

Horizontal load beam	$S12 \times 50$
Vertical support beam	$S12 \times 50$
Cross brace	$S6 \times 12.5$

All members A36 structural steel

- The required maximum deflection governed the selection of the material section size, as smaller sizing would be lighter and more than adequate to support the load of 6000 lb, but provide too much deflection than the required 0.400 in.
- The force in the cross brace (21066 lbf) does not yield to buckling as shown in the first set of calculations and with a $S6 \times 12.5$ section the cross brace is designed above the imposed load of 6500 lb
- The horizontal load beam is designed to withstand above the imposed bending moment of 516000 lbf·in. The minimum required section of $S10 \times 25.4$ was exceeded, as shown in the second set of calculations, to accommodate the required deflection constraint. Also, the excessive section will allow additional safety against failure from overloading.

5.61 Design the support members, *AB* and *CD*, for the platform lift shown in Figure P5-61. Select a mild steel and choose suitable cross-sectional shapes with no more than a 4 : 1 ratio of moments of inertia between the two principal directions of the cross section. You may choose two different cross sections to make up each arm to reduce weight. The actual structure has four support arms, but the loads shown are for one side of the platform with the two arms shown. The loads shown are under operating conditions. Use a factor of safety of 2 for human safety. In developing the finite element model, remove the platform and replace it with statically equivalent loads at the joints at *B* and *D*. Use truss elements or beam elements with low bending stiffness to model the arms from *B* to *D*, the intermediate connection, *E* to *F*, and the hydraulic actuator. The allowable stresses are $0.66S_y$ in bending and $0.60S_y$ in tension. Check buckling using either Euler's method or Johnson's method as appropriate. Also check maximum deflections. Any deflection greater than $\frac{1}{360}$ of the length of member *AB* is considered too large.

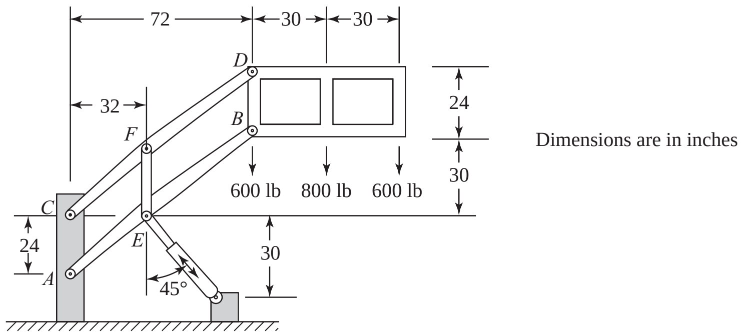


Figure P5-61

- Many viable solutions are possible.
- This design recommends 1020 steel with cross sections of 5 in. $\times$ 3 in. $\times$ $\frac{3}{8}$ in. rectangular tubing.
- The maximum deflection is then 0.244 in. which is less than the maximum allowable of 0.25 in.
- The bending stress is 11,418 psi which is less than the allowable of 31,600 psi.
- The axial stress is 2520 psi which is less than the allowable of 28,700 psi.

5.62 A two-story building frame is to be designed as shown in Figure P5-62. The members are all to be I-beams with rigid connections. We would like the floor joists beams to have a 15-in. depth and the columns to have a 10 in. width. The material is to be A36 structural steel. Two horizontal loads and vertical loads are shown. Select members such that the allowable bending in the beams is 24,000 psi. Check buckling in the columns using Euler's or Johnson's method as appropriate. The allowable deflection in the beams should not exceed $\frac{1}{360}$ of each beam span. The overall sway of the frame should not exceed 0.5 in.

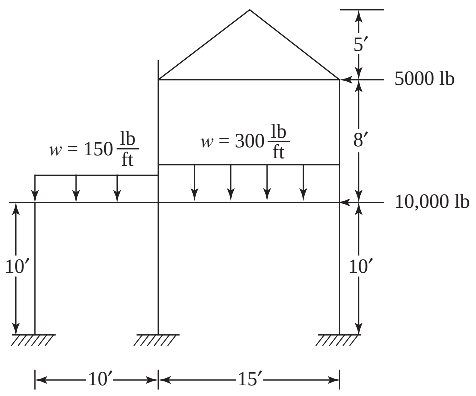


Figure P5-62

- Many viable solutions are possible.
- This design recommends A36 structural steel I-beams.
- W 16×26 beams are recommended for the horizontal and diagonal members with largest bending stress of 7000 psi which is less than the allowable of 24,000 psi.
- W 10×49 sections are recommended for the vertical members. Column buckling was verified to be satisfied.
- Maximum sway in the horizontal direction is 0.246 in. which is less than the allowable of 0.50 in.
- Another satisfactory solution is W 10×26 beams for horizontal and diagonal members and W 10×33 sections for the vertical members. The sway then becomes 0.417 in.

5.63 A pulpwood loader as shown in Figure P5–63 is to be designed to lift 2.5 kip. Select a steel and determine a suitable tubular cross section for the main upright member *BF* that has attachments for the hydraulic cylinder actuators *AE* and *DG*. Select a steel and determine a suitable box section for the horizontal load arm *AC*. The horizontal load arm may have two different cross sections *AB* and *BC* to reduce weight. The finite element model should use beam elements for all members except the hydraulic cylinders, which should be truss elements. The pinned joint at *B* between the upright and horizontal beam is best modeled with end release of the end node of the top element on the upright member. The allowable bending stress is 0.66 S_y in members *AB* and *BC*. Member *BF* should be checked for buckling. The allowable deflection at *C* should be less than $\frac{1}{360}$ of the length of *BC*. As a bonus, the client would like you to select the size of the hydraulic cylinders *AE* and *DG*.

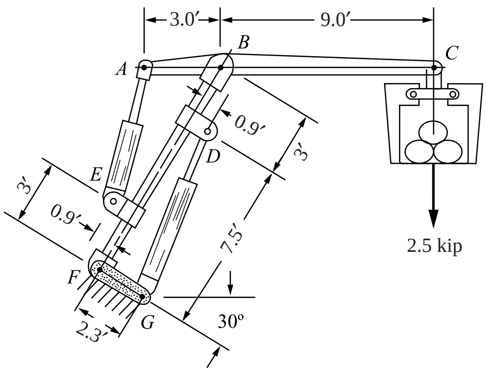


Figure P5–63

- Many viable solutions are possible.
- The design recommends AISI 1020 rolled steel.
- The horizontal beam, AC, is recommended to be a rectangular tube 4 in. by 16 in. with 0.25 in. thickness. The maximum bending stress in member AC is 7736 psi less than the allowable of 3100 psi. The maximum deflection is 0.299 in. less than the allowable of 0.300 in.
- The vertical member, BF, is recommended to be a square tube 10 in. by 10 in. with 0.5 in. thickness.

5.65 A small hydraulic floor crane as shown in Figure P5-65 carries a 5000 lb load. Determine the size of the beam and column needed. Select either a standard box section or a wide-flange section. Assume a rigid connection between the beam and column. The column is rigidly connected to the floor. The allowable bending stress in the beam is $0.60S_y$. The allowable deflection is $\frac{1}{360}$ of the beam length. Check the column for buckling.

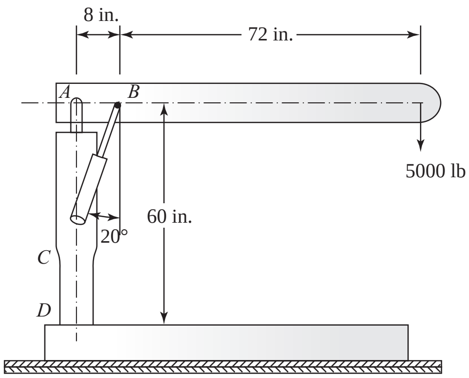


Figure P5-65

- Many viable solutions are possible.
- The design recommends A36 structural steel.
- The horizontal and vertical members are recommended to be W 10×68.
- The largest bending stress in the horizontal beam is 4756 psi less than the allowable of 21,600 psi. The maximum deflection is 0.215 in. less than the allowable of 0.222 in.
- The column, ACD, has a bending stress of 5284 psi.
- The column should be checked for buckling.

5.68 Design the gabled frame subjected to the external wind load shown (comparable to an 80 mph wind speed) for an industrial building. Assume this is one of a typical frame spaced every 20 feet. Select a wide flange section based on allowable bending stress of 20 ksi and an allowable compressive stress of 10 ksi in any member. Neglect the possibility of buckling in any members. Use ASTM A36 steel.

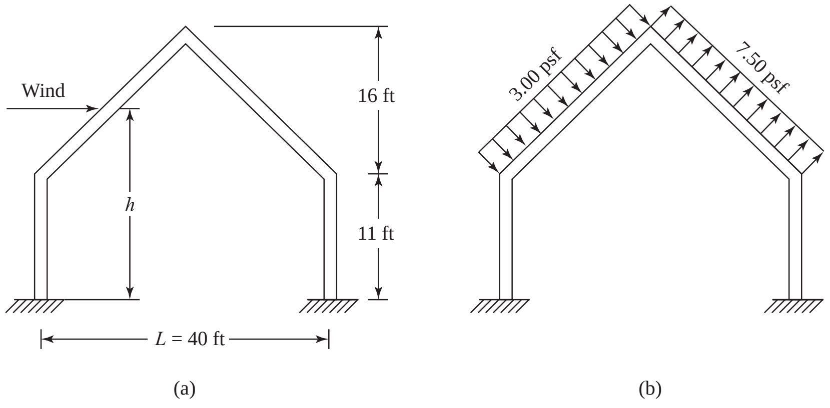


Figure P5-68

- Many viable solutions are possible.
- The design recommends A36 structural steel.
- The frame members are recommended to be W 10×12 wide flange shapes.
- The maximum worst stress (combined bending and compression) in any member is 19.5 ksi.
- The maximum displacement is 0.719 in.

5.69 Design the gabled frame shown for a balanced snow load shown (typical of the Midwest) for an apartment building. Select a wide flange section for the frame. Assume the allowable bending stress not to exceed 140 MPa. Use ASTM A36 steel.

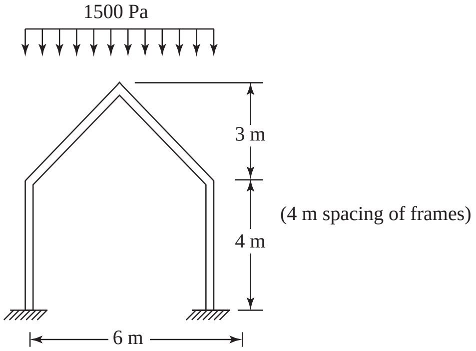


Figure P5-69

- Many viable solutions are possible.
- The table below lists some W sections that were considered.
- The recommended W 6×12 with bending stress of 120.4 MPa is less than the allowable value of 140 MPa.
- The maximum displacement is 0.0147 m.

Table 1: Beam Trial Runs

Beam Section	Bending Stress (Local 3)	Displacement
W30 × 173	1.6 MPa	6.26*10^–5
W12 × 45	15.2 MPa	0.0010 m
W8 × 13	88.7 MPa	0.0082 m
W6 × 12	120.4 MPa	0.0147 m

5.70 Design a gantry crane that must be able to lift 10 tons as it must lift compressors, motors, heat exchangers, and controls. This load should be placed at the center of one of the main 12-foot-long beams as shown in Figure P5-70 by the hoisting device location. Note that this beam is on one side of the crane. Assume you are using ASTM A36 structural steel.

- Many viable solutions are possible.
- The design is based on the load being applied to the center span of a 12 ft long beam.
- The design recommends W 10×100 for the horizontal beams.
- The bracing members are recommended to be W 4×13.
- The vertical columns are recommended to be 4 in. $\times$ 4 in. $\times$ $\frac{1}{4}$ in. thick hollow square tubes.
- The table below is a summary of the final member sizes and deflection, stress, and buckling calculated and allowable results.

Table 1: This table shows all the members with corresponding material and size.

Member	Quantity	Material	Size (in. $\times \frac{\text{lb}}{\text{ft}}$)
loaded 12 ft beam	1	ASTM A36 St. Steel	$W10 \times 100$
unloaded 12 ft beam	1	ASTM A36 St. Steel	$W10 \times 100$
8 ft Beams	2	ASTM A36 St. Steel	$W10 \times 100$
Corner Braces	8	ASTM A36 St. Steel	$W4 \times 13$
Columns	4	ASTM A36 St. Steel	4×4 hollow $\times \frac{1}{4}$ thick (in.)

Table 2: This table shows that the maximum deflections are less than the allowable deflection, and that the calculated bending stresses are less than the allowable stresses in the beams.

	Calculations				
	Maximum Deflection (in.)		Allowable Deflection (in.)	Bending Stress (psi)	
Member	By Hand	Using Algor		Calculated	Allowable
Loaded 12 ft Beam	0.0722	0.0847	0.2667	6405	7200
Unloaded 12 ft Beam	–	0.0141	0.2667	23.145	7200
8 ft Beams	–	0.01977	0.2667	1.286	7200
Corner Braces	–	0.02921	0.1333	256.654	7200
Columns	–	0.2863	0.4000	–	–

Table 3: This table shows that the corner braces and the columns have loads smaller than the load that would cause buckling.

	Buckling Strength (lb)	
Member	Calculated load	Allowable load
Loaded 12 ft Beam	–	–
Unloaded 12 ft Beam	–	–
8 ft Beams	–	–
Corner Braces	24000	330000
Columns	24572	60000

5.71 Design the rigid highway bridge frame structure shown in Figure P5-71 for a moving truck load (shown below) simulating a truck moving across the bridge. Use the load shown and place it along the top girder at various locations. Use the allowable stresses in bending and compression and allowable deflection given in the *Standard Specification for Highway Bridges*, American Association of State Highway and Transportation Officials (AASHTO), Washington, D.C. or use some other reasonable values.

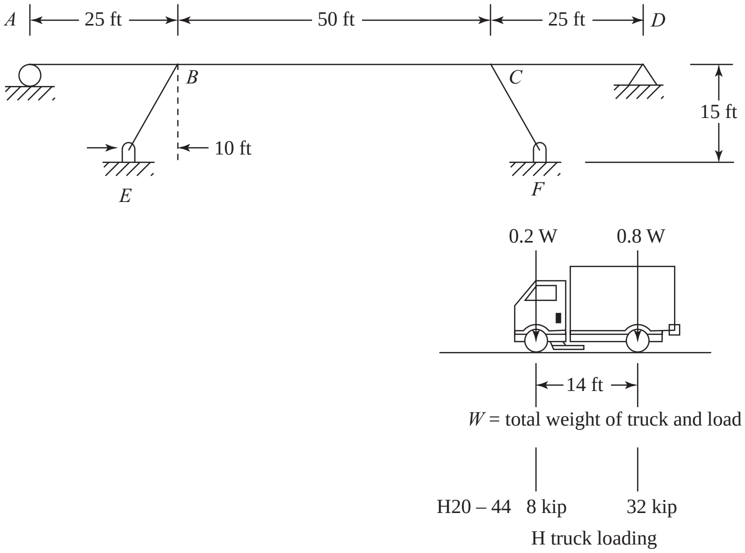


Figure P5-71

- Many viable solutions are possible.
- A36 structural steel is chosen in the design.
- After some iteration, W 24×94 wide flange sections were selected for all members.
- The largest bending stress of 12960 psi with the truck in the center span location is less than the allowable of 20,000 psi.
- The largest deflection of 0.731 in. is less than the allowable of 0.75 in. ($\frac{1}{800}$ of the span length).

5.73 The curved semi-circular frame shown in Figure P 5-73 is supported by a pin on the left end and a roller on the right end and is subjected to a load P = 1000 lb at its apex. The frame has a radius to centerline of cross section of R = 120 in. Select a structural steel W shape from Appendix F such that the maximum stress does not exceed 20 ksi. Perform a finite element analysis using 4, 8, and then 16 elements in your finite element model. Also determine the maximum deflection for each model. It is suggested that the finite element answers for deflection be compared to the solution obtained by classical methods, such as using Castigliano's theorem. The expression for deflection under the load is given by using Castigliano's theorem as

$$\delta_y = \frac{0.178\, PR^3}{EI} + \frac{0.393\, PR}{AE} + \frac{0.393\, PR}{A_v G}$$

where A is the cross sectional area of the W shape, A_v is the shear area of the W shape (use depth of web times thickness of web for the shear area), $E = 30 \times 10^6$ psi, and $G = 11.5 \times 10^6$ psi.

Now change the radius of the frame to 20 in. and repeat the problem. Run the finite element model with the shear area included in your computer program input and then without. Comment on the difference in results and compare to the predicted analytical deflection by using the equation above for δ_y.

For R = 20 in. A = 8.79 in. t_{web} = 0.260 in. depth = 12.34 in.

$SA2 = t_{\text{web}}$ depth $SA2 = 3.2084 \times 10^0$ in.

$$I_x = 238 \text{ in.}^4 \quad E = 29 \times 10^6 \text{ psi} \quad G = 11.6 \times 10^6 \text{ psi} \quad P = 1000 \text{ lb}$$

$$\delta_m = \frac{0.178\, PR^3}{EI_x} \qquad \delta_m = 2.06317 \times 10^{-4} \text{ in.}$$

$$\delta_n = \frac{0.393\, PR}{AE} \qquad \delta_n = 3.08344 \times 10^{-5} \text{ in.}$$

$$\delta_v = \frac{0.393\, PR}{SA2G} \qquad \delta_v = 2.11191 \times 10^{-4} \text{ in.}$$

$$\delta_{\max} = \delta_m + \delta_n + \delta_v$$

$$\delta_{\max} = 4.48343 \times 10^{-4} \text{ in.}$$

$$\delta_{\text{maxnoshear}} = \delta_m + \delta_n$$

$$\delta_{\text{maxnoshear}} = 2.37151 \times 10^{-4} \text{ in.}$$

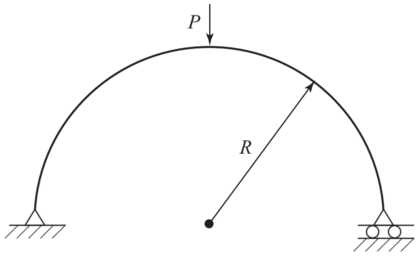


Figure P5-73

- The table below shows the results for the 2 – 16 element models without shear area and with shear area included in some cases for both radii.

Radius = 120 in.

No. of Elements	Max Def. (in.)	Max. Stress (psi)
2	5.92E-02	1595
4	4.69E-02	1576
8	4.52E-02	1566
16	4.48E-02	1560
8 (SA2 incl.)	4.65E-02	1566
16 (SA2 incl.)	4.62E-02	1560
Longhand	4.48E-02	
Longhand (SA2 incl.)	4.60E-02	

Radius = 20 in.

No. of Elements	Max Def. (in.)	Max. Stress (psi)
2	3.01E-04	299
4	2.47E-04	281
8	2.39E-04	270
16	2.39E-04	265
8 (SA2 incl.)	4.55E-04	270
16 (SA2 incl.)	4.55E-04	265
Longhand	2.37E-04	
Longhand (SA2 incl.)	4.48E-04	

Chapter 6

6.1 For sketch of N_i see Figure 6.8. Others follow similarly

By Equation (6.2.18)

$$N_i + N_j + N_m = \frac{1}{2A}\ [(\alpha_i + \alpha_j + \alpha_m) + (\beta_i + \beta_j + \beta_m)\ x + [\gamma_i + \gamma_j + \gamma_m)y] \qquad (1)$$

By Equation (6.2.10)

$$\alpha_i + \alpha_j + \alpha_m = x_j\,y_m - y_j\,x_m + y_i\,x_m - x_i\,y_m + x_i\,y_j - y_i\,x_j$$

$$= 2\text{A (by Equation (6.2.9))} \qquad (2)$$

$$\beta_i + \beta_j + \beta_m = y_j - y_m + y_m - y_i + y_i - y_j = 0 \qquad (3)$$

$$\gamma_i + \gamma_j + \gamma_m = x_m - x_j + x_i - x_m + x_j - x_i = 0 \qquad (4)$$

By using (2)–(4) in (1), we obtain

$$N_i + N_j + N_m = 1 \text{ identically}$$

6.2 By Equation (6.2.47)

$$\pi_p = \frac{1}{2}\ \{d\}^T \iiint_v\ [B]^T [D]\ [B]\ dV\{d\} - \{d\}^T\ \{f\}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix},\ \{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\therefore \quad \pi_p = \frac{1}{2}\ [u_i\ v_i\ u_j\ v_j\ u_m\ v_m] \iiint_v \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_j & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix}$$

$$\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_i & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} dV \times \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$-\ [u_i\ v_i\ u_j\ v_j\ u_m\ v_m]\ \{\ f\}$$

$$\frac{\partial\pi}{\partial u_i} = \underbrace{2\,\frac{1}{2}\left(\frac{1}{2A}\right)\left(\frac{E}{1-v^2}\right)\frac{1}{2A}}_{C} \int_v [\beta_i \;\; 0 \;\; \gamma_i] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_i \\ 0 \\ \gamma_i \end{Bmatrix} dV\, u_i - f_{1x}$$

$$\frac{\partial\pi}{\partial v_i} = 2C \int_v [0 \;\; \gamma_i \;\; \beta_i] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_i \\ \beta_i \end{Bmatrix} dV\, v_i - f_{1y}$$

$$\frac{\partial\pi}{\partial u_i} = 2C \int_v [\beta_j \;\; 0 \;\; \gamma_j] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_i \\ 0 \\ \gamma_j \end{Bmatrix} dV\, u_j - f_{2x}$$

$$\frac{\partial\pi}{\partial v_j} = 2C \int_v [0 \;\; \gamma_j \;\; \beta_j] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_i \\ \beta_i \end{Bmatrix} dV\, v_j - f_{2y}$$

$$\frac{\partial\pi}{\partial u_m} = 2C \int_v [\beta_m \;\; 0 \;\; \gamma_m] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \beta_m \\ 0 \\ \gamma_m \end{Bmatrix} dV\, u_m - f_{3x}$$

$$\frac{\partial\pi}{\partial v_m} = 2C \int_v [0 \;\; \gamma_m \;\; \beta_m] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ \gamma_m \\ \beta_m \end{Bmatrix} dV\, v_m - f_{3y}$$

$$\therefore \quad \frac{\partial\pi}{\partial\{d\}} = 2C \int_v \begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_i & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{bmatrix} \beta_i & 0 & \beta_i & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_m & \gamma_m & \beta_m \end{bmatrix}$$

$$\times\, dV \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} - \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

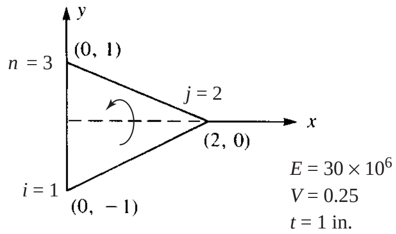
From Equation (3.10.27) or Equation (6.2.48)

$$\frac{\partial\pi}{\partial\{d\}} = 0$$

$$\therefore \quad \frac{\partial\pi}{\partial\{d\}} = \int_v [B]^T\, [D]\, [B]\, dV\{d\} - \{f\} = 0$$

6.3

(a)



(a)

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$$x_i = 0,\ y_i = -1,\ x_j = 2,\ y_j = 0,\ x_m = 0,\ y_m = 1$$

$$A = \frac{1}{2}\,b\,h = \frac{1}{2}(2)(2) = 2\text{ in.}^2$$

$$\beta_i = y_j - y_m = 0 - 1 = -1$$

$$\beta_j = y_m - y_i = 1 - (-1) = 2$$

$$\beta_m = y_i - y_j = -1 - 0 = -1$$

$$\gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\gamma_m = x_j - x_i = 2 - 0 = 2$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Since it is plane stress $[D] = \dfrac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$

$$\text{So } [B]^T\,[D] = \frac{30\times10^6}{4(0.9375)}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -0.25 & -0.75 \\ -0.5 & -2 & -0.375 \\ 2 & 0.5 & 0 \\ 0 & 0 & 0.75 \\ -1 & -0.25 & 0.75 \\ 0.5 & 2 & -0.375 \end{bmatrix}\frac{30\times10^6}{4(0.9375)}$$

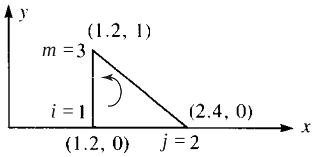
$$[k] = t\,A[B]^T\,[D]\,[B]$$

$$\Rightarrow \quad [k] = (1\,\text{in.})(2)\frac{30\times 10^6}{4(0.9375)}\begin{bmatrix} -1 & -0.25 & -0.75 \\ -0.5 & -2 & -0.375 \\ 2 & 0.5 & 0 \\ 0 & 0 & 0.75 \\ -1 & -0.25 & 0.75 \\ 0.5 & 2 & -0.375 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$[k] = 4.0\times 10^6 \begin{array}{c} \begin{array}{ccc} \quad i=1 \quad\quad & \quad j=2 \quad\quad & \quad m=3 \end{array} \\ \begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} \end{array}$$

(b) $x_i = 1.2$, $y_i = 0$, $x_j = 2.4$, $y_j = 0$, $x_m = 1.2$, $y_m = 1$



(b)

$$\beta_i = y_j - y_m = 0 - 1 = -1$$

$$\beta_j = y_m - y_j = 1 - 0 = 1$$

$$\beta_m = y_i - y_j = 0 - 0 = 0$$

$$\gamma_i = x_m - x_j = 1.2 - 2.4 = -1.2$$

$$\gamma_j = x_i - x_m = 1.2 - 1.2 = 0$$

$$\gamma_m = x_j - x_i = 2.4 - 1.2 = 1.2$$

$$A = \frac{1}{2}(1.2)(1) = 0.6\ \text{in.}^2$$

$$\text{So } [B]^T[D] = \frac{30\times 10^6}{(1.2)(0.9375)}\begin{bmatrix} -1 & 0 & -1.2 \\ 0 & -1.2 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1.2 \\ 0 & 1.2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= \frac{25\times10^6}{0.9375}\begin{bmatrix} -1 & -0.25 & -0.45 \\ -0.3 & -1.2 & -0.375 \\ 1 & 0.25 & 0 \\ 0 & 0 & 0.375 \\ 0 & 0 & 0.45 \\ 0.3 & 1.2 & 0 \end{bmatrix}$$

$$[k] = t\,A[B]^T\,[D]\,[B]$$

$$[k] = \frac{(1\text{in.})(6\text{in.}^2)\,25\times10^6}{2(0.6)\,(0.9375)}\begin{bmatrix} -1 & -0.25 & -0.45 \\ -0.3 & -1.2 & -0.375 \\ 1 & 0.25 & 0 \\ 0 & 0 & 0.375 \\ 0 & 0 & 0.45 \\ 0.3 & 1.2 & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1.2 & 0 & 0 & 0 & 1.2 \\ -1.2 & -1 & 0 & 1 & 1.2 & 0 \end{bmatrix}$$

$$[k] = \frac{25\times10^6}{1.875}\begin{array}{c} \begin{array}{cccccc} \quad i=1 & & \quad j=2 & & \quad m=3 & \end{array} \\ \begin{bmatrix} 1.54 & 0.75 & -1 & -0.45 & -0.54 & -0.3 \\ 0.75 & 1.815 & -0.3 & -0.375 & -0.45 & -1.44 \\ -1 & -0.3 & 1 & 0 & 0 & 0.3 \\ -0.45 & -0.375 & 0 & 0.375 & 0.45 & 0 \\ -0.54 & -0.45 & 0 & 0.45 & 0.54 & 0 \\ -0.3 & -1.44 & 0.3 & 0 & 0 & 1.44 \end{bmatrix} \end{array}$$

(c) $E = 30\times10^6 \qquad \nu = 0.25 \qquad t = 1$

Triangle coordinate definition

$i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x = 0 \\ y = 1 \end{matrix}$ This defines an array variable
x coordinate is the top
y coordinate is the bottom

$j = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ Area of triangle = $\frac{1}{2}$ base × height

$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A = \frac{1}{2}\,(j_x - i_x)\,(m_y - i_y)$

$A = 1$

Develop stiffness matrix

$\beta_i = j_y - m_y \qquad \beta_i = -1 \qquad \gamma_i = m_x - j_x \qquad \gamma_i = -2$

$\beta_j = m_y - i_y \qquad \beta_j = 1 \qquad \gamma_j = i_x - m_x \qquad \gamma_j = 0$

$\beta_m = i_y - j_y \qquad \beta_m = 0 \qquad \gamma_m = j_x - i_x \qquad \gamma_m = 2$

$$[B_i] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix} \qquad [B_j] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_j \end{pmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{pmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{pmatrix}$$

Gradient matrix

$$[B] = \text{augment}\ (B_i, B_j, B_m)$$

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -0.5 & 0 & 0.5 & 1 & 0 \end{pmatrix}$$

Plane stress

Constitutive matrix

$$[D] = \frac{E}{1-v^2}\begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{pmatrix}$$

$$[D] = \begin{pmatrix} 3.2\times10^7 & 8\times10^6 & 0 \\ 8\times10^6 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix}$$

$[k] = t\,A\,[B]^T\,[D]\,[B]$ Constant-strain triangular element stiffness matrix

$$[k] = \begin{pmatrix} 2\times10^7 & 1\times10^7 & -8\times10^6 & -6\times10^6 & -1.2\times10^7 & -4\times10^6 \\ 1\times10^7 & 3.5\times10^7 & -4\times10^6 & -3\times10^6 & -6\times10^6 & -3.2\times10^7 \\ -8\times10^6 & -4\times10^6 & 8\times10^6 & 0 & 0 & 4\times10^6 \\ -6\times10^6 & -3\times10^6 & 0 & 3\times10^6 & 6\times10^6 & 0 \\ -1.2\times10^7 & -6\times10^6 & 0 & 6\times10^6 & 1.2\times10^7 & 0 \\ -4\times10^6 & -3.2\times10^7 & 4\times10^6 & 0 & 0 & 3.2\times10^7 \end{pmatrix}$$

6.4 In general we know that

$$\{\sigma\} = \frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \times \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(a) For the first element we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30\times 10^6}{(1-0.25^2)} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0.0025 \\ 0.0012 \\ 0 \\ 0.0 \\ 0.0025 \end{bmatrix} = \begin{Bmatrix} 19200 \text{ psi} \\ 4800 \text{ psi} \\ -15000 \text{ psi} \end{Bmatrix}$$

The principal stresses are given by the equations

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}}$$

and the plane that are acting upon is

$$\theta_p = \frac{1}{2} \tan^{-1} \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2} \right)}$$

$$\sigma_1 = \frac{19200 + 4800}{2} + \left[\left(\frac{19200 - 4800}{2} \right)^2 + (-15000)^2 \right]^{\frac{1}{2}}$$

$$= 28639 \text{ psi}$$

$$\sigma_2 = \frac{19200 + 4800}{2} - \left[\left(\frac{19200 - 4800}{2} \right)^2 + (-15000)^2 \right]^{\frac{1}{2}}$$

$$= -4639 \text{ psi}$$

$$\theta_p = \frac{1}{2} \tan^{-1} \frac{-15000}{\left(\frac{19200 - 4800}{2} \right)} = -32.2°$$

(b) For the second element we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30\times 10^6}{(1-0.25)^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \frac{1}{2(0.6)} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1.2 & 0 & 0 & 0 & 1.2 \\ -1.2 & -1 & 0 & 1 & 1.2 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 0.0025 \\ 0.0012 \\ 0 \\ 0.0 \\ 0.0025 \end{bmatrix}$$

$$= \begin{Bmatrix} \sigma_x = 32000 \text{ psi} \\ \sigma_y = 8000 \text{ psi} \\ \tau_{xy} = -25000 \text{ psi} \end{Bmatrix}$$

$$\sigma_1 = \frac{32000+8000}{2} + \left[\left(\frac{32000-8000}{2}\right)^2 + (-25000)^2\right]^{\frac{1}{2}}$$

$$\sigma_1 = 47731 \text{ psi}$$

$$\sigma_2 = \frac{32000+8000}{2} - \left[\left(\frac{32000-8000}{2}\right)^2 + (-25000)^2\right]^{\frac{1}{2}}$$

$$\sigma_2 = -7731 \text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{-25000}{\left(\frac{32000-8000}{2}\right)}$$

$$\theta_p = -32.2°$$

(c) For third element we have

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -0.5 & 0 & 0.5 & 1 & 0 \end{pmatrix}\frac{1}{\text{in.}}$$

$$[D] = \begin{pmatrix} 3.2\times10^7 & 8\times10^6 & 0 \\ 8\times10^6 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix}\frac{\text{lb}}{\text{in.}^2}$$

$u_1 = 0.0$ in. $\qquad v_1 = 0.0025$ in.

$u_2 = 0.0012$ in. $\qquad v_2 = 0.0$ in.

$u_3 = 0.0$ in. $\qquad v_3 = 0.0025$ in.

$$\{d\} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \text{ Displacement matrix}$$

Stress evaluation

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} 1.92\times10^4 \\ 4.8\times10^3 \\ -1.5\times10^4 \end{pmatrix}\frac{\text{lb}}{\text{in.}^2}$$

Principal stresses

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{max} = 2.864 \times 10^4 \ \frac{\text{lb}}{\text{in.}^2} = \sigma_1$$

$$\sigma_{min} = -4.639 \times 10^3 \ \frac{\text{lb}}{\text{in.}^2} = \sigma_2$$

Principal angle

$$\theta_p = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_p = -32.179 \text{ deg.}$$

6.5 Von Mises stress for biaxial stress state

(a) $\sigma_1 = 28639$ $\qquad$ $\sigma_2 = -4639$

$\sigma_e = \sqrt{\sigma_1^{\,2} + \sigma_2^{\,2} - \sigma_1\sigma_2}$ $\qquad$ $\sigma_e = 3.122 \times 10^4$ psi

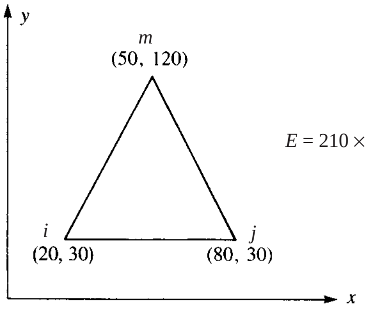
(b) $\sigma_1 = 47731$ $\qquad$ $\sigma_2 = -7731$

$\sigma_e = \sqrt{\sigma_1^{\,2} + \sigma_2^{\,2} - \sigma_1\sigma_2}$ $\qquad$ $\sigma_e = 5.203 \times 10^4$ psi

(c) $\sigma_e = \sqrt{\sigma_1^{\,2} + \sigma_2^{\,2} - \sigma_1\sigma_2}$

$$\sigma_e = 3.122 \times 10^4 \ \frac{\text{lb}}{\text{in.}^2}$$

6.6



(a)

$E = 210 \times 10^9$ Pa, $\nu = 0.25$, $t = 0.010$ m

$\beta_i = y_j - y_m = 30 - 120 = -90$ $\qquad$ $\gamma_i = x_m - x_i = 50 - 80 = -30$

$\beta_j = y_m - y_i = 120 - 30 = 90$ $\qquad$ $\gamma_i = x_i - x_m = 20 - 50 = -30$

$\beta_m = y_i - y_j = 30 - 30 = 0$ $\qquad$ $\gamma_m = x_j - x_i = 80 - 20 = 60$

$$2\text{A} = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$

$$= 20(-90) + 80(90) + 50(0) = 5400 \text{ mm}^2$$

$$[B] = \frac{1}{5400}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[D] = \frac{210 \times 10^9}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} = 2.24 \times 10^{11}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$$[k] = (0.01)\left(\frac{5.4\times10^{-3}}{2}\right)\left(\frac{1}{5.4\times10^{-3}}\right)\begin{bmatrix} -90 & 0 & -30 \\ 0 & -30 & -90 \\ 90 & 0 & -30 \\ 0 & -30 & 90 \\ 0 & 0 & 60 \\ 0 & 60 & 0 \end{bmatrix}(2.24\times10^{11})\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}[B]$$

$$[k] = 1.12\times10^{9}\begin{bmatrix} -90 & -22.5 & -11.25 \\ -7.5 & -30 & -33.75 \\ 90 & 22.5 & -11.25 \\ -7.5 & -30 & 33.75 \\ 0 & 0 & 22.5 \\ 15 & 60 & 0 \end{bmatrix}\frac{1}{5.4\times10^{-3}}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0 \end{bmatrix}$$

$$[k] = 2.074\times10^{5}\begin{bmatrix} 8437.5 & 1687.5 & -7762.5 & -337.5 & -675 & -1350 \\ 1687.5 & 3937.5 & 337.5 & -2137.5 & -2025 & -1800 \\ -7762.5 & 337.5 & 8437.5 & -1687.5 & -675 & 1350 \\ -337.5 & -2137.5 & -1687.5 & 3937.5 & 2025 & -1800 \\ -675 & -2025 & -675 & 2025 & 1350 & 0 \\ -1350 & -1800 & 1350 & -1800 & 0 & 3600 \end{bmatrix}$$

(b) Similarly

$$\beta_i = -5 \qquad \gamma_i = 0$$

$$\beta_j = 2.5 \qquad \gamma_j = -5$$

$$\beta_m = 2.5 \qquad \gamma_m = 5$$

$$[k] = 4.48\times10^{7}\begin{bmatrix} 25.0 & 0 & -12.5 & 6.25 & -12.5 & -6.25 \\ & 9.375 & 9.375 & -4.6875 & -9.375 & -4.6875 \\ & & 15.625 & -7.8125 & -3.125 & -1.5625 \\ & & & 27.343 & 1.5625 & -3.125 \\ & & & & 15.625 & 7.8125 \\ \text{Symmetry} & & & & & 27.343 \end{bmatrix}$$

Now solve P6-6c for stiffness matrix

$$t = 0.01$$

$$A = \frac{tA}{2} \qquad\qquad A = 5\times10^{-5}$$

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$$[k] = \begin{pmatrix} 1.225\times10^9 & 3.5\times10^8 & -1.015\times10^9 & -7\times10^7 & -2.1\times10^8 & -2.8\times10^8 \\ 3.5\times10^8 & 7\times10^8 & 7\times10^7 & -1.4\times10^8 & -4.2\times10^8 & -5.6\times10^8 \\ -1.015\times10^9 & 7\times10^7 & 1.225\times10^9 & -3.5\times10^8 & -2.1\times10^8 & 2.8\times10^8 \\ -7\times10^7 & -1.4\times10^8 & -3.5\times10^8 & 7\times10^8 & 4.2\times10^8 & -5.6\times10^8 \\ -2.1\times10^8 & -4.2\times10^8 & -2.1\times10^8 & 4.2\times10^8 & 4.2\times10^8 & 0 \\ -2.8\times10^8 & -5.6\times10^8 & 2.8\times10^8 & -5.6\times10^8 & 0 & 1.12\times10^9 \end{pmatrix}$$

6.7 (a) By Equation (6.2.36)

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

Using results of Problem 6.5 (a)

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \frac{2.24\times10^{11}}{5400\times10^{-3}}\begin{bmatrix}1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375\end{bmatrix}\begin{bmatrix}-90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & -30 & 90 & 60 & 0\end{bmatrix}$$

$$\times\,10^{-3}\ \frac{\text{m}}{\text{mm}}\begin{Bmatrix}0.002 \\ 0.001 \\ 0.0005 \\ 0 \\ 0.003 \\ 0.001\end{Bmatrix}$$

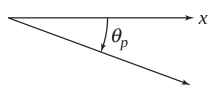
$$= \begin{Bmatrix}-5.29\ \text{GPa} \\ -0.766\ \text{GPa} \\ 0.233\ \text{GPa}\end{Bmatrix}$$

$$\sigma_{1,2} = \frac{-5.29 + (-\,0.156)}{2} \pm \sqrt{\left(\frac{-5.29 + 0.156}{2}\right)^2 + 0.233^2}$$

$$= -\,2.72 \pm 2.58$$

$$\sigma_1 = -\,0.14\ \text{GPa} \qquad \sigma_2 = -\,5.30\ \text{GPa}$$

$$\tan 2\theta_{p_1} = \frac{2(0.233)}{-5.29 + 0.156} = -\,0.091$$



$$\theta_p = -\,2.59°$$

(b) From Problem 6.5 (b) β's and γ's given

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \frac{2.24\times10^{11}}{25\times10^{-6}}\begin{bmatrix}1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375\end{bmatrix}\begin{bmatrix}-5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5\end{bmatrix}$$

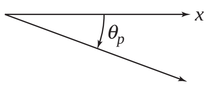
$$\times\,10^{-3}\ \frac{\text{m}}{\text{mm}}\begin{Bmatrix}0.002 \\ 0.001 \\ 0.0005 \\ 0 \\ 0.003 \\ 0.001\end{Bmatrix} = \begin{Bmatrix}0 \\ 42.0\,\text{GPa} \\ 33.6\,\text{GPa}\end{Bmatrix}$$

$$\sigma_{1,2} = \frac{0+42.0}{2} \pm \sqrt{\left(\frac{-42}{2}\right)^2 + 33.6^2}$$

$$= 21 \pm 39.6$$

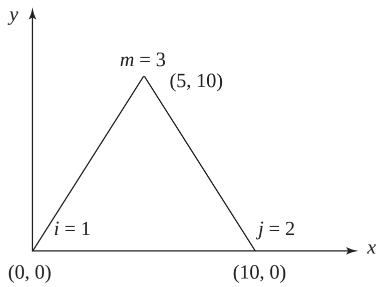
$$\sigma_1 = 60.6 \text{ GPa} \qquad \sigma_2 = -18.6 \text{ GPa}$$

$$\tan 2\theta_p = \frac{2(33.6)}{0-42.0}$$



$$\theta_p = -29°$$

(c)



Plane stress also find von Mises stress

$$\beta_i = -10 \text{ mm} \qquad \gamma_i = -5 \text{ mm}$$

$$\beta_j = 10 \text{ mm} \qquad \gamma_j = -5 \text{ mm}$$

$$\beta_m = 0 \qquad \gamma_m = 10 \text{ mm}$$

$$A = 2.5 \times 10^{-5} \text{ m}^2$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$[D] = \frac{(210\times 10^9 \, \frac{\text{N}}{\text{m}^2})}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 224 \times 10^9 \, \frac{\text{N}}{\text{m}^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Determine the stresses in the element with nodal displacements listed

$u_1 = 0.002$ $v_1 = 0.001$ $u_2 = 0.0005$ $v_2 = 0$ $u_3 = 0.003$ $v_3 = 0.001$

Here are the coordinates for the element

$x_1 = 0$ $y_1 = 0$ $x_2 = 0.01$ $y_2 = 0$ $x_3 = 0.005$ $y_3 = 0.01$

$$E = 210 \times 10^9 \quad v = 0.25$$

$$\{d\} = \begin{pmatrix} u_1 \\ v_2 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \qquad [D] = \begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{pmatrix} \qquad [D] = [D]\frac{E}{1-v^2}$$

Equation (6.1.8)

$$[D] = \begin{pmatrix} 2.24 \times 10^{11} & 5.6 \times 10^{10} & 0 \\ 5.6 \times 10^{10} & 2.24 \times 10^{11} & 0 \\ 0 & 0 & 8.4 \times 10^{10} \end{pmatrix}$$

Equation (6.2.10)

$\beta_1 = y_2 - y_3$ $\beta_1 = -0.01$ $\gamma_1 = x_3 - x_2$ $\gamma_1 = -5 \times 10^{-3}$

$\beta_2 = y_3 - y_1$ $\beta_2 = 0.01$ $\gamma_2 = x_1 - x_3$ $\gamma_2 = -5 \times 10^{-3}$

$\beta_3 = y_1 - y_2$ $\beta_3 = 0$ $\gamma_3 = x_2 - x_1$ $\gamma_3 = 0.01$

$TA = x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)$

$TA = 1 \times 10^{-4}$ twice the area, Equation (6.2.9)

Equation (6.2.32) combined

$$[B] = \frac{1}{TA}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_2 & \beta_3 \end{pmatrix}$$

$$[B] = \begin{pmatrix} -100 & 0 & 100 & 0 & 0 & 0 \\ 0 & -50 & 0 & -50 & 0 & 100 \\ -50 & -100 & -50 & 100 & 100 & 0 \end{pmatrix}$$

In-plane stresses

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\{\sigma\} = \begin{pmatrix} -3.08 \times 10^{10} \\ 2.8 \times 10^9 \\ 6.3 \times 10^9 \end{pmatrix} \qquad \begin{matrix} \sigma_x = \sigma_0 & \sigma_x = -3.08 \times 10^{10} \\ \sigma_y = \sigma_1 & \sigma_y = 2.8 \times 10^9 \\ \tau_{xy} = \sigma_2 & \tau_{xy} = 6.3 \times 10^9 \end{matrix}$$

Note: use the left bracket after the sigma then the 0, or 1 or 2 for the values in the sigma matrix

Principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left[\left[\left[\frac{(\sigma_x - \sigma_y)}{2}\right]^2 + \tau_{xy}{}^2\right]\right]^{\frac{1}{2}}$$

$$\sigma_1 = 3.942 \times 10^9 \ \frac{\text{N}}{\text{m}^2} \qquad \text{sqrt} = \left[\left[\left[\frac{(\sigma_x - \sigma_y)}{2}\right]^2 + \tau_{xy}{}^2\right]\right]^{\frac{1}{2}}$$

$$\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_2 = \sigma_{\text{ave}} - \text{sqrt} \qquad \theta_p = \frac{1}{2}\ \text{a}\tan\left(2\,\frac{\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

$$\sigma_2 = -\,3.194 \times 10^{10} \ \frac{\text{N}}{\text{m}^2} \qquad \theta_p = -\,0.179$$

$\theta_p = -\,10.278°$ Principal angle

6.8 Von Mises stress

(a) $\sigma_1 = -\,0.14$ $\sigma_2 = -\,5.30$

$$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2} \qquad \sigma_e = 5.231 \text{ GPa}$$

(b) $\sigma_1 = 60.6$ $\sigma_2 = -\,18.6$

$$\sigma_e = \sqrt{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1\sigma_2} \qquad \sigma_e = 71.732 \text{ GPa}$$

(c) $\sigma_1 = 3.94$ GPa $\sigma_2 = -\,31.9$ GPa

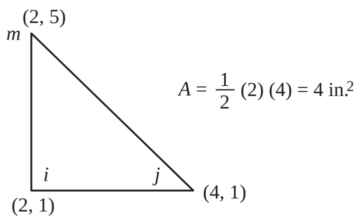
P6.8c von Mises stress, Equation (6.5.37a)

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\left[\left[(\sigma_1 - \sigma_2)^2\right] + \sigma_2{}^2 + \sigma_1{}^2\right]^{\frac{1}{2}}$$

$$\sigma_{vm} = 3.409 \times 10^{10} \ \frac{\text{N}}{\text{m}^2}$$

6.9

(a)



Plane strain

$$\beta_i = y_j - y_m = 1\text{–}5 = -\,4 \qquad \gamma_i = x_m - x_j = 2 - 4 = -\,2$$

$$\beta_j = y_m - y_i = 5 - 1 = 4 \qquad \gamma_j = x_i - x_m = 2 - 2 = 0$$

$$\beta_m = y_i - y_j = 1 - 1 = 0 \qquad \gamma_m = x_j - x_i = 4 - 2 = 2$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30 \times 10^6}{(1 + 0.25)[1 - 2(0.25)]}\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$\times\left(\frac{1}{8}\right)\begin{bmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & 0 & 2\\ -2 & -4 & 0 & 4 & 2 & 0\end{bmatrix}\begin{Bmatrix}0.001\\ 0.005\\ 0.001\\ 0.0025\\ 0\\ 0\end{Bmatrix}$$

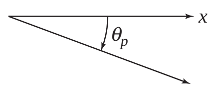
$$=\begin{Bmatrix}-15000\\ -45000\\ -18000\end{Bmatrix}\text{ psi}$$

$$\sigma_{1,2} = \frac{-15000+(-45000)}{2} \pm \sqrt{\left(\frac{-15000-(-45000)}{2}\right)^2 (-18000)^2}$$

$$= -30000 \pm 23430$$

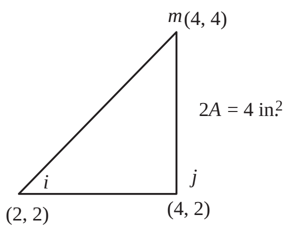
$\sigma_1 = -\,6570$ psi $\qquad\qquad \sigma_2 = -53{,}430$ psi

$$\tan 2\theta_p = \frac{2(-18000)}{-15000-(-45000)}$$



$$\theta_p = -\,25.1°$$

(b)



$\beta_i = 2 - 4 = -\,2 \qquad \gamma_i = 4 - 4 = 0$

$\beta_j = 4 - 2 = 2 \qquad \gamma_j = 2 - 4 = -\,2$

$\beta_m = 2 - 2 = 0 \qquad \gamma_m = 4 - 2 = 2$

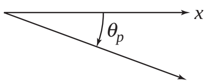
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = 48\times10^6\begin{bmatrix}0.75 & 0.25 & 0\\ 0.25 & 0.75 & 0\\ 0 & 0 & 0.25\end{bmatrix}\left(\frac{1}{4}\right)\begin{bmatrix}-2 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -2 & -2 & 2 & 2 & 0\end{bmatrix}$$

$$\times\begin{Bmatrix}0.001\\ 0.005\\ 0.001\\ 0.0025\\ 0\\ 0\end{Bmatrix} = \begin{Bmatrix}-15{,}000\\ -45{,}000\\ -21{,}000\end{Bmatrix}\text{ psi}$$

$$\sigma_{1,2} = \frac{-15000-45000}{2} \pm \sqrt{\left[\frac{-15000-(-45000)}{2}\right]^2 + (-21000)^2}$$

$\sigma_1 = -\,4193$ psi $\qquad\qquad \sigma_2 = -55{,}800$ psi

$$\tan 2\theta_p = \frac{2(-21000)}{-15000-(-45000)}$$



$$\theta_p = -27.2°$$

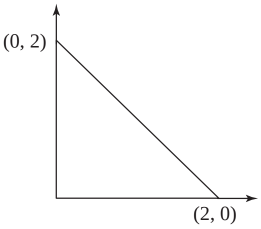
(c) Given displacements (in.)

$u_1 = 0.001$ $v_1 = 0.005$

$u_2 = 0.001$ $v_2 = 0.0025$

$u_3 = 0$ $v_3 = 0$

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$



Material definition

$E = 30 \times 10^6$ psi $v = 25$

Geometry description

$\beta_i = 0 - 2 \quad (y_j - y_m)$

$\beta_j = 2 - 0 \quad (y_m - y_i)$

$\beta_m = 0 - 0 \quad (y_i - y_j)$

$\gamma_i = 0 - 2 \quad (x_m - x_j)$

$\gamma_j = 0 - 0 \quad (x_i - x_m)$

$\gamma_m = 2 - 0 \quad (x_j - x_i)$

$A = \frac{1}{2} \times 2 \times 2$ $A = 2$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{pmatrix}$$

Plane strain constitutive matrix

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

Stress matrix

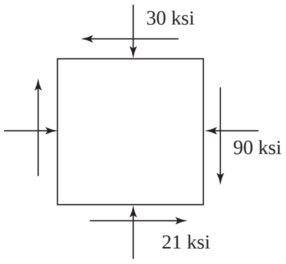
$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\sigma = \begin{Bmatrix} -3\times 10^4 \\ -9\times 10^4 \\ -2.1\times 10^4 \end{Bmatrix}$$

$$\sigma_x = -3 \times 10^4 \text{ (psi)}$$

$$\sigma_y = -9 \times 10^4 \text{ (psi)}$$

$$\tau_{xy} = -2.1 \times 10^4 \text{ (psi)}$$



$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 = -2.338 \times 10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = -9.662 \times 10^4 \text{ psi}$$

$$\theta_p = \frac{1}{2}\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

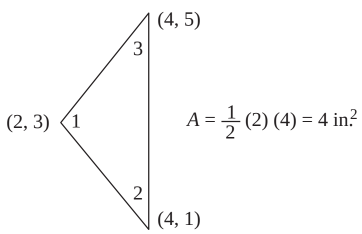
$$\theta_p = -0.305 \text{ (rad)}$$

$$\sigma_3 = 0$$

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\sigma_{vm} = 8.731 \times 10^4 \text{ (psi)}$$

(d)



$$\beta_1 = -4 \qquad \gamma_1 = 0$$

$$\beta_2 = 2 \qquad \gamma_2 = -2$$

$$\beta_3 = 2 \qquad \gamma_3 = 2$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 48 \times 10^6 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \left(\frac{1}{8}\right) \begin{bmatrix} -4 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -4 & -2 & 2 & 2 & 2 \end{bmatrix}$$

$$\left\{\begin{matrix}0.001\\0.005\\0.001\\0.0025\\0\\0\end{matrix}\right\} = \left\{\begin{matrix}-16.5\\-25.5\\-25.5\end{matrix}\right\} \text{ ksi}$$

$$\sigma_{1,2} = \frac{-16.5-25.5}{2} \pm \sqrt{\left(\frac{-16.5+25.5}{2}\right)^2 + (-25.5)^2}$$

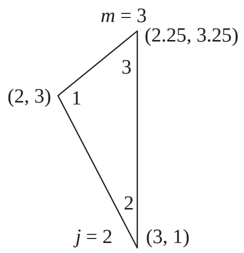
$$\sigma_1 = 4.89 \text{ ksi} \qquad \sigma_2 = -46.9 \text{ ksi}$$

$$\tan 2\theta_p = \frac{2\,(-25.5)}{-16.5+25.5}$$

$$\theta_p = -40.0°$$

(e)

$$A = \frac{1}{2}\begin{vmatrix}1 & 1 & 1\\2 & 3 & 2.25\\3 & 1 & 3.25\end{vmatrix}$$



$$= 0.375 \text{ in.}^2$$

$$\beta_i = 1 - 3.25 = -2.25 \qquad \gamma_i = 2.25 - 3 = -0.75$$

$$\beta_j = 3.25 - 3 = 0.25 \qquad \gamma_j = 2 - 2.25 = -0.25$$

$$\beta_m = 3 - 1 = 2 \qquad \gamma_m = 3 - 2 = 1$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\left\{\begin{matrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{matrix}\right\} = 4.8 \times 10^7 \begin{bmatrix}0.75 & 0.25 & 0\\0.25 & 0.75 & 0\\0 & 0 & 0.25\end{bmatrix}$$

$$\times \frac{1}{0.75}\begin{bmatrix}-2.25 & 0 & 0.25 & 0 & 2 & 0\\0 & -0.75 & 0 & -0.25 & 0 & 1\\-0.75 & -2.25 & -0.25 & 0.25 & 1 & 2\end{bmatrix}\left\{\begin{matrix}0.001\\0.005\\0.001\\0.0025\\0\\0\end{matrix}\right\}$$

$$\left\{\begin{matrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{matrix}\right\} = \left\{\begin{matrix}-166\\-242\\-186\end{matrix}\right\} \text{ ksi}$$

$$\sigma_{1,2} = \frac{-166+(-242)}{2} \pm \sqrt{\left(\frac{-166+242}{2}\right)^2 + (-186)^2}$$

$$\sigma_1 = -14.2 \text{ ksi} \qquad \sigma_2 = -394 \text{ ksi}$$

$$\theta_p = \frac{1}{2}\left(\tan^{-1}\frac{2(-186)}{-166+242}\right) = -39.2°$$

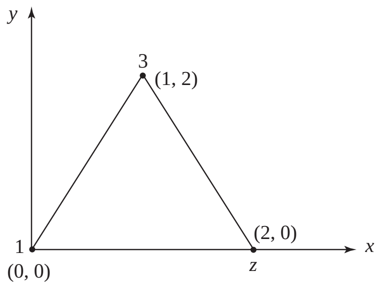
(f) Material properties

$$E = 30 \times 10^6 \text{ psi} \qquad \text{Modulus of elasticity.}$$

$$v = 0.25 \qquad \text{Poisson ratio}$$

Nodal coordinates (coordinates defined CCW around element)

$$x_1 = 0 \text{ in.}$$
$$y_1 = 0 \text{ in.}$$
$$x_2 = 2 \text{ in.}$$
$$y_2 = 0 \text{ in.}$$
$$x_3 = 1 \text{ in.}$$
$$y_3 = 2 \text{ in.}$$



Nodal displacements

$$u_1 = 0.001 \text{ in.}$$
$$v_1 = 0.005 \text{ in.}$$
$$u_2 = 0.001 \text{ in.}$$
$$v_2 = 0.0025 \text{ in.}$$
$$u_3 = 0 \text{ in.}$$
$$v_3 = 0 \text{ in.}$$

Set-up displacement vector

$$\{d\} = (u_1\ v_1\ u_2\ v_2\ u_3\ v_3)^T$$

$$\{d\}^T = (0.001 \quad 0.005 \quad 0.0001 \quad 0.0025 \quad 0 \quad 0) \text{ in.}$$

Area of triangular element ($\frac{1}{2} \times \text{base} \times \text{height}$)

$$A = \frac{1}{2}\ (x_2 - x_1)\ (y_3 - y_1)$$

$$A = 2 \text{ in.}^2$$

Calculate gradient matrix, B, as given in text Equation (6.2.32)

Elements of B given by text Equation (6.2.10).

$$\beta_1 = y_2 - y_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$$

$$\gamma_1 = x_3 - x_2 \qquad \gamma_2 = x_1 - x_3 \qquad \gamma_3 = x_2 - x_1$$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix}$$

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ -0.25 & -0.5 & -0.25 & 0.5 & 0.5 & 0 \end{pmatrix}\frac{1}{\text{in.}}$$

Calculate constitutive matrix for plane strain

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[D] = \begin{pmatrix} 3.6\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 3.6\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{pmatrix} \text{psi}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} -22500.00 \\ -67500.00 \\ -21000.00 \end{pmatrix} \text{psi}$$

Principal stresses

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 = -\,1.422 \times 10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_2 = -\,75.777 \times 10^3 \text{ psi}$$

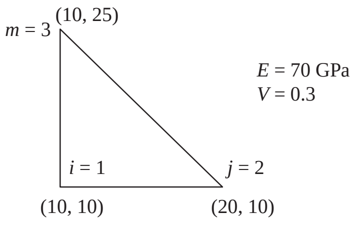
Angular location of principal stress plane

$$\theta_p = \frac{\operatorname{atan}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_p = -\,21.513°$$

6.10

(a)



$\beta_i = -\,15$ $\quad\quad$ $\gamma_i = -\,10$

$\beta_j = 15$ $\quad\quad$ $\gamma_j = 0$

$\beta_m = 0$ $\quad\quad$ $\gamma_m = 10$

$2A = 150 \text{ mm}^2 = 150 \times 10^{-6} \text{ m}^2$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{70\times10^6}{(1+0.3)[1-2(0.3)]}\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\times \frac{1}{150}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -15 & 0 & 15 & 10 & 0 \end{bmatrix}\begin{Bmatrix} 5.0 \\ 2.0 \\ 0 \\ 0 \\ 5.0 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -52.5 \\ -32.8 \\ -5.38 \end{Bmatrix} \text{ MPa}$$

$$\sigma_{1,2} = \frac{-52.5 + (-32.8)}{2} \pm \sqrt{\left(\frac{-52.5 + 32.8}{2}\right)^2 + (-5.38)^2}$$

$$= -42.65 \pm 11.22$$

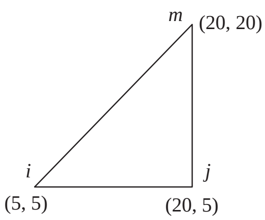
$$\sigma_1 = -31.4 \text{ MPa}$$

$$\sigma_2 = -53.9 \text{ MPa}$$

$$2\theta_p = \tan^{-1} \frac{2(-5.38)}{-52.5 - (-32.8)} = 28.64°$$

$$\theta_p = 14.32°$$

(b)



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 10^9 \begin{bmatrix} 94.2 & 40.4 & 0 \\ & 94.2 & 0 \\ & & 26.9 \end{bmatrix} \frac{1}{225 \times 10^{-6}}$$

$$\times \begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & -15 & 0 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix}\begin{Bmatrix} 0.005 \\ 0.002 \\ 0 \\ 0 \\ 0.005 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -31.4 \\ -13.5 \\ 5.38 \end{Bmatrix} \text{ MPa}$$

$$\sigma_{1,2} = \frac{-31.4 - 13.5}{2} \pm \sqrt{\left(\frac{-31.4 + 13.5}{2}\right)^2 + (5.38)^2}$$

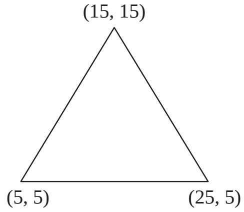
$$= -22.44 \pm 10.46$$

$\sigma_1 = -\,11.98$ MPa $\qquad \sigma_2 = -\,32.9$ MPa

$$\tan 2\theta_p = \frac{2(5.38)}{-31.4+13.46}$$

$\theta_p = -\,15.5°$

(c)



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \left(\frac{70}{1.3\times 0.4}\right)\begin{bmatrix}0.7 & 0.3 & 0\\ 0.3 & 0.7 & 0\\ 0 & 0 & 0.2\end{bmatrix}\frac{1}{200}$$

$$\times\begin{bmatrix}-10 & 0 & 10 & 0 & 0 & 0\\ 0 & -10 & 0 & -10 & 0 & 20\\ -10 & -10 & -10 & 10 & 20 & 0\end{bmatrix}\begin{Bmatrix}5\\2\\0\\0\\5\\0\end{Bmatrix}\times 10^{-3}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{Bmatrix}-27.6\\ -19.5\\ 4.04\end{Bmatrix}\text{MPa}$$

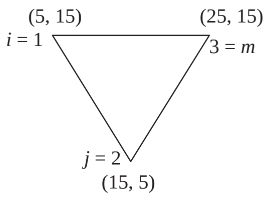
$$\sigma_{1,2} = \frac{-27.6-19.5}{2} \pm \sqrt{\left(\frac{-27.6+19.5}{2}\right)^2 + 4.04^2}$$

$\sigma_1 = -\,17.9$ MPa $\qquad \sigma_2 = -\,29.3$ MPa

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\times 4.04}{-27.6+19.5}\right)$$

$\theta_p = -\,22.5°$

(d)



$E = 70 \times 10^9 \qquad v = 3 \qquad t = 1$

$y_1 = 0.015 \qquad y_2 = 0.005 \qquad y_3 = 0.015$

$x_1 = 0.005 \qquad x_2 = 0.015 \qquad x_3 = 0.025$

$\beta_1 = y_2 - y_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2 \qquad \gamma_2 = x_1 - x_3 \qquad \gamma_3 = x_2 - x_1$

$$A = \frac{1}{2}\ [x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)]$$

$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{pmatrix} \quad [D] = \frac{E}{(1+v)\,(1-2v)}\begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = t\,A\,[B]^T\,[D]\,[B]$$

$u_1 = 0.000005$ $\qquad$ $u_2 = 0$ $\qquad$ $u_3 = 0.000005$

$v_1 = 0.000002$ $\qquad$ $v_2 = 0$ $\qquad$ $v_3 = 0$

$$\{d\} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}$$

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left(\frac{70}{1.3\times 0.4}\right)\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}\frac{1}{200}$$

$$\times\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & 0 & 10 & 10 & 10 & -10 \end{bmatrix}\begin{Bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 5 \\ 0 \end{Bmatrix}\times 10^{-3}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -1.05 \\ 0.70 \\ 3.5 \end{Bmatrix}\ \text{MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{1,2} = \frac{-1.05 - 0.70}{2} \pm \sqrt{\left(\frac{-1.05 + 0.70}{2}\right)^2 + (-3.5)^2}$$

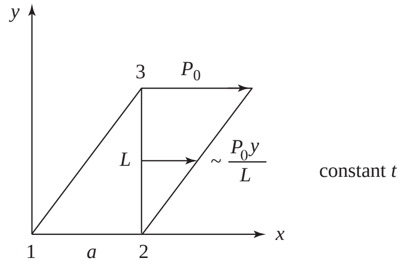
$$\sigma_1 = -3.43\ \text{MPa} \qquad \sigma_2 = -\,3.78\ \text{MPa}$$

$$\theta_p = \frac{1}{2}\ \tan^{-1}\left(\frac{2\,(-3.5)}{-1.05 - 0.70}\right)$$

$$\theta_p = 38°$$

6.11

(a)



Equation (6.3.7)

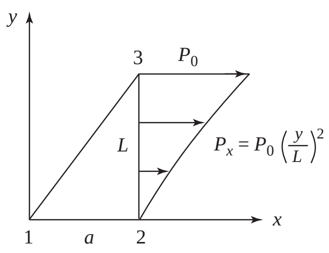
$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{\substack{\text{evaluated of } x=a \\ y=y}} \begin{Bmatrix} \frac{P_0\,y}{L} \\ 0 \end{Bmatrix} dz\,dy = t\int_0^L \begin{Bmatrix} \frac{N_1\,P_0\,y}{L} \\ 0 \\ \frac{N_2\,P_0\,y}{L} \\ 0 \\ \frac{N_3\,P_0\,y}{L} \\ 0 \end{Bmatrix} dy$$

By Equations (6.3.12) – (6.3.17)

$$N_1 = \frac{L(a-x)}{2A},\ N_2 = \frac{Lx-ay}{2A},\ N_3 = \frac{ay}{2A}$$

$$\{f_s\} = \frac{t\,P_0}{2\left(\frac{aL}{2}\right)} \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{6}aL^2 \\ 0 \\ \frac{1}{3}aL^2 \\ 0 \end{Bmatrix} \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{6}P_0Lt \\ 0 \\ \frac{1}{3}P_0Lt \\ 0 \end{Bmatrix}$$

(b)



By Equation (6.3.11)

$$\{f_s\} = t\int_0^L \begin{Bmatrix} N_1\,P_x \\ 0 \\ N_2\,P_x \\ 0 \\ N_3\,P_x \\ 0 \end{Bmatrix}_{\substack{x=a \\ y=y}} dy$$

Now $N_1 = 0,\ N_2 = \dfrac{L_x - ay}{2A},\ N_3 = \dfrac{ay}{2A}$

$$\{f_s\} = t\int_0^L \left\{\begin{array}{c} 0 \\ 0 \\ \left(\frac{Lx-ay}{2A}\right) P_0 \left(\frac{y}{L}\right)^2 \\ 0 \\ \frac{ay}{2A}\left(\frac{y}{L}\right)^2 P_0 \\ 0 \end{array}\right\}_{\substack{x=a \\ y=y}} dy$$

Simplifying and integrating

$$\{f_s\} = \frac{P_0\, t}{2\left(\frac{1}{2}aL\right)} \left\{\begin{array}{c} 0 \\ 0 \\ \left(\frac{Lay^3}{3L^2} - \frac{ay^4}{4L^2}\right)\Big|_0^L \\ 0 \\ \frac{ay^4}{4L^2}\Big|_0^L \\ 0 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ \frac{P_0\, tL}{12} \\ 0 \\ \frac{P_0\, tL}{4} \\ 0 \end{array}\right\}$$

or

$$f_{2x} = \frac{P_0\, tL}{12}$$

$$f_{3x} = \frac{P_0\, tL}{4}$$

6.12

(a)

$$P_y(x) = ax^2 + bx + c$$

Given

$$a\left(\frac{L}{2}\right)^2 + b\,\frac{L}{2} + p_1 = p_2$$

$$aL^2 + bL + p_1 = p_3$$

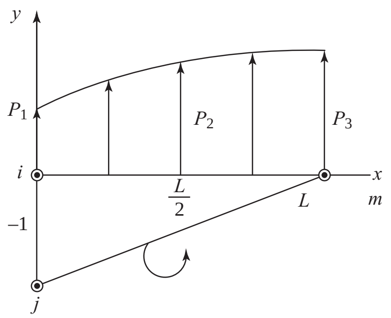
$$c = p_1$$

Find $\quad (\text{a, b, c}) \rightarrow \begin{bmatrix} 2\,\dfrac{(p_1 + p_3 - 2p_2)}{L^2} \\ \dfrac{(-3p_1 - p_3 + 4p_2)}{L} \\ p_1 \end{bmatrix}$

$$a(p_1, p_2, p_3) = 2\,\frac{(p_1 + p_3 - 2p_2)}{L^2}$$

$$b(p_1, p_2, p_3) = \frac{(-3p_1 - p_3 + 4p_2)}{L}$$

$$c(p_1, p_2, p_3) = p_1$$



Forces in y direction at nodes 1 and 3 are

$N_1 = N_i$

$$f_{s_{1y}} = \int_0^L \left(1-\frac{x}{L}\right)\left(a(p_1,\,p_2,\,p_3)x^2 + b(p_1,\,p_2,\,p_3)x + c\,(p_1,\,p_2,\,p_3)\right)dx \to \frac{1}{6}Lp_1 + \frac{1}{3}Lp_2$$

$$= \int_{x=0}^{L} N_1\, P_y(x)\, dx$$

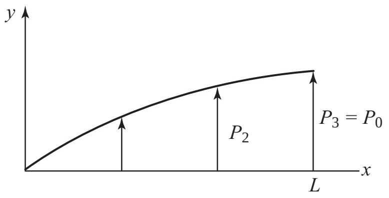
$N_3 = N_m$

$$f_{s_{3y}} = \int_0^L \frac{x}{L}\left(a(p_1,\,p_2,\,p_3)x^2 + b(p_1,\,p_2,\,p_3)x + c\,(p_1,\,p_2,\,p_3)\right)\,dx \to \frac{1}{6}Lp_3 + \frac{1}{3}Lp_2$$

$$= \int_0^L N_m\, P_y(x)$$

(Special case)

If $p_1 = 0 \qquad p_3 = p_0$



$$p(x) = ax^2 + bx + c$$

Want

$$p(x) = p_0\left(\frac{x}{L}\right)^2 = ax^2 + \cancel{bx} + \cancel{c}$$

$$\therefore\ a = \frac{p_0}{L^2} = \frac{2}{L^2}\ (p_0 - 2p_2)$$

$$\frac{2p_0}{L^2} - \frac{p_0}{L^2} = \frac{4p_2}{L^2}$$

$$\frac{p_0}{\cancel{L^2}} = \frac{4p_2}{\cancel{L^2}}$$

$$p_2 = \frac{p_0}{4}$$

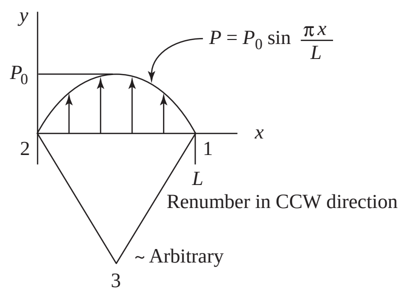
Then

$$f_{s_{14}} = \frac{1}{6}\, L\, \cancel{p_1}^{0} + \frac{1}{3}\, Lp_2 = \frac{1}{3}\, L\, \frac{p_0}{4} = \frac{p_0 L}{12} \rightarrow \text{Like 6.11 b with } f_{sx} = \frac{pL}{12}$$

$$f_{s_{34}} = \frac{1}{6}\, L\overset{\overset{p_0}{\|}}{p_3} + \frac{1}{3}\, L\overset{\overset{\frac{p_0}{4}}{\|}}{p_2} = \frac{3}{12} Lp_0 = \frac{p_0 L}{4} \quad \Rightarrow \quad f_{3x} \text{ in 6.11b}$$

(These answers match P6.11 for special case)

(b)



$$\{f_s\} = \int_s \int [N_s]^T \; [T_s]\; ds$$

$$[T_s] = \text{Surface tractions}$$

$$= \begin{Bmatrix} T_{sx} \\ T_{sy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_0 \sin \pi \frac{N}{L} \end{Bmatrix}$$

$$[N_s] = \text{Shape function matrix evaluated along edge 1-2}$$

$$= \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Let $i = 1$

$j = 2$

$m = 3$

$$N_i - N_1 = \frac{1}{2A}\; (\alpha_i + \beta_i x + \gamma_i y)$$

$$N_1(y = 0) = \frac{1}{2A}\; (\alpha_i + \beta_i x)$$

$$\alpha_i = x_j y_m - y_j x_m$$

$$= 0(ym) - 0(xm) = 0$$

$$\beta_i = y_j - y_m$$

$$= 0 - y_m = - y_m$$

$$N_1(y = 0) = \frac{1}{2A}\; (0 - y_m x) = \frac{-y_m x}{2A}$$

$$N_j = N_2 = \frac{1}{2A}\; (\alpha_j + \beta_j x + \gamma_j y)$$

$$N_2(y=0) = \frac{1}{2A}\ (\alpha_i + \beta_i x)$$

$$\alpha_j = y_j x_m - x_j y_m$$

$$= 0(x_m) - Ly_m = -Ly_m$$

$$\beta_j = y_m - y_i$$

$$= y_m - 0 = y_m$$

$$N_2(y=0) = \frac{1}{2A}\ [-Ly_m + y_m x] = \frac{y_m}{2A}\ [x - L]$$

$$N_m = N_3 = \frac{1}{2A}\ [\alpha_m + \beta_m x + \gamma_m y]$$

$$N_3(y=0) = \frac{1}{2A}\ [\alpha_m + \beta_m x]$$

$$\alpha_m = x_i y_j - y_i x_j = L(0) - 0(0) = 0$$

$$\beta_m = y_i - y_j = 0$$

$\therefore N_m(y=0) = 0$ As expected

$$\{f_s\} = \int_0^{x=L}\int_0^{z=t} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} 0 \\ P_0 \sin\frac{\pi x}{L} \end{Bmatrix}\ dz\, dx$$

$$= t\int_0^{x=L} \begin{bmatrix} 0 \\ N_1 P_0 \sin\frac{\pi x}{L} \\ 0 \\ N_2 P_0 \sin\frac{\pi x}{L} \\ 0 \\ N_3 P_0 \sin\frac{\pi x}{L} \end{bmatrix} dx = t\int_0^{x=L} \begin{bmatrix} 0 \\ \frac{-y_m x}{2A} P_0 \sin\left(\frac{\pi x}{L}\right) \\ 0 \\ \frac{y_m}{2A}(x-L) P_0 \sin\frac{\pi x}{L} \\ 0 \\ 0 \end{bmatrix} dx$$

2nd term in (A) ($y_m = y_3$)

$$f_{s1y} = \frac{-t\, y_3 P_0}{2A}\int_0^{x=L} x \sin\left(\frac{\pi x}{L}\right) dx$$

$$\int u\,dv = u - \int v\,du$$

$$u = x \quad du = dx$$

$$dv = \sin\frac{\pi x}{L}\, dx \quad v = -\cos\frac{\frac{\pi x}{L}}{\frac{\pi}{L}}$$

$$= \frac{-t\, y_3 P_0}{2A}\left[-\frac{xL}{\pi}\cos\pi\frac{x}{L} + \int_0^{x=L}\frac{L}{\pi}\cos\left(\pi\frac{x}{L}\right)dx\right]$$

$$= \frac{-t\, y_3 P_0}{2A}\left[-\frac{xL}{\pi}\cos\left(\pi\frac{x}{L}\right) + \frac{L^2}{\pi^2}\sin\left(\pi\frac{x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{-t\, y_3 P_0}{2A}\left[\frac{-L^2}{\pi}(-1) + 0 - 0 - 0\right]$$

$$f_{s1y} = \frac{-t\, y_3 P_0}{2A}\left(\frac{L^2}{\pi}\right) = \frac{t\, P_0 L}{\pi}$$

$$A = \frac{-1}{2}\, L y_3$$

4th term in (A)

$$f_{s2y} = t \int_0^L \left[\frac{y_3}{2A} P_0\, x \sin\frac{\pi x}{L} - \frac{y_3}{2A} P_0\, L \sin\left(\frac{x\pi}{L}\right)\right] dx$$

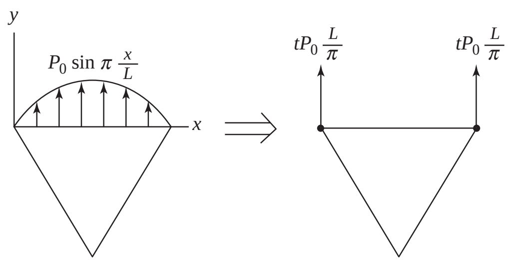
$$= \frac{t\, y_3}{2A} P_0 \underbrace{\int_0^L x \sin\frac{\pi x}{L}\, dx}_{\text{DONE IN 2nd TERM}} - \frac{t\, y_3}{2A} P_0 L \int_0^L \sin\frac{\pi x}{L}\, dx$$

$$= \frac{t\, y_3 P_0}{2A}\left[\frac{L^2}{\pi}\right] + \frac{t\, y_3 P_0 L}{2A}\left[\frac{L}{\pi}\cos\pi\frac{x}{L}\right]\Bigg|_0^L$$

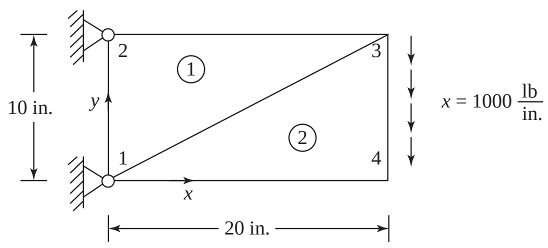
$$= \frac{t\, y_3 P_0}{2A}\frac{L^2}{\pi} + \frac{t\, y_3 P_0 L}{2A}\frac{L}{\pi}\,[-1 + 1]$$

$$f_{s2y} = \frac{t\, y_3 P_0\, L^2}{2A\,\pi} = \frac{t\, \cancel{y_3}\, L^{\cancel{2}} P_0}{\cancel{2}\left[\frac{1}{\cancel{2}}\cancel{L}\,\cancel{y_3}\right]\pi} = \frac{t\, L\, P_0}{\pi}$$

$$\therefore \quad \vec{f}_s = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{t\, P_0\, L}{\pi} \\ 0 \\ \frac{t\, P_0\, L}{\pi} \\ 0 \\ 0 \end{Bmatrix}$$



6.13



Refer to Section 6.5 for [K]

Since $\quad u_1 = v_1 = 0,\ u_2 = v_2 = 0$

$$\begin{Bmatrix} 0 \\ -5{,}000 \\ 0 \\ -5{,}000 \end{Bmatrix} = \frac{75000\times 5}{0.91}\begin{bmatrix} 48 & 0 & -28 & 14 \\ & 87 & 12 & -80 \\ & & 48 & -26 \\ \text{Symmetry} & & & 87 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Solving

$$u_3 = 0.50\times 10^{-3}\text{ in.} \qquad v_3 = -0.275\times 10^{-2}\text{ in.}$$
$$u_4 = -0.609\times 10^{-3}\text{ in.} \qquad v_4 = -0.293\times 10^{-2}\text{ in.}$$

Using element (1)

By Equation (6.2.36), $\{\sigma\} = [D]\,[B]\,\{d\}$

$$\{\sigma\} = \frac{30\times 10^6}{(0.91)(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$
$$\times\begin{Bmatrix} 0 \\ 0 \\ 0.5\times 10^{-3} \\ -0.275\times 10^{-2} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 824 \\ 247 \\ -1586 \end{Bmatrix}\text{ psi}$$

$$\sigma_{1,2} = \frac{824+247}{2} \pm \sqrt{\left(\frac{824-247}{2}\right)^2 + (-1586)^2}$$

$$\sigma_1 = 2149\text{ psi} \qquad \sigma_2 = -1077\text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{-2\times 1586}{824-247}\right)$$

$$\theta_p = -40^\circ$$

Using element (2)

$$\{\sigma\} = \frac{30\times 10^6}{0.91(200)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix}$$
$$\times\begin{Bmatrix} 0 \\ 0 \\ -0.609\times 10^{-3} \\ -0.293\times 10^{-2} \\ 0.5\times 10^{-3} \\ -0.275\times 10^{-2} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -825 \\ 292 \\ -411 \end{Bmatrix} \text{ psi}$$

$$\sigma_{1,2} = \frac{-825+292}{2} \pm \sqrt{\left(\frac{-825-292}{2}\right)^2 + (-411)^2}$$

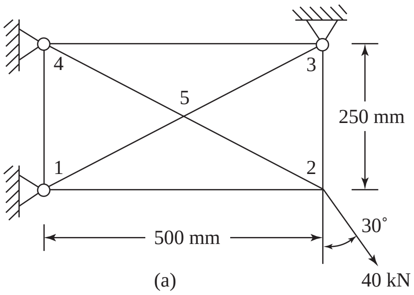
$$\sigma_1 = 426 \text{ psi} \qquad \sigma_2 = -960 \text{ psi}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{-2\times 411}{-825-292}\right)$$

$$\theta_p = 18.15°$$

6.14

(a)



(a)

INPUT TABLE 1. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.2100E+12	0.3000E+00	0.0000E+00	0.5000E–02

INPUT TABLE 3. NODAL POINT DATA

NODAL POINT	TYPE	X	Y
1	3	0.0000E+00	0.0000E+00
2	0	0.5000E+00	0.0000E+00
3	3	0.5000E+00	0.2500E+00
4	3	0.0000E+00	0.2500E+00
5	0	0.2500E+00	0.1250E+00

X-DISP. OR LOAD	Y-DISP. OR LOAD
0.0000E+00	0.0000E+00
0.2000E+05	– 0.3464E+05
0.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00

INPUT TABLE 4. ELEMENT DATA

	GLOBAL INDICES OF ELEMENT NODES				
ELEMENT	1	2	3	4	MATERIAL
1	1	5	4	4	1
2	1	2	5	5	1
3	5	2	3	3	1
4	4	5	3	3	1

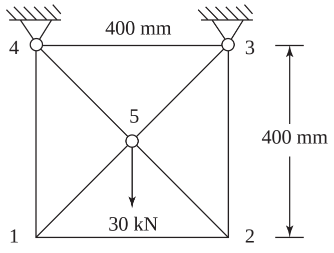
OUTPUT TABLE 1. NODAL DISPLACEMENTS, m

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.28124740E–04	– 0.32985310E–04
3	0.00000000E+00	0.00000000E+00
4	0.00000000E+00	0.00000000E+00
5	0.11497950E–04	– 0.10347600E–04

OUTPUT TABLE 2. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA (X)	SIGMA (Y)	TAU (X, Y)
1	0.083	0.125	0.0613E+07	3.1840E+06	3.3431E+06
2	0.250	0.042	1.6384E+07	1.5239E+07	–6.9854E+06
3	0.417	0.125	1.1502E+07	3.1158E+07	–1.1072E+07
4	0.250	0.208	5.7310E+06	1.9103E+07	–7.4294E+06
			SIGMA (1)	SIGMA (2)	ANGLE
			1.1896E+07	1.9012E+06	2.0993E+01
			2.2820E+07	8.8026E+06	–4.2657E+01
			3.6135E+07	6.5251E+06	–6.5798E+01
			2.2412E+07	2.4221E+06	–6.5993E+01

(c)



(b)

INPUT TABLE 1. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2. MATERAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO,	MATERIAL DENISTY	MATERIAL THICKNESS
1	0.2100E+12	0.3000E+00	0.0000E+00	0.5000E–02

INPUT TABLE 3. NODAL POINT DATA

POINT	TYPE	X OR LOAD	Y OR LOAD
1	0	0.0000E+00	0.0000E+00
2	0	0.4000E+00	0.0000E+00
3	3	0.4000E+00	0.4000E+00
4	3	0.0000E+00	0.4000E+00
5	0	0.2000E+00	0.2000E+00
		0.0000E+00	0.0000E+00
		0.0000E+00	0.0000E+00
		0.0000E+00	0.0000E+00
		0.0000E+00	0.0000E+00
		0.0000E+00	–0.3000E+05

INPUT TABLE 4. ELEMENT DATA

ELEMENT	GLOBAL INDICES OF ELEMENT NODES 1	2	3	4	MATERIAL
1	1	2	5	5	1
2	2	3	5	5	.1
3	3	4	5	5	1
4	4	1	5	5	–.1

OUTPUT TABLE 1. NODAL DISPLACEMENTS (m)

NODE	U = X–DISP.	V = Y–DISP.
1	–0.16515414E–05	–0.12504538E–04
2	0.16515442E–05	– 0.12504535E–04
3	0.00000000E+00	0.00000000E+00
4	0.00000000E+00	0.00000000E+00
5	0.27411561E–12	–0.16279491E–04

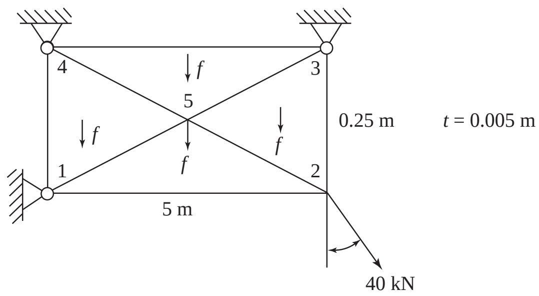
OUTPUT TABLE 2. STRESSES AT ELEMENT CENTROIDS $\left(\dfrac{\text{N}}{\text{m}^2}\right)$

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.20	0.07	5.9891E+05	–3.7840E+06	4.0454E–01
2	0.33	0.20	3.1171E+06	7.5000E+06	3.7160E+06
3	0.20	0.33	5.6352E+06	1.8784E+07	–1.1070E–01
4	0.07	0.20	3.1171E+06	7.5000E+06	–3.7160E+06

SIGMA(1)	SIGMA(2)	ANGLE
5.9891E+05	–3.7840E+06	5.2883E–06
9.6226E+06	9.9449E+05	6.0265E+01
1.8784E+07	5.6352E+06	–9.0000E+01
9.6226E+06	9.9449E+05	–6.0265E+01

6.15

(a)



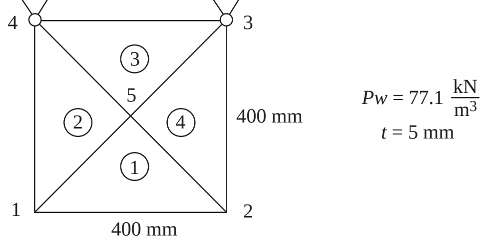
$$F = (0.5\ \cancel{\text{m}})\ (0.25\ \cancel{\text{m}})\ (0.005\ \cancel{\text{m}})\ \left(77.1\frac{\text{kN}}{\cancel{\text{m}^3}}\right) = 0.0482\ \text{kN}$$

$$f_1 = f_2 = f_3 = f_4 = \frac{-\frac{1}{2}(0.25)(0.25)(0.005)\left(77.1\ \frac{\text{kN}}{\text{m}^3}\right)\left[\frac{1000\ \text{N}}{1\ \text{kN}}\right](2)}{3} = -\ 8.031\ \text{N}$$

$$f_5 = (2) \times f_1 = 16.063\ \text{N} \downarrow$$

$$f_{5y} = -16.063\ \text{N} \downarrow$$

(c)



Equation (6.3.6)

$$\begin{Bmatrix} f_{B1x}^{(1)} \\ f_{B1y}^{(1)} \\ f_{B2x}^{(1)} \\ f_{B2y}^{(1)} \\ f_{B5x}^{(1)} \\ f_{B5y}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 77.1 \\ 0 \\ 77.1 \\ 0 \\ 77.1 \end{Bmatrix} \frac{(0.4\,\text{m})(0.2\,\text{m})(0.005\,\text{m})}{(2)\,(3)} = \begin{Bmatrix} 0 \\ 5.14 \\ 0 \\ 5.14 \\ 0 \\ 5.14 \end{Bmatrix} 10^{-3}\ \text{kN}$$

All body force matrices for each element identical to above

Adding the 4 element body force matrices

$$\{F_B\} = \begin{Bmatrix} f_{B1x} \\ f_{B1y} \\ f_{B2x} \\ f_{B2y} \\ f_{B3x} \\ f_{B3y} \\ f_{B4x} \\ f_{B4y} \\ f_{B5x} \\ f_{B5y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 10.28 \\ 0 \\ 20.56 \end{Bmatrix} \text{N}$$

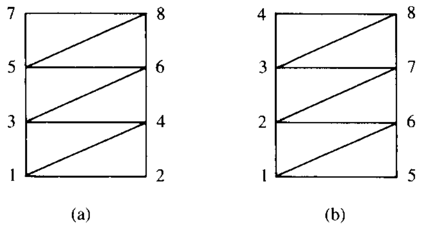
6.16 The triangular element is called a constant strain triangle (CST) because the strain is constant throughout the element.

6.17 The stresses are also constant as the strains are constant.

6.18
a. No, bending in the plane takes place
b. Yes, loads in-plane of the wall
c. Yes, a plane stress problem
d. Yes, if loads in-plane of the bar
e. Yes, a plane strain problem
f. Yes, a plane stress problem
g. No, loads out of the plane of the wrench
h. Yes, as loads in the plane
i. No, bending in the plane takes place

6.19 We must connect the beam element to two or more nodes of a plane stress element. The beam must be along the edge of the plane stress element.

6.20



$$n_b = n_0\,(m + 1)$$

(a) $n_b = 2(3 + 1) = 8$

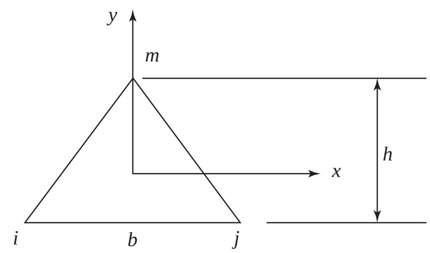
(b) $n_b = 2(5 + 1) = 12$

for model (a) model(b)

$\lfloor n_b = 8 \rfloor$ $\lfloor n_b = 12 \rfloor$

$$
\begin{array}{cc}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
\times & \times & \times & \times & & & & \\
 & \times & 0 & \times & 0 & & & \\
 & & \times & \times & \times & \times & & \\
 & & & \times & 0 & \times & 0 & \\
 & & & & \times & \times & \times & \times \\
 & \text{Symmetry} & & & & \times & 0 & \times \\
 & & & & & & \times & \times \\
 & & & & & & & \times
\end{bmatrix}
&
\begin{bmatrix}
\times & \times & 0 & 0 & \times & \times & & \\
 & \times & \times & 0 & 0 & \times & \times & \\
 & & \times & \times & 0 & 0 & \times & \times \\
 & & & \times & 0 & 0 & 0 & \times \\
 & & & & \times & \times & 0 & 0 \\
 & & & & & \times & \times & 0 \\
 & & & & & & \times & \times \\
\text{Symmetry} & & & & & & & \times
\end{bmatrix}
\end{array}
$$

6.21



By (6.2.10)

$$\alpha_i = x_j y_m - y_j x_m = \frac{b}{2}\left(\frac{2b}{3}\right) - \left(-\frac{h}{3}\right)(0) = \frac{bh}{3} = \frac{2A}{3}$$

$$\alpha_j = \left(-\frac{h}{3}\right)(0) - \left(-\frac{b}{2}\right)\left(\frac{2h}{3}\right) = \frac{bh}{3} = \frac{2A}{3}$$

$$\alpha_m = \left(-\frac{b}{2}\right)\left(-\frac{h}{3}\right) - \left(-\frac{h}{3}\right)\left(\frac{b}{2}\right) = \frac{bh}{3} = \frac{2A}{3}$$

$$\therefore \quad \{f_B\} = \int_v [N]^T \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} dV \qquad N_i = \frac{1}{2A}\left(\frac{2A}{3}\right) = \frac{1}{3} \text{ and } N_j = N_m = \frac{1}{3}$$

or

$$\{f_{b_i}\} = \int_v \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} dV$$

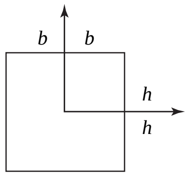
$$= \int_v \begin{bmatrix} \frac{1}{3} & \\ 0 & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} t dA$$

$$\{f_{b_i}\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \frac{V}{3}$$

Similarly

$$\{f_{b_j}\} = \{f_{bm}\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \frac{V}{3} \tag{6.3.6}$$

6.24



$$N_1 = \frac{(b-x)(h-y)}{4bh},\; N_2 = \frac{(b+x)(h-y)}{4bh}$$

$$N_3 = \frac{(b+x)(h+y)}{4bh},\; N_4 = \frac{(b-x)(h+y)}{4bh} \qquad (1)$$

at center ($x = 0, y = 0$)

$$N_1 = \frac{1}{4},\; N_2 = \frac{1}{4},\; N_3 = \frac{1}{4},\; N_4 = \frac{1}{4}$$

$$N_1 + N_2 + N_3 + N_4 = 1$$

at point $\left(x = \dfrac{b}{2}, y = \dfrac{h}{2}\right)$

$$N_1 = \frac{\left(b - \frac{b}{2}\right)\left(h - \frac{h}{2}\right)}{4bh} = \frac{1}{16}$$

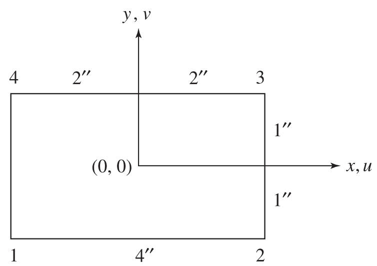
$$N_2 = \frac{3}{16},\; N_3 = \frac{9}{16},\; N_4 = \frac{3}{16}$$

$$\therefore\; N_1 + N_2 + N_3 + N_4 = 1$$

In general add the functions in Equation (1) and you get for all x and y on the element

$$N_1 + N_2 + N_3 + N_4 = 1$$

6.25



$$\{\sigma\} = [D]\,[B]\,\{d\}$$

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & h-y & 0 & h+y & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & b+x & 0 & b-x \\ -(b-x) & -(h-y) & -(b+x) & h-y & b+x & h+y & b-x & -(h+y) \end{bmatrix}$$

At center ($x = 0, y = 0$)

$$[B] = \frac{1}{8}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$\{\varepsilon\} = \frac{1}{8}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.005 \\ 0.0025 \\ 0.0025 \\ -0.0025 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 0.0009375 \\ -0.00125 \\ -0.000625 \end{Bmatrix} \frac{\text{in.}}{\text{in.}}$$

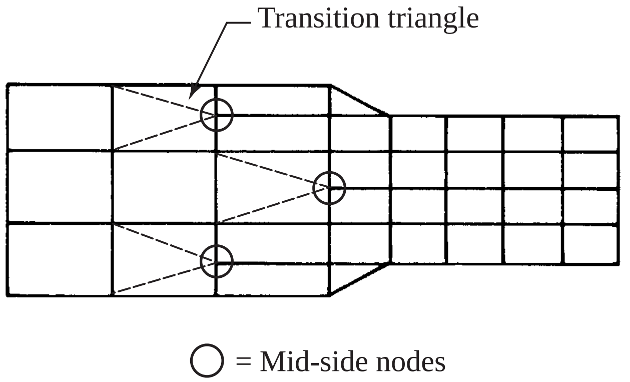
$$\{\sigma\} = [D]\,\{\varepsilon\}$$

$$= \frac{30 \times 10^6}{1 - 0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 0.0009375 \\ -0.00125 \\ -0.000625 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 18.54 \\ -31.94 \\ -7.21 \end{Bmatrix} \text{ ksi}$$

Chapter 7

7.1 For the simple 4 noded elements it is a violation of displacement compatibility to have a mid-side node. Some of the elements have mid-side nodes in this model. Use ‘transition’ triangle to go from smaller to larger rectangular elements.



7.2 The mesh sizing is not fine enough in the reentrant corner region at *C*. We need smaller elements near point *C* and small radius at *C*.

7.3 Based on the formulation used here we can not have $\mu = 0.5$ for the plane strain case as the denominator in the material property matrices [*D*] (see Equation (6.1.10) and [*K*] (see Equation (6.4.3) becomes zero. A penalty formulation see Reference [7] can be used to avoid this problem.

7.4 The structure is plane strain if this section represents a cross section of a long structure in which the loads do not vary in the *z* direction.
The structure is a plane stress problem if this section is a thin plate type structure with loads in the plane of the structure only.

Also see Section 6.1 for descriptions of plane stress and plane strain and examples of each.

7.5 When abrupt changes in thickness at *E*’s occur from element to element.

7.6 Unit thickness/7.7 (a) best aspect ratio.

7.9 (a) No, as replacing a portion of the patch by a different material with different mechanical properties will in general produce non-uniform strain under constant state of applied stress. For rigid body mode tests, however, different mechanical property materials still result in rigid body displacement.

(b) Yes, the patch can be arbitrary in shape. If we apply a test displacement field of $u_x = 1$, $u_y = 0$ at the external nodes of a patch of say 4 elements and set the internal nodal force to zero, then solve for the displacement components at internal node *i*, these displacement components should agree with the value of the displacement function at that node. Also the strain function or field should vanish identically at any point over each element.

(c) Yes, we can mix triangular and quadrilateral elements in a 2-d patch test as long as the material properties are the same.

(d) No. Mixing bars with plane elements would alter the constant strain states as the plane element and bar are of different structural types.

(e) The patch test should be applied when developing new finite elements, to determine if the element can represent rigid body motion as well as states of constant strain when these conditions occur.

7.10 Using Mathcad

$A = 1 \times 10^{-4}$ $E = 200 \times 10^{9}$ $L_1 = 0.6$ $[L_2] = 1.4$

$$[k_1] = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [k_1] = [k_1]\, A\, \frac{E}{L_1} \qquad [k_1] = \begin{pmatrix} 3.333\times 10^{7} & -3.333\times 10^{7} & 0 \\ -3.333\times 10^{7} & 3.333\times 10^{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[k_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad [k_2] = [k_2]\, A\, \frac{E}{L_2} \qquad [k_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1.429\times 10^{7} & -1.429\times 10^{7} \\ 0 & -1.429\times 10^{7} & 1.429\times 10^{7} \end{pmatrix}$$

$$[k] = [k_1] + [k_2] \qquad [k] = \begin{pmatrix} 3.333\times 10^{7} & -3.333\times 10^{7} & 0 \\ -3.333\times 10^{7} & 4.762\times 10^{7} & -1.429\times 10^{7} \\ 0 & -1.429\times 10^{7} & 1.429\times 10^{7} \end{pmatrix}$$

Set these 3 values to defined quantities of $u_1 = u_3 = 1$ for the rigid body patch test

$u_1 = 1$ $u_3 = 1$ $F_2 = 0$

Guess at F_1, F_3, and u_2 as shown below.

$F_1 = 1$ $u_2 = 0$ $F_3 = 1$

Given Use the given command to create a solve block.

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = [k] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Use control and equal sign here.

$$\begin{pmatrix} F_1 \\ u_2 \\ F_3 \end{pmatrix} = \text{Find}\,(F_1, u_2, F_3)$$

Use the 'Find' command to find F_1, u_2, and F_3.

$F_1 = 0$ $u_2 = 1$ $F_3 = 0$

The rigid body motion patch test is satisfied as $u_2 = 1$.

Now check the constant strain test. Let $u(x) = x$ for the nodes at the boundaries, i.e., $u_1 = 0$ and $u_3 = 2$, Verify that u_2 $(x = 0.6) = 0.6$.

$u_1 = 0$ $u_3 = 2$ $F_2 = 0$ Initial these values

$F_1 = 1$ $F_3 = 1$ $u_2 = 0$ Guesses for these values.

Given

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = [k] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

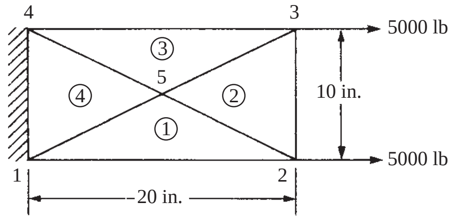
$$\begin{pmatrix} F_1 \\ u_2 \\ F_3 \end{pmatrix} = \text{Find}\,(F_1, u_2, F_3)$$

Use the 'Find' command to solve for F_1, u_2, and F_3.

$F_1 = -2 \times 10^{7}$ $F_3 = 2 \times 10^{7}$ $u_2 = 0.6$

Now upon solving the system of equations $u_2 = 0.6$ as it should to satisfy the patch test for constant strain.

7.12



INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	5
NUMBER OF ELEMENTS.	4
NUMBER OF DIFFERENT MATERIALS.	1
NUMBER OF SURFACE LOAD CARDS.	0
1 = PLANE STRAIN, 2 = PLANE STRESS	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO,	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.3000E+08	0.3000E+00	0.0000E+00	0.1000E+01

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0	0.2000E+02	0.0000E+00	0.5000E+04	0.0000E+00
3	0	0.2000E+02	0.1000E+02	0.5000E+04	0.0000E+00
4	3	0.0000E+00	0.1000E+02	0.0000E+00	0.0000E+00
5	0	0.1000E+02	0.5000E+01	0.0000E+00	0.0000E+00

INPUT TABLE 4.. ELEMENT DATA

	GLOBAL	INDICES OF	ELEMENT	NODES	
ELEMENT	1	2	3	4	MATERIAL
1	1	2	5	5	1
2	2	3	5	5	1
3	5	3	4	4	1
4	1	5	4	4	1

OUTPUT TABLE 1.. NODAL DISPLACEMENTS

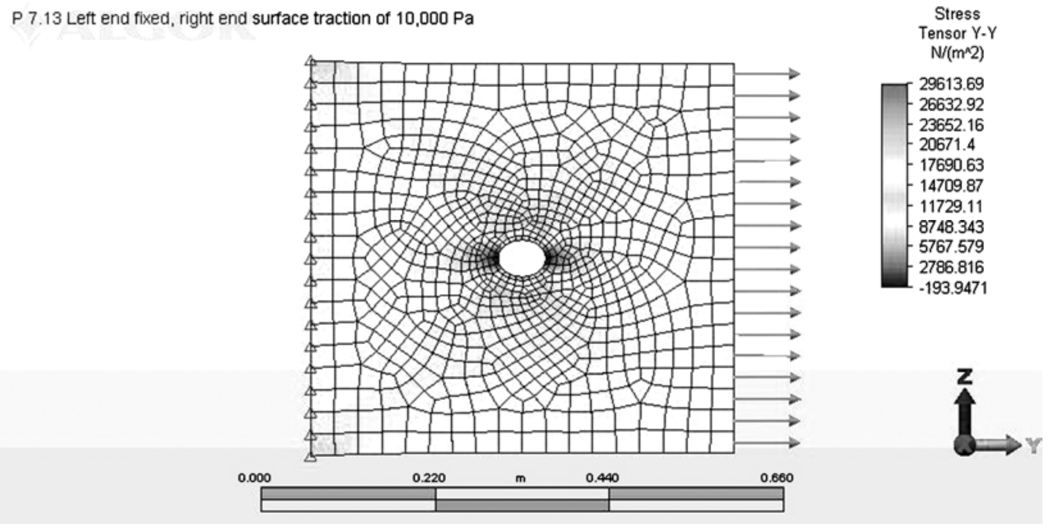
NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.64664544E–03	0.66631597E–04
3	0.61664509E–03	–0.66630528E–04
4	0.00000000E+00	0.000000000E+00
5	0.30527671E–03	0.24373945E–09

OUTPUT TABLE 2.. STRESSES AT ELEMENT CENTROIDS

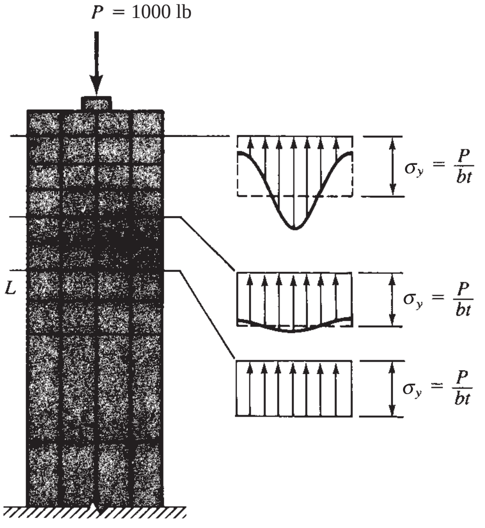
ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	10.00	1.67	1.0000E+03	1.0011E+02	–3.2032E+00
2	16.67	5.00	9.9359E+02	–1.0171E+02	1.2599E–05
3	10.00	8.33	1.0000E+03	1.0011E+02	3.2035E+00
4	3.33	5.00	1.0064E+03	3.0192E+02	2.8124E–04

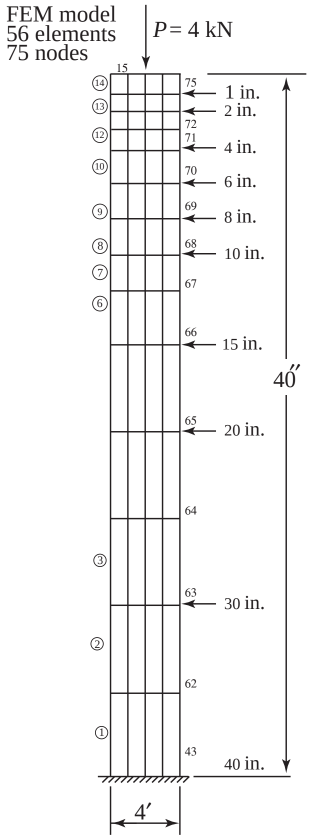
SIGMA(1)	SIGMA(2)	ANGLE
1.0000E+03	1.0010E+02	–2.0395E–01
9.9359E+02	–1.0171E+02	6.5906E–07
1.0000E+03	1.0010E+02	2.0396E–01
1.0064E+03	3.0192E+02	2.2873E–05

7.13



7.14

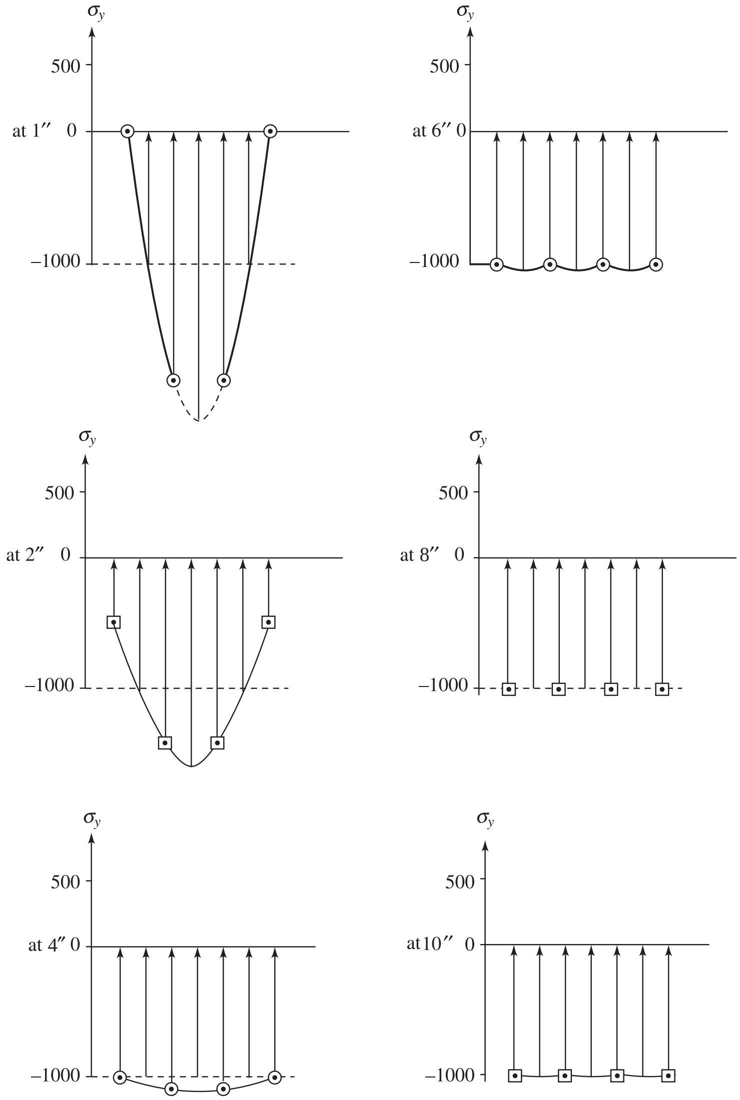




STRESS IN PSI AT VARIOUS DISTANCES
ALONG THE TENSILE PLATE WITH A 1000 # LOAD

ELEMENT #	14	28	42	56	
STRESS	26	–2026	–2026	26	(AT 1″)
ELEMENT #	13	27	41	55	
STRESS	–563	–1437	–1437	–563	(AT 2″)
ELEMENT #	11	25	39	53	
STRESS	–969	–1031	–1031	–969	(AT 4″)
ELEMENT #	10	24	38	52	
STRESS	–1002	–998	–998	–1002	(AT 6″)
ELEMENT #	9	23	37	51	
STRESS	–1002	–998	–998	–1002	(AT 8″)
ELEMENT #	8	22	36	50	
STRESS	–1001	–1000	–1000	–1001	(AT 10″)
ELEMENT #	6	20	34	48	
STRESS	–1000	–1000	–1000	–1000	(AT 15″)
ELEMENT #	5	21	33	47	
STRESS	–1000	–1000	–1000	–1000	(AT 20″)
ELEMENT #	3	17	31	45	
STRESS	–1002	–998	–998	–1002	(AT 30″)
ELEMENT #	1	15	29	43	
STRESS	–1005	–995	–995	–1005	(AT 40″)

Stress in plate at various depths (psi)



7.15

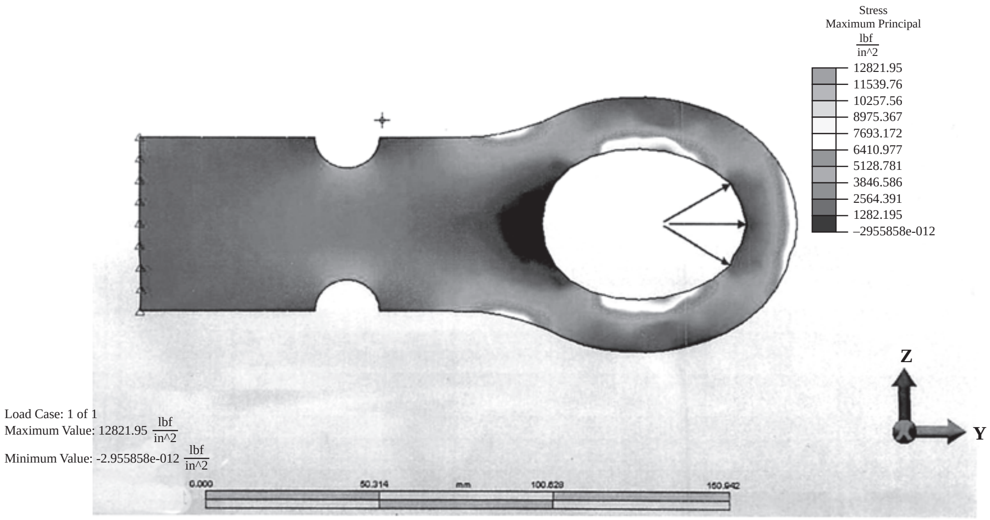
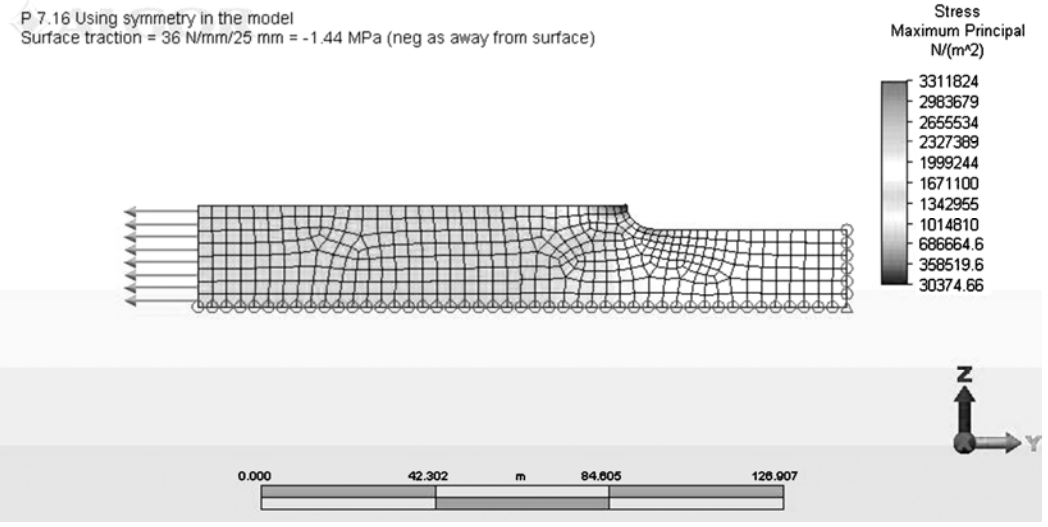


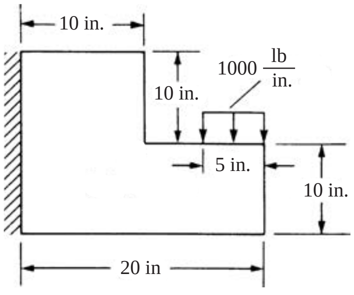
Figure. 8

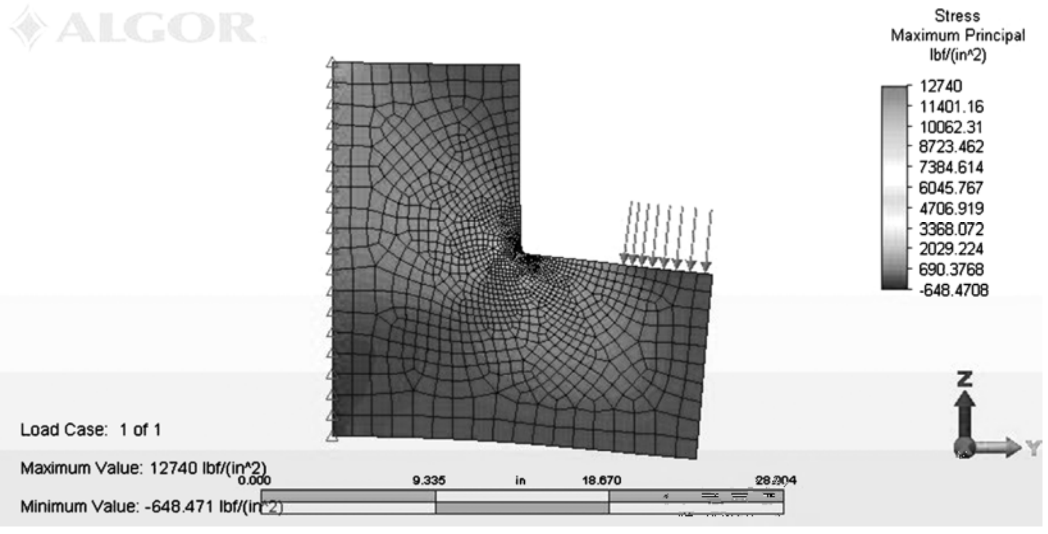
The maximum principal stresses are at the top and bottom of the circular opening of the connecting rod. The maximum stresses can be seen in Figure 8 above. Using a mesh density of only 400 (Figure 1- Figure 4) the precision in the area of interest (i. e., where the maximum stress occurs); see Figure 1 and Figure 4, the precision was about 0.28 further refinement was required. The mesh was refined to 800 and 1200 then finally to 1600. With the mesh density being 1600 the precision was less than 0.1 in. the place of interest and can be assumed correct. The maximum principal stress of about 12822 $\frac{\text{lb}}{\text{in.}^2}$ was determined.

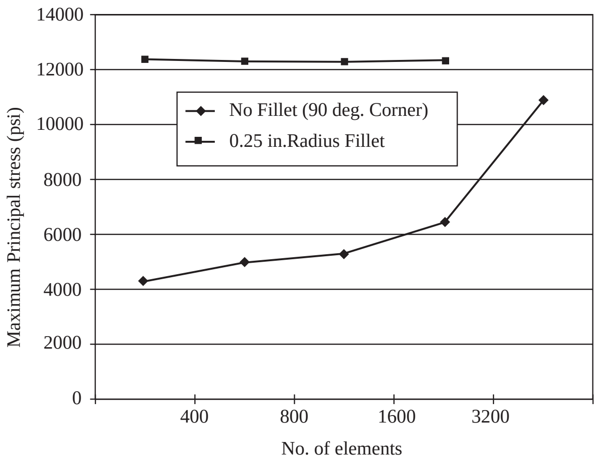
7.16



7.17

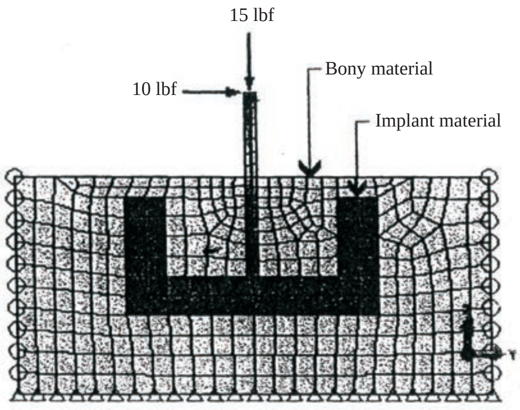






Maximum principal stress vs no. of elements

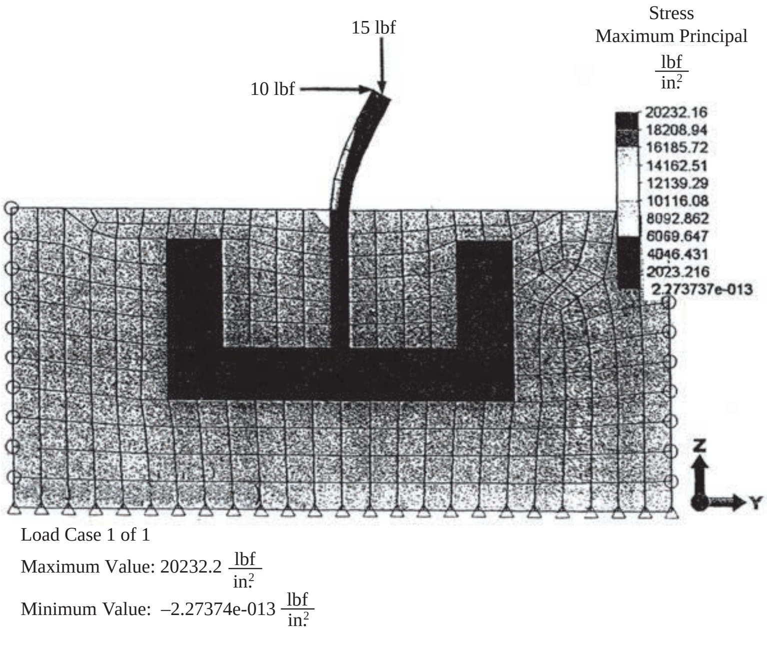
7.19



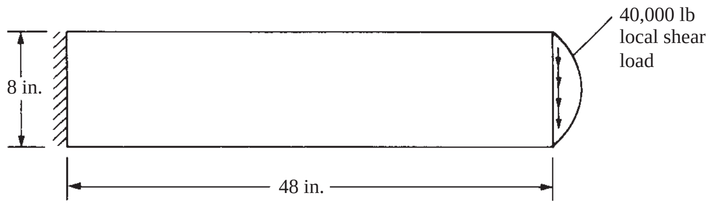
Model Parameters

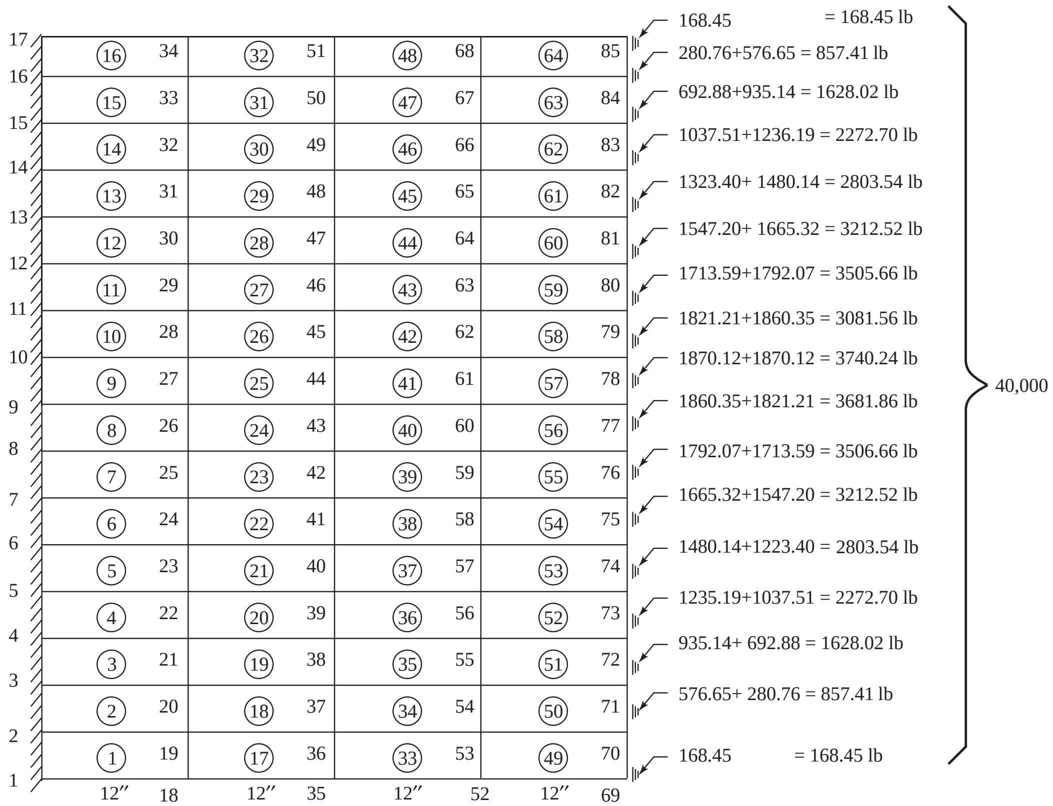
Implant material modulus of elasticity 1.6×10^6 psi
Bony material modulus of elasticity 1.0×10^6 psi
Implant depth below bony material 0.100 in.

400 Mesh Density



7.20





64 – 12 in. by $\frac{1}{2}$ in. elements

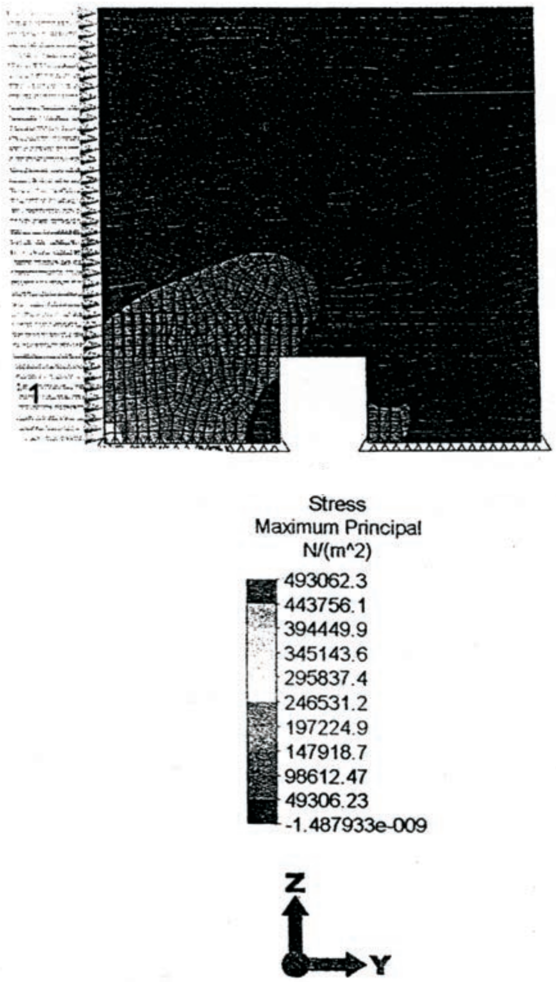
From computer program output

$$Y_{\text{maxA}} = -0.4993 \text{ in.} \quad Y_{\text{max exact}} = -1.152 \text{ in.} = \frac{PL^3}{3\,EI}$$

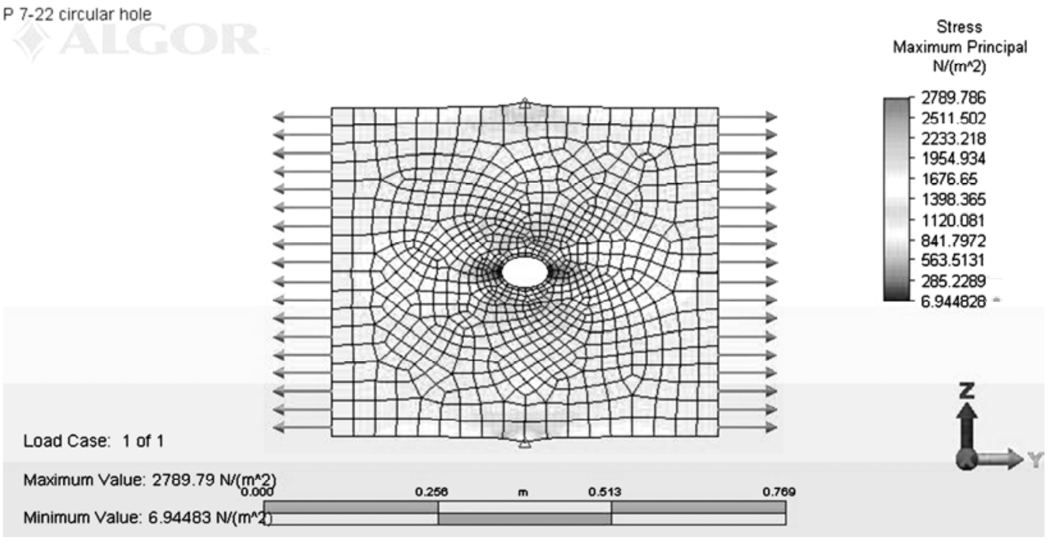
$$AR = \frac{12}{\frac{1}{2}} = 24 \text{ (56\% error due to large aspect ratio)}$$

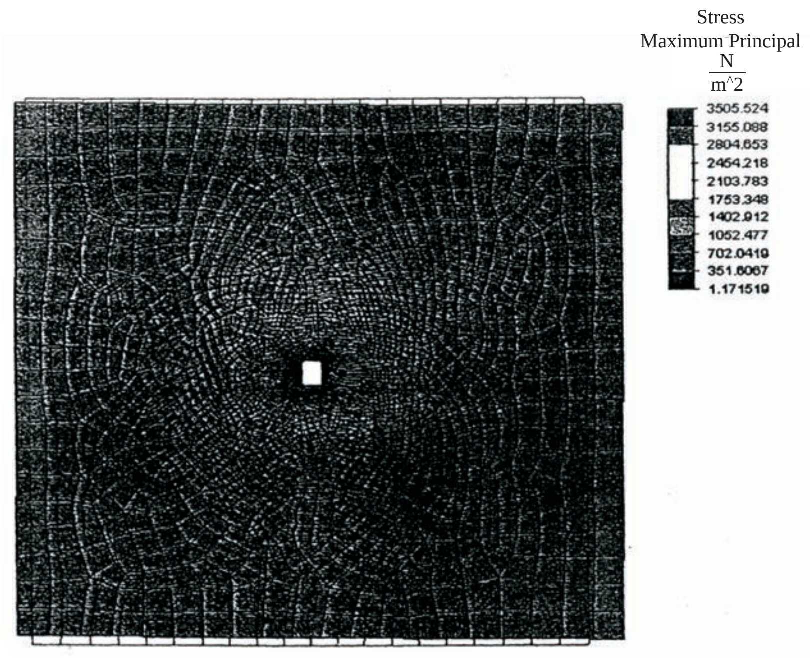
For other results see Example in Section 7.1, Table 7.1

7.21


Stress
Maximum Principal
N/(m^2)
493062.3
443756.1
394449.9
345143.6
295837.4
246531.2
197224.9
147918.7
98612.47
49306.23
-1.487933e-009
Z
Y

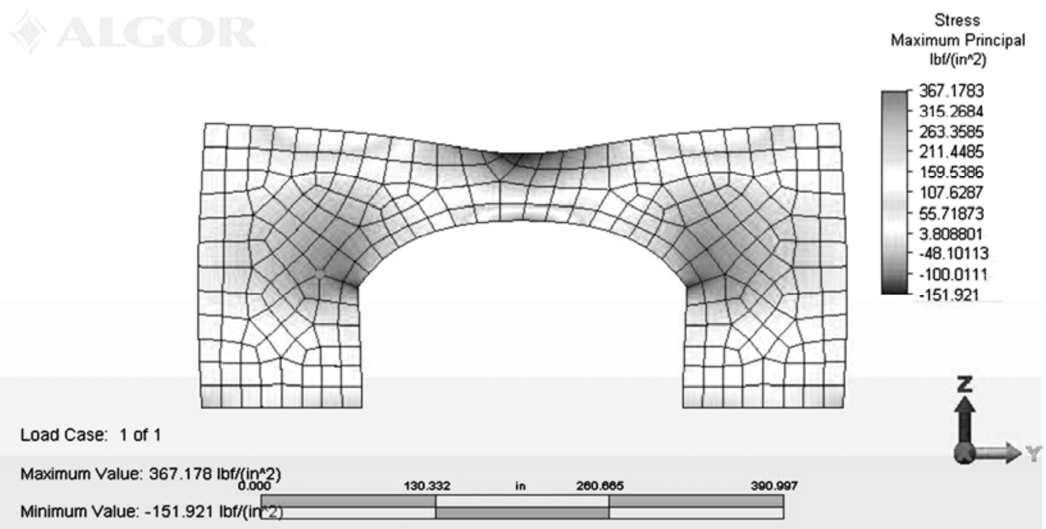
7.22


P 7-22 circular hole
ALGOR
Stress
Maximum Principal
N/(m^2)
2789.786
2511.502
2233.218
1954.934
1676.65
1398.365
1120.081
841.7972
563.5131
285.2289
6.944828
Z
Y
Load Case: 1 of 1
Maximum Value: 2789.79 N/(m^2)
Minimum Value: 6.94483 N/(m^2)
0.000
0.256
m
0.513
0.769

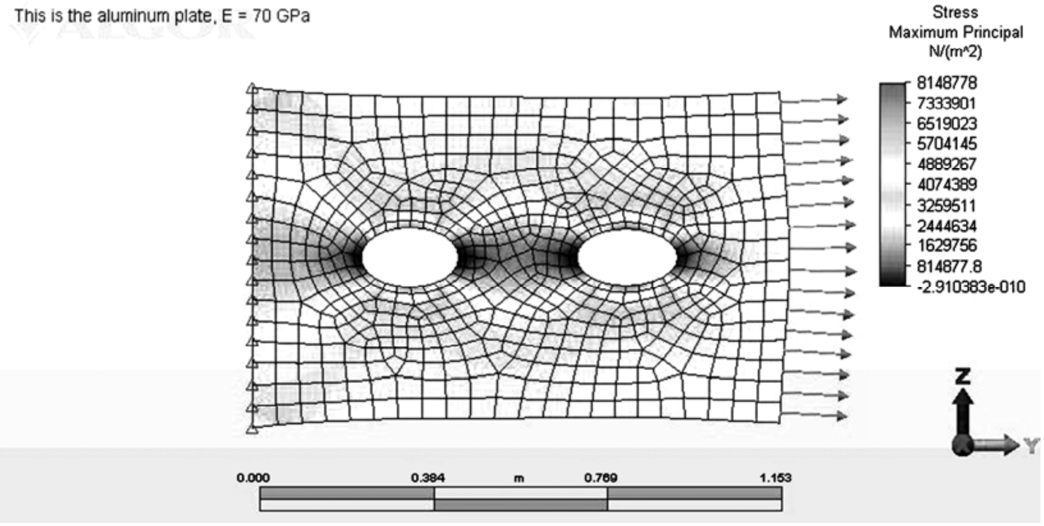


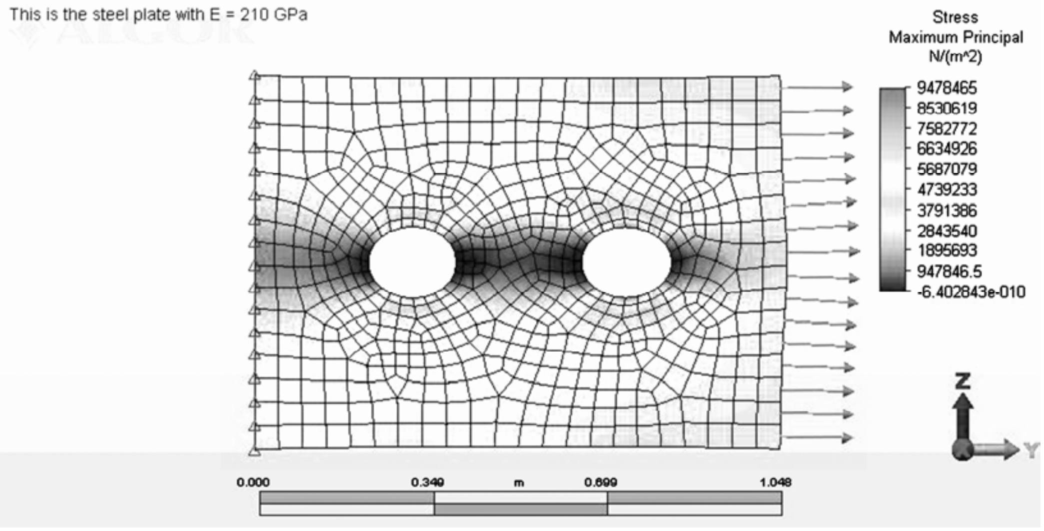
The figure above is the maximum principal stress. The maximum is 3505 $\frac{N}{m^{\wedge}2}$ The location of maximum stress occurs at the corners of the hole (with 1 mm radii)

7.23

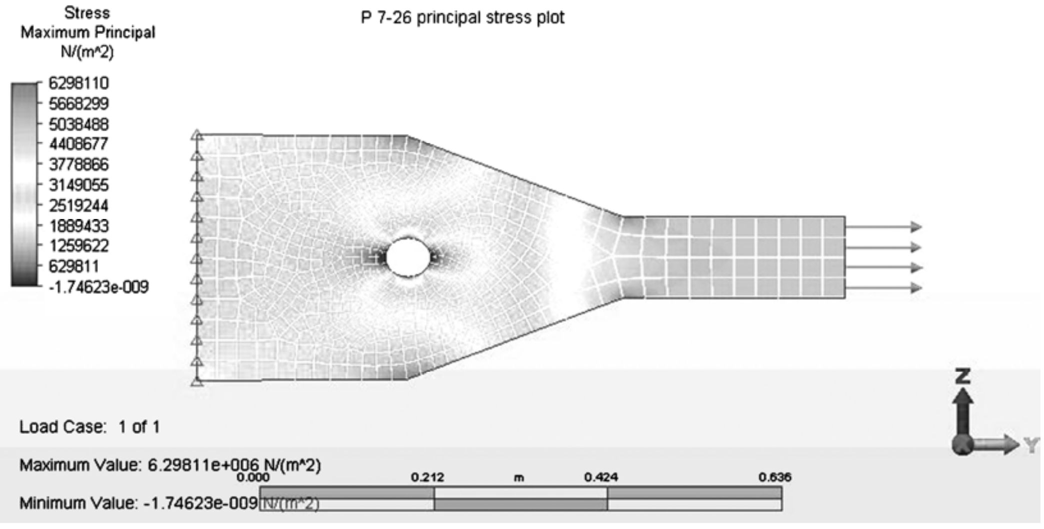


7.25


This is the aluminum plate, E = 70 GPa
Stress
Maximum Principal
N/(m^2)
8148778
7333901
6519023
5704145
4889267
4074389
3259511
2444634
1629756
814877.8
-2.910383e-010
Z
Y
0.000
0.384
m
0.769
1.153

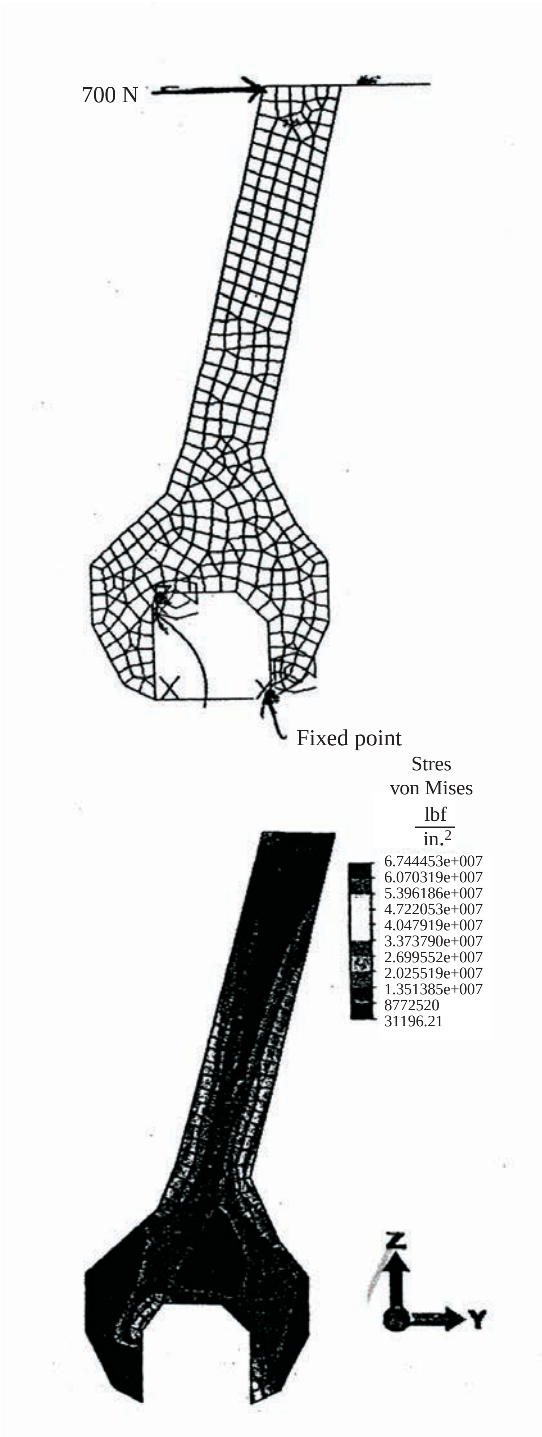

This is the steel plate with E = 210 GPa
Stress
Maximum Principal
N/(m^2)
9478465
8530619
7582772
6634926
5687079
4739233
3791386
2843540
1895693
947846.5
-6.402843e-010
Z
Y
0.000
0.349
m
0.699
1.048

7.26


Stress
Maximum Principal
N/(m^2)
P 7-26 principal stress plot
6298110
5668299
5038488
4408677
3778866
3149055
2519244
1889433
1259622
629811
-1.74623e-009
Load Case: 1 of 1
Maximum Value: 6.29811e+006 N/(m^2)
Minimum Value: -1.74623e-009 N/(m^2)
Z
Y
0.000
0.212
m
0.424
0.636

The largest principal stress of 6.29 MPa occurs at the top and bottom inside edges of the hole. The second largest principal stress of 5.67 MPa occurs at the elbow between the smallest cross section and where the taper begins.

7.28



7.31

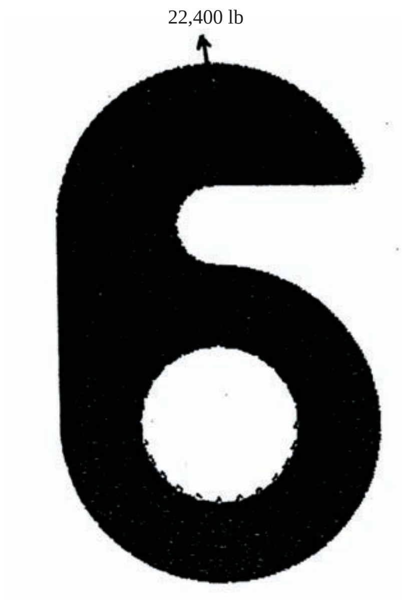


Figure 1: Mesh with Boundary Conditions and Nodal Force

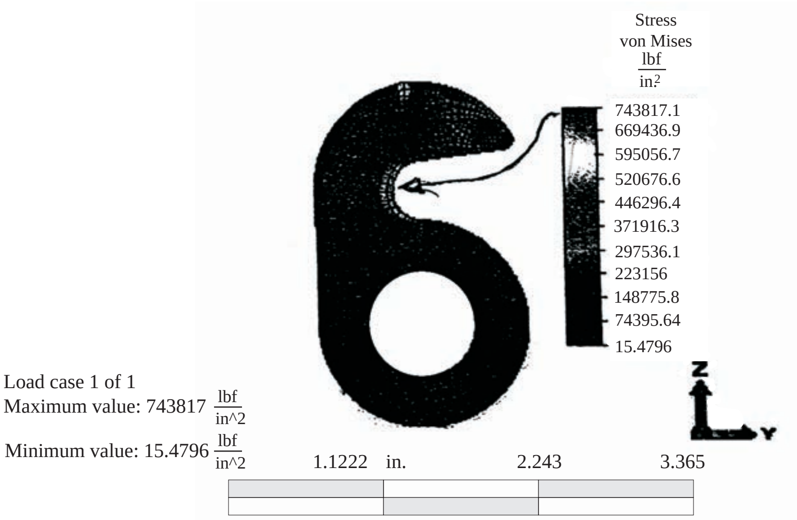
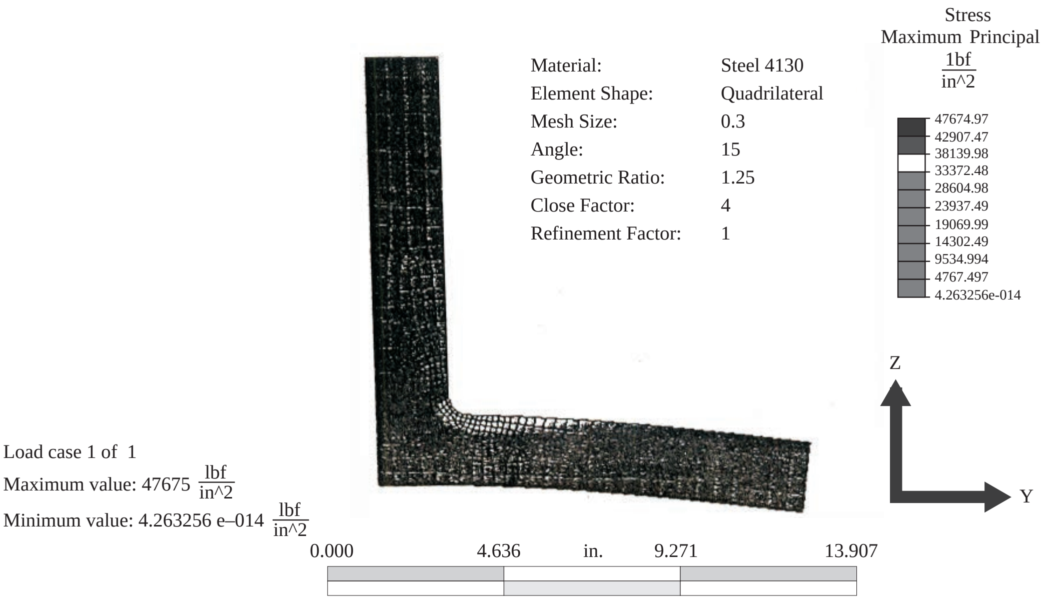


Figure 2: von Mises stresses (psi)

7.32

600 Mesh Density

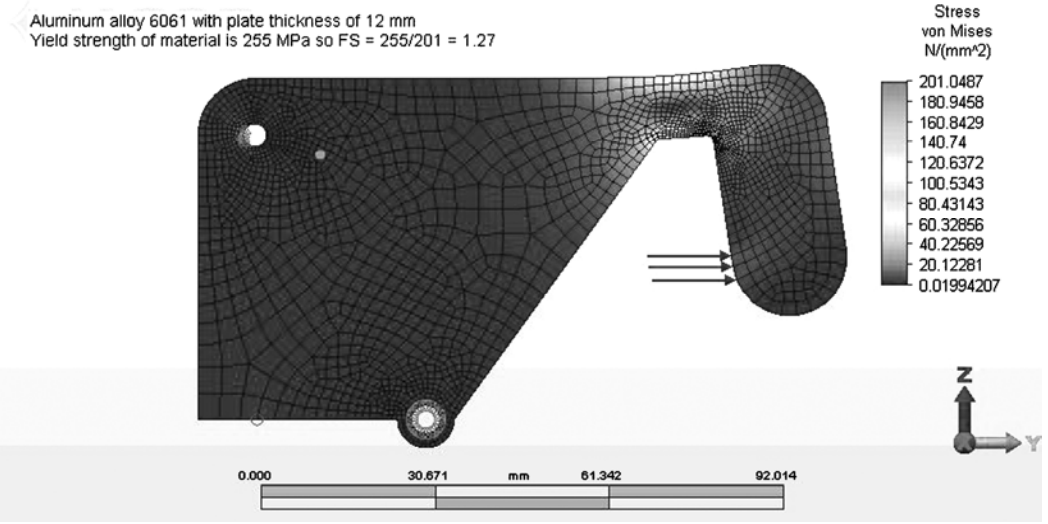


Mesh Density: 25-600

Bracket without Fillet	Maximum Principal Stress $\frac{\text{lb}}{\text{in.}^2}$
25 Mesh Density	22811.17
50	26114.15
100	27050.65
150	28179.32
200	28967.93
300	28800.52
400	35102.97
500	32852.23
600	33678.14
Bracket with Fillet	
25 Mesh Density	47481.11
50	47492.06
100	47502.16
150	47511.98
200	47521.59
300	47832.01
400	47688.08
500	47658.56
600	47674.97

In the FEA world re-entrant corners are a bad thing. These represent an infinite change in stiffness inside the part, which will result in an infinite stress concentration.

7.33



7.37

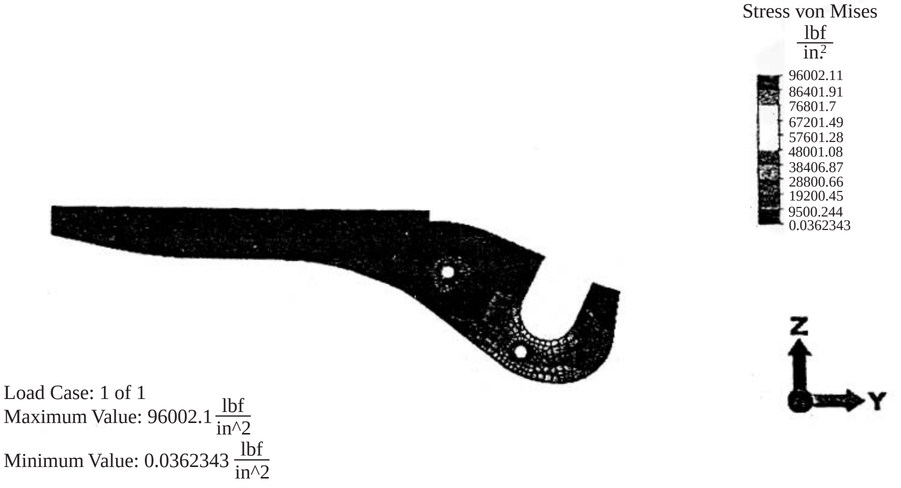
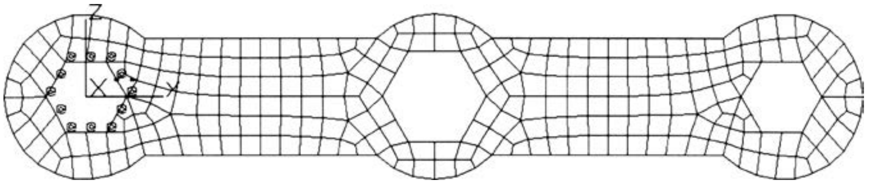


Figure 3: Maximum Stress Plot

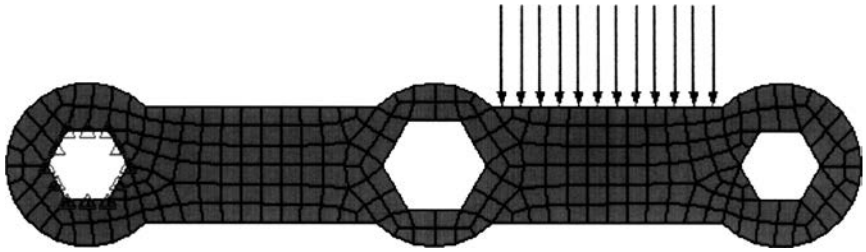
- Maximum von Mises stress occurred at the base of the notch in the crimper tool. The value was 96,002 psi.

7.38

The model is shown first with the boundary and loading conditions were then applied. The nodes of the far left hex were constrained from all movement. The red surface in the second figure below was selected and changed to surface 2. This allow the 100 $\frac{\text{N}}{\text{cm}^2}$ = 10,000,000 $\frac{\text{dyn}}{\text{cm}^2}$ force to be applied to the surface as shown.



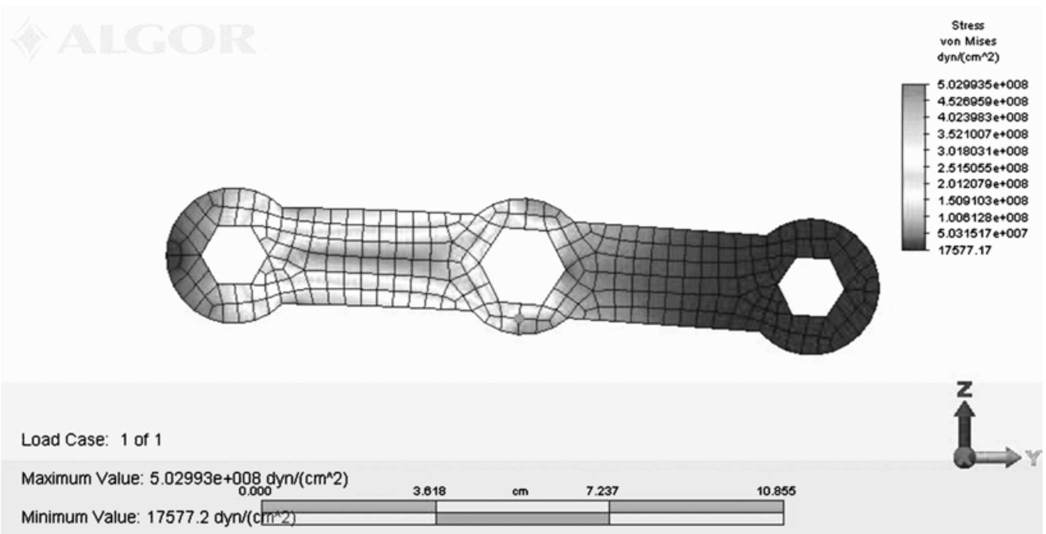
Next a material was chosen and an initial guess at the thickness *t* was made. ASTM A-514 was chosen, as this is a quenched and tempered steel with a high yield strength and will allow for the thickness to be minimized. A thickness of 1 cm was chosen as the initial guess, as this is an easy number to work with and it is compatible with the other wrench dimensions. A check was then performed to insure that the model properly reflected the problem. This check is shown below.



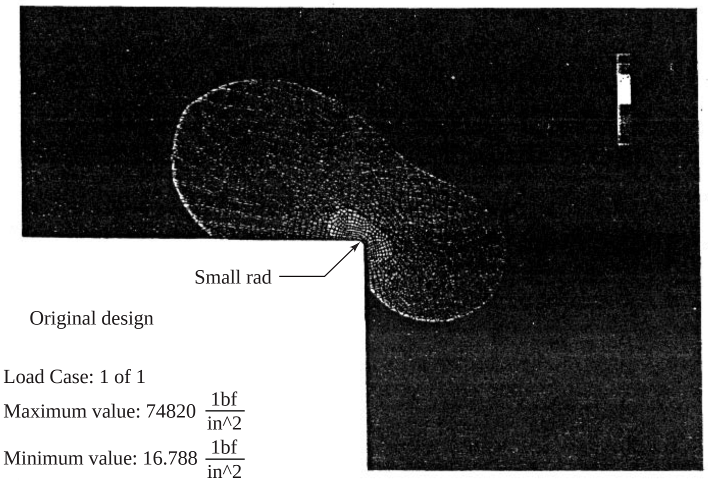
Material	Thickness (cm)	Pressure $\left(\frac{\text{N}}{\text{cm\^{}2}}\right)$	Stress $\left(\frac{\text{N}}{\text{cm\^{}2}}\right)$
ASTM A 514	1	100	5.03E+08
ASTM A 514	0.1	1000	5.03E+09
Al 3003-H16	0.3175	314.96	1.58E+09
Al 3003-H16	0.4	250	1.26E+09
Al 3003-H16	0.5	200	1.01E+09
Al 3003-H16	0.47625	209.97	1.06E+09

$$\frac{2}{3}S_y = \frac{2}{3} * 1.72\text{e}9 = 1.15\text{e}9 \frac{\text{dyn}}{\text{cm}^2}$$

With a thickness of 0.47625 cm, the stress was found to be $1.06 * 10^9 \ \frac{\text{dyn}}{\text{cm}^2}$.



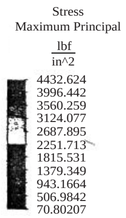
7.39 Zoomed in of the previous. To simulate a real cut, I inserted a very small radius at the point of concern.



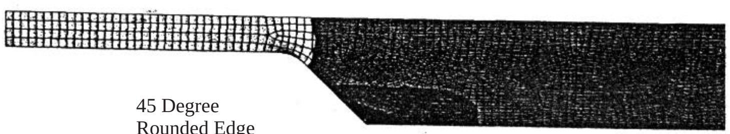
Load Case: 1 of 1

Maximum value: 74820 $\frac{1bf}{in\text{^}2}$

Minimum value: 16.788 $\frac{1bf}{in\text{^}2}$



Maximum principal and von Mises look very similar

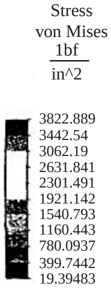


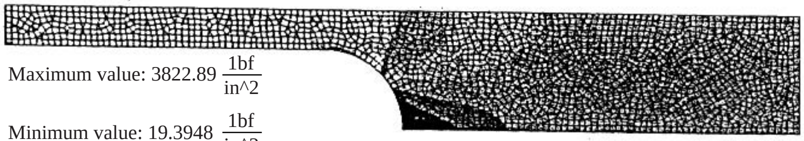
Load Case: 1 of 1

Maximum Value: 4432.62 $\frac{1bf}{in\text{^}2}$

Minimum Value: 70.8021 $\frac{1bf}{in\text{^}2}$

This geometry gave me the best result and had a better precision than the other configurations.



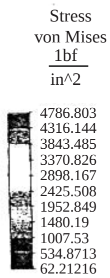


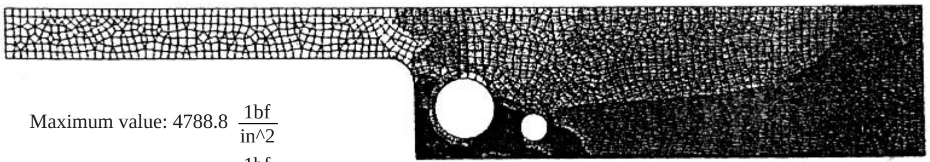
Maximum value: 3822.89 $\frac{1bf}{in^2}$

Minimum value: 19.3948 $\frac{1bf}{in^2}$

Radius Model

This geometry had some interesting results but the overall stress was still higher than the rounded off configuration





Maximum value: 4788.8 $\frac{1bf}{in^2}$

Minimum value: 62.2122 $\frac{1bf}{in^2}$

Relief Holes Model

Chapter 8

8.1 Triangular element

From Section 8.2

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{b} + \frac{2x^2}{b^2} + \frac{4xy}{bh} + \frac{2y^2}{h^2}$$

$$N_2 = \frac{-x}{b} + \frac{2x^2}{b^2},\ N_3 = \frac{-y}{h} + \frac{2y^2}{h^2}$$

$$N_4 = \frac{4xy}{bh},\ N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2}$$

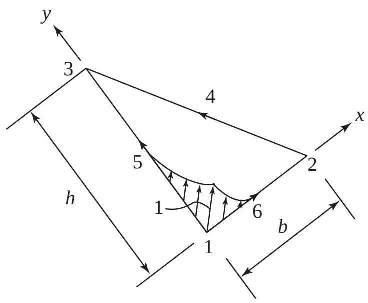
$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

(a)

At $x = 0$ evaluate N's

$y = 0$

$N_1 = 1,\ N_2 = 0 = N_3 = N_4 = N_5 = N_6$



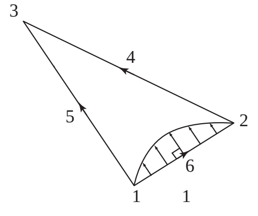
(b)

At $x = \frac{b}{2}$

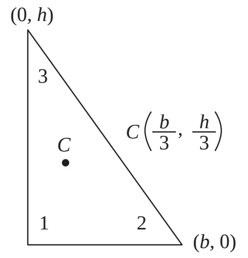
$y = 0$

$$N_1 = 1 - \frac{3(\frac{b}{2})}{b} - \frac{3(0)}{b} + \frac{2\left(\frac{b}{2}\right)^2}{b^2} + \frac{4\,(0)\,(0)}{bh} + \frac{2(0)^2}{h^2} = 1 - \frac{3}{2} + \frac{1}{2} = 0$$

and $N_2 = N_3 = N_4 = N_5 = 0,\quad N_6 = \frac{4\left(\frac{b}{2}\right)}{b} - \frac{4\left(\frac{b}{2}\right)^2}{b^2} = 1$



8.2



Strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ v_6 \end{Bmatrix} \quad (1)$$

Evaluate β's and γ's at centroid

$$\beta_1 = -3h + \frac{4hx}{b} + 4y = -3h + \frac{4h\left(\frac{b}{3}\right)}{b} + 4\left(\frac{h}{3}\right) = -\frac{h}{3}$$

$$\beta_2 = -h + \frac{4hx}{b} = -h + \frac{4h\left(\frac{b}{3}\right)}{b} = \frac{1}{3}h$$

$$\beta_3 = 0,\ \beta_4 = 4y = 4\left(\frac{h}{3}\right) = \frac{4h}{3}$$

$$\beta_5 = -4y = \frac{-4h}{3}$$

$$\beta_6 = 4h - \frac{-8hx}{b} - 4y = 4h - \frac{8h\left(\frac{b}{3}\right)}{b} - 4\left(\frac{h}{3}\right) = 0$$

$$\gamma_1 = -3b + 4x + \frac{4by}{h} = -3b + 4\left(\frac{b}{3}\right) + \frac{4b\left(\frac{h}{3}\right)}{h} = \frac{-b}{3}$$

$$\gamma_2 = 0,\ \gamma_3 = -b + \frac{4by}{h} = -b + \frac{4b\left(\frac{h}{3}\right)}{h} = \frac{b}{3}$$

$$\gamma_4 = 4x = \frac{4b}{3},\ \gamma_5 = 4b - 4x - \frac{8by}{h} = 4b - 4\left(\frac{b}{3}\right) - \frac{8b\left(\frac{h}{3}\right)}{h} = 0$$

$$\gamma_6 = \frac{-4b}{3} \quad (2)$$

Performing the multiplications in (1)
(After substituting β's and γ's from (2))

$$\varepsilon_x = \left(\frac{-h}{3}u_1 + \frac{h}{3}u_2 + \frac{4h}{3}u_4 - \frac{4h}{3}u_5\right)\frac{1}{bh}$$

$$\varepsilon_y = \left(\frac{-b}{3}v_1 + \frac{b}{3}v_3 + \frac{4b}{3}v_4 - \frac{4b}{3}v_6\right)\frac{1}{bh}$$

$$\gamma_{xy} = \left(\frac{-b}{3}u_1 - \frac{h}{3}v_1 + \frac{h}{3}v_2 + \frac{b}{3}u_3 + \frac{4b}{3}u_4 + \frac{4h}{3}v_4 - \frac{4h}{3}v_5 - \frac{4b}{3}u_6\right)\frac{1}{bh}$$

$$\varepsilon_x = \frac{h}{3}\ [-u_1 + u_2 + 4u_4 - 4u_5]\left(\frac{1}{bh}\right)$$

$$\varepsilon_y = \frac{b}{3}\ [-v_1 + v_3 + 4v_4 - 4v_6]\left(\frac{1}{bh}\right)$$

$$\gamma_{xy} = \left\{\frac{b}{3}[-u_1 + u_3 + 4u_4 - 4u_6] + \frac{h}{3}[-v_1 + v_2 + 4v_4 - 4v_5]\right\}\left(\frac{1}{bh}\right)$$

Stresses

$$\{\sigma\} = [D]\ \{\varepsilon\}$$

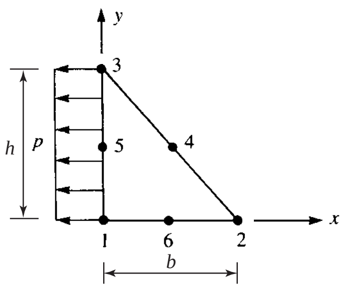
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\sigma_x = \frac{E}{1-v^2}\left[\frac{h}{3}(-u_1 + u_2 + 4u_4 - 4u_5) + v\,\frac{b}{3}(-v_1 + v_3 + 4v_4 - 4v_6)\right]\frac{1}{bh}$$

$$\sigma_y = \frac{E}{1-v^2}\left[v\,\frac{h}{3}(-u_1 + u_2 + 4u_4 - 4u_5) + \frac{b}{3}(-v_1 + v_3 + 4v_4 - 4v_6)\right]\frac{1}{bh}$$

$$\tau_{xy} = \frac{E}{2(1+v)}\left[\frac{b}{3}(-u_1 + u_3 + 4u_4 - 4u_6) + \frac{h}{3}(-v_1 + v_2 + 4v_4 - 4v_5)\right]\frac{1}{bh}$$

8.3



The equation is $\{f_s\} = \int_s [N_s]^T \{T\}\ ds$ (1)

$$\{T\} = \begin{Bmatrix}p_x\\ p_y\end{Bmatrix} = \begin{Bmatrix}p\\ 0\end{Bmatrix}$$ …. is the surface traction (2)

$$[N] = \begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6\end{bmatrix} \quad (3)$$

Substituting (2) and (3) in Equation (1), we have

$$\{f_s\} = \int_0^t \int_0^h \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ \vdots & \vdots \\ N_6 & 0 \\ 0 & N_6 \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dy\, dz \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix}$$

$$\{f_s\} = t \int_0^h \begin{Bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \\ \vdots \\ N_6 p \\ 0 \end{Bmatrix} dy \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix} \tag{4}$$

From Section 8.2 for this particular element we have

$$N_1 = 1 - \frac{3x}{b} - \frac{3y}{h} + 2x^2 + 4xy + \frac{2y^2}{h^2}$$

$$N_2 = \frac{-x}{b} + \frac{2x^2}{b^2},\ N_3 = \frac{-y}{h} + \frac{2y^2}{h^2}$$

$$N_4 = \frac{4xy}{bh},\ N_5 = \frac{4y}{h} - \frac{4xy}{bh} - \frac{4y^2}{h^2} \tag{5}$$

$$N_6 = \frac{4x}{b} - \frac{4x^2}{b^2} - \frac{4xy}{bh}$$

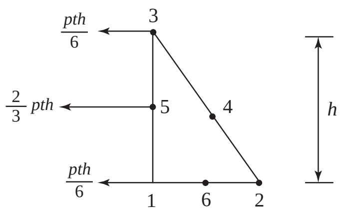
Substitute (5) into (4) and evaluating N's at $x = 0$, $y = y$, we have

$$\{f_s\} = t\int_0^h \begin{Bmatrix} \left(1-\frac{3y}{h}+\frac{2y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(\frac{-y}{h}+\frac{2y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(\frac{4y}{h}-\frac{4y^2}{h^2}\right)p \\ 0 \\ 0 \\ 0 \end{Bmatrix} dy$$

$$f_{s1x} = pt\left(y - \frac{3y^2}{2h} + \frac{2y^3}{3h^2}\right)\Bigg|_0^h = \frac{pth}{6}$$

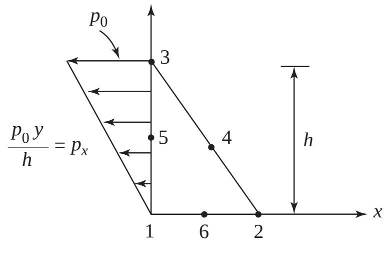
$$f_{s3x} = \left(\frac{-y^2}{2h} + \frac{2y^3}{3h^2}\right) pt = \frac{pth}{6}$$

$$f_{s5x} = pt\left(\frac{4y^2}{2h} - \frac{4y^3}{3h^2}\right)\Bigg|_0^h = \frac{2\ pth}{3}$$



Nodal equivalent forces

8.4



$$\{f_s\} = \int_s [N_s]^T \{T\} ds \tag{1}$$

$$\{T\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} \frac{p_0\, y}{h} \\ 0 \end{Bmatrix} \tag{2}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \quad (3)$$

Substituting (2) and (3) in (1)

$$\{f_s\} = \int_0^t \int_0^h \left\{ \begin{matrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ \vdots & \\ N_6 & 0 \\ 0 & N_6 \end{matrix} \right\} \left\{ \begin{matrix} \frac{p_0 y}{h} \\ 0 \end{matrix} \right\} dy\, dz \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix}$$

$$\{f_s\} = t \int_0^h \left\{ \begin{matrix} N_1 \frac{p_0 y}{h} \\ 0 \\ N_2 \frac{p_0 y}{h} \\ 0 \\ N_3 \frac{p_0 y}{h} \\ 0 \\ N_4 \frac{p_0 y}{h} \\ 0 \\ N_5 \frac{p_0 y}{h} \\ 0 \\ N_6 \frac{p_0 y}{h} \\ 0 \end{matrix} \right\} dy \quad \begin{matrix} \text{at } x = 0 \\ y = y \end{matrix} \quad (4)$$

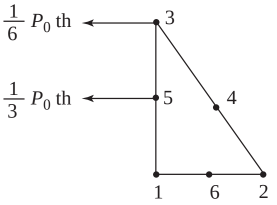
(Shape functions are same as in Problem 8.3). Upon substituting shape functions into (4) evaluating the N's at $x = 0$, $y = y$, we have

$$\{f_s\} = t \int_0^h \left\{ \begin{matrix} \left(1 - \frac{3y}{h} + \frac{2y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \\ \left(\frac{-y}{h} + \frac{2y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \\ \left(\frac{4y}{h} - \frac{4y^2}{h^2}\right) \frac{p_0 y}{h} \\ 0 \\ 0 \\ 0 \end{matrix} \right\} dy$$

$$f_{s1x} = \frac{p_0\,t}{h}\left(\frac{y^2}{2} - \frac{3y^3}{3h} + \frac{2y^4}{4h^2}\right)\Bigg|_0^h = 0$$

$$f_{s3x} = \frac{p_0\,t}{h}\left(\frac{-y^3}{3h} + \frac{2y^4}{4h^2}\right)\Bigg|_0^h = \frac{p_0\,th}{6}$$

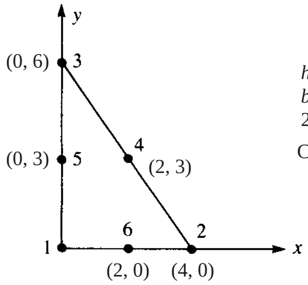
$$f_{s5x} = \frac{p_0\,t}{h}\left(\frac{4y^3}{3h} - \frac{4y^4}{4h^2}\right)\Bigg|_0^h = \frac{p_0\,th}{3}$$



Note: Different result in Problem 6.9

Nodal equivalent forces

8.5 (a)



(a)

$h = 6$ in.
$b = 4$ in.
$2A = 24$ in.^{2}
Centroid at $\left(\frac{4}{3}, 2\right)$

$$\{\varepsilon\} = [B]\,\{d\}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 & \beta_4 & 0 & \beta_5 & 0 & \beta_6 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 & 0 & \gamma_4 & 0 & \gamma_5 & 0 & \gamma_6 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \gamma_5 & \beta_5 & \gamma_6 & \beta_6 \end{bmatrix}$$

Element is oriented as in Section 9.2

$\therefore$ β's and γ's as in Section 9.2

$$\beta_1 = -3h + \frac{4hx}{6} + 4y = 6x + 4y - 18$$

$$\beta_2 = -h + \frac{4hx}{6} = 6x - 6,\ \beta_3 = 0$$

$$\beta_4 = 4y,\ \beta_5 = -4y$$

$$\beta_6 = 4h - \frac{8hx}{6} - 4y = -12x - 4y + 24$$

$$\gamma_1 = -3b + 4x + \frac{4by}{h} = 4x + \frac{8}{3}y - 12$$

$$\gamma_2 = 0$$

$$\gamma_3 = -b + \frac{4by}{h} = \frac{8}{3}y - 4$$

$$\gamma_4 = 4x$$

$$\gamma_5 = 4b - 4x - \frac{8by}{h} = -4x - \frac{16}{3}y + 16$$

$$\gamma_6 = -4x$$

$$\therefore \quad 2A\,\varepsilon_x = \beta_2 u_2 + \beta_4 u_4 + \beta_6 u_6$$

$$= 0.001\,(6x - 6) + 0.0002\,(4y) + 0.0005\,(-12x - 4y + 24)$$

$$2A\,\varepsilon_x = -0.0012y + 0.006$$

$$\therefore \quad \varepsilon_x = -5 \times 10^{-5}\,y + 2.5 \times 10^{-4}$$

$$2A\,\varepsilon_y = \gamma_3 v_3 + \gamma_4 v_4 + \gamma_5 v_5 + \gamma_6 v_6$$

$$= 0.0002\left(\frac{8}{3}y - 4\right) + 0.0001\,(4x) + 0.0001\left(-4x - \frac{16}{3}y + 16\right) + 0.001\,(-4x)$$

$$2A\,\varepsilon_y = -0.004x + 0.0008$$

$$\therefore \quad \varepsilon_y = -1.67 \times 10^{-4}x + 3.33 \times 10^{-5}$$

$$2A\,\gamma_{xy} = 0.002\,(6x - 6) + 0.0005\left(\frac{8}{3}y - 4\right) + 0.0002\,(4x) + 0.0001(4y)$$

$$+ 0.0001\,(-4y) + 0.0005\,(-4x) + 0.001\,(-12x - 4y + 24)$$

$$2A\,\gamma_{xy} = -0.0012x - 0.00267y + 0.01$$

$$\therefore \quad \gamma_{xy} = -5 \times 10^{-5}\,x - 1.11 \times 10^{-4}\,y + 4.167 \times 10^{-4}$$

Evaluate stresses at centroid

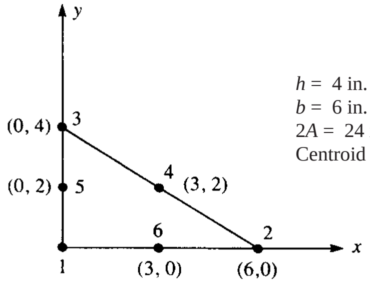
$$\{\sigma\} = [D]\,\{\varepsilon\}$$

$$\{\varepsilon\}\Big|_{\left(\frac{4}{3},\,2\right)} = \begin{Bmatrix} 0.00015 \\ -1.89 \times 10^{-4} \\ 0.000128 \end{Bmatrix}$$

$$\{\sigma\} = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} 1.5 \times 10^{-4} \\ -1.89 \times 10^{-4} \\ 1.28 \times 10^{-4} \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3288 \\ -4848 \\ 1536 \end{Bmatrix} \text{ psi}$$

(b)



(b)

Using expressions for β's and γ's from part (a) with h = 4 in. and b = 6 in. now

$$\beta_1 = \frac{8}{3}x + 4y - 12,\ \beta_2 = \frac{8}{3}x - 4,\ \beta_3 = 0$$

$$\beta_4 = 4y,\ \beta_5 = -4y,\ \beta_6 = \frac{-16}{3}x - 4y + 16$$

$$\gamma_1 = 4x + 6y - 18,\ \gamma_2 = 0,\ \gamma_3 = 6y - 6$$

$$\gamma_4 = 4x,\ \gamma_5 = -4x - 12y + 24,\ \gamma_6 = -4x$$

$$2A\,\varepsilon_x = 0.001\left(\frac{8}{3}x - 4\right) + 0.0002\,(4y) + 0.0005\left(-\frac{16}{3}x - 4y + 16\right)$$

$$2A\,\varepsilon_x = -0.0012y + 0.004$$

$$\therefore \quad \varepsilon_x = -5\times10^{-5}y + 1.67\times10^{-4}$$

$$2A\,\varepsilon_y = 0.0002\,(6y - 6) + 0.0001\,(4x)$$

$$+\,0.0001\,(-4x - 12y + 24) + (0.001)\,(-4x)$$

$$2A\,\varepsilon_y = -0.004\,x + 0.0012$$

$$\therefore \quad \varepsilon_y = -1.67\times10^{-4}x + 5\times10^{-5}$$

$$2A\,\gamma_{xy} = 0.002\left(\frac{8}{3}x - 4\right) + 0.0005\,(6y - 6)$$

$$+\,0.0002\,(4x) + 0.0001\,(4y) + 0.0001\,(-4y)$$

$$+\,0.0005\,(-4x) + 0.001\left(-\frac{16}{3}x - 4y + 16\right)$$

$$2A\,\gamma_{xy} = -0.0012x - 0.001y + 0.005$$

$$\therefore \quad \gamma_{xy} = -5\times10^{-5}x - 4.167\times10^{-5}y - 2.083\times10^{-4}$$

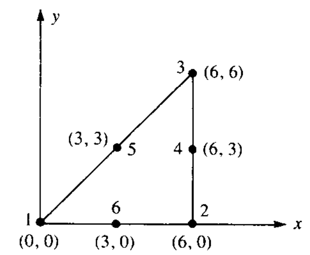
$$\{\sigma\} = [D]\,\{\varepsilon\} \text{ at centroid } \left(2, \frac{4}{3}\right)$$

$$\{\varepsilon\}\Big|_{(2,\,\frac{4}{3})} = \begin{Bmatrix} 0.0001 \\ -0.000284 \\ -0.0000527 \end{Bmatrix}$$

$$\{\sigma\} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} 0.0001 \\ -0.000284 \\ -0.0000527 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 928 \\ -8288 \\ 632 \end{Bmatrix} \text{ psi}$$

8.6



Using Equation (8.1.14) in (8.1.13)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} [X]^{-1}\{d\} \qquad (1)$$

where by Equation (8.1.7) $\{a\} = [X]^{-1}\{d\}$ and

$$\{a\} = \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & & & & & & \\ 1 & 6 & 0 & 36 & 0 & 0 & & & & & & \\ 1 & 6 & 6 & 36 & 36 & 36 & & & \underline{O}_{6\times 6} & & & \\ 1 & 6 & 3 & 36 & 18 & 9 & & & & & & \\ 1 & 3 & 3 & 9 & 9 & 9 & & & & & & \\ 1 & 3 & 0 & 9 & 0 & 0 & & & & & & \\ \hline & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 6 & 0 & 36 & 0 & 0 \\ & & & & & & 1 & 6 & 6 & 36 & 36 & 36 \\ & & \underline{O}_{6\times 6} & & & & 1 & 6 & 3 & 36 & 18 & 9 \\ & & & & & & 1 & 3 & 3 & 9 & 9 & 9 \\ & & & & & & 1 & 3 & 0 & 9 & 0 & 0 \end{array}\right]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} \qquad (2)$$

Using computer, we invert $[X]$ in Equation (2) and reorder $\{d\}$ to normal form $[u_1\, v_1\, u_2\, v_2 \ldots]^T = \{d\}^T$

$\therefore \quad [X]^{-1}\{d\} =$

$$\left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.667 & 0 \\ 0 & 0 & 0.167 & 0 & -0.167 & 0 & 0 & 0 & 0.667 & 0 & -0.667 & 0 \\ 0.056 & 0 & 0.056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.111 & 0 \\ 0 & 0 & -0.111 & 0 & 0 & 0 & 0.111 & 0 & -0.111 & 0 & -0.111 & 0 \\ 0 & 0 & 0.056 & 0 & -0.056 & 0 & 0.111 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.667 \\ 0 & 0 & 0 & 0.167 & 0 & -0.167 & 0 & 0 & 0 & 0.667 & 0 & -0.667 \\ 0 & 0.056 & 0 & 0.056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.111 \\ 0 & 0 & 0 & -0.111 & 0 & 0 & 0 & 0.111 & 0 & -0.111 & 0 & 0.111 \\ 0 & 0 & 0 & 0.056 & 0 & -0.056 & 0 & -0.111 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_6 \\ v_6 \end{Bmatrix} \tag{3}$$

at centroid ($x = 4$, $y = 2$)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 & 0 & 1 & 0 & 8 & 2 & 0 \end{bmatrix} [X]^{-1} \{d\} \tag{4}$$

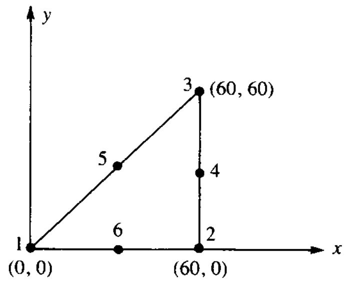
Multiplying matrices in Equation (4) yields

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} (-0.052\,u_1 + 0.059\,u_2 + 0.222\,u_4 - 0.222\,u_5 + 0.001\,u_6) \\ (-0.053\,v_2 - 0.391\,v_3 + 0.223\,v_5 - 0.223\,v_6) \\ (-0.052\,v_1 + 0.059\,v_2 + 0.222\,v_4 - 0.222\,v_5 + 0\,v_6 - \\ 0.053\,u_2 - 0.391\,u_3 + 0.223\,u_5 - 0.223\,u_6) \end{Bmatrix}$$

Then

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - v^2} \begin{Bmatrix} \varepsilon_x + v\,\varepsilon_y \\ \varepsilon_y + v\,\varepsilon_x \\ \left(\frac{1-v}{2}\right) \gamma_{xy} \end{Bmatrix}$$

8.7



$$u_1 = u(0, 0) = a_1 \tag{1}$$

$$u_2 = u(60, 0) = a_1 + 60\,a_2 + 3600\,a_4 \tag{2}$$

$$u_3 = u(60, 60) = a_1 + 60\,a_2 + 60\,a_3 + 3600\,a_4 + 3600\,a_5 + 3600\,a_6 \tag{3}$$

$$u_4 = u(60, 30) = a_1 + 60\,a_2 + 30\,a_3 + 3600\,a_4 + 1800\,a_5 + 800\,a_6 \tag{4}$$

$$u_5 = u(30, 30) = a_1 + 30\,a_2 + 30\,a_3 + 900\,a_4 + 900\,a_5 + 900\,a_6 \tag{5}$$

$$u_6 = u(30, 0) = a_1 + 30\,a_2 + 900\,a_4 \tag{6}$$

By (1) $\Rightarrow a_1 = u_1$

$$\text{By (2) – 2 (6)} \Rightarrow a_4 = \frac{u_2 - 2u_6 + u_1}{1800}$$

$$\text{By – (2) + 4 (6)} \Rightarrow a_2 = \frac{4u_6 - u_2 + 3u_1}{60}$$

$$\text{By 2(4) – (3)} \Rightarrow a_6 = \frac{u_2 + u_3 - 2u_4}{1800}$$

$$\text{(4) – (5)} \Rightarrow a_5 = \frac{-u_2 + u_4 - u_5 + u_6}{900}$$

$$\text{(4)} \Rightarrow a_3 = \frac{u_2 - u_3 + 4u_5 - 4u_6}{60}$$

Can verify by substituting all a's into Equation (3)

$$\therefore \quad u = u_1 + \left(\frac{4u_6 - u_2 + 3u_1}{60}\right) x + \left(\frac{u_2 - u_3 + 4u_5 - 4u_6}{60}\right) y$$

$$+ \left(\frac{u_2 - 2u_6 + u_1}{1800}\right) x^2 + \left(\frac{-u_2 + u_4 - u_5 + u_6}{900}\right) xy$$

$$+ \left(\frac{u_2 + u_3 - 2u_4}{1800}\right) y^2$$

$\therefore$ Shape functions are

$$N_1 = 1 - \frac{3x}{60} + \frac{x^2}{1800} \text{ (From all } u_1 \text{ coefficient)}$$

$$N_2 = \frac{-x}{60} + \frac{y}{60} + \frac{x^2}{1800} - \frac{xy}{900} + \frac{y^2}{1800}$$

$$N_3 = \frac{-y}{60} + \frac{y^2}{1800}$$

$$N_4 = \frac{xy}{900} - \frac{2y^2}{1800}$$

$$N_5 = \frac{4y}{60} - \frac{xy}{900}$$

$$N_6 = \frac{4x}{60} - \frac{4y}{60} - \frac{2x^2}{1800} + \frac{xy}{900}$$

$$2A = 2\left(\frac{1}{2}\right)(60)(60) = 3600$$

$$\beta_1 = 2A\left(\frac{\partial N_1}{\partial x}\right) = 3600\left(-\frac{3}{60} + \frac{2x}{1800}\right) = -180 + 4x$$

$$\beta_2 = 3600\left[-\frac{1}{60} + \frac{2x}{1800} - \frac{y}{900}\right] = -60 + 4x - 4y$$

$$\beta_3 = 0,\ \beta_4 = 3600\left(\frac{y}{900}\right) = 4y$$

$$\beta_5 = 3600\left(\frac{-y}{900}\right) = -4y$$

$$\beta_6 = 3600\left[\frac{4}{60} - \frac{4x}{1800} + \frac{y}{900}\right] = 240 - 8x + 4y$$

$$\gamma_1 = 2A\ \frac{\partial N_1}{\partial y} = 0,\ \gamma_5 = 3600\left(\frac{4}{60} - \frac{x}{900}\right) = 240 - 4x$$

$$\gamma_2 = 3600\left(\frac{1}{60} - \frac{x}{900} + \frac{2y}{1800}\right) = 60 - 4x + 4y$$

$$\gamma_3 = 3600\left(-\frac{1}{60} + \frac{2y}{1800}\right) = -60 + 4y$$

$$\gamma_4 = 3600\left(\frac{x}{900} - \frac{4y}{1800}\right) = 4x - 8y$$

$$\gamma_6 = 3600\left(\frac{-4}{60} + \frac{x}{900}\right) = -240 + 4x$$

Chapter 9

9.1

(a)

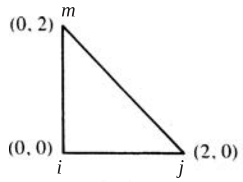


Figure 9.1a

$$[k] = 2\,\pi\,\underline{r}\,A\,[\bar{B}]^T\,[D]\,[\bar{B}]$$

$$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 0,\ z_m = 2$$

$$\alpha_i = r_j\,z_m - z_j\,r_m = 2.2 - 0 = 4$$

$$\alpha_j = r_m\,z_i - z_m\,r_i = 0.0 - 2.0 = 0$$

$$\alpha_m = r_i\,z_j - z_i\,r_j = 0.0 - 0.2 = 0$$

$$\beta_i = z_j - z_m = 0 - 2 = -2$$

$$\beta_j = z_m - z_i = 2 - 0 = 2$$

$$\beta_m = z_i - z_j = 0 - 0 = 0 \qquad \bar{r} = \frac{1}{3}(2) = \frac{2}{3}$$

$$\gamma_i = r_m - r_j = 0 - 2 = -2 \qquad \bar{z} = \frac{1}{3}(2) = \frac{2}{3}$$

$$\gamma_j = r_i - r_m = 0 - 0 = 0 \qquad A = \frac{1}{2}\,(2)\,(2) = 2$$

$$\gamma_m = r_j - r_i = 2 - 0 = 2$$

$$[\bar{B}] = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 \\ -2 & -2 & 0 & 2 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{30 \times 10^6}{(1+0.25)(1-0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[\bar{B}]^T\ [D] = \frac{30\times10^6}{2.5}\begin{bmatrix} -1 & 0 & 1 & -0.5 \\ -0.5 & -1.5 & -0.5 & -0.5 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 1.5 & 0.5 \\ 0.5 & 1.5 & 0.5 & 0 \end{bmatrix}$$

$$[k] = 25.1327\times10^6\begin{bmatrix} 5 & 1 & 0 & -1 & 1 & 0 \\ 1 & 4 & -2 & -1 & -2 & -3 \\ 0 & -2 & 8 & 0 & 4 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -2 & 4 & 1 & 4 & 1 \\ 0 & -3 & 2 & 0 & 1 & 3 \end{bmatrix}$$

(b)

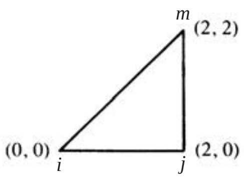


Figure 9.1b

$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 2,\ z_m = 2$

$\alpha_i = 4,\ \alpha_j = 0,\ \alpha_m = 0$

$\beta_i = -2,\ \beta_j = 2,\ \beta_m = 0$

$\gamma_i = 0,\ \gamma_j = -2,\ \gamma_m = 2$

$$\bar{r} = 2\times\frac{2}{3} = 1.333,\ \bar{z} = \frac{2}{3},\ A = 2$$

$$[\bar{B}] = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 2 & 2 & 0 \end{bmatrix}$$

$$[D] = \frac{30\times10^6}{(1+0.25)\left(1-\frac{0.5}{2}\right)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[\bar{B}]^T[D] = \frac{30\times10^6}{2.5}\begin{bmatrix} -1.25 & -0.25 & 0.25 & 0 \\ 0 & 0 & 0 & -0.5 \\ 1.75 & 0.75 & 1.25 & -0.5 \\ -0.5 & -1.5 & -0.5 & 0.5 \\ 0.25 & 0.25 & 0.75 & 0.5 \\ 0.5 & 1.5 & 0.5 & 0 \end{bmatrix}$$

$$[k] = 50.265 \times 10^6 \begin{bmatrix} 2.75 & 0 & -2.25 & 0.5 & 0.25 & -0.5 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -2.25 & 1 & 5.75 & -2.5 & 0.25 & 1.5 \\ 0.5 & -1 & -2.5 & 4 & 0.5 & -3 \\ 0.25 & -1 & 0.25 & 0.5 & 1.75 & 0.5 \\ -0.5 & 0 & 1.5 & -3 & 0.5 & 3 \end{bmatrix}$$

(c) $E = 30 \times 10^6 \ \dfrac{\text{lb}}{\text{in.}^2}$ $\quad v = 0.25$ (Mathcad used here)

Triangle coordinate definition

$i = \begin{pmatrix} 0 \text{ in.} \\ 0 \text{ in.} \end{pmatrix} \quad \begin{matrix} r = 0 \\ z = 1 \end{matrix}$ This defines an array variable

x coordinate is the top
y coordinate is the bottom

$j = \begin{pmatrix} 2 \text{ in.} \\ 0 \text{ in.} \end{pmatrix}$ Area of triangle

$\dfrac{1}{2}$ base $\times$ height

$m = \begin{pmatrix} 1 \text{ in.} \\ 2 \text{ in.} \end{pmatrix}$ $\quad A = \dfrac{1}{2}\,(j_r - i_r)\,(m_z - i_z) \quad A = 2 \text{ in.}^2$

Develop stiffness matrix

$\alpha_i = j_r\, m_z - j_z\, m_r \quad \alpha_i = 4 \text{ in.}^2 \quad \beta_i = j_z - m_z \quad \beta_i = -2 \text{ in.} \quad \gamma_i = m_r - j_r \quad \gamma_i = -1 \text{ in.}$

$\alpha_j = m_r\, i_z - m_z\, i_r \quad \alpha_j = 0 \quad \beta_j = m_z - i_z \quad \beta_j = 2 \text{ in.} \quad \gamma_j = i_r - m_r \quad \gamma_j = -1 \text{ in.}$

$\alpha_m = i_r\, j_z - i_z\, j_r \quad \alpha_m = 0 \quad \beta_m = i_z - j_z \quad \beta_m = 0 \text{ in.} \quad \gamma_m = j_r - i_r \quad \gamma_m = 2 \text{ in.}$

Evaluate $[B]$ at centroid of element

$r_{\text{bar}} = \dfrac{i_r + j_r + m_r}{3} \quad z_{\text{bar}} = \dfrac{i_z + j_z + m_z}{3} \quad r_{\text{bar}} = 1 \text{ in.} \quad z_{\text{bar}} = 0.667 \text{ in.}$

$$[B_i] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r_{\text{bar}}} + \beta_i + \frac{\gamma_i z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_i & \beta_i \end{pmatrix} \qquad [B_j] = \frac{1}{2A}\begin{pmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \frac{\alpha_j}{r_{\text{bar}}} + \beta_j + \frac{\gamma_j z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_j & \beta_j \end{pmatrix}$$

$$[B_m] = \frac{1}{2A}\begin{pmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \frac{\alpha_m}{r_{\text{bar}}} + \beta_m + \frac{\gamma_m z_{\text{bar}}}{r_{\text{bar}}} & 0 \\ \gamma_m & \beta_m \end{pmatrix}$$

$B = \text{augment}(B_i, B_j, B_m)$

Gradient matrix at centroid of element

$$[B] = \begin{pmatrix} -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.3333 & 0 & 0.3333 & 0 & 0.3333 & 0 \\ -0.25 & -0.5 & -0.25 & 0.5 & 0.5 & 0 \end{pmatrix} \frac{1}{\text{in.}}$$

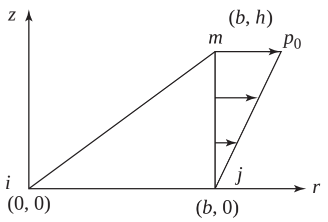
$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix} \text{Axisymmetric stress constitutive matrix}$$

$$[D] = \begin{bmatrix} 3.6\times10^7 & 1.2\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 3.6\times10^7 & 1.2\times10^7 & 0 \\ 1.2\times10^7 & 1.2\times10^7 & 3.6\times10^7 & 0 \\ 0 & 0 & 0 & 1.2\times10^7 \end{bmatrix} \frac{\text{lb}}{\text{in.}^2}$$

$[k] = 2\pi\ r_{\text{bar}}\ A\ [B]^T\ [D]\ [B]$ Axisymmetric element stiffness matrix

$$[k] = \begin{bmatrix} 1.225\times10^8 & 2.513\times10^7 & -5.341\times10^7 & -1.257\times10^7 & 6.283\times10^6 & -1.257\times10^7 \\ 2.513\times10^7 & 6.597\times10^7 & -1.257\times10^7 & -9.425\times10^6 & -5.027\times10^7 & -5.655\times10^7 \\ -5.341\times10^7 & -1.257\times10^7 & 2.231\times10^8 & -5.027\times10^7 & 5.655\times10^7 & 6.283\times10^7 \\ -1.257\times10^7 & -9.425\times10^6 & -5.027\times10^7 & 6.597\times10^7 & 2.513\times10^7 & -5.655\times10^7 \\ 6.283\times10^6 & -5.027\times10^7 & 5.655\times10^7 & 2.513\times10^7 & 8.796\times10^7 & 2.513\times10^7 \\ -1.257\times10^7 & -5.655\times10^7 & 6.283\times10^7 & -5.655\times10^7 & 2.513\times10^7 & 1.131\times10^8 \end{bmatrix} \frac{\text{lb}}{\text{in.}}$$

9.2



$$\{f_s\} = \int_s [N_s]^T \begin{Bmatrix} p_r \\ p_2 \end{Bmatrix} ds = \int_s \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix}_{\substack{@\ \text{Evaluated} \\ r=b \\ z=z}} \begin{Bmatrix} \frac{p_0 z}{h} \\ 0 \end{Bmatrix} 2\pi r dz$$

Now $\quad N_i = \dfrac{1}{2A}\ (\alpha_i + \beta_i r + \gamma_i z),\ N_j = \dfrac{1}{2A}\ (\alpha_j + \beta_j r + \gamma_j z)$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m r + \gamma_m z)$$

$$r_i = 0,\ r_j = b,\ r_m = b,\ z_i = 0,\ z_j = 0,\ z_m = h$$

$$\alpha_i = r_j\, z_m - z_j\, r_m = bh - 0b = bh,\ \alpha_j = r_m\, z_i - z_m\, r_i = 0$$

$$\beta_i = z_j - z_m = 0 - h = -h,\ \beta_j = z_m - z_i = h - 0 = h$$

$$\gamma_i = r_m - r_j = b - b = 0,\ \gamma_j = r_i - \gamma_m = 0 - b = -b$$

$$\alpha_m = r_i\, z_j - z_i\, r_j = 0,\ \beta_m = z_i - z_j = 0 - 0 = 0$$

$$\gamma_m = r_j - r_i = b - 0 = b$$

So the shape functions evaluated at $r = b$ and $z = z$

$$N_i = \frac{1}{bh}\ (bh + (-h)\, b + 0\ \ z) = 0$$

$$N_j = \frac{1}{bh}\ (0 + h\, b + (-b)z) = \frac{1}{bh}\ (hb - bz)$$

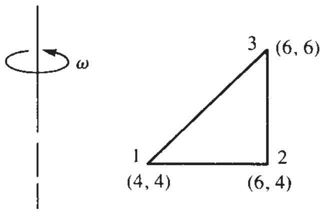
$$N_m = \frac{1}{bh}\ (0 + 0\ b + b\ z) = \frac{1}{bh}\ (bz)$$

$$\{f_s\} = \int_0^h \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{bh}(hb - bz) & 0 \\ 0 & \frac{1}{bh}(hb - bz) \\ \frac{1}{bh}(bz) & 0 \\ 0 & \frac{1}{bh}(bz) \end{bmatrix} \begin{Bmatrix} \frac{p_0\, z}{h} \\ 0 \end{Bmatrix} 2\pi\, dz$$

$$= \frac{2\pi b}{bh} \int_0^h \begin{Bmatrix} 0 \\ 0 \\ p_0 bz - \frac{p_0 bz^2}{h} \\ 0 \\ \frac{p_0 bz^2}{h} \\ 0 \end{Bmatrix} dz = \frac{2\pi}{h} \begin{Bmatrix} 0 \\ 0 \\ \frac{p_0 bz^2}{2} - \frac{p_0 bz^3}{3h} \\ 0 \\ \frac{p_0 bz^3}{3h} \\ 0 \end{Bmatrix}$$

$$= \frac{2\pi}{h} \begin{Bmatrix} 0 \\ 0 \\ \frac{p_0\, bh^2}{6} \\ 0 \\ \frac{p_0\, bh^2}{3} \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = 2\pi b \begin{Bmatrix} 0 \\ 0 \\ \frac{p_0\, h}{6} \\ 0 \\ \frac{p_0\, h}{3} \\ 0 \end{Bmatrix}$$

9.3



Equation to be evaluated is $\{f_B\} = \dfrac{2\pi\bar{r}A}{3}\begin{Bmatrix} \bar{R}_B \\ Z_B \\ \bar{R}_B \\ Z_B \\ \bar{R}_B \\ Z_B \end{Bmatrix}$

$$\bar{r} = 4 + 2 \times \frac{2}{3} \quad \Rightarrow \bar{r} = 5.333 \text{ in.}$$

$$Z_B = 0.283 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\bar{R}_B = w^2 \rho \ \bar{r} = \left[20\,\frac{\text{rev.}}{\text{min}}\left(2\pi\frac{\text{rad}}{\text{rev}}\right)\frac{1 \text{ min}}{60\ s}\right]^2 \frac{\left(0.283\,\frac{\text{lb}}{\text{in.}^3}\right)}{\left(32.2\times 12\,\frac{\text{lb}}{\text{in.}^3}\right)}\ [5.333 \text{ m}]$$

$$\bar{R}_B = 0.01712 \ \frac{\text{lb}}{\text{in.}^3}$$

$$\frac{2\pi\bar{r}A}{3} = \frac{2}{3}\pi\,(5.333 \text{ in.})\ (2 \text{ in.}^2) = 22.34 \text{ in.}^3$$

So

$$f_{B1r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B1z} = (–22.34)\ (0.283) = –\ 6.32 \text{ lb}$$
$$f_{B2r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B3z} = (–\ 22.34)\ (0.283) = –\ 6.32 \text{ lb}$$
$$f_{B3r} = (22.34)\ (0.01712) = 0.382 \text{ lb}$$
$$f_{B3z} = (–\ 22.34)\ (0.283) = –\ 6.32 \text{ lb}$$

9.4

(a) Element Figure 9.4 a

The equation to be evaluated is $\{\sigma\} = [D]\ [B]\ \{d\}$

$r_i = 0,\ z_i = 0,\ r_j = 2,\ z_j = 0,\ r_m = 1,\ z_m = 3$

$\alpha_i = 6,\ \alpha_j = 0,\ \alpha_m = 0,\ \beta_i = –3,\ \beta_j = 3,\ \beta_m = –3,$

$\gamma_i = –1,\ \gamma_j = –1,\ \gamma_m = 2$

$\bar{r} = 1,\ \bar{z} = 1,\ A = \dfrac{1}{2}\ (3)\ (2) = 3 \text{ in.}^2$

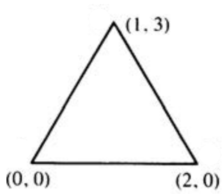


Figure 9.4a

$$[\bar{B}] = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 & 0 & -3 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ 2 & 0 & 2 & 0 & -1 & 0 \\ -1 & -3 & -1 & 3 & 2 & -3 \end{bmatrix}$$

$$[D] = \frac{30\times 10^6}{(1+0.25)(1-0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \frac{30\times10^6}{(1.25)(0.5)}\begin{bmatrix}0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}\left(\frac{1}{6}\right)$$

$$\times\begin{bmatrix}-3 & 0 & 3 & 0 & -3 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ 2 & 0 & 2 & 0 & -1 & 0\\ -1 & -3 & -1 & 3 & 2 & -3\end{bmatrix}\begin{Bmatrix}1\\2\\5\\6\\0\\0\end{Bmatrix}\times10^{-4}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \begin{Bmatrix}80.0\times10^2\\ -17.8\times10^{-10}\\ 80.0\times10^2\\ 12.0\times10^2\end{Bmatrix}\text{psi}$$

(b) Element Figure 9.4b

$r_i = 1,\ z_i = 0,\ r_j = 3,\ z_j = 0,\ r_m = 3,\ z_m = 3$

$\alpha_i = 9,\ \alpha_j = -3,\ \alpha_m = 0,\ \beta_i = -3,\ \beta_j = 3,\ \beta_m = 0$

$\gamma_i = 0,\ \gamma_j = -2,\ \gamma_m = 2$

$\bar{r} = 2.333,\ \bar{z}_m = 0.666,\ A = 3$

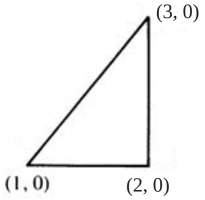


Figure 9.4 b

$$[\bar{B}] = \frac{1}{6}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0.858 & 0 & 1.14 & 0 & 0.512 & 0\\ 0 & -3 & -2 & 3 & 2 & 0\end{bmatrix}$$

$$[\bar{D}] = \frac{30\times10^6}{(1+0.25)(1-0.5)}\begin{bmatrix}0.75 & 0.25 & 0.25 & 0\\ 0.25 & 0.75 & 0.25 & 0\\ 0.25 & 0.25 & 0.75 & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}$$

$$\{\sigma\} = [D]\,[\bar{B}]\,\{d\}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\end{Bmatrix} = \begin{Bmatrix}5830\\ -3770\\ 3090\\ 400\end{Bmatrix}\text{psi}$$

(c) $u_1 = 0.0001$ in. $w_1 = 0.0002$ in.

$u_2 = 0.0005$ in. $w_2 = 0.0006$ in.

$u_3 = 0.0$ in. $w_3 = 0.0$ in.

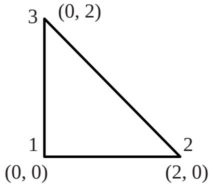


Figure 9.4c

$$d = \begin{pmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{pmatrix} \quad \begin{pmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{pmatrix} = [D]\,[B]\,\{d\} = \begin{Bmatrix} 9.6\times10^3 \\ 2.4\times10^3 \\ 1.2\times10^4 \\ 1.8\times10^3 \end{Bmatrix} \frac{\text{lb}}{\text{in.}^2}$$

9.5 By Equation (9.1.35)

$$\{f_{si}\} = \int_{z_j}^{z_m} \frac{1}{2A}\begin{bmatrix} \alpha_j + \beta_j r_j + \gamma_j z & 0 \\ 0 & \alpha_j + \beta_j\ r_j + \gamma_j z \end{bmatrix}\begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r_j\, dz$$

Now

$$\alpha_j = r_m z_i - r_i z_m = r_j z_j - r_i z_m \qquad \text{Since } r_j = r_m$$

$$\beta_j = z_m - z_j \qquad \gamma_j = r_i - r_j \qquad z_i = z_m$$

$$A = \frac{1}{2}\,(r_j - r_i)\,(z_m - z_j)$$

$$\therefore \quad \{f_{sj}\} = \int_{z_j}^{z_m} \frac{2\pi r_j}{2A}\begin{Bmatrix} p_r[r_j\, z_j - r_i\, z_m + (z_m - z_j) r_j + (r_i - r_j) z \\ p_z[r_j\, z_j - r_i\, z_m + (z_m - z_j) r_j + (r_i - r_j) z \end{Bmatrix} dz$$

Integrating, we obtain

$$\{f_{sj}\} = \frac{2\pi r_j}{2A}\begin{Bmatrix} P_r\left[(r_j\, z_j - r_i\, z_m)\,(z_m - z_j) + r_j\,(z_m - z_j)^2 + (r_i - r_j)\frac{(z_m^2 - z_j^2)}{2}\right] \\ P_z\left[(r_j\, z_j - r_i\, z_m)\,(z_m - z_j) + r_j\,(z_m - z_j)^2 + (r_i - r_j)\frac{(z_m^2 - z_j^2)}{2}\right] \end{Bmatrix}$$

Factoring out $z_m - z_j$ and simplifying

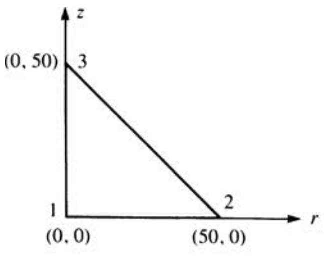
$$\{f_{si}\} = \frac{2\pi r_j (z_m - z_j)}{2A}\begin{Bmatrix} p_r\left[\frac{-r_j\, z_j}{2} - \frac{r_i\, z_m}{2} + \frac{r_j\, z_j}{2} + \frac{r_i\, z_m}{2}\right] \\ p_z\left[\frac{-r_j\, z_j}{2} - \frac{r_j\, z_m}{2} + \frac{r_j\, z_j}{2} + \frac{r_i\, z_m}{2}\right] \end{Bmatrix}$$

$$= \frac{2\pi r_j (z_m - z_j)}{2A}\begin{Bmatrix} \left[p_r\left[\frac{1}{2}(z_m - z_j)(r_j - r_i)\right]\right] \\ p_z\left[\frac{1}{2}(z_m - z_j)(r_j - r_i)\right) \end{Bmatrix}$$

$$= \frac{2\pi r_j (z_m - z_j)}{2A}\begin{Bmatrix} p_r\ A \\ p_z A \end{Bmatrix}$$

$$\{f_{sj}\} = \frac{2\pi r_j (z_m - z_j)}{2}$$

9.6 (a)



(a)

E = 210 GPa
υ = 0.25
$A = \frac{1}{2}$ (50)(50)
A = 1250 mm^2
$\bar{r}$ = 16.67 mm
$\bar{z}$ = 16.67 mm

$$\alpha_i = 50\ (50) - (0)\ (0) = 2500\ \text{mm}^2$$

$$\beta_i = -50\ \text{mm},\ \gamma_i = -50\ \text{mm}$$

$$\beta_j = 50\ \text{mm},\ \gamma_j = 0\ \text{mm}$$

$$\beta_m = 0\ \text{mm},\ \gamma_m = 50\ \text{mm}$$

$$[\bar{B}] = \frac{1}{2(1250)}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ 50 & 0 & 50 & 0 & 50 & 0 \\ -50 & -50 & 0 & 50 & 50 & 0 \end{bmatrix}\frac{1}{\text{mm}}$$

$$[D] = \frac{210 \times 10^9}{(1.25)(0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}\frac{\text{N}}{\text{m}^2}$$

$$[D]\ [\bar{B}] = \frac{210 \times 10^9}{1250(1.25)}\begin{bmatrix} -25 & -12.5 & 50 & 0 & 12.5 & 12.5 \\ 0 & -37.5 & 25 & 0 & 12.5 & 37.5 \\ 25 & -12.5 & 50 & 0 & 37.5 & 12.5 \\ -12.5 & -12.5 & 0 & 12.5 & 12.5 & 0 \end{bmatrix}$$

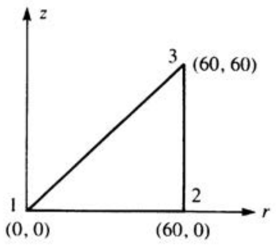
$$[k] = 2\pi\ \bar{r}\ A\ [\bar{B}]^T\ [D]\ [\bar{B}]$$

$$= \frac{2\pi(0.01667\ \text{m})(1250)(210 \times 10^9) \times 10^{-6}}{(1250)(1.25)}\ [\bar{B}]^T\ [D]\ [\bar{B}]$$

Multiplying $[\bar{B}]^T$ times $[D]\ [\bar{B}]$, we obtain

$$[k] = 7.039 \times 10^6\begin{bmatrix} 3125 & 625 & 0 & -625 & 625 & 0 \\ & 2500 & -1250 & -625 & -1250 & -1875 \\ & & 5000 & 0 & 2500 & 1250 \\ & & & 625 & 625 & 0 \\ & & & & 2500 & 625 \\ \text{Symmetry} & & & & & 1875 \end{bmatrix}\frac{\text{N}}{\text{m}}$$

9.6 (b)



(b)

$\bar{r} = 40$ mm
$\bar{z} = 20$ mm

$$A = 1800\ \text{mm}^2$$

$$\beta_i = -60\ \text{mm},\ \beta_j = 60\ \text{mm},\ \beta_m = 0$$

$$\gamma_i = 0,\ \gamma_j = -60\ \text{mm},\ \gamma_m = 60\ \text{mm}$$

$$[k] = 2\pi\,\bar{r}\,A\,[\bar{B}]^T[D]\,[\bar{B}]$$

$$[\bar{B}] = \frac{1}{2(1800)}\begin{bmatrix} -60 & 0 & 60 & 0 & 0 & 0 \\ 0 & 0 & 0 & -60 & 0 & 60 \\ 30 & 0 & 30 & 0 & 30 & 0 \\ 0 & -60 & -60 & 60 & 60 & 0 \end{bmatrix}\frac{1}{\text{mm}}$$

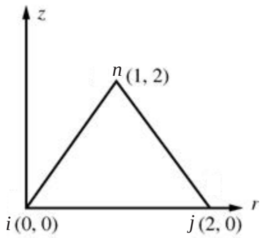
$[D]$ as in 9.6 (a)

$$[D]\,[\bar{B}] = \frac{210\times10^9}{1.25(1800)}\begin{bmatrix} -37.5 & 0 & 52.5 & -15 & 7.5 & 15 \\ -7.5 & 0 & 22.5 & -45 & 7.5 & 45 \\ 7.5 & 0 & 37.5 & -15 & 22.5 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix}$$

$$[k] = \frac{2\pi(0.04\,\text{m})(\cancel{1800})(210\times10^9)}{(1.25)(1800)(2)(\cancel{1800})}[\bar{B}]^T[D][\bar{B}]$$

$$[k] = 11.73\times10^6\begin{bmatrix} 2475 & 0 & -2025 & 450 & 225 & -450 \\ & 900 & 900 & -900 & -900 & 0 \\ & & 5175 & -2250 & 225 & 1350 \\ & & & 3600 & 450 & -2700 \\ & & & & 1575 & 450 \\ \text{Symmetry} & & & & & 2700 \end{bmatrix}$$

(c)



$r_i = 0$ $\quad z_i = 0$

$r_j = 0.002$ $\quad z_j = 0$

$r_m = 0.001$ $\quad z_m = 0.002$

$$A = \frac{1}{2}(r_j - r_i)(z_m - z_i),\ E = 210\times10^9,\ z = \frac{1}{3}\times0.002,\ r = \frac{0.002}{2},\ v = 0.25$$

$\alpha_i = r_j z_m - z_j r_m \qquad \alpha_j = r_m z_i - z_m r_i \qquad \alpha_m = r_i z_j - z_i r_j$

$\beta_i = z_j - z_m \qquad \beta_j = z_m - z_i \qquad \beta_m = z_i - z_j$

$\gamma_i = r_m - r_j \qquad \gamma_j = r_i - r_m \qquad \gamma_m = r_j - r_i$

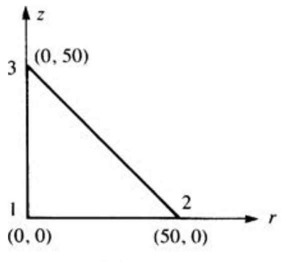
$$[B] = \frac{1}{2A}\begin{pmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r}+\beta_i+\frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r}+\beta_j+\frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r}+\beta_m+\frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{pmatrix}$$

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{pmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = 2\pi r\, A\, [B]^T [D]\, [B]$$

$$[k] = \begin{pmatrix} 8.577\times10^8 & 1.759\times10^8 & -3.738\times10^8 & -8.796\times10^7 & 4.398\times10^7 & -8.796\times10^7 \\ 1.759\times10^8 & 4.618\times10^8 & -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & -3.958\times10^8 \\ -3.738\times10^8 & -8.796\times10^7 & 1.561\times10^9 & -3.519\times10^8 & 3.958\times10^8 & 4.398\times10^8 \\ -8.796\times10^7 & -6.597\times10^7 & -3.519\times10^8 & 4.618\times10^8 & 1.759\times10^8 & -3.958\times10^8 \\ 4.398\times10^7 & -3.519\times10^8 & 3.958\times10^8 & 1.759\times10^8 & 6.158\times10^8 & 1.759\times10^8 \\ -8.796\times10^7 & -3.958\times10^8 & 4.398\times10^8 & -3.958\times10^8 & 1.759\times10^8 & 7.917\times10^8 \end{pmatrix}$$

9.7 (a)



(a)

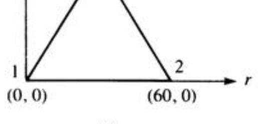
From Problem 9.6 (a), we have $[D]\ [\overline{B}]$

$$\therefore\quad \{\sigma\} = [D]\ [\overline{B}]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210\times10^3\,\text{MPa}}{(1250)(1.25)}\begin{bmatrix} -25 & -12.5 & 50 & 0 & 12.5 & 12.5 \\ 0 & -37.5 & 25 & 0 & 12.5 & 37.5 \\ 25 & -12.5 & 50 & 0 & 37.5 & 12.5 \\ -12.5 & -12.5 & 0 & 12.5 & 12.5 & 0 \end{bmatrix}\begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -84 \\ -84 \\ 252 \\ -101 \end{Bmatrix} \text{MPa}$$

(b)



(b)

From Problem 9.6 (b), we have $[D]\ [\overline{B}]$

$$\therefore \quad \begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{210 \times 10^3}{(1500)(1.25)} \begin{bmatrix} -37.5 & 0 & 52.5 & -15 & 7.5 & 15 \\ -7.5 & 0 & 22.5 & -45 & 7.5 & 45 \\ 7.5 & 0 & 37.5 & -15 & 22.5 & 15 \\ 0 & -15 & -15 & 15 & 15 & 0 \end{bmatrix} \begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -103 \\ -103 \\ 112 \\ -73 \end{Bmatrix} \text{MPa}$$

(c) $u_i = 0.00005 \qquad w_i = 0.00003$

$u_j = 0.00002 \qquad w_j = 0.00002$

$u_m = 0 \qquad w_m = 0$

$$\{d\} = \begin{pmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{pmatrix}$$

$$\begin{pmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{pmatrix} = [D]\,[B]\,\{d\}$$

$$\sigma_r = -2.87 \times 10^9 \text{ Pa}$$

$$\sigma_z = -2.45 \times 10^9 \text{ Pa}$$

$$\sigma_\theta = 3.57 \times 10^9 \text{ Pa}$$

$$\tau_{rz} = -1.89 \times 10^9 \text{ Pa}$$

9.8 No, not in general, as the axisymmetric elements are rings, not plane quadrilaterals or triangles. So axisymmetric nodes are actually nodal circles whereas plane stress elements have node points.

9.9 No, the element circumferential strain is a function of *r* and *z* (see Equation (9.1.15).

9.10 Make $u_r = 0$ for all nodes acting on the axis of symmetry.

9.11 How would you evaluate circumferential strain ε_θ at $r = 0$?

From text Equation (9.1.15)

$$\varepsilon_\theta = \frac{a_1}{r} + a_2 + \frac{a_3 z}{r} \qquad (1)$$

$$\varepsilon_r = a_2 \qquad (2)$$

Also from text Equation (9.1.1e)

$$\varepsilon_\theta = \frac{u}{r} \qquad (3)$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{4}$$

$$\therefore \quad u = \varepsilon_\theta \, r \tag{5}$$

Substituting (1) into (5)

$$\Rightarrow \quad u = \left[\frac{a_1}{r} + a_2 + \frac{a_3 \, z}{r}\right] r = a_1 + a_2 r + a_3 z \tag{6}$$

Partial of (6) with reference to r

$$\Rightarrow \frac{\partial u}{\partial r} = a_2 \qquad \text{Compare to (2)} \tag{7}$$

$\therefore \varepsilon_\theta\big|_{r=0} = \varepsilon_r = a_2$ as stated in problem statement

9.12 What will be the stresses σ_r and σ_θ at $r = 0$?

From Equation (9.1.2)

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left(\underset{\varepsilon_r(1-v)}{\overset{a_2}{\parallel}} + \underset{\varepsilon_z(v)}{\overset{a_6}{\parallel}} + \underset{\varepsilon_\theta(v)}{\overset{a_2}{\parallel}}\right)$$

$$= \frac{E}{(1+v)(1-2v)}\,(a_z - \cancel{a_2 v} + a_6 v + \cancel{a_2 v})$$

$$\sigma_r = \boxed{\frac{E}{(1+v)(1-2v)}\,(a_2 + a_6\, v)}$$

$$\therefore \quad \sigma_\theta = \frac{E}{(1+v)(1-2v)}\left(\underset{\varepsilon_r(v)}{\overset{a_2}{||}} + \underset{\varepsilon_z(v)}{\overset{a_6}{||}} + \underset{\varepsilon_\theta(1-v)}{\overset{a_2}{||}}\right)$$

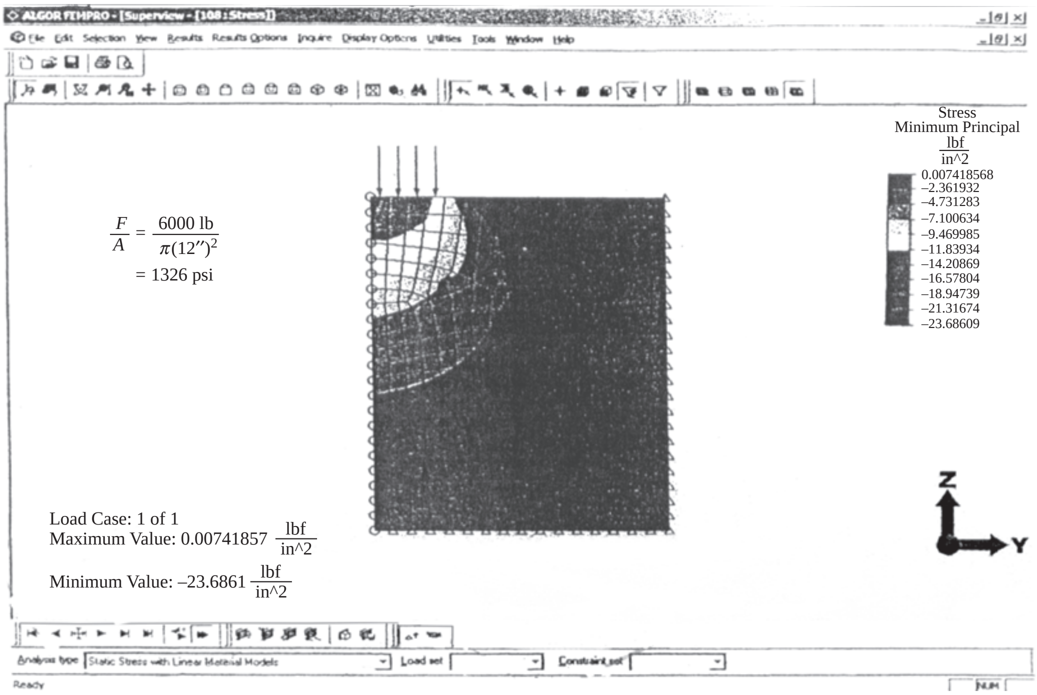
$$= \frac{E}{(1+v)(1-2v)}\,(\cancel{a_2 v} + a_6 v + a_2 - \cancel{a_2 v})$$

$$\sigma_\theta = \boxed{\frac{E}{(1+v)(1-2v)}\,(a_2 + a_6\, v)}$$

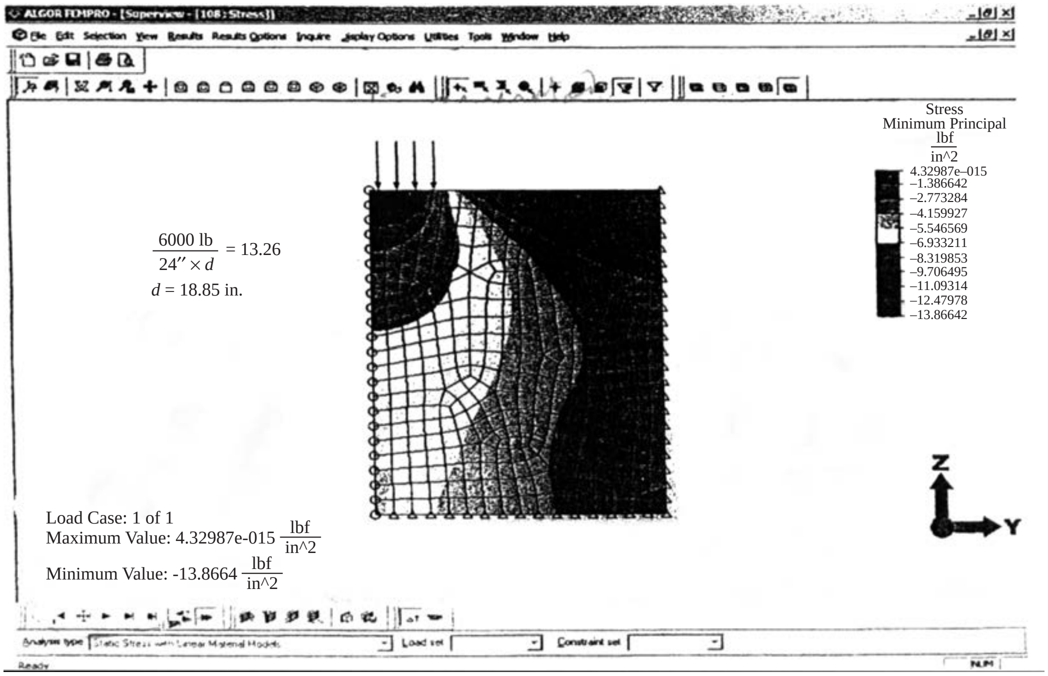
$$\therefore \boxed{\text{at } \; r = 0, \quad \sigma_\theta = \sigma_r}$$

9.13

Axisymmetric model pressure load of 13.26 $\dfrac{\text{lb}}{\text{in.}^2}$

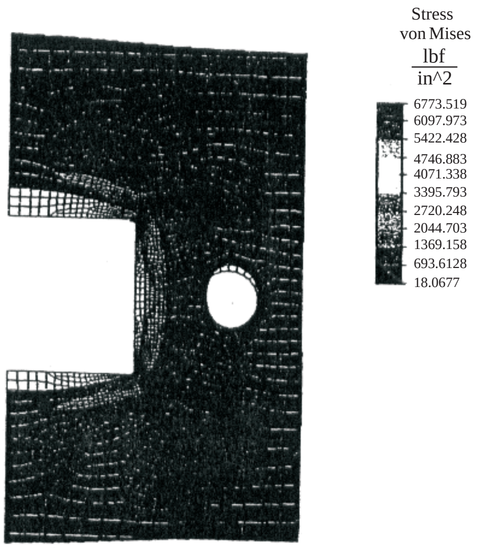


Plane stress with a thickness of 18.85 inches



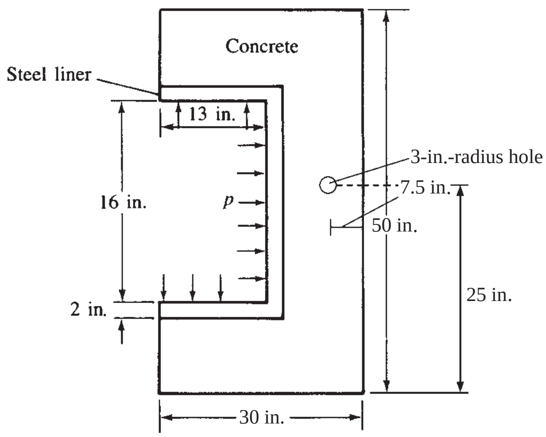
9.14 von Mises stresses (with filleted corners)

von Mises stresses (with filleted corners)

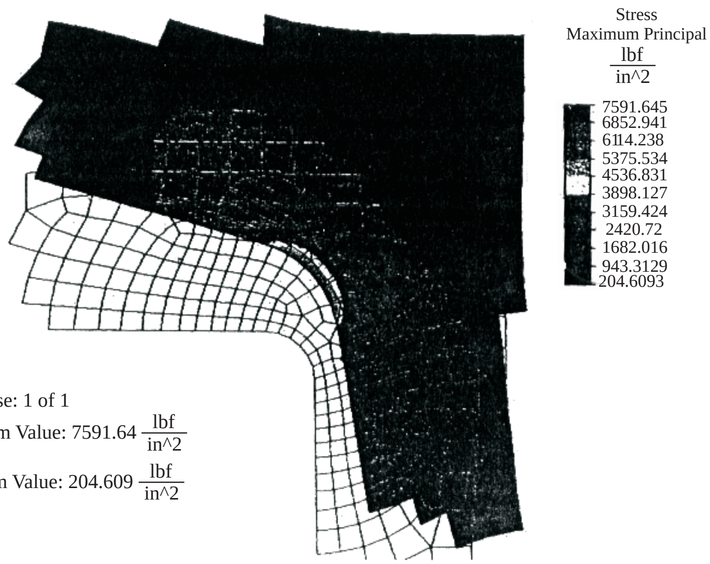


Load Case: 1 of 1
Maximum Value: 6773.52 lbf/in^2
Minimum Value: 18.0677 lbf/in^2

Model of a nuclear reactor

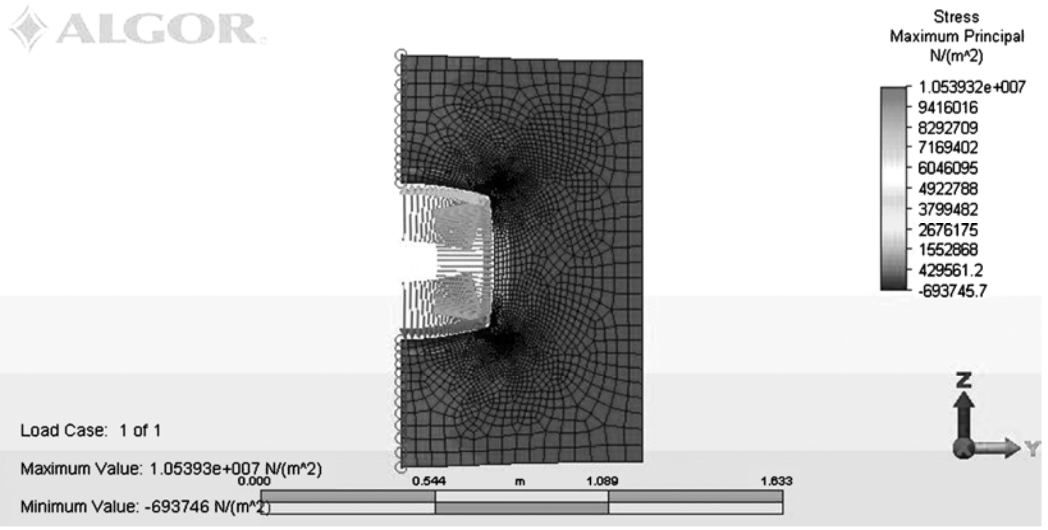


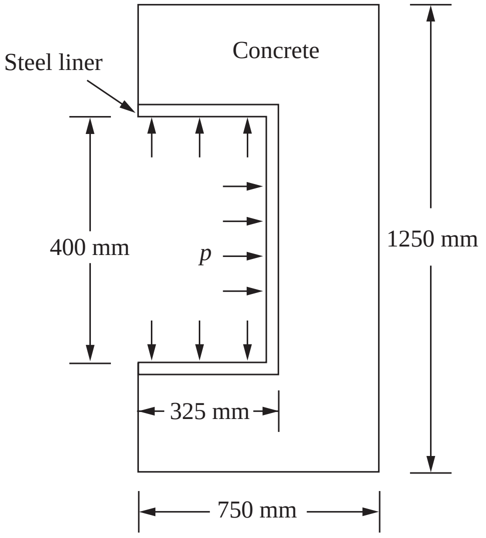
Zoomed maximum principal stresses in filleted corner



Load Case: 1 of 1
Maximum Value: 7591.64 lbf/in^2
Minimum Value: 204.609 lbf/in^2

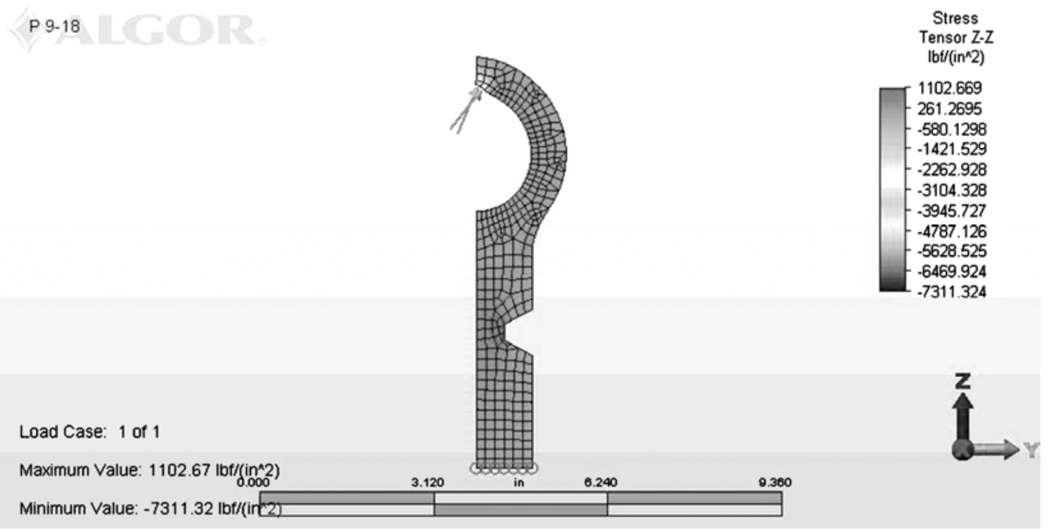
9.15





Note: Without the arc (inside radius), we have a 90° re-entrant corner where stress is approaching infinity. We have a singularity in the linear-elastic solution based on linear theory of elasticity. Therefore, we need the arc as in good practice or elastic-plastic model where an upper bound on the corner stress is the yield strength of the material.

9.18



9.19

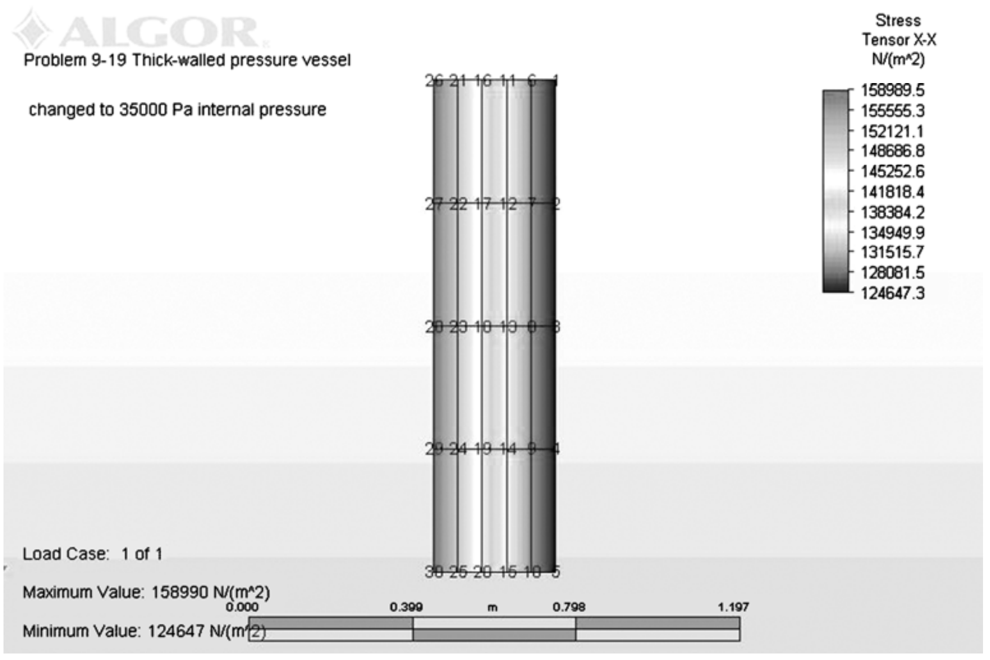


Figure 1 Thick walled open-ended cylinder

Theoretical Solution for hoop stress at inner radius

$$q = 35 \times 10^6 \quad a = 1.5 \quad b = 1.2 \quad r = 1.2$$

$$\sigma_\theta = \frac{qb^2}{r^2} \cdot \frac{a^2 + r^2}{a^2 - b^2}$$

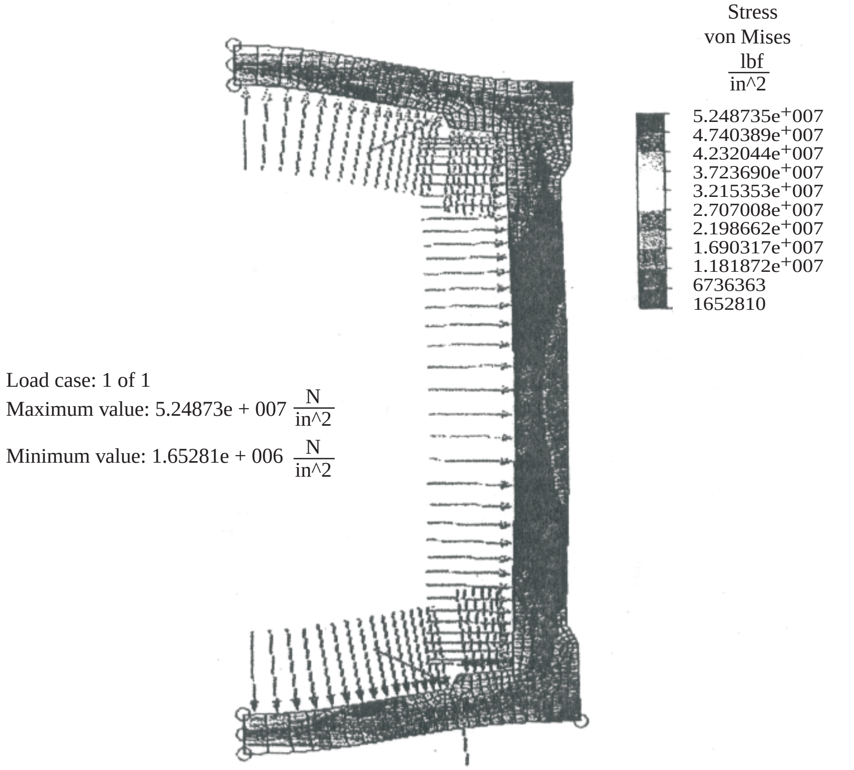
$$\sigma_\theta = 1.594 \times 10^8 \text{ Pa}$$

	Algor Results	Theoretical Results
Hoop Stress	159.5 MPa	159.4MPa
Maximum Principal Stress	159.5 MPa	-
Minimum Principal Stress	–35 MPa	-
Deflection in *y*-direction	0.93mm	-

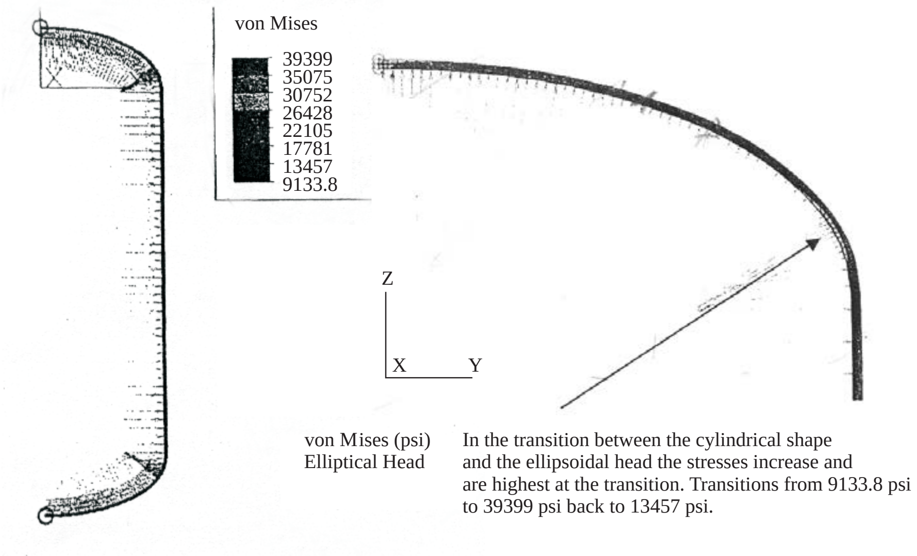
The Algor results for hoop stress and the theoretical solution for hoop stress are very close which proves that the Algor model is correct. The pipe has a very minimum internal and external deflection, less than 1mm on the inner radius. The stresses are also manageable at 159 MPa.

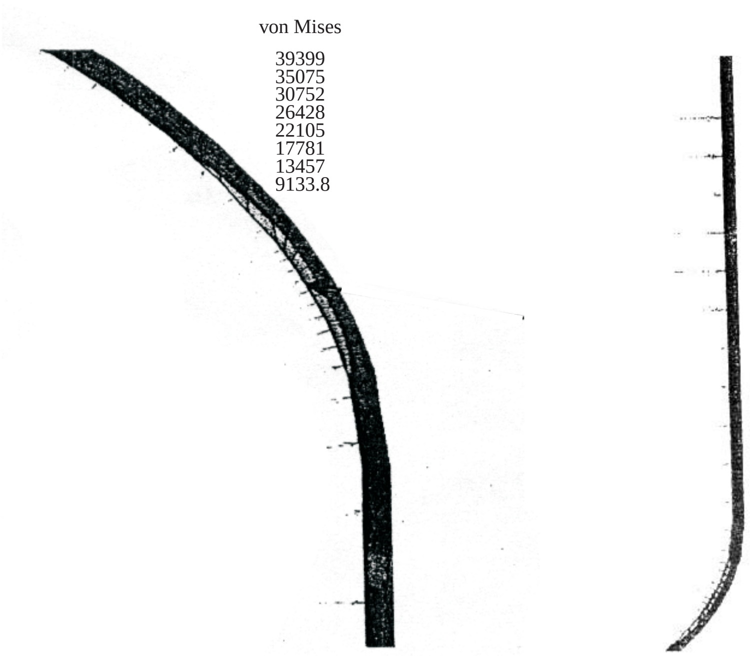
9.20 A steel cylindrical pressure vessel with flat plate end caps is shown in the figure with vertical axis of symmetry. Addition of thickened sections helps to reduce stress concentrations in the corners. Analyze the design and identify the most critically stressed regions. Note that inside sharp re-entrant corners produce infinite stress concentration zones, so refining the mesh in these regions will not help you get a better answer unless you use an inelastic theory or place small fillet radii there. Recommend any design changes in your report. Let the pressure inside be 1000 kPa.

503 elements and 645 nodes. Stresses are highest at sharp corners and the middle of the top and bottom of the pressure vessel. The design is acceptable as the von Mises stresses do not reach the yield strength of the material.



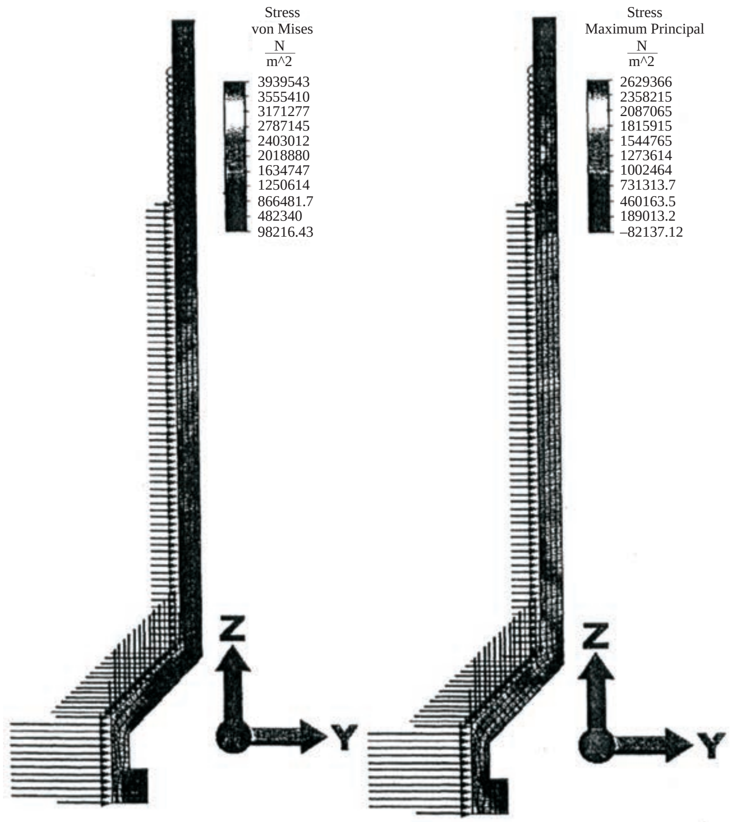
9.22





The recommended head shape of the hemispherical ends versus the ellipsoidal ends would be the hemispherical ends due to a lower stress concentration at the transition between the head and the cylindrical body.

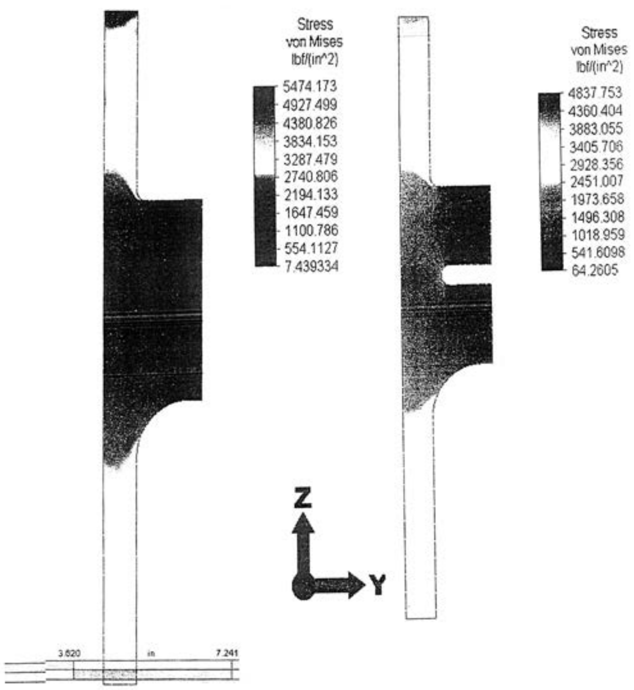
9.23 According to the von Mises stress analysis, the average stress through the glass is around $\frac{1}{5}$ of the tensile strength of the glass. If the maximum force used with this syringe is 45 N, the design should be fine. However, if 45 N is the normal operating force which may increase, I would recommend analyzing this again with a safety factor of 4 (180 N force) to make sure it will still be under 5 MPa. As for the maximum principal stresses, they are well below the tensile strength of the glass and do not appear to be an issue.



Another analysis with a safety factor of 4 (28.64789 $\frac{\text{N}}{\text{radian}}$) reveals that this syringe is still within the tensile strength of glass is all areas. With this information, I would conclude that the syringe design is indeed safe with this material specification.

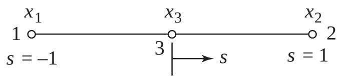
9.25 Steel hole punch

No side groove | With side groove



Chapter 10

10.1



$$J = \frac{dx}{ds} \qquad u = a_1 + a_2 s + a_3 s^2$$

$$x = a_1 + a_2 s + a_3 s^2$$

$$x_1 = a_1 + a_2(-1) + a_3(-1)^2 \tag{1}$$

$$x_2 = a_1 + a_2(1) + a_3(1)^2 \tag{2}$$

(1) – (2) gives

$$x_1 - x_2 = -2a_2,\ a_2 = \frac{x_2 - x_1}{2} \tag{3}$$

$$x_3 = a_1 + a_2(0) + a_3(0)$$

$$\therefore \quad a_1 = x_3 \tag{4}$$

(3) and (4) into (1)

$$a_3 = x_1 - x_3 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2 - 2x_3}{2}$$

$$\therefore \quad x = x_3 + \frac{x_2 + x_1}{2}\, s + \frac{(x_1 + x_2 - 2x_3)}{2} s^2 \tag{5}$$

$$J = \frac{dx}{ds} = \frac{x_2 + x_1}{2} + (x_1 + x_2 - 2x_3)s \tag{6}$$

Now $\quad x_3 = \dfrac{x_1 + x_2}{2}$ for x_3 at $s = 0$

and $\quad x_2 - x_1 = L$

$$\therefore \quad J = \frac{L}{2} + [x_1 + x_2 - (x_1 + x_2)]s$$

$$J = \frac{L}{2}$$

10.2 Using Equation (10.1.1 b)

(a)

$$s = [x - \frac{(x_1 + x_2)}{2}] \left(\frac{2}{x_2 - x_1}\right)$$

At $A \Rightarrow x = x_A = 14$ in.

$$s = [14 - \frac{(10 + 20)}{2}] \left(\frac{2}{20 - 10}\right)$$

$$s = (14 - 15)\,\frac{2}{10} = -\,\frac{1}{5} = -\,0.2$$

By Equation (10.1.5)

$$N_1 = \frac{1+0.2}{2} = 0.6,\ N_2 = \frac{1-0.2}{2} = 0.4$$

By Equation (10.1.1 b)

(b)

At $A = 7$ in.

$$s = \left[7 - \left(\frac{5+10}{2}\right)\right]\left(\frac{2}{10-5}\right)$$

$$s = [7 - 7.5]\left(\frac{2}{5}\right)$$

$$s = -0.2$$

By Equation (10.1.5)

$$N_1 = \frac{1-(-0.2)}{2} = 0.6$$

$$N_2 = \frac{1-(0.2)}{2} = 0.4$$

10.3 (a) Using Equation (10.1.1 b)

$$x = x_A = 40 \text{ mm}$$

$$s = \left[40 - \left(\frac{20+60}{2}\right)\right]\left(\frac{2}{60-20}\right)$$

$$s = [40 - 40]\left(\frac{2}{40}\right)$$

$$s = 0$$

$$N_1 = \frac{1-0}{2} = \frac{1}{2},\ N_2 = \frac{1+0}{2} = \frac{1}{2}$$

(b)

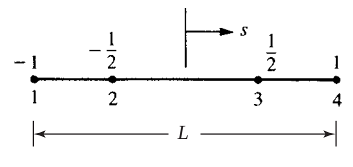
$$s = \left[20 - \left(\frac{10+30}{2}\right)\right]\left(\frac{2}{30-10}\right)$$

$$s = [20 - 20]\ \frac{2}{20}$$

$$s = 0$$

$$N_1 = \frac{1\quad 0}{2} = \frac{1}{2},\ N_2 = \frac{1+0}{2} = \frac{1}{2}$$

10.4



(1) $$u = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \quad \text{(A)}$$

$$x = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \quad \text{(B)}$$

$$x_1 = a_1 + a_2(-1) + a_3(-1)^2 + a_4(-1)^3 \quad (1)$$

$$x_2 = a_1 + a_2 \left(\frac{-1}{2}\right) + a_3 \left(\frac{-1}{2}\right)^2 + a_4 \left(\frac{-1}{2}\right)^3 \quad (2)$$

$$x_3 = a_1 + a_2 \left(\frac{1}{2}\right) + a_3 \left(\frac{1}{2}\right)^2 + a_4 \left(\frac{1}{2}\right)^3 \quad (3)$$

$$x_4 = a_1 + a_2(1) + a_3(1)^2 + a_4(1)^3 \quad (4)$$

$$(1) + (4) \Rightarrow x_1 + x_4 = 2a_1 + 2a_3 \quad (5)$$

$$(2) + (3) \Rightarrow x_2 + x_3 = 2a_1 + \frac{a_3}{2} \quad (6)$$

(5) – (6) gives

$$x_1 + x_4 - (x_2 + x_3) = 2a_1 + 2a_3 - \left(2a_1 + \frac{a_3}{2}\right)$$

or $$a_3 = \frac{2}{3}(x_1 + x_4 - x_2 - x_3) \quad (7)$$

(7) into (5)

$$x_1 + x_4 = 2a_1 + 2\left(\frac{2}{3}\right)(x_1 + x_4 - x_2 - x_3)$$

or $$a_1 = \frac{-\frac{1}{3}(x_1 + x_4) + \frac{4}{3}(x_2 + x_3)}{2} \quad (8)$$

$$(1) - (4) \Rightarrow x_1 - x_4 = -2a_2 - 2a_4 \quad (9)$$

$$(2) - (3) \Rightarrow x_2 - x_3 = -a_2 - \frac{a_4}{4} \quad (10)$$

(9) – 2 (10) gives

$$x_1 - x_4 - 2(x_2 - x_3) = \frac{-3a_4}{2}$$

$$\therefore \quad a_4 = \frac{2}{3}[2(x_2 - x_3) - (x_1 - x_4)] \quad (11)$$

(11) into (9) yields

$$a_2 = \frac{\frac{1}{3}(x_1 - x_4) - \frac{8}{3}(x_2 - x_3)}{2} \quad (12)$$

Substituting (7), (8), (11) and (12) into (B)

$$x = \frac{4(x_2 + x_3) - (x_1 + x_4)}{6} + \frac{[(x_1 - x_4) - 8(x_2 - x_3)]s}{6}$$

$$+ \frac{4(x_1 + x_4 - x_2 - x_3)}{6} s^2 + \frac{[8(x_2 - x_3) - 4(x_1 - x_4)]s^3}{6} \quad (13)$$

Combine like x_1, x_2, x_3 and x_4 coefficients

$$x = \left(-\frac{2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}\right)x_1 + \left(\frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}\right)x_2$$

$$+ \left(-\frac{4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}\right)x_3 + \left(\frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}\right)x_4 \qquad (14)$$

By (14) then

$$\{x\} = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$\therefore \quad N_1 = \frac{-2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}$$

$$N_2 = \frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}$$

$$N_3 = \frac{-4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}$$

$$N_4 = \frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}$$

(2) $$\frac{du}{ds} = \left[-2s^2 + \frac{4}{3}s + \frac{1}{6} \quad 4s^2 - \frac{4}{3}s - \frac{4}{3} \quad -4s^2 - \frac{4}{3}s + \frac{4}{3} \quad 2s^2 + \frac{4}{3}s - \frac{1}{6}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Differentiating (13)

$$\frac{dx}{ds} = \left(-2s^2 + \frac{4}{3}s + \frac{1}{6}\right)x_1 + \left(4s^2 + \frac{4}{3}s - \frac{4}{3}\right)x_2$$

$$+ \left(-4s^2 - \frac{4}{3}s + \frac{4}{3}\right)x_3 + \left(2s^2 + \frac{4}{3}s - \frac{1}{6}\right)x_4$$

Simplifying

$$= 2s^2(x_4 - x_1) + \frac{4}{3}s(x_4 + x_1) - \frac{1}{6}(x_4 - x_1)$$

$$-4s^2(x_3 - x_2) - \frac{4}{3}s(x_3 + x_2) + \frac{4}{3}(x_3 - x_2)$$

$$= 2s^2 L + \frac{8}{3}s\frac{(x_4 + x_1)}{2} - \frac{1}{6}L - 4s^2\left(\frac{L}{2}\right) - \frac{8}{3}s\left(\frac{x_3 + x_2}{2}\right) + \frac{4}{3}\left(\frac{L}{2}\right)$$

$$= 2s^2 L + \frac{8}{3}s\,x_c - \frac{L}{6} - 2s^2 L - \frac{8}{3}s\,x_c + \frac{2}{3}L$$

$$\frac{dx}{ds} = \frac{L}{2}$$

Now

$$\frac{du}{dx} = \frac{\frac{du}{ds}}{\frac{dx}{ds}} \quad \text{and} \quad \frac{du}{dx} = \varepsilon_x = [B]\{d\}$$

$$\therefore \quad \varepsilon_x = \left[\frac{-12s^2+8s+1}{3L} \quad \frac{12s^2-4s-4}{\frac{3L}{2}} \quad \frac{-12s^2-4s+4}{\frac{3L}{2}} \quad \frac{12s^2+8s-1}{3L}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore \quad [B] = \left[\frac{-12s^2+8s+1}{3L} \quad \frac{12s^2-4s-4}{\frac{3L}{2}} \quad \frac{-12s^2-4s+4}{\frac{3L}{2}} \quad \frac{12s^2+8s-1}{3L}\right]$$

10.5 (a) Using Equation (10.5.6)

$$x = x_A = 13 = 15 + \left(\frac{20-10}{2}\right)s + \frac{10+20-2\,(15)}{2}s^2$$

$$\therefore\ 0s^2 + 5\text{s} + 2 = 0$$

$$s = \frac{-2}{5} = -0.4$$

$$N_1 = \frac{s(s-1)}{2} = \frac{-0.4\,(-0.4-1)}{2}$$

$$= 0.28$$

$$N_2 = \frac{s(s+1)}{2} = \frac{-0.4\,(-0.4+1)}{2}$$

$$= -0.12$$

$$N_3 = 1 - s_2 = 1 - (-0.4)^2$$

$$= 0.84$$

$$\Sigma N\text{'s} = 0.28 - 0.12 + 0.84 = 1$$

$$u = a_1 + a_2\,s + a_3\,s^2$$

$$u_1 = 0.006 = a_1 + a_2(-1) + a_3(-1)^2$$

$$u_3 = 0 = a_1 + a_2(0) + a_3(0)$$

$$u_2 = -0.006 = a_1 + a_2(1) + a_3(1)^2$$

$$\therefore \quad a_3 = 0,\ a_2 = -0.006,\ a_1 = 0$$

$$\therefore \quad u = -0.006s \quad \text{and} \quad s = -0.4 \quad \text{at} \quad x_A = 13$$

$$\therefore \quad u = -0.006\,(-0.4) = 0.0024\ \text{in.}$$

$$\varepsilon_x = \frac{2s-1}{L}\,u_1 + \frac{2s+1}{L}\,u_2 - \frac{4s}{L}\,u_3$$

$$\varepsilon_x = \frac{2s-1}{L}\,(0.006) + \frac{2s+1}{L}\,(-0.006) = 0$$

$$\varepsilon_x = \frac{-0.12}{L}\ (L = 10^{11})$$

$$\therefore \quad \varepsilon_x = -0.012\ \frac{\text{in.}}{\text{in.}}$$

10.6 (a) Using Equation (10.5.6)

$$x = x_A = 1.5 = 1 + \left(\frac{2-0}{2}\right)s + \left(\frac{0+2-2\,(1)}{2}\right)s^2$$

$$1.5 = 1 + s + 0s^2$$

$$s - 0.5 = 0$$

$$s = 0.5$$

$$N_1 = \frac{s(s-1)}{2} = \frac{0.5(0.5-1)}{2}$$

$$= -0.125$$

$$N_2 = \frac{s(s+1)}{2} = \frac{0.5(0.5+1)}{2}$$

$$= 0.375$$

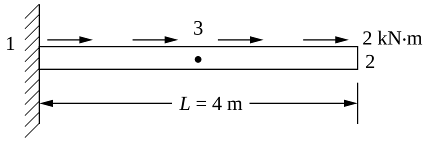
$$N_3 = 1 - s^2 = 1 - 0.5^2$$

$$= 0.75$$

$$\Sigma N\text{'s} = -0.125 + 0.375 + 0.75$$

$$= 1.0$$

10.8



$A = 2 \times 10^{-4}$ m², $E = 205$ GPa
$L = 4$ m

$$\{F\} = [K]\{d\} \tag{A}$$

where by Equation (10.5.22)

$$[K] = \frac{AE}{L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ & 4.67 & -5.33 \\ \text{Symmetry} & & 10.67 \end{bmatrix} \tag{1}$$

By Equation (10.5.9) for N's and using Equation (10.1.21)

$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\}\, dx, \ \{T\} = \left\{2\,\frac{\text{kN}}{\text{m}}\right\}, \ dx = \frac{L}{2}\,ds$$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{s(s-1)}{2} \\ \dfrac{s(s+1)}{2} \\ 1 - s^2 \end{Bmatrix} \left(2\,\frac{\text{kN}}{\text{m}}\right) \frac{L}{2}\, ds \tag{2}$$

Upon integrating Equation (2)

$$\{f_s\} = \begin{Bmatrix} \left(\dfrac{s^3}{6} - \dfrac{s^2}{4}\right)_{-1}^{1} \\ \left(\dfrac{s^3}{6} - \dfrac{s^2}{4}\right)_{-1}^{1} \\ \left(s - \dfrac{s^3}{3}\right)_{-1}^{1} \end{Bmatrix} (2)\ \frac{4}{2} \tag{3}$$

$$\{f_s\} = \begin{Bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{4}{3} \end{Bmatrix} (4) = \begin{Bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{16}{3} \end{Bmatrix} \text{kN} \tag{4}$$

Using Equations (4) and (1) in (A) and applying boundary condition $u_1 = 0$,

$$\begin{Bmatrix} \frac{4}{3} \\ \frac{16}{3} \end{Bmatrix} = \begin{bmatrix} 2.393 & -2.732 \\ -2.732 & 5.468 \end{bmatrix} \times 10^4 \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{5}$$

Solving (5) for u_2 and u_3

$$u_2 = 3.885 \times 10^{-4} \text{ m},\ u_3 = 2.916 \times 10^{-4} \text{ m} \tag{6}$$

Stress in bar

$$E = 205 \times 10^9$$

At $s = -1$ or $x = 0$

$$\sigma_1 = E\left[\frac{2(-1)-1}{4} \ \frac{2(-1)+1}{4} \ \frac{-4(-1)}{4}\right] \begin{Bmatrix} 0 \\ 3.885 \times 10^{-4} \\ 2.916 \times 10^{-4} \end{Bmatrix}, \quad \underbrace{\sigma_1 = (3.987 \times 10^7)}_{\substack{\parallel \\ 39.87 \text{ MPa}}} \frac{\text{N}}{\text{m}}$$

At $s = 1$ or $x = L$

$$\sigma_2 = E\left[\frac{2\times 1-1}{4} \ \frac{2+1}{4} \ \frac{-4}{4}\right] \begin{Bmatrix} 0 \\ 3.885 \times 10^{-4} \\ 2.916 \times 10^{-4} \end{Bmatrix}, \ \sigma_2 = (-4.357 \times 10^4)$$

– 0.04357 MPa

Note small number compared to stress at fixed end.

10.9

$$x = \frac{1}{4}\,[(1-s)(1-t)\,x_1 + (1+s)(1-t)\,x_2 + (1+s)(1+t)\,x_3 + (1-s)(1+t)\,x_4]$$

$$y = \frac{1}{4}\,[(1-s)(1-t)\,y_1 + (1+s)(1-t)\,y_2 + (1+s)(1+t)\,y_3 + (1-s)(1+t)\,y_4]$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \text{ Equation (10.2.10)}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}\,[-x_1 + tx_1 + x_2 - tx_2 + x_3 + t\,x_3 - x_4 - tx_4]$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}\,[-x_1 + sx_1 - x_2 - sx_2 + x_3 + sx_3 + x_4 - sx_4]$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\ [-y_1 + ty_1 + y_2 - ty_2 + y_3 + ty_3 - y_4 - ty_4]$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}\ [-y_1 + sy_1 - y_2 - sy_2 + y_3 + sy_3 + y_4 - sy_4]$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where

$$J_{11} = \frac{1}{4}\ (-x_1 + tx_1 + x_2 - tx_2 + x_3 + tx_3 - x_4 - tx_4)$$

$$J_{12} = \frac{1}{4}\ (-y_1 + ty_1 + y_2 - ty_2 + y_3 + ty_3 - y_4 - ty_4)$$

$$J_{21} = \frac{1}{4}\ (-x_1 + sx_1 - x_2 - sx_2 + x_3 + sx_3 + x_4 - sx_4)$$

$$J_{22} = \frac{1}{4}\ (-y_1 + sy_1 - y_2 - sy_2 + y_3 + sy_3 + y_4 - sy_4)$$

Find determinate $|J|$

$$|J| = J_{11}J_{22} - J_{21}J_{12}$$

Multiplying and collecting terms

$$\begin{aligned}|J| = \frac{1}{16}\ [&2x_1y_2 - 2tx_1y_2 + 2tx_1y_3 - 2sx_1y_3 + 2sx_1y_4 - 2x_1y_4 - 2x_2y_1 + 2tx_2y_3 \\ &+ 2x_2y_3 + 2sx_2y_3 - 2sx_2y_4 - 2tx_2y_4 + 2sx_3y_1 - 2tx_3y_1 - 2x_3y_2 - 2sx_3y_2 \\ &+ 2x_3y_4 + 2tx_3y_4 + 2x_4y_1 - 2sx_4y_1 + 2sx_4y_2 + 2x_4y_2 + 2tx_4y_2 - 2x_4y_3 \\ &- 2tx_4y_3]\end{aligned}$$

Factor out x_i's

$$\begin{aligned}|J| = \frac{1}{8}\ [&x_1(y_2 - ty_2 + ty_3 - sy_3 + sy_4 - y_4) \\ &+ x_2(-y_1 + ty_1 + y_3 + sy_3 - sy_4 - ty_4) + x_3(sy_1 - ty_1 - y_2 - sy_2 + y_4 + ty_4) \\ &+ x_4(y_1 - sy_1 + sy_2 + ty_2 - y_3 - ty_3)]\end{aligned}$$

$$|J| = \frac{1}{8}\ [x_1\ x_2\ x_3\ x_4] \begin{bmatrix} y_2 - t\,y_2 + ty_3 - s\,y_3 + s\,y_4 - y_4 \\ -y_1 + t\,y_1 + y_3 + s\,y_3 - s\,y_4 - t\,y_4 \\ s\,y_1 - t\,y_1 - y_2 - s\,y_2 + y_4 + t\,y_4 \\ y_1 - s\,y_1 + s\,y_2 + t\,y_2 - y_3 - t\,y_3 \end{bmatrix}$$

$$|J| = \frac{1}{8}\ [x_1\ x_2\ x_3\ x_4] \begin{bmatrix} y_1(0) + y_2(1-t) + y_3(t-s) + y_4(s-1) \\ y_1(-1+t) + y_2(0) + y_3(0) + y_4(1+t) \\ y_1(s-t) + y_2(-1-s) + y_3(0) + y_4(1+t) \\ y_1(1-s) + y_2(s+t) + y_3(-1-t) + y_4(0) \end{bmatrix}$$

$$|J| = \frac{1}{8}\ [x_1\ x_2\ x_3\ x_4] \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ -1+t & 0 & s+1 & -s-t \\ s-t & -1-s & 0 & 1+t \\ 1-s & s+t & -1-t & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

10.10

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

x and y from Problem 10.9

$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)x_1 + \frac{1}{4}(1-t)x_2 + \frac{1}{4}(1+t)x_3 + \frac{1}{4}(-1)(1+t)x_4$$

$$= N_{1,s}\,x_1 + N_{2,s}\,x_2 + N_{3,s}\,x_3 + N_{4,s}\,x_4$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}(-1)(1-s)x_1 + \frac{1}{4}(-1)(1+s)x_2 + \frac{1}{4}(1+s)x_3 + \frac{1}{4}(1-s)x_4$$

$$= N_{1,t}\,x_1 + N_{2,t}\,x_2 + N_{3,t}\,x_3 + N_{4,t}\,x_4$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(-1)(1-t)y_1 + \frac{1}{4}(1-t)y_2 + \frac{1}{4}(1+t)y_3 + \frac{1}{4}(-1)(1+t)y_4$$

$$= N_{1,s}\,y_1 + N_{2,s}\,y_2 + N_{3,s}\,y_3 + N_{4,s}\,y_4$$

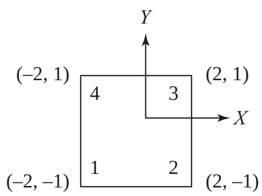
$$\frac{\partial y}{\partial t} = \frac{1}{4}(-1)(1-s)y_1 + \frac{1}{4}(-1)(1+s)y_2 + \frac{1}{4}(1+s)y_3 + \frac{1}{4}(1-s)y_4$$

$$= N_{1,t}\,y_1 + N_{2,t}\,y_2 + N_{3,t}\,y_3 + N_{4,t}\,y_4$$

$$[J] = \begin{bmatrix} N_{1,s} & N_{2,s} & N_{3,s} & N_{4,s} \\ N_{1,t} & N_{2,t} & N_{3,t} & N_{4,t} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

10.11 (a)

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{pmatrix} x_1 = -z,\ x_2 = z \\ x_3 = z,\ x_4 = -z \end{pmatrix}$$



$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)(-2) + \frac{1}{4}(1-t)(2) + \frac{1}{4}(1+t)(2) + \frac{1}{4}(-1)(1+t)(-2) = z$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}(-1)(1-s)(-2) + \frac{1}{4}(-1)(1+s)(2) + \frac{1}{4}(1+s)(2)$$

$$+ \frac{1}{4}(1-s)(-2) = 0$$

$y_1 = 1$, $y_2 = -1$

$y_3 = 1$, $y_4 = 1$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(-1)(1-t)(-1) + \frac{1}{4}(1-t)(-1) + \frac{1}{4}(1+t)(1)$$

$$+ \frac{1}{4}(-1)(1+t)(1) = 0$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}\ (-1)\,(1-s)\,(-1) + \frac{1}{4}\ (-1)\,(1+s)\,(-1) + \frac{1}{4}\ (1+s)\,(1)$$

$$+ \frac{1}{4}\ (1-s)\,(1) = 1$$

$$|[J]| = \left|\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 2 - 0 = 2$$

By Equation (10.2.22)

$$|[J]| = \frac{1}{8}\ [-2\ \ 2\ \ 2\ -2] \times \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix} \qquad \text{(A)}$$

Symplying by multiplying the matrices in Equation (A) yields

$|[J]| = 2$ also

and $\quad |[J]| = \dfrac{A}{4}$ as

$A = 4 \times 2 = 8$ (area of element)

$\therefore \quad = |[J]|\ \dfrac{8}{4} = 2$

10.12

By Equation (10.2.18)

$$[B(s,t)] = \frac{1}{|[J]|}\ [B_1] \quad [B_2] \quad [B_3] \quad [B_4]$$

By Equation (10.2.3)

$$x = \frac{1}{4}\ [(1-s)\,(1-t)x_1 + (1+s)\,(1-t)x_2 + (1+s)\,(1+t)x_3 + (1-s)$$
$$(1+t)x_4] \qquad (1)$$

$$y = \frac{1}{4}\ [(1-s)\,(1-t)y_1 + (1+s)\,(1-t)y_2 + (1+s)\,(1+t)y_3 + (1-s)$$
$$(1+t)y_4]$$

By Equation (10.2.16)

$$[D'] = \frac{1}{|[J]|}\begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\,)}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\,)}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\,)}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\,)}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\,)}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\,)}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\,)}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\,)}{\partial t} \end{bmatrix} \qquad (2)$$

Let

$$a = \frac{\partial y}{\partial t} = \frac{1}{4}\ [y_1(s-1) + y_2(-1-s) + y_3(1+s) + y_4(1-s)]$$

$$b = \frac{\partial y}{\partial s} = \frac{1}{4}\ [y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)]$$

$$c = \frac{\partial x}{\partial s} = \frac{1}{4}\ [x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)]$$

$$d = \frac{\partial x}{\partial t} = \frac{1}{4}\ [x_1(s-1) + x_2(-1-s) + x_3(1+s) + x_4(1-s)]$$

By Equation (10.3.5)

$$N_1 = \frac{(1-s)(1-t)}{4},\ N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4},\ N_4 = \frac{(1-s)(1+t)}{4}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Let

$$N_{i,s} = \frac{\partial N_i}{\partial s}, \quad N_{i,t} = \frac{\partial N_i}{\partial t}$$

Now

$$[B] = [D'][N]$$

Using $[D']$ from Equation (2) above, we obtain

$$[B] = \frac{1}{|[J]|}\begin{bmatrix} a\frac{\partial(\ )}{\partial s} - b\frac{\partial(\ )}{\partial t} & 0 \\ 0 & c\frac{\partial(\ )}{\partial t} - d\frac{\partial(\ )}{\partial s} \\ c\frac{\partial(\ )}{\partial t} - d\frac{\partial(\ )}{\partial s} & a\frac{\partial(\ )}{\partial s} - b\frac{\partial(\ )}{\partial t} \end{bmatrix}\begin{bmatrix} \frac{(1-s)(1-t)}{4} & 0\cdots 0 \\ 0 & \frac{(1-s)(1-t)}{4} & \cdots \end{bmatrix}$$

$$\begin{array}{cccc} \text{col}(1) & \text{col}(2) & \cos(7) & \cos(8) \end{array}$$

$$[B] = \frac{1}{|[J]|}\begin{bmatrix} \frac{a(t-1)}{4} - \frac{b(s-1)}{4} & 0 \quad \cdots & \cdots\frac{a(1-t)}{4} - \frac{b(1-s)}{4} & 0 \\ 0 & \frac{c(s-1)}{4} - \frac{d(t-1)}{4}\cdots & 0 & \frac{c(1-s)}{4} - \frac{d(-1-t)}{4} \\ \frac{c(s-1)}{4} - \frac{d(t-1)}{4} & \frac{a(t-1)}{4} - \frac{b(s-1)}{4}\cdots & \cdots\frac{c(1-s)}{4} - \frac{d(-1-t)}{4} & \frac{a(-1-t)}{4} - \frac{b(1-s)}{4} \end{bmatrix}$$

By Equation (10.2.19)

$$[B] = \frac{1}{|[J]|} = [B_1] \quad [B_2] \quad [B_3] \quad [B_4]$$

where the submatrices are

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

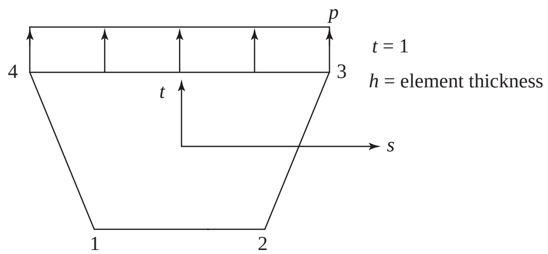
where, for instance

$$N_{1,s} = \frac{\partial N_1}{\partial s} = \frac{t-1}{4}$$

$$N_{1,t} = \frac{\partial N_1}{\partial t} = \frac{s-1}{4}$$

etc.

10.13



$$\{f_s\} = \int_{-1}^{1} [N_s]^T \{T\}\, h\, \frac{L}{2}\, ds$$

At $\qquad t = 1$

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} h\, \frac{L}{2}\, ds$$

$$p_s = 0,\ p_t = p$$

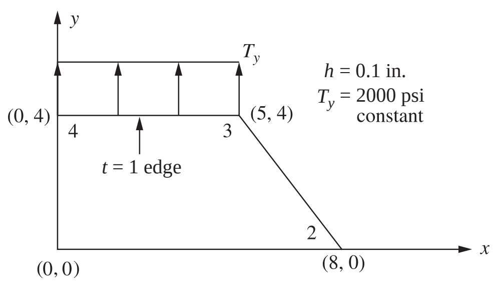
$$\therefore \quad \{f_s\} = \int_{-1}^{1} \begin{Bmatrix} 0 \\ N_3 p \\ 0 \\ N_4 p \end{Bmatrix} h\, \frac{L}{2}\, ds$$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} 0 \\ \frac{(1+s)\,(1+t)}{4}\, p \\ 0 \\ \frac{(1-s)\,(1+t)}{4}\, p \end{Bmatrix}_{t=1} h\, \frac{L}{2}\, ds$$

$$= \left. \begin{bmatrix} 0 \\ \frac{ps}{2} + \frac{ps^2}{4} \\ 0 \\ \frac{ps}{2} - \frac{ps^2}{4} \end{bmatrix} \right|_{-1}^{1} \frac{Lh}{2} = \begin{matrix} f_{s3s} = 0 \\ f_{s3t} = \dfrac{pLh}{2} \\ f_{s4s} = 0 \\ f_{s4t} = \dfrac{pLh}{2} \end{matrix}$$

10.14

(a)



$$\{ f_s\} = \int_{-1}^{1} [N_s]^T \{T\} \frac{L}{2} \, h \, dt$$

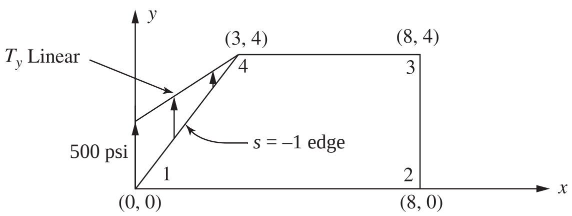
$$L = 5 \text{ in.}, \quad p_t = 2000 \text{ psi}, \quad p_s = 0$$

$$N_3 = \frac{1+s}{2} \text{ and } N_4 = \frac{1-s}{2} \text{ for } t = 1$$

$$\therefore \quad \begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{Bmatrix} p_s = 0 \\ p_t = 2000 \end{Bmatrix} h \, \frac{L}{2} \, dt$$

$$= \begin{Bmatrix} 0 \\ \int_{-1}^{1} \left(\frac{1+s}{2}\right)(2000)(0.1)\left(\frac{5}{2}\right) dt \\ 0 \\ \int_{-1}^{1} \left(\frac{1-s}{2}\right)(2000)(0.1)\left(\frac{5}{2}\right) dt \end{Bmatrix} \begin{Bmatrix} 0 \\ 500 \\ 0 \\ 500 \end{Bmatrix} \text{ lb}$$

(b)



h = 0.1 in.
s = –1
L = 5 in.

$$\begin{Bmatrix} f_{s1s} \\ f_{s1t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{Bmatrix} p_s = 0 \\ p_t = -250t + 250 \end{Bmatrix} h \, \frac{L}{2} \, dt$$

$$f_{s1s} = f_{s4s} = 0 \quad N_1 = \frac{1-t}{2}, \; N_4 = \frac{1+t}{2} \text{ for } s = -1$$

$$f_{s1t} = \int_{-1}^{1} \left(\frac{1-t}{2}\right)(-250\,t + 250)\,(0.1)\left(\frac{5}{2}\right) dt$$

$$= 83.33 \text{ lb}$$

$$f_{s4t} = \int_{-1}^{1} \left(\frac{1+t}{2}\right)(-250\,t + 250)\,(0.1)\left(\frac{5}{2}\right) dt$$

$$= 41.67 \text{ lb}$$

10.15

(a) $\int_{-1}^{1} \cos \frac{s}{2} \, ds$ Use Table 10.1

$$I = \sum_{i=1}^{3} W_i \cos \frac{s_i}{2} = W_1 \cos \frac{s_1}{2} + W_2 \cos \frac{s_2}{2} + W_3 \cos \frac{s_3}{2}$$

$$= \frac{5}{9}\cos\left(\frac{0.7746}{2}\right) + \frac{8}{9}\cos(0) + \frac{5}{9}\cos\left(\frac{-0.7746}{2}\right)$$

$$I = 1.918 \qquad \text{(Analytical } I = 1.918\text{)}$$

That is

$$\int_{-1}^{1} \cos\frac{s}{2}\, ds = 2\sin\frac{s}{2}\Big|_{-1}^{1} = 2\sin\frac{1}{2} - 2\sin\left(-\frac{1}{2}\right)$$

$$= 4\sin\frac{1}{2} = 4(0.47)$$

$$= 1.918$$

(b) $\int_{-1}^{1} s^2\, ds$

$$I = \sum_{i=1}^{3} W_i s_i^2 = W_1 s_1^2 + W_2 s_2^2 + W_3 s_3^2$$

$$= \frac{5}{9}(0.7746)^2 + \frac{8}{9}(0)^2 + \frac{5}{9}(-0.7746)^2$$

$$I = 0.667 \qquad \text{(Analytical } I = 0.667\text{)}$$

(c) $\int_{-1}^{1} s^4\, ds = \frac{5}{9}(0.7746)^4 + \frac{8}{9}(0)^4 + \frac{5}{9}(-0.7746)^4$

$$= 0.400$$

(d) $\int_{-1}^{1} \frac{\cos s}{1-s^2}\, ds = \frac{5}{9}\left(\frac{\cos(0.7746)}{1-0.7746^2}\right) + \frac{8}{9}\left(\frac{\cos 0}{1-0^2}\right) + \frac{5}{9}\frac{\cos(-0.7746)}{1-(-0.7746)^2} = 2.873$

(Exact is 3.86)

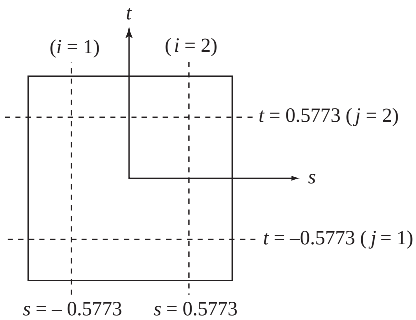
(e) $\int_{-1}^{1} s^3\, ds = \frac{5}{9}(0.7746)^3 + \frac{8}{9}(0)^3 + \frac{5}{9}(-0.7746)^3 = 0$

(f) $\int_{-1}^{1} s\cos s\, ds = \left(\frac{5}{9}\right)(0.7746)\cos(0.7746) + \left(\frac{8}{9}\right)(0)\cos(0) + \frac{5}{9}(-0.7746)\cos(-0.7746)$

$$= 0.30756 + 0 - 0.30756$$

$$= 0$$

10.16



$$[k] = [B]^T(s_1, t_1)\,[D]\,[B](s_1, t_1)\,|[J](s_1, t_1)|\,h\,W_1\,W_1$$
$$+ [B]^T(s_2, t_1)\,[D]\,[B](s_2, t_1)\,|[J](s_2, t_1)|\,h\,W_2\,W_1$$

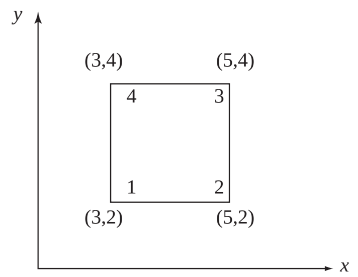
$$+ [B]^T (s_1, t_2) [D] [B] (s_1, t_2) \mid [J] (s_1, t_2) \mid h W_1 W_2$$
$$+ [B]^T (s_2, t_2) [D] [B] (s_2, t_2) \mid [J] (s_2, t_2) \mid h W_2 W_2$$

where

$$s_1 = -0.5773, s_2 = 0.5773$$
$$t_1 = -0.5773, t_2 = 0.5773$$

Using computer program

(a)



ENTER THE GAUSS POINTS S AND T FOR POINT 1
–0.57735, –0.57735
ENTER THE WEIGHT FOR POINT 1
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 1
3.0, 2.0
ENTER THE GAUSS POINTS S AND T FOR POINT 2
0.57735, –0.57735
ENTER THE WEIGHT FOR POINT 2
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 2
5.0, 2.0
ENTER THE GAUSS POINTS S AND T FOR POINT 3
0.57735, 0.57735
ENTER THE WEIGHT FOR POINT 3
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 3
5.0, 4.0
ENTER THE GAUSS POINTS S AND T FOR POINT 4
–0.57734, 0.57734
ENTER THE WEIGHT FOR POINT 4
1.0
ENTER THE NODAL VALUES X AND Y FOR POINT 4
3.0, 4.0
ENTER THE VALUE FOR YOUNGS MODULUS
30000000.0
ENTER THE VALUE FOR POISSONS RATIO
0.25
ENTER THE VALUE FOR THE THICKNESS, h
1.0

THE GAUSS VALUES S AND T AND WEIGHTS ARE

POINT	S	T	WEIGHT
1	–5.773500E–001	–5.773500E–001	1.0000000
2	5.773500E–001	–5.773500E–001	1.0000000
3	5.773500E–001	5.773500E–001	1.0000000
4	–5.773400E–001	5.773400E–001	1.0000000

THE NODAL COORDINATE VALUES ARE

NODE	X	Y
1	3.0000000	2.0000000
2	5.0000000	2.0000000
3	5.0000000	4.0000000
4	3.0000000	4.0000000

THE ELEMENT PARAMETERS ARE

YOUNGS MODULUS	POISSON'S RATIO	THICKNESS
30000000.0000000	2.500000E–001	1.0000000

DO YOU WISH TO VIEW THE VALUES OF J (Y/N)?
THE VALUES OF I J I ARE
THE VALUE OF J 1
1.0000000
THE VALUE OF J 2
1.0000000
THE VALUE OF J 3
1.0000000
THE VALUE OF J 4
1.0000000
DO YOU WITH TO VIEW THE B MATRIX (Y/N)?
THE B MATRIX VALUES ARE ($[B]_{3\times 8}$)

–1.0566 E–1	0	1.0566 E–1	0	0	–3.943 E–1
				0	–1.0566 E–1
– 3.943 E–1		–1.0566 E–1	–1.0566 E–1		1.0566 E–1
		3.943 E–1	0	–3.943 E–1	0
0		1.0566 E–1	0	3.943E–1	1.0566 E–1
		3.943 E–1		3.943 E–1	–3.943 E–1

} = one row of the 3×8 $[B]$

DO YOU WISH TO VIEW THE D MATRIX (Y/N)?
THE VALUES OF THE D MATRIX ARE

32000000.0000000	8000000.0000000	0.0000000
8000000.0000000	32000000.0000000	0.0000000
0.0000000	0.0000000	12000000.0000000

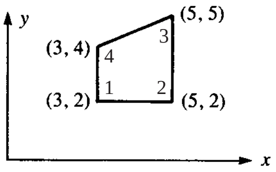
DO YOU WISH TO VIEW THE K MATRIX (Y/N)?
THE K MATRIX VALUES ARE

14666660.0000000	5000015.0000000	–8666672.0000000	–1000005.0000000
5000015.0000000	14666610.0000000	1000001.0000000	1333353.0000000
–8666672.0000000	1000001.0000000	14666690.0000000	–5000011.0000000
–1000005.0000000	1333353.0000000	–5000011.0000000	14666690.0000000
–7333369.0000000	–4999991.0000000	1333353.0000000	1000001.0000000

–4999981.0000000	–7333369.0000000	–1000005.0000000	–8666672.0000000
1333383.0000000	–1000025.0000000	–7333369.0000000	5000015.0000000
999970.6000000	–8666592.0000000	5000015.0000000	–7333369.0000000

–7333369.0000000	–4999981.0000000	1333383.0000000	999970.6000000
–4999991.0000000	–7333369.0000000	–1000025.0000000	–8666592.0000000
1333353.0000000	–1000005.0000000	–7333369.0000000	5000015.0000000
1000001.0000000	–8666672.0000000	5000015.0000000	–7333369.0000000
14666610.0000000	5000015.0000000	–8666592.0000000	–1000025.0000000
5000015.0000000	14666660.0000000	999970.6000000	1333383.0000000
–8666592.0000000	999970.6000000	14666580.0000000	–4999961.0000000
–1000025.0000000	1333383.0000000	–4999961.0000000	14666580.0000000

(b)



THE K MATRIX VALUES ARE

8-1 Column	8-2
14990860.0000000	3483641.0000000
3483641.0000000	13670480.0000000
–11385300.0000000	370145.5000000
–1626729.0000000	–565783.1000000
–4661870.0000000	–4327808.0000000
–4309764.0000000	–5451907.0000000
1056312.0000000	474022.0000000
2452853.0000000	–7652789.0000000
8-3	8-4
–11385300.0000000	–1626729.0000000
370145.6000000	–565783.1000000
19631590.0000000	–6267461.0000000
–6267461.0000000	14127760.0000000
4403077.0000000	2237624.0000000
222705.5000000	–5114752.0000000
–12649370.0000000	5656566.0000000
5674610.0000000	–8447218.0000000
8-5	8-6
–4661870.0000000	–4309764.0000000
–4327808.0000000	–5451907.0000000
4403076.0000000	222705.4000000
2237624.0000000	–5114752.0000000
11571920.0000000	3807131.0000000
3807131.0000000	11105380.0000000
–11313130.0000000	279927.3000000
–1716948.0000000	–538717.6000000
8-7	8-8
1056312.0000000	2452853.0000000
474021.9000000	–7652789.0000000

–12649370.0000000	5674610.0000000
5656566.0000000	–8447218.0000000
–11313130.0000000	–1716948.0000000
279927.4000000	–538717.6000000
22906180.0000000	–6410515.0000000
–6410515.0000000	16638720.0000000

10.18

$$[B\,(s,t)] = \frac{1}{|[J]|}\;[B_1]\;\;[B_2]\;\;[B_3]\;\;[B_4]\;\;[B_5]\;\;[B_6]\;\;[B_7]\;\;[B_8]$$

$$N_{1,\,s} = \frac{1}{4}\,(1-t)\,(s+t+1) - \frac{1}{4}\,(1-s)\,(1-t)$$

$$N_{2,\,s} = \frac{1}{4}\,(1-t)\,(s-t-1) + \frac{1}{4}\,(1+s)\,(1-t)$$

$$N_{3,\,s} = \frac{1}{4}\,(1+t)\,(s+t-1) - \frac{1}{4}\,(1+s)\,(1+t)$$

$$N_{4,\,s} = \frac{1}{4}\,(1+t)\,(s-t+1) - \frac{1}{4}\,(1-s)\,(1+t)$$

$$N_{5,\,s} = (t-1)s$$

$$N_{6,\,s} = \frac{1}{2}\,(1-t^2)$$

$$N_{7,\,s} = -\,(1+t)s$$

$$N_{8,\,s} = \frac{1}{2}\,(t^2-1)$$

$$N_{1,\,t} = \frac{1}{4}\,(1-s)\,(s+t+1) + \frac{1}{4}\,(1-s)\,(t-1)$$

$$N_{2,\,t} = \frac{1}{4}\,(1+s)\,(t+1-s) - \frac{1}{4}\,(1+s)\,(1-t)$$

$$N_{3,\,t} = \frac{1}{4}\,(1+s)\,(s+t-1) + \frac{1}{4}\,(1+s)\,(1+t)$$

$$N_{4,\,t} = \frac{1}{4}\,(1-s)\,(-s+t-1) + \frac{1}{4}\,(1-s)\,(1+t)$$

$$N_{5,\,t} = \frac{1}{2}\,(1+s)\,(s-1)$$

$$N_{6,\,t} = -\,(1+s)\,t$$

$$N_{7,\,t} = \frac{1}{2}\,(1-s^2)$$

$$N_{8,\,t} = (s-1)\,t$$

$$|[J]| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

$$= [N_{1,s}\,x_1 + N_{2,s}\,x_2 + \ldots + N_{8,s}\,x_8]$$

$$\times [N_{1,t}\,y_1 + N_{2,t}\,y_2 + \ldots + N_{8,t}\,y_8]$$

$$- [N_{1,s}\,y_1 + N_{2,s}\,y_2 + \ldots + N_{8,s}\,y_8]$$

$$\times [N_{1,t}\,x_1 + N_{2,t}\,x_2 + \ldots + N_{8,t}\,x_8]$$

$$[B_i] = \begin{bmatrix} a\,(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c\,(N_{i,t}) - d\,(N_{i,s}) \\ c\,(N_{i,t}) - d\,(N_{i,s}) & a\,(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

where

$$a = \frac{\partial y}{\partial t} = N_{1,t}\,y_1 + N_{2,t}\,y_2 + \ldots + N_{8,t}\,y_8$$

$$b = \frac{\partial y}{\partial s} = N_{1,s}\,y_1 + N_{2,s}\,y_2 + \ldots + N_{8,s}\,y_8$$

$$c = \frac{\partial x}{\partial s} = N_{1,s}\,x_1 + N_{2,s}\,x_2 + \ldots + N_{8,s}\,x_8$$

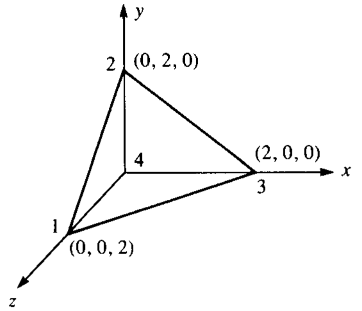
$$d = \frac{\partial x}{\partial t} = N_{1,t}\,x_1 + N_{2,t}\,x_2 + \ldots + N_{8,t}\,x_8$$

10.21 The 2-pt rule works as we have a 2^{nd} order in s for the integrand see Equation (10.6.19) and for integrand of order $2n - 1 = 2 \times 2 - 1 = 3$ we get exact solution.

Chapter 11

11.1

(a)



$$[B] = \frac{1}{6V}\begin{bmatrix} \beta_1 & 0 & 0 & \beta_2 & 0 & 0 & \beta_3 & 0 & 0 & \beta_4 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 & \gamma_2 & 0 & 0 & \gamma_3 & 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 & \delta_2 & 0 & 0 & \delta_3 & 0 & 0 & \delta_4 \\ \gamma_1 & \beta_1 & 0 & \gamma_2 & \beta_2 & 0 & \gamma_3 & \beta_3 & 0 & \gamma_4 & \beta_4 & 0 \\ 0 & \delta_1 & \gamma_1 & 0 & \delta_2 & \gamma_2 & 0 & \delta_3 & \gamma_3 & 0 & \delta_4 & \gamma_4 \\ \delta_1 & 0 & \beta_1 & \delta_2 & 0 & \beta_2 & \delta_3 & 0 & \beta_3 & \delta_4 & 0 & \beta_4 \end{bmatrix}$$

By Equations (12.2.4) to (12.2.8)

$$\beta_1 = -\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \beta_2 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \beta_3 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4$$

$$\beta_4 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -4,\ \gamma_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\gamma_2 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4,\ \gamma_3 = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0,\ \gamma_4 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -4$$

$$\delta_1 = -\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 4,\ \delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix},\ \delta_3 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\delta_4 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4$$

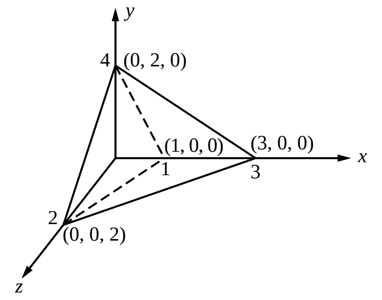
$$6V = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = (1)\,(-1)^2\begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + 2\,(-1)^3\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 8$$

$$[B] = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & -4 & -4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -4 & -4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & -4 \end{bmatrix}$$

Problem 11-1: 'B' matrix for tetrahedral solid element

(b)

$$x_1 = 1 \quad y_1 = 0 \quad z_1 = 0$$
$$x_2 = 0 \quad y_2 = 0 \quad z_2 = 2$$
$$x_3 = 3 \quad y_3 = 0 \quad z_3 = 0$$
$$x_4 = 0 \quad y_4 = 2 \quad z_4 = 0$$



Geometry description (m)

$$\alpha_1 = \begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_1 = -\begin{pmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_1 = -\begin{pmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_2 = \begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_2 = -\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{pmatrix} \quad \beta_3 = -\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{pmatrix} \quad \delta_3 = -\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{pmatrix}$$

$$\alpha_4 = -\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \quad \beta_4 = \begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{pmatrix} \quad \gamma_4 = -\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{pmatrix} \quad \delta_4 = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$V = \frac{1}{6}\left|\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}\right| \qquad V = 1.333$$

$$[B_1] = \frac{1}{6V}\begin{pmatrix} |\beta_1| & 0 & 0 \\ 0 & |\gamma_1| & 0 \\ 0 & 0 & |\delta_1| \\ |\gamma_1| & |\beta_1| & 0 \\ 0 & |\delta_1| & |\gamma_1| \\ |\delta_1| & 0 & |\beta_1| \end{pmatrix} \qquad [B_2] = \frac{1}{6V}\cdot\begin{pmatrix} |\beta_2| & 0 & 0 \\ 0 & |\gamma_2| & 0 \\ 0 & 0 & |\delta_2| \\ |\gamma_2| & |\beta_2| & 0 \\ 0 & |\delta_2| & |\gamma_2| \\ |\delta_2| & 0 & |\beta_2| \end{pmatrix}$$

$$[B_3] = \frac{1}{6V}\begin{pmatrix} |\beta_3| & 0 & 0 \\ 0 & |\gamma_3| & 0 \\ 0 & 0 & |\delta_3| \\ |\gamma_3| & |\beta_3| & 0 \\ 0 & |\delta_3| & |\gamma_3| \\ |\delta_3| & 0 & |\beta_3| \end{pmatrix} \qquad [B_4] = \frac{1}{6V}\begin{pmatrix} |\beta_4| & 0 & 0 \\ 0 & |\gamma_4| & 0 \\ 0 & 0 & |\delta_4| \\ |\gamma_4| & |\beta_4| & 0 \\ 0 & |\delta_4| & |\gamma_4| \\ |\delta_4| & 0 & |\beta_4| \end{pmatrix}$$

$[B]$ = augment ($[B_1]$, $[B_2]$, $[B_3]$, $[B_4]$)

$[B]$ =

−0.5	0	0	0	0	0	0.5	0	0	0	0	0
0	−0.75	0	0	0	0	0	0.25	0	0	0.5	0
0	0	−0.75	0	0	0.5	0	0	0.25	0	0	0
−0.75	−0.5	0	0	0	0	0.25	0.5	0	0.5	0	0
0	−0.75	−0.75	0	0.5	0	0	0.25	0.25	0	0	0.5
−0.75	0	−0.5	0.5	0	0	0.25	0	0.5	0	0	0

11.2(a) Use Equation (11.2.18) and substitute $[B]$ from 11.1 (a) and $[D]$ from Equation (11.1.5) into Equation (11.2.18)

$$\therefore\quad [k] = [B]^T\,[D]\,[B]\,V$$

[B] =

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	0.5	0	0	–0.5	0	0
2	0	0	0	0	0.5	0	0	0	0	0	–0.5	0
3	0	0	0.5	0	0	0	0	0	0	0	0	–0.5
4	0	0	0	0.5	0	0	0	0.5	0	–0.5	–0.5	0
5	0	0.5	0	0	0	0	0	0	0.5	0	–0.5	–0.5
6	0.5	0	0	0	0	0	0	0	0.5	–0.5	0	–0.5

$$v = 0.3 \qquad E = 30 \times 10^6$$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ 0 & 1-v & v & 0 & 0 & 0 \\ 0 & 0 & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = [B]^T \; [D] \; [B] \; V$$

[k] =

	1	2	3	4	5	6	7	8	9	10	11	12
1	3.846	0	0	0	0	0	0	0	3.846	–3.846	0	–3.846
2	0	3.846	0	0	0	0	0	0	3.846	0	–3.846	–3.846
3	0	0	13.462	0	0	0	0	0	0	0	0	13.462
4	0	0	0	3.846	0	0	0	3.846	0	– 3.846	– 3.846	0
5	0	0	5.769	0	13.462	0	0	0	0	0	13.462	–5.769
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	5.769	0	5.769	0	13.462	0	0	13.462	–5.769	–5.769
8	0	0	0	3.846	0	0	0	3.846	0	–3.846	–3.846	0
9	3.846	3.846	0	0	0	0	0	0	7.692	–3.846	–3.846	–7.692
10	–3.846	0	–5.769	–3.846	–5.769	0	13.462	–3.846	–3.846	21.154	9.615	9.615
11	0	–3.846	–5.769	–3.846	13.462	0	0	–3.846	–3.846	3.846	21.154	9.615
12	–3.846	–3.846	13.462	0	0	0	0	0	–7.692	3.846	3.846	21.154

$$\times 10^6 \; \frac{\text{lb}}{\text{in.}}$$

(b) Evaluate the stiffness matrix for the element shown

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ 0 & 1-v & v & 0 & 0 & 0 \\ 0 & 0 & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$$[k] = [B]^T \; [D] \; [B] \; V$$

$[k] =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	$3.077\cdot10^{7}$	$1.442\cdot10^{7}$	$1.442\cdot10^{7}$	$-5.769\cdot10^{6}$	0	$-5.769\cdot10^{6}$	$-1.923\cdot10^{7}$	$-8.654\cdot10^{6}$	$-8.654\cdot10^{6}$	$-5.769\cdot10^{6}$	$-5.769\cdot10^{6}$	0
1	$1.442\cdot10^{7}$	$4.279.10^{7}$	$2.163\cdot10^{7}$	0	$-5.769\cdot10^{6}$	$-8.654\cdot10^{6}$	$-1.058\cdot10^{7}$	$-1.683\cdot10^{7}$	$-7.212\cdot10^{6}$	$-3.846\cdot10^{6}$	$-2.019\cdot10^{7}$	$-5.769\cdot10^{6}$
2	$1.442\cdot10^{7}$	$2.163\cdot10^{7}$	$4.279\cdot10^{7}$	$-3.846\cdot10^{6}$	$-5.769\cdot10^{6}$	$-2.019\cdot10^{7}$	$-1.058\cdot10^{7}$	$-7.212\cdot10^{6}$	$-1.683\cdot10^{7}$	0	$-8.654\cdot10^{6}$	$-5.769\cdot10^{6}$
3	$-5.769\cdot10^{6}$	0	$-3.846\cdot10^{6}$	$3.846\cdot10^{6}$	0	0	$1.923\cdot10^{6}$	0	$3.846\cdot10^{6}$	0	0	0
4	0	$-5.769\cdot10^{6}$	$-5.769\cdot10^{6}$	0	$3.846\cdot10^{6}$	0	0	$1.923\cdot10^{6}$	$1.923\cdot10^{6}$	0	0	$3.846\cdot10^{6}$
5	$-5.769\cdot10^{6}$	$-8.654\cdot10^{6}$	$-2.019\cdot10^{7}$	0	0	$1.346\cdot10^{7}$	$5.769\cdot10^{6}$	$2.885\cdot10^{6}$	$6.731\cdot10^{6}$	0	$5.769\cdot10^{6}$	0
6	$-1.923\cdot10^{7}$	$-1.058\cdot10^{7}$	$-1.058\cdot10^{7}$	$1.923\cdot10^{6}$	0	$5.769\cdot10^{6}$	$1.538\cdot10^{7}$	$4.808\cdot10^{6}$	$4.808\cdot10^{6}$	$1.923\cdot10^{6}$	$5.769\cdot10^{6}$	0
7	$-8.654\cdot10^{6}$	$-1.683\cdot10^{7}$	$-7.212\cdot10^{6}$	0	$1.923\cdot10^{6}$	$2.885\cdot10^{6}$	$4.808\cdot10^{6}$	$8.173\cdot10^{6}$	$2.404\cdot10^{6}$	$3.846\cdot10^{6}$	$6.731\cdot10^{6}$	$1.923\cdot10^{6}$
8	$-8.654\cdot10^{6}$	$-7.212\cdot10^{6}$	$-1.683\cdot10^{7}$	$3.846\cdot10^{6}$	$1.923\cdot10^{6}$	$6.731\cdot10^{6}$	$4.808\cdot10^{6}$	$2.404\cdot10^{6}$	$8.173\cdot10^{6}$	0	$2.885\cdot10^{6}$	$1.923\cdot10^{6}$
9	$-5.769\cdot10^{6}$	$-3.846.10^{6}$	0	0	0	0	$1.923\cdot10^{6}$	$3.846\cdot10^{6}$	0	$3.846\cdot10^{6}$	0	0
10	$-5.769\cdot10^{6}$	$-2.019\cdot10^{7}$	$-8.654\cdot10^{6}$	0	0	$5.769\cdot10^{6}$	$5.769\cdot10^{6}$	$6.731\cdot10^{6}$	$2.885\cdot10^{6}$	0	$1.346\cdot10^{7}$	0
11	0	$-5.769\cdot10^{6}$	$-5.769\cdot10^{6}$	0	$3.846\cdot10^{6}$	0	0	$1.923\cdot10^{6}$	$1.923\cdot10^{6}$	0	0	$3.846\cdot10^{6}$

11.3 (a) $\{\varepsilon\} = [B]\{d\}$

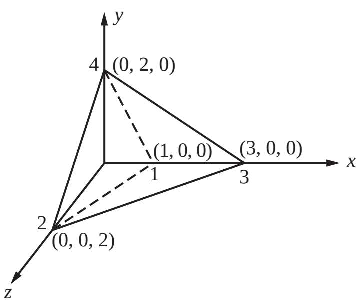
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & -4 & -4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -4 & -4 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & -4 \end{bmatrix}$$

$$\times \begin{Bmatrix} 0.005 \\ 0.0 \\ 0.0 \\ 0.001 \\ 0.0 \\ 0.001 \\ 0.005 \\ 0.0 \\ 0.0 \\ -0.001 \\ 0.0 \\ 0.005 \end{Bmatrix} = \begin{Bmatrix} 0.003 \\ 0.0 \\ -0.0025 \\ 0.001 \\ -0.002 \\ 0.0005 \end{Bmatrix} \frac{\text{in.}}{\text{in.}}$$

$\{\sigma\} = [D]\{\varepsilon\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{30 \times 10^3}{(1+0.3)(1-2(0.3))}\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}\begin{Bmatrix} 0.003 \\ 0.0 \\ -0.0025 \\ 0.001 \\ -0.002 \\ 0.0005 \end{Bmatrix} = \begin{Bmatrix} 77.9 \\ 8.65 \\ -49.0 \\ 11.5 \\ -23.1 \\ 5.77 \end{Bmatrix} \text{ ksi}$$

(b)



Nodal displacements (in.)

$u_1 = 0.005 \quad v_1 = 0 \quad w_1 = 0$

$u_2 = 0.001 \quad v_2 = 0 \quad w_2 = 0.001$

$u_3 = 0.005 \quad v_3 = 0 \quad w_3 = 0$

$u_4 = -0.001 \quad v_4 = 0 \quad w_4 = 0.005$

$$\{d_1\} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} \quad \{d_2\} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} \quad \{d_3\} = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} \quad \{d_4\} = \begin{pmatrix} u_4 \\ v_4 \\ w_4 \end{pmatrix}$$

$$\{d\} = \text{stack}\,(\{d_1\}, \{d_2\}, \{d_3\}, \{d_4\})$$

Material properties

$$E = 30 \times 10^6 \qquad v = 0.3$$

Element strain matrix ([B] from P11.b)

$$\{\varepsilon\} = [B]\,\{d\}$$

$$\varepsilon = \begin{pmatrix} 0 \\ 0 \\ 5\times 10^{-4} \\ -3\times 10^{-3} \\ 2.5\times 10^{-3} \\ -2\times 10^{-3} \end{pmatrix} \frac{\text{in.}}{\text{in.}}$$

Determine constitutive matrix

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

Determine element stresses

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

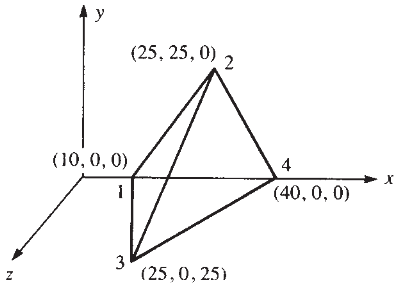
$$\{\sigma\} = \begin{Bmatrix} 8.654\times 10^{3} \\ 8.654\times 10^{3} \\ 2.019\times 10^{4} \\ -3.462\times 10^{4} \\ 2.885\times 10^{4} \\ -2.308\times 10^{4} \end{Bmatrix} \text{(psi)}$$

11.4 The strains and stress are constant in the 4-noded tetrahedral element.

11.5 Use Equation (11.2.10) for the shape functions $N_1 - N_4$. Substitute $[N]_{3\times 12}$ from Equation (11.2.9) and $\{X\}$ from Equation (11.2.20a) into Equation (11.2.19), $\{f_b\}_{12\times 1} = \iint_V\int [N]^T_{12\times 3} \begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix}$ to show at node i, $\{f_{b_i}\} = \dfrac{V}{4}\begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix}$.

11.6

(a)



$$\beta_1 = -\begin{vmatrix} 1 & 25 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = -625, \qquad \beta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\beta_3 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 25 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0, \qquad \beta_4 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 25 & 0 \\ 1 & 0 & 25 \end{vmatrix} = 625$$

$$\gamma_1 = \begin{vmatrix} 1 & 25 & 0 \\ 1 & 25 & 25 \\ 1 & 40 & 0 \end{vmatrix} = -375, \qquad \gamma_2 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 0 & 25 \\ 1 & 40 & 0 \end{vmatrix} = 750$$

$$\gamma_3 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = 0, \qquad \gamma_4 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 25 & 25 \end{vmatrix} = -375$$

$$\delta_1 = -\begin{vmatrix} 1 & 25 & 25 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = -375, \ \delta_2 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 0 \\ 1 & 40 & 0 \end{vmatrix} = 0$$

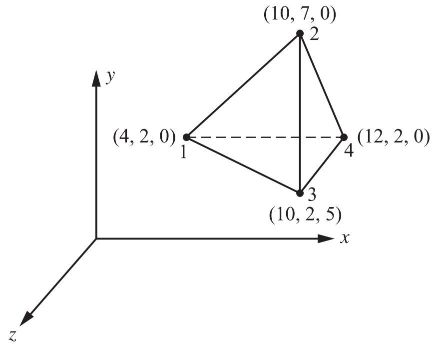
$$\delta_3 = -\begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 25 \\ 1 & 40 & 0 \end{vmatrix} = 750, \ \delta_4 = \begin{vmatrix} 1 & 10 & 0 \\ 1 & 25 & 25 \\ 1 & 25 & 0 \end{vmatrix} = -375$$

$$6V = \begin{vmatrix} 1 & 10 & 0 & 0 \\ 1 & 25 & 25 & 0 \\ 1 & 25 & 0 & 25 \\ 1 & 40 & 0 & 0 \end{vmatrix} = 1\,(-1)^2 \begin{vmatrix} 25 & 25 & 0 \\ 25 & 0 & 25 \\ 40 & 0 & 0 \end{vmatrix}$$

$$+\,10(-1)^3 \begin{vmatrix} 1 & 25 & 0 \\ 1 & 0 & 25 \\ 1 & 0 & 0 \end{vmatrix} = 18750$$

$$[B] = \frac{1}{18750}\begin{bmatrix} -625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 625 & 0 & 0 \\ 0 & -375 & 0 & 0 & 750 & 0 & 0 & 0 & 0 & 0 & -375 & 0 \\ 0 & 0 & -375 & 0 & 0 & 0 & 0 & 0 & 750 & 0 & 0 & -375 \\ -375 & -625 & 0 & 750 & 0 & 0 & 0 & 0 & 0 & -375 & 625 & 0 \\ 0 & -375 & -375 & 0 & 0 & 750 & 0 & 750 & 0 & 0 & -375 & -375 \\ -375 & 0 & -625 & 0 & 0 & 0 & 750 & 0 & 0 & -375 & 0 & 625 \end{bmatrix}$$

(b)



$$x_1 = 4 \qquad x_2 = 10 \qquad x_3 = 10 \qquad x_4 = 12$$

$$y_1 = 2 \qquad y_2 = 7 \qquad y_3 = 2 \qquad y_4 = 2$$

$$z_1 = 0 \qquad z_2 = 0 \qquad z_3 = 5 \qquad z_4 = 0$$

$$v = \frac{1}{6}\left|\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}\right|$$

$$\alpha_1 = \left|\begin{pmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \; \beta_1 = -\left|\begin{pmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \; \gamma_1 = \left|\begin{pmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \; \delta_1 = -\left|\begin{pmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_2 = -\left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \; \beta_2 = \left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \; \gamma_2 = -\left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \; \delta_2 = \left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_3 = \left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{pmatrix}\right| \; \beta_3 = -\left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{pmatrix}\right| \; \gamma_3 = \left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{pmatrix}\right| \; \delta_3 = -\left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{pmatrix}\right|$$

$$\alpha_4 = -\left|\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\right| \; \beta_4 = \left|\begin{pmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{pmatrix}\right| \; \gamma_4 = -\left|\begin{pmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{pmatrix}\right| \; \delta_4 = \left|\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right|$$

$$[B_1] = \frac{1}{6V}\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \delta_1 \\ \gamma_1 & \beta_1 & 0 \\ 0 & \delta_1 & \gamma_1 \\ \delta_1 & 0 & \beta_1 \end{pmatrix} [B_2] = \frac{1}{6V}\begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \delta_2 \\ \gamma_2 & \beta_2 & 0 \\ 0 & \delta_2 & \gamma_2 \\ \delta_2 & 0 & \beta_2 \end{pmatrix} \quad [B_3] = \frac{1}{6V}\begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \gamma_3 & 0 \\ 0 & 0 & \delta_3 \\ \gamma_3 & \beta_3 & 0 \\ 0 & \delta_3 & \gamma_3 \\ \delta_3 & 0 & \beta_3 \end{pmatrix}$$

$$[B_4] = \frac{1}{6v}\begin{pmatrix} \beta_4 & 0 & 0 \\ 0 & \gamma_4 & 0 \\ 0 & 0 & \delta_4 \\ \gamma_4 & \beta_4 & 0 \\ 0 & \delta_4 & \gamma_4 \\ \delta_4 & 0 & \beta_4 \end{pmatrix}$$

$$[B] = ([B_1]\ \ [B_2]\ \ [B_3]\ \ [B_4])$$

$$[B] = \begin{bmatrix} \begin{pmatrix} -0.125 & 0 & 0 \\ 0 & -0.05 & 0 \\ 0 & 0 & -0.05 \\ -0.05 & -0.125 & 0 \\ 0 & -0.05 & -0.05 \\ -0.05 & 0 & -0.125 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0.2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.125 & 0 & 0 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.15 \\ -0.15 & 0.125 & 0 \\ 0 & -0.15 & -0.15 \\ -0.15 & 0 & 0.125 \end{pmatrix} \end{bmatrix}$$

11.7 **(a)** $\{\varepsilon\} = [B]\ \{d\}$

(see $[B]$ from Problem 11.6 above)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = [B] \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.01 \\ 0.02 \\ 0.01 \\ 0.02 \\ 0.01 \\ 0.005 \\ 0.0 \\ 0.01 \\ 0.01 \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.0006 \\ 0.0 \\ 0.000733 \\ 0.0004 \\ 0.00113 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{210 \times 10^9}{(1+0.3)(1-2(0.3))} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.0006 \\ 0 \\ 0.00073 \\ 0.0004 \\ 0.00113 \end{Bmatrix}$$

$$= \begin{Bmatrix} 72.7 \\ 169.6 \\ 72.7 \\ 59.2 \\ 32.3 \\ 91.5 \end{Bmatrix} \text{ MPa}$$

11.7 (b)

$u_1 = 0$ $\quad v_1 = 0$ $\quad w_1 = 0$

$u_2 = 0.00001$ $\quad v_2 = 0.00002$ $\quad w_2 = 0.00001$

$u_3 = 0.00002$ $\quad v_3 = 0.00001$ $\quad w_3 = 0.000005$

$u_4 = 0$ $\quad v_4 = 0.00001$ $\quad w_4 = 0.00001$

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix}$$

$[B] =$

	0	1	2	3	4	5	6	7	8	9
0	–125	0	0	0	0	0	0	0	0	125
1	0	–50	0	0	200	0	0	0	0	0
2	0	0	–50	0	0	0	0	0	200	0
3	–50	–125	0	200	0	0	0	0	0	–150
4	0	–50	–50	0	0	200	0	200	0	0
5	–50	0	–125	0	0	0	200	0	0	–150

$$\text{disp} = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{pmatrix} \qquad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = [D]\ [B]\ \{\text{disp}\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{Bmatrix} 1.963\times10^3 \\ 5.236\times10^3 \\ 1.309\times10^3 \\ 2.127\times10^3 \\ 654.48 \\ 3.436\times10^3 \end{Bmatrix}$$

11.8

$$u = a_1 + a_2 x + a_3 y + a_4 z$$
$$+ a_5 xy + a_6 xz + a_7 yz$$
$$+ a_8 x^2 + a_9 y^2 + a_{10} z^2$$

Similar expressions for v and w.

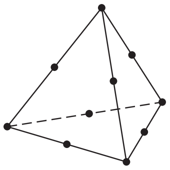
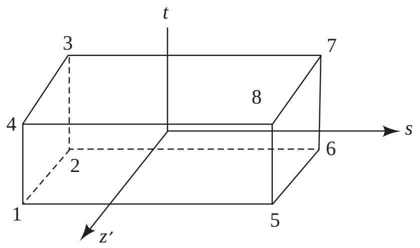


Figure P11-8

11.9 Loads must be in the y-z plane (in the plane of the plane elements).

11.10



Using Equation (11.3.3) and Figure 11.5

$$N_i = \frac{(1+ss_i)(1+tt_i)(1+z'z_i')}{8}$$

$$N_1 = \frac{(1-s)(1-t)(1+z')}{8}, \ (s_1 = -1,\ t_1 = -1,\ z_1 = 1)$$

$$N_2 = \frac{(1-s)(1-t)(1-z')}{8}, \ (s_2 = -1,\ t_2 = -1,\ z'_2 = -1)$$

$$N_3 = \frac{(1-s)(1+t)(1-z')}{8}, \ (s_3 = -1,\ t_3 = 1,\ z_3 = -1)$$

$$N_4 = \frac{(1-s)(1+t)(1+z')}{8}, \ (s_4 = -1,\ t_4 = 1,\ z'_4 = 1)$$

$$N_5 = \frac{(1+s)(1-t)(1+z')}{8}, \ (s_5 = 1,\ t_5 = -1,\ z'_5 = 1)$$

$$N_6 = \frac{(1+s)(1+t)(1-z')}{8}, \ (s_6 = 1,\ t_6 = -1,\ z'_6 = -1)$$

$$N_7 = \frac{(1+s)(1+t)(1-z')}{8}, \ (s_7 = 1,\ t_7 = 1,\ z'_7 = -1)$$

$$N_8 = \frac{(1+s)(1+t)(1+z')}{8}, (s_8 = 1, t_8 = 1, z'_8 = 1)$$

11.11 Quadratic hexahedral element (see Figure 11.6)

By Equation (11.3.11)

$$N_i = \frac{(1+ss_i)(1+tt_i)(1+z'z'_i)}{8} \ (ss_i + tt_i + z'z'_i - 2)$$

Node 1

$$s_1 = -1, t_1 = -1, z'_1 = 1$$

$$N_1 = \frac{(1-s)(1-t)(1+z')}{8} \ (-s - t + z' - 2)$$

Node 2

$$s_2 = -1, t_2 = -1, z'_2 = -1$$

$$N_2 = \frac{(1-s)(1-t)(1-z')}{8} \ (-s - t - z' - 2)$$

Node 3

$$s_3 = -1, t_3 = 1, z'_3 = -1$$

$$N_3 = \frac{(1-s)(1+t)(1-z')}{8} \ (-s + t - z' - 2)$$

Node 4

$$s_4 = -1, t_4 = 1, z'_4 = 1$$

$$N_4 = \frac{(1-s)(1+t)(1+z')}{8} \ (-s + t + z' - 2)$$

Node 5

$$s_5 = 1, t_5 = -1, z'_5 = 1$$

$$N_5 = \frac{(1+s)(1-t)(1+z')}{8} \ (s - t + z' - 2)$$

Node 6

$$s_6 = 1, t_6 = -1, z'_6 = -1$$

$$N_6 = \frac{(1+s)(1-t)(1-z')}{8} \ (s - t - z' - 2)$$

Node 7

$$s_7 = 1, t_7 = 1, z'_7 = -1$$

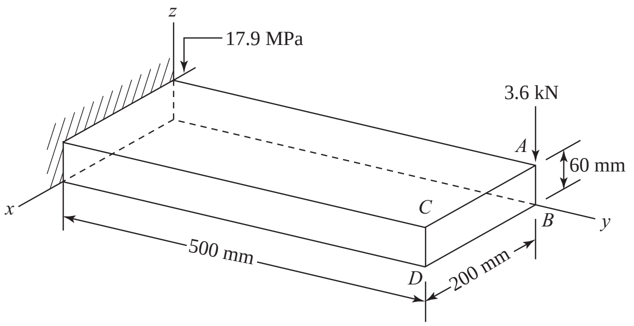
$$N_7 = \frac{(1+s)(1+t)(1-z')}{8} \ (s + t - z' - 2)$$

Node 8

$s_8 = 1,\ t_8 = 1,\ z'_8 = 1$

$$N_8 = \frac{(1+s)(1+t)(1+z')}{8}\ (s + t + z' - 2)$$

11.13



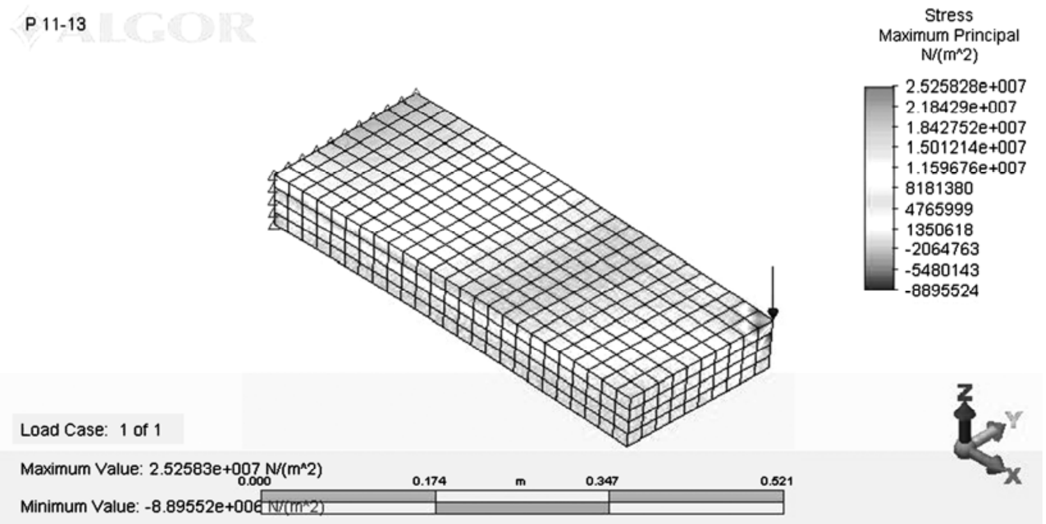
Outer free face corner node deflections (z-direction)

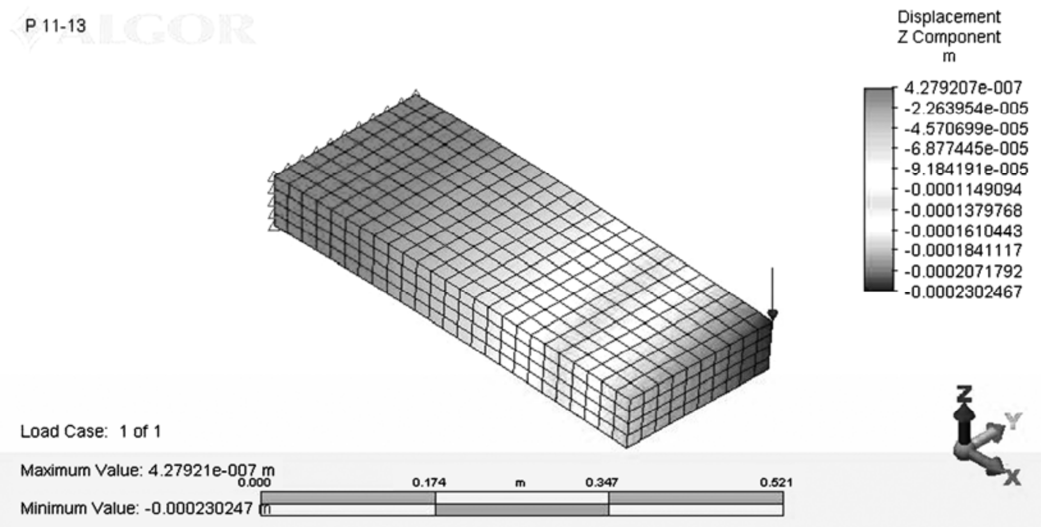
Pt A = – 0.000231 m Pt B = – 0.000224 m Pt C = – 0.000187 m Pt D = – 0.000187 m

Mechanics of materials, $\delta = \dfrac{PL^3}{3EI}$ = – 0.000208 m, where $I = \dfrac{(0.2)\,(0.06)^3}{12} = 3.6 \times 10^{-6}\ \text{m}^4$

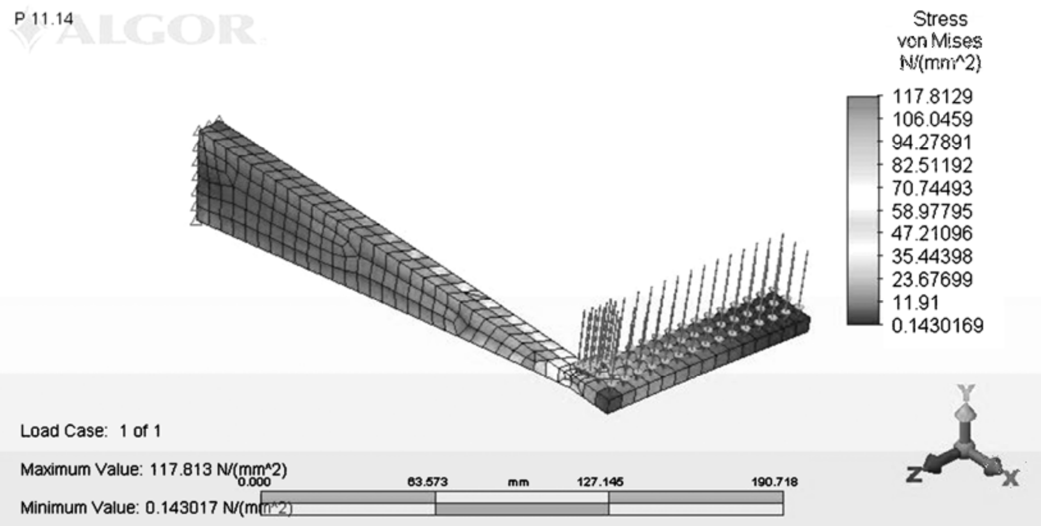
$L = 0.5$ m, $P = 3600$ N, $E = 200 \times 10^9\ \dfrac{\text{N}}{\text{m}^2}$

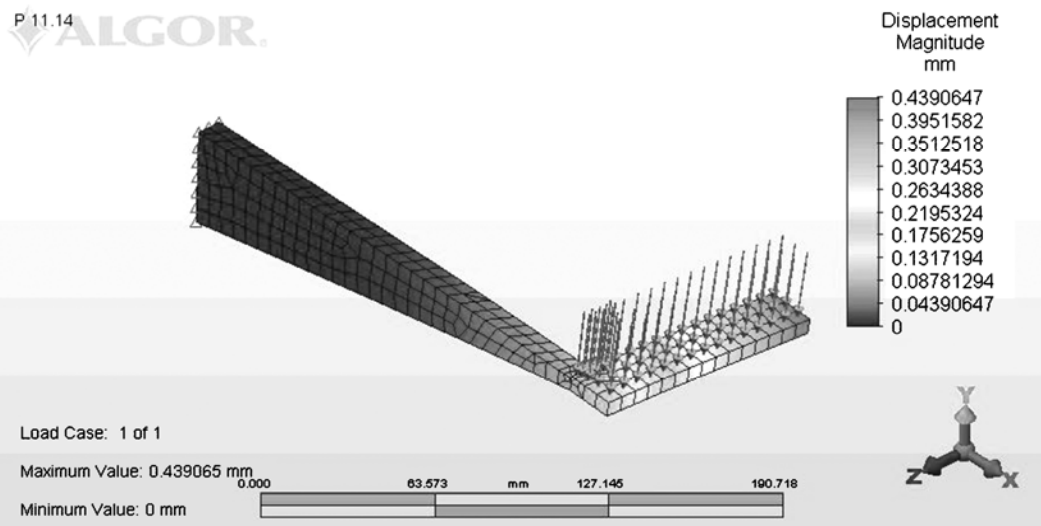
The corner node answers from computer program Algor. Note the classical mechanics of materials solution gives the maximum deflection for a load applied through the centroid not offset.





11.14 (Load replaced with concentrated end load)





11.15

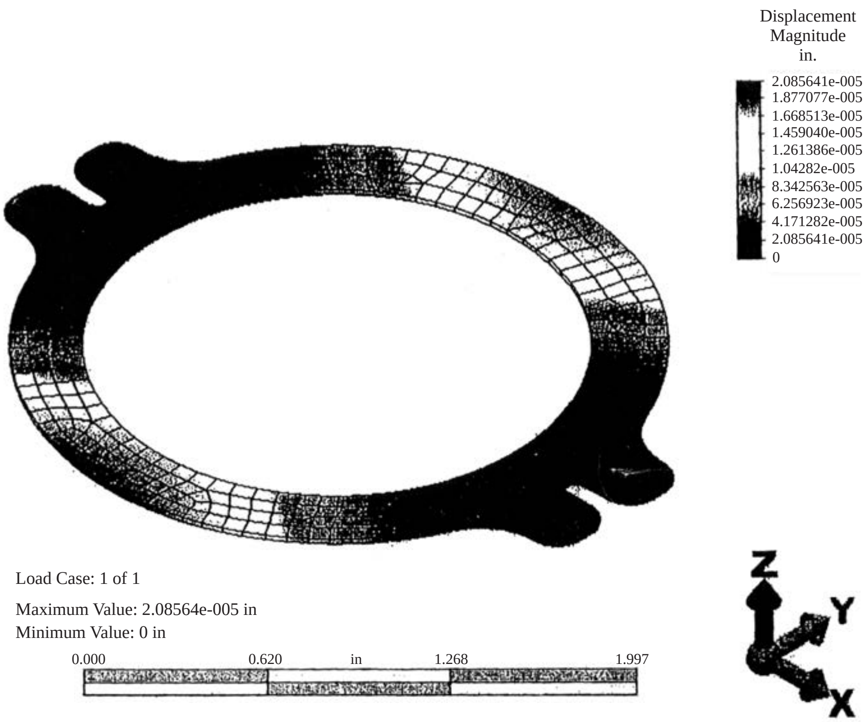


Figure 5 Flap valve with maximum displacement of 2.09×10^{-5} in.

Conclusion and recommendations

The applied pressure of 2.318 psi on the annular region of the flap valve with the clip ears supported on both sides, resulted in a maximum von Mises stress of 31,001 psi. The safety factor with the applied pressure is 2.0. It is recommended to use a pressure no greater than 2,318 psi during the operation of the compressor with this flap valve.

11.16

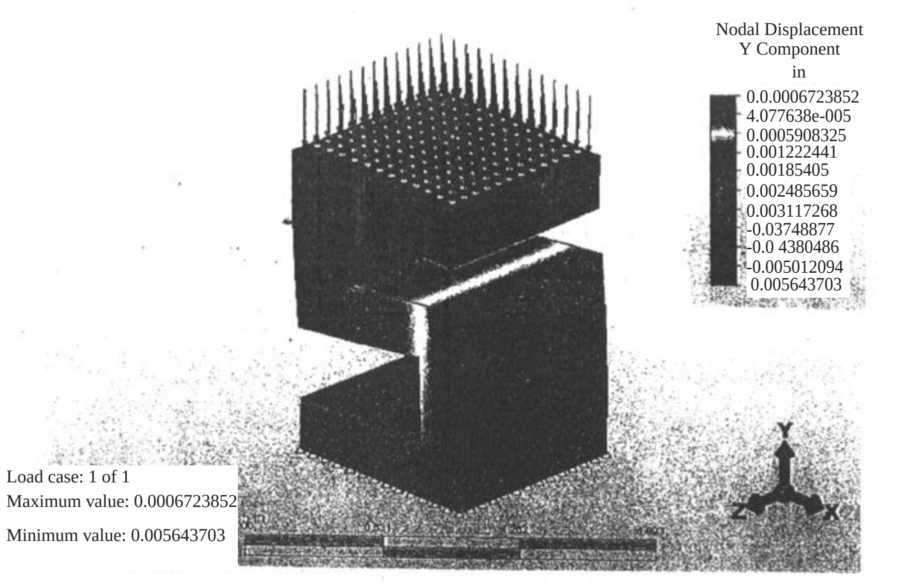


Figure 1 Displacement of our designed s-block

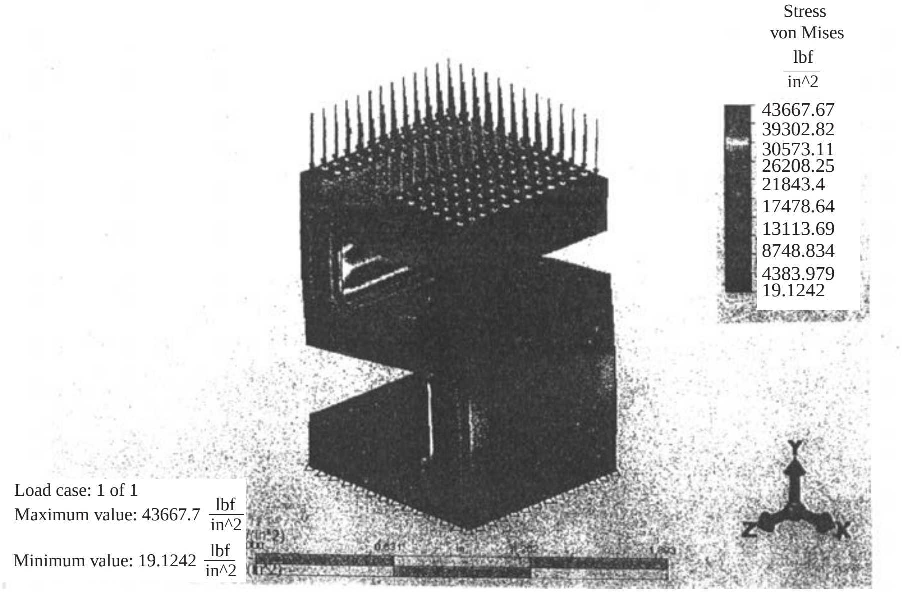


Figure 2 von Mises stress on our designed s-block
(Final thickness = 0.25 in.)

11.17

Determine the thickness of the device such that the maximum deflection is 0.1 in. vertically.

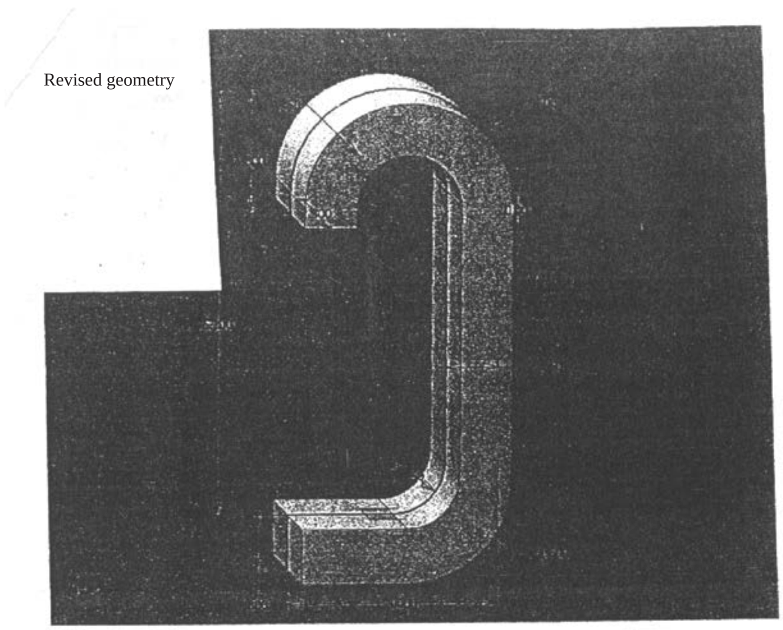


Figure 1 Inventor Model Dimensions
(Final thickness = 0.75 in.)

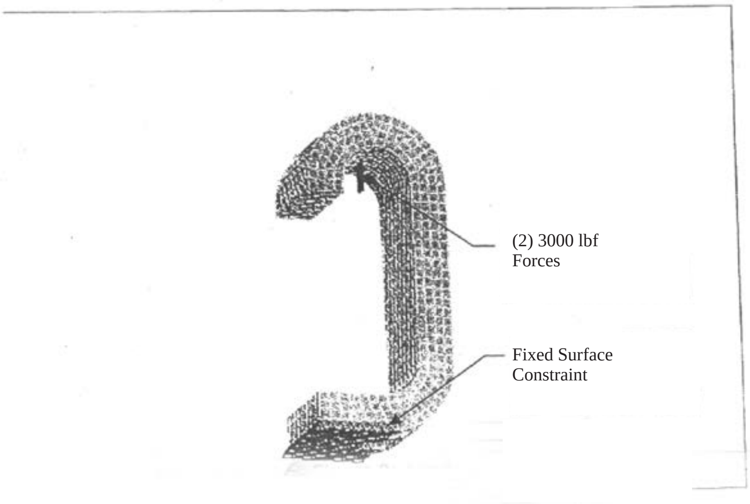


Figure 2 Loads and Constraints

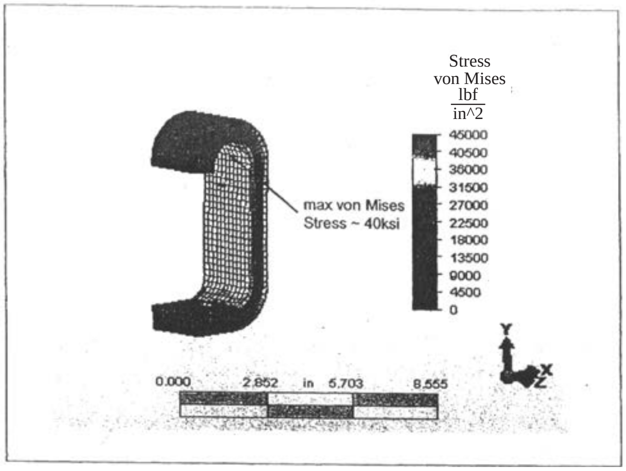


Figure 3 von Mises Stress Distribution (0.75 in. thick)

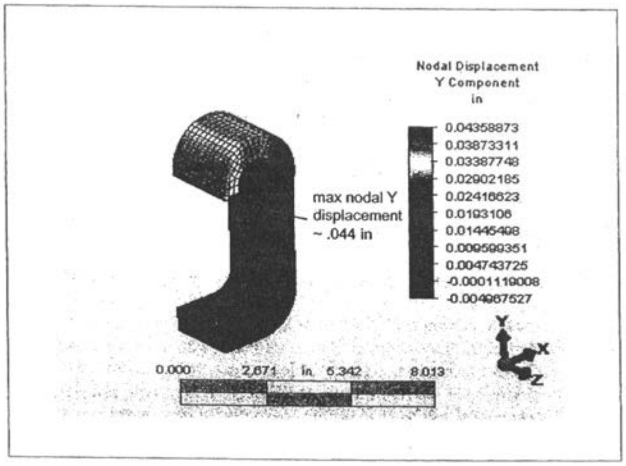


Figure 4 Vertical Displacement Distributions
Maximum displacement = 0.0436 in.

11.18

Model Variables

Variable	Value
Material	1035 quenched & tempered steel
Modulus of Elasticity	200 MPa
Force	150 N
Yield Strength	615 MPa
Maximum von Mises Stress	758 MPa
Maximum Displacement	4.13 mm

Algor Results

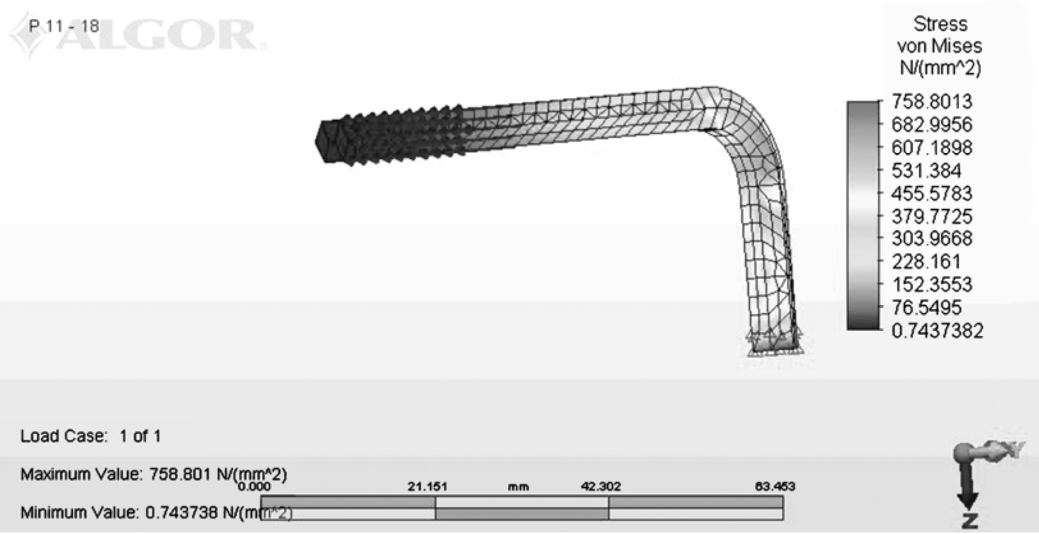


Figure 4 von Mises Stress (MPa)

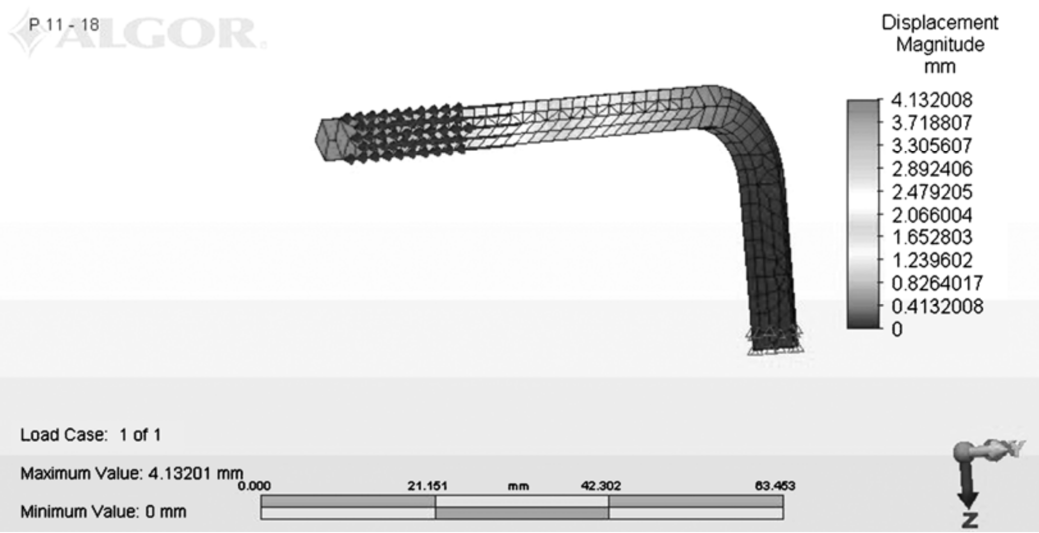
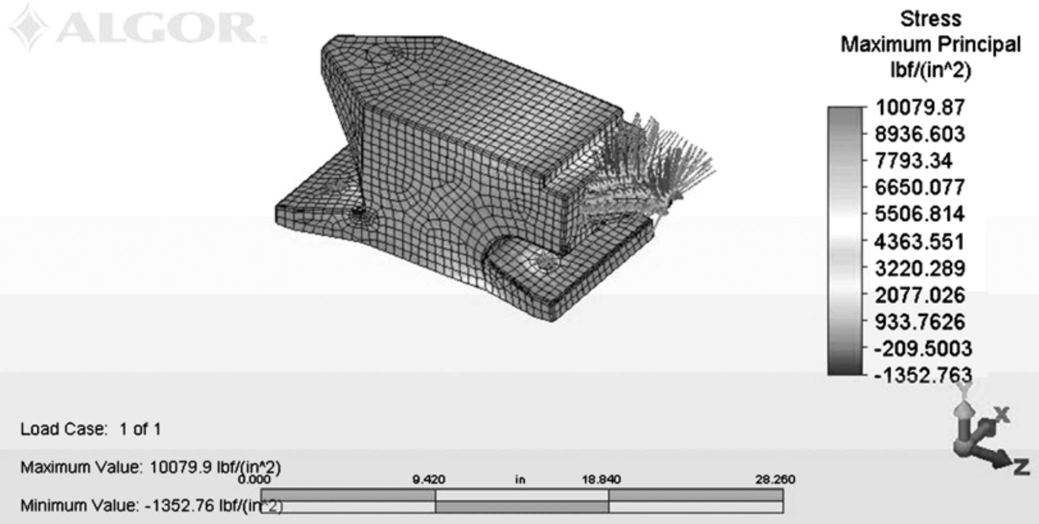


Figure 5 Displacement (mm)

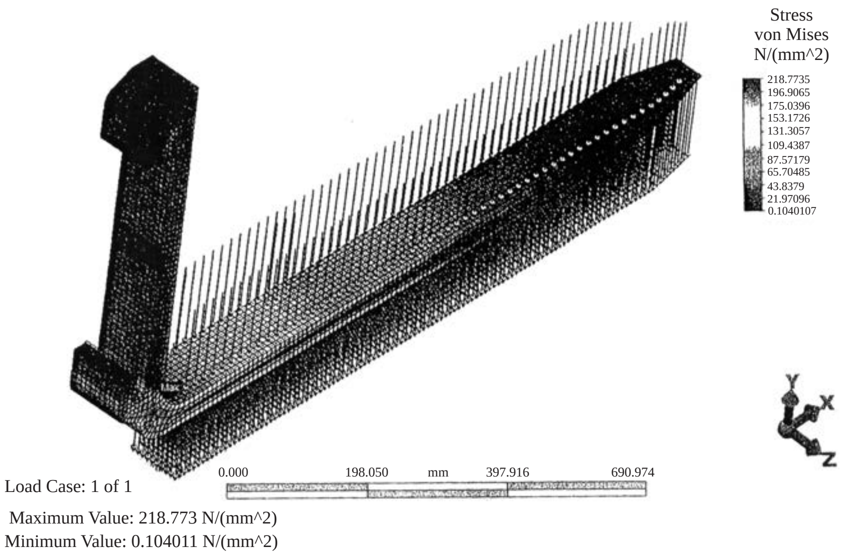
11.19



Maximum principal stress is 10080 psi.

11.20

von Mises Stress



The yield strength of AISI 4130 is approximately 360.69 MPa (from eFunda). Converting the maximum von Mises stress of 218.8 $\frac{\text{N}}{\text{mm}^2}$ into MPa gives approximately 218.8 MPa. This is over 100 MPa below the yield strength, so this will not fail due to static loading.

11.21

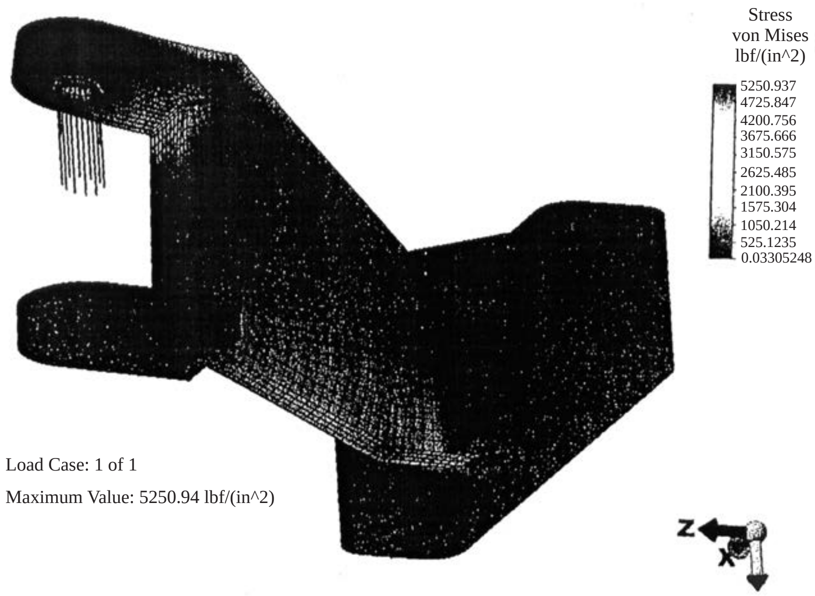


Figure 1 von Mises Stress of part.

With yield strength of 6,000 psi and a max von Mises stress of 5250 psi, it is getting close to failing due to the total weight of the entire car placed on one wheel. Under normal operation conditions the actual weight placed on a front wheel is less than one quarter of the entire car weight.

11.22

With 6,282 elements Algor calculates higher stresses. Figure 7 shows the von Mises stress for analysis with 6282 elements.

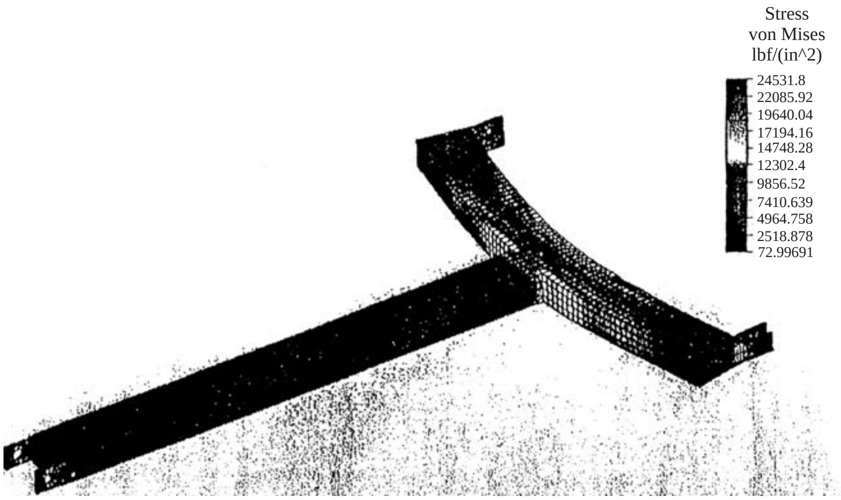


Figure 7

The yield strength of the material is 53,700 psi and the maximum von Mises stress is 24,530 psi so the hitch has a factor of safety of 2.18 when analyzed with 6282 elements.

11.23

Solution: C bracket

Maximum von Mises stress = 1681.273 $\dfrac{\text{lb}}{\text{in.}^2}$

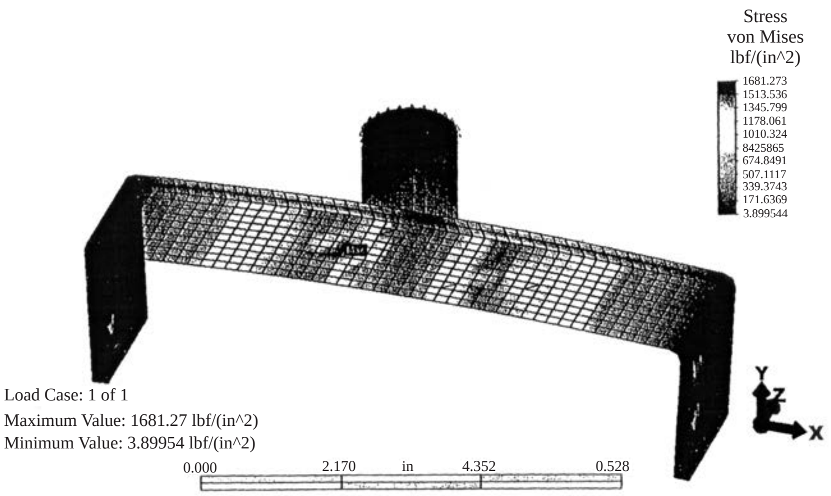


Figure 2 von Mises Stress

Maximum deflection = 0.0041 in.

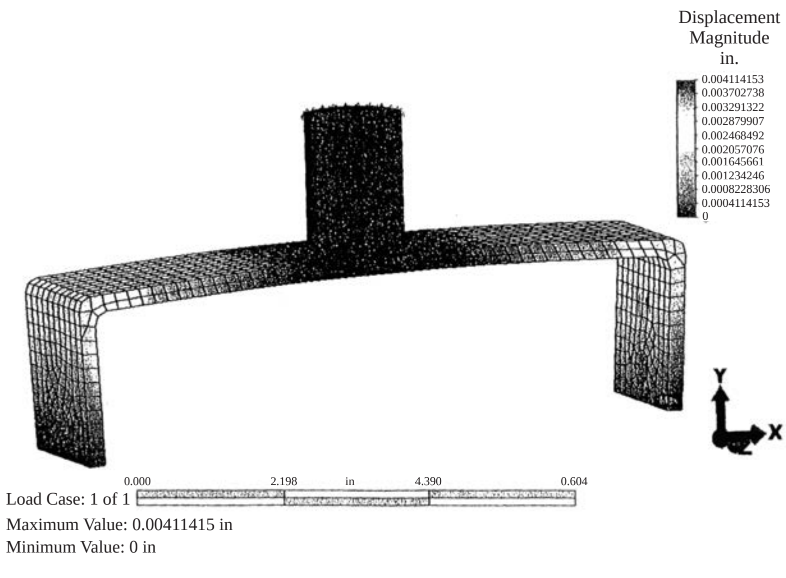


Figure 3 Maximum Deflection

11.24

The von Mises plot of the loader under loading is shown below in Figure 2. I am looking for the load that caused the loader to break. For this to happen, there is no factor of safety, However, based on the von Mises plot, you can see that much of the loader has a near infinite safety factor.

Figure 2 von Mises Plot

I found that a load of 2200 pounds applied on three sides of the right lower arm (left in the above figure) in a counter clockwise direction caused the loader to fail. This loader put the loader under 44251 psi, which is just over the yield strength. It appears it failed roughly six to eight inches above the end of the arm. This is very near where the loader broke this spring.

11.25

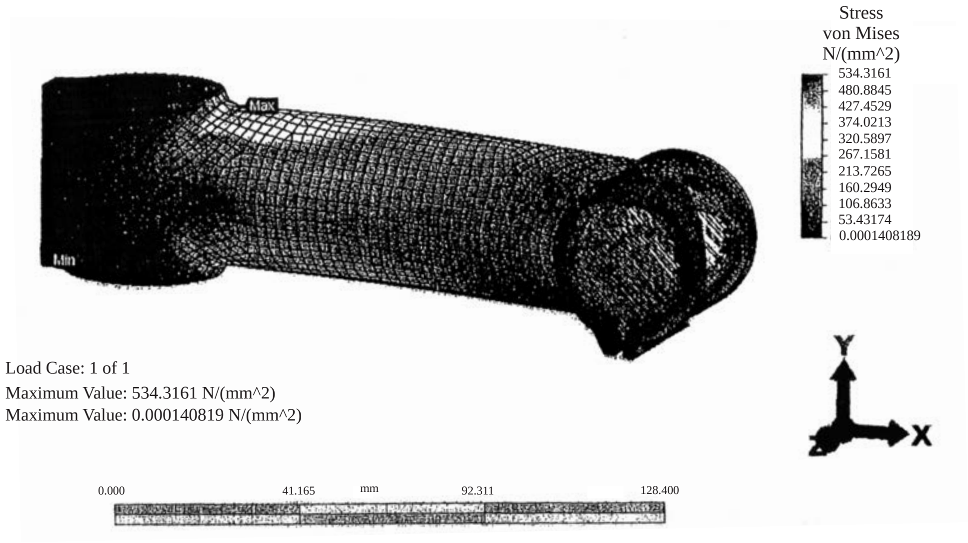
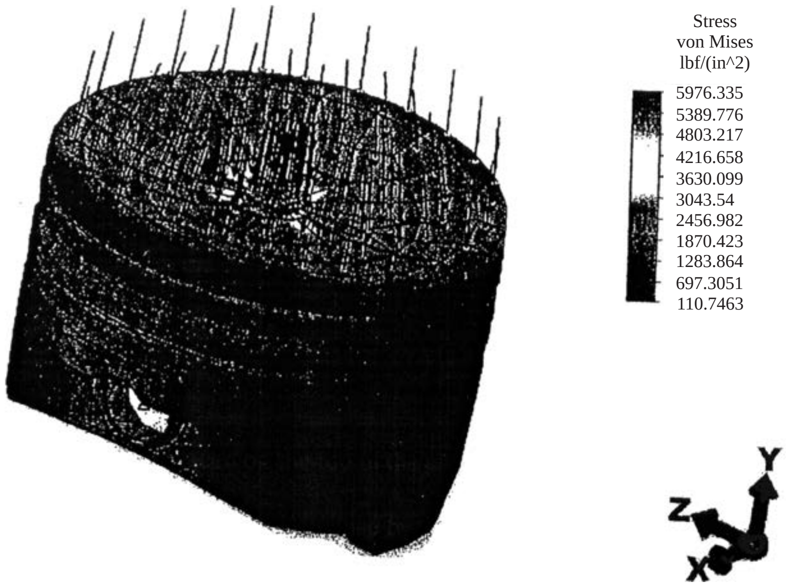


Figure 3 von Mises plot due to 250 lb rider (maximum stress is 534.3 MPa)

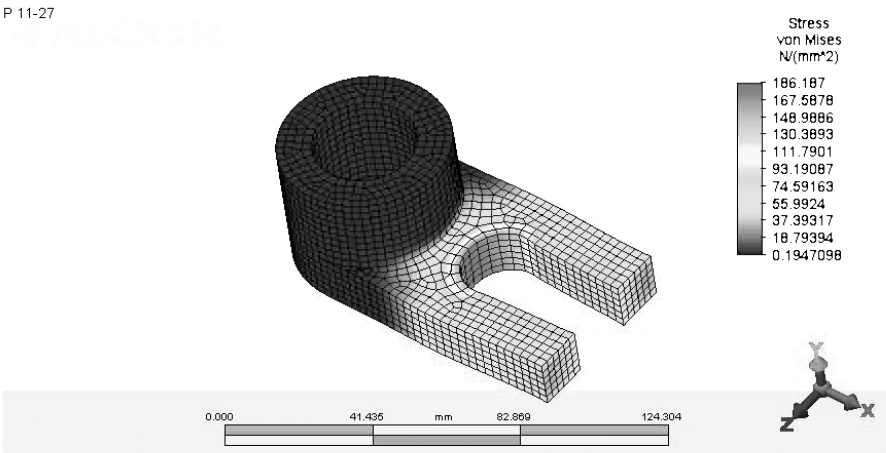
11.26

The load of 185 psi placed on the top of the piston head with the wrist-pin hole fixed on the top side to represent resistance on the piston head by the connecting rod. This is ample constraint because in a real internal combustion engine, the pistons are not frozen, but allowed to travel up and down with minimal friction resistance.



The maximum von Mises stress of 5976 psi is less than the yield strength of the material (σ_y = 44,200 psi)

11.27



(Largest von Mises stress is 186 psi) located inside surface of hole.

Chapter 12

Solve these problems using the plate element from a computer program.

12.1 A square steel plate of dimensions 20 in. by 20 in. with thickness of 0.1 is clamped all around. The plate is subjected to a uniformly distributed loading of $1\ \frac{\text{lb}}{\text{in.}^2}$. Using a 2 by 2 mesh and then a 4 by 4 mesh, determine the maximum deflection and maximum stress in the plate. Compare the finite element solution to the classical one in [1].

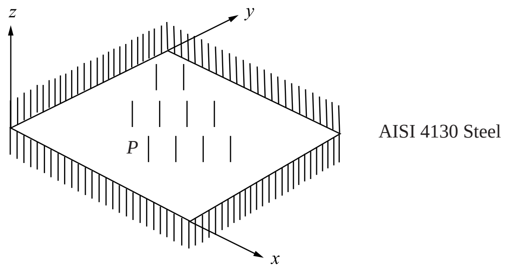
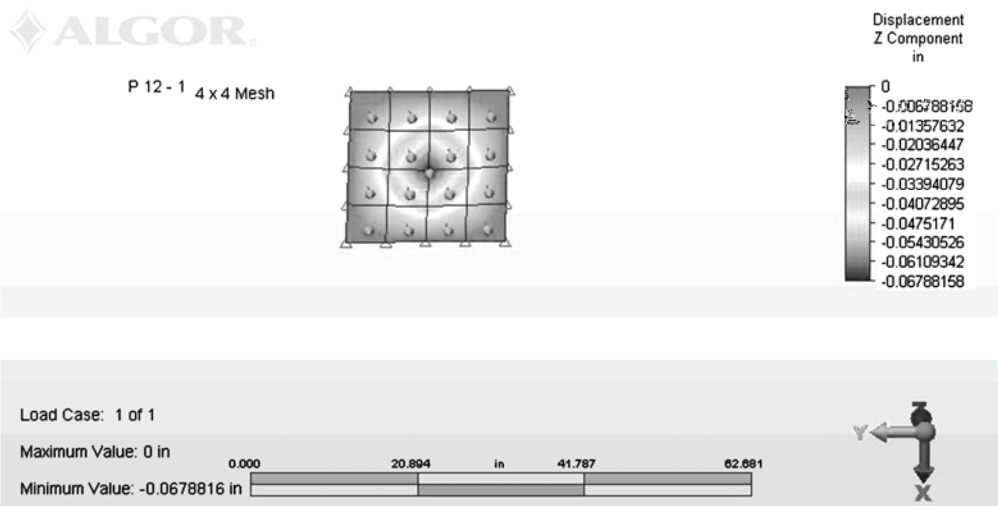


Figure P12–1 AISI 4130 Steel



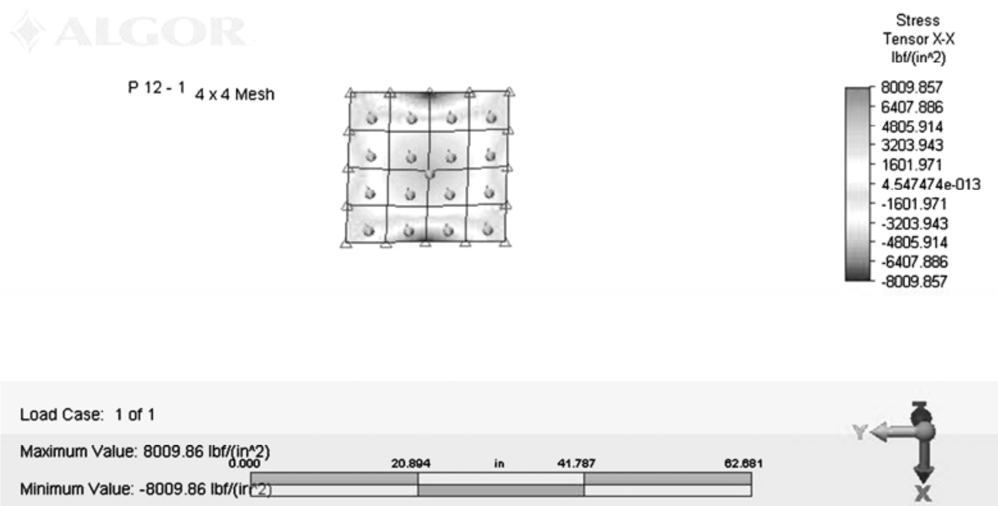


Plate bending

$20'' \times 20'' \times 0.1''$ 4130 steel plate

Clamped all the way around with a 1 psi load.

Mesh size	Minimum stress	Maximum stress	Maximum deflection
2×2	3440.01 psi	3573.76 psi	0.0785 in.
4×4	668.06 psi	7046.06 psi	0.06788 in.

Analytical solution: (A.C. ugural, plates and shells McGraw Hill)

$$W_{\max} = \frac{Pa^4}{D}\,(0.00126) = \frac{(1\text{ psi})\,(20'')^4}{2.7470 \times 10^3}\,(0.00126) = 0.07339\text{ in.}$$

where

$$D = \frac{Et^3}{12\,(1 - v^2)} = \frac{(30 \times 10^6)\,(0.1)^3}{12\,(1 - 3^2)} = \frac{0.03 \times 10^6}{(0.91)\,12} = 2.747 \times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$\sigma_{\max} = \frac{6\,M_{\max}}{t^2} = \frac{\sigma\,(0.0513\,p_0\,a^2)}{t^2}$$

$$= 12312\text{ psi}$$

12.2 An *L*-shaped plate with thickness 0.1 in. is made of ASTM A-36 steel. Determine the deflection under the load and the maximum principal stress and its location using the plate element. Then model the plate as a grid with two beam elements with each beam having the stiffness of each *L*-portion of the plate and compare your answer.

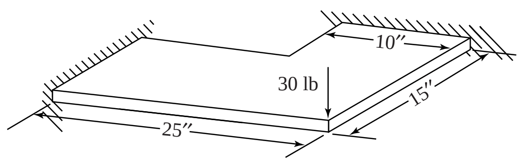


Figure P12–2

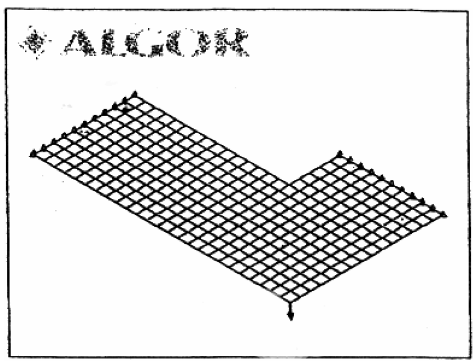


Figure 1: Plate Bending Mesh with Boundary Conditions and Nodal Force

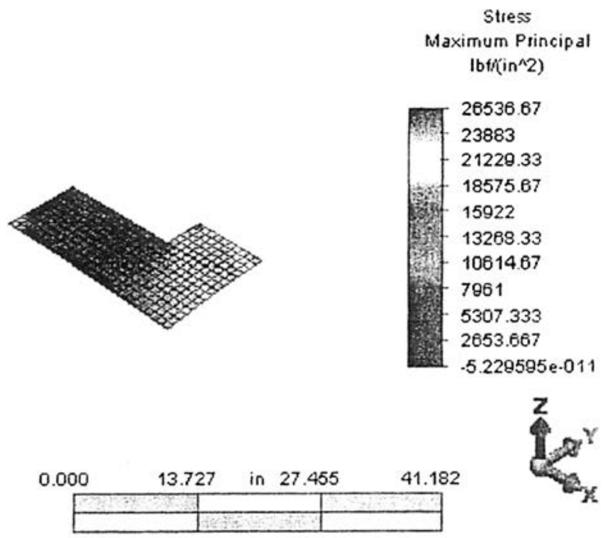


Figure 2: Plate Bending Maximum Principal stresses (psi)

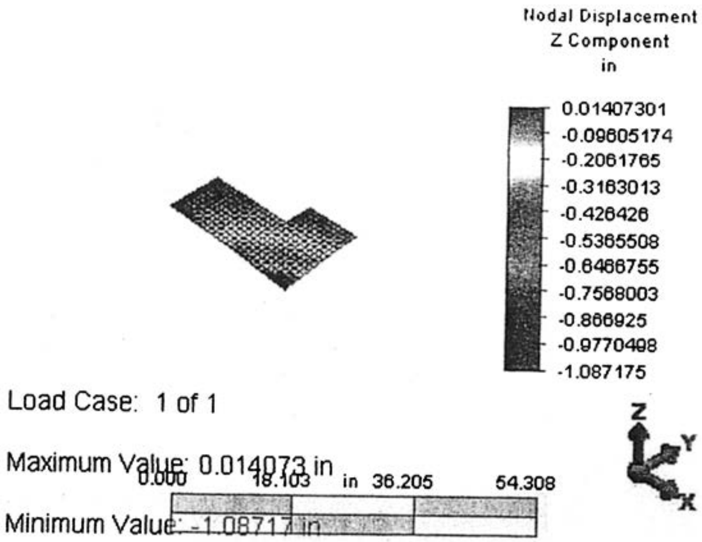


Figure 3 Nodal Z Displacement (in.)

12.3 A square simply supported 20 in. by 20 in. steel plate with thickness 0.15 in. has a round hole of 4 in. diameter drilled through its center. The plate is uniformly loaded with a load of 2 $\frac{\text{lbf}}{\text{in.}^2}$. Determine the maximum principal stress in the plate.

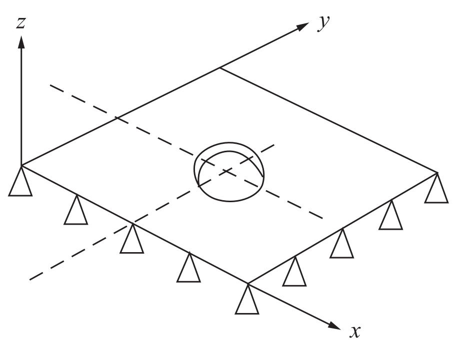
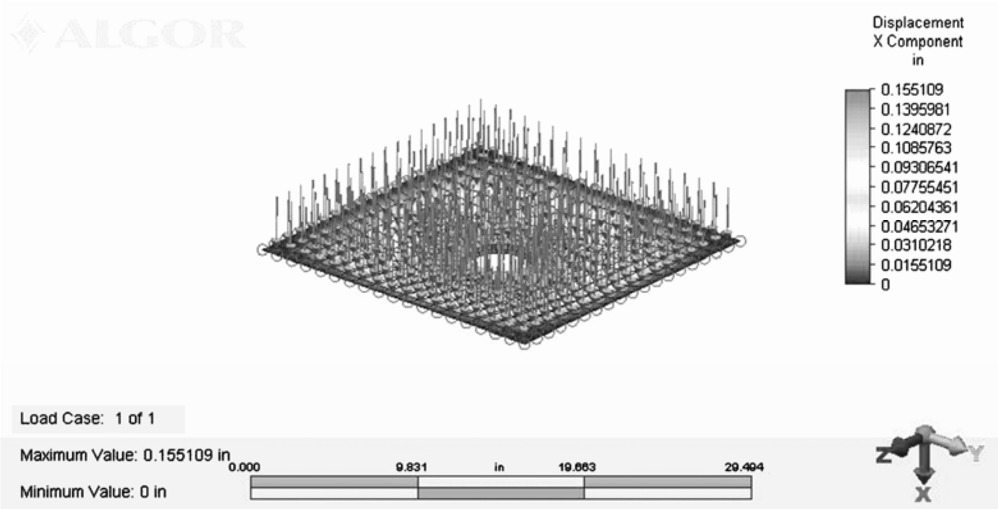
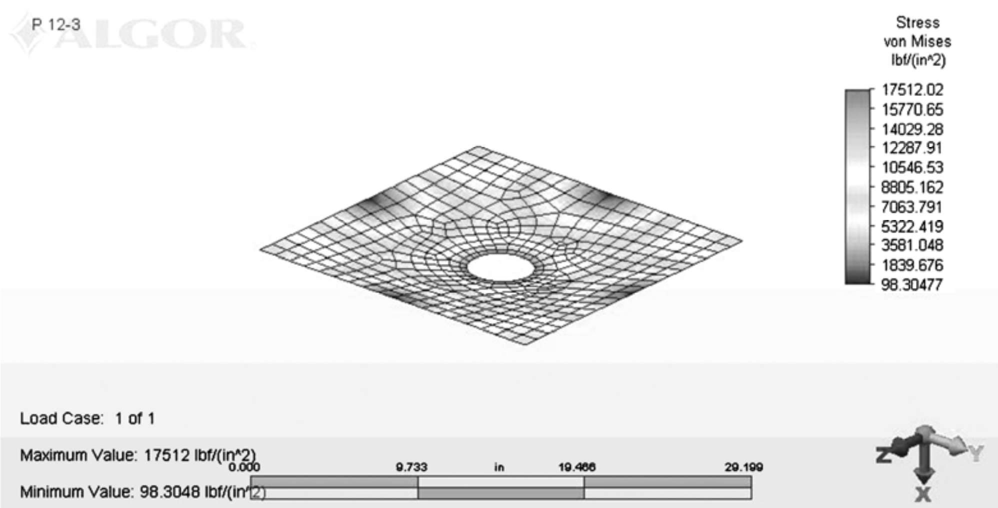


Figure P12–3





From the Algor analysis, the maximum principal stress was determined to be 10152.22 psi.

The precision of the von Mises stress was very close to 0.1, therefore, the results were deemed feasible.

	Stress (psi)		Displacement (in.)			
	von Mises	Maximum Principal	X	Y	Z	Magnitude
Maximum Value	9041.787	10152.22	–0.0468	0	0	–0.0468

12.4 A *C*-channel section structural steel beam of 2 in. wide flanges, 3 in. depth and thickness of both flanges and web of 0.25 in. is loaded as shown with 100 lb acting in the y direction on the free end. Determine the free end deflection and angle of twist. Now move the load in the z direction until the rotation (angle of twist) becomes zero. This distance is called the shear center (the location where the force can be placed so that the cross section will bend but not twist).

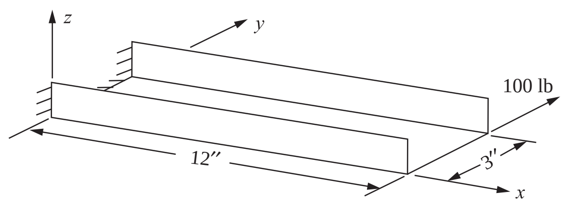
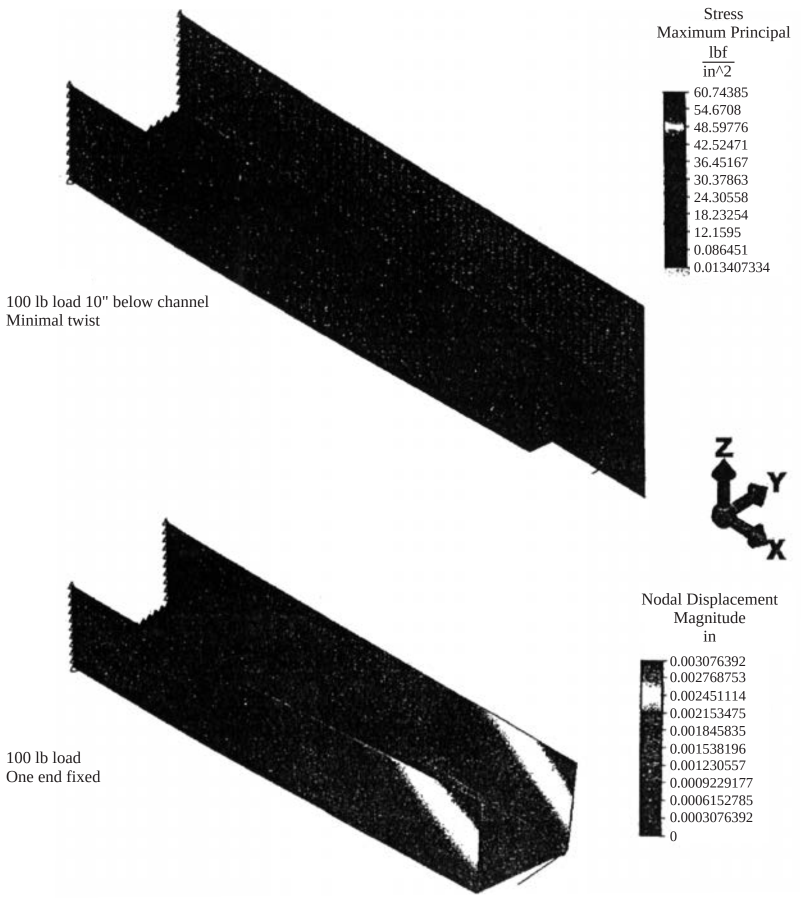


Figure P12–4



Need to apply load at shear center. See Table 5-1, Chapter 5, (P-263).

12.5 For the simply supported structural steel $W\ 14 \times 61$ wide flange beam shown, compare the plate element model results with the classical beam bending results for deflection and bending stress. The beam is subjected to a central vertical load of 22 kip. The cross-sectional area is 17.9 in.^{2}, depth is 13.89 in., flange width is 9.995 in., flange thickness of 0.645 in., web thickness of 0.375 in., and moment of inertia about the strong axis of 640 in.^{4}

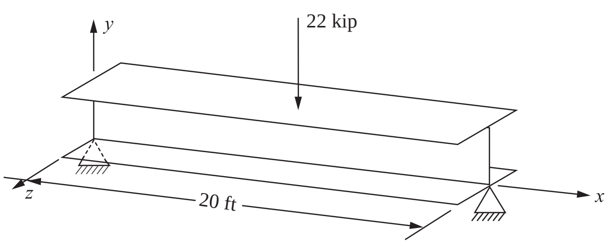
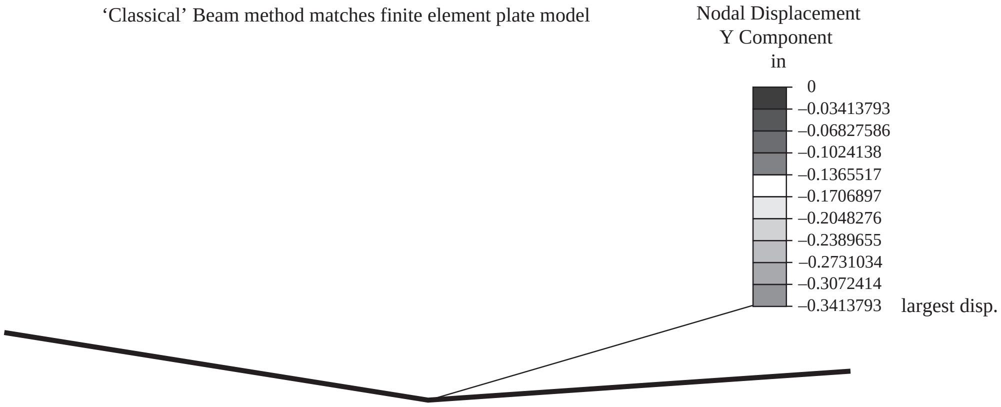


Figure P12–5

$$\delta = \frac{PL^3}{48EI} = \frac{(22\,K)(240'')^3}{48.29 \times 10^3\,\text{ksi} \times 640\,\text{in.}^4}$$

$$= -\,0.34\ \text{in.}$$



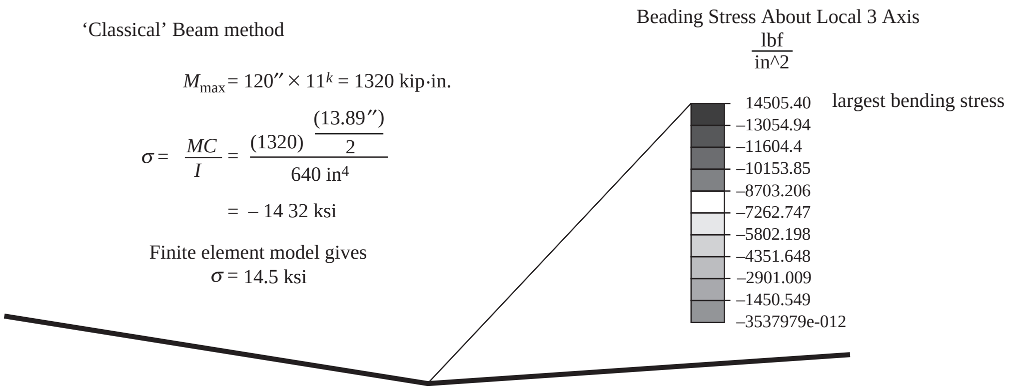
'Classical' Beam method

$$M_{\max} = 120'' \times 11^k = 1320\ \text{kip·in.}$$

$$\sigma = \frac{MC}{I} = \frac{(1320)\,\dfrac{(13.89'')}{2}}{640\ \text{in}^4}$$

$$= -\,14\ 32\ \text{ksi}$$

Finite element model gives
$\sigma = 14.5$ ksi



12.6 For the structural steel plate structure shown, determine the maximum principal stress and its location. If the stresses are unacceptably high, recommend any design changes. The initial thickness of each plate is 0.25 in. The left and right edges are simply supported. The load is a uniformly applied pressure of 10 $\frac{\text{lb}}{\text{in.}^2}$ over the top plate.

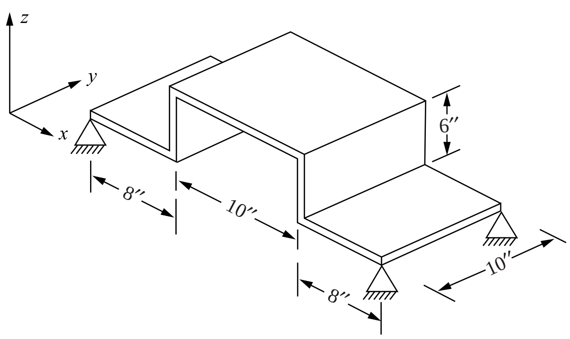
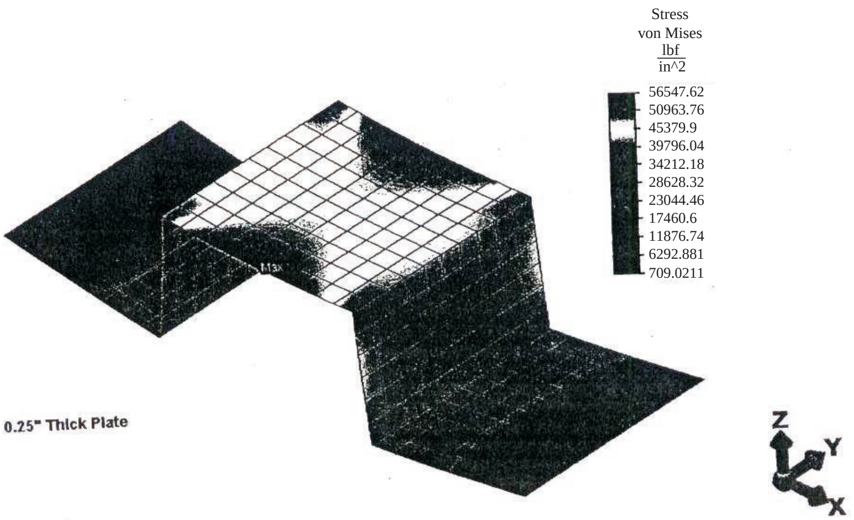


Figure: P12–6



This problem was a plate element analysis problem with plates that I assumed were made from ASTM A-36 steel. We were asked to determine the maximum principal stress and its location for the structure. We were asked to start with a plate thickness of 0.25 in., and if the stresses were found to be unacceptably high, we were to recommend design changes. A comparison of the results for a range of thickness is as follows.

	Maximum von Mises Stress $\left(\dfrac{\text{lb}}{\text{in}^{\wedge}2}\right)$
0.25 in. thick plate	56,547.62
0.375 in. thick plate	25,194.28
0.50 in. thick plate	14,205.43

ASTM A-36 steel has the following strength properties.

- S_t = 58 to 80 ksi
- S_y = 36 ksi minimum

Therefore, the stress levels in the structure would lead to failure with the 0.25 in. thick plate using the MDET (maximum distortional energy theory). The 0.375 in. thick plate could be used allowing a factor of safety equal to 1.43 using the MDET. This is a small safety factor, and I would recommend using the 0.50 in. thick plate to construct the structure, which allows a more comfortable safety factor of 2.53 using the MDET.

12.7 Design a steel box structure 4 ft wide by 8 ft long made of plates to be used to protect construction workers while working in a trench. That is, determine a recommended thickness of each plate. The depth of the structure must be 8 ft. Assume the loading is from a side load acting along the long sides due to a wet soil (density of 62.4 $\frac{\text{lb}}{\text{ft}^3}$) and varies linearly with the depth. The allowable deflection of the plate type structure is 1 in. and the allowable stress is 20 ksi.

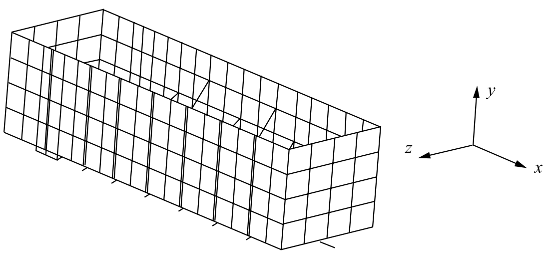


Figure P12–7

Solution method

I first attempted to solve the problem by examining the long side and short side of the box individually, rather than as a welded assembly. It was felt that once believable results were obtained from modeling each side as an individual plate, the model could be expanded to include the entire box.

However, Algor appears to have a glitch so that its 'surface hydrostatic pressure' feature does not work. Therefore, I was unable to easily apply a linearly varying pressure along the surface of a plate.

To get around this problem, I tried to break the side plate of the box into 4 vertically stacked sections (different Algor parts) and then applying a different pressure to the surface of each section so that it approximated the linear pressure distribution of the soil. However, Algor failed to recognize the applied surface pressure on 3 of the 4 sections. So this model was deficient also.

To solve the problem, I was forced to apply loads directly to the nodes. To facilitate the selection of the nodes and to make the nodal forces regular and repeatable, the plate was manually meshed into a convenient rectangular pattern. Four different nodal loads were used to approximate the linear pressure distribution.

Results

For the short side of the box (48″ wide × 96″ deep), it was found that a $\frac{3''}{8}$ thick plate was as thin as could be used and still meet the given design criteria. Using simply supported boundary conditions, the maximum von Mises stress was 19.9 ksi and the maximum deflection was 0.76″. The maximum stress was located at the bottom corners, which is typically an area of high stress for rectangular plates.

These results were than compared to equations found in Roark's 'Formulas for Stress and Strain'. From Roark, the maximum stress was 18.2 ksi and the maximum deflection was 0.68 in.

The same analysis was performed for the long side of the box (96″ square). It was found that a 0.75″ plate was right at the margins of acceptability for deflection. The Algor and hand calculations for both sides of the box are summarized below.

	Algor model		Hand Calc.	
	von Mises (ksi)	Deflection (in.)	von Mises (ksi)	Deflection (m)
Long side of box ($\frac{3}{4}''$ thk × 96″ sq)	14.5	1.06	11.36	0.93
Short side of box ($\frac{3}{8}''$ thk × 48″ × 96″ dp)	19.9	0.76	18.18	0.68

Comments

For both sides, the Algor model predicted higher stress and higher deflection than the hand calculations. I suspect the difference in results can be attributed to two sources. One, the stepwise variation of pressure that I was forced to use in the Algor model to approximate the hydrostatic pressure. The other is due to the coarseness of the mass.

Nodal loads to approximate hyprostatic pressure variation as function of depth

$$q_0 = \gamma\ a = \text{Maximum pressure}$$

$$= 62.4\ \frac{\text{lb}}{\text{ft}^3}\ 8\,\text{ft}\left(\frac{1\,\text{ft}}{12\ \text{in.}}\right)^2$$

$$= 3.467 \text{ psi soil pressure at } 8' \text{ depth}$$

Assume linear pressure profile

$$x = \text{depth}$$

$$q(x) = q_0 - q_0\ \frac{x}{a}$$

$$= q_0 \left(1 - \frac{x}{a}\right)$$

For Step 1

$$q\left(x = \frac{1}{8}a\right) = q_0 \frac{7}{8} = 3.04 \text{ psi}$$

For Step 2

$$q\left(x = \frac{3}{8}a\right) = \frac{5}{8} q_0 = 2.17 \text{ psi}$$

For Step 3

$$q\left(x = \frac{5}{8}a\right) = \frac{3}{8} q_0 = 1.30 \text{ psi}$$

For Step 4

$$q\left(x = \frac{7}{8}a\right) = \frac{1}{8} q_0 = 0.43 \text{ psi}$$

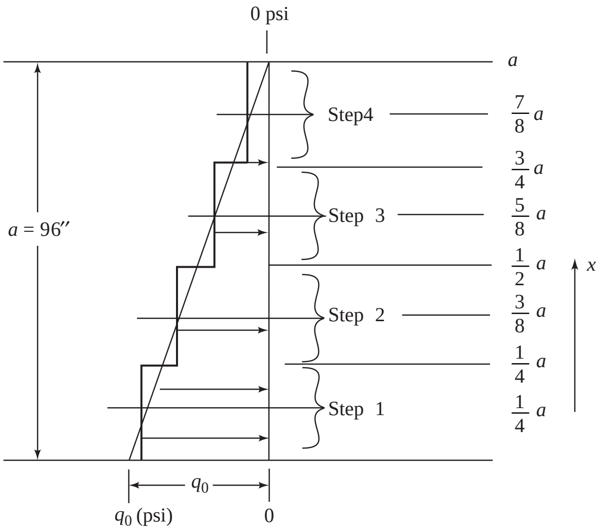
For a $g'' \times g''$ mesh, the area supported by a interior node is, $A_I = g^2$

For a $g'' \times g''$ mesh, the area supported by a perimeter node is $A_P = \dfrac{g^2}{2}$

Let $g = 4'' \Rightarrow$ nodal force for step 1 interior node $= (3.04)\,(4)^2 = \dfrac{48.64 \text{ lb}}{\text{node}}$

Let $g = 4'' \Rightarrow$ nodal force for step 1 perimeter node $= (3.04)\,\dfrac{4^2}{2} = 24.32\ \dfrac{\text{lb}}{\text{node}}$

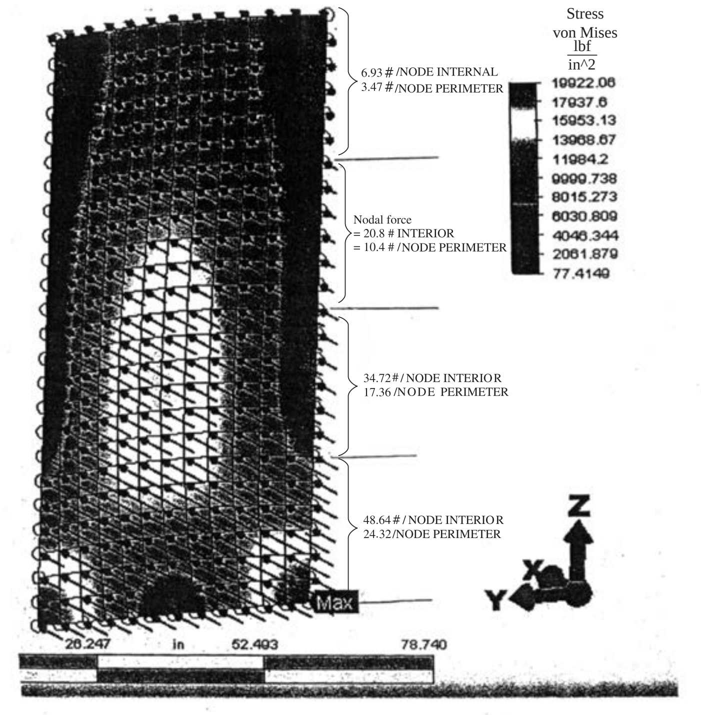
etc. …



Manual mesh
Stepwise approximation of soil pressure
Simple support boundary conditions
Short side

$48''$ wide $\times$ $96''$ Deep $\times$ $\dfrac{7''}{8}$ thick

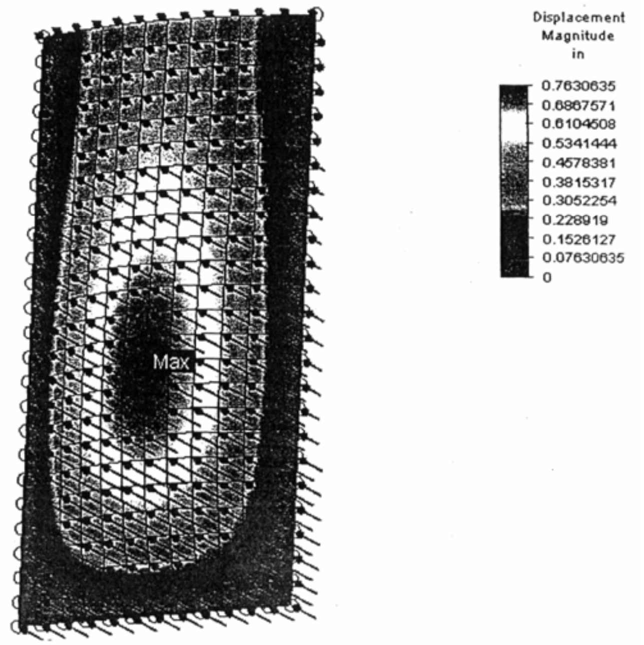
$4''$ square mesh



Manual mesh
Stepwise approximation of soil pressure
Simple support boundary conditions
Short side of box

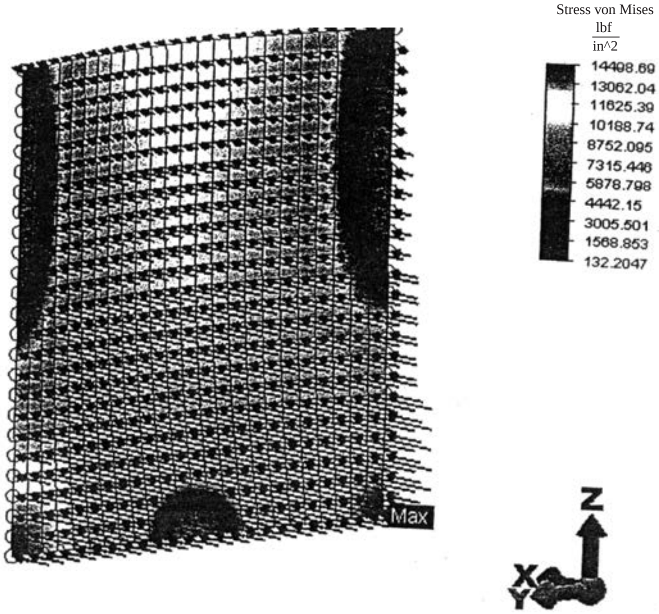
$48'' \text{ wide} \times 96'' \text{ deep} \times \dfrac{3''}{8} \text{ thick}$

4″ square mesh

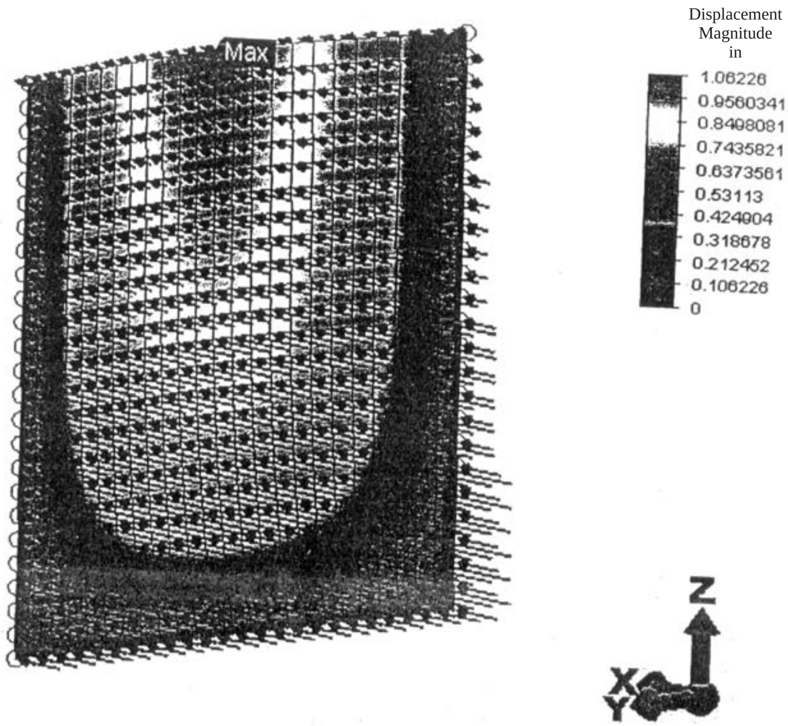


Manual 4″ square mesh
Stepwise approximation of soil pressure

Simple boundary conditions
Long side (96″ square)
0.75″ thick



Manual 4″ square mesh
Stepwise approximation of soil pressure
Simple boundary conditions
Long side (96″ square)
0.75″ thick



12.8 Determine the maximum deflection and maximum principal stress of the circular plate shown in Figure P12–8. The plate is subjected to a uniform pressure p = 700 kPa and fixed along its outer edge. Let E = 200 GPa, v = 0.3, radius r = 500 mm, and thickness t = 12 mm.

Analytical solution

$$W_{\max} = \frac{Pr^4}{64\,D} = \frac{(700,000)\,(0.5)^4}{64\,(2.013\times10^6)} = 0.0217 \text{ m}$$

$$\begin{aligned} D &= 0.091\,E\,t^3 \\ &= 0.091\,(200\times10^9)\,(0.012)^3 \\ &= 2.013\times106 \text{ lb}\cdot\text{in.} \end{aligned}$$

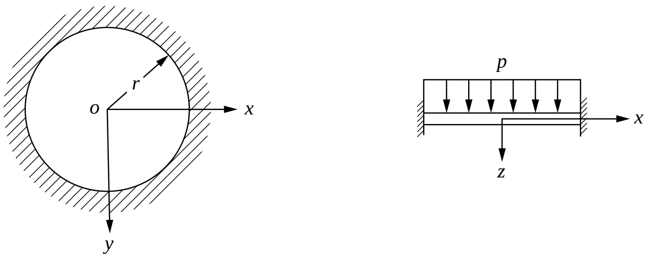
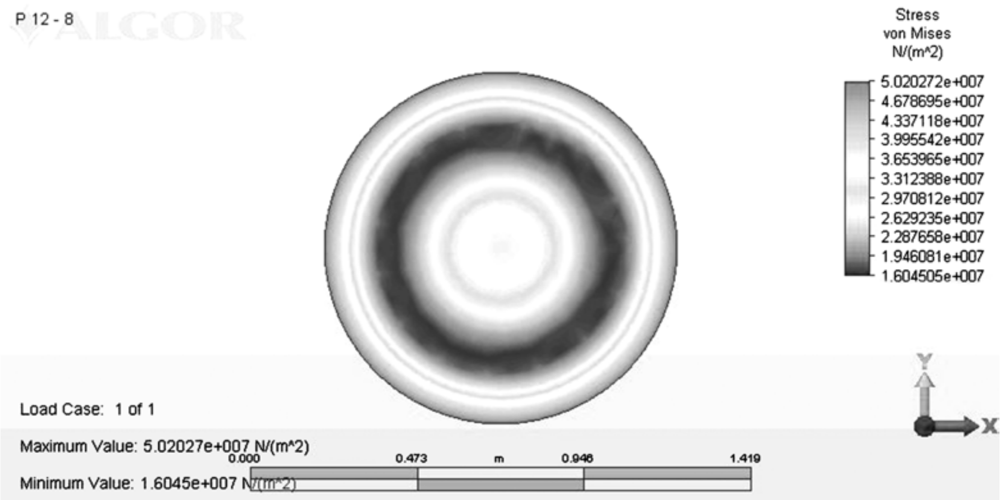


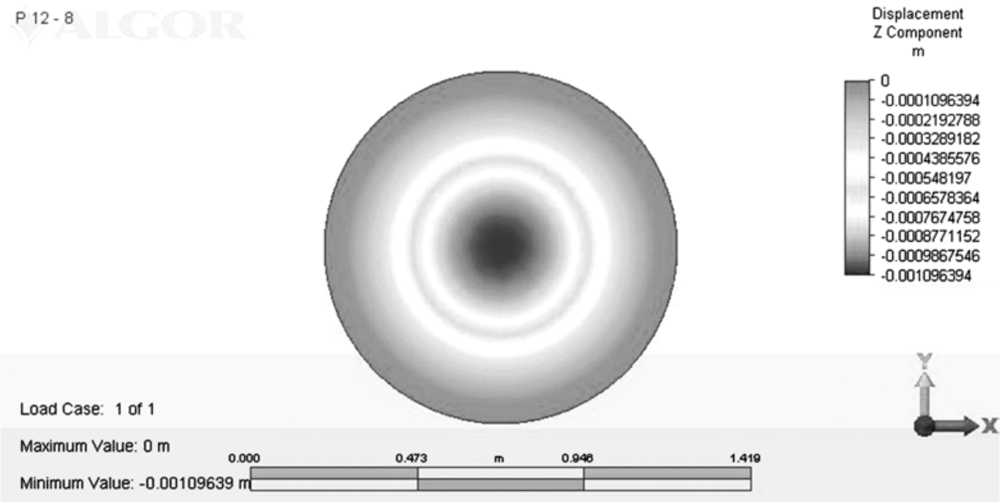
Figure P12–8



Algor

$$\begin{aligned} W_{\max} &= -0.001096 \text{ m} \\ &= -1.096\times10^{-3} \text{ m} \end{aligned}$$

Compares closely to analytical solution



12.9 Determine the maximum deflection and maximum stress for the plate shown. The plate is fixed along three sides. A uniform pressure of 70 kPa is applied to the surface. The plate is made of steel with E = 200 GPa, ν = 0.3, and t = 12 mm and sides equal to a = 0.75 m and b = 1 m.

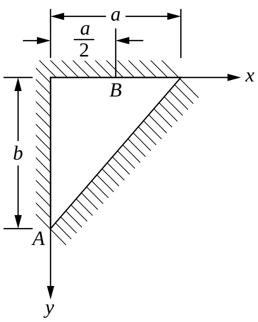
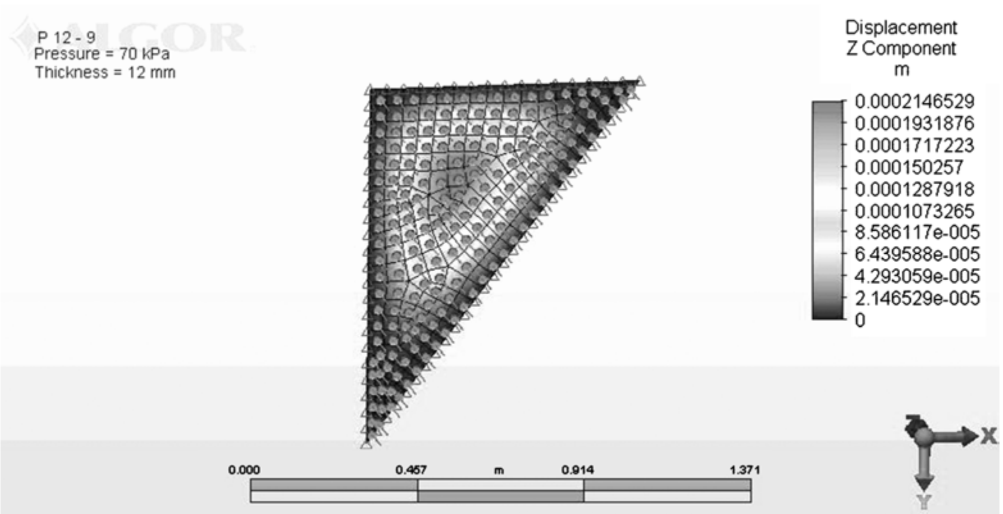
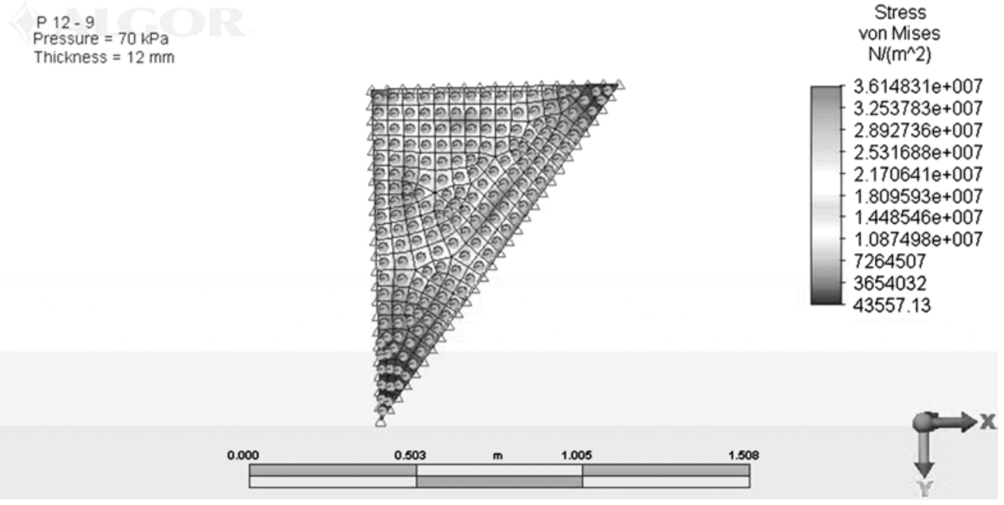


Figure P12–9





12.11 A square steel plate 2 m by 2 m and 10 mm thick at the bottom of a tank must support salt water at a height of 3 m, as shown in Figure P12–11. Assume the plate to be built in (fixed all around). The plate allowable stress is 100 MPa. Let E = 200 GPa, v = 0.3 for the steel properties. The weight density of salt water is 10.054 $\frac{\text{kN}}{\text{m}^3}$. Determine the maximum principal stress in the plate and compare to the yield strength.

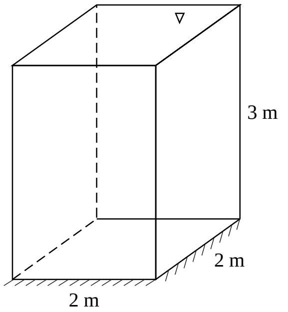
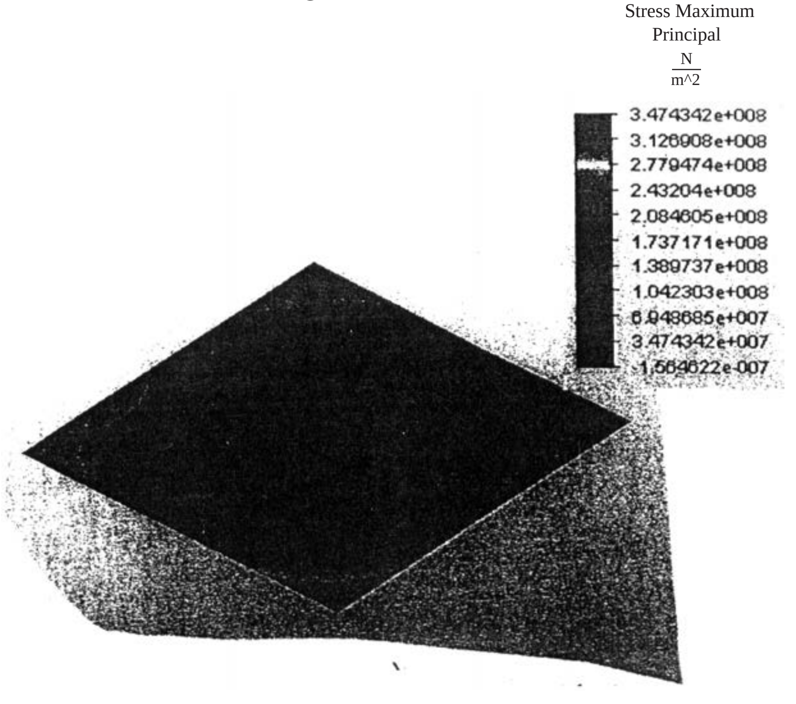
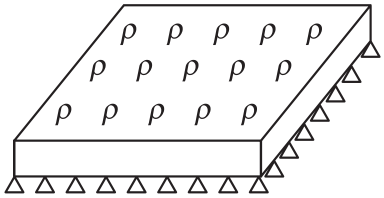


Figure P12–11



$$F_{\text{tot}} = (V)\,(\rho) = (3\,\cancel{m})(2\,\cancel{m})(2\,\cancel{m})(10.054E^{3}\,\frac{\text{N}}{\cancel{m^3}}) = 120{,}648\ \text{N}$$

$$P = \frac{F}{A_{\text{plate}}} = \frac{120{,}648\,\text{N}}{(2\text{m})(2\text{m})} = 30{,}162\ \frac{\text{N}}{\text{m}^2}$$



Find

Maximum principal stress = ? 347.43 $\frac{\text{MN}}{\text{m}^2}$

Safe or not safe = ?

Since stress allowable = 100 MPa < actual stress 347.43 MPa

Tank plate is not safe.

12.12 A stockroom floor carries a uniform load of $p = 80\ \frac{\text{lb}}{\text{ft}^2}$ over half the floor as shown in Figure P12–12. The floor has opposite edges clamped and remaining edges and mid-span simply supported. The dimensions are 40 ft by 20 ft. The floor thickness is 6 in. The floor is made of reinforced concrete with $E = 3 \times 10^6$ psi and $\nu = 0.25$. Determine the maximum deflection and maximum principal stress in the floor.

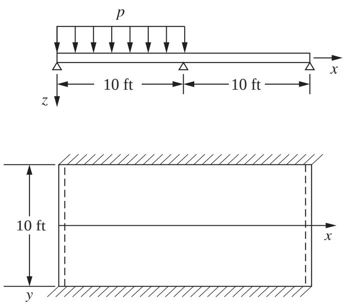
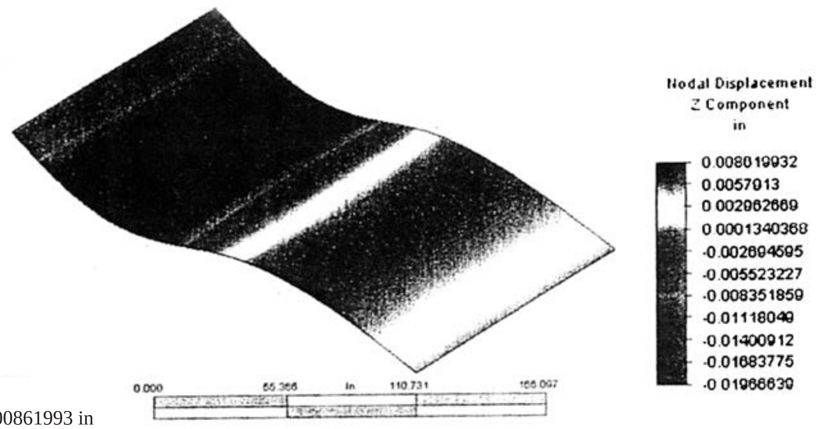
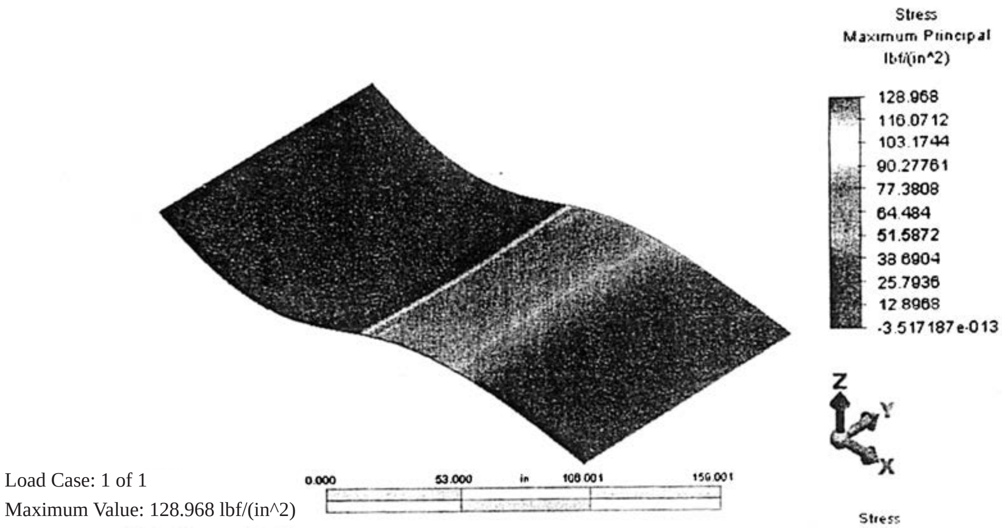


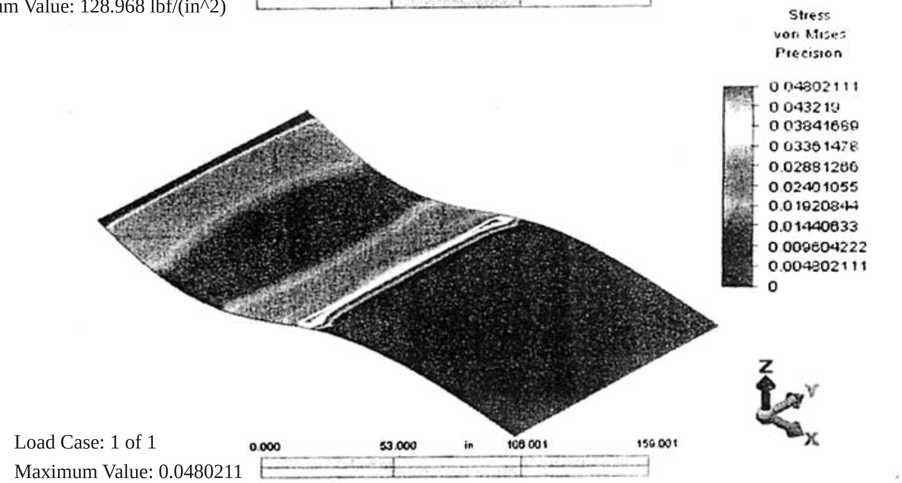
Figure P12–12



Load Case: 1 of 1

Maximum Value: 0.00861993 in

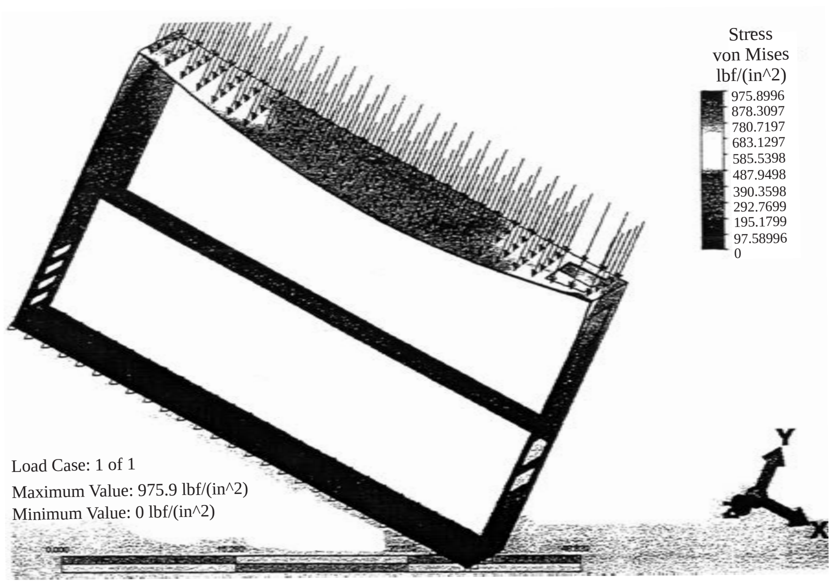


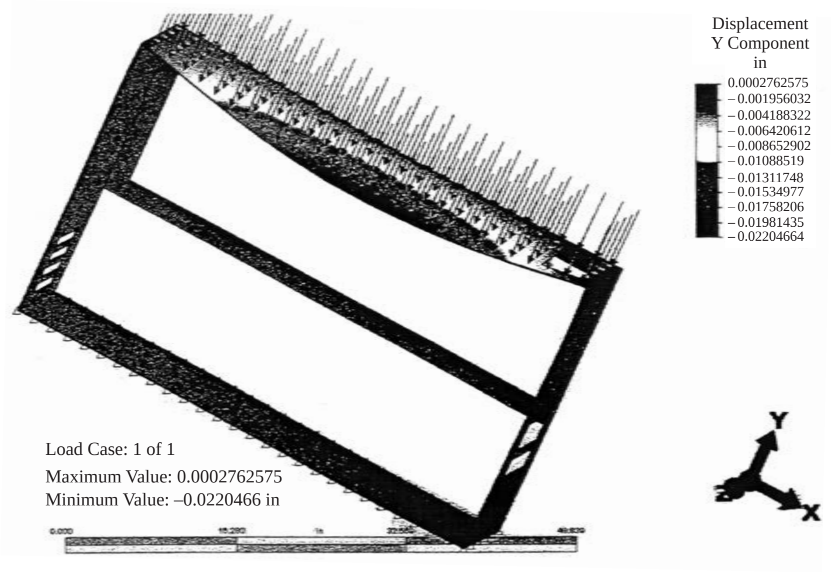


12.13

Material used: AISI 4130 Steel

Uniform load of 0.1 psi applied to top surface.



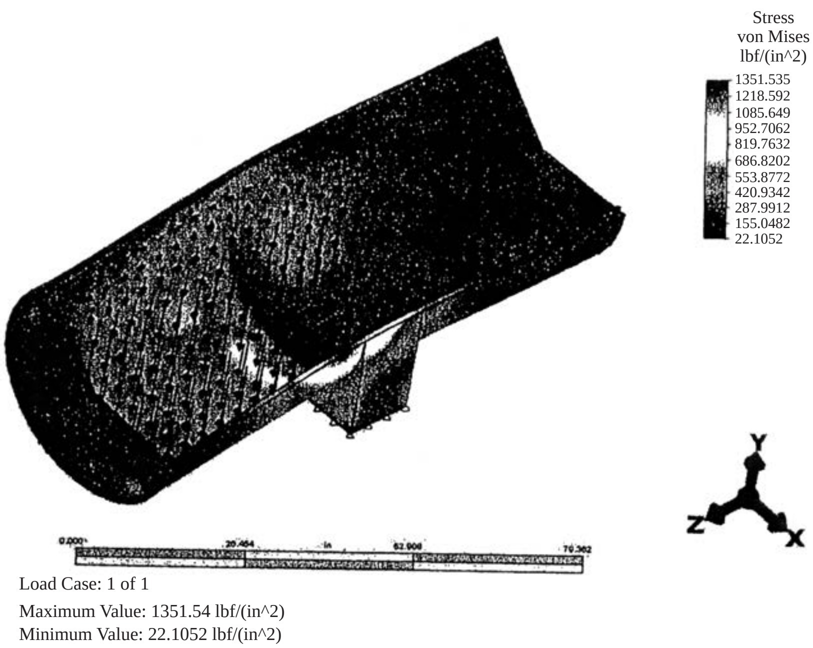


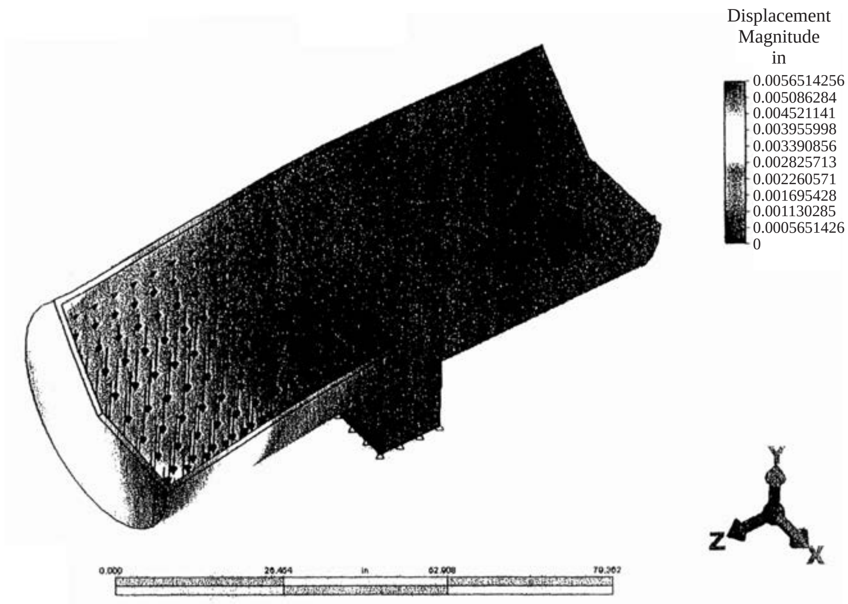
12.15

Model variables

Variable	Value
Material	1010 cold rolled
Modulus of Elasticity	29×10^6 psi
Maximum von Mises Stress	1351 psi
Maximum Displacement	0.00565 in.

Algor results





Load Case: 1 of 1

Maximum Value: 0.00565143 in
Minimum Value: 0 in

Figure 6 Displacement (mm)

12.16

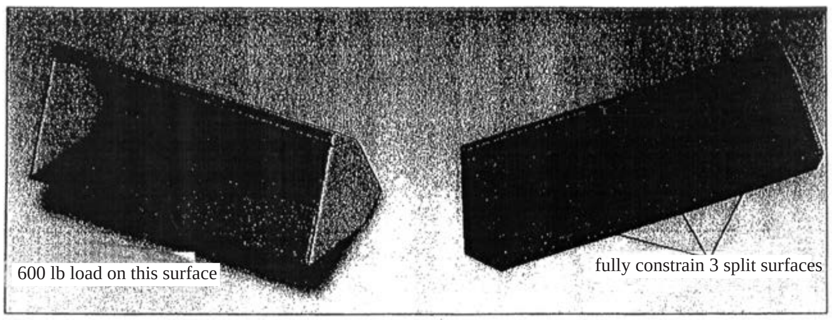


Figure 2 The Boundary Conditions on the bucket.

The results of the analysis are shown below in Figure 3.

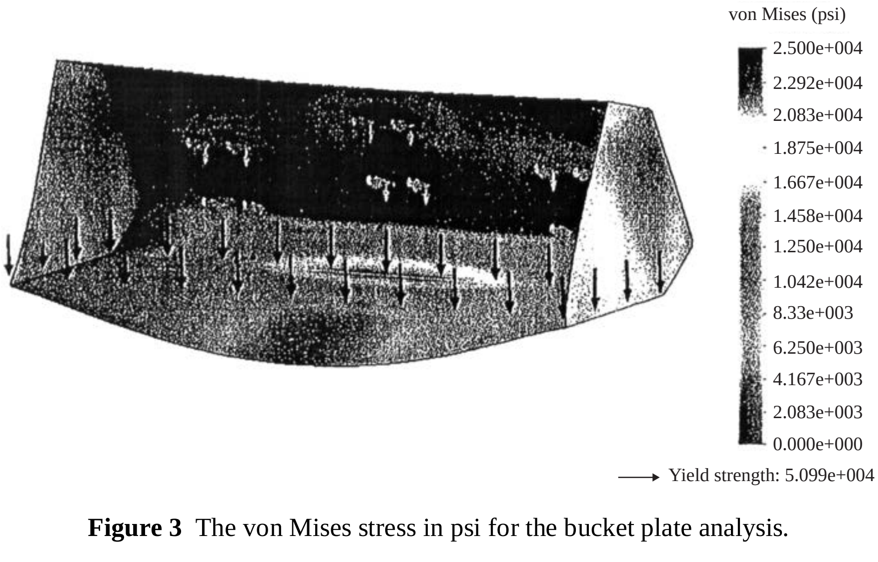
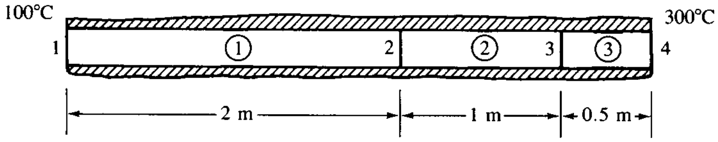


Figure 3 The von Mises stress in psi for the bucket plate analysis.

Chapter 13

13.1



Element $[K]$'s

$$[k^{(1)}] = \frac{(0.1\text{m}^2)(200\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{2\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(2)}] = \frac{(0.1\text{m}^2)(100\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{1\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}$$

$$[k^{(3)}] = \frac{(0.1\text{m}^2)(50\,\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{0.5\,\text{m}} = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}10 & -10\\ -10 & 10\end{bmatrix}$$

$$q^* = 0 \qquad Q = 0$$

$$\therefore \quad \{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0$$

Assemble equations

$$\begin{bmatrix}10 & -10 & 0 & 0\\ -10 & 20 & -10 & 0\\ 0 & -10 & 20 & -10\\ 0 & 0 & -10 & 10\end{bmatrix}\begin{Bmatrix}t_1\\ t_2\\ t_3\\ t_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ 0\\ 0\\ F_4\end{Bmatrix}$$

Boundary conditions $t_1 = 100°\text{C}$, $t_y = 300°\text{C}$

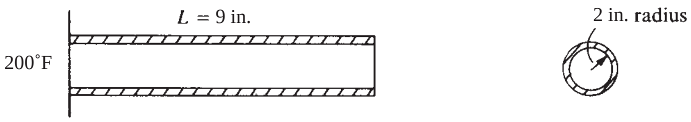
$$\therefore \quad \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 20 & -10 & 0\\ 0 & -10 & 20 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}t_1\\ t_2\\ t_3\\ t_4\end{Bmatrix} = \begin{Bmatrix}100\\ 1000\\ 3000\\ 300\end{Bmatrix}$$

Solving

$$t_2 = 166.7°\text{C}$$

$$t_3 = 233.3°\text{C}$$

13.2



$$[k^{(1)}] = [k^{(2)}] = \frac{\pi(2'')^2\, 3\,\frac{\text{Btu}}{\text{h}\cdot\text{in.}\cdot{}^\circ\text{F}}}{3\text{ in.}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 12.57 \end{bmatrix}\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

Now there is convection through right end

$$[k^{(3)}] = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 12.57 \end{bmatrix} + \left(1\frac{\text{Btu}}{\text{kip}\cdot\text{in.}^2\cdot{}^\circ\text{F}}\right)\pi(2'')^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 12.57 & -12.57 \\ -12.57 & 25.13 \end{bmatrix}$$

$$\text{Now } q^* = 0, \quad Q = 0 \quad \therefore \quad \{f^{(1)}\} = \{f^{(2)}\} = 0$$

$$\{f^{(3)}\} = h\,T_\infty\,A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1)\,(0°\text{F})\,\pi\,2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assemble equations and boundary condition $t_1 = 200°\text{F}$

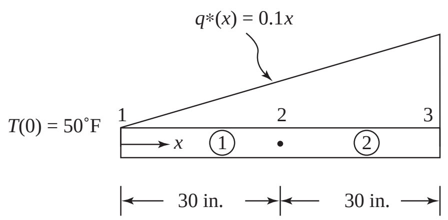
$$\begin{bmatrix} 12.57 & -12.57 & 0 & 0 \\ & 25.13 & -12.57 & 0 \\ & & 25.13 & -12.57 \\ \text{Symmetry} & & & 25.13 \end{bmatrix}\begin{Bmatrix} t_1 = 200°\text{F} \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solve Equations (2-4) of above

$$\begin{bmatrix} 25.13 & -12.57 & 0 \\ & 25.13 & -12.57 \\ \text{Symmetry} & & 25.13 \end{bmatrix}\begin{Bmatrix} t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 2513 \\ 0 \\ 0 \end{Bmatrix}$$

$$t_2 = 150°\text{F},\; t_3 = 100°\text{F},\; t_4 = 50°\text{F}$$

13.3



$$\frac{AK_{xx}}{L} = \frac{(2\,\text{in.}^2)\,(3\,\frac{\text{Btu}}{\text{h}\cdot\text{in.}\cdot{}^\circ\text{F}})}{30\text{ in.}} = \frac{1}{5}\;\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

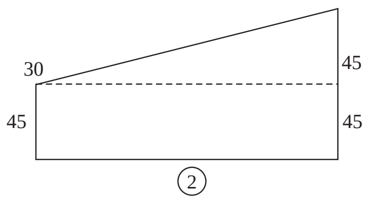
$$h = 0 \quad \therefore \quad \frac{hPL}{6} = 0, \quad hA = 0, \quad h\,T_\infty\,P\,L = 0$$

$$[k^{(1)}] = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [k^{(2)}]$$

$$\{f_q^{(1)}\} = \int_0^L q^*\,[N]^T\,dx = \int_0^{L\,=\,30}(0.1x)\begin{Bmatrix} 1-\frac{X}{L} \\ \frac{X}{L} \end{Bmatrix}dx$$

$$= 0.1 \begin{Bmatrix} \frac{L^2}{6} \\ \frac{L^2}{3} \end{Bmatrix} \frac{\text{Btu}}{\text{h}} = \begin{Bmatrix} 15 \\ 30 \end{Bmatrix}$$

Then



$$\{f_q^{(2)}\} = \begin{Bmatrix} 15+45 \\ 30+45 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 75 \end{Bmatrix}$$

Solve $\{F\} = [K]\,\{t\}$

Heat flow

$$\begin{Bmatrix} 15+F_1 \\ 30+60 \\ 75 \end{Bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} t_1 = 50°\text{F} \\ t_2 \\ t_3 \end{Bmatrix} \qquad \text{(A)}$$

Solve the 2nd and 3rd equations

$$90 + 10 = \frac{1}{5}\,(2\,t_2 - t_3) \qquad (1)$$

$$+\,75 = \frac{1}{5}\,(-\,t_2 + t_3) \qquad (2)$$

Adding (1) and (2)

$$175 = \frac{2}{5}\,t_2 - \frac{1}{5}\,t_2$$

$$175 = \frac{1}{5}\,t_2$$

$$t_2 = 875°\text{F}$$

Back-substitution into (1) yields

$$t_3 = 1750 - 500$$

$$t_3 = 1250\ °\text{F}$$

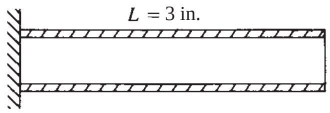
Solve for F_1 using Equation (A) above

$$15 + F_1 = \frac{1}{5}\,(t_1 - t_2)$$

$$F_1 = -\,15 + \frac{1}{5}\,(50 - 875)$$

$$= -\,180\ \frac{\text{Btu}}{\text{h}} = \text{heat flow out left end.}$$

13.4



$$Q = 10000\ \frac{\text{Btu}}{\text{h}\cdot\text{ft}^3}$$

$$[k^{(1)}] = [k^{(2)}] = \frac{AK_{xx}}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{\pi\left(\frac{1}{12}\right)^2 12}{\left(12\,\frac{\text{in.}}{\text{ft}}\right)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \pi\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$[k^{(3)}] = [k^{(1)}] + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= \pi\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 100\ \pi\left(\frac{1}{12}\right)^2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$[k^{(3)}] = \pi\begin{bmatrix}1 & -1\\ -1 & 1.694\end{bmatrix}$$

$$\{f^{(1)}\} = \frac{QAL}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = \frac{(10000)\pi\left(\frac{1}{12}\right)^2\frac{1}{12}}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$= \pi\begin{Bmatrix}2.894\\2.894\end{Bmatrix}$$

$$\{f^{(2)}\} = \{f^{(1)}\}$$

$$\{f^{(3)}\} = \{f^{(1)}\} + \{f_{h_{\text{end}}}\} = \pi\begin{Bmatrix}2.894\\2.894\end{Bmatrix} + h\,T_\infty\,A\begin{Bmatrix}0\\1\end{Bmatrix}$$

$$= \pi\begin{Bmatrix}2.894\\2.894\end{Bmatrix} + (100)\,(100)\,\pi\left(\frac{1}{12}\right)^2\begin{Bmatrix}0\\1\end{Bmatrix}$$

$$\{f^{(3)}\} = \pi\begin{Bmatrix}2.894\\2.894\end{Bmatrix} + \pi\begin{Bmatrix}0\\69.4\end{Bmatrix}$$

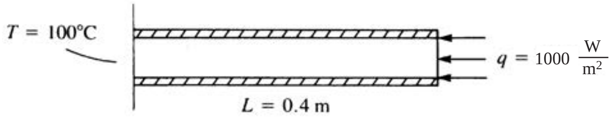
Assemble global equations

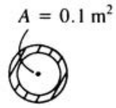
$$\pi\begin{bmatrix}1 & -1 & 0 & 0\\ & 2 & -1 & 0\\ & & 2 & -1\\ & & & 1.694\end{bmatrix}\begin{Bmatrix}t_1\\t_2\\t_3\\t_4\end{Bmatrix} = \begin{Bmatrix}2.894\\2.894+2.894\\2.894+2.894\\2.894+69.4\end{Bmatrix}$$

Solving simultaneously

$t_1 = 151°\text{F}$, $t_2 = 148°\text{F}$, $t_3 = 140°\text{F}$, $t_4 = 125°\text{F}$,

13.5





$$\frac{AK_{xx}}{L} = \frac{(0.1\text{m}^2)(6\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}})}{0.1\text{m}} = 6\ \frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \frac{\text{W}}{{}^\circ\text{C}}$$

$[k^{(4)}] = [k^{(1)}]$ also

$\{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0$ as $Q = 0$, $q^* = 0$

$$\{f^{(4)}\} = qA\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left(1000\frac{\text{W}}{\text{m}^2}\right)(0.1\text{m}^2)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= 100\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\text{W}$$

Assemble global equations

$$\begin{bmatrix} 6 & -6 & 0 & 0 & 0 \\ & 12 & -6 & 0 & 0 \\ & & 12 & -6 & 0 \\ & & & 12 & -6 \\ \text{Symmetry} & & & & 6 \end{bmatrix}\begin{Bmatrix} t_1 = 100^\circ\text{C} \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ 0 \\ 100 \end{Bmatrix} \qquad (1)$$

Now $t_1 = 100^\circ\text{C}$ into Equation (1)

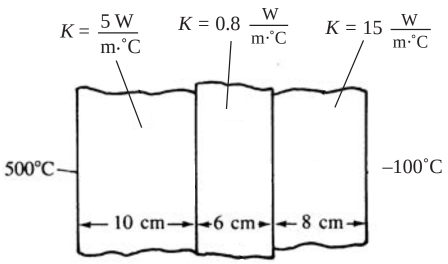
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 12 & -6 & 0 & 0 \\ 0 & -6 & 12 & -6 & 0 \\ 0 & 0 & -6 & 12 & -6 \\ 0 & 0 & 0 & -6 & 6 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 600 \\ 0 \\ 0 \\ 100 \end{Bmatrix} \qquad (2)$$

Solving Equations (2-5) of Equation (2)

$t_2 = 116.7^\circ\text{C}$, $t_3 = 133.3^\circ\text{C}$

$t_4 = 150^\circ\text{C}$, $t_5 = 166.7^\circ\text{C}$

13.6



Area = A (Can use unit A)

$$[k^{(1)}] = \frac{A(5)}{(0.1\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$[k^{(2)}] = \frac{A(0.8)}{(0.06\,\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 13.3 & -13.3 \\ -13.3 & 13.3 \end{bmatrix}$$

$$[k^{(3)}] = \frac{A(15)}{(0.08\,\text{m})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 187.5 & -187.5 \\ -187.5 & 187.5 \end{bmatrix}$$

$$\{f\}\text{'s} = 0$$

Assemble global equations with $t_1 = 500°\text{C}$ and $t_4 = 100°\text{C}$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = A \begin{bmatrix} 50 & -50 & 0 & 0 \\ & 50+13.3 & -13.3 & 0 \\ & & 13.3+187.5 & -187.5 \\ \text{Symmetry} & & & 187.5 \end{bmatrix} \begin{Bmatrix} t_1 = 500\,°\text{C} \\ t_2 \\ t_3 \\ t_4 = 100\,°\text{C} \end{Bmatrix}$$

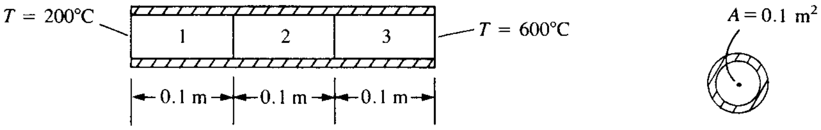
Solving the 2nd and 3rd equations above

$$t_2 = 420.5°\text{C},\ t_3 = 121.2°\text{C}$$

$$q^{(3)} = -K_{xx}\,\frac{(100-121.2)°\text{C}}{0.08\,\text{m}}$$

$$q^{(3)} = 3975\ \frac{\text{W}}{\text{m}^2}$$

13.7



$$\frac{AK_{xx}^{(1)}}{L} = \frac{(0.1\text{m}^2)5}{0.1\text{m}} = 5$$

$$[k^{(1)}] = \frac{AK_{xx}^{(1)}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\quad [k^{(3)}] = 15\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assemble global equations

$$\begin{bmatrix} 5 & -5 & 0 & 0 \\ & 15 & -10 & 0 \\ & & 25 & -15 \\ \text{Symmetry} & & & 15 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ 0 \\ 0 \\ F_{4x} \end{Bmatrix} \tag{1}$$

Boundary conditions

$t_1 = 200°\text{C}$, $t_4 = 600°\text{C}$

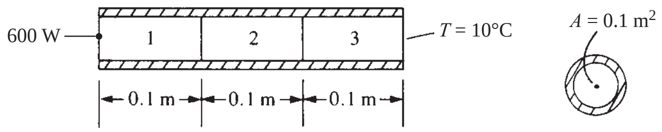
$\therefore$ Equation (1) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 15 & -10 & 0 \\ & & 25 & 0 \\ \text{Symmetry} & & & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 5(200) \\ 15(600) \\ 600 \end{Bmatrix} \tag{2}$$

Solving the 2nd and 3rd of Equations (2)

$$t_2 = 418.2°\text{C},\quad t_3 = 527.3°\text{C}$$

13.8 A composite wall is shown below. For element 1, let $K_{xx} = 5\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$, for element 2 let $K_{xx} = 10\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$, for element 3 let $K_{xx} = 15\ \frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}$. The left end has a heat source of 600 W applied to it. The right end is held at 10°C. Determine the left end temperature and the interface temperatures and the heat flux through element 3.



$$[k^{(1)}] = \frac{(0.1)(5)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$[k^{(2)}] = \frac{(0.1)(10)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$[k^{(3)}] = \frac{(0.1)(15)}{0.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$\{f^{(2)}\} = \{f^{(3)}\} = 0 \quad \{f^{(1)}\} = 600\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 600 \\ 0 \\ 0 \\ F_{4x} \end{bmatrix} = \left[\begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 15 & -10 & 0 \\ 0 & -10 & 25 & -15 \\ 0 & 0 & -15 & 15 \end{bmatrix}\right]\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix}$$

$$400 = 5t_1 - 5t_2 \qquad t_1 = 230^\circ\text{C}$$

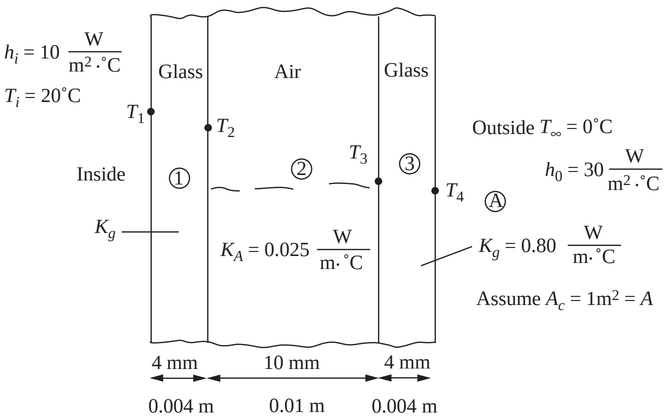
$$0 = -5t_1 + 15t_2 - 10\,t_3 \qquad t_2 = 110^\circ\text{C}$$

$$0 = -10t_2 + 25t_3 - 15\,(10) \qquad t_3 = 50^\circ\text{C}$$

$$F_{yx} = 15\,(50) + 15\,(10) = -\,600 \text{ W}$$

$$q_x = -15\left[-\frac{1}{0.1}\ \frac{1}{0.1}\right]\begin{Bmatrix} 50 \\ 10 \end{Bmatrix} = 6000\frac{\text{W}}{\text{m}^2}$$

13.9



Find T_1 T_2, T_3, T_4, Q (heat transfer through the double pane) (use $A = 1\text{ cm}^2$)

$$[k^{(1)}] = \frac{AK_g}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + h_i A\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \frac{1\text{m}^2\left(0.80\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.004\,\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 10\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}(1\text{m}^2)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}210 & -200\\ -200 & 200\end{bmatrix}\frac{\text{W}}{\text{K}}$$

$$[k^{(2)}] = \frac{AK_A}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1\text{m}^2\left(0.025\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.01\,\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}2.5 & -2.5\\ -2.5 & 2.5\end{bmatrix}\frac{\text{W}}{\text{K}}$$

$$[k^{(3)}] = \frac{AK_g}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + h_0 A\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \frac{1\text{m}^2\left(0.80\,\frac{\text{W}}{\text{m}\cdot\text{K}}\right)}{0.004\,\text{m}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 30\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}(1\text{m}^2)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}200 & -200\\ -200 & 230\end{bmatrix}$$

$$\{f^{(1)}\} = h_i\,T_\infty\,A\begin{Bmatrix}1\\0\end{Bmatrix} = 10\frac{\text{W}}{\text{m}^2\cdot\text{K}}\,(293\text{ K})\,(1\text{m}^2)\begin{Bmatrix}1\\0\end{Bmatrix} = 2930\text{ W}\begin{Bmatrix}1\\0\end{Bmatrix}$$

$$\{f^{(2)}\} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

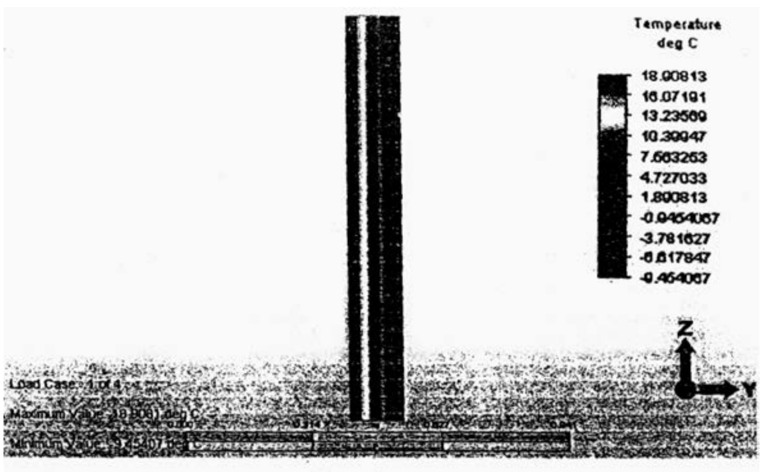
$$\{f^{(3)}\} = h_0\,T_\infty\,A\begin{Bmatrix}0\\1\end{Bmatrix} = 30\frac{\text{W}}{\text{m}^2\cdot\text{K}}\,(273\text{ K})\,(1\text{m}^2)\begin{Bmatrix}0\\1\end{Bmatrix} = 8190\text{ W}\begin{Bmatrix}0\\1\end{Bmatrix}$$

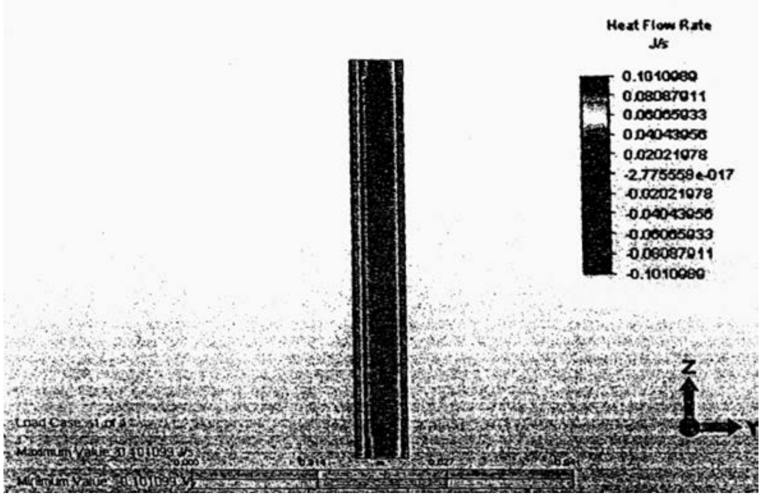
$$\{F\} = [K]\,[T]$$

$$\begin{Bmatrix}F_1 = 2930\,\text{W}\\ F_2 = 0\\ F_3 = 0\\ F_4 = 8190\text{ W}\end{Bmatrix} = \begin{bmatrix}210 & -200 & 6 & 0\\ -200 & 200+2.5 & -2.5 & 0\\ 0 & -2.5 & 2.5+200 & -200\\ 0 & 0 & -200 & 230\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}$$

$$\begin{aligned}T_1 &= 289.3\,K = 16.3^\circ\text{C}\\ T_2 &= 289.1\,K = 16.1^\circ\text{C}\\ T_3 &= 279.4\,K = 1.4^\circ\text{C}\\ T_4 &= 274.2\,K = 1.2^\circ\text{C}\end{aligned}$$

13.10





13.11

Unit definition

Given in problem statement (Mathcad used to solve this one)

$$L_{\text{total}} = 20 \text{ cm}$$

$$K_{xx} = 15 \ \frac{\text{W}}{\text{m} \cdot {}^\circ\text{C}}$$

$$h_1 = 50 \ \frac{\text{W}}{\text{m} \cdot {}^\circ\text{C}}$$

$$h_2 = 80 \ \frac{\text{W}}{\text{m} \cdot {}^\circ\text{C}}$$

$$T_{\text{left end}} = 100^\circ\text{C}$$

$$T_{\text{inf}} = 20^\circ\text{C}$$

$$\text{dia} = 0.5 \text{ cm}$$

$$\text{radius} = \frac{\text{dia}}{2}$$

$$P = \pi \ \text{dia} \qquad A = \pi \ \text{radius}^2$$

$$L = \frac{L_{\text{total}}}{4}$$

$$P = 0.016 \text{ m}$$

$$A = 0.00002 \text{ m}^2$$

$$\text{radius} = 0.0025 \text{ m}$$

$$M = 150 \ \frac{\text{W}}{\text{m}^3 \cdot {}^\circ\text{C}}$$

$$y_1(x) = Mx + h_1$$

$$k_{h_1} = \begin{bmatrix} P\int_0^{0.05\text{m}} y_1(x)\left(1 - \frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x}{L} - \frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x}{L} - \frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_1(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

Develop stiffness matricies for each element

$$x = \frac{L_{\text{total}}}{4}$$

$$[k_{h_1}] = \begin{pmatrix} 0.0136 & 0.007 \\ 0.007 & 0.0146 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_1] = \frac{AK_{xx}}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_1}]$$

$$[k_1] = \begin{pmatrix} 0.0195 & 0.0011 \\ 0.0011 & 0.0205 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_2(x) = Mx + y_1(x)$$

$$[k_{h_2}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_2(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_2(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_2}] = \begin{pmatrix} 0.0141 & 0.0075 \\ 0.0075 & 0.016 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_2] = \frac{AK_{xx}}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_2}]$$

$$[k_2] = \begin{pmatrix} 0.02 & 0.0016 \\ 0.0016 & 0.0219 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_3(x) = Mx + y_2(x)$$

$$[k_{h_3}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_3(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_3(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_3}] = \begin{pmatrix} 0.015 & 8.018\times10^{-3} \\ 8.018\times10^{-3} & 0.018 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_3] = \frac{AK_{xx}}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_3}]$$

$$[k_3] = \begin{pmatrix} 0.02 & 2.127\times10^{-3} \\ 2.127\times10^{-3} & 0.023 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$y_4(x) = Mx + y_3(x)$$

$$[k_{h_4}] = \begin{bmatrix} P\int_0^{0.05\text{m}} y_4(x)\left(1-\frac{x}{L}\right)^2 dx & P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx \\ P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x}{L}-\frac{x^2}{L^2}\right)dx & P\int_0^{0.05\text{m}} y_4(x)\left(\frac{x^2}{L^2}\right)dx \end{bmatrix}$$

$$[k_{h_4}] = \begin{pmatrix} 0.0151 & 0.0085 \\ 0.0085 & 0.019 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[k_4] = \frac{AK_{xx}}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + [k_{h_4}] + h_2 A \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[k_4] = \begin{pmatrix} 0.0209 & 0.0026 \\ 0.0026 & 0.0264 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

$$[K] = \begin{pmatrix} k_{1_{0,0}} & k_{1_{0,1}} & 0 & 0 & 0 \\ k_{1_{0,0}} & k_{1_{1,1}} + k_{2_{0,0}} & k_{2_{0,1}} & 0 & 0 \\ 0 & k_{2_{1,0}} & k_{2_{1,1}} + k_{3_{0,0}} & k_{3_{0,1}} & 0 \\ 0 & 0 & k_{3_{1,0}} & k_{3_{1,1}} + k_{4_{0,0}} & k_{4_{0,1}} \\ 0 & 0 & 0 & k_{4_{1,0}} & k_{4_{1,1}} \end{pmatrix}$$

Global $[K]$

$$[K] = \begin{pmatrix} 0.0195 & 0.0011 & 0 & 0 & 0 \\ 0.0011 & 0.0404 & 0.0016 & 0 & 0 \\ 0 & 0.0016 & 0.0424 & 0.0021 & 0 \\ 0 & 0 & 0.0021 & 0.0443 & 0.0026 \\ 0 & 0 & 0 & 0.0026 & 0.0264 \end{pmatrix} \frac{\text{W}}{°\text{C}}$$

Element force matricies for each element

$$y_1(x) = 57.5\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{1\text{ave}} = \frac{h_1 + y_1(x)}{2} \qquad h_{1\text{ave}} = 53.75\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_2(x) = 65\,\frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{2\text{ave}} = \frac{y_1(x) + y_2(x)}{2} \qquad h_{2\text{ave}} = 61.25\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_3(x) = 72.5\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{3\text{ave}} = \frac{y_2(x) + y_3(x)}{2} \qquad h_{3\text{ave}} = 68.75\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$y_4(x) = 80\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}} \qquad h_{4\text{ave}} = \frac{y_3(x) + y_4(x)}{2} \qquad h_{4\text{ave}} = 76.25\ \frac{\text{W}}{\text{m}^2 \cdot °\text{C}}$$

$$\{f_1\} = \frac{h_{1\text{ave}}\,T_{\text{inf}}\ P\,L}{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad f_1 = \begin{pmatrix}6.188\\6.188\end{pmatrix}\ \text{W}$$

$$\{f_2\} = \frac{h_{2\text{ave}}\,T_{\text{inf}}\ P\,L}{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad f_2 = \begin{pmatrix}7.051\\7.051\end{pmatrix}\ \text{W}$$

$$\{f_3\} = \frac{h_{3\text{ave}}\,T_{\text{inf}}\ P\,L}{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad f_3 = \begin{pmatrix}7.914\\7.914\end{pmatrix}\ \text{W}$$

$$\{f_4\} = \left[\frac{h_{4\text{ave}}\,T_{\text{inf}}\ P\,L}{2}\begin{pmatrix}1\\1\end{pmatrix}\right] + \left[h_2\,T_{\text{inf}}\ A\begin{pmatrix}0\\1\end{pmatrix}\right] \qquad f_4 = \begin{pmatrix}8.778\\9.238\end{pmatrix}\ \text{W}$$

$$F_1 = f_{1_{0,0}}$$

$$F_2 = f_{1_{1,0}} + f_{2_{0,0}}$$

$$F_3 = f_{2_{1,0}} + f_{3_{0,0}}$$

$$F_4 = f_{3_{1,0}} + f_{4_{0,0}}$$

$$F_5 = f_{4_{1,0}}$$

$F_1 = 6.188$ W, $F_2 = 13.239$ W, $F_3 = 14.966$ W, $F_4 = 16.692$ W, $F_5 = 9.238$ W

Set up equations to solve for temperature

$$[K_{\text{mod}}] = \begin{pmatrix} k_{1_{1,1}} + k_{2_{0,0}} & k_{2_{0,1}} & 0 & 0 \\ k_{2_{1,0}} & k_{2_{1,1}} + k_{3_{0,0}} & k_{3_{0,1}} & 0 \\ 0 & k_{3_{1,0}} & k_{3_{1,1}} + k_{4_{0,0}} & k_{4_{0,1}} \\ 0 & 0 & k_{4_{1,0}} & k_{4_{1,1}} \end{pmatrix}$$

Guess

$$t_2 = 75°C$$

$$t_3 = 60°C$$

$$t_4 = 50°C$$

$$t_5 = 25°C$$

Given

$$\begin{bmatrix} F_2 + (-K_{1,0}\, T_{\text{left end}}) \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = K_{\text{mod}} \begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}$$

$$\begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \text{Find}\,(t_2, t_3, t_4, t_5)$$

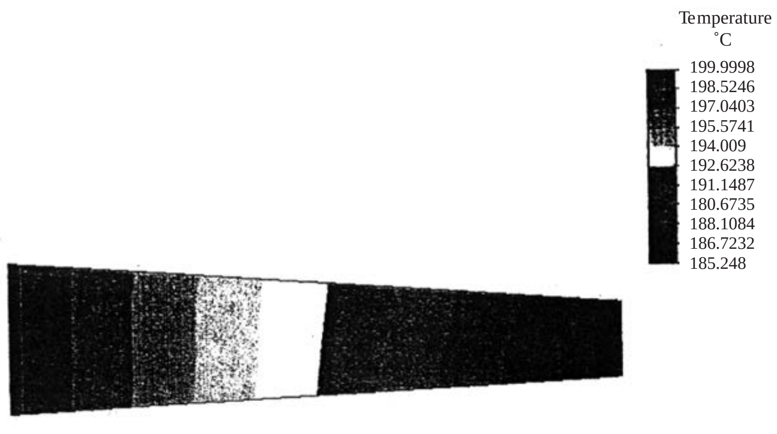
$$\begin{pmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \begin{pmatrix} 30.717 \\ 51.077 \\ 69.113 \\ 42.348 \end{pmatrix} °C$$

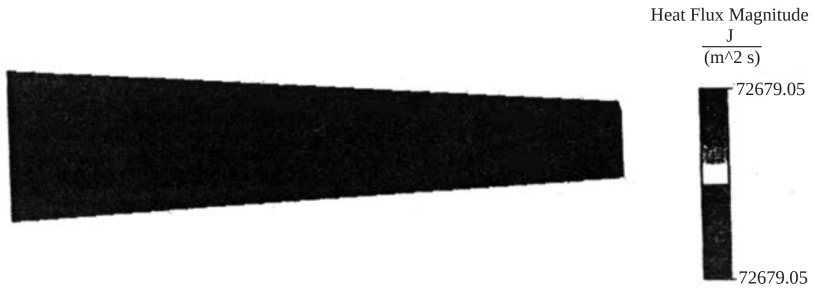
$$F_1 = (K_{0,\,0}\; K_{0,\,1}) \begin{pmatrix} T_{\text{left end}} \\ t_2 \end{pmatrix}$$

$$F_1 = 7.614\text{ W}$$

13.12 A tapered aluminum fin ($k = \frac{200\,\text{W}}{\text{m}\cdot°\text{C}}$) shown in Figure P13-12, has a circular cross-section with base diameter of 1 cm and tip diameter of 0.5 cm. The base is maintained at 200°C and looses heat by convection to the surrounding at T_∞ = 10°C, $h = 150 \frac{\text{W}}{\text{m}^2\cdot°\text{C}}$. The tip of the fin is insulated. Assume one-dimensional heat flow and determine the temperatures at the quarter points along the fin. What is the rate of heat loss in Watts through each element? Use four elements with an average cross-sectional area for each element.

(Algor results)





13.13

$$[k^{(1)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ Use unit area, } A = 1\text{ ft}^2$$

$$= \frac{(1\,\text{ft}^2)\left(0.10\,\dfrac{\text{Btu}}{\text{h}\cdot\text{ft}\cdot{}^\circ\text{F}}\right)}{\dfrac{0.50\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 1.5\,\frac{\text{Btu.1ft}^2}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix} 3.9 & -2.4 \\ -2.4 & 2.4 \end{bmatrix}$$

$$[k^{(2)}] = \frac{(1\,\text{ft}^2)(0.02)}{\dfrac{5\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.048 & -0.048 \\ -0.048 & 0.048 \end{bmatrix}$$

$$[k^{(3)}] = \frac{(1\,\text{ft}^2)(0.8)}{\dfrac{0.5\text{ in.}}{12\text{ in.}}(1\,\text{ft})}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 19.2 & -19.2 \\ -19.2 & 23.2 \end{bmatrix}$$

$$\{f^{(1)}\} = h\,T_\infty A\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \left(1.5\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\right)(1\,\text{ft}^2)(65^\circ\text{F})\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 97.5 \\ 0 \end{Bmatrix}$$

$$\{f^{(3)}\} = h\,T_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left(4.0\,\frac{\text{Btu}}{\text{h}\cdot\text{ft}^2\cdot{}^\circ\text{F}}\right)(1\text{ft}^2)(0^\circ\text{F})\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{f^{(2)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

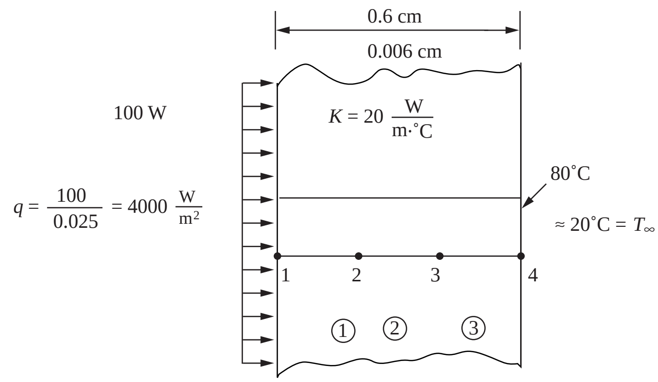
$$\{F\} = [K]\,\{t\}$$

$$\begin{Bmatrix} 97.5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3.9 & -2.4 & 0 & 0 \\ -2.4 & 2.448 & -0.048 & 0 \\ 0 & -0.048 & 19.248 & -19.2 \\ 0 & 0 & -19.2 & 23.2 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix}$$

Solving for t's

$$t_1 = 63.05°\text{F}, \quad t_2 = 61.83°\text{F}, \quad t_3 = 0.884°\text{F}, \quad t_4 = 0.731°\text{F}$$

13.15 Base plate of an iron is 0.6 cm thick. The plate is subjected to 600 W of power over a base surface area of 250 cm^2 resulting in a uniform flux generated on the inside surface. The thermal conductivity of the metal base plate is $k = 20\ \frac{\text{W}}{\text{m}\cdot°\text{C}}$. The outside ambient temperature of plate is 20°C at steady state. Assume 1-D heat transfer through the plate thickness. Using 3 elements, model the plate to determine the temperatures at the inner surface and interior $\frac{1}{3}$ points.



From Mathcad solution

$A = 0.025$ $\qquad Kxx = 20$ $\qquad h = 20$ $\qquad L = 20$

$q = \dfrac{100}{\text{A}}$ $\qquad q = 4 \times 10^3$ $\qquad T_{\text{inf}} = 20$ $\qquad qA = 100$

$$[k_1] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad k_1 = A\frac{K_{xx}}{L}k_1 \qquad [k_1] = \begin{pmatrix} 25 & -25 & 0 & 0 \\ -25 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[k_{2c}] = A\frac{K_{xx}}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad k_2 = k_2\,c \qquad [k_2] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 25 & -25 & 0 \\ 0 & -25 & 25 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[k_{3c}] = A\frac{K_{xx}}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad [k_{3h}] = h\,A\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad h\,A = 0.5$$

$$[k_3] = [k_{3c}] + [k_{3h}] \qquad [k_3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & -25 \\ 0 & 0 & -25 & 25.5 \end{pmatrix}$$

$$\{f_q\} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} q\,A \qquad \{f_q\} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \{f_h\} = h\,T_{\text{inf}}\ A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \{f_h\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10 \end{pmatrix}$$

$$\{f\} = \{f_q\} + \{f_h\} \qquad \{f\} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 10 \end{pmatrix} \qquad [K] = [k_1] + [k_2] + [k_3]$$

$$[k] = \begin{pmatrix} 25 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 25.5 \end{pmatrix}$$

$$\text{temps} = 1\ \text{solve}\,(k, f) \qquad \text{temps} = \begin{pmatrix} 232 \\ 228 \\ 224 \\ 220 \end{pmatrix} \qquad \text{temps}_1 = 228$$

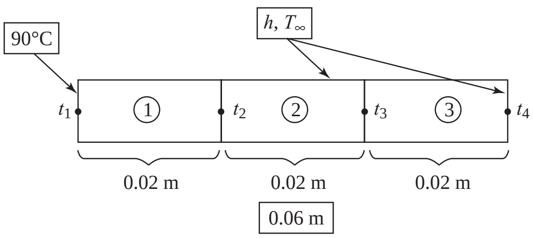
$$q_1 = -\ Kxx \begin{pmatrix} \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \cdot \begin{pmatrix} \text{temps}_1 \\ \text{temps}_2 \end{pmatrix}$$

Remember to use the left bracket key to get the subscript temps 1 and temps 2.

$$q_1 = 4 \times 10^3$$

13.16 A hot surface of a plate is cooled by attaching fins (called pin fins) to it. The surface of the plate (left end of the fin) is at 90°C. The typical fin is 6 cm (0.06 m) long and has a cross-sectional area of 5×10^{-6} m^2 with a perimeter of 0.006 m. The fin is made of copper with $k = 400\ \frac{\text{W}}{\text{m}\cdot°\text{C}}$. The temperature of the surrounding air is $T_\infty = 20°\text{C}$ with heat transfer coefficient on the surface (including the right end) estimated to be 10 $\frac{\text{W}}{\text{m}^2\cdot°\text{C}}$.

Use three elements in your model to estimate the temperature distribution along the fin length.



$$[k^{(1)}] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{5\times10^{-6}\cdot 400}{0.02}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$+ \frac{10\cdot 0.006\cdot 0.02}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.1004 \end{bmatrix} \frac{\text{W}}{°\text{C}} = [k^{(2)}]$$

$$[k^{(3)}] = [k^{(1)}] + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.1004 \end{bmatrix} + 10.5 \times 10^{-6} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad [k^{(3)}] = \begin{bmatrix} 0.1004 & -0.0998 \\ -0.0998 & 0.10045 \end{bmatrix} \left(\frac{\text{W}}{°\text{C}}\right)$$

$$\{f^{(1)}\} = \frac{h T_\infty PL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{10 \cdot 20°\text{C} \cdot 0.006\,\text{m} \cdot 0.02\,\text{m}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.012 \\ 0.012 \end{Bmatrix} \text{W}$$

$$\{f^{(2)}\} = \{f^{(1)}\}$$

$$\{f^{(3)}\} = \{f^{(1)}\} + h\,T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \{f^{(1)}\} + 10.20°\text{C} \cdot 5 \times 10^{-6}\,\text{m}^2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

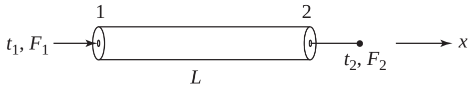
$$\{f^{(3)}\} = \begin{Bmatrix} 0.012 \\ 0.012 \end{Bmatrix} \text{W}$$

$$\begin{bmatrix} 0.1004 & -0.0998 & 0 & 0 \\ & 2(0.1004) & -0.0998 & 0 \\ & & 2(0.1004) & -0.0998 \\ \text{Symmetry} & & & 0.10045 \end{bmatrix} \begin{Bmatrix} t_1 = 90°\text{C} \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 0.012 + h \\ 2(0.012) \\ 2(0.012) \\ 0.013 \end{Bmatrix}$$

Solving Equations (2-4),

$$t_2 = 87.95°\text{C}, \quad t_3 = 86.72°\text{C}, \quad t_4 = 86.24°\text{C}$$

13.17



F's in. Btu's, t's in °F

Fourier's law

$$q = -K_{xx} \frac{dT}{dx} \left(\frac{\text{Btu}}{\text{ft}^2}\right) \tag{1}$$

Want to link thermal inputs F_1 and F_2 to nodal temperatures t_1 and t_2

$$[T] = [N]\{t\} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{2}$$

$$\frac{d[T]}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = [B]\{t\} \tag{3}$$

Total heat input at node 1 is

$$F_1 = q\,A \tag{4}$$

and at node 2

$$F_2 = -\,qA \text{ (negative sign accounts for the positive direction of } F_2 \text{ being an output at node 2)} \quad (5)$$

Using Equations (1) and (3) in (4) and (5), we have

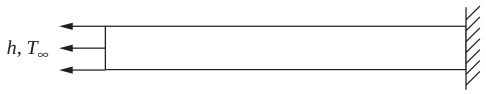
$$F_1 = -\,K_{xx}\,[B]\,\{t\}\,A \qquad F_2 = K_{xx}\,[B]\,\{t\}\,A \quad (6)$$

or

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \quad (7)$$

$$\therefore\ [k] = \frac{K_{xx}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{element conductivity matrix}$$

13.18



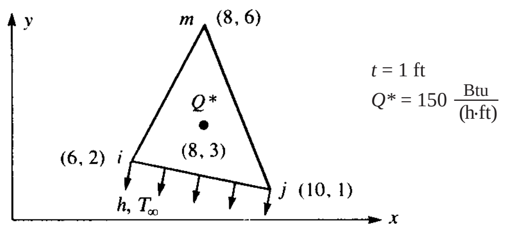
See Equation (13.4.28)

Now convection from left end

$$[K_{h_{\text{left}}}] = hA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\{f_{h_{\text{left}}}\} = h\,T_\infty\,A\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

13.19



$$[k] = [B]^T\,[D]\,[B]\,t\,A + \int_s h[N]^T\,[N]\,ds$$

$$\alpha_i = x_j\,y_m - x_m\,y_j = (10)\,(6) - (8)\,(1) = 52$$

$$\alpha_j = x_m\,y_i - x_i\,y_m = (8)\,(2) - (6)\,(6) = -\,20$$

$$\alpha_m = x_i\,y_j - x_j\,y_i = (6)\,(1) - (10)\,(2) = -\,14$$

$$\beta_i = y_j - y_m = 1 - 6 = -5$$

$$\beta_j = y_m - y_i = 6 - 2 = 4$$

$$\beta_m = y_i - y_j = 2 - 1 = 1$$

$$\gamma_i = x_m - x_j = 8 - 10 = -\,2$$

$$\gamma_j = x_i - x_m = 6 - 8 = -\,2$$

$$\gamma_m = x_j - x_i = 10 - 6 = 4$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_i & y_i \\ 1 & x_m & y_m \end{vmatrix} = \begin{vmatrix} 1 & 6 & 2 \\ 1 & 10 & 1 \\ 1 & 8 & 6 \end{vmatrix} = 18 \text{ ft}^2$$

Conduction part of $[k] = [k_c]$

$$[k_c] = \frac{1}{2A}\begin{bmatrix} \beta_i & \gamma_i \\ \beta_i & \gamma_j \\ \beta_m & \gamma_m \end{bmatrix}\begin{bmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{bmatrix}\frac{1}{2A}\begin{bmatrix} \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} t\,A$$

$$= \frac{1}{18}\begin{bmatrix} -5 & -2 \\ 4 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}\frac{1}{18}\begin{bmatrix} -5 & 4 & 1 \\ -2 & -2 & 4 \end{bmatrix} (1 \text{ ft})\,(9 \text{ ft}^2)$$

$$[k_c] = \begin{bmatrix} 12.08 & -6.67 & -5.42 \\ & 8.33 & -1.67 \\ \text{Symmetry} & & 7.08 \end{bmatrix}$$

Convection part of $[k] = [k_h]$

$$L_{ij} = 4.123 \text{ ft}$$

$$[k_h] = \frac{hL_{ij}t}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{(20)(4.123)(1)}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 27.49 & 13.74 & 0 \\ & 27.49 & 0 \\ \text{Symmetry} & & 0 \end{bmatrix}$$

$$\text{Total } [k] = [k_c] + [\underline{k_h}]$$

$$[k] = \begin{bmatrix} 39.57 & 7.08 & -5.42 \\ & 35.82 & -1.67 \\ \text{Symmetry} & & 7.08 \end{bmatrix} \frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

Force matrix

$$\{f\} = \int_V [N]^T Q\, dV + \int_{S_2} [N]^T q\, ds + \int_{S_3} [N]^T h\, T_\infty ds$$

$$q = 0$$

For point source $Q^* = \bar{Q}$

$$\{f_Q\} = \bar{Q}t\begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}\Bigg|_{\substack{x=8 \\ y=3}}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)\Big|_{\substack{x=8 \\ y=3}}$$

$$N_i = \frac{1}{18}(52 + (-5)(8) + (-2)(3)) = 0.333$$

$$N_j = \frac{1}{18}(-20 + (4)(8) + (-2)(3)) = 0.333$$

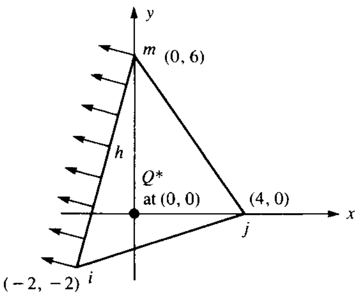
$$N_m = \frac{1}{18}(-14 + (1)(8) + (4)(3)) = 0.333$$

$$\therefore \ \{f_Q\} = 150\ (1) \begin{Bmatrix} 0.333 \\ 0.333 \\ 0.333 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 50 \\ 50 \end{Bmatrix} \frac{\text{Btu}}{\text{h}}$$

$$\{f_h\} = \frac{hT_\infty L_{ij} t}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \frac{(20)(70)(4.123)1}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\{f_h\} = \begin{Bmatrix} 2886 \\ 2886 \\ 0 \end{Bmatrix} \ \therefore \ \{f\} = \begin{Bmatrix} 50+2886 \\ 50+2886 \\ 50+0 \end{Bmatrix} = \begin{Bmatrix} 2936 \\ 2936 \\ 50 \end{Bmatrix} \frac{\text{Btu}}{\text{h}}$$

13.20



$$[k] = t\,A\,[B]^T\,[D]\,[B] + \frac{h\,L_{im}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad (1)$$

$$2A = \begin{vmatrix} 1 & -2 & -2 \\ 1 & 4 & 0 \\ 1 & 0 & 6 \end{vmatrix} = 44$$

$L_{im} = 8.246$ m

$\beta_i = -6,\ \beta_j = 8,\ \beta_m = -2$

$\gamma_i = -4,\ \gamma_j = -2,\ \gamma_m = 6$

$$N_i = \frac{1}{44}[24 + (-6)(0) + 0] = 0.545$$

$$N_j = \frac{1}{44}[12 + 8(0) + 0] = 0.278$$

$$N_m = \frac{1}{44}[8 + (-2)(0) + 0] = 0.181$$

By (1)

$$[K] = \frac{1}{4(22)}\begin{bmatrix} -6 & -4 \\ 8 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}\begin{bmatrix} -6 & 8 & -2 \\ -4 & -2 & 6 \end{bmatrix}$$

$$+ \frac{20(8.246)}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} 63.9 & -6.82 & 25.45 \\ & 11.6 & -4.77 \\ \text{Symmetry} & & 61.82 \end{bmatrix} \frac{\text{W}}{°\text{C}}$$

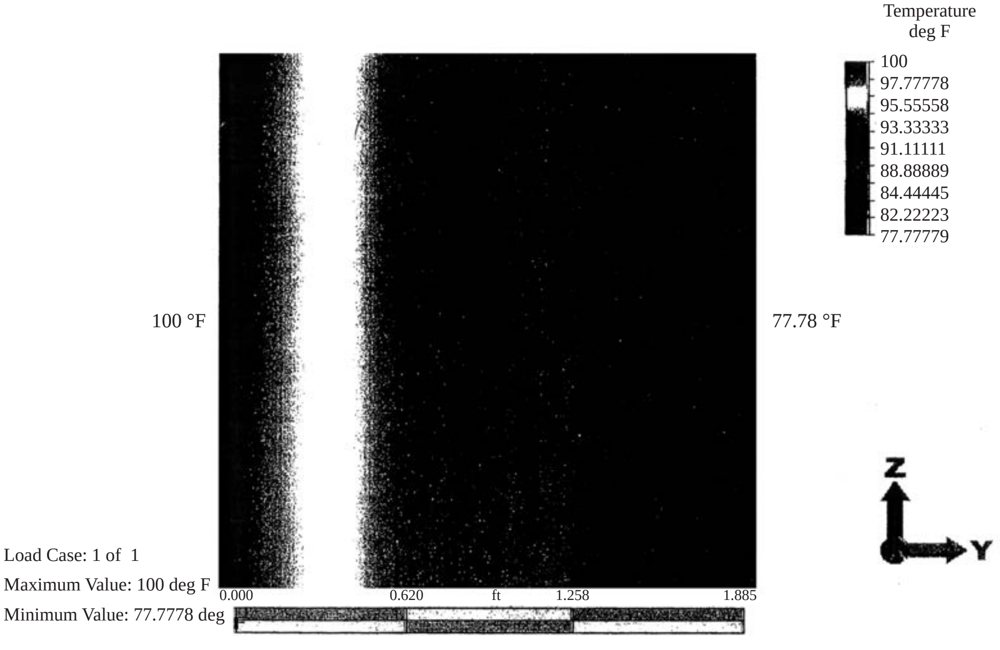
$$\{f\} = \int_v [N]^T \big|_{(0,0)} \; Q^* \, dV + \int_{S_3} hT_\infty [N]^{\text{T}}_{\text{Along } S_3} \, ds$$

$$= Q^* \times \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}_{(0,0)} + \frac{hT_\infty L_{im}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

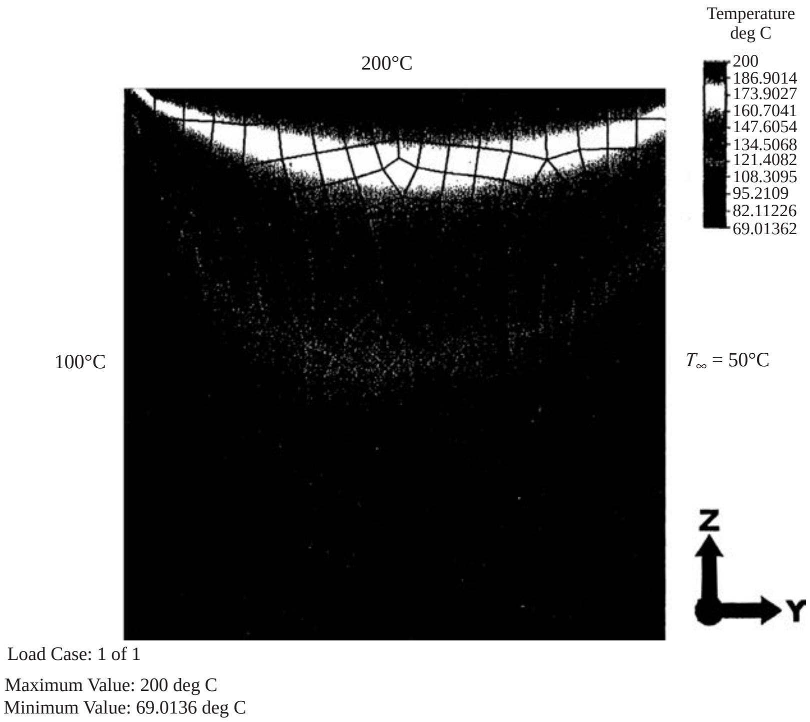
$$= 100 \begin{Bmatrix} 0.545 \\ 0.273 \\ 0.183 \end{Bmatrix} + \frac{(20)(15)(8.246)}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\{f\} = \begin{Bmatrix} 1291 \\ 27.3 \\ 1254 \end{Bmatrix} \text{W}$$

13.21



13.22 For the square plate in figure P13-22, determine the temperature distribution. Let $K_{xx} = K_{yy} = 10 \frac{\text{W}}{\text{m}\cdot{}^{\circ}\text{C}}$, and $h = 20 \frac{\text{W}}{\text{m}^2\cdot{}^{\circ}\text{C}}$. The temperature along the left side is maintained at 100 °C and that along the top side is maintained at 200 °C.



The maximum temperature is along the top edge of the plate and is 200 °C. The smallest temperature is at the lower right edge of the plate and is 69.01 °C.

13.23

HEAT—Problem 13–23
KXX = 1.0 *KYY* = 1.0
CONVECTION COEFF = 0.0
FLUID TEMPERATURE = 0.0
SEMI–BANDWIDTH = 4

NEL	NODE	NUMBER		X(1)	Y(1)
1	1	2	3	0.0000	2.0000
2	2	5	3	0.0000	0.0000
3	3	5	4	0.7500	1.0000
4	1	3	4	0.0000	2.0000
5	4	5	6	1.5000	2.0000
6	5	e	6	1.5000	0.0000
7	6	8	7	2.2500	1.0000
8	4	6	7	1.5000	2.0000

X(2)	Y(2)	X(3)	Y(3)
0.0000	0.0000	0.7500	1.0000
1.5000	0.0000	0.7500	1.0000

1.5000	0.0000	1.5000	2.0000
0.7500	1.0000	1.5000	2.0000
1.5000	0.0000	2.2500	1.0000
3.0000	0.0000	2.2500	1.0000
3.0000	0.0000	3.0000	2.0000
2.2500	1.0000	3.0000	2.0000

PRESCRIBED NODAL TEMPERATURE VALUES

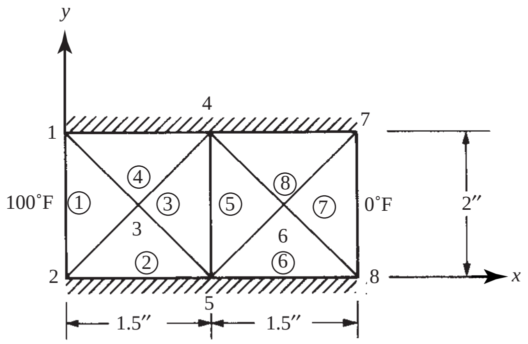
1	0.10000E+03
2	0.10000E+03
7	0.00000E+00
8	0.00000E+00

RESULTING NODAL TEMPERATURE VALUES

1	0.10000E+03	2	0.10000E+03
3	0.75000E+02	4	0.50000E+02
5	0.50000E+02		
6	0.25000E+02	7	0.00000E+00
8	0.00000E+00		

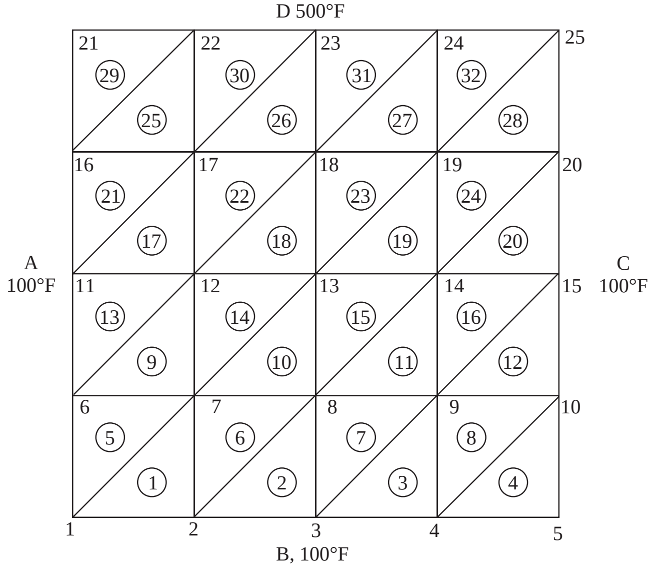
ELEMENT RESULTANTS

ELEMENT	GRAD (X)	GRAD (Y)	AVE TEMP
1	–0.3333E+02	0.0000E+00	0.9167E+02
2	–0.3333E+02	0.0000E+00	0.7500E+02
3	–0.3333E+02	–0.1907E–05	0.5833E+02
4	–0.3333E+02	–0.1907E–05	0.7500E+02
5	–0.3333E+02	–0.1907E–05	0.4167E+02
6	–0.3333E+02	0.0000E+00	0.2500E+02
7	–0.3333E+02	0.0000E+00	0.8333E+01
8	–0.3333E+02	–0.1907E+05	0.2500E+02



5	0.20000E + 03
10	0.10647E + 03
15	0.15418E + 03
20	0.85000E + 02
25	0.13588E + 03
30	0.16754E + 03
35	0.13246E + 03
40	0.85000E + 02

13.24

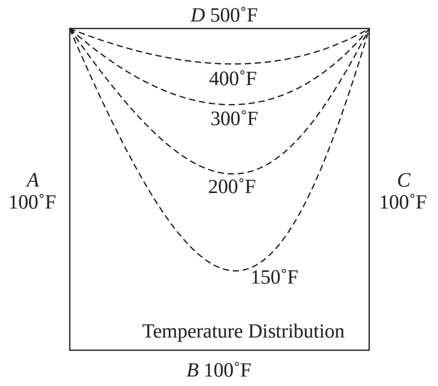


PRESCRIBED NODAL VALUES

1	0.10000E+03	15	0.10000E+03
2	0.10000E+03	16	0.10000E+03
3	0.10000E+03	20	0.10000E+03
4	0.10000E+03	21	0.50000E+03
5	0.10000E+03	22	0.50000E+03
6	0.10000E+03	23	0.50000E+03
10	0.10000E+03	24	0.50000E+03
11	0.10000E+03	25	0.50000E+03

NODAL VALUES, LOADING CASE 1

1	0.10000E+03	2	0.10000E+03
3	0.10000E+03	4	0.10000E+03
5	0.10000E+03		
6	0.10000E+03	7	0.12857E+03
8	0.13929E+03	9	0.12857E+03
10	0.10000E+03		
11	0.10000E+03	12	0.17500E+03
13	0.20000E+03	14	0.17500E+03
15	0.10000E+03		
16	0.10000E+03	17	0.27143E+03
18	0.31071E+03	19	0.27143E+03
20	0.10000E+03		
21	0.50000E+03	22	0.50000E+03
23	0.50000E+03	24	0.50000E+03
25	0.50000E+03		



13.25 Same model as in Problem 13.24

Now convection from right side and bottom.

Convection from side 1 of element 1

Convection from side 1 of element 2

Convection from side 1 of element 3

Convection from side 1 of element 4

Convection from side 2 of element 4

Convection from side 2 of element 12

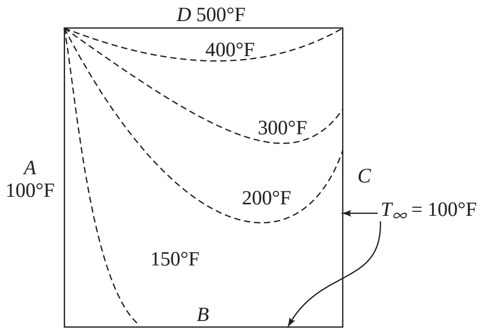
Convection from side 2 of element 20

Convection from side 2 of element 28

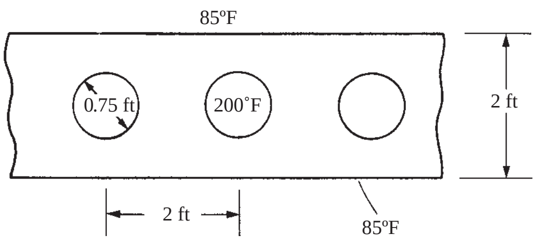
ELEMENT RESULTANTS

ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.1618E+03	0.6971E+02	0.1328E+03
2	0.9794E+02	0.1099E+03	0.1659E+03
3	–0.1227E+00	0.1318E+03	0.1759E+03
4	0.4868E+02	–0.5679E+02	0.1473E+03
5	0.2315E+03	0.0000E+00	0.1193E+03
6	0.1381E+03	0.6971E+02	0.1636E+03
7	0.2178E+02	0.1099E+03	0.1851E+03
8	–0.6407E+02	0.1318E+03	0.1815E+03
9	0.2315E+03	0.1631E+03	0.1522E+03
10	0.1381E+03	0.2262E+03	0.1997E+03
11	0.2178E+02	0.2176E+03	0.2142E+03
12	–0.6407E+02	–0.5051E+02	0.1830E+03
13	0.3946E+03	0.0000E+00	0.1329E+03
14	0.2012E+03	0.1631E+03	0.2018E+03
15	0.1319E+02	0.2262E+03	0.2312E+03
16	–0.3322E+03	0.2176E+03	0.2064E+03
17	0.3946E+03	0.3565E+03	0.1955E+03
18	0.2012E+03	0.4142E+03	0.2667E+03
19	0.1319E+02	0.4841E+03	0.2915E+03
20	–0.3322E+03	0.1633E+03	0.3072E+03
21	0.7512E+03	0.0000E+00	0.1626E+03

22	0.2589E+03	0.3565E+03	0.2797E+03
23	0.8312E+02	0.4142E+03	0.3249E+03
24	0.6042E+02	0.4841E+03	0.3380E+03
25	0.7512E+03	0.3488E+03	0.2959E+03
26	0.2539E+03	0.5900E+03	0.3801E+03
27	0.8312E+02	0.5068E+03	0.4086E+03
28	0.6042E+02	0.4464E+03	0.4206E+03
29	0.0000E+00	0.1600E+04	0.3667E+03
30	0.0000E+00	0.8488E+03	0.4293E+03
31	0.0000E+00	0.5900E+03	0.4508E+03
32	0.0000E+00	0.5068E+03	0.4579E+03



13.26

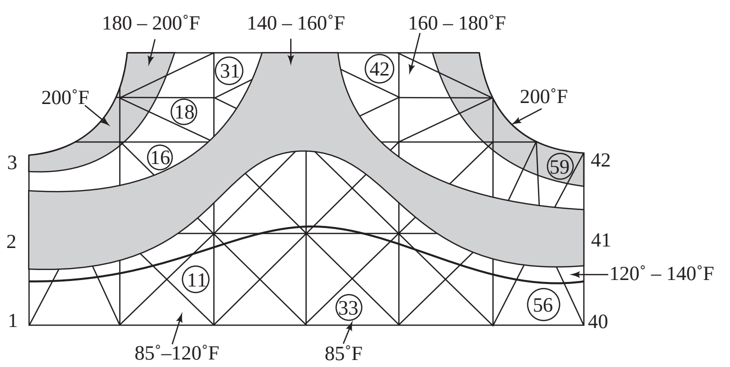


PRESCRIBED NODAL TEMPERATURE VALUES

1	0.85000E+02
3	0.20000E+03
5	0.20000E+03
6	0.85000E+02
9	0.20000E+03
12	0.20000E+03
13	0.85000E+02
20	0.85000E+02
26	0.85000E+02
33	0.20000E+03
34	0.85000E+02
37	0.20000E+03
39	0.20000E+03
40	0.85000E+02
42	0.20000E+03

RESULTING NODAL TEMPERATURE VALUES

1	0.85000E+02	2	0.13937E+03	3	0.20000E+03	4	0.13737E+03
6	0.85000E+02	7	0.13246E+03	8	0.18216E+03	9	0.20000E+03
11	0.14805E+03	12	0.20000E+03	13	0.85000E+02	14	0.12341E+03
16.	0.16387E+03	17	0.16754E+03	18	0.10324E+03	19	0.13588E+03
21	0.11956E+03	22	0.14636E+03	23	0.15741E+03	24	0.10324E+03
26	0.85000E+02	27	0.12341E+03	28	0.15418E+03	29	0.16387E+03
31	0.10647E+03	32	0.14805E+03	33	0.20000E+03	34	0.85000E+02
36	0.18216E+03	37	0.20000E+03	38	0.13737E+03	39	0.20000E+03
41	0.13937E+03	42	0.20000E+03				



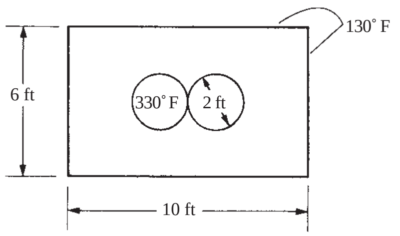
Finite element model
(62 elements, 42 nodes)
(Element resutlants are given below)

ELEMENT RESULTANTS

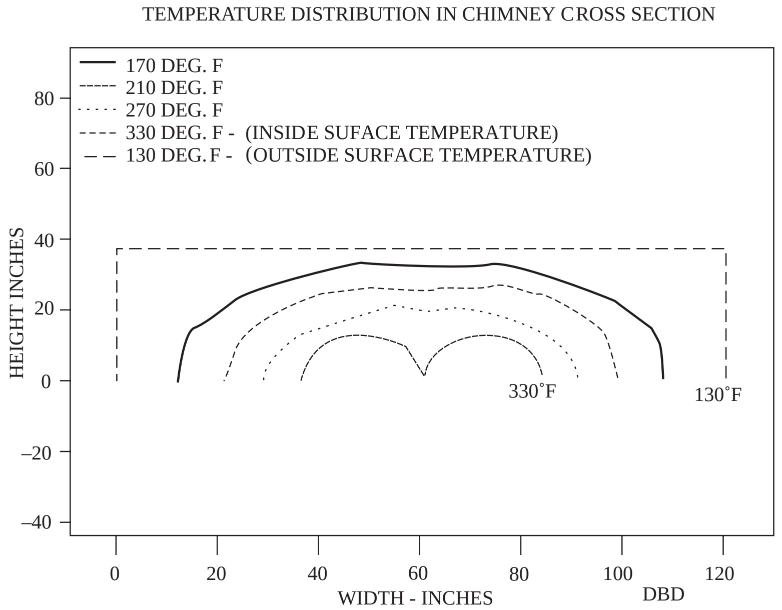
ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	–0.1194E+02	0.1631E+03	0.1206E+03
2	–0.1194E+02	0.2079E+03	0.1589E+03
3	–0.4697E+02	0.1879E+03	0.1791E+03
4	–0.1071E+03	0.1071E+03	0.1941E+03
5	–0.1071E+03	0.1491E+03	0.1715E+03
6	–0.2951E+02	0.1879E+03	0.1566E+03
7	0.2787E–05	0.1571E+03	0.1025E+03
8	–0.2951E+02	0.1424E+03	0.1183E+03
9	–0.1357E+02	0.1424E+03	0.1080E+03
10	–0.2092E–04	0.1288E+03	0.9216E+02
11	–0.1357E+02	0.1152E+03	0.1050E+03
12	–0.2714E+02	0.1288E+03	0.1208E+03
13	–0.5554E+02	0.1491E+03	0.1542E+03
14	–0.2714E+02	0.1207E+03	0.1346E+03
15	–0.5554E+02	0.9231E+02	0.1419E+03
16	–0.8394E+02	0.1207E+03	0.1615E+03
17	–0.8394E+02	0.1071E+03	0.1788E+03
18	–0.1084E+03	0.5818E+02	0.1727E+03

19	–0.1084E+03	0.2197E+02	0.1771E+03
20	–0.1113E+03	0.2783E+02	0.1892E+03
21	–0.5775E+01	0.1152E+03	0.1039E+03
22	0.2190E–04	0.1095E+03	0.9108E+02
23	–0.5775E+01	0.1037E+03	0.1026E+03
24	–0.1155E+02	0.1095E+03	0.1154E+03
25	–0.1751E+02	0.9231E+02	0.1378E+03
26	–0.1155E+02	0.8635E+02	0.1263E+03
27	–0.1751E+02	0.8039E+02	0.1339E+03
28	–0.2347E+02	0.8635E+02	0.1455E+03
29	–0.2347E+02	0.5818E+02	0.1548E+03
30	–0.3597E+02	0.3317E+02	0.1559E+03
31	–0.3037E+02	0.2197E+02	0.1629E+03
32	0.5775E+01	0.1037E+03	0.1026E+03
33	0.6646E–05	0.1095E+03	0.9108E+02
34	0.5775E+01	0.1152E+03	0.1039E+03
35	0.1155E+02	0.1095E+03	0.1154E+03
36	0.1751E+02	0.8039E+02	0.1339E+03
37	0.1155E+02	0.8635E+02	0.1263E+03
38	0.1751E+02	0.9231E+02	0.1378E+03
39	0.2347E+02	0.8635E+02	0.1455E+03
40	0.2347E+02	0.5818E+02	0.1548E+03
41	0.3597E+02	0.3317E+02	0.1559E+03
42	0.3037E+02	0.2197E+02	0.1629E+03
43	0.1357E+02	0.1152E+03	0.1050E+03
44	0.3809E–05	0.1288E+03	0.9216E+02
45	0.1357E+02	0.1424E+03	0.1080E+03
46	0.2715E+02	0.1288E+03	0.1208E+03
47	0.5554E+02	0.9231E+02	0.1419E+03
48	0.2715E+02	0.1207E+03	0.1346E+03
49	0.5554E+02	0.1491E+03	0.1542E+03
50	0.8394E+02	0.1207E+03	0.1615E+03
51	0.8394E+02	0.1071E+03	0.1788E+03
52	0.1084E+03	0.5818E+02	0.1727E+03
53	0.1084E+03	0.2197E+02	0.1771E+03
54	0.1113E+03	0.2783E+02	0.1892E+03
55	0.2951E+02	0.1424E+03	0.1183E+03
56	0.9467E–06	0.1571E+03	0.1025E+03
57	0.1194E+02	0.1631E+03	0.1206E+03
58	0.1194E+02	0.2079E+03	0.1589E+03
59	0.4697E+02	0.1879E+03	0.1791E+03
60	0.2951E+02	0.1879E+03	0.1566E+03
61	0.1071E+03	0.1491E+03	0.1715E+03
62	0.1071E+03	0.1071E+03	0.1941E+03

13.27

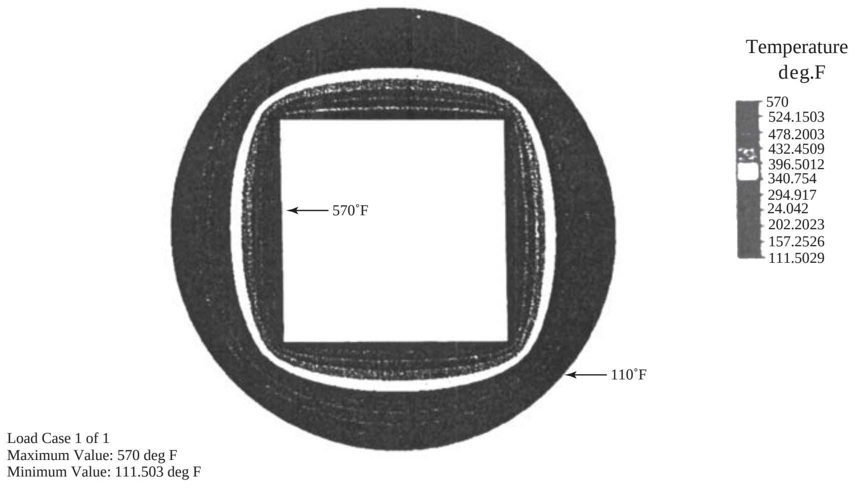


130˚ F
6 ft
330˚ F
2 ft
10 ft

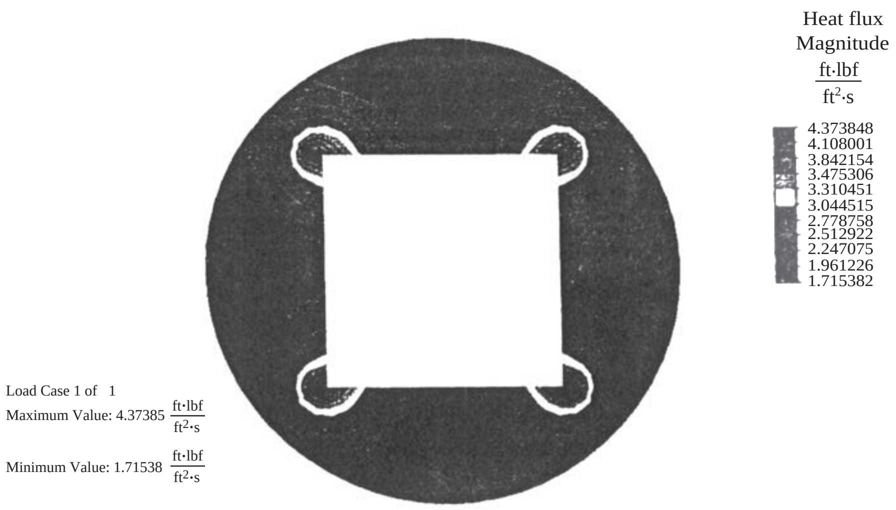


TEMPERATURE DISTRIBUTION IN CHIMNEY CROSS SECTION
170 DEG. F
210 DEG. F
270 DEG. F
330 DEG. F - (INSIDE SUFACE TEMPERATURE)
130 DEG. F - (OUTSIDE SURFACE TEMPERATURE)
HEIGHT INCHES
WIDTH - INCHES
330˚F
130˚F
DBD

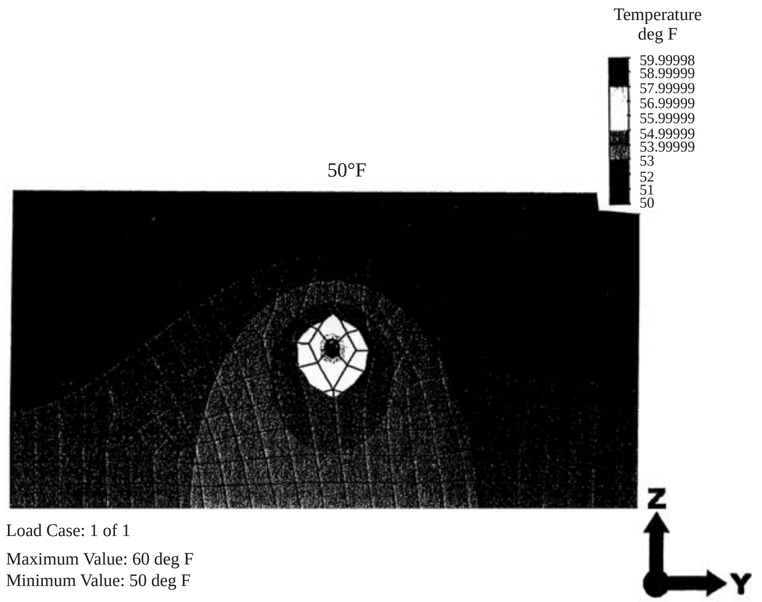
13.28



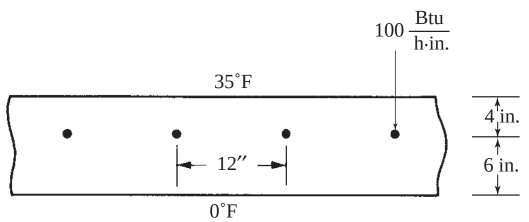
570˚F
110˚F
Temperature
deg.F
570
524.1503
478.2003
432.4509
396.5012
340.754
294.917
24.042
202.2023
157.2526
111.5029
Load Case 1 of 1
Maximum Value: 570 deg F
Minimum Value: 111.503 deg F

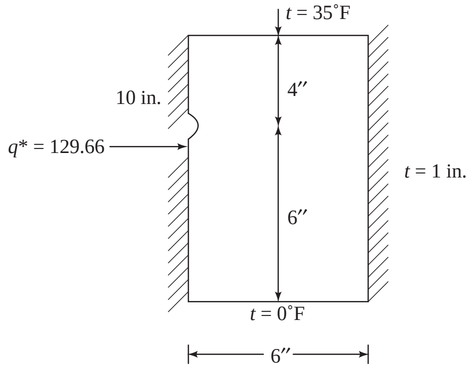


13.29 The temperature distribution of the earth is shown below with a 60 °F oil pipe 15 ft under the earth's surface at 50 °F.



13.30



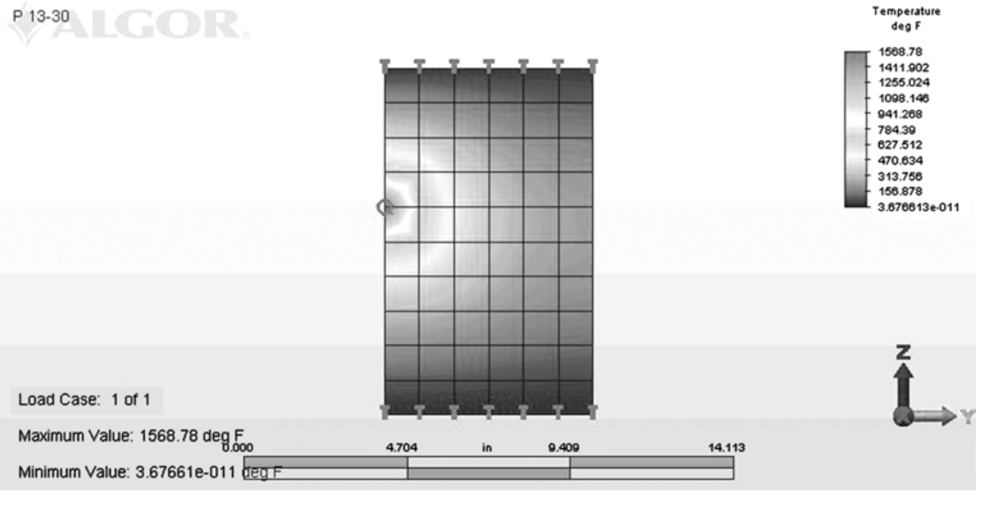


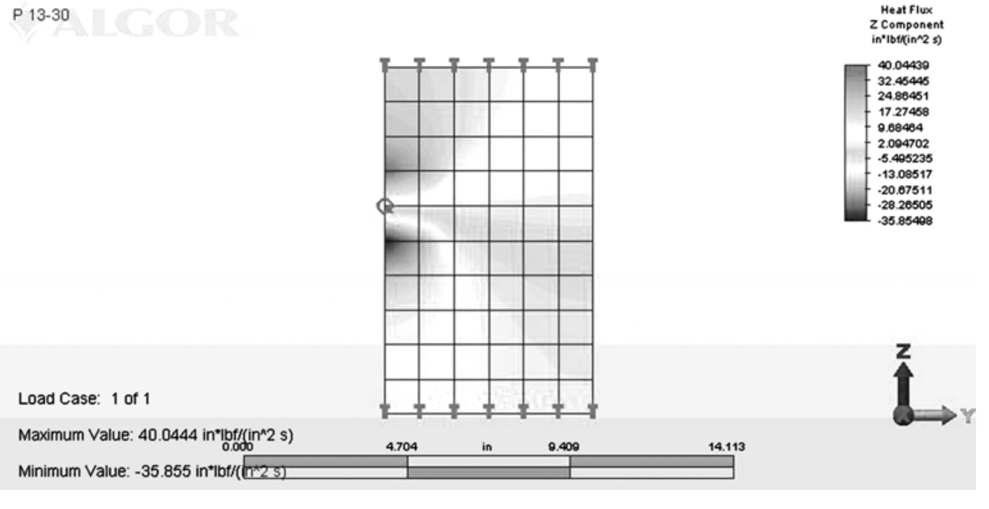
$$K = 0.50 \; \frac{\text{Btu}}{\text{h}\cdot\text{ft}\cdot{}^\circ\text{F}} \; \frac{778\ \text{lb}\cdot\text{ft}}{1\,\text{Btu}} \; \frac{1\,\text{h}}{3600\,\text{s}} = 0.108056 \; \frac{\text{in.}\cdot\text{lb}}{\text{s}\cdot\text{in.}\cdot{}^\circ\text{F}}$$

$$q^* = 50 \; \frac{\cancel{\text{Btu}}}{\cancel{\text{hr}}}\left[\frac{1\,\cancel{\text{hr}}}{60\,\cancel{\text{min}}}\right]\left[\frac{1\,\cancel{\text{min}}}{60\text{s}}\right]\left[\frac{778\,\text{lbf}\ \cancel{\text{ft}}}{1\,\cancel{\text{Btu}}}\right]\left[\frac{12\,\text{in}}{1\,\cancel{\text{ft}}}\right] = 129.66 \; \frac{\text{lbf}\cdot\text{in.}}{\text{s}}$$

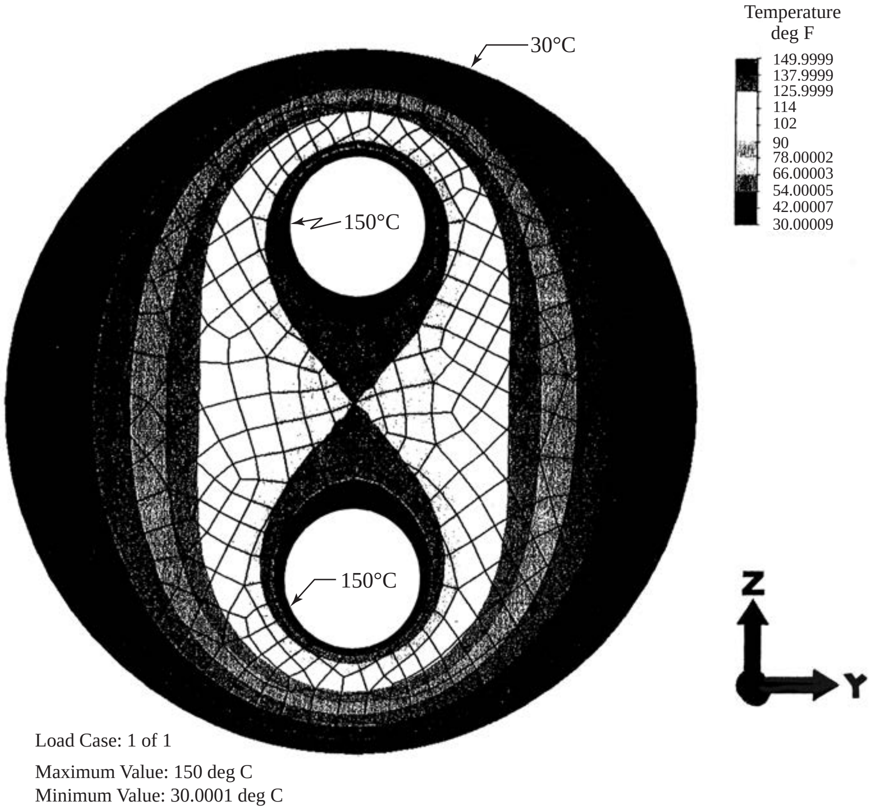
Heat source

$$q^* = 129.66 \; \frac{\text{lbf}\cdot\text{in.}}{\text{s}}$$

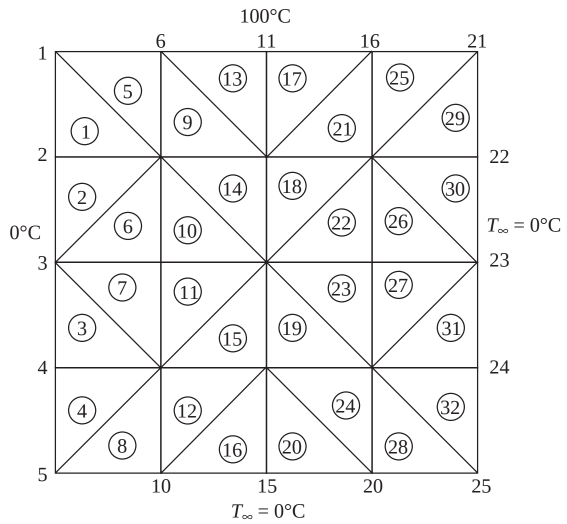




13.31



13.32



KXX = 10.0 KYY = 10.0

CONVECTION COEFF = 10.0

FLUID TEMPERATURE = 0.0

SEMI-BANDWIDTH = 7

NEL	NODE	NUMBER		X(1)	Y(1)	X(2)	Y(2)	X(3)	Y(3)
1	1	2	7	0.0000	1.0000	0.0000	0.8000	0.2000	0.8000
2	2	3	8	0.0000	0.8000	0.0000	0.5000	0.2000	0.5000

3	3	4	8	0.0000	0.5000	0.0000	0.2000	0.2000	0.5000
4	4	5	9	0.0000	0.2000	0.0000	0.0000	0.2000	0.2000
5	1	7	6	0.0000	1.0000	0.2000	0.8000	0.2000	1.0000
6	2	8	7	0.0000	0.8000	0.2000	0.5000	0.2000	0.8000
7	4	9	8	0.0000	0.2000	0.2000	0.2000	0.2000	0.5000
8	5	10	9	0.0000	0.0000	0.2000	0.0000	0.2000	0.2000
9	6	7	12	0.2000	1.0000	0.2000	0.8000	0.5000	0.8000
10	7	8	13	0.2000	0.8000	0.2000	0.5000	0.5000	0.5000
11	8	9	13	0.2000	0.5000	0.2000	0.2000	0.5000	0.5000
12	9	10	14	0.2000	0.2000	0.2000	0.0000	0.5000	0.2000
13	6	12	11	0.2000	1.0000	0.5000	0.8000	0.5000	1.0000
14	7	13	12	0.2000	0.8000	0.5000	0.5000	0.5000	0.8000
15	9	14	13	0.2000	0.2000	0.5000	0.2000	0.5000	0.5000
16	10	15	14	0.2000	0.0000	0.5000	0.0000	0.5000	0.2000
17	11	12	16	0.5000	1.0000	0.5000	0.8000	0.8000	1.0000
18	12	13	17	0.5000	0.8000	0.5000	0.5000	0.8000	0.8000
19	13	14	19	0.5000	0.5000	0.5000	0.2000	0.8000	0.2000
20	14	15	20	0.5000	0.2000	0.5000	0.0000	0.8000	0.0000
21	12	17	16	0.5000	0.8000	0.8000	0.8000	0.8000	1.0000
22	13	18	17	0.5000	0.5000	0.8000	0.5000	0.8000	0.8000
23	13	19	18	0.5000	0.5000	0.8000	0.2000	0.8000	0.5000
24	14	20	19	0.5000	0.2000	0.8000	0.0000	0.8000	0.2000
25	16	17	21	0.8000	1.0000	0.8000	0.8000	1.0000	1.0000
26	17	18	22	0.8000	0.8000	0.8000	0.5000	1.0000	0.8000
27	18	19	24	0.8000	0.5000	0.8000	0.2000	1.0000	0.2000
28	19	20	25	0.8000	0.2000	0.8000	0.0000	1.0000	0.0000
29	17	22	21	0.8000	0.8000	1.0000	0.8000	1.0000	1.0000
30	18	23	22	0.8000	0.5000	1.0000	0.5000	1.0000	0.8000
31	18	24	23	0.8000	0.5000	1.0000	0.2000	1.0000	0.5000
32	19	25	24	0.8000	0.2000	1.0000	0.0000	1.0000	0.2000

CONVECTION FROM SIDE 1 OF ELEMENT 8
CONVECTION FROM SIDE 1 OF ELEMENT 16
CONVECTION FROM SIDE 2 OF ELEMENT 20
CONVECTION FROM SIDE 2 OF ELEMENT 28
CONVECTION FROM SIDE 3 OF ELEMENT 29
CONVECTION FROM SIDE 3 OF ELEMENT 30
CONVECTION FROM SIDE 2 OF ELEMENT 31
CONVECTION FROM SIDE 2 OF ELEMENT 32

PRESCRIBED NODAL TEMPERATURE VALUES

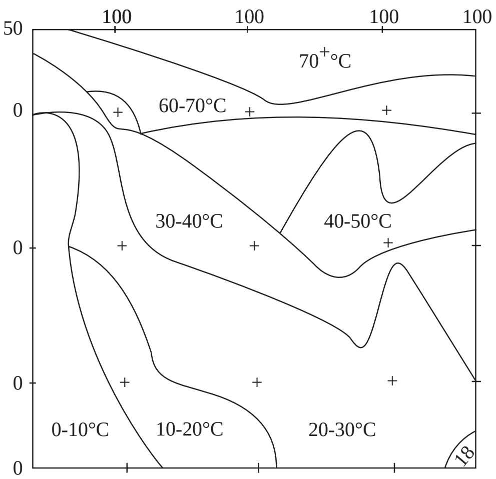
2	0.00000E+00
3	0.00000E+00
4	0.00000E+00
5	0.00000E+00
6	0.10000E+03
11	0.10000E+03
16	0.10000E+03
21	0.10000E+03

RESULTING NODAL TEMPERATURE VALUES

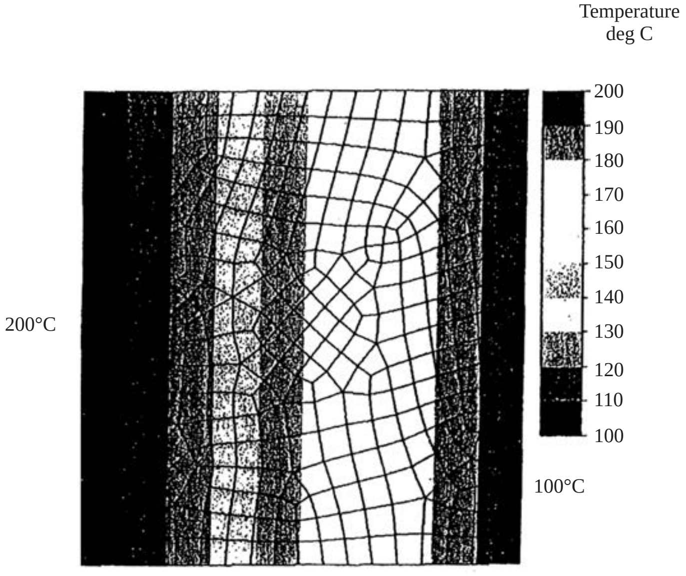
1	0.50000E+02	2	0.00000E+00	3	0.00000E+00	4	0.00000E+00
6	0.10000E+03	7	0.47910E+02	8	0.21010E+02	9	0.11377E+02
11	0.10000E+03	12	0.68538E+02	13	0.38137E+02	14	0.22992E+02
16	0.10000E+03	17	0.69017E+02	18	0.40007E+02	19	0.25421E+02
21	0.10000E+03	22	0.66669E+02	23	0.39117E+02	24	0.22699E+02
10	0.85870E+01	15	0.18000E+02	20	0.20040E+02	25	0.17598E+02

ELEMENT RESULTANTS

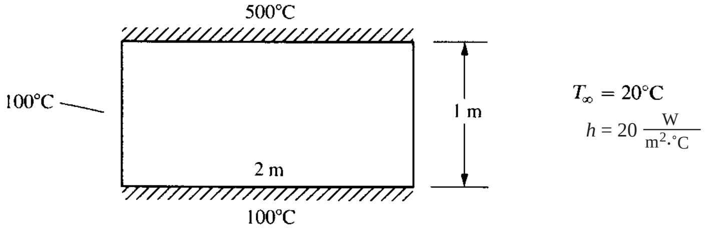
ELEMENTS	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.2395E+03	0.2500E+03	0.3264E+02
2	0.1051E+03	0.0000E+00	0.7003E+01
3	0.1051E+03	0.0000E+00	0.7003E+01
4	0.5688E+02	0.0000E+00	0.3792E+01
5	0.2500E+03	0.2605E+03	0.6597E+02
6	0.2395E+03	0.8967E+02	0.2297E+02
7	0.5688E+02	0.3211E+02	0.1080E+02
8	0.4293E+02	0.1395E+02	0.6655E+01
9	0.6876E+02	0.2605E+03	0.7215E+02
10	0.5709E+02	0.8967E+02	0.3569E+02
11	0.5709E+02	0.3211E+02	0.2351E+02
12	0.3872E+02	0.1395E+02	0.1432E+02
13	0.2862E–04	0.1573E+03	0.8951E+02
14	0.6876E+02	0.1013E+03	0.5153E+02
15	0.3872E+02	0.5048E+02	0.2417E+02
16	0.3138E+02	0.2496E+02	0.1653E+02
17	0.3336E–04	0.1573E+03	0.8951E+02
18	0.1595E+01	0.1013E+03	0.5856E+02
19	0.8097E+01	0.5048E+02	0.2885E+02
20	0.6800E+01	0.2496E+02	0.2034E+02
21	0.1595E+01	0.1549E+03	0.7918E+02
22	0.6234E+01	0.9670E+02	0.4905E+02
23	0.6234E+01	0.4862E+02	0.3452E+02
24	0.8097E+01	0.2691E+02	0.2282E+02
25	– 0.3310E–04	0.1549E+03	0.8967E+02
26	– 0.1174E+02	0.9670E+02	0.5856E+02
27	– 0.1361E+02	0.4862E+02	0.2938E+02
28	– 0.1221E+02	0.2691E+02	0.2102E+02
29	– 0.1174E+02	0.1667E+03	0.7856E+02
30	– 0.4451E+01	0.9184E+02	0.4860E+02
31	– 0.4451E+01	0.5473E+02	0.3394E+02
32	– 0.1361E+02	0.2550E+02	0.2191E+02

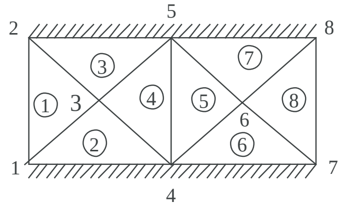


13.34



13.35





KXX = 10.0 KYY = 10.0

CONVECTION COEFF = 20.0

FLUID TEMPERATURE = 20.0

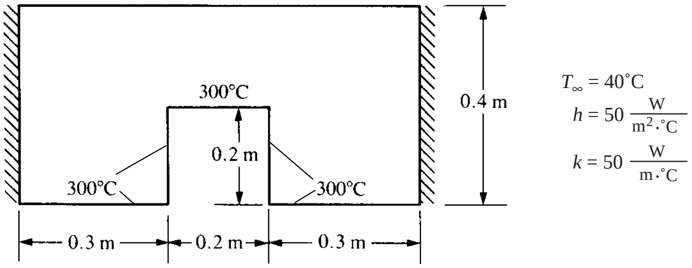
SEMI-BANDWIDTH = 4

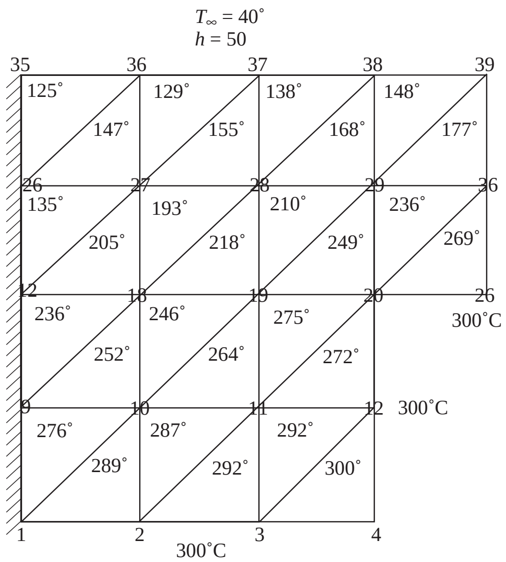
NEL	NODE	NUMBER		X(I)	Y(1)	X(2)	Y(2)	X(3)	Y(3)
1	1	3	2	0.0000	0.0000	0.5000	0.5000	0.0000	1.0000
2	1	4	3	0.0000	0.0000	1.0000	0.0000	0.5000	0.5000
3	2	3	5	0.0000	1.0000	0.5000	0.5000	1.0000	1.0000
4	3	4	5	0.5000	0.5000	1.0000	0.0000	1.0000	1.0000
5	4	6	5	1.0000	0.0000	1.5000	0.5000	1.0000	1.0000
6	4	7	6	1.0000	0.0000	2.0000	0.0000	1.5000	0.5000
7	5	6	8	1.0000	1.0000	1.5000	0.5000	2.0000	1.0000
8	6	7	8	1.5000	0.5000	2.0000	0.0000	2.0000	1.0000

ELEMENT RESULTANTS

ELEMENT	GRAD(X)	GRAD(Y)	AVE TEMP
1	0.1000E+03	0.2000E+03	0.2167E+03
2	0.0000E+00	0.3000E+03	0.1500E+03
3	0.2000E+03	0.3000E+03	0.3500E+03
4	0.1000E+03	0.4000E+03	0.2833E+03
5	– 0.1867E+03	0.4000E+03	0.2689E+03
6	0.1333E+02	0.2000E+03	0.1400E+03
7	– 0.3867E+03	0.2000E+03	0.2733E+03
8	– 0.1867E+03	0.7629E–05	0.1444E+03

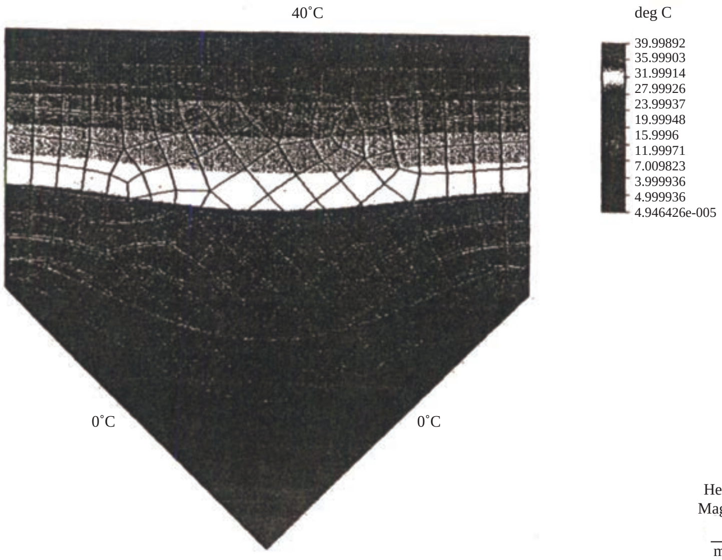
13.36

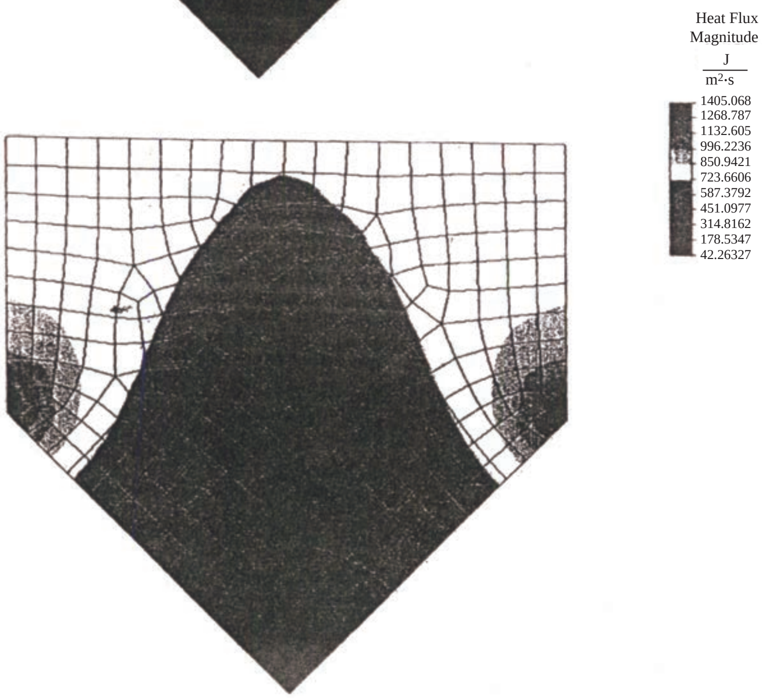




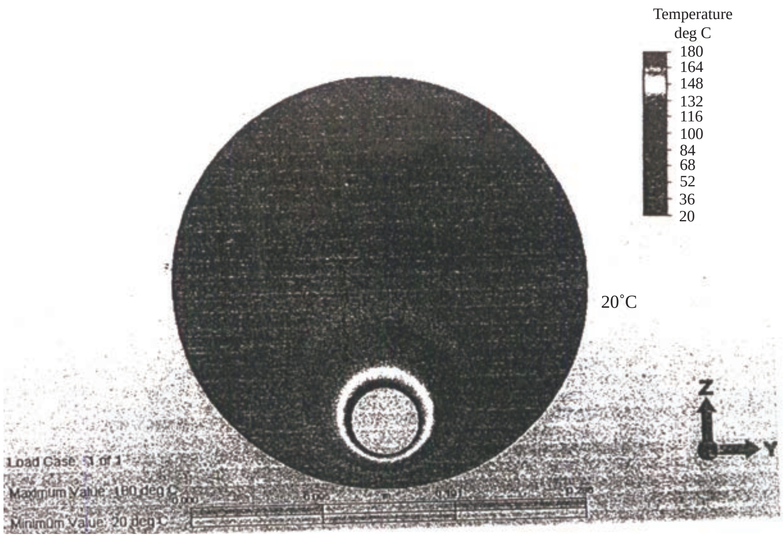
Results showing one-half of model

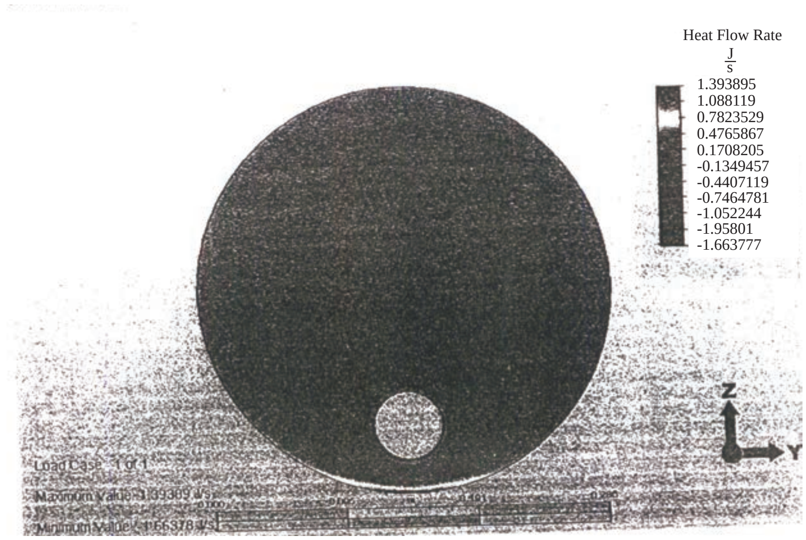
13.37 Determine the temperature distribution and rate of heat flow through the plain carbon steel ingot shown in Figure P13-36. Let $k = 60 \frac{W}{m \cdot K}$ for the steel. The top surface is held at 40C, while the underside surface is held at 0C. Assume that no heat is lost from the sides.


40˚C
deg C
39.99892
35.99903
31.99914
27.99926
23.99937
19.99948
15.9996
11.99971
7.009823
3.999936
4.999936
4.946426e-005
0˚C
0˚C

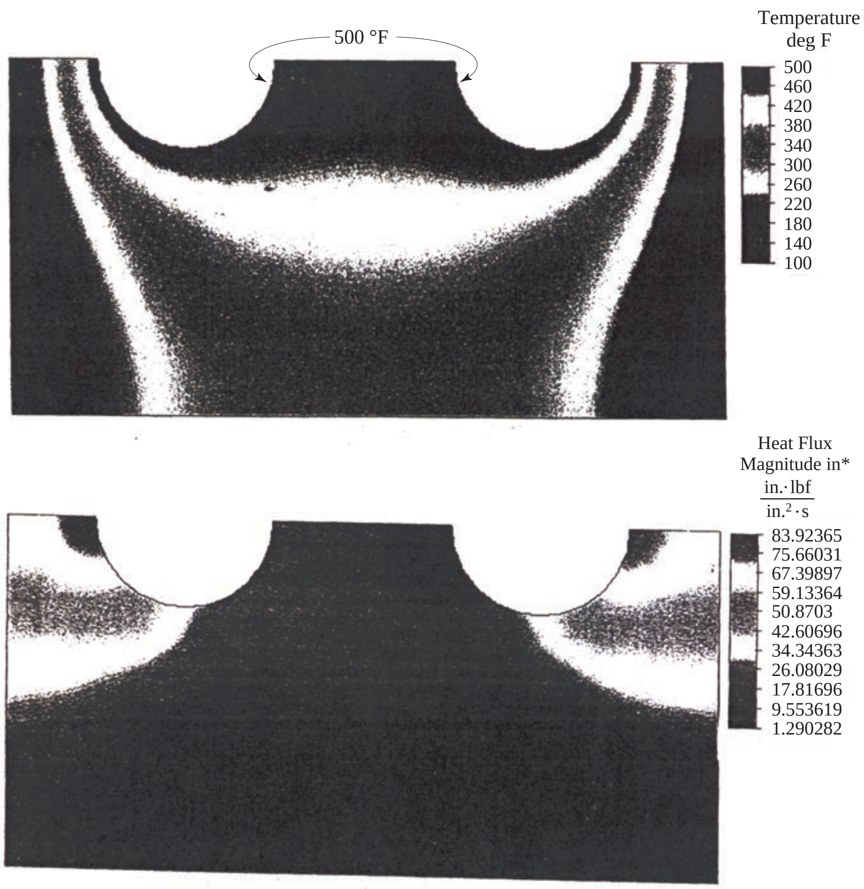

Heat Flux
Magnitude
J / m²·s
1405.068
1268.787
1132.605
996.2236
850.9421
723.6606
587.3792
451.0977
314.8162
178.5347
42.26327

13.38

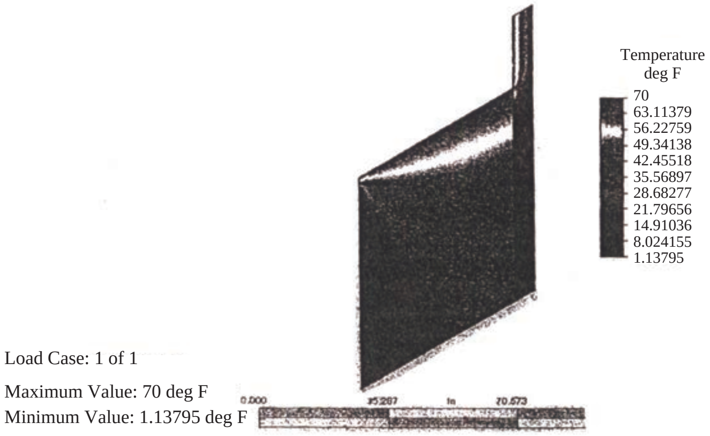

Temperature
deg C
180
164
148
132
116
100
84
68
52
36
20
20˚C
Z
Y
Load Case: 1 of 1
Maximum Value: 180 deg C
Minimum Value: 20 deg C

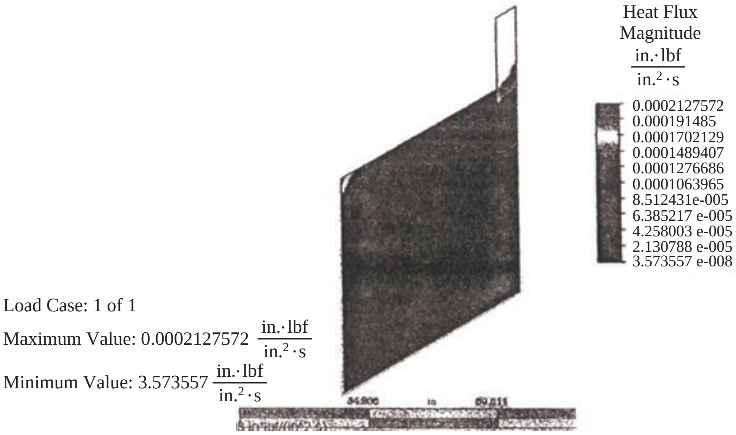


13.39

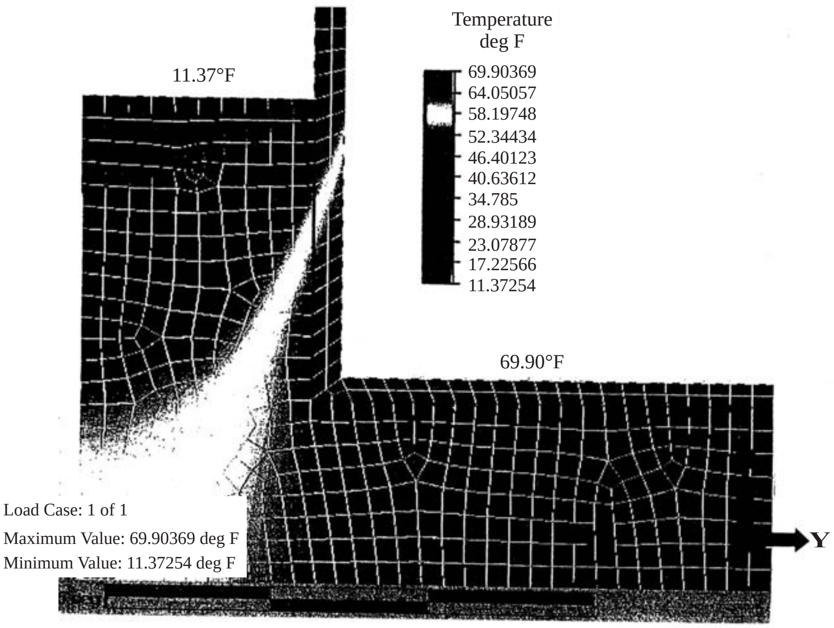


13.40 For the basement wall, determine the temperature distribution and the heat transfer through the wall and soil.

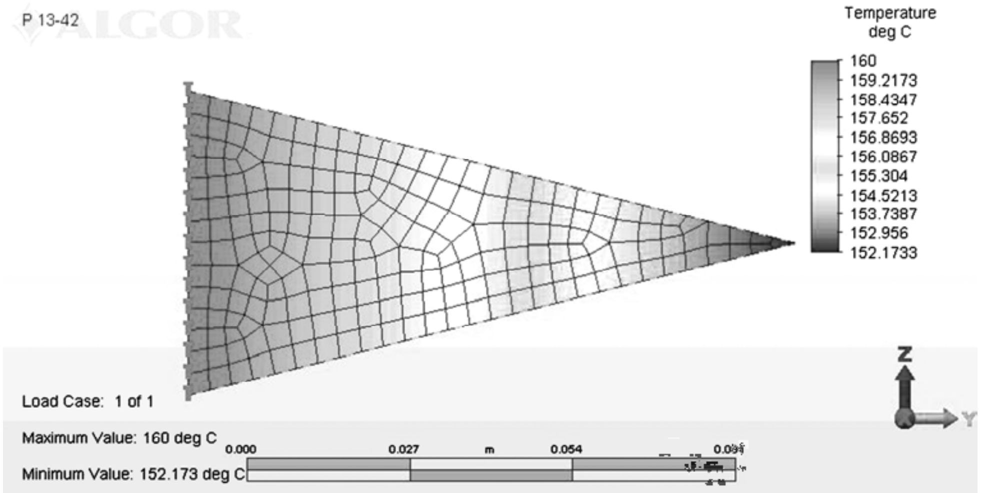

Temperature
deg F
70
63.11379
56.22759
49.34138
42.45518
35.56897
28.68277
21.79656
14.91036
8.024155
1.13795
Load Case: 1 of 1
Maximum Value: 70 deg F
Minimum Value: 1.13795 deg F


Heat Flux
Magnitude
$\frac{\text{in.}\cdot\text{lbf}}{\text{in.}^2\cdot\text{s}}$
0.0002127572
0.000191485
0.0001702129
0.0001489407
0.0001276686
0.0001063965
8.512431e-005
6.385217 e-005
4.258003 e-005
2.130788 e-005
3.573557 e-008
Load Case: 1 of 1
Maximum Value: 0.0002127572 $\frac{\text{in.}\cdot\text{lbf}}{\text{in.}^2\cdot\text{s}}$
Minimum Value: 3.573557 $\frac{\text{in.}\cdot\text{lbf}}{\text{in.}^2\cdot\text{s}}$

13.41


Temperature
deg F
69.90369
64.05057
58.19748
52.34434
46.40123
40.63612
34.785
28.93189
23.07877
17.22566
11.37254
11.37°F
69.90°F
Load Case: 1 of 1
Maximum Value: 69.90369 deg F
Minimum Value: 11.37254 deg F
Y

13.42 Aluminum fins ($k = 170\ \frac{\text{W}}{\text{m}\cdot\text{K}}$) with triangular profiles shown in Figure P13.41 are used to remove heat from a surface with temperature of 160°C. The temperature of the surrounding air is 25°C. The natural convection coefficient is $h = 25\ \frac{\text{W}}{\text{m}^2\cdot\text{K}}$. Determine the temperature distribution throughout and the heat loss from a typical fin.



13.44 The Allen Wrench, shown in Figure 1, is exposed at one end to at temperature of 300 K, while the other end has a heat flux of 10 $\frac{\text{W}}{\text{m\^{}2}}$. Determine the temperature distribution throughout the wrench. It has a thermal conductivity of 43.6 $\frac{\text{W}}{\text{m}\cdot\text{K}}$ and a specific heat capacity of 0.000486 $\frac{\text{J}}{\text{g}\cdot\text{K}}$.

Part dimensions

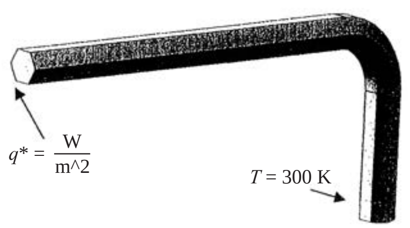


Figure 1: Original Model with forces

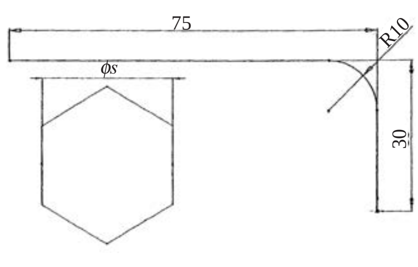


Figure 2: Dimension

Boundary condition

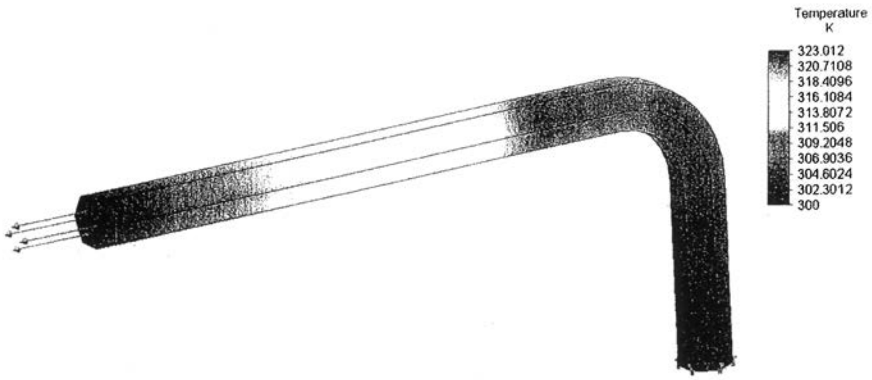
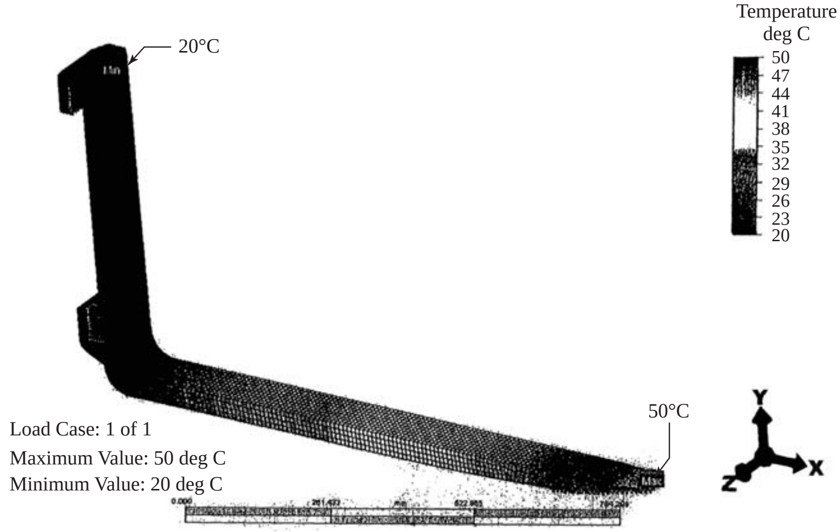
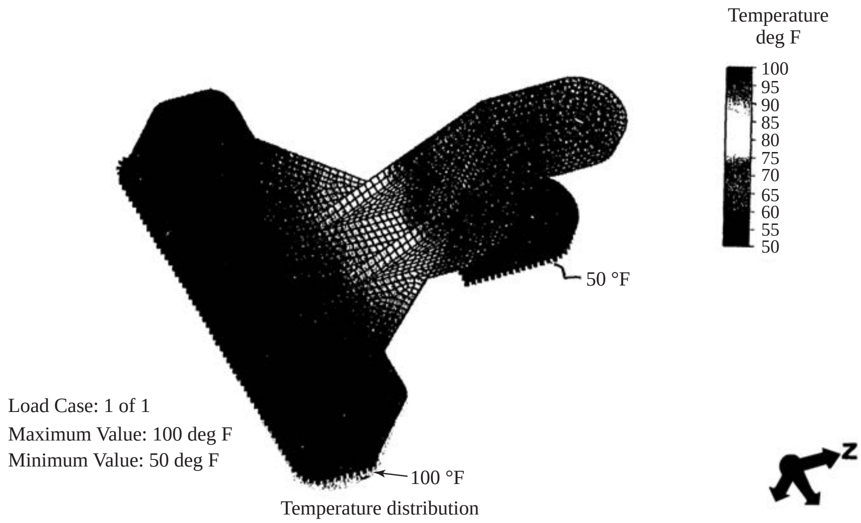


Figure 3: Temperature Distribution (K)

13.45 Temperature distribution



13.46 The thermal aspect of this component is that the base has an applied temperature of 100 °F. This is theoretically due to the warmed aluminum plate being heated by the electric motor. The lower surface of the lower finger has an applied surface temperature of 50 °F.



Temperature distribution

13.48

$$\frac{K_{xx}A}{L} = \frac{(0.017)\frac{\pi(1.5'')^2}{4\times 144}}{\frac{10''}{4}/\frac{12''}{1'}} = 0.001\frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}} \text{ Small so neglect}$$

$$\dot{m}c = 10\ (0.24) = 2.4\ \frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$\frac{hPL}{6} = \frac{3}{6}\left(\frac{\pi\,(1.5'')}{\frac{12''}{1'}}\right)\frac{\frac{10''}{4}}{\frac{12''}{1'}} = 0.04096\ \frac{\text{Btu}}{\text{h}\cdot{}^\circ\text{F}}$$

$$hPLT_\infty = (3)\ \pi\ \frac{(1.5)}{12}\ \left(\frac{\frac{10}{4}}{12}\right)(200^\circ\text{F}) = 49.087\ \frac{\text{Btu}}{\text{h}}$$

$$[k^{(1)}] = \frac{2.4}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix} + 0.04096\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix}-1.118 & 1.24091\\ -1.159 & 1.2818\end{bmatrix}$$

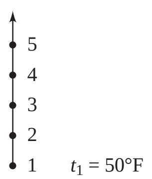
$$[k^{(2)}] = [k^{(3)}] = [k^{(4)}] = [k^{(1)}] \text{ also}$$

$$[K] = \begin{pmatrix}1.2818-1.118 & 1.2409 & 0 & 0\\ -1.159 & 1.2818-1.118 & 1.2409 & 0\\ 0 & -1.159 & 1.2818-1.118 & 1.2409\\ 0 & 0 & -1.159 & 1.2818\end{pmatrix}$$

$$[K] = \begin{pmatrix}0.164 & 1.241 & 0 & 0\\ -1.159 & 0.164 & 1.241 & 0\\ 0 & -1.159 & 0.164 & 1.241\\ 0 & 0 & -1.159 & 1.282\end{pmatrix}$$

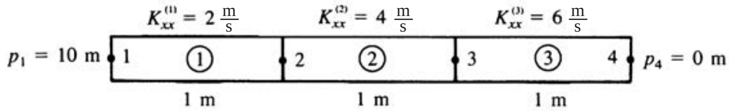
$$\{F\} = \begin{pmatrix}49.087 + 1.159(50)\\ 49.087\\ 49.087\\ 24.5435\end{pmatrix}$$

$$[K^{-1}]\ \{F\} = \begin{pmatrix}64.719\\ 77.715\\ 89.747\\ 100.296\end{pmatrix} = \begin{Bmatrix}t_2\\ t_3\\ t_4\\ t_5\end{Bmatrix}$$



Chapter 14

14.1



Global [K]

$$[K] = 4\frac{\text{m}^2}{\text{s}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\{P\} = \begin{Bmatrix} 10 \\ p_2 \\ p_3 \\ 0 \end{Bmatrix}$$

Accounting for the boundary conditions

$$p_1 = 10, \quad p_4 = 0, \text{ we get}$$

$$\begin{bmatrix} 12 & -8 \\ -8 & 20 \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 40 \\ 0 \end{Bmatrix}$$

Solving

$$p_2 = 4.545 \text{ m}, \quad p_3 = 1.818 \text{ m}$$

$$v_x^{(1)} = -k_{xx}^{(1)}\left[-\frac{1}{L}\ \frac{1}{L}\right]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$= -2\,[-1\ 1]\begin{Bmatrix} 40 \\ 4.545 \end{Bmatrix} = 10.91\ \frac{\text{m}}{\text{s}}$$

$$v_x^{(2)} = -4\,[-4.545 + 1.818] = 10.91\ \frac{\text{m}}{\text{s}}$$

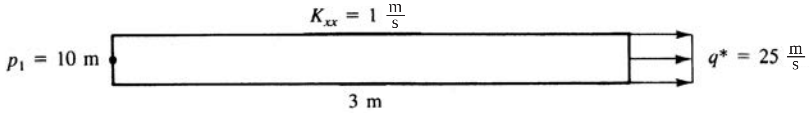
$$v_x^{(3)} = -6(-1.818) = 10.91\ \frac{\text{m}}{\text{s}}$$

$$Q_f^{(1)} = Av_x^{(1)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

$$Q_f^{(2)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

$$Q_f^{(3)} = 21.82\ \frac{\text{m}^3}{\text{s}}$$

14.2



$$[\underline{k}^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{1\times 2}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[f^{(3)}] = q^* A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -25\ \frac{\text{m}}{\text{s}}\ (2\text{m}^2) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50 \end{Bmatrix}$$

Global equations

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} p_1 = 10 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \end{Bmatrix}$$

Rewriting

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ -50 \end{Bmatrix}$$

Solving

$$p_2 = -15\text{ m},\quad p_3 = -40\text{ m},\quad p_4 = -65\text{ m}$$

$$v_x^{(1)} = -K_{xx}[B] \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} 10 \\ -15 \end{Bmatrix} \ (L = 1)$$

$$= 25\ \frac{\text{m}}{\text{s}}$$

$$v_x^{(2)} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{Bmatrix} -15 \\ -40 \end{Bmatrix} = 25\ \frac{\text{m}}{\text{s}}$$

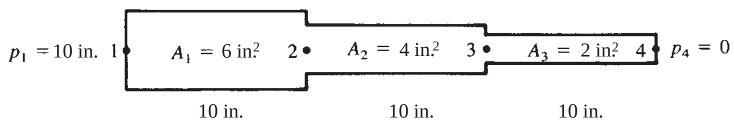
$$v_x^{(3)} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix} \begin{Bmatrix} -40 \\ -65 \end{Bmatrix} = 25\ \frac{\text{m}}{\text{s}}$$

Volumetric flow rates

$$Q_1 = Q_2 = Q_3 = v_x^{(1)} A_1 = v_x^{(2)} A_2 = v_x^{(3)} A_3$$

$$= 25\ (2) = 50\ \frac{\text{m}^3}{\text{s}}$$

14.3



$$[k^{(1)}] = 0.6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(2)}] = 0.4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(3)}] = 0.2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 & -0.6 & 0 & 0 \\ -0.6 & 1.0 & -0.4 & 0 \\ 0 & -0.4 & 0.6 & -0.2 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix} \begin{Bmatrix} 10\text{ in.} \\ p_2 \\ p_3 \\ p_4 = 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Using the 2nd and 3rd equations above

$$\begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.6 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$$

$$p_2 = 8.182 \text{ in.}, \quad p_3 = 5.455 \text{ in.}$$

$$v_x^{(1)} = -K_{xx}[B]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 10 \\ 8.182 \end{Bmatrix}$$

$$v_x^{(1)} = 0.182 \ \frac{\text{in.}}{\text{s}}$$

$$v_x^{(2)} = -1\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 8.182 \\ 5.455 \end{Bmatrix}$$

$$v_x^{(2)} = 0.273 \ \frac{\text{in.}}{\text{s}}$$

$$v_x^{(3)} = -\begin{bmatrix} -\frac{1}{10} & \frac{1}{10} \end{bmatrix}\begin{Bmatrix} 5.455 \\ 0 \end{Bmatrix}$$

$$= 0.545 \ \frac{\text{in.}}{\text{s}}$$

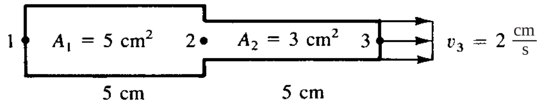
$$Q_f^{(1)} = A\,v_x^{(1)} = (6 \text{ in.}^2)\left(0.182 \ \frac{\text{in.}}{\text{s}}\right)$$

$$= 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(2)} = 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(3)} = 1.091 \ \frac{\text{in.}^3}{\text{s}}$$

14.4



$$[k^{(1)}] = \frac{2 \times 5}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[k^{(2)}] = \frac{2 \times 3}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & -\frac{6}{5} \\ -\frac{6}{5} & \frac{6}{5} \end{bmatrix}$$

$$F_3 = -2 \times 3 = -6 \ \frac{\text{cm}^3}{\text{s}}, \quad \text{Negative, as water flows out of right edge}$$

Assemble global equations

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3.2 & -1.2 \\ 0 & -1.2 & 1.2 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \end{Bmatrix}$$

Now assume $p_1 = 0$

$$\begin{bmatrix} 3.2 & -1.2 \\ -1.2 & 1.2 \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -6 \end{Bmatrix}$$

Solving for p_2 and p_3

$$p_2 = -3 \qquad p_3 = -8$$

Velocities

$$v_x^{(1)} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$= -2\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \end{bmatrix}\begin{Bmatrix} 0 \\ -3 \end{Bmatrix}$$

$$v_x^{(1)} = 1.2\ \frac{\text{cm}}{\text{s}} \rightarrow \text{(to right)}$$

$$v_x^{(2)} = -2\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} \end{bmatrix}\begin{Bmatrix} -3 \\ -8 \end{Bmatrix}$$

$$v_x^{(2)} = 2\ \frac{\text{cm}}{\text{s}} \rightarrow \text{(to right)}$$

Flow rates

$$Q_f^{(1)} = v_x^{(1)}\, A = \left(1.2\frac{\text{cm}}{\text{s}}\right)(5\ \text{cm}^2)$$

$$Q_f^{(1)} = 6\ \frac{\text{cm}^3}{\text{s}}$$

$$Q_f^{(2)} = v_x^{(2)}\, A = \left(\frac{2\ \text{cm}}{\text{s}}\right)(3\ \text{cm}^2)$$

$$Q_f^{(2)} = 6\ \frac{\text{cm}^3}{\text{s}}$$

14.5

$$[k] = \int_V [B]^T [D] [B]\, dV$$

For 1-D formulation

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad [D] = [K_{xx}]$$

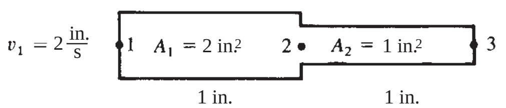
$$\therefore \qquad [k] = \int_v \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [K_{xx}] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dV$$

For element with constant cross sectional area A

$$[k] = \int_0^L K_{xx}\begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} A\, dx$$

$$[k] = \frac{K_{xx}\, A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

14.6



$$K_{xx} = \frac{1}{10}\frac{\text{in.}}{\text{s}}$$

$$[k^{(1)}] = \frac{2\times 1}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(2)}] = \frac{1\times 1}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

'Force' at node 1

$$F_{1x} = \left(\frac{2\text{ in.}}{\text{s}}\right)(2\text{ in.}^2) = 4\ \frac{\text{in.}^3}{\text{s}}$$

Assume $\quad p_3 = 0$

Assemble equations

$$\begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2+0.1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}$$

Solving

$$p_1 = 60\text{ in.}, \quad p_2 = 40\text{ in.}$$

Velocities

$$v_x^{(1)} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$$v_x^{(1)} = -\frac{1}{10}\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 60 \\ 40 \end{Bmatrix} = 2.0\ \frac{\text{in.}}{\text{s}}$$

$$v_x^{(2)} = -\frac{1}{10}\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 40 \\ 0 \end{Bmatrix} = 4.0\ \frac{\text{in.}}{\text{s}}$$

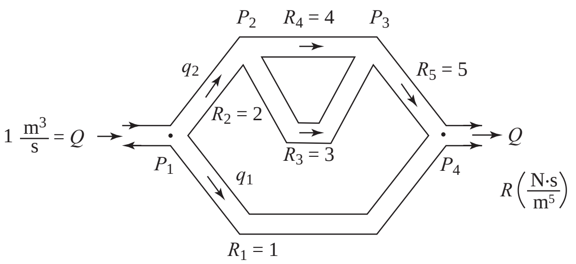
Volumetric flow rates

$$Q_f^{(1)} = v_x^{(1)}\,A^{(1)} = \left(2\,\frac{\text{in.}}{\text{s}}\right)(2\text{ in.}^2) = 4\ \frac{\text{in.}^3}{\text{s}}$$

$$Q_f^{(2)} = v_x^{(2)}\,A^{(2)} = \left(4\,\frac{\text{in.}}{\text{s}}\right)(1\text{ in.}^2) = 4\ \frac{\text{in.}^3}{\text{s}}$$

14.7

(a)



$$[k^{(1)}] = \frac{1}{1}\begin{array}{c} \begin{matrix} 1 & \quad 4 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \frac{1}{2}\begin{array}{c} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{1}{3}\begin{array}{c} \begin{matrix} 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(4)}] = \frac{1}{4}\begin{array}{c} \begin{matrix} 2 & \quad 3 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \frac{1}{5}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble $[k]$'s

$$\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1+\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}+\frac{1}{3}+\frac{1}{4} & -\frac{1}{2}-\frac{1}{4} \\ 0 & -\frac{1}{3}-\frac{1}{4} & \frac{1}{3}+\frac{1}{4}+\frac{1}{5} \end{bmatrix} \\ \frac{\text{m}^5}{\text{N}\cdot\text{s}} \end{array} \begin{array}{c} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} \\ \frac{\text{N}}{\text{m}^2} \end{array} = \begin{array}{c} \begin{Bmatrix} 1 = Q_1 \\ 0 \\ 0 \end{Bmatrix} \\ \frac{\text{m}^3}{\text{s}} \end{array}$$

Solve in Mathcad for p_1, p_2, p_3

$p_1 = 0.897 \frac{\text{N}}{\text{m}^2}$, $\quad p_2 = 0.691 \frac{\text{N}}{\text{m}^2}$, $\quad p_3 = 0.515 \frac{\text{N}}{\text{m}^2}$

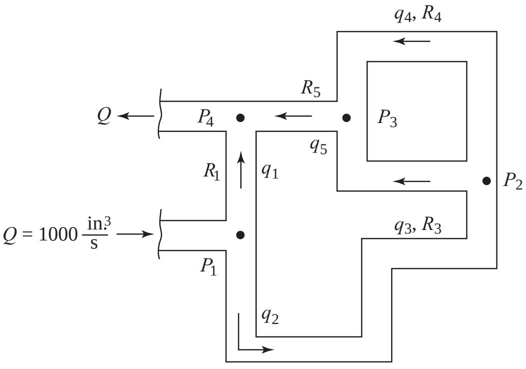
$q_1 = \frac{\Delta P}{R_1} = \frac{P_1 - P_4}{R_1} = 0.897$, $\quad q_2 = \frac{\Delta P}{R_2} = \frac{P_1 - P_2}{R_2} = 0.103$, $\quad q_3 = \frac{P_2 - P_3}{R_3} = 0.059$

$q_4 = \frac{P_2 - P_3}{R_4} = 0.044$, $\quad q_5 = \frac{P_3 - 0}{R_5} = 0.103$ $\quad$ (all qs in $\frac{\text{m}^3}{\text{s}}$)

$q_1 + q_4 = 0.103$ (check equals q_2)

$q_1 + q_5 = 1$ (check equals $Q = 1$)

(b)



$R_1 = 10 \ \frac{\text{lb}\cdot\text{s}}{\text{in.}^5}$

$R_2 = 20$

$R_3 = 30$

$R_4 = 40$

$R_5 = 50$

$$[k^{(1)}] = \frac{1}{10}\begin{array}{c} \begin{array}{cc} 1 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \frac{1}{20}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \frac{1}{30}\begin{array}{cc} 2 & 3 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(4)}] = \frac{1}{40}\begin{array}{cc} 2 & 3 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(5)}] = \frac{1}{50}\begin{array}{cc} 3 & 4 \\ \end{array}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assemble

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\begin{bmatrix} \frac{1}{10}+\frac{1}{20} & -\frac{1}{20} & 0 \\ -\frac{1}{20} & \frac{1}{20}+\frac{1}{30}+\frac{1}{40} & -\frac{1}{30}-\frac{1}{40} \\ 0 & -\frac{1}{30}-\frac{1}{40} & \frac{1}{30}+\frac{1}{40}+\frac{1}{50} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{\text{in.}^5}{\text{lb}\cdot\text{s}} \qquad \frac{\text{lb}}{\text{in.}^2} \qquad \frac{\text{in.}^3}{\text{s}}$$

Solve for p_1, p_2, p_3 using Mathcad as

$p_1 = 8971$ psi, $\quad p_2 = 6912$ psi, $\quad p_3 = 5147$ psi,

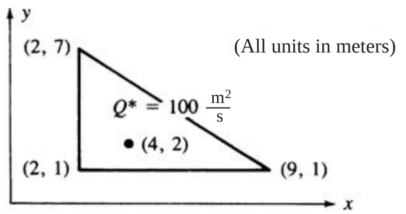
$q_1 = \dfrac{P_1 - P_4 = 0}{R_1} = 897$, $\quad q_2 = \dfrac{P_1 - P_2}{R_2} = 103$, $\quad q_3 = \dfrac{P_2 - P_3}{R_3} = 59$

$q_4 = \dfrac{P_2 - P_3}{R_4} = 44$, $\quad q_5 = \dfrac{P_3 - 0}{R_5} = 103$ $\quad$ (all qs in $\dfrac{\text{in.}^3}{\text{s}}$)

$q_3 + q_4 = 103$ (check equals q_2)

$q_1 + q_5 = 1000$ (check equals Q)

14.8



$$\begin{Bmatrix} f_{Qi} \\ f_{Qs} \\ f_{Qm} \end{Bmatrix} = Q^*\, t \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix}\Bigg|_{\substack{x=4 \\ y=2}}$$

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$\alpha_i = x_i y_m - y_j x_m = 61$$

$$\alpha_j = y_j x_m - x_i y_m = -12$$

$$\alpha_m = x_i y_j - y_i x_j = -7$$

$$\beta_i = y_i - y_m = -6 \quad \gamma_i = x_m - x_j = -7$$

$$\beta_j = y_m - y_i = 6 \qquad \gamma_j = x_i - x_m = 0$$

$$\beta_m = y_i - y_j = 0 \qquad \gamma_m = x_j - x_i = 7$$

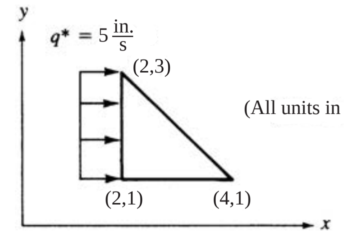
$$N_i\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(61 - 6(4) - 7(2)) = \frac{23}{42}$$

$$N_j\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(-12 + 6(4)) = \frac{12}{42}$$

$$N_m\Big|_{\substack{x=4\\y=2}} = \frac{1}{52}\,(-7 + 7(2)) = \frac{7}{42}$$

$$\therefore \quad \{f_Q\} = \left(100\frac{\text{m}^2}{\text{s}}\right)\left(\frac{I_m}{42}\right)\begin{Bmatrix}23\\12\\7\end{Bmatrix} = \begin{Bmatrix}54.76\\28.57\\16.67\end{Bmatrix}\frac{\text{m}^3}{\text{s}}$$

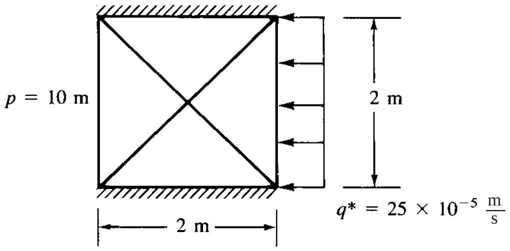
14.9



$$\{f\} = \frac{q^* L t}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}f_1\\f_2\\f_3\end{Bmatrix} = \frac{5\times 2\times 1}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} = \begin{Bmatrix}5\\0\\5\end{Bmatrix}\frac{\text{in.}^3}{\text{s}}$$

14.10



From Equations (14.3.16)

$$[K] = \begin{bmatrix}25 & 0 & 0 & 0 & -25\\ & 25 & 0 & 0 & -25\\ & & 25 & 0 & -25\\ & & & 25 & -25\\ \text{Symmetry} & & & & 100\end{bmatrix}\times 10^{-5}$$

$$F = \begin{Bmatrix} 0 \\ 25 \\ 25 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-5} \ \frac{\text{m}^3}{\text{s}}$$

Boundary conditions $p_1 = p_4 = 10$ m

$$\therefore \quad \begin{bmatrix} 25 & 0 & -25 \\ & 25 & -25 \\ \text{Symmetry} & & 100 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \\ p_5 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 \\ 500 \end{Bmatrix}$$

Solving

$$p_2 = 12 \text{ m}, \ p_3 = 12 \text{ m}, \ p_5 = 11 \text{ m}$$

14.11

(33 nodes, 56 elements)

FLUIDS PROBLEM 14–11

BOUNDARY VALUES

NODAL FORCES

LOADING	CASE 1
0	0.00000E+00
LOADING	CASE 2
1	0.50000E+04

PRESCRIBED NODAL VALUES

26	0.50000E+03
27	0.50000E+03
28	0.50000E+03
29	0.50000E+03
30	0.50000E+03
31	0.50000E+03
32	0.50000E+03
33	0.50000E+03

FLUIDS PROBLEM 14–11

NODAL VALUES, LOADING CASE 1

1	– 0.46404E+03	2	0.37597E+02
5	0.14359E+07	6	– 0.26094E+03
9	0.45441E+02	10	0.25835E+03
13	0.25077E+03	14	– 0.57844E+02
17	0.26088E+03	18	0.39845E+03
3	– 0.17302E+02	4	– 0.16157E+03
7	– 0.75913E+02	8	0.13183E+03
11	0.23328E+03	12	0.19417E+03
15	0.16262E+03	16	0.23693E+03
19	0.38851E+03	20	0.37488E+03
21	0.38550E+03	22	0.28400E+03
25	0.39925E+03	26	0.50000E+03
29	0.50000E+03	30	0.50000E+03
33	0.50000E+03		
23	0.35570E+03	24	0.38834E+03
27	0.50000E+03	28	0.50000E+03
31	0.50000E+03	32	0.50000E+03

FLUIDS PROBLEM 14–11

NODAL VALUES, LOADING CASE 2

1	– 0.49446E+03	2	0.23005E+02
5	0.14476E+07	6	– 0.28970E+03
9	0.30588E+02	10	0.25067E+03
13	0.23855E+03	14	– 0.84934E+02
17	0.25296E+03	18	0.39522E+03
21	0.38006E+03	22	0.27361E+03
25	0.39590E+03	26	0.50000E+03
29	0.50000E+03	30	0.50000E+03
33	0.50000E+03		

3	– 0.33072E+02	4	– 0.18077E+03
7	– 0.96780E+02	8	– 0.34748E+01
11	0.22496E+03	12	0.18421E+03
15	0.14912E+03	16	0.22760E+03
19	0.38500E+03	20	0.37068E+03
23	0.34982E+03	24	0.38433E+03
27	0.50000E+03	28	0.50000E+03
31	0.50000E+03	32	0.50000E+03

FLUIDS PROBLEM 14–11

ELEMENT VELOCITY COMPONENTS

ELEMENT	VEL(X)	VEL(Y)
1	– 0.20066E+05	– 0.52061E+04
2	– 0.13173E+05	– 0.12099E+05
3	0.81240E+04	– 0.12099E+05
4	0.81240E+04	– 0.12094E+05
5	0.81240E+04	0.13833E+05
6	0.28676E+04	0.19089E+05
7	– 0.97320E+04	0.19089E+05
8	– 0.20066E+05	0.87555E+04
9	– 0.88300E+04	0.41014E+04
10	– 0.88300E+04	– 0.29504E+04
11	– 0.12222E+05	– 0.19572E+04
12	– 0.49369E+04	– 0.92425E+04
13	– 0.14055E+05	– 0.14229E+05
14	0.74983E+04	– 0.14229E+05
15	0.81240E+04	– 0.12092E+05
16	0.81240E+04	– 0.12097E+05
17	0.81240E+04	– 0.12099E+05
18	0.81240E+04	0.96035E+04
19	0.21425E+04	0.11355E+05
20	0.24686E+04	0.11029E+05
21	– 0.13329E+04	0.89500E+04
22	– 0.43866E+04	0.89500E+04
23	– 0.48610E+04	0.73297E+04
24	– 0.85681E+04	0.36225E+04
25	– 0.56043E+04	0.23942E+04
26	– 0.56043E+04	– 0.21350E+04
27	– 0.67102E+04	– 0.20719E+04
28	– 0.27549E+04	– 0.60273E+04
29	– 0.41023E+04	– 0.72286E+04
30	0.31961E+04	– 0.72286E+04
31	0.33656E+04	– 0.42572E+04
32	0.40359E+04	– 0.35868E+04
33	0.13674E+05	– 0.14397E+05
34	0.13674E+05	0.70190E+04
35	0.33137E+04	0.76100E+04
36	0.27655E+04	0.81583E+04

37	0.40783E+03	0.60563E+04
38	– 0.27156E+04	0.60563E+04
39	– 0.27727E+04	0.50557E+04
40	– 0.55241E+04	0.23043E+04
41	– 0.40618E+04	0.16983E+04
42	– 0.40618E+04	– 0.16833E+04
43	– 0.45923E+04	– 0.17156E+04
44	– 0.18482E+04	– 0.44597E+04
45	– 0.23311E+04	– 0.50048E+04
46	0.20741E+04	– 0.50048E+04
47	0.20397E+04	– 0.44382E+04
48	0.45798E+04	– 0.18980E+04
49	0.86399E+04	– 0.54948E+04
50	0.86399E+04	0.35806E+04
51	0.48159E+04	0.33480E+04
52	0.23920E+04	0.57719E+04
53	0.12355E+04	0.44664E+04
54	– 0.18510E+04	0.44664E+04
55	– 0.18156E+04	0.38847E+04
56	– 0.40301E+04	0.16702E+04

RESULTANT NODAL VALUES

1	– 0.49446E+03
2	0.23005E+02
3	– 0.33072E+02
4	– 0.18077E+03
5	0.14476E+07
6	– 0.28970E+03
7	– 0.96780E+02
8	– 0.34748E+01
9	0.30588E+02
10	0.25067E+03
11	0.22496E+03
12	0.18421E+03
13	0.23855E+03
14	0.84934E+02
15	0.14912E+03
16	0.22760E+03
17	0.25296E+03
18	0.39522E+03
19	0.38500E+03
20	0.37068E+03
21	0.38006E+03
22	0.27361E+03
23	0.34982E+03
24	0.38433E+03
25	0.39590E+03
26	0.50000E+03
27	0.50000E+03
28	0.50000E+03
29	0.50000E+03
30	0.50000E+03
31	0.50000E+03
32	0.50000E+03
33	0.00000E+00

14.12

BOUNDARY VALUES

NODAL FORCES

LOADING CASE 1

0	0.00000E+00

PRESCRIBED NODAL VALUES

1	0.60000E+01
2	0.60000E+01
3	0.60000E+01
6	0.30000E+01
7	0.30000E+01
8	0.30000E+01

NODAL VALUES, LOADING CASE 1

1	0.60000E+01	2	0.60000E+01
5	0.45000E+01	6	0.30000E+01
3	0.60000E+01	4	0.45000E+01
7	0.30000E+01	8	0.30000E+01

14.13 Use $\frac{1}{4}$ of the whole system due to symmetry

(25 nodes, 32 elements)

NODAL FORCES

LOADING CASE 1

0	0.00000E+00

LOADING CASE 2

25	0.12500E+03 (pumping rate)

PRESCRIBED NODAL VALUES

1	0.10000E+03
6	0.10000E+03
11	0.10000E+03
16	0.10000E+03
21	0.10000E+03

FLUID PROBLEM 14-12

NODAL VALUES, LOADING CASE 1

1	0.10000E+03	2	0.10000E+03
5	0.10000E+03	6	0.10000E+03
9	0.10000E+03	10	0.10000E+03
13	0.10000E+03	14	0.10000E+03
17	0.10000E+03	18	0.10000E+03
21	0.10000E+03	22	0.10000E+03
25	0.10000E+03		
3	0.10000E+03	4	0.10000E+03
7	0.10000E+03	8	0.10000E+03
11	0.10000E+03	12	0.10000E+03
15	0.10000E+03	16	0.10000E+03
19	0.10000E+03	20	0.10000E+03
23	0.10000E+03	24	0.10000E+03

NODAL VALUES, LOADING CASE 2

1	0.10000E+03	2	0.10062E+03
5	0.10202E+03	6	0.10000E+03
9	0.10177E+03	10	0.10209E+03
13	0.10125E+03	14	0.10185E+03
17	0.10063E+03	18	0.10128E+03
21	0.10000E+03	22	0.10063E+03
25	0.10363E+03		
3	0.10121E+03	4	0.10174E+03
7	0.10062E+03	8	0.10122E+03
11	0.10000E+03	12	0.10062E+03
15	0.10231E+03	16	0.10000E+03
19	0.10198E+03	20	0.10277E+03
23	0.10129E+03	24	0.10207E+03

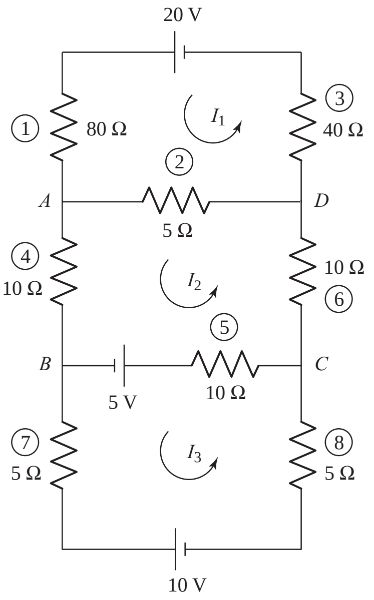
RESULTANT NODAL VALUES

1	0.10000E+03
2	0.10062E+03
3	0.10121E+03
4	0.10174E+03
5	0.10202E+03
6	0.10000E+03
7	0.10062E+03
8	0.10122E+03
9	0.10177E+03
10	0.10209E+03
11	0.10000E+03
12	0.10062E+03
13	0.10125E+03
14	0.10185E+03
15	0.10231E+03
16	0.10000E+03
17	0.10063E+03
18	0.10128E+03
19	0.10198E+03
20	0.10277E+03
21	0.10000E+03
22	0.10063E+03
23	0.10129E+03
24	0.10207E+03
25	0.00000E+00

ELEMENT VELOCITY COMPONENTS

ELEMENT	VEL (X)	VEL (Y)
1	0.12207E–03	0.57220E–05
2	0.79155E–04	0.21935E–04
3	0.26226E–04	0.47684E–05
4	0.17643E–04	0.41962E–04
5	0.11921E–03	0.00000E+00
6	0.92506E–04	0.57220E–05
7	0.12875E–04	0.21935E–04
8	0.33379E–04	0.47684E–05
9	0.12207E–03	0.57220E–05
10	0.94414E–04	– 0.19073E–05
11	0.14305E–04	0.47684E–05
12	0.32902E–04	0.24796E–04
13	0.11921E–03	0.00000E+00
14	0.92506E–04	0.57220E–05
15	0.12875E–04	– 0.19073E–05
16	0.45300E–04	0.47684E–05
17	0.12207E–03	0.57220E–05
18	0.94414E–04	0.28610E–04
19	0.14305E–04	0.35286E–04
20	0.48161E–04	0.95367E–06
21	0.11921E–03	0.00000E+00
22	0.10443E–03	0.57220E–05
23	0.95367E–05	0.28610E–04
24	0.30041E–04	0.35266E–04
25	0.12207E–03	0.57220E–05
26	0.10967E–03	– 0.56267E–04
27	0.17643E–04	– 0.19073E–04
28	0.36240E–04	0.95367E–06
29	0.11921E–03	0.00000E+00
30	0.80585E–04	0.57220E–05
31	0.28133E–04	– 0.56267E–04
32	0.45300E–04	– 0.19073E–04

14.17



$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} I_1 & 0 \end{array} \\ \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} I_1 & 0 \end{array} \\ \begin{bmatrix} 40 & -40 \\ -40 & 40 \end{bmatrix} \end{array}, \quad [k^{(4)}] = \begin{array}{c} \begin{array}{cc} I_2 & 0 \end{array} \\ \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \end{array}$$

$$[k^{(5)}] = \begin{array}{c} \begin{array}{cc} I_2 & I_3 \end{array} \\ \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \end{array}, \quad [k^{(6)}] = \begin{array}{c} \begin{array}{cc} I_2 & 0 \end{array} \\ \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \end{array}$$

$$[k^{(7)}] = \begin{array}{c} \begin{array}{cc} I_3 & 0 \end{array} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}, \quad [k^{(8)}] = \begin{array}{c} \begin{array}{cc} I_3 & 0 \end{array} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{bmatrix} 80+5+40 & -5 & 0 \\ -5 & 5+10+10+10 & -10 \\ 0 & -10 & 10+5+5 \end{bmatrix} \end{array} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix}$$

$$\begin{bmatrix} 125 & -5 & 0 \\ -5 & 35 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 5 \\ -10 \end{Bmatrix}$$

$$I_{AD} = I_1 - I_2$$

$$I_{BC} = I_2 - I_3$$

Using Mathcad

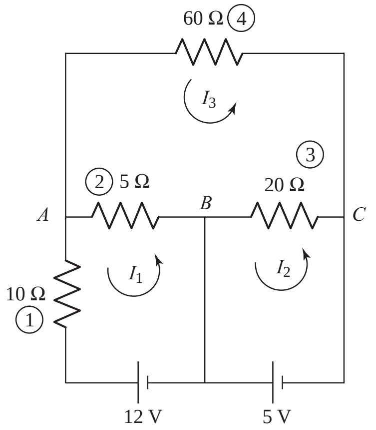
$$[K] = \begin{pmatrix} 125 & -5 & 0 \\ -5 & 35 & -10 \\ 0 & -10 & 20 \end{pmatrix} \qquad [V] = \begin{pmatrix} 20 \\ 5 \\ -10 \end{pmatrix}$$

$$\text{AMPS} = \text{1solve}(K, V) \qquad \text{AMPS} = \begin{pmatrix} 0.161 \\ 0.027 \\ -0.487 \end{pmatrix}$$

$$I_{AD} = \text{AMPS}_0 - \text{AMPS}_1 \qquad I_{AD} = 0.134$$

$$I_{BC} = \text{AMPS}_1 - \text{AMPS}_2 \qquad I_{BC} = 0.513$$

14.18



Resistor element stiffness matrices.

$$[k^{(1)}] = 10\begin{array}{c} \begin{array}{cc} I_1 & 0 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(2)}] = 5\begin{array}{c} \begin{array}{cc} I_1 & I_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$[k^{(3)}] = 20\begin{array}{c} \begin{array}{cc} I_2 & I_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \quad [k^{(4)}] = 60\begin{array}{c} \begin{array}{cc} I_3 & 0 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble equations

$$\begin{array}{c} \begin{array}{ccc} I_1 & I_2 & I_3 \end{array} \\ \begin{bmatrix} 10+5 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 5+20+60 \end{bmatrix} \end{array} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -6 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 15 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 85 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \\ I_3 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -6 \\ 0 \end{Bmatrix}$$

Solve in Mathcad

$I_{AB} = I_1 - I_3$

$I_{BC} = I_2 - I_3$

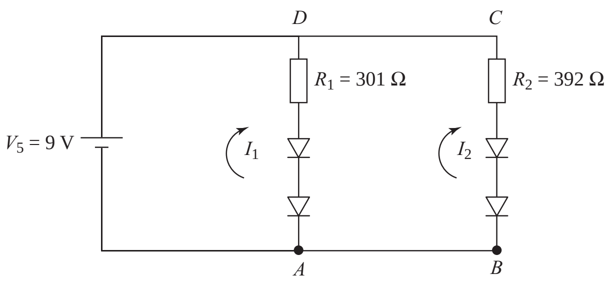
$$[K_1] = \begin{pmatrix} 15 & 0 & -5 \\ 0 & 20 & -20 \\ -5 & -20 & 85 \end{pmatrix} \qquad [V_1] = \begin{pmatrix} -12 \\ -6 \\ 0 \end{pmatrix}$$

$$\text{AMPS}_1 = \text{1solve}\,(K_1, V_1) \qquad \text{AMPS}_1 = \begin{pmatrix} -0.853 \\ -0.458 \\ -0.158 \end{pmatrix}$$

$$I_{AB_1} = \text{AMPS}_{1_0} - \text{AMPS}_{1_2} \qquad I_{AB_1} = -0.695$$

$$I_{BC_1} = \text{AMPS}_{1_1} - \text{AMPS}_{1_2} \qquad I_{BC_1} = -0.3$$

14.19



$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{bmatrix} 301 & -301 \\ -301 & 301 \end{bmatrix} \end{array}, \quad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} I_2 & 0 \end{array} \\ \begin{bmatrix} 392 & -392 \\ -392 & 392 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{bmatrix} 301 & -301 \\ -301 & 301+392 \end{bmatrix} \end{array} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 0 \end{Bmatrix}$$

And upon solving in Mathcad

$$I_{AD} = I_1 - I_2 \qquad \text{if} < 0.015 \text{ amp } R_1 \qquad \text{acceptable}$$

$$I_{BC} = I_2 \qquad \text{if} < 0.015 \text{ amp } R_2 \qquad \text{acceptable}$$

$$[k] = \begin{pmatrix} 301 & -301 \\ -301 & 301+392 \end{pmatrix} \qquad [v] = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

I[st] iteration

$$\text{AMPS} = \text{1solve}\,(k, v) \qquad \text{AMPS} = \begin{pmatrix} 0.053 \\ 0.023 \end{pmatrix}$$

$$I_{AD} = \text{AMPS}_0 - \text{AMPS}_1 \qquad I_{AD} = 0.03$$

$$I_{BC} = \text{AMPS}_1 \qquad I_{BC} = 0.023$$

Note: amps through diodes must be < 0.015 AMPS, so try larger resistors. Try changing 301 to 392 as well.

$$[k_2] = \begin{pmatrix} 392 & -392 \\ -392 & 784 \end{pmatrix}$$

$$\text{AMPS}_2 = \text{1solve}\,(k_2, v) \qquad \text{AMPS}_2 = \begin{pmatrix} 0.046 \\ 0.023 \end{pmatrix}$$

No good. Try larger ohm resistor (Try 523 ohms)

$$[k_3] = \begin{pmatrix} 523 & -523 \\ -523 & 1046 \end{pmatrix}$$

$$\text{AMPS}_3 = \text{1solve}\,(k_3, v) \qquad \text{AMPS}_3 = \begin{pmatrix} 0.034 \\ 0.017 \end{pmatrix}$$

$$\text{I}_{\text{AD}_3} = \text{AMPS}_{3_0} - \text{AMPS}_{3_1} \qquad \text{I}_{\text{AD}_3} = 0.03 > 0.015 \quad \therefore \text{ no good}$$

Try 549 ohms for both resistors.

$$[k_4] = \begin{pmatrix} 549 & -549 \\ -549 & 1098 \end{pmatrix}$$

$$\text{AMPS}_4 = \text{1solve}\,(k_4, v) \qquad \text{AMPS}_4 = \begin{pmatrix} 0.033 \\ 0.016 \end{pmatrix}$$

Final iteration

Try 715 for R_1 and 806 for R_2

$$[k_5] = \begin{pmatrix} 715 & -715 \\ -1521 & 1521 \end{pmatrix}$$

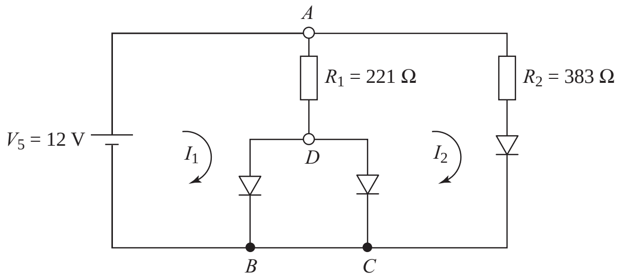
$$\text{AMPS}_5 = \text{1solve}\,(k_5, v) \qquad \text{AMPS}_5 = \begin{pmatrix} 0.024 \\ 0.011 \end{pmatrix}$$

$$\text{I}_{\text{AD}_5} = \text{AMPS}_{5_0} - \text{AMPS}_{5_1} \qquad \text{I}_{\text{AD}_5} = 0.013$$

$$\text{I}_{\text{BC}_5} = \text{AMPS}_{5_1} \qquad \text{I}_{\text{BC}_5} = 0.011 \qquad \text{This works. Now amps} < 0.015 \text{ amps}$$

$\therefore$ Let $R_1 = 715\ \Omega$, $R_2 = 806\ \Omega$

14.20



$$\begin{array}{ccc} & I_1 & I_2 \\ [k^{(1)}] = & \multicolumn{2}{c}{\begin{bmatrix} 221 & -221 \\ -221 & 221 \end{bmatrix}} \end{array}, \quad \begin{array}{ccc} & I_2 & 0 \\ [k^{(2)}] = & \multicolumn{2}{c}{\begin{bmatrix} 383 & -383 \\ -383 & 383 \end{bmatrix}} \end{array}$$

$$\begin{bmatrix} 221 & -221 \\ -221 & 221+383 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 0 \end{Bmatrix}$$

$I_{AD} = I_1 - I_2$ if < 0.015 amp R_1 acceptable

If $I_2 < 0.015$ amp R_2 acceptable

$I_{BD} = I_1$

$I_{CD} = I_2$

Try 2000 Ω for R_1

1270 Ω for R_2

Then $I_1 = 0.015$ amp, $I_2 = 0.00945$ amp

$$[k_{20}] = \begin{pmatrix} 221 & -221 \\ -221 & 221+383 \end{pmatrix} \qquad \{v_{20}\} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\text{AMPS}_{2_0} = \text{1solve}\,(k_{20}, v_{20}) \qquad \text{AMPS}_{2_0} = \begin{pmatrix} 0.086 \\ 0.031 \end{pmatrix}$$

Must increase ohms size so amps < 0.015. Try previous final R's of 715 and 806 ohms.

$$\text{AMPS}_{2_0} = \text{1solve}\,(k_5, v_{20}) \qquad \text{AMPS}_{2_0} = \begin{pmatrix} 0.032 \\ 0.015 \end{pmatrix}$$

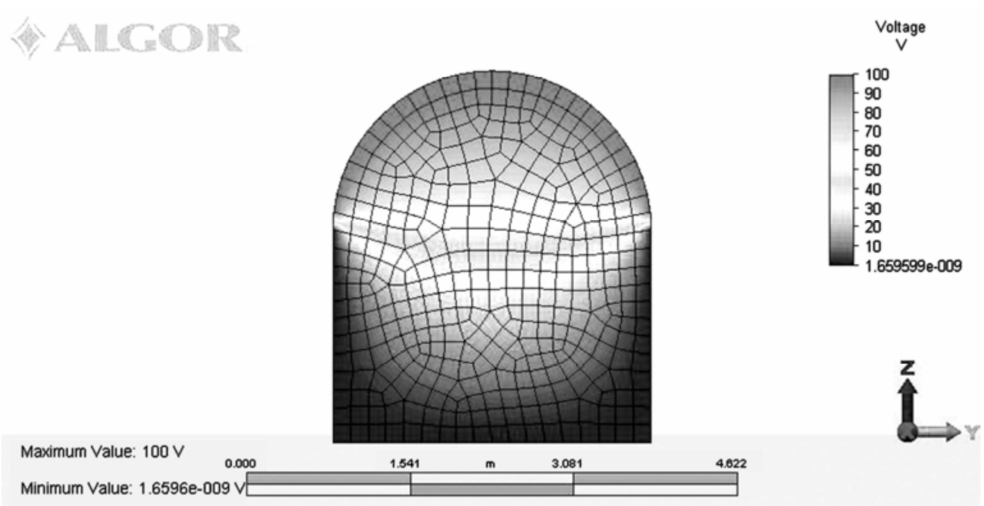
Still amps too large in diodes under $R1$ (Try 20000 ohm for $R1$ and 1270 ohms for $R2$)

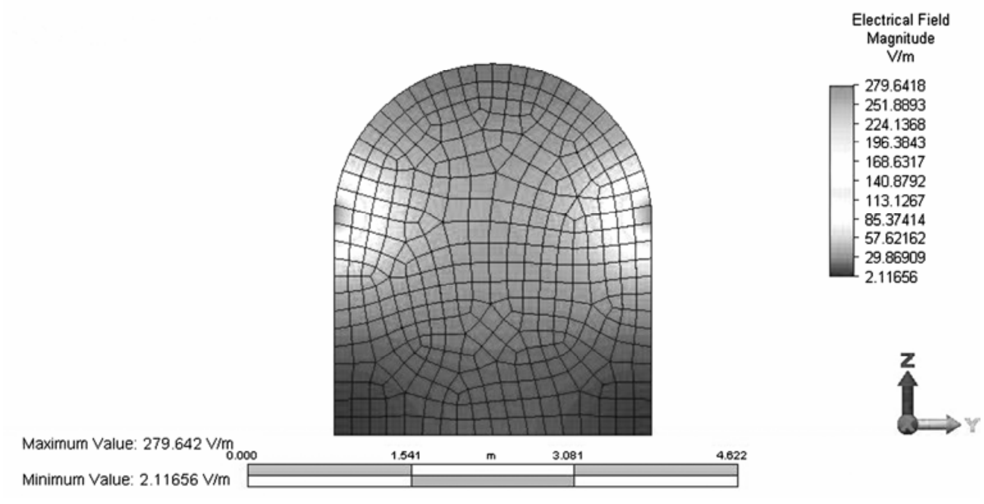
$$k_{20}f = \begin{pmatrix} 2000 & -2000 \\ -2000 & 3270 \end{pmatrix}$$

$$\text{AMPS}_{2_{0_f}} = \text{1solve}\,(k_{20}f, v_{20}) \qquad \text{AMPS}_{2_{0_f}} = \begin{pmatrix} 0.015 \\ 9.449 \times 10^{-3} \end{pmatrix}$$

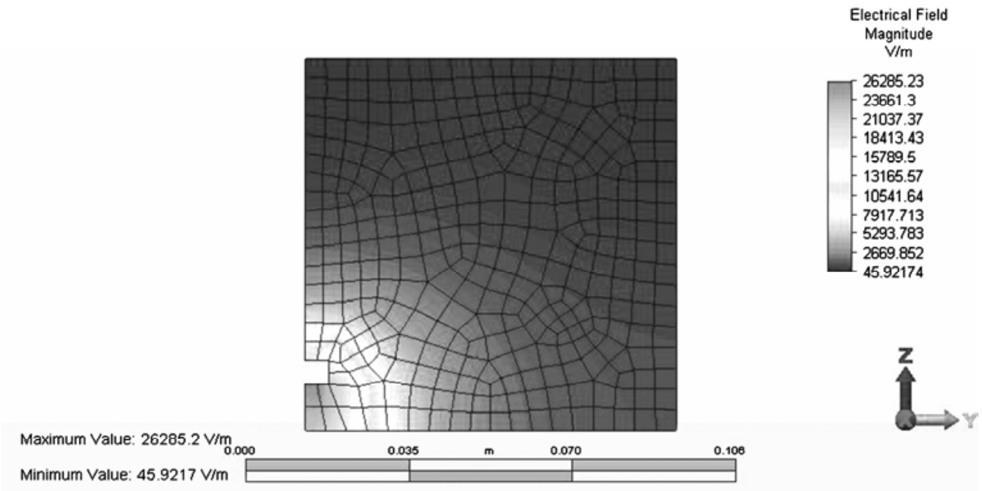
$\therefore$ Let $R_1 = 2000\ \Omega$, $R_2 = 1270\ \Omega$

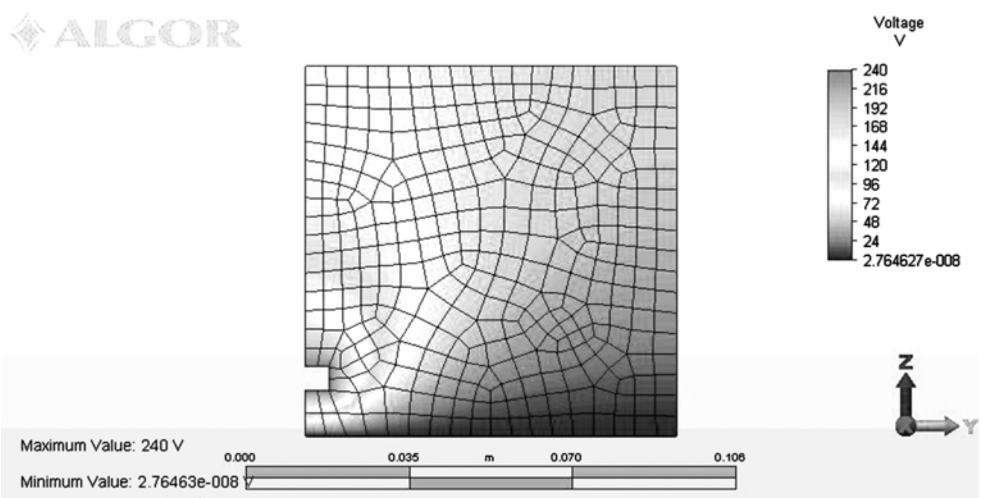
14.22





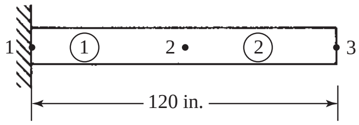
14.23


Electrical Field Magnitude V/m
26285.23
23661.3
21037.37
18413.43
15789.5
13165.57
10541.64
7917.713
5293.783
2669.852
45.92174
Z
Y
Maximum Value: 26285.2 V/m
Minimum Value: 45.9217 V/m
0.000
0.035
m
0.070
0.106


ALGOR
Voltage V
240
216
192
168
144
120
96
72
48
24
2.764627e-008
Z
Y
Maximum Value: 240 V
Minimum Value: 2.76463e-008 V
0.000
0.035
m
0.070
0.106

Chapter 15

15.1



$T = 50^\circ\text{F}$, $E = 30 \times 10^4$ psi
$\alpha = 7 \times 10^{-6}/^\circ\text{F}$, $A = 4\ \text{in}^2$

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ [k^{(1)}] = \dfrac{AE}{60} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}, \qquad \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ [k^{(2)}] = \dfrac{AE}{60} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}, \qquad \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$\{F\} = [K]\{d\}$ becomes

$$\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix} = \frac{AE}{60}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

Solving

$$\begin{aligned} u_2 &= \alpha T L \\ &= (7 \times 10^{-6})\,(50^\circ\text{F})\,(60\ \text{in.}) \\ &= 0.021\ \text{in.} \end{aligned}$$

$$u_3 = 2\,\alpha T L = 0.042\ \text{in.}$$

Reactions and actual nodal forces

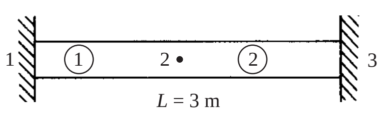
$$\{F\} = [K]\{d\} - \{F_0\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\left(\frac{AE}{L}\right)\begin{Bmatrix} 0 \\ \alpha TL \\ 2\alpha TL \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = \sigma^{(2)} = \frac{0}{4\ \text{in.}^2} = 0$$

15.2



$T = -20^\circ\text{C}$, $\alpha = 12 \times 10^{-6}/^\circ\text{C}$
$E = 210$ GPa, $A = 1 \times 10^{-2}\ \text{m}^2$
$L = 3$ m

$$[k^{(1)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [k^{(2)}] = \frac{AE}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \{f^{(2)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Global equations

$$\frac{AE}{1.5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \end{Bmatrix}$$

Solving

$$\frac{AE}{1.5}\ (2u_2) = 0$$

$$u_2 = 0$$

Forces in elements

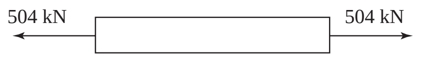
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} E\alpha TA \\ -E\alpha TA \end{Bmatrix}$$

$$E\ \alpha\, T\, A = (210 \text{ GPa})\ (12 \times 10^{-6}/°\text{C})\ (-20°\text{C}) \times (1 \times 10^{-2} \text{ m}^2)$$

$$= -504 \text{ kN}$$

$$\therefore\ f_{1x}^{(1)} = -504 \text{ kN},\ \ f_{2x}^{(1)} = 504 \text{ kN}$$

FBD element 1



$$\sigma^{(1)} = \frac{504 \text{ kN}}{1 \times 10^{-2} \text{ m}^2} = 50{,}400 \text{ KPa}$$

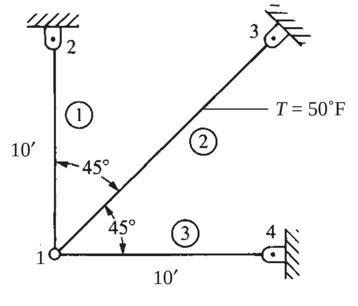
$$= 50.4 \text{ MPa}$$

Similarly

$$\sigma^{(2)} = 50.4 \text{ MPa}$$

$$F_{1x} = -504 \text{ kN},\ F_{3x} = 504 \text{ kN}$$

15.3



$\alpha = 7 \times 10^{-6}/°\text{F}$, $E = 30 \times 10^6$ psi
$A = 2$ in.^{2}, $f_{1x} = -E\alpha TA$

$$\{f\} = [T]^T\ \{f'\}$$

$$\begin{Bmatrix} f'^{(2)}_{1x} \\ f'^{(2)}_{3x} \end{Bmatrix} = \begin{Bmatrix} -21000 \\ 21000 \end{Bmatrix}$$

$$\{f\} = [T]^T\ \{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} 0.707 & -0.707 & 0 & 0 \\ 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & -0.707 \\ 0 & 0 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} f'_{1x} = -21,000 \\ 0 \\ 21,000 \\ 0 \end{Bmatrix}$$

$f_{1x} = -14,850$ lb, $f_{1y} = -14,850$ lb

$f_{3x} = 14,850$ lb, $f_{3y} = -14,850$ lb

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

$$\begin{Bmatrix} F_{1x} = -14,850 \\ F_{1y} = \ 14,850 \end{Bmatrix} = 500000 \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving

$$-u_1 = v_1 = -0.01753 \text{ in.}$$

By Equation (14.1.57)

$$\sigma^{(1)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - \{0\}$$

$$\sigma^{(1)} = \frac{30 \times 10^6}{120 \text{ in.}} [0 \ \ -1 \ \ 0 \ \ 1] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^{(1)} = 4350 \text{ psi (T)}$$

$$\sigma^{(2)} = \frac{E}{L} [-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix} - E\alpha T$$

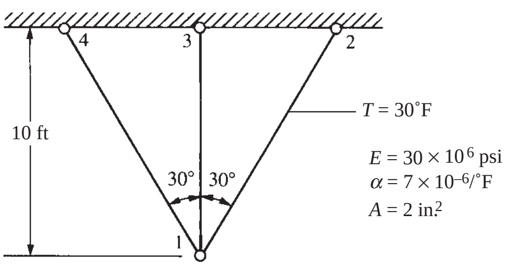
$$= \frac{30 \times 10^6}{120\sqrt{2}} [-0.707 \ \ -0.707 \ \ 0.707 \ \ 0.707] \begin{Bmatrix} 0.01753 \\ -0.01753 \\ 0 \\ 0 \end{Bmatrix}$$

$$= -10500$$

$$\sigma^{(2)} = -6150 \text{ psi (C)}$$

$$\sigma^{(3)} = 4350 \text{ psi (T)}$$

15.4



$$\{f'^{(1)}\} = \begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix} = \begin{Bmatrix} -12600 \\ 12600 \end{Bmatrix} \text{lb}$$

$$\{f\} = [T]^T \{f'\}$$

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} f'_{1x} = -12600 \\ f'_{1y} = 0 \\ f'_{2x} = 12600 \\ f'_{2y} = 0 \end{Bmatrix}$$

$$f_{1x} = -\,6300 \text{ lb}$$

$$f_{1y} = -\,10912 \text{ lb}$$

$$f_{2x} = 6300 \text{ lb}$$

$$f_{2y} = 10912 \text{ lb}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Global equations

$$\begin{Bmatrix} F_{1x} = -6300 \\ F_{1y} = -10912 \end{Bmatrix} = \frac{(2 \text{ in.}^2)(30 \times 10^6)}{120 \text{ in.}} \begin{bmatrix} \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 + \frac{3\sqrt{3}}{4} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$-6300 = 216{,}506\, u_1 \qquad u_1 = -\,0.0291 \text{ in.}$$

$$-10912 = 1{,}149{,}519\, v_1 \qquad v_1 = -0.0095 \text{ in.}$$

$$\sigma^{(1)} = \frac{E}{L}\,[-C \quad -S \quad C \quad S] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} - E\alpha T$$

$$= \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[-\frac{1}{2} \quad \frac{-\sqrt{3}}{2} \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix}$$

$$-\,30 \times 10^6 \times 7 \times 10^{-6} \times 30°\text{F}$$

$$\sigma^{(1)} = (216506)\,(0.0228) - 6300$$

$$= -\,1370 \text{ psi (C)}$$

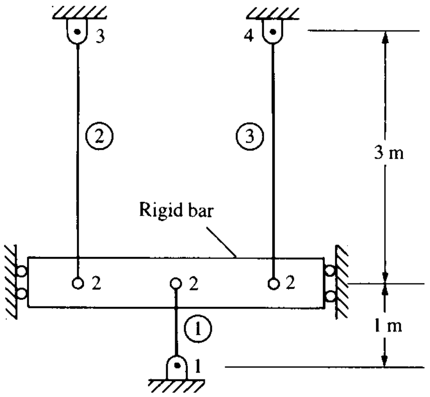
$$\sigma^{(2)} = \frac{30 \times 10^6}{120}\,[0 \;\; -1 \;\; 0 \;\; 1] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$\sigma^{(2)} = 2375 \text{ psi (T)}$$

$$\sigma^{(3)} = \frac{30 \times 10^6}{\frac{2(120)}{\sqrt{3}}} \left[\frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2} \quad \frac{\sqrt{3}}{2} \right] \begin{Bmatrix} -0.0291 \\ -0.0095 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$\sigma^{(3)} = -\,1370 \text{ psi (C)}$$

15.5



$E = 210$ GPa
$\alpha = 12 \times 10^{-6}/°\text{C}$
$T^{(1)} = 20°\text{C}$
$A = 2 \times 10^{-2}\ \text{m}^2$
$L = 1$ m

$$[k^{(1)}] = \frac{AE}{L^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad [k^{(2)}] = \frac{AE}{L^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \frac{AE}{L^{(3)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

Global equations

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+\frac{1}{3}+\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -E\alpha TA + F_{1x} \\ E\alpha TA \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

Solving equation (2) above

$$\frac{AE}{L}\left(\frac{5}{3}\right) u_2 = E\alpha TA$$

$$u_2 = \frac{3\alpha T L}{5} = \frac{3(12 \times 10^{-6})(20°)(1\text{m})}{5}$$

$$u_2 = 1.44 \times 10^{-4}\ \text{m}$$

Global nodal forces

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{\left(2 \times 10^{-2}\right)\left(210 \times 10^{9}\right)}{1\text{m}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} 0 \\ 1.44 \times 10^{-4} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \\ 0 \\ 0 \end{Bmatrix}$$

$$E\alpha TA = (210 \times 10^{9})\ (12 \times 10^{-6})\ (20°)\ (2 \times 10^{-2})$$

$$= 1{,}008{,}000 \text{ N} = 1{,}008 \text{ kN}$$

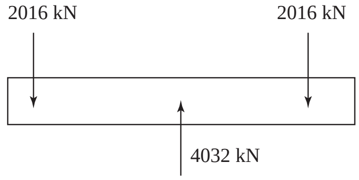
$$F_{1x} = -604.8 + 1{,}008 = 403.2 \text{ kN}$$

$$F_{2x} = 1{,}008 - 1{,}008 = 0$$

$$F_{3x} = -2{,}016 \text{ kN}$$

$F_{4x} = -2{,}016$ kN

FBD



$$\sigma^{(1)} = [C']\{d\} - E\alpha T$$

$$= \frac{E}{L}\,[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 1.44 \times 10^{-4} \\ v_2 = 0 \end{Bmatrix} - E\alpha T$$

$$= \frac{E}{L}\,(1.44 \times 10^{-4}) - E\alpha T$$

$$= 30.2 - 50.4$$

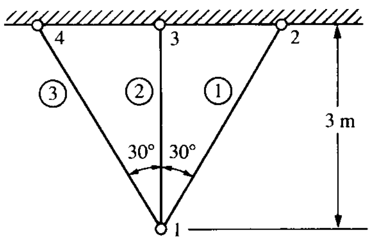
$$\sigma^{(1)} = -20.2 \text{ MPa (C)}$$

$$\sigma^{(2)} = \frac{210 \times 10^{-9}}{3\,\text{m}}\,[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix} 1.44 \times 10^{-4} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - 0$$

$$= -100.8 \times 10^{5}\ \frac{\text{N}}{\text{m}^2}$$

$$\sigma^{(2)} = -10.08 \text{ MPa (C)} = \sigma^{(3)}$$

15.6



$E = 70$ GPa $\qquad \alpha = 23 \times 10^{-6}/{}^\circ\text{C}$

$A = 4 \times 10^{-2}$ m^2 $\qquad T^{(2)} = -20^\circ\text{C}$

$$[k^{(1)}] = \frac{AE}{3.46\,\text{m}}\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ & \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ & & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \text{Symmetry} & & & \frac{3}{4} \end{bmatrix}$$

$$[k^{(2)}] = \frac{AE}{3\,\text{m}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[k^{(3)}] = \frac{AE}{3.46\,\text{m}}\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ & \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ & & \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ & \text{Symmetry} & & \frac{3}{4} \end{bmatrix}$$

Boundary conditions

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Thermal forces

$$f'^{(2)}_{1x} = -E\alpha TA,\ f'^{(2)}_{3x} = E\alpha TA$$

Convert to global forces using

$$\{f\} = [T]^T \{f'\}$$

$$\begin{Bmatrix} f^{(2)}_{1x} \\ f^{(2)}_{1y} \\ f^{(2)}_{3x} \\ f^{(2)}_{3y} \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ E\alpha TA \\ 0 \\ -E\alpha TA \end{Bmatrix}$$

$$\therefore\quad f^{(2)}_{1y} = E\alpha TA = (70 \times 10^6)\,(23 \times 10^{-6})\,(-\,20°\text{C}) \times (4 \times 10^{-2}\ \text{m}^2)$$

$$= 1288\ \text{kN}$$

$$f^{(2)}_{3y} = -\,1288\ \text{kN}$$

Assemble equations $\{F\} = [K]\,\{d\} - \{F_0\}$

$$\frac{AE}{3.46}\begin{bmatrix} 0.5 & 0 \\ 0 & 2.65 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1288 \end{Bmatrix}$$

Solving

$$u_1 = 0$$

$$v_1 = \frac{1288(3.46)}{2.65\left(4 \times 10^{-2}\right)\left(70 \times 10^6\right)} = 6 \times 10^{-4}\ \text{m}$$

Element forces

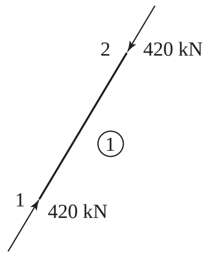
$$\{f'\} = [k']\,\{d'\} - \{f'_0\} = [k']\,[T]^*\,\{d\} - \{f'_0\}$$

$$\begin{Bmatrix} \hat{f}^{(1)}_{1x} \\ \hat{f}^{(1)}_{2x} \end{Bmatrix} = \frac{AE}{3.46}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{AE}{3.46}\begin{bmatrix} C & S & -C & -S \\ -C & -S & C & S \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 6\times 10^{-4} \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix}$$

$$= \frac{\left(4\times 10^{-2}\right)\left(70\times 10^{6}\right)}{3.46}\begin{Bmatrix} \frac{\sqrt{3}}{2}\times 6\times 10^{-4} \\ -\frac{\sqrt{3}}{2}\times 6\times 10^{-4} \end{Bmatrix}$$

$$= \begin{Bmatrix} 420 \\ -420 \end{Bmatrix} \text{ kN}$$



Stresses

$$\underline{\sigma} = [C'] \{d\} - E\,\alpha\,T \quad \text{or} \quad \sigma^{(1)} = \frac{f'_{2x}}{A}$$

Element 1

$$\sigma^{(1)} = \frac{-420}{4\times 10^{-2}} = -10.5 \text{ MPa (C)}$$

Element 2

$$\sigma^{(2)} = \frac{70\times 10^{6}}{3.0\,\text{m}}\ [0\ \ -1\ \ 0\ \ 1]\begin{Bmatrix} u_1 = 0 \\ v_1 = 6\times 10^{-4} \\ 0 \\ 0 \end{Bmatrix}$$

$$- 70 \times 10^{6}\,(23 \times 10^{-6})\,(-20°\text{C})$$

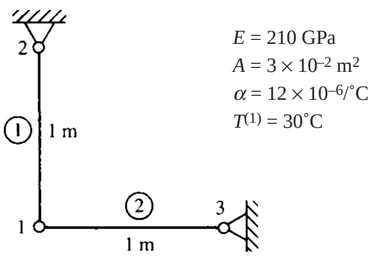
$$= -14000 + 32200$$

$$= 18200 \text{ kPa}$$

$$\sigma^{(2)} = 18.2 \text{ MPa (T)}$$

$$\sigma^{(3)} = \sigma^{(1)} = -10.5 \text{ MPa (C)}$$

15.7



$$E\,\alpha\,T\,A = \left(210\times 10^{6}\,\frac{\text{kN}}{\text{m}^2}\right)\times 12\times \frac{10^{-6}}{°\text{C}} \times 30°\text{C} \times (3\times 10^{-2})$$

$$= 2.27\times 10^{3} \text{ kN}$$

$$[k^{(1)}] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\left\{ f_0'^{(1)} \right\} = \begin{Bmatrix} -E\alpha TA \\ E\alpha TA \end{Bmatrix}$$

$$[k^{(2)}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ f_0'^{(2)} \right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Transform initial forces to global forces

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} -E\alpha TA \\ 0 \\ E\alpha TA \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ E\alpha TA \\ 0 \\ -E\alpha TA \end{Bmatrix}$$

Assemble global equations

$$\frac{\left(3\times10^{-2}\,\text{m}^2\right)\left(210\times10^6\,\frac{\text{kN}}{\text{m}^2}\right)}{1\text{m}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.27\times10^3 \end{Bmatrix}$$

where $u_2 = v_2 = u_3 = v_3 = 0$

Solving

$$u_1 = 0 \qquad v_1 = -\,3.6\times10^{-4}\text{ m}$$

Stresses

$$\sigma^{(1)} = \frac{E}{L}\left[\overset{0}{\underset{\|}{-C}} \quad \overset{-1}{\underset{\|}{-S}} \quad \overset{0}{\underset{\|}{C}} \quad \overset{1}{\underset{\|}{S}}\right]\begin{Bmatrix} 0 \\ -3.6\times10^{-4} \\ 0 \\ 0 \end{Bmatrix} - E\alpha T$$

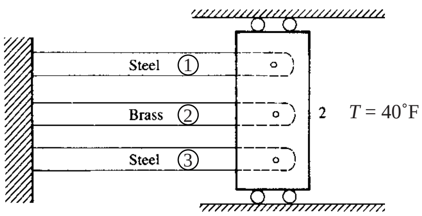
$$= \frac{210\times10^{-9}}{1\text{m}}\ (3.6\times10^{-4}\text{ m}) - 210\times10^9\times12\times10^{-6}\times30$$

$$\sigma^{(1)} = 0$$

$$\sigma^{(2)} = \frac{210\times10^{-9}}{1\text{m}}\ (0) - 0 = 0$$

For statically determinate structure thermal stresses are zero.

15.8



$$[k^{(1)}] = \frac{(2)\left(30\times10^6\right)}{60}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [k^{(3)}]$$

$$[k^{(2)}] = \frac{2\left(15\times10^6\right)}{60}\begin{array}{c} \begin{matrix} 1 & \;\;2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Global equations

$$10^6\begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} - 15600\times 2 - 12000 \\ 15600\times 2 + 12000 \end{Bmatrix} \qquad (1)$$

where

$$f^{(1)} = -E\alpha TA = -\,30\times10^6\times6.5\times10^{-6}\times40\times2$$

$$= -15600 \text{ lb}$$

$$\{f^{(1)}\} = \begin{Bmatrix} -15600 \\ 15600 \end{Bmatrix} = \{f^{(3)}\}$$

$$f^{(2)} = 15\times10^6\times10\times10^{-6}\times40\times2$$

$$= -\,12000 \text{ lb}$$

$$\{f^{(2)}\} = \begin{Bmatrix} -12000 \\ 12000 \end{Bmatrix} \text{ lb}$$

Solving Equation (1) above

$$u_2 = 0.01728 \text{ in.}$$

Stresses

$$\sigma_{\text{st}} = \frac{AE}{L}\frac{u_2}{A} - E\alpha T$$

$$= \frac{\left(30\times10^6\right)}{60}(0.01728) - 30\times10^6\ (6.5\times10^{-6})\ 40$$

$$= 8640 - 7800$$

$$\sigma_{\text{st}} = 840 \text{ psi (T)}$$

$$\sigma_{\text{sr}} = \frac{E}{L}u_2 - E\alpha T$$

$$= \frac{15\times10^6}{60}(0.01728) - 15\times10^6\,(10\times10^{-6})\times40$$

$$= 4320 - 6000 = -\,1680 \text{ psi}$$

15. 9 A uniform temperature increase of 10°C in each element yields zero stress for this special symmetric arrangement of the truss elements. See the table and figure 1 below showing the stresses from Algor to be zero in each element of the truss.

Note that if the truss is not symmetric as shown in figure 2 and then is uniformly heated, the middle element has a stress of –3.46 MPa in it, while the top element has a stress of 2.83 MPa in it.

**** 3-D truss elements

Number of elements = 3

Number of materials = 1

**** Nodal stresses for 3-D truss elements

El. #	LC	ND	Stress	Force
1	1	I	–1.118E-08	–1.341E-11
1	1	J	1.118E-08	1.341E-11
2	1	I	–1.118E-08	–1.341E-11
2	1	J	1.118E-08	1.341E-11
3	1	I	–1.118E-08	–1.341E-11
3	1	J	1.118E-08	1.341E-11

When we uniformly heat the truss the stresses go to zero for this symmetric structure.

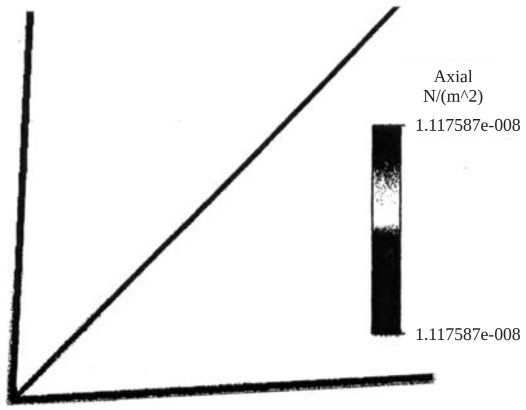


Figure 1

When we uniformly heat the truss the stresses are not zero for this unsymmetric statically indeterminate structure.

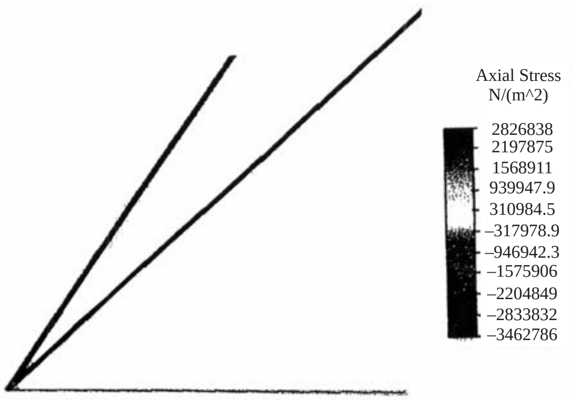
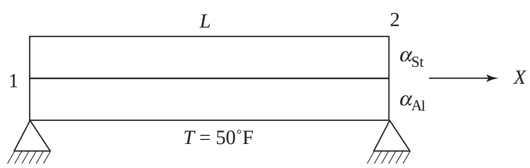


Figure 2

15.10 Bodies that are statically indeterminate will have stress due to uniform temperature change. (see figure 2 in solution to P 15.9) except in special symmetry cases (see figure 1 and table of results in P15.9). Also see, for instance, example 15.1, figure 15-5 and P 15.3, P 15.4 and P 15.6.

15.11



$$[k_{st}] = \frac{AE_{st}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ [k_{A1}] = \frac{AE_{Al}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{A}{L}\begin{bmatrix} E_{st}+E_{Al} & -E_{st}-E_{Al} \\ -E_{st}-E_{Al} & E_{st}+E_{Al} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} - E_{st}\,\alpha_{st}\,TA - E_{Al}\,\alpha_{Al}\,TA \\ (E_{st}\,\alpha_{st} + E_{A}\alpha_{Al})TA \end{Bmatrix}$$

Boundary conditions $u_1 = 0$

$$\frac{\cancel{A}}{L}\ (E_{st} + E_{A1})\ u_2 = (E_{st} + \alpha_{st} + E_{A1}\ \alpha_{A1})\ T\,\cancel{A}$$

$$u_2 = \frac{(E_{st}\alpha_{st} + E_{A1}\alpha_{A1})TL}{E_{st} + E_{A1}}$$

$$u_2 = \frac{(30\times10^{6}\times6.5\times10^{-6} + 10\times10^{6}\times13\times10^{-6})TL}{(30+10)\times10^{6}}$$

$$u_2 = 8.125\times10^{-6}\ TL$$

$$= (8.125\times10^{-6})\ (50°\text{F})\ L$$

$$u_2 = 406.25\times10^{-6}\ L$$

$$\sigma_{st} = [C']\ \{d\} - \sigma_T$$

$$= \frac{E_{st}}{L}\ (-1 \quad 0 \quad 1 \quad 0)\begin{Bmatrix} 0 \\ 0 \\ 406.25\times10^{-6}L \\ 0 \end{Bmatrix} - E_{st}\ \alpha_{st}\ T$$

$$= 30\times10^{6}\times406.25\times10^{-6} - 30\times10^{6}\times6.5\times10^{-6}\times50$$

$$= 12187.5 - 9750$$

$$\sigma_{st} = 2437.5\text{ psi (T)}$$

$$\sigma_{A1} = 10\times10^{6}\times406.25\times10^{-6} - 10\times10^{6}\times13\times10^{-6}\times50$$

$$= 4062.5 - 6500$$

$$\sigma_{A1} = -24375\text{ psi (C)}$$

$$\sigma_{st} = -\sigma_{A1}$$

15.12 To close gap of 0.005 in.

$$\delta_{gap} = \alpha_{br}\ \Delta TL_{br} + \alpha_m\ \Delta T\ L_m$$

$$\Delta T = \frac{\delta_{gap}}{\alpha_{br}L_{br} + \alpha_m L_m}$$

$$= \frac{0.005\text{ in.}}{11.3\times10^{-6}\times1\text{ in.} + 14.5\times10^{-6}\times1.5\text{ in.}}$$

$$\Delta T = 151.3°\text{F to close gap}$$

$$\delta_{br_{close}} = \alpha_{br}\ \Delta T\ L_{br}$$

$$= 11.3\times10^{-6}\times151.3°\times1\text{ in.}$$

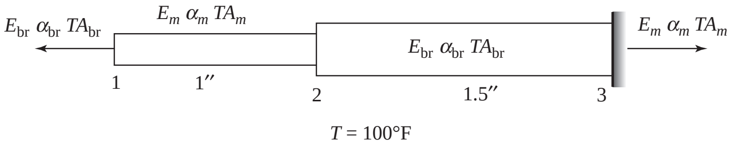
$$= 0.0017097\text{ in.}$$

$$\delta_{m_{close}} = \alpha_m\ \Delta_T\ L_m$$

$$= 14.5 \times 10^{-6} \times 151.3 \times 15 \text{ in.}$$

$$= 0.0032908 \text{ in.}$$

(a)



$$[k_{br}] = \frac{A_{br}E_{br}}{L_{br}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ [k_m] = \frac{A_m E_m}{L_m}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{A_{br}E_{br}}{L_{br}} & \frac{-A_{br}E_{br}}{L_{br}} & 0 \\ \frac{-A_{br}E_{br}}{L_{br}} & \frac{A_{br}E_{br}}{L_{br}} + \frac{A_m E_m}{L_m} & \frac{-A_m E_m}{L_m} \\ 0 & \frac{-A_m E_m}{L_m} & \frac{A_m E_m}{L_m} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} F_{1x} - E_{br}\alpha_{br}\ TA_{br} \\ -E_m\alpha_m\ TA_m + E_{br}\alpha_{br}\ TA_{br} \\ F_{3x} + E_m\alpha_m\ TA_m \end{Bmatrix}$$

$$\left(\frac{A_{br}E_{br}}{L_{br}} + \frac{A_m E_m}{L_m}\right) u_2 = -E_m\ \alpha_m\ TA_m + E_{br}\ \alpha_{br}\ TA_{br}$$

$$u_2 = \left(\frac{-E_m\alpha_m TA_m + E_{br}\alpha_{br}TA_{br}}{\frac{A_{br}E_{br}}{L_{br}} + \frac{A_m E_m}{L_m}}\right)$$

$$u_2 = \left(\frac{-4.5\times10^6 \times 14.5\times10^{-6} \times 0.15 \text{ in.}^2 + 15\times10^6 \times 11.3\times10^{-6} \times 0.1 \text{ in.}^2}{\frac{0.10\times15\times10^6}{1''} + \frac{0.15\times4.5\times10^6}{1.5''}}\right)100$$

$$u_2 = 3.673 \times 10^{-4} \text{ in.} \rightarrow$$

$$\sigma_{br} = \frac{E_{br}}{L_{br}}\ u_2 - E_{br}\,\alpha_{br} T$$

$$= \frac{E_{br}}{L_{br}}\left(\frac{-E_m\alpha_m TA_m + E_{br}\alpha_{br}TA_{br}}{\frac{A_{br}E_{br}}{L_{br}} + \frac{A_m E_m}{L_m}}\right) - E_{br}\ \alpha_{br}\ T$$

$$= \frac{1.5\times10^6}{1''}\left(\frac{-4.5\times10^6 \times 14.5\times10^{-6} \times 0.15 \text{ in.}^2 + 15\times10^6 \times 11.3\times10^{-6} \times 0.1 \text{ in.}^2}{\frac{0.10\times15\times10^6}{1} + \frac{0.15\times4.5\times10^6}{1.5}}\right) 100°$$

$$- 15 \times 10^6 \times 11.3 \times 10^{-6} \times 100$$

$$= 15 \times 10^6 \left(\frac{-9.7875 + 16.95}{(1.5 + 0.45)\times10^6}\right) 100 - 16950$$

$$= \frac{107.4375 \times 100}{1.95} - 16950$$

$$= \frac{10743.75}{1.95} - 16950$$

$$= 5509.6 - 16950$$

$$\sigma_{br} = -11440 \text{ psi (C)}$$

$$F_{br} = (-11440 \text{ psi})(0.1 \text{ in.}^2) = -1144 \text{ lb}$$

$$\sigma_m = [C'] \{d\} - \sigma_m$$

$$= \frac{E_m}{L_m} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 3.673 \times 10^{-4} \text{ in.} = u_2 \\ 0 = v_2 \\ 0 = u_3 \\ 0 = v_3 \end{Bmatrix} - E_m \alpha_m T$$

$$= \frac{-4.5 \times 10^6}{1.5 \text{ in.}} (-3.673 \times 10^{-4} \text{ in.}) - 4.5 \times 10^6 \times 14.5 \times 10^{-6} \times 100°\text{F}$$

$$= -1100 - 6525$$

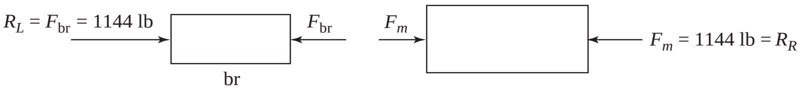
$$\sigma_m = -7625 \text{ psi (C)}$$

Check equations

$$F_m = \sigma_m A_m = (-7625)(0.15 \text{ in.}^2)$$

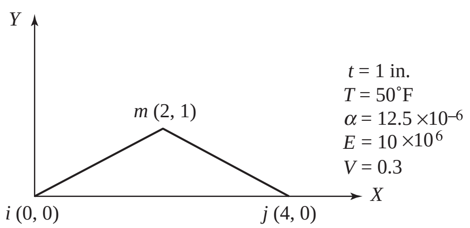
$$F_m = -1144 \text{ lb}$$

Same as $F_b = -1144$ lb. So equations check satisfied.



$\therefore$ Reactions R_L and R_m equal but opposite in direction.

15.13



$$\{f_T\} = \frac{\alpha E t T}{2(1-v)} \begin{Bmatrix} \beta_i \\ \gamma_i \\ \beta_j \\ \gamma_j \\ \beta_m \\ \gamma_m \end{Bmatrix}$$

$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 2 - 4 = -2$$

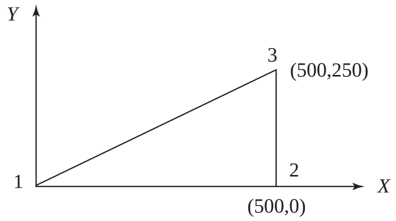
$\beta_j = y_m - y_i = 1 - 0 = 1 \qquad \gamma_j = x_i - x_m = 0 - 2 = -2$

$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_j - x_i = 4 - 0 = 4$

$$\{f_T\} = \frac{(12.5\times10^{-6})(10\times10^{6})(1)(50)}{2(1-0.3)}\begin{Bmatrix}-1\\-2\\1\\-2\\0\\4\end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix}-4464\\-8929\\4464\\-8929\\0\\17857\end{Bmatrix}\text{ lb}$$

15.14



$E = 70$ GPa, $\nu = 0.3$

$T = 30°C$, $\alpha = 23\times10^{-6}/°C$ $t = 5$ mm

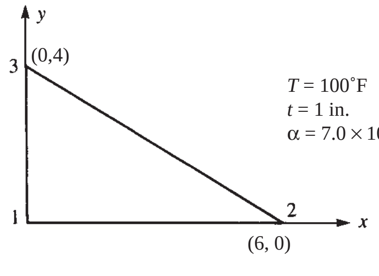
$\beta_1 = -250$ mm, $\beta_2 = 250$ mm, $\beta_3 = 0$

$\gamma_1 = 0$, $\gamma_2 = -500$ mm, $\gamma_3 = 500$ mm

$$\{f_T\} = \frac{(23\times10^{-6})(70\times10^{9})(0.005\,\text{m})30°\text{C}}{2(1-0.3)}\begin{Bmatrix}-0.25\,\text{m}\\0\\0.25\,\text{m}\\-0.50\,\text{m}\\0\\0.50\,\text{m}\end{Bmatrix}$$

$$\{f_T\} = 43125\begin{Bmatrix}-1\\0\\1\\-2\\0\\2\end{Bmatrix}\text{ N}$$

15.15



$T = 100°F$
$t = 1$ in.
$\alpha = 7.0\times10^{-6}/°F$

$\beta_i = y_j - y_m = 0 - 4 = -4, \qquad \gamma_i = x_m - x_j = 0 - 6 = -6$

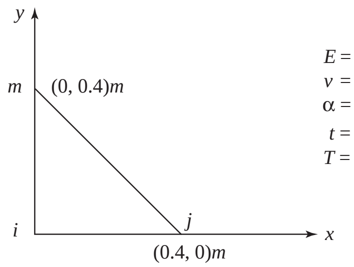
$\beta_j = y_m - y_i = 4 - 0 = 4, \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$

$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 6 - 0 = 6$

$$\{f_T\} = \frac{(7.0\times10^{-6})(30\times10^{6})(1)(100°\text{F})}{2(1-0.3)}\begin{Bmatrix}-4\\-6\\4\\0\\0\\6\end{Bmatrix}$$

$$= \begin{Bmatrix}-60,000\\-90,000\\60,000\\0\\0\\90,000\text{ lb}\end{Bmatrix}$$

15.16



E = 210 GPa
ν = 0.25
$\alpha = 12 \times 10^{-6}/°\text{C}$
t = 0.01 m
T = –20˚C

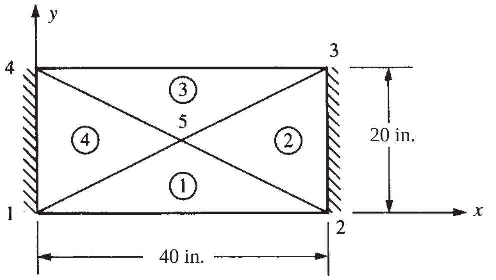
$$\{f_T\} = \frac{\alpha E t T}{2(1-\nu)}\begin{Bmatrix}\beta_i\\\gamma_i\\\beta_j\\\gamma_j\\\beta_m\\\gamma_m\end{Bmatrix}$$

$\beta_i = -0.4$ m $\qquad \gamma_i = -0.4$ m
$\beta_j = 0.4$ m $\qquad \gamma_j = 0$
$\beta_m = 0 \qquad \gamma_m = 0.4$ m

$$\{f_T\} = \frac{(12\times10^{-6})(210\times10^{9})(0.01)(-20°\text{C})}{2(1-0.25)}\begin{Bmatrix}-0.4\\-0.4\\0.4\\0\\0\\0.4\end{Bmatrix}$$

$$\{f_T\} = \begin{Bmatrix}134.4\\134.4\\-134.4\\0\\0\\-134.4\end{Bmatrix}\text{ kN}$$

15.17



Thermal force matrix

Element 1

$$i = 1,\ j = 2,\ m = 5$$

$$\beta_i = y_j - y_m = 0 - 10 = -10, \qquad \gamma_i = x_m - x_j = 20 - 40 = -20$$

$$\beta_j = y_m - y_i = 10 - 0 = 10, \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0, \qquad \gamma_m = x_j - x_i = 40 - 0 = 40$$

$$\{f_T^{(1)}\} = \frac{(12.5\times10^{-6})(10\times10^{6})(1)(50)}{2(1-0.3)}\begin{Bmatrix}-10\\-20\\10\\-20\\0\\40\end{Bmatrix}$$

$$\{f_T^{(1)}\} = \begin{Bmatrix}-44643\\89286\\44643\\-89286\\0\\178572\end{Bmatrix}$$

Element 2

$$i = 2,\ j = 3,\ m = 5$$

$$\beta_i = 20 - 10 = 10, \qquad \gamma_i = 20 - 40 = -20$$

$$\beta_j = 10 - 0 = 10, \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 0 - 20 = -20, \qquad \gamma_m = 40 - 40 = 0$$

$$\{f_T^{(2)}\} = 4464.3\begin{Bmatrix}10\\-20\\10\\20\\-20\\0\end{Bmatrix} = \begin{Bmatrix}44643\\-89286\\44643\\89286\\-89286\\0\end{Bmatrix}$$

Element 3

$$i = 3,\ j = 4,\ m = 5$$

$$\beta_i = 20 - 10 = 10 \qquad \gamma_j = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 40 - 20 = 20$$

$$\beta_m = 20 - 20 = 0 \qquad \gamma_m = 0 - 40 = -40$$

$$\{f_T^{(3)}\} = 4464.3\begin{Bmatrix} 10 \\ 20 \\ -10 \\ 20 \\ 0 \\ -40 \end{Bmatrix} = \begin{Bmatrix} 44643 \\ 89286 \\ -44643 \\ 89286 \\ 0 \\ -178572 \end{Bmatrix}$$

Element 4

$$i = 4,\ j = 1,\ m = 5$$

$$\beta_i = 0 - 10 = -10 \qquad \gamma_i = 20 - 0 = 20$$

$$\beta_j = 10 - 20 = -10 \qquad \gamma_j = 0 - 20 = -20$$

$$\beta_m = 20 - 0 = 20 \qquad \gamma_m = 0 - 0 = 0$$

$$\{f_T^{(4)}\} = 4464.3\begin{Bmatrix} -10 \\ 20 \\ -10 \\ -20 \\ 20 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -44643 \\ 89286 \\ -44643 \\ -89286 \\ 89286 \\ 0 \end{Bmatrix}$$

$$\{F_0\} = [K]\,\{d\}$$

By direct superposition, we have

$$\begin{Bmatrix} -89,286 \\ -178,572 \\ 89,286 \\ -178,572 \\ 89,286 \\ 178,572 \\ -89,286 \\ 178,572 \\ 0 \\ 0 \end{Bmatrix} = \frac{10\times10^6}{4.16}\begin{bmatrix} 3 & 2 & 0.1 & 0.2 & 0 & 0 & -0.1 & -0.2 & -3 & -2 \\ & 6 & -0.2 & 2.6 & 0 & 0 & 0.2 & -2.6 & -2 & -6 \\ & & 3 & -2 & -0.1 & 0.2 & 0 & 0 & -3 & 2 \\ & & & 6 & -0.2 & -2.6 & 0 & 0 & 2 & -6 \\ & & & & 3 & 2 & 0.1 & 0.2 & -3 & -2 \\ & & & & & 6 & -0.2 & 2.6 & -2 & -6 \\ & & & & & & 3 & -2 & -3 & 2 \\ & & & & & & & 6 & 2 & -6 \\ & & & & & & & & 12 & 0 \\ & & \text{Symmetry} & & & & & & & 24 \end{bmatrix}$$

$$\times \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ \vdots \\ \vdots \\ u_5 \\ v_5 \end{Bmatrix}$$

Solving

$$0 = \frac{10\times10^6}{4.16}\ 12\,u_s \Rightarrow u_s = 0$$

$$0 = \frac{10\times10^6}{4.16}\ 24\,v_s \Rightarrow v_s = 0$$

Stresses

$$\{\sigma\} = \{\sigma_L\} - \{\sigma_T\}$$

$$\{\sigma_L\} = [D]\,[B]\,\{d\} = 0 \text{ as } \{d\} = \underline{0}$$

$$\therefore \quad \{\sigma\} = -\{\sigma_T\} = -[D]\,\{\varepsilon_T\}$$

Element 1

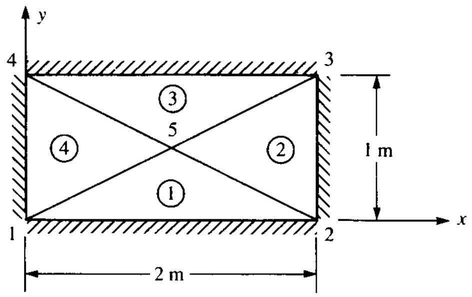
$$\{\sigma\} = -\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \tau_{xy} \end{Bmatrix} = \frac{-10\times10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 6.25\times10^{-4} \\ 6.25\times10^{-4} \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} -8929 \\ -8929 \\ 0 \end{Bmatrix}\text{psi}$$

Since $[D]$ and $\{\varepsilon_T\}$ are same for all elements, all element stresses are equal.

15.18



Based on use of symmetry

$$u_s = v_s = 0 \text{ (Also see solution to Problem 15.17)}$$

$$\therefore \quad \{\sigma\} = \{\sigma_L\} - \{\sigma_T\} = [D]\,[B]\,\{d\} - [D]\,\{\varepsilon_T\}$$

$$\{\sigma\} = -[D]\,\{\varepsilon_T\}$$

All stresses in elements are equal

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{-E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}\begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$= \frac{-210\times10^9}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{0} \end{bmatrix}\begin{Bmatrix} 12\times10^{-6}(-20) \\ 12\times10^{-6}(-20) \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 67.2 \\ 67.2 \\ 0 \end{Bmatrix} \text{ MPa}$$

15.19

For bar with $\alpha = \alpha_0 \left(1 + \dfrac{X}{L}\right)$

$T =$ constant

By Equation (15.1.18)

$$\{f_T\} = A \int_0^L [B]^T \, [D] \, \varepsilon_T \, dx$$

L, A, E, T constant $\varepsilon_T = \alpha T$

$$\{f_T\} = A \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \, \alpha_0 \left(1 + \frac{X}{L}\right) T \, dx$$

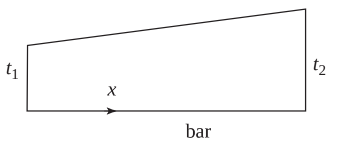
$$= \frac{AE\alpha_0 T}{L} \int_0^L \begin{Bmatrix} -\left(1 + \frac{X}{L}\right) \\ \left(1 + \frac{X}{L}\right) \end{Bmatrix} dx$$

$$= \frac{AE\alpha_0 T}{L} \begin{Bmatrix} -\left(X + \frac{X^2}{2L}\right)\Big|_0^L \\ \left(X + \frac{X^2}{2L}\right)\Big|_0^L \end{Bmatrix}$$

$$= \frac{AE\alpha_0 T}{L} \begin{Bmatrix} -\left(L + \frac{L}{2}\right) \\ \left(L + \frac{L}{2}\right) \end{Bmatrix}$$

$$\{f_T\} = \frac{3\,AE\alpha_0 T}{2} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

15.20



$T = t_1 + t_2 \, x$

By Equation (15.1.18)

$$\{f_T\} = A \int_0^L [B]^T \, [D] \, \{\varepsilon_T\} \, dx$$

$$T = t_1 + t_2 x \qquad [N] = \begin{bmatrix} 1 - \dfrac{x}{L} & \dfrac{x}{L} \end{bmatrix}$$

$$\{f_T\} = A \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E\alpha \, [N] \, \{t\} \, dx$$

$$= AE\ \alpha \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix}\ dx$$

$$= \frac{AE\alpha}{L} \int_0^L \begin{Bmatrix} -1+\frac{x}{L} & \frac{-x}{L} \\ 1-\frac{x}{L} & \frac{x}{L} \end{Bmatrix} \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix}\ dx$$

$$= \frac{AE\alpha}{L} \int_0^L \begin{Bmatrix} \left(-1+\frac{x}{L}\right) t_1 - \frac{x}{L} t_2 \\ \left(1-\frac{x}{L}\right) t_1 + \frac{x}{L} t_2 \end{Bmatrix} dx$$

$$= \frac{AE\alpha}{L} \begin{Bmatrix} \left(-x+\frac{x^2}{2L}\right)\Big|_0^L\ t_1 - \frac{x^2}{2L} t_2\Big|_0^L \\ \left(x-\frac{x^2}{2L}\right)\Big|_0^L\ t_1 + \frac{x^2}{2L} t_2\Big|_0^L \end{Bmatrix}$$

$$\{f_T\} = \frac{AE\alpha}{L} \begin{Bmatrix} \left(-L+\frac{L}{2}\right) t_1 - \frac{L}{2} t_2 \\ \left(L-\frac{L}{2}\right) t_1 + \frac{L}{2} t_2 \end{Bmatrix}$$

For $t_1 = t_2 = T$ (constant temperature over element)

$$\{f_T\} = \frac{AE\alpha}{L} \begin{Bmatrix} -TL \\ TL \end{Bmatrix} = \begin{Bmatrix} -AE\alpha T \\ AE\alpha T \end{Bmatrix}$$

Equation (15.1.18)

15.21

$$\{f_T\} = \int_s [B]^T\ [D]\ \{\varepsilon_T\}\ d\ s \tag{1}$$

$$\{\varepsilon_T\} = \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \tag{2}$$

Using centroidal approximation

$$\{f_T\} = \int_s [B]^T\ [D]\ \{\varepsilon_T\}\ d\ s$$

$$= 2\pi\ \bar{r}\ \ A\ \alpha T\ \left[\bar{B}\right]^T [D] \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{3}$$

where for axisymmetric case

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \tag{4}$$

substituting (4) into (3) and multiplying

$$\{f_T\} = \frac{2\pi \bar{r} A E \alpha T \left[\bar{B}\right]^T}{1-2v} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

15.22 Using modified CSFEP to account for thermal stress due to element temperature change.

INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	4
NUMBER OF ELEMENTS.	2
NUMBER OF DIFFERENT MATEIRALS.	1
NUMBER OF SURFACE LOAD CARDS	0
1 = PLANE STRAIN, 2 = PLANE STRESS. . . .	2
BODY FORCES (1 = IN – Y DIREC., 0 = NONE)	0

INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS
1	0.3000E+00	0.3333E+00	0.7800E-01	0.1000E+00
				ALPHA
				0.1000E–04

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	2	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00
4	0	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00

INPUT TABLE 4.. ELEMENT DATA

	GLOBAL INDICES OF ELEMENT NODES					
ELEMENT	1	2	3	4	MATERIAL	TEMP.
1	1	2	3	3	1	0.800E+02
2	2	4	3	3	1	0.800E+02

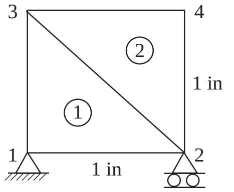
OUTPUT TABLE 1.. NODAL DISPLACEMENTS

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.80000000E–03	0.00000000E+00
3	0.14551920E–10	0.80000000E–03
4	0.80000000E–03	0.80000000E–03

TABLE 2.. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.33	0.33	0.0000E+00	0.0000E+00	1.6371E–04
2	0.67	0.67	6.5484E–04	1.9645E–03	6.5484E–04
			SIGMA(1)	SIGMA(2)	ANGLE
			1.6371E–04	–1.6371E–04	0.0000E+00
			2.2358E–03	3.8359E–04	6.7500E+01

15.23



Data File 0

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Verification.5
4, 2, 2, 0, 2, 0
1, 1
0.30E+8, 0.33333, 0., 0.1, 1.E–5
.15E+8, 0.25, 0., 0.1, 5.E–5
1, 3, 0., 0., 0., 0.
2, 2, 1., 0., 0., 0.
3, 0, 0., 1., 0., 0.
4, 0, 1., 1., 0., 0.
1, 1, 2. 3, 3. 1. 80
2, 2, 4, 3, 3, 2. 50.
```

Run using CSFEP modified for temperature changes in elements

verification.5

0 INPUT TABLE 1.. BASIC PARAMETERS

NUMBER OF NODAL POINTS.	4
NUMBER OF ELEMENTS.	2
NUMBER OF DIFFERENT MATERIALS.	2
NUMBER OF SURFACE LOAD CARDS	0
1 = PLANE STRAIN, 2 = PLANE STRESS. . . .	2
BODY FORCE (1 = IN – Y DIREC., 0 = NONE)	0

0 INPUT TABLE 2.. MATERIAL PROPERTIES

MATERIAL NUMBER	MODULUS OF ELASTICITY	POISSON'S RATIO	MATERIAL DENSITY	MATERIAL THICKNESS	ALPHA
1	0.3000E+08	0.3333E+00	0.0000E+00	0.1000E+00	0.1000E–04
2	0.1500E+08	0.2500E+00	0.0000E+00	0.1000E+00	0.5000E–04

INPUT TABLE 3.. NODAL POINT DATA

NODAL POINT	TYPE	X	Y	X-DISP. OR LOAD	Y-DISP. OR LOAD
1	3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	2	0.1000E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0	0.0000E+00	0.1000E+01	0.0000E+00	0.0000E+00
4	0	0.1000E+01	0.1000E+01	0.0000E+00	0.0000E+00

0 INPUT TABLE 4.. ELEMENT DATA

	GLOBAL INDICES OR ELEMENT NODES				
ELEMENT	1	2	3	4	MATERIAL TEMP
1	1	2	3	3	1 0.800E+02
2	2	4	3	3	2 0.500E+02

0 OUTPUT TABLE 1.. NODAL DISPLACEMENTS

NODE	U = X-DISP.	V = Y-DISP.
1	0.00000000E+00	0.00000000E+00
2	0.98888970E–03	0.00000000E+00
3	–0.75555370E–03	0.98888970E–03
4	0.13194460E–02	0.20750000E–02

1OUTPUT TABLE 2.. STRESSES AT ELEMENT CENTROIDS

ELEMENT	X	Y	SIGMA(X)	SIGMA(Y)	TAU(X, Y)
1	0.33	0.33	8.5000E+03	8.5000E+03	–8.5000E+03
2	0.67	0.67	–8.5000E+03	–8.5000E+03	8.5000E+03
			SIGMA(1)	SIGMA(2)	ANGLE
			1.7000E+04	–3.9063E–03	0.0000E+00
			–1.9531E–03	–1.7000E+04	4.5000E+01

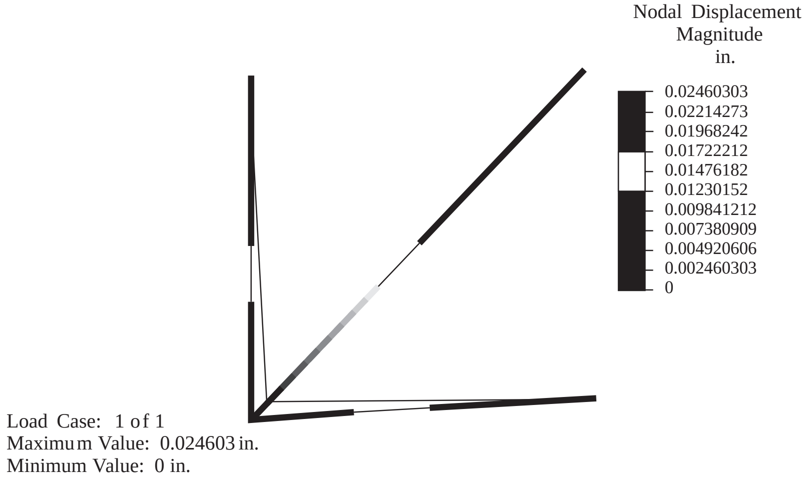
15.24 Solve Problem 15.3 using the Algor Program.

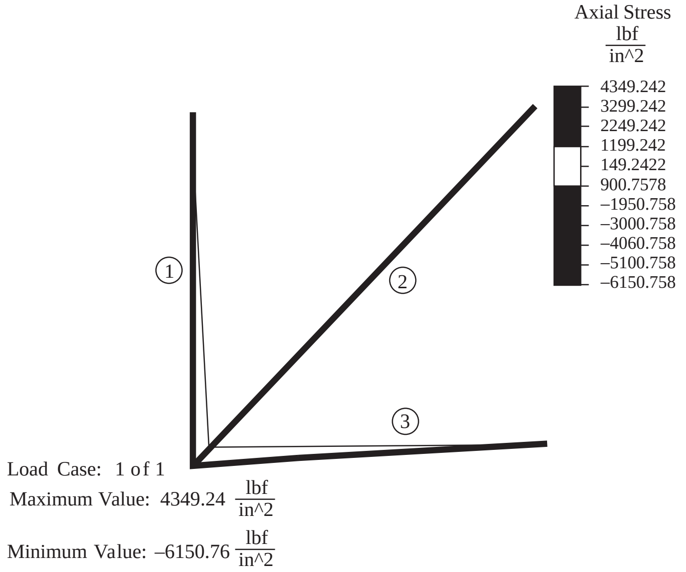
Displacements/Rotations (degrees) of nodes

NODE number	X-translation	Y-translation	Z-translation	X-rotation	Y-rotation	Z-rotation
1	–1.7397E–02	–1.7397E–02	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
4	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

**** Nodal stresses for 3-D truss elements

El. #	LC	ND	Stress	Force
----	---	--	---------	----------
1	1	I	–4.349E+03	–8.698E+03
1	1	J	4.349E+03	8.698E+03
2	1	I	–4.349E+03	–8.698E+03
2	1	J	4.349E+03	8.698E+03
3	1	I	6.151E+03	1.230E+04
3	1	J	–6.151E+03	–1.230E+04





15.25 For the plane truss shown in Figure P15-6, bar element 2 is subjected to a uniform temperature drop of T = 20°C. Let E = 70 GPa, A = 4 × 10^–4m^2, and alpha = $23 \times 10^{-6}\ \frac{\frac{\text{mm}}{\text{mm}}}{°\text{C}}$. Determine the stresses in each bar and the displacement of node 1.

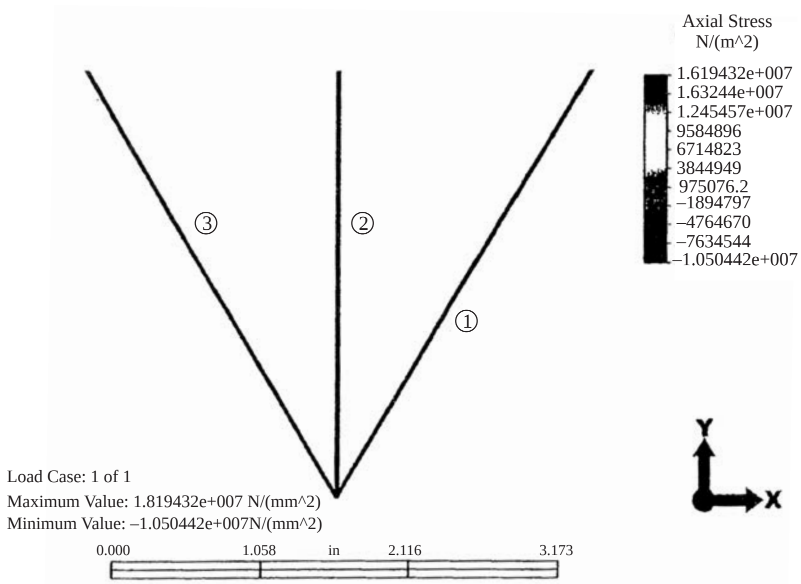


Figure 1 Axial Stress

As shown in Figure 1, the stress in bar 1 and 3 is 10.5 MPa (C) and the stress in bar 2 is 18.2 MPa (T).

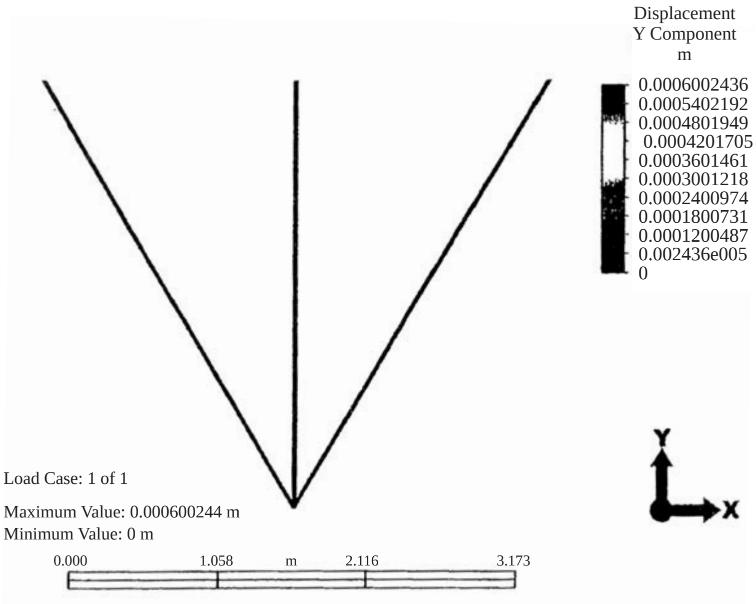
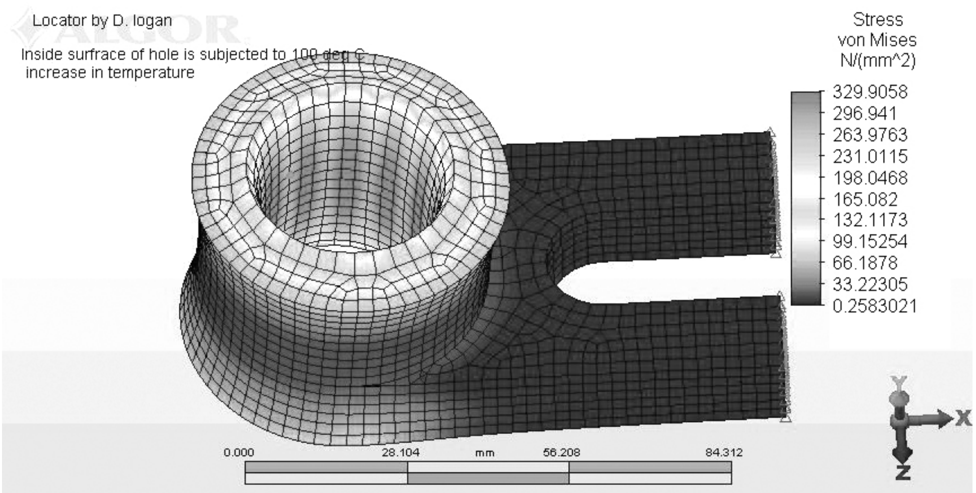


Figure 2 Displacement in Y Direction

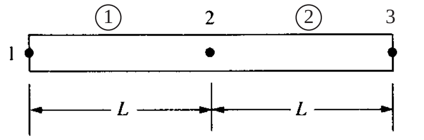
Figure 2 shows the *Y* displacement at node 1 is 0.0006 m in the positive *Y* direction. There is no displacement at node 1 in the *X* direction.

15.27



Chapter 16

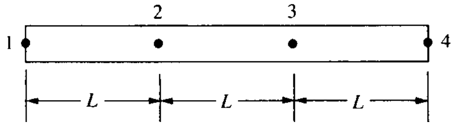
16.1



$$[m^{(1)}] = \frac{\rho AL}{6}\,, \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}, \quad [m^{(2)}] = \frac{\rho AL}{6} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

16.2



(a) Lumped mass matrix

$$[m^{(1)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad [m^{(2)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$[m^{(3)}] = \frac{\rho AL}{2} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{\rho AL}{2}$$

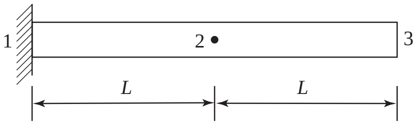
(b) Consistent mass matrix

$$[m^{(1)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [m^{(2)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m^{(3)}] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

16.3



$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$([K] - \omega^2\,[M])\,\{X\} = 0$ with $x_1 = 0$

$$\left(\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2\,\frac{\rho AL}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\qquad \omega^2 = \lambda$

divide by ρAL and let $\mu = \dfrac{E}{\rho L^2}$

$$\therefore \qquad \left|\mu\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} \\ -\mu - \frac{\lambda}{6} & \mu - \frac{\lambda}{3} \end{vmatrix} = 0$$

$$\left(2\mu - \frac{2}{3}\lambda\right)(\mu - \frac{\lambda}{3}) - (-\mu - \frac{\lambda}{6})^2 = 0$$

$$2\mu^2 - \frac{4}{3}\,\mu\lambda + \frac{2}{9}\lambda^2 - \mu^2 - \frac{\mu\lambda}{3} - \frac{\lambda^2}{36} = 0$$

$$\mu^2 - \frac{5}{3}\,\mu\lambda + \frac{7}{36}\,\lambda^2 = 0$$

or $$\lambda^2 - \frac{60}{7}\,\mu\lambda + \frac{36}{7}\,\mu^2 = 0$$

$$\lambda_{1,2} = \frac{\frac{60}{7}\mu \pm \sqrt{\left(\frac{60}{7}\mu\right)^2 - 4\,(1)\left(\frac{36}{7}\right)\mu^2}}{2}$$

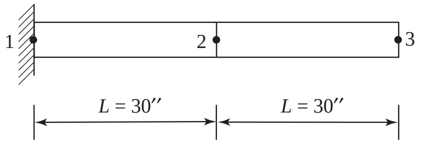
$$= \frac{8.571\,\mu \pm 7.273\,\mu}{2}$$

$$\lambda_1 = 0.649\,\mu \qquad \lambda_2 = 7.922\,\mu$$

$$\therefore \qquad \omega_1 = \lambda_1^{\frac{1}{2}} = 0.806\sqrt{\mu}$$

$$\omega_2 = \lambda_2^{\frac{1}{2}} = 2.815\sqrt{\mu}$$

16.4 (a) Two equal length elements



From Problem 16.3 results

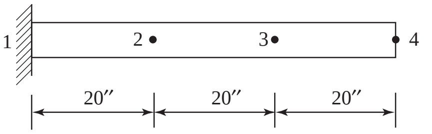
$$\omega_1 = 0.806\,\mu^{\frac{1}{2}},\ \omega_2 = 2.815\,\mu^{\frac{1}{2}}$$

$$\mu = \frac{E}{\rho L^2} = \frac{30\times 10^6}{(0.00073)\,(30)^2} = 45.66 \times \frac{10^6}{\text{s}^2}$$

$$\omega_1 = 0.806\,\sqrt{45.66\times 10^6} = 5.446 \times 10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_2 = 2.815\,\sqrt{45.66\times 10^6} = 19.02 \times 10^3\ \frac{\text{rad}}{\text{s}}$$

(b) 3 equal length elements



$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \frac{AE}{L}$$

$$[M] = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now $\quad x_1 = 0 \quad ([K] - \omega^2\,[M])\,\{X\} = 0$

$$\therefore \quad \left| \frac{AE}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\,\frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

or

$$\left| \mu \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\lambda}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} & 0 \\ -\mu - \frac{\lambda}{6} & 2\mu - \frac{2}{3}\lambda & -\mu - \frac{\lambda}{6} \\ 0 & -\mu - \frac{\lambda}{6} & -\mu - \frac{\lambda}{3} \end{vmatrix} = 0$$

$$\left(2\mu-\frac{2}{3}\lambda\right)\left(2\mu-\frac{2}{3}\lambda\right)\left(\mu-\frac{\lambda}{3}\right)-$$

$$\left[\left(-\mu-\frac{\lambda}{6}\right)\left(-\mu-\frac{\lambda}{6}\right)\left(2\mu-\frac{2}{3}\lambda\right)+\left(-\mu-\frac{\lambda}{6}\right)\left(-\mu-\frac{\lambda}{6}\right)\left(\mu-\frac{\lambda}{3}\right)\right]=0$$

$$\left(\mu-\frac{\lambda}{3}\right)\left[4\mu^2-\frac{8}{3}\mu\lambda+\frac{4}{9}\lambda^2-3\mu^2-\mu\lambda-\frac{\lambda^2}{12}\right]=0$$

$$\left(\mu-\frac{\lambda}{3}\right)=0,\ \lambda_1=3\mu$$

or

$$\mu^2-\frac{11}{3}\mu\lambda+\frac{13\lambda^2}{36}=0$$

$$\lambda_{2,3}=\frac{132\mu\pm\sqrt{(132\mu)^2-4(13)\,36\mu^2}}{2(13)}$$

$$\lambda_2=9.873\ \mu,\quad \lambda_3=0.2805\ \mu$$

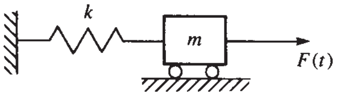
$$\mu=\frac{E}{\rho L^2}=\frac{30\times10^6}{(0.00073)\,(20'')^2}=1.0274\times10^8$$

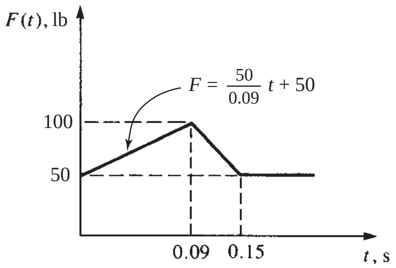
$$\omega_1=\sqrt{\lambda_1}=\sqrt{0.2805\mu}=5.368\times10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_2=\sqrt{3\mu}=17.556\times10^3\ \frac{\text{rad}}{\text{s}}$$

$$\omega_3=\sqrt{9.873\mu}=31.85\times10^3\ \frac{\text{rad}}{\text{s}}$$

16.5





$$t_0=0\ d_0=d_{0\text{dot}}=0$$

$$d_{0\text{dotdot}}=\frac{1}{2}(50-0)=25\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{-1}=0-0+\frac{(0.03)^2}{2}\times25=0.01125\text{ ft}$$

$$t_1=0.03\text{ s}$$

$$d_1=\frac{1}{2}\ \{(0.03)^2\,(50)+(2\times2-0.03^2\times2000)\,0-2\times0.01125\}$$

$$=0.01125\text{ ft}$$

$$d_2 = \frac{1}{2}\{(0.03)^2\,(66.67) + (2 \times 2 - 0.03^2 \times 2000 \times 0.01125\}$$

$$d_2 = 0.04238 \text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{2}\{66.67 - 2000 \times 0.01125\}$$

$$d_{2\text{dotdot}} = 22.09 \ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = \frac{0.04238 - 0}{2\,(0.03)} = 0.71 \ \frac{\text{ft}}{\text{s}}$$

$$t_2 = 0.06 \text{ s}$$

$$d_3 = \frac{1}{2}\{(0.03)^2\,(83.33) + (2 \times 2 - (0.03)^2 \times 2000) \times (0.04238) - 2 \times 0.01125\}$$

$$d_3 = 0.07287 \text{ ft}$$

$$d_{2\text{dotdot}} = \frac{1}{2}\{83.33 - 2000 \times 0.04238\}$$

$$d_{2\text{dotdot}} = -\,0.715 \ \frac{\text{ft}}{\text{s}^2}$$

$$d_{2\text{dot}} = \frac{0.07287 - 0.01125}{2(0.03)} = 1.03 \ \frac{\text{ft}}{\text{s}}$$

$$t_3 = 0.09 \text{ s}$$

$$d_4 = \frac{1}{2}\{0.03^2 \times 100 + (2 \times 2 - 0.03^2 \times 2000) \times (0.07287) - 2 \times 0.04238\}$$

$$d_4 = 0.08278 \text{ ft}$$

$$d_{3\text{dotdot}} = \frac{1}{2}(100 - 2000 \times 0.07287)$$

$$d_{3\text{dotdot}} = -\,22.87 \ \frac{\text{ft}}{\text{s}^2}$$

$$d_{3\text{dot}} = \frac{0.08278 - 0.04238}{2(0.03)} = 0.67 \ \frac{\text{ft}}{\text{s}}$$

$$t_4 = 0.12 \text{ s}$$

$$d_5 = \frac{1}{2}\{(0.03)^2\,75 + (2 \times 2 - 0.03^2 \times 2000)\,(0.08278) - 2 \times 0.07287\}$$

$$d_5 = 0.05194 \text{ ft}$$

$$d_{4\text{dotdot}} = \frac{1}{2}(75 - 2000 \times 0.08278)$$

$$d_{4\text{dotdot}} = -\,45.28 \ \frac{\text{ft}}{\text{s}^2}$$

$$d_{4\text{dot}} = \frac{0.05194 - 0.07287}{2(0.03)} = -\,0.35 \ \frac{\text{ft}}{\text{s}}$$

$$t_5 = 0.15 \text{ s}$$

$$d_6 = \frac{1}{2}\{(0.03)^2\,50 + (2 \times 2 - 0.03^2 \times 2000)\,(0.5194) - 2 \times 0.08278\}$$

$$d_6 = -\,3.146 \times 10^{-3} \text{ ft}$$

$$d_{5\text{dotdot}} = \frac{1}{2}\{50 - 2000\ (0.05194)\}$$

$$d_{5\text{dotdot}} = -26.94\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{5\text{dot}} = \frac{-3.146\times10^{-3} - 0.08278}{2\ (0.03)} = -1.43\,\frac{\text{ft}}{\text{s}}$$

Summary

t, s	$F(t)$ lb	d_i, ft	$d_{i\text{dotdot}}\ \frac{\text{ft}}{\text{s}^2}$	$d_{i\text{dot}}\ \frac{\text{ft}}{\text{s}}$
0	50	0	25	0
0.03	66.67	0.01125	22.09	0.71
0.06	83.33	0.04238	–0.715	1.03
0.09	100	0.07287	–22.87	0.67
0.12	75	0.08278	–45.28	–0.35
0.15	50	0.05194	–26.94	–1.43

(b) By Newmark's method

$$\beta = \frac{1}{6},\ \gamma = \frac{1}{2} \qquad M = 2 \text{ slugs}$$

$$F = \frac{50}{0.09}\,t + 50 \qquad K = 2000\ \frac{\text{lb}}{\text{ft}}$$

$$F(0.03) = 555.6\ (0.03) + 50 = 66.67 \text{ lb}$$

$$F'_{i+1} = F_{i+1} + \frac{M}{\beta\,\Delta t^2}\left[d_i + \Delta t\ d_{i\text{dot}} + \left(\frac{1}{2}-\beta\right)\Delta t^2\ d_{i\text{dotdot}}\right]$$

$$F'_i = 66.67 + \frac{2}{\frac{1}{6}(0.03)^2}\left[0 + (0.03)(0) + \left(\frac{1}{2}-\frac{1}{6}\right)(0.03)^2\ d_{0\text{dotdot}}\right]$$

and $$d_{0\text{dotdot}} = M^{-1}\ (F_0 - K\,d_0) = \frac{50 - 2000\ (0)}{2} = 25\ \frac{\text{ft}}{\text{s}^2}$$

$$\therefore\quad F'_i = 66.67 + \frac{2}{\frac{1}{6}(0.03)^2}\left[\left(\frac{1}{2}-\frac{1}{6}\right)(0.03)^2\ (25)\right]$$

$$= 166.67 \text{ lb}$$

$$\therefore\quad d_1 = \frac{F'_1}{K'} = \frac{166.67}{K'}$$

and

$$K' = K + \frac{1}{\beta(\Delta t)^2}\ M$$

$$= 2000 + \frac{1}{\left(\frac{1}{6}\right)(0.03)^2}\ (2) = 15333$$

$$\therefore\quad d_1 = \frac{166.67}{15333} = 0.01087 \text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_1 - d_0 - \Delta t\left(d_{0\text{dot}}\right) - (\Delta t)^2\left(\frac{1}{2}-\beta\right)d_{0\text{dotdot}}\right]$$

$$= \frac{1}{\left(\frac{1}{6}\right)(0.03)^2}\left[0.01087 - 0 - 0 - (0.03)^2\left(\frac{1}{2} - \frac{1}{6}\right)25\right]$$

$$d_{1\text{dotdot}} = 22.47\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = d_{0\text{dot}} + \Delta t\ [(1 - \gamma)\ d_{0\text{dotdot}} + r\ d_{1\text{dot}}]$$

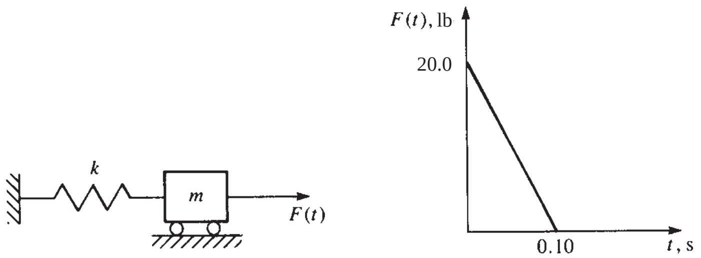
$$= 0 + (0.03)\left[\left(1 - \frac{1}{2}\right)25 + \frac{1}{2}(22.47)\right]$$

$$d_{1\text{dot}} = 0.71205\ \frac{\text{ft}}{\text{s}}$$

Table below summarizes the results using Newmark's method

Time	Displacement	Velocity	Acceleration	Force	Force prime
0	0	0	25	50	
0.03	0.01087	0.711957	22.46377	66.66667	166.6667
0.06	0.039319	1.084121	2.347196	83.33333	602.8986
0.09	0.069606	0.825234	–19.6063	100	1067.297
0.12	0.081832	–0.13384	–44.3317	75	1254.753
0.15	0.059363	–1.31425	–34.3627	50	910.2281
0.18	0.011632	–1.62917	13.3684	50	178.3512

16.6



(a) Using central difference

$$d_0 = 0, \quad d_{0\text{dot}} = 0$$

$$\Delta t = 0.02\ \text{s}$$

Step 1 $\quad t = 0.02\text{s} \qquad F_1 = 16$ lb

$$[M] = m = 2\ \text{slugs},\ [M^{-1}] = \frac{1}{2}$$

$$[K] = k = 1200\ \frac{\text{lb}}{\text{ft}}$$

$$d_{0\text{dotdot}} = \frac{1}{2}[20 - 1200\ (0)] = 10\ \frac{\text{ft}}{\text{s}^2}$$

$$\{d_{-1}\} = 0 - (0.02)\ (0) + \frac{0.02^2}{2}(10) = 0.002\ \text{ft}$$

$$\{d_1\} = \frac{1}{2}\ [(0.02)^2\ (20) + \{2(2) + (0.02)^2\ (1200)\}\ (0) - 2\ (0.002)]$$

$$= 0.002\ \text{ft}$$

$$\{d_2\} = \frac{1}{2}\ [0.02^2\ (16) + \{2(2) - 0.02^2\ (1200)\}\ (0.002) - 2(0)]$$

$$= 0.00672 \text{ ft}$$

$$d_{1\text{dotdot}} = \frac{1}{2}\ (16 - 1200\ (0.002)) = 6.8\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{1\text{dot}} = \frac{0.00672 - 0}{2(0.02)} = 0.168\ \frac{\text{ft}}{\text{s}}$$

Step 2 $\quad t = 0.04$ s $\quad F_2 = 12$ lb

$$d_3 = \frac{1}{2}[0.02^2\ (12) + \{2(2) - 0.02^2(1200)\}\ (0.00672) - 2\ (0.002)^2]$$

$$d_3 = 0.01223 \text{ ft}$$

$$d_{2\text{dotdot}} = \frac{1}{2}\ (12 - 1200\ (0.00672)) = 1.968\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{2\text{dot}} = \frac{0.01223 - 0.002}{2(0.02)} = 0.2558\ \frac{\text{ft}}{\text{s}}$$

Step 3 $\quad t = 0.06$ s $\quad F_3 = 8$ lb

$$d_4 = \frac{1}{2}\ [0.02^2\ (8) + 3.52\ (0.01223) - 2\ (0.00672)]$$

$$= 0.0164 \text{ ft}$$

$$d_{3\text{dotdot}} = \frac{1}{2}\ (8 - 1200\ (0.01223)) = -3.338\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{3\text{dot}} = \frac{0.0164 - 0.00672}{2(0.02)} = 0.242\ \frac{\text{ft}}{\text{s}}$$

Step 4 $\quad t = 0.08$ s $\quad F_4 = 4$ lb

$$d_5 = \frac{1}{2}\ [0.02^2\ (4) + 3.52\ (0.0164) - 2\ (0.01223)$$

$$= 0.01743 \text{ ft}$$

$$d_{4\text{dotdot}} = \frac{1}{2}\ (4 - 1200\ (0.01640)) = -7.84\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{4\text{dot}} = \frac{0.01743 - 0.01223}{2(0.02)} = 0.13\ \frac{\text{ft}}{\text{s}}$$

Step 5 $\quad t = 0.10$ s $\quad F_5 = 0$

$$d_6 = \frac{1}{2}\ [0.02^2(0) + 3.52\ (0.01743) - 2\ (0.01640)]$$

$$= 0.01428 \text{ ft}$$

$$d_{5\text{dotdot}} = \frac{1}{2}\ (0 - 1200\ (0.01743)) = -\ 10.46\ \frac{\text{ft}}{\text{s}^2}$$

$$d_{5\text{dot}} = \frac{0.01428 - 0.01640}{2(0.02)} = -\ 0.053\ \frac{\text{ft}}{\text{s}}$$

Summary

t, s	d, ft	$d_{\text{dot}}, \frac{\text{ft}}{\text{s}}$	$d_{\text{dotdot}}, \frac{\text{ft}}{\text{s}^2}$
0	0	0	10
0.02	0.002	0.168	6.8
0.04	0.00672	0.2558	1.968
0.06	0.01223	0.242	–3.338
0.08	0.01640	0.130	–7.89
0.10	0.01743	–0.053	–10.46

(b) Newmark's time integration method (Mathcad solution)

$$F_0 = 20\ \text{lb} \qquad K = 1200\ \frac{\text{lb}}{\text{ft}} \qquad M = 2\ \text{slug} \qquad M = 2\ \frac{\text{lb}\cdot\text{s}^2}{\text{ft}} \qquad \Delta t = 0.02\text{s}$$

Assume $\beta = \frac{1}{6}$ $\quad \gamma = \frac{1}{2}$ for linear acceleration within each time step

At time $t = 0$

$$d_0 = 0\ \text{ft} \qquad d_{0\text{dot}} = 0\ \frac{\text{ft}}{\text{s}}$$

Acceleration at $t = 0$

$$d_{0\text{dotdot}} = \frac{F_0 - K\,d_0}{M} \qquad d_{0\text{dotdot}} = 10\ \frac{\text{ft}}{\text{s}^2}$$

Displacement at $t = 0.02$

$$K_{\text{prime}} = K + \frac{1}{\beta\,(\Delta t)^2}\,M \qquad K_{\text{prime}} = 2.6 \times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$F_1 = \frac{4}{5}\ F_0 \qquad F_1 = 16\ \text{lb}$$

$$F_{1\text{prime}} = F_1 + \frac{M}{\beta(\Delta t)^2}\left[d_0 + (\Delta t)\,d_{0\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\ d_{0\text{dotdot}}\right]$$

$$F_{1\text{prime}} = 56\ \text{lb}$$

$$d_1 = \frac{F_{1\text{prime}}}{K_{\text{prime}}} \qquad d_1 = 1.795 \times 10^{-3}\ \text{ft}$$

Acceleration at $t = 0.02$

$$d_{1\text{dotdot}} = \frac{1}{\beta\,(\Delta t)^2}\left[d_1 - d_0 - (\Delta t)\,d_{0\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right)d_{0\text{dotdot}}\right]$$

$$d_{1\text{dotdot}} = 6.923\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.02$

$$d_{1\text{dot}} = d_{0\text{dot}} + (\Delta t)\ [(1 - \gamma)\ d_{0\text{dotdot}} + \gamma\ d_{1\text{dotdot}}] \qquad d_{1\text{dot}} = 0.169\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.04$ $\qquad F_2 = \frac{3}{5}\ F_0 \qquad F_2 = 12\ \text{lb}$

$$F_{2\text{prime}} = F_2 + \frac{M}{\beta(\Delta t)^2}\left[d_1 + (\Delta t)\, d_{1\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{1\text{dotdot}}\right]$$

$F_{2\text{prime}} = 195.07$ lb

$$d_2 = \frac{F_{2\text{prime}}}{K_{\text{prime}}} \qquad d_2 = 6.252 \times 10^{-3} \text{ ft}$$

Acceleration at $t = 0.04$

$$d_{2\text{dotdot}} = \frac{1}{\beta(\Delta t)^2}\left[d_2 - d_1 - (\Delta t)\, d_{1\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{1\text{dotdot}}\right]$$

$$d_{2\text{dotdot}} = 2.249 \ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.04$

$$d_{2\text{dot}} = d_{1\text{dot}} + (\Delta t)\ [(1 - \gamma)\, d_{1\text{dotdot}} + \gamma\ d_{2\text{dotdot}}]$$

$$\text{d}_{2\text{dot}} = 0.0261 \ \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.06$

$$F_3 = \frac{2}{5}\ F_0 \qquad F_3 = 8 \text{ lb}$$

$$F_{3\text{prime}} = F_3 + \frac{M}{\beta(\Delta t)^2}\left[d_2 + (\Delta t)\, d_{2\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{2\text{dotdot}}\right]$$

$F_{3\text{prime}} = 361.136$ lb

$$d_3 = \frac{F_{3\text{prime}}}{K_{\text{prime}}} \qquad d_3 = 0.012 \text{ ft}$$

Acclertaion at $t = 0.06$

$$d_{3\text{dotdot}} = \frac{1}{\beta(\Delta t)^2}\left[d_3 - d_2 - (\Delta t)\, d_{2\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{2\text{dotdot}}\right]$$

$$d_{3\text{dotdot}} = -2.945 \ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.06$

$$d_{3\text{dot}} = d_{2\text{dot}} + (\Delta t)\ [(1 - \gamma)\, d_{2\text{dotdot}} + \gamma\ d_{3\text{dotdot}}]$$

$$d_{3\text{dot}} = 0.254 \ \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.08$

$$F_4 = \frac{1}{5}\ F_0 \qquad F_4 = 4 \text{ lb}$$

$$F_{4\text{prime}} = F_4 + \frac{M}{\beta(\Delta t)^2}\left[d_3 + (\Delta t)\, d_{3\text{dot}} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\, d_{3\text{dotdot}}\right]$$

$F_{4\text{prime}} = 491.85$ lb

$$d_4 = \frac{F_{4\text{prime}}}{K_{\text{prime}}} \qquad d_4 = 0.016 \text{ ft}$$

Acceleration at $t = 0.08$

$$d_{4\text{dotdot}} = \frac{1}{\beta(\Delta t)^2}\left[d_4 - d_3 - (\Delta t)\, d_{3\text{dot}} - (\Delta t)^2\left(\frac{1}{2} - \beta\right) d_{3\text{dotdot}}\right]$$

$$d_{4\text{dotdot}} = -7.459 \ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.08$

$$d_{4dot} = d_{3dot} + (\Delta t)\ [(1-\gamma)\ d_{3dotdot} + \gamma\ d_{4dotdot}] \qquad d_{4dot} = 0.15\ \frac{\text{ft}}{\text{s}}$$

Displacement at $t =\ \ 0.10$

$$F_5 = \frac{0}{5}\ F_0 \qquad F_5 = 0\ \text{lb}$$

$$F_{5prime} = F_5 + \frac{M}{\beta\,(\Delta t)^2}\left[d_4 + (\Delta t)\,d_{4dot} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\,d_{4dotdot}\right]$$

$$F_{5prime} = 533.071\ \text{lb}$$

$$d_5 = \frac{F_{5prime}}{K_{prime}} \qquad d_5 = 0.017\ \text{ft}$$

Acceleration at $t = 0.10$

$$d_{5dotdot} = \frac{1}{\beta\,(\Delta t)^2}\left[d_5 - d_4 - (\Delta t)\,d_{4dot} - (\Delta t)^2\left(\frac{1}{2} - \beta\right)d_{4dotdot}\right]$$

$$d_{5dotdot} = -\,10.251\ \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.10$

$$d_{5dot} = d_{4dot} + (\Delta t)\ [(1-\gamma)\ d_{4dotdot} + \gamma\,d_{5dotdot}] \qquad d_{5dot} = -\,0.027\ \frac{\text{ft}}{\text{s}}$$

(c) Wilson's method

$$F_0 = 20\ \text{lb} \qquad K = 1200\ \frac{\text{lb}}{\text{ft}} \qquad M = 2\ \text{slug} \qquad M = 2\,\frac{\text{lb}\cdot\text{s}^2}{\text{ft}} \quad \Delta t = 0.02\text{s}$$

Assume $\qquad \Theta = 1$

At time $t = 0$

$$d_0 = 0\ \text{ft} \qquad d_{0dot} = 0\,\frac{\text{ft}}{\text{s}}$$

Acceleration at $\ t = 0$

$$d_{0dotdot} = \frac{F_0 - K\ d_0}{M} \qquad d_{0dotdot} = 10\ \frac{\text{ft}}{\text{s}^2}$$

Displacement at $t = 0.02$

$$K_{prime} = K + \frac{6}{(\Theta\ \Delta t)^2}\,M \qquad K_{prime} = 2.6 \times 10^3\ \frac{\text{lb}}{\text{in.}}$$

$$F_1 = \frac{4}{5}\ F_0 \qquad F_1 = 16\ \text{lb}$$

$$F_{1prime} = F_1 + \frac{M}{(\Theta\ \Delta t)^2}\ [6\ d_0 + 6\ \Theta\ (\Delta t)\ d_{0dot} + 2\ (\Theta\ \Delta t)^2\ d_{0dotdot}]$$

$$F_{1prime} = 56\ \text{lb}$$

$$d_1 = \frac{F_{1prime}}{K_{prime}} \qquad d_1 = 1.795 \times 10^{-3}\ \text{ft}$$

Acceleration at $t = 0.02$

$$d_{1dotdot} = \frac{6}{\Theta^2\,(\Delta t)^2}\,(d_1 - d_0) - \frac{6}{\Theta\,(\Delta t)}\,d_{0dot} - 2\,d_{0dotdot} \qquad d_{1dotdot} = 6.923\,\frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.02$

$$d_{1dot} = \frac{3}{\Theta\,(\Delta t)}(d_1 - d_0) - 2\,d_{0dot} - \frac{\Theta\,(\Delta t)}{2}\,d_{0dotdot} \qquad d_{1dot} = 0.169\,\frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.04$

$$F_2 = \frac{3}{5}\,F_0 \qquad F_2 = 12\text{ lb}$$

$$F_{2prime} = F_2 + \frac{3}{\Theta^2\,(\Delta t)^2}\,[6\,d_1 + 6\,\Theta\,(\Delta t)\,d_{1dot} + 2\,(\Theta\,\Delta t)^2\,d_{1dotdot}]$$

$$F_{2prime} = 195.077\text{ lb}$$

$$d_2 = \frac{F_{2prime}}{K_{prime}} \qquad d_2 = 6.252 \times 10^{-3}\text{ ft}$$

Acceleration at $t = 0.04$

$$d_{2dotdot} = \frac{6}{\Theta^2\,(\Delta t)^2}\,(d_2 - d_1) - \frac{6}{\Theta\,(\Delta t)}\,d_{1dot} - 2\,d_{1dotdot} \qquad d_{2dotdot} = 2.249\,\frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.04$

$$d_{2dot} = \frac{3}{\Theta\,(\Delta t)}\,(d_2 - d_1) - 2\,d_{1dot} - \frac{\Theta\,\Delta t}{2}\,d_{1dotdot} \qquad d_{2dot} = 0.261\,\frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.06$

$$F_3 = \frac{2}{5}\,F_0 \qquad F_3 = 8\text{ lb}$$

$$F_{3prime} = F_3 + \frac{M}{\theta^2\,(\Delta t)^2}\,[6\,d_2 + 6\,\Theta\,(\Delta t)\,d_{2dot} + 2\,(\Theta\,\Delta t)^2\,d_{2dotdot}]$$

$$F_{3prime} = 361.136\text{ lb}$$

$$d_3 = \frac{F_{3prime}}{K_{prime}} \qquad d_3 = 0.012\text{ ft}$$

Acceleration at $t = 0.06$

$$d_{3dotdot} = \frac{6}{\Theta^2\,(\Delta t)^2}(d_3 - d_2) - \frac{6}{\Theta\,(\Delta t)}\,d_{2dot} - 2\,d_{2dotdot} \qquad d_{3dotdot} = -2.945\,\frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.06$

$$d_{3dot} = \frac{3}{\Theta\,\Delta t}(d_3 - d_2) - 2\,d_{2dot} - \frac{\Theta\,\Delta t}{2}\,d_{2dotdot} \qquad d_{3dot} = 0.254\,\frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.08$

$$F_4 = \frac{1}{5}\,F_0 \qquad F_4 = 4\text{ lb}$$

$$F_{4prime} = F_4 + \frac{M}{(\Theta\,\Delta t)^2}[6\,d_3 + 6\,\Theta\,(\Delta t)\,d_{3dot} + 2\,(\Theta\,\Delta t)^2\,d_{3dotdot}]$$

$F_{4\text{prime}} = 491.865$ lb

$$d_4 = \frac{F_{4\text{prime}}}{K_{\text{prime}}} \qquad d_4 = 0.016 \text{ ft}$$

Acceleration at $t = 0.08$

$$d_{4\text{dotdot}} = \frac{6}{\Theta^2 (\Delta t)^2}(d_4 - d_3) - \frac{6}{\Theta(\Delta t)} d_{3\text{dot}} - 2\, d_{3\text{dotdot}}$$

$$d_{4\text{dotdot}} = -7.459 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.08$

$$d_{4\text{dot}} = \frac{3}{\Theta\,\Delta t}(d_4 - d_3) - 2\, d_{3\text{dot}} - \frac{\Theta\,\Delta t}{2} d_{3\text{dotdot}} \qquad d_{4\text{dot}} = 0.15 \frac{\text{ft}}{\text{s}}$$

Displacement at $t = 0.10$

$$F_5 = \frac{0}{5} F_0 \qquad F_5 = 0 \text{ lb}$$

$$F_{5\text{prime}} = F_5 + \frac{M}{(\Theta\ \ \Delta t)^2}[6\, d_4 + 6\, \Theta\, (\Delta t)\, d_{4\text{dot}} + 2\, (\Theta\, \Delta t)^2\ d_{4\text{dotdot}}]$$

$F_{5\text{prime}} = 533.071$ lb

$$d_5 = \frac{F_{5\text{prime}}}{K_{\text{prime}}} \qquad d_5 = 0.017 \text{ ft}$$

Acceleration at $t = 0.10$

$$d_{5\text{dotdot}} = \frac{6}{\Theta^2 (\Delta t)^2}(d_5 - d_4) - \frac{6}{\Theta(\Delta t)} d_{4\text{dot}} - 2\, d_{4\text{dotdot}}$$

$$d_{5\text{dotdot}} = -10.251 \frac{\text{ft}}{\text{s}^2}$$

Velocity at $t = 0.10$

$$d_{5\text{dot}} = \frac{3}{\Theta\,\Delta t}(d_5 - d_4) - 2\, d_{4\text{dot}} - \frac{\Theta\,\Delta t}{2} d_{4\text{dotdot}} \qquad d_{5\text{dot}} = -0.027 \frac{\text{ft}}{\text{s}}$$

Newmark's time integration method. Wilson's method (Linear acceleration)

Assume linear acceleration within each time step $\Theta = 1$

$\beta = 0.167$ $K' = 31200$

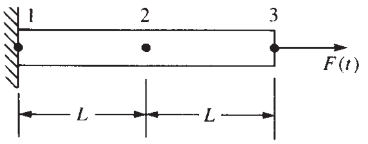
$\gamma = 0.5$

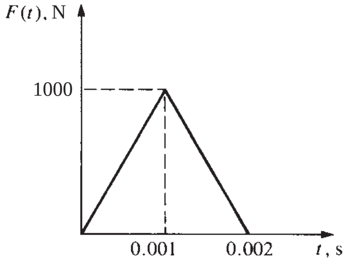
$K' = 31200$

Summary table Newmark and Wilson same results.

di	*time*	$F(t)$ (lb)	d_i (ft)	d_i *velocity* ($\frac{\text{ft}}{\text{s}}$)	d_i *accel* ($\frac{\text{ft}}{\text{s}^2}$)	F'
0	0	20	0	0	10	
1	0.02	16	0.00179	0.169	6.923	56
2	0.04	12	0.00625	0.261	2.249	195
3	0.06	8	0.01157	0.254	–2.945	361
4	0.08	4	0.01576	0.150	–7.459	492
5	0.1	0	0.01709	– 0.027	–10.251	533

16.7





Use time step $\Delta t = 2.5 \times 10^{-4}$ s

$$[u_1] = 0,\ u_{1\text{dot}} = u_{1\text{dotdot}} = 0$$

$$\{d_{0\text{dotdot}}\} = [M^{-1}]\,(\{F_0\} - [K]\{d_0\})$$

$$\{d_{0\text{dotdot}}\} = \frac{\rho AL}{2}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$t = 0.00025 \text{ s}$$

$$\{d_1\} = \frac{F_1'}{[K']} = [K']^{-1}\ F_i'$$

$$[K'] = [K] + \frac{1}{\beta\,\Delta t^2}\ [M]$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\rho AL}{2\beta\Delta t^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K']^{-1} = \begin{bmatrix} 1.319\times 10^{-7} & 1.040\times 10^{-8} \\ 1.040\times 10^{-8} & 2.637\times 10^{-7} \end{bmatrix}$$

$$[F_1'] = \begin{Bmatrix} 0 \\ 250 \end{Bmatrix} + \frac{\rho AL}{2\beta\Delta t^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\{d_0\} + \Delta t\{d_{0\text{dot}}\} + \left(\frac{1}{2} - \beta\right)\{d_{0\text{dotdot}}\}\right\}$$

$$[F_1'] = \begin{Bmatrix} 0 \\ 250 \end{Bmatrix}$$

$$\therefore\quad \{d_1\} = \begin{Bmatrix} 2.6\times 10^{-6} \\ 6.593\times 10^{-5} \end{Bmatrix} \text{ in.}$$

$$\{d_{1\text{dotdot}}\} = \frac{1}{\beta\Delta t^2}\left[\begin{Bmatrix} 2.6\times 10^{-6} \\ 6.593\times 10^{-5} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \Delta t\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \Delta t^2\left(\frac{1}{2} - \beta\right)\left\{\cancelto{0}{d_{0\text{dotdot}}}\right\}\right]$$

$$\{d_{1\text{dotdot}}\} = \begin{Bmatrix} 249.56 \\ 6328.8 \end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

Computer program solution

Lumped mass contribution

3504000	0	0
0	7008000	0
0	0	3504000
Stiffness		
300000	–300000	0
–300000	600000	–300000
0	–300000	300000
3804000	–300000	0

$K_{\text{prime}} =$ –300000 7608000 –300000
0 –300000 3804000

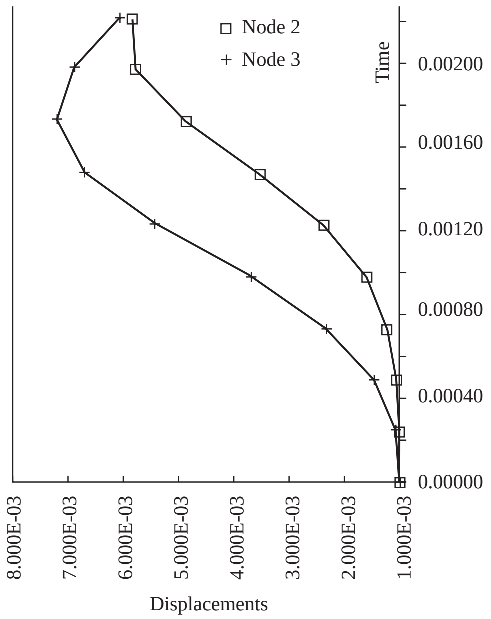
Inverse of $[K]_{\text{prime}}$ (after introducing boundary condition)

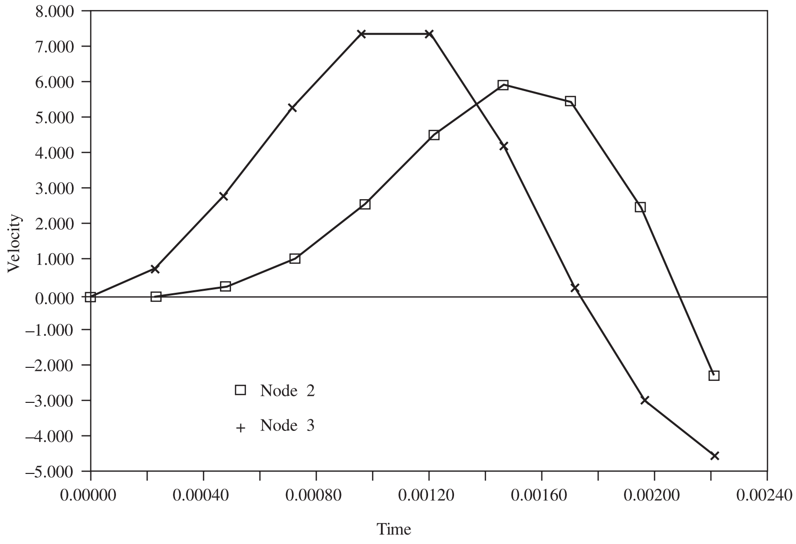
1.319E–07 1.040E–08

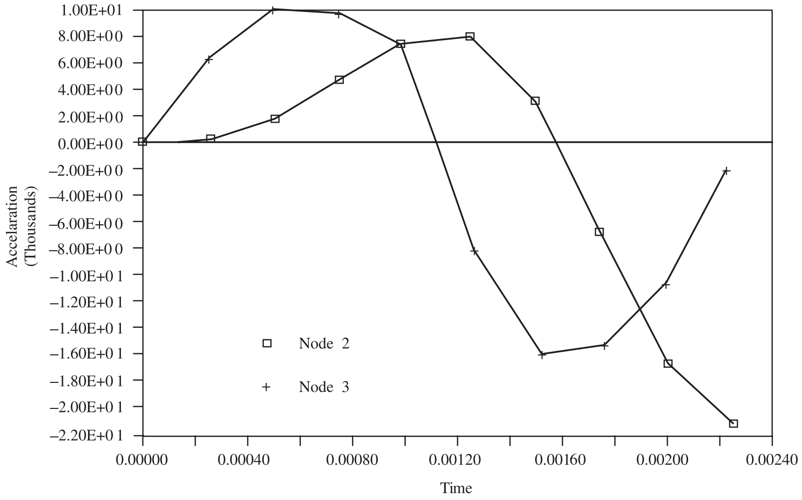
1.040E–08 2.637E–07

			Displacement (inches)		Velocity $\left(\frac{\text{in.}}{\text{s}}\right)$	
Step	Time	Node:	2	3	2	3
0	0		0	0	0	0
1	0.00025		2.600E–06	6.593E–05	0.031	0.791
2	0.0005		3.402E–05	4.985E–04	0.284	2.817
3	0.00075		1.901E–04	1.510E–03	1.085	5.265
4	0.001		6.364E–04	3.102E–03	2.605	7.369
5	0.00125		1.528E–03	5.009E–03	4.548	7.252
6	0.0015		2.865E–03	6.487E–03	5.941	4.235
7	0.00175		4.345E–03	7.050E–03	5.488	0.302
8	0.002		5.400E–03	6.694E–03	2.537	–2.951
9	0.00225		5.460E–03	5.713E–03	–2.248	–4.540

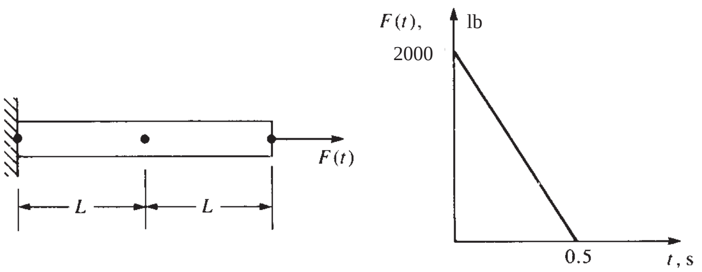
Acceleration $\left(\frac{\text{in.}}{\text{s}^2}\right)$		*F* prime		Force	
2	3	2	3	2	3
0	0	0	0	0	0
249.560	6328.830	0.000	250.000	0	250
1768.905	9881.174	109.307	1886.014	0	500
4641.891	9701.659	993.395	5686.004	0	750
7519.349	7128.285	3910.631	11610.659	0	1000
8027.620	–8065.184	10121.342	18596.407	0	750
3112.573	–16071.468	19848.150	23815.881	0	500
–6735.088	–15389.137	30938.266	25515.703	0	250
–16876.200	–10634.974	39078.413	23845.304	0	0
–21398.085	–2081.767	39826.784	20095.845	0	0







16.8



$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 50 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K]\{d\} + [M]\{d_{\text{dotdot}}\} = \{F(t)\}$$

Find proper time step

$$[M]\{d_{\text{dotdot}}\} + [K]\{d\} = 0$$

$$([K] - \omega^2 [M])\{d'\} = 0$$

$$d_1' = 0$$

$$\left(10^4 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} 50\right) \begin{Bmatrix} u_2' \\ u_3' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 400 - 2\omega^2 & -200 \\ -200 & 200 - \omega^2 \end{vmatrix} = 0$$

$$\omega^2 = 300 \pm 100\sqrt{13}$$

$$\omega = 25.7\ \frac{\text{rad}}{\text{s}}$$

$$\Delta t \leqq \frac{3}{4}\left(\frac{2}{\omega_{\max}}\right) = \frac{3}{4}\left(\frac{2}{25.7}\right) = 0.058 \text{ s}$$

$$\therefore \ \text{use } \Delta t = 0.05 \text{ s} \qquad \beta = \frac{1}{6},\ \gamma = \frac{1}{2}$$

Step 1 $\quad t = 0.05$ s $\qquad F_0 = 2000$ lb

$$d_0 = 0$$

$$\{d_{0\text{dotdot}}\} = [M^{-1}](\{F_0\} - [K]\{d_0\})$$

$$= \frac{1}{50}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2000 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 40 \end{Bmatrix}$$

$$[K'] = [K] + \frac{1}{\beta \Delta t^2}[M] = \begin{bmatrix} 20000 & -10000 \\ -10000 & 10000 \end{bmatrix} + 2400 \times 50 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K'] = \begin{bmatrix} 260{,}000 & -10{,}000 \\ -10{,}000 & 130{,}000 \end{bmatrix}$$

$$\{F_1'\} = \{F_1\} + \frac{[M]}{\beta \Delta t^2}\left(\{d_0\} + \Delta t\{d_{0\text{dot}}\} + \frac{1}{3}\Delta t^2\{d_{0\text{dotdot}}\}\right)$$

$$= \begin{Bmatrix} 0 \\ 1800 \end{Bmatrix} + 2 \times 50 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 40 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5800 \end{Bmatrix}$$

$$\{d_1\} = [K']^{-1}\{F_1'\} = \frac{1}{10^4} \times \frac{1}{3.37}\begin{bmatrix} 13 & 1 \\ 1 & 26 \end{bmatrix} \begin{Bmatrix} 0 \\ 5800 \end{Bmatrix}$$

$$\{d_1\} = \begin{Bmatrix} 0.001721 \\ 0.0448 \end{Bmatrix} \text{ in.}$$

$$\{d_{1\text{dotdot}}\} = \frac{1}{\beta \Delta t^2}\left[\{d_1\} - \{d_0\} - \Delta t\{d_{0\text{dot}}\} - \Delta t^2 \frac{1}{3}\{d_{0\text{dotdot}}\}\right]$$

$$= 2400\left(\begin{Bmatrix}0.001721\\0.0448\end{Bmatrix} - 0.05^2 \times \frac{1}{3} \times \begin{Bmatrix}0\\40\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}4.131\\27.39\end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

$$\{d_{1\text{dot}}\} = \{d_{0\text{dot}}\} + \Delta t\left(\frac{1}{2}\{d_{0\text{dotdot}}\} + \frac{1}{2}\{d_{1\text{dotdot}}\}\right)$$

$$\{d_{1\text{dot}}\} = 0.05 \times \frac{1}{2}\left(\begin{Bmatrix}0\\40\end{Bmatrix} + \begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right) = \begin{Bmatrix}0.103\\1.685\end{Bmatrix}$$

Step 2 $\quad t = 0.1$ s

$$\{F_2\} = \begin{Bmatrix}0\\2000(1-2(0.1))\end{Bmatrix} = \begin{Bmatrix}0\\1600\end{Bmatrix}$$

$$\{F_2'\} = \begin{Bmatrix}0\\1600\end{Bmatrix} + 2400 \times 50 \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\left(\begin{Bmatrix}0.001721\\0.0448\end{Bmatrix} + 0.05\begin{Bmatrix}0.103\\1.685\end{Bmatrix} + \frac{1}{3}(0.05)^2\begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right)$$

$$\{F_2'\} = \begin{Bmatrix}0\\1600\end{Bmatrix} + 120000\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}0.01031\\0.1518\end{Bmatrix} = \begin{Bmatrix}2475\\19818\end{Bmatrix}$$

$$\{d_2\} = [K']^{-1}\{F_2'\} = \frac{1}{337 \times 10^4}\begin{bmatrix}13 & 1\\1 & 26\end{bmatrix}\begin{Bmatrix}2475\\19818\end{Bmatrix}$$

$$\{d_2\} = \begin{Bmatrix}0.01544\\0.1536\end{Bmatrix}\text{in.}$$

$$\{d_{2\text{dotdot}}\} = 2400\left[\begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} - \begin{Bmatrix}0.00172\\0.0448\end{Bmatrix} - 0.05\begin{Bmatrix}0.103\\1.685\end{Bmatrix} - \frac{0.05^2}{3}\begin{Bmatrix}4.131\\27.39\end{Bmatrix}\right]$$

$$\{d_{2\text{dotdot}}\} = \begin{Bmatrix}12.27\\4.37\end{Bmatrix}\frac{\text{in.}}{\text{s}^2}$$

$$\{d_{2\text{dot}}\} = \begin{Bmatrix}0.103\\1.685\end{Bmatrix} + 0.05 \times \frac{1}{2}\left[\begin{Bmatrix}4.131\\27.39\end{Bmatrix} + \begin{Bmatrix}12.27\\4.37\end{Bmatrix}\right]$$

$$= \begin{Bmatrix}0.513\\0.2479\end{Bmatrix}\frac{\text{in.}}{\text{s}}$$

Step 3 $\quad t = 0.15$ s

$$\{F_3\} = \begin{Bmatrix}0\\2000(1-2(0.15))\end{Bmatrix} = \begin{Bmatrix}0\\1400\end{Bmatrix}$$

$$\{F_3'\} = \begin{Bmatrix}0\\1400\end{Bmatrix} + 12 \times 10^4\begin{Bmatrix}2 & 0\\0 & 1\end{Bmatrix} \times \left(\begin{Bmatrix}0.01544\\0.1536\end{Bmatrix} + 0.05\begin{Bmatrix}0.513\\2.479\end{Bmatrix} + \frac{0.05^2}{3}\begin{Bmatrix}12.27\\4.37\end{Bmatrix}\right)$$

$$= \begin{Bmatrix}12316\\35143\end{Bmatrix}$$

$$\{d_3\} = \frac{1}{337 \times 10^4}\begin{bmatrix} 13 & 1 \\ 1 & 26 \end{bmatrix}\begin{matrix} 12316 \\ 35143 \end{matrix} = \begin{Bmatrix} 0.0579 \\ 0.2745 \end{Bmatrix} \text{ in.}$$

$$\{d_{3\text{dotdot}}\} = 2400\left(\begin{Bmatrix} 0.0579 \\ 0.2745 \end{Bmatrix} - \begin{Bmatrix} 0.01544 \\ 0.1536 \end{Bmatrix} - 0.05\begin{Bmatrix} 0.513 \\ 2.479 \end{Bmatrix} - \frac{0.05^2}{3}\begin{Bmatrix} 12.27 \\ 4.37 \end{Bmatrix}\right)$$

$$= \begin{Bmatrix} 15.80 \\ -16.06 \end{Bmatrix} \frac{\text{in.}}{\text{s}^2}$$

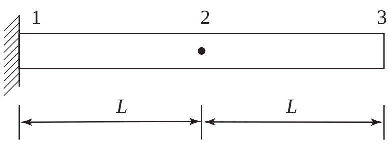
$$\{d_{3\text{dot}}\} = \begin{Bmatrix} 0.513 \\ 2.479 \end{Bmatrix} + \frac{0.05}{2}\left(\begin{Bmatrix} 12.27 \\ 4.37 \end{Bmatrix} + \begin{Bmatrix} 15.80 \\ -16.06 \end{Bmatrix}\right)$$

$$= \begin{Bmatrix} 1.2149 \\ 2.1868 \end{Bmatrix} \frac{\text{in.}}{\text{s}}$$

Steps 4 and 5 follow similar procedures as above
Table below summarizes results

t, s	F, lb	d_i, in. Node 2	Node 3	d_i, $\frac{\text{in.}}{\text{s}}$		$d_{i\text{dot}}$, $\frac{\text{in.}}{\text{s}^2}$	
0	2000	0	0	0	0	0	40
0.05	1800	0.00172	0.0448	0. 103	1.685	4.131	27.39
0.10	1600	0.01544	0.1536	0.513	2.479	12.27	4.37
0.15	1400	0.0579	0.2745	1.2149	2.187	15.80	– 16.06
0.20	1200	0.1356	0.3616	1.836	1.255	9.042	– 21.20
0.25	1000	0.2323	0.401	1.905	0.383	–6.376	– 13.71

16.11



Global stiffness matrix

$$[k] = \frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L \end{bmatrix} \end{array}$$

Lumped mass matrix

$$
[m] = \frac{\rho A L}{2}
\begin{array}{c}
\begin{array}{cccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Invoking boundary conditions $v_1 = \phi_1 = 0$

$$
\frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0
$$

Let $\omega^2 = \lambda$ and divide ρAL

and $\mu = \dfrac{EI}{\rho A}$

(a)
$$
\begin{vmatrix}
\frac{24\mu}{L^4} - \lambda & 0 & \frac{-12\mu}{L^4} & \frac{6\mu}{L^2} \\
0 & \frac{8\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{2\mu}{L^2} \\
\frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \frac{12\mu}{L^4} - \frac{\lambda}{2} & \frac{-6\mu}{L^3} \\
\frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{4\mu}{L^2}
\end{vmatrix} = 0
$$

$$
\left(\frac{24\mu}{L^4} - \lambda\right)(-1)^2
\begin{vmatrix}
\frac{8\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{-2\mu}{L^2} \\
\frac{-6\mu}{L^3} & \frac{12\mu}{L^3} - \frac{\lambda}{2} & \frac{-6\mu}{L^3} \\
\frac{2\mu}{L^2} & \frac{-6\mu}{L^3} & \frac{4\mu}{L^2}
\end{vmatrix}
+ \left(\frac{-12\mu}{L^4}\right)(-1)^{1+3}
\begin{vmatrix}
0 & \frac{8\mu}{L^2} & \frac{2\mu}{L^2} \\
\frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \frac{-6\mu}{L^3} \\
\frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{4\mu}{L^2}
\end{vmatrix}
$$

$$
+ \frac{6\mu}{L^3}(-1)^{1+4}
\begin{vmatrix}
0 & \frac{8\mu}{L^2} & \frac{-6\mu}{L^3} \\
\frac{-12\mu}{L^4} & \frac{-6\mu}{L^3} & \left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right) \\
\frac{6\mu}{L^3} & \frac{2\mu}{L^2} & \frac{-6\mu}{L^3}
\end{vmatrix} = 0
$$

$$
\left(\frac{24\mu}{L^4} - \lambda\right)\left[\frac{32\mu^2}{L^4}\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right) + \frac{36\mu^2}{L^6}\frac{2\mu}{L^2} + \frac{36\mu^2}{L^6}\frac{2\mu}{L^2}\right.
$$

$$
\left. - \left\{\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right)\frac{4\mu^2}{L^4} + \frac{36\mu^2}{L^6}\frac{8\mu}{L^2} + \frac{36\mu^2}{L^6}\frac{4\mu}{L^2}\right\}\right]
$$

$$
- \frac{12\mu}{L^4}\left[\frac{-36\mu^2}{L^6}\frac{8\mu}{L^2}\ \frac{-12\mu}{L^2}\frac{4\mu^2}{L^4} - \left\{\frac{-36\mu^2}{L^6}\frac{2\mu}{L^2}\ \frac{-12\mu}{L^4}\frac{32\mu^2}{L^4}\right\}\right]
$$

$$
\frac{-6\mu}{L^3}\left[\left(\frac{12\mu}{L^4} - \frac{\lambda}{2}\right)\frac{48\mu^2}{L^5} + \frac{12\mu}{L^4}\frac{12\mu^2}{L^5} - \left\{\frac{6\mu}{L^3}\frac{36\mu^2}{L^6} + \frac{48\mu^2}{L^5}\frac{12\mu}{L^4}\right\}\right] = 0
$$

$$\left(\frac{24\mu}{L^4}-\lambda\right)\left[\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{28\mu^2}{L^4}-\frac{288\mu^3}{L^8}\right]-\frac{12\mu}{L^4}\left[\frac{120\mu^3}{L^8}\right]$$

$$-\frac{6\mu}{L^3}\left[\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{48\mu^2}{L^5}-\frac{648\mu^3}{L^9}\right]=0$$

$$2\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)^2\frac{28\mu^2}{L^4}-2\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{288\mu^3}{L^6}+\frac{2448\mu^4}{L^{12}}-\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\left(\frac{288\mu^3}{L^8}\right)=0$$

$$\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)^2\frac{56\mu^2}{L^4}-\left(\frac{12\mu}{L^4}-\frac{\lambda}{2}\right)\frac{864\mu^3}{L^8}+\frac{2448\mu^4}{L^{12}}=0$$

Let $\frac{12\mu}{L^4}-\frac{\lambda}{2}=x$

$$\frac{56\mu^2}{L^4}x^2-\frac{864\mu^3}{L^8}x+\frac{2448\mu^4}{L^{12}}=0$$

$$56x^2-\frac{864\mu}{L^4}x+\frac{2448\mu^2}{L^8}=0$$

$$x_{1,2}=\frac{\frac{864\mu}{L^4}\pm\sqrt{\left(\frac{864\mu}{L^4}\right)^2-4(56)\left(\frac{2448\mu^2}{L^8}\right)}}{2(56)}=\frac{\frac{864\mu}{L^4}\pm\sqrt{\frac{198144\mu^2}{L^8}}}{112}$$

$$x_1=\frac{\frac{864\mu}{L^4}+\frac{445.1\mu}{L^4}}{112}\quad\Rightarrow\quad x_1=\frac{11.69\mu}{L^4}$$

$$x_2=\frac{\frac{864\mu}{L^4}-\frac{445.1\mu}{L^4}}{112}\quad\Rightarrow\quad x_2=\frac{3.74\mu}{L^4}$$

$$x_1=\frac{12\mu}{L^4}-\frac{\lambda_1}{2}=\frac{11.69\mu}{L^4}\quad\Rightarrow\quad\lambda_1=\frac{0.62\mu}{L^4}$$

$$x_2=\frac{12\mu}{L^4}-\frac{\lambda_2}{2}=\frac{3.74\mu}{L^4}\quad\Rightarrow\quad\lambda_2=\frac{16.52\mu}{L^4}$$

$$\omega_1^2=\frac{0.62\mu}{L^4}=\frac{0.62\,EI}{\rho AL^4}$$

$$\omega_1=\sqrt{\frac{0.62EI}{\rho AL^4}}\quad\Rightarrow\quad\boxed{\omega_1=\frac{0.787}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}}$$

$$\omega_2^2=\frac{16.52EI}{\rho A\,L^4}\quad\Rightarrow\quad\omega_2=\sqrt{\frac{16.52\,EI}{\rho A\,L^4}}$$

$$\Rightarrow\quad\boxed{\omega_2=\frac{4.06}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}}$$

The exact solution from simple beam theory yields

$$\omega_1=\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}\left(\frac{1.875}{2L}\right)^2=\frac{0.879}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

(a) $$\omega_2=\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}\left(\frac{4.694}{2L}\right)^2=\frac{5.5}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

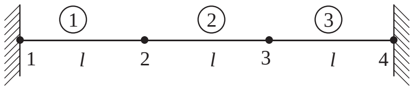
Note: L = one-half actual length of beam in the analysis.

Expressing answers in terms of full length $l = 2L$, we obtain

$$\omega_1 = \frac{0.787}{\left(\frac{l}{2}\right)^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}} = \frac{3.15}{l^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

and $$\omega_2 = \frac{16.24}{l^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

(b)



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{Symmetry} & & & 4l^2 \end{bmatrix}$$

$$[m^{(1)}] = [m^{(2)}] = [m^{(3)}] = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Boundary conditions $v_1 = \phi_1 = v_4 = \phi_4 = 0$

$$|\,[K] - \omega^2\,[M]\,| = 0$$

$$\left| \frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l \\ & 8l^2 & -6l & 2l^2 \\ & & 24 & 0 \\ \text{Symmetry} & & & 8l^2 \end{bmatrix} - \rho Al\omega^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ \text{Symmetry} & & & 0 \end{bmatrix} \right| = 0$$

Let $$\mu = \frac{EI}{\rho Al^4}$$

$$\begin{vmatrix} 24\mu - \omega^2 & 0 & -12\mu & 6l\mu \\ & 8l^2\mu & -6l\mu & 2l^2\mu \\ & & 24\mu - \omega^2 & 0 \\ \text{Symmetry} & & & 8l^2\mu \end{vmatrix} = 0$$

Rewrite 4×4 determinent as

$$\begin{vmatrix} 24\mu - \omega^2 & 0 & -12\mu & 6l\mu \\ -24l\mu & 0 & -6l\mu & -30l^2\mu \\ 6\mu & 0 & 24\mu - \omega^2 & 24l\mu \\ 6l\mu & 2l^2\mu & 0 & 8l^2\mu \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} 24\mu-\omega^2 & -12\mu & 6l\mu \\ -24l\mu & -6l\mu & -30l^2\mu \\ 6\mu & 24\mu-\omega^2 & 24l\mu \end{vmatrix} = 0$$

Evaluate

$$972\ l\mu^2 - 192\ L\mu\omega^2 + 5\ l\omega^4 = 0$$

$$\omega^2 = \frac{192\ l\mu \pm \sqrt{(192l\mu)^2 - 4\times 5l^2\mu^2\ 972}}{2(10l)} = \frac{192\ l\mu \pm 132l\mu}{10l}$$

$$\omega_1^2 = 6\mu \qquad \omega_2^2 = 32.4\ \mu$$

$$\omega_1 = 2.45\sqrt{\frac{EI}{\rho A\left(\frac{l}{3}\right)^4}} = \frac{22.04}{l^2}\sqrt{\frac{EI}{\rho A}}$$

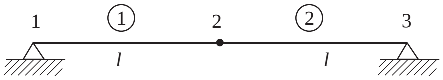
$$\omega_2 = 5.69\sqrt{\frac{EI}{\rho A\left(\frac{l}{3}\right)^4}} = \frac{51.23}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Let $\quad 3l = L \qquad L =$ whole length

$$\therefore \quad \omega_1 = \frac{22.04}{\left(\frac{L}{3}\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{198.4}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = \frac{51.23}{\left(\frac{L}{3}\right)^2}\sqrt{\frac{EI}{\rho A}} = \frac{461.07}{L^2}\sqrt{\frac{EI}{\rho A}}$$

(c)



Boundary conditions $v_1 = v_5 = 0$

$$[K] = \frac{EI}{l^3}\begin{array}{c}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ & 24 & 0 & 6l \\ & & 8l^2 & 2l^2 \\ & & & 4l^2 \end{bmatrix}\end{array}\begin{array}{l}\phi_1 \\ v_2 \\ \phi_2 \\ \phi_3\end{array}$$

$$[M] = \frac{\rho Al}{2}\begin{bmatrix} & 0 & 0 & 0 \\ 0 & & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{l}\phi_1 \\ v_2 \\ \phi_2 \\ \phi_3\end{array}$$

$$I = \frac{1}{3}\ \rho A\frac{l^3}{8}$$

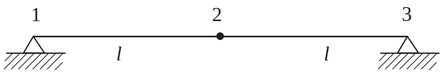
$$|\,[K] - \omega^2\,[M]\,| = 0$$

$$\left|\frac{\rho AE}{24}\begin{bmatrix}4l^2 & -6l & 2l^2 & 0\\ & 24 & 0 & 6l\\ & & 8l^2 & 2l^2\\ & & & 4l^2\end{bmatrix} - \omega^2\frac{\rho AE}{24}\begin{bmatrix}0 & 0 & 0 & 0\\ & \frac{24l}{E} & 0 & 0\\ & & 0 & 0\\ & & & 0\end{bmatrix}\right| = 0$$

$$\begin{vmatrix}2l^2 & -3l & l^2 & 0\\ & 12\left(1-\frac{l\omega^2}{E}\right) & 0 & 3l\\ & & 4l^2 & l^2\\ \text{Symmetry} & & & 2l^2\end{vmatrix} = 0$$

$$\omega = \frac{2.45}{l^2}\sqrt{\frac{EI}{\rho A}}$$

(d)



Boundary conditions $v_1 = \phi_1 = v_3 = 0$

$$[K] = \frac{EI}{l^3}\begin{bmatrix}24 & 0 & 6l\\ 0 & 8l^2 & 2l^2\\ 6l & 2l^2 & 4l^2\end{bmatrix}$$

$$[M] = \frac{\rho AL}{2}\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\left|\frac{EI}{l^2}\begin{bmatrix}24 & 0 & 6l\\ & 8l^2 & 2l^2\\ & & 4l^2\end{bmatrix} - \omega^2\frac{\rho AL}{2}\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right| = 0$$

Divide by ρAl and let $\omega^2 = \lambda$ and $\dfrac{EI}{\rho AL^4} = \mu$

Determinant becomes

$$\begin{vmatrix}24\mu - \lambda & 0 & 6l\mu\\ & 8l^2\mu & 2l^2\mu\\ & & 4l^2\mu\end{vmatrix} = 0$$

Simplifying

$$672 - 288\, l^4\,\mu^3 = 28\, l^4\,\mu^2\,\lambda$$

$$\lambda = 13.71\,\mu$$

$$\omega = \sqrt{\lambda} = 3.703\,\sqrt{\mu}\,,\ 2l = L$$

$$\omega = \frac{14.81}{L^2}\left(\frac{EI}{\rho A}\right)^{\frac{1}{2}}$$

16.13

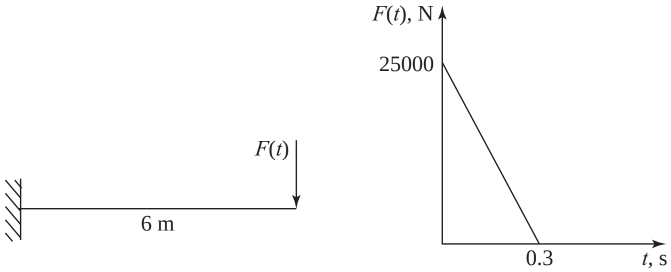
Problem 16.13 using DFRAME
2, 3
1, 1, 1, 0, 0, 0, 0, 0, 0
2, 0, 0, 0, 4, 0, 0, – 1.0, 0
3, 0, 1, 0, 8, 0, 0, 0, 0
1, 1, 2, 210*e*+9, 2*e*–2, 4*e*–4, 1.352*e*–9
2, 2, 3, 210*e*+9, 2*e*–2, 4*e*–4, 1.352*e*–9
0.03, 12
0.25, 0.50
2
–16666.667, 5000, 0, 0, 0.3

Using DFRAME that properly calculates initial acceleration.

		Displacement		Velocity		Acceleration	
Node	Time	'Y'	'Z'	'Y'	'Z'	'Y'	'Z'
1	0	0	0	0	0	0	0
1	0.03	0	–0.00045	0	–0.03016	0	–2.011
1	0.06	0	0.000047	0	0.06349	0	8.254
1	0.09	0	–0.00040	0	–0.09365	0	–18.73
1	0.12	0	0.000095	0	0.127	0	33.44
1	0.15	0	–0.00035	0	–0.1571	0	–52.38
1	0.18	0	0.000142	0	0.1905	0	75.56
1	0.21	0	–0.00030	0	–0.2206	0	–103
1	0.24	0	0.000190	0	0.254	0	134.6
1	0.27	0	–0.00026	0	–0.2841	0	–170.5
1	0.3	0	0.000238	0	0.3175	0	210.6
1	0.33	0	–0.00023	0	–0.3492	0	–255
1	0.36	0	0.000238	0	0.381	0	303.7
2	0	0	0	0	0	–4.6E+13	0
2	0.03	–0.00120	0	–0.08051	0	4.6E+13	0
2	0.06	0.000127	1.2E–21	0.1695	8.3E–20	–4.6E+13	5.5E–18
2	0.09	–0.00107	1.0E–20	–0.25	5.0E–19	4.6E+13	2.2E–17
2	0.12	0.000254	2.5E–21	0.3389	–1.0E–18	–4.6E+13	–1.2E–16
2	0.15	–0.00095	0	–0.4194	8.3E–19	4.6E+13	2.4E–16
2	0.18	0.000381	0	0.5084	–8.3E–19	–4.6E+13	–3.5E–16
2	0.21	–0.00082	–1.0E–20	–0.5889	1.7E–19	4.6E+13	4.2E–16
2	0.24	0.000507	0	0.6778	5.0E–19	–4.6E+13	–4.0E–16
2	0.27	–0.00069	–5.0E–21	–0.7583	–8.3E–19	4.6E+13	3.1E–16
2	0.3	0.000634	0	0.8473	1.2E–18	–4.6E+13	–1.8E–16
2	0.33	–0.00063	–5.0E–21	–0.932	–1.5E–18	4.6E+13	–5.9E–31
2	0.36	0.000634	0	1.017	1.8E–18	–4.6E+13	2.2E–16
3	0	0	0	0	0	0	0
3	0.03	0	0.000452	0	0.03016	0	2.011
3	0.06	0	–0.00004	0	–0.06349	0	–8.254
3	0.09	0	0.000404	0	0.09365	0	18.73
3	0.12	0	–0.00009	0	–0.127	0	–33.44
3	0.15	0.	0.000357	0	0.1571	0	52.38
3	0.18	0	–0.00014	0	–0.1905	0	–75.56
3	0.21	0	0.000309	0	0.2206	0	103

3	0.24	0	–0.00019	0	–0.254	0	–134.6
3	0.27	0	0.000261	0	0.2841	0	170.5
3	0.3	0	–0.00023	0	–0.3175	0	–210.6
3	0.33	0	0.000238	0	0.3492	0	255
3	0.36	0	–0.00023	0	–0.381	0	–303.7

16.14 Note that even though damping data was not given in the problem. This solution includes damping.

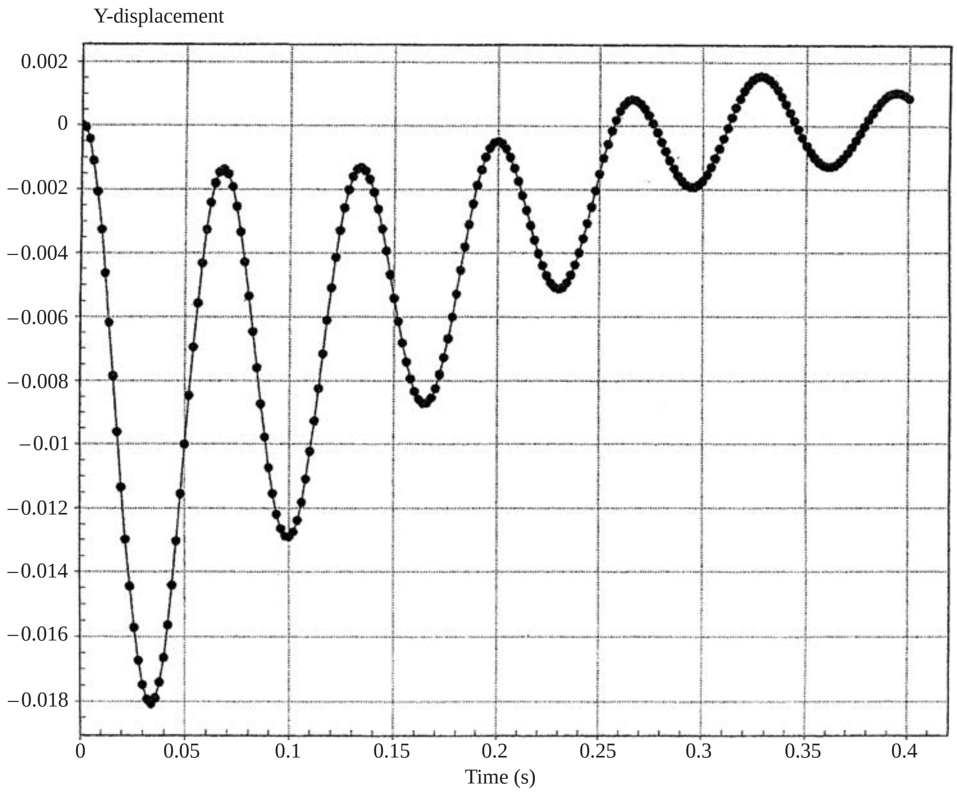


Let the time step or increment = 0.002 s and the number of time steps = 200.
Also use the following data in the Algor program

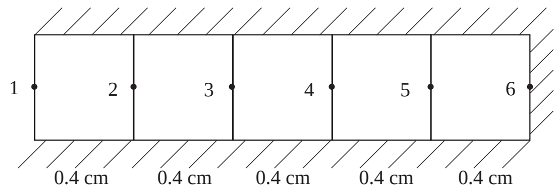
$\rho = 7800 \dfrac{\text{kg}}{\text{m}^3}$, $E = 210$ GPa, $Y = 0.3$, $\sigma_{ys} = 250$ MPa

$A = 2 \times 10^{-2}\ \text{m}^2$, $\text{J1} = 16 \times 10^{-4}\ \text{m}^4$, $\text{I2} = \text{I3} = 8 \times 10^{-4}\ \text{m}^4$, $\text{S2} = \text{S3} = 16 \times 10^{-4}\ \text{m}^3$

Damping is to be included so use mass damping coefficient $C_m = \alpha = 3.00$ and stiffness damping coefficient $C_k = \beta = 0.001$



16.17



$K = 2\ \dfrac{\text{W}}{(\text{m}\cdot{}^\circ\text{C})}$

$\rho = 2500\ \dfrac{\text{kg}}{\text{m}^3}$

$C = 800\ \dfrac{\text{W}\cdot\text{s}}{(\text{kg}\cdot{}^\circ\text{C})}$

$T_1^0 = 200^\circ\text{C}$ $T_1' = 0^\circ\text{C}$ (sudden temperature change to $T_1' = 0^\circ\text{C}$)

Using Equation (16.8.16)

$$\left(\frac{1}{\Delta t}[M] + \beta[K]\right)\{T_i\}_{+1} = \left(\frac{1}{\Delta t}[M] - (1-\beta)[K]\right)\{T_i\} + (1-\beta)\{F_i\} + \beta\{F_i\}_{+1}$$

$$h = 0 \text{ assume unit area}$$

$$[K] = \frac{AK}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(l\text{m}^2)2\frac{\text{W}}{(\text{m}\cdot{}^\circ\text{C})}}{0.004\ \text{m}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 500\ \frac{\text{W}}{{}^\circ\text{C}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M = \frac{C\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\left(800\frac{\text{W}\cdot\text{s}}{\text{kg}\cdot{}^\circ\text{C}}\right)\left(2500\frac{\text{kg}}{\text{m}^3}\right)(l\text{m}^2)(0.4\ \text{cm})}{2\left(100\frac{\text{cm}}{\text{m}}\right)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 4000\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\text{W}\cdot\text{s}}{{}^\circ\text{C}}$$

$$\text{Let } [C] = \left(\frac{1}{\Delta t}[M] + \beta[K]\right)^{-1}\left\{\frac{1}{\Delta t}[M] - (1-\beta)[K]\right\}$$

Then

$$\{T_i\}_{+1} = C\{T_i\} \text{ as } \{F_i\} = 0, \{F_i\}_{+1} = 0$$

For $[T_1]$ ($t = 8$ s) and eliminating 1st row and column for boundary condition $t_1 = 0$ we have

$$\{T_1\} = \begin{Bmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{Bmatrix} = C'\begin{Bmatrix} 200 \\ 200 \\ 200 \\ 200 \\ 200 \end{Bmatrix}$$

where $[C']$ is $[C]$ with 1st row and column deleted

with row and column one deleted

$$[K] = \begin{bmatrix} 1000 & -500 & 0 & 0 & 0 \\ & 1000 & -500 & 0 & 0 \\ & & 1000 & -500 & 0 \\ & & & 1000 & -500 \\ \text{Symmetry} & & & & 500 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 8000 & 0 & 0 & 0 & 0 \\ & 8000 & 0 & 0 & 0 \\ & & 8000 & 0 & 0 \\ & & & 8000 & 0 \\ \text{Symmetry} & & & & 4000 \end{bmatrix}$$

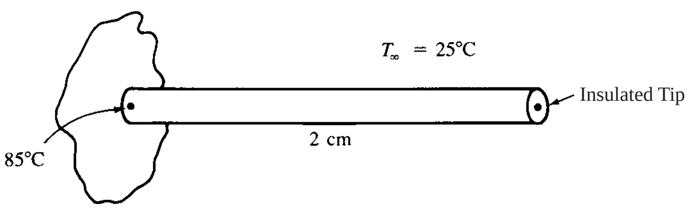
$$\frac{M}{\Delta t} + \beta[K] = \begin{bmatrix} 1666.7 & -333.3 & 0 & 0 & 0 \\ & 1666.7 & -333.3 & 0 & 0 \\ & & 1666.7 & -333.3 & 0 \\ & & & 1666.7 & -333.3 \\ \text{Symmetry} & & & & 833.3 \end{bmatrix}$$

$$\frac{[M]}{\Delta t} - (1-\beta)\,[K] = \begin{bmatrix} 666.7 & 166.7 & 0 & 0 & 0 \\ & 666.7 & 166.7 & 0 & 0 \\ & & 666.7 & 166.7 & 0 \\ & & & 666.7 & 166.7 \\ \text{Symmetry} & & & & 333.3 \end{bmatrix}$$

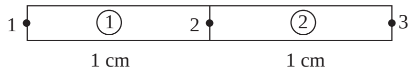
$$[C'] = \begin{bmatrix} 0.5247 & 0.2139 & 0.04472 & 0.00972 & 0.00194 \\ & 0.4839 & 0.2057 & 0.04472 & 0.00894 \\ & & 0.4839 & 0.2139 & 0.0428 \\ & & & 0.5247 & 0.2049 \\ \text{Symmetry} & & & & 0.4820 \end{bmatrix}$$

	Node / t_1 s	Temperatures, °C 1	2	3	4	5	6
	0	200	200	200	200	200	200
1	8	0	159	191	198	199.6	199.8
2	16	0	135	178	193	198.2	199.1
3	24	0	120	165	187	195.5	197.5
4	32	0	109	155	180	191.7	194.8
5	40	0	101	146	173	187.1	191.1
6	48	0	94	138	167	182.0	186.7
7	56	0	88	131	160	176.5	181.6
8	64	0	84	125	154	170.8	176.3
9	72	0	79	119	148	165.1	170.7
10	80	0	76	114	142	159.3	165.0

16.18



Two element solutions



0.4 cm diameter

$$[K] = \frac{AK}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\rho L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{\text{W}}{{}^\circ\text{C}}$$

$$= (0.12567\ \text{cm}^2)\left(400\frac{\text{W}}{\text{m}\cdot{}^\circ\text{C}}\right)\left(\frac{1\ \text{m}}{100\ \text{cm}}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$+ \left(150\frac{\text{W}}{\text{m}^2\cdot{}^\circ\text{C}}\right)\frac{(1.2567\ \text{cm})(1\ \text{cm})}{6\left(100\ \frac{\text{cm}}{\text{m}}\right)^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5027 & -0.5027 \\ -0.5027 & 0.5027 \end{bmatrix} + \begin{bmatrix} 0.00628 & 0.00314 \\ 0.00314 & 0.00628 \end{bmatrix}$$

Using consistent mass

$$[m] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \left(375\frac{\text{W}\cdot\text{s}}{\text{kg}\cdot{}^\circ\text{C}}\right)\left(8900\frac{\text{kg}}{\text{m}^3}\right)\left(0.12567\frac{\text{cm}^2}{100^2}\right)\frac{\frac{1\ \text{cm}}{100}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1398 & 0.0699 \\ 0.0699 & 0.1398 \end{bmatrix}\frac{\text{W}\cdot\text{s}}{{}^\circ\text{C}}$$

$$\{f\} = \frac{hT_\infty PL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.23561 \\ 0.23561 \end{bmatrix}\text{W}$$

Global $[K]$ and $[M]$

$$[K] = \begin{bmatrix} 0.50893 & -0.49951 & 0 \\ & 1.01786 & -0.49951 \\ \text{Symmetry} & & 0.50893 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 0.1398 & 0.06991 & 0 \\ & 0.2796 & 0.06991 \\ \text{Symmetry} & & 0.1398 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Using Equation (16.8.16) with

$$[T_0] = \begin{Bmatrix} 25^\circ\text{C} \\ 25^\circ\text{C} \\ 25^\circ\text{C} \end{Bmatrix} \text{ (initial temperature of rod)}$$

Suddenly end (left) temperature becomes 85°C ($\beta = \frac{2}{3}$)

For $t = 0.1$ s

$$\left(\frac{1}{\Delta t}[M]+\beta[K]\right)\{T_1\} = \left(\frac{1}{\Delta t}[M]-(1-\beta)[K]\right)\{T_0\} + \{F_0\}$$

As $\{F_i\} = \{F_i\}_{+1}$ for all time t

$$\begin{bmatrix} 1.7374 & 0.3660 & 0 \\ & 3.4747 & 0.3660 \\ \text{Symmetry} & & 1.7374 \end{bmatrix} \begin{Bmatrix} 85°\text{C} \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 1.228 & 0.865 & 0 \\ & 2.457 & 0.865 \\ & & 1.228 \end{bmatrix} \begin{Bmatrix} 25 \\ 25 \\ 25 \end{Bmatrix} + \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Since $t_1 = 85°$ boundary condition adjust equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.4747 & 0.366 \\ 0 & 0.366 & 1.37 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 105.17 \times 85(0.3660) \\ 52.585 \end{Bmatrix}$$

Solving

$$t_1 = 85°, \quad t_2 = 18.536°\text{C}, \quad t_3 = 26.362°\text{C}$$

For $t = 0.2$ s

$$\begin{bmatrix} 1.737 & 0.366 & 0 \\ & 3.475 & 0.366 \\ \text{Symmetry} & & 1.737 \end{bmatrix} \begin{Bmatrix} 85 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} 1.228 & 0.865 & 0 \\ & 2.457 & 0.865 \\ & & 1.228 \end{bmatrix} \begin{Bmatrix} 85 \\ 18.536 \\ 26.362 \end{Bmatrix} + \begin{Bmatrix} 0.2356 \\ 0.4712 \\ 0.2356 \end{Bmatrix}$$

Solving

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 141.93 - 0.865(85) \\ 48.427 \end{Bmatrix}$$

$$t_2 = 29.613°\text{C}, \quad t_3 = 21.635°\text{C}$$

For $t = 0.3$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 134.0 \\ 52.687 \end{Bmatrix}$$

$$t_2 = 36.404°\text{C}, \quad t_3 = 22.662°\text{C}$$

For $t = 0.4$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 152.0 \\ 59.58 \end{Bmatrix}$$

$$t_2 = 41.03°\text{C}, \quad t_3 = 25.655°\text{C}$$

For $t = 0.5$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 165.9 \\ 67.26 \end{Bmatrix}$$

$$t_2 = 44.665°\text{C}, \quad t_3 = 29.31°\text{C}$$

For $t = 0.6$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 178.03 \\ 74.90 \end{Bmatrix}$$

$$t_2 = 47.75°\text{C}, \quad t_3 = 33.06°\text{C}$$

For $t = 0.7$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 188.85 \\ 82.17 \end{Bmatrix}$$

$$t_2 = 50.48°\text{C}, \quad t_3 = 36.67°\text{C}$$

For $t = 0.8$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 198.7 \\ 88.97 \end{Bmatrix}$$

$$t_2 = 52.96°\text{C}, \quad t_3 = 40.06°\text{C}$$

For $t = 0.9$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 207.7 \\ 95.28 \end{Bmatrix}$$

$$t_2 = 55.22°\text{C}, \; t_3 = 43.22°\text{C}$$

For $t = 1.0$ s

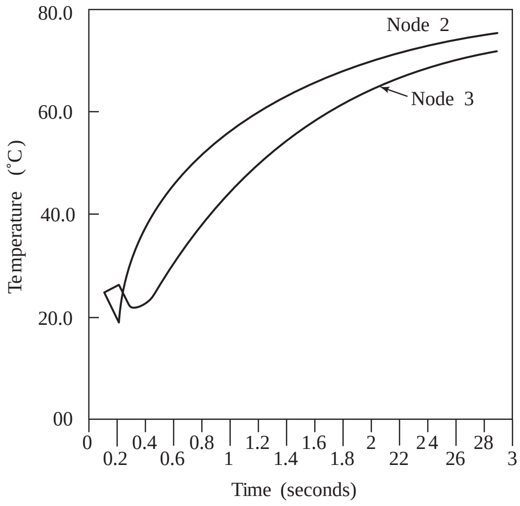
$$\begin{bmatrix} 1 & 0 & 0 \\ & 3.475 & 0.366 \\ & & 1.737 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 215.99 \\ 101.11 \end{Bmatrix}$$

$$t_2 = 57.30°\text{C}, \; t_3 = 46.14°\text{C}$$

etc. (A computer solution follows)

Time	NODE		
(s)	1	2	3
0	25	25	25
0.1	85	18.53611	26.36189
0.2	85	29.61303	21.63526
0.3	85	36.18435	22.42717
0.4	85	40.72491	25.30428
0.5	85	44.27834	28.85201
0.6	85	47.29072	32.49614
0.7	85	49.95809	36.01157
0.8	85	52.37152	39.31761
0.9	85	54.57756	42.39278
1	85	56.60353	45.23933
1.1	85	58.46814	47.86852
1.2	85	60.1859	50.29457

1.3	85	61.76908	52.53218
1.4	85	63.22852	54.59557
1.5	85	64.574	56.49814
1.6	85	65.81448	58.25235
1.7	85	66.95818	59.86974
1.8	85	68.01265	61.36096
1.9	85	68.98485	62.73586
2	85	69.88121	64.0035
2.1	85	70.70765	65.17226
2.2	85	71.46961	66.24984
2.3	85	72.17214	67.24336
2.4	85	72.81986	68.15938
2.5	85	73.41705	69.00393
2.6	85	73.96766	69.78261
2.7	85	74.47531	70.50053
2.8	85	74.94336	71.16246
2.9	85	75.3749	71.77274
3	85	75.77277	72.33542



Appendix A

A.1

(a) $[A] + [B] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 12 \end{bmatrix}$

(b) $[A] + [C]$, Nonsense, $[A]$ and $[C]$ not same order

(c) $[A]\,[C]^T$, Nonsense, columns $[A] \neq$ rows $[C]^T$

(d) $$[D]\,[E] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 3(1) + 1(2) + 2(3) \\ 1(1) + 4(2) + 0(2) \\ 2(1) + 0(2) + 3(3) \end{Bmatrix} = \begin{Bmatrix} 11 \\ 9 \\ 11 \end{Bmatrix}$$

(e) $[D]\,[C]$, Nonsense, columns $[D] \neq$ rows $[C]$

(f) $$[C]\,[D] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) + (1)(1) + 0 & 3 + 4 + 0 & 6 + 0 + 0 \\ -3 + 0 + 6 & -1 + 0 + 0 & -2 + 0 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 6 \\ 3 & -1 & 7 \end{bmatrix}$$

A.2 $[A] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad [A]^{-1} = \dfrac{[C]^T}{|[A]|}$

$$C_{11} = (-1)^{1+1}(4) = 4, \qquad C_{12} = (-1)^{1+2}(-1) = 1$$

$$C_{21} = (-1)^{2+1}(0) = 0, \qquad C_{22} = (-1)^{2+2}(1) = 1$$

$$[C] = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|[A]| = A_{11}C_{11} + A_{12}C_{12}$$

$$= (1)(4) + (0)(1) = 4$$

$$[C]^T = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad [A]^{-1} = \frac{\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}}{4} = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Verify by multiplying $[A]\,[A]^{-1} = [I]$

A.3 $[D]^{-1} = \dfrac{[C]^T}{|[D]|}$

$$[D] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

$$|\,[D]\,| = 12(3) + (-3)(1) + (-8)(2) = 17$$

$$[D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.4 Nonsense

A.5 $[B] = \begin{bmatrix} 2 & 0 \\ -2 & 8 \end{bmatrix}$

(1) $\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ divide 1st row by 2

(2) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ -2 & 8 & 0 & 1 \end{array}\right]$ multiply 1st row by 2 and add to row 2

(3) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 8 & 1 & 1 \end{array}\right]$ divide 2nd row by 8

(4) $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{array}\right]$ $\therefore [B]^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$

A.6 $[D]^{-1}$ by row reduction

$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ divide row 1 by 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ subtract row 1 from 2

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$ multiply row 1 by 2 and subtract from row 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & \frac{-2}{3} & \frac{5}{3} & \frac{-2}{3} & 0 & 1 \end{array}\right]$ multiple row 2 by $\frac{2}{11}$ and add to row 3

$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 3\frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} & 1 & 0 \\ 0 & 0 & \frac{51}{33} & \frac{-24}{33} & \frac{2}{11} & 1 \end{array}\right]$ multiply row 2 by $\frac{3}{11}$ and row 3 by $\frac{33}{51}$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{-2}{11} & \frac{-1}{11} & \frac{3}{11} & 0 \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 3 by } \tfrac{2}{11} \text{ and add to row 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 3 by } \tfrac{2}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 0 & \frac{11}{17} & \frac{-4}{51} & \frac{-22}{51} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-24}{51} & \frac{6}{51} & \frac{33}{51} \end{array}\right]. \text{ multiply row 2 by } \tfrac{1}{3} \text{ and subtract from row 1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{12}{17} & \frac{-3}{17} & \frac{-8}{17} \\ 0 & 1 & 0 & \frac{-3}{17} & \frac{5}{17} & \frac{2}{17} \\ 0 & 0 & 1 & \frac{-8}{17} & \frac{2}{17} & \frac{11}{17} \end{array}\right]$$

$$\therefore [D]^{-1} = \frac{1}{17}\begin{bmatrix} 12 & -3 & -8 \\ -3 & 5 & 2 \\ -8 & 2 & 11 \end{bmatrix}$$

A.7 Show that $([A]\,[B])^T = [B]^T\,[A]^T$ by using

$$[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad [B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$([A]\,[B]) = \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{11}(b_{12}) + a_{12}(b_{22}) & a_{11}(b_{13}) + a_{12}(b_{23}) \\ a_{21}(b_{11}) + a_{22}(b_{21}) & a_{21}(b_{12}) + a_{22}(b_{22}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

$$([A]\,[B])^T = \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{21}(b_{11}) + a_{22}(b_{21}) \\ a_{11}(b_{12}) + a_{12}(b_{22}) & a_{21}(b_{12}) + a_{22}(b_{22}) \\ a_{11}(b_{13}) + a_{12}(b_{23}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{bmatrix} \qquad [A]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$[B]^T\,[A]^T = \begin{bmatrix} b_{11}(a_{11}) + b_{21}(a_{12}) & b_{11}(a_{21}) + b_{21}(a_{22}) \\ b_{12}(a_{11}) + b_{22}(a_{12}) & b_{12}(a_{21}) + b_{22}(a_{22}) \\ b_{13}(a_{11}) + b_{23}(a_{12}) & b_{13}(a_{21}) + b_{23}(a_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}) + a_{12}(b_{21}) & a_{21}(b_{11}) + a_{22}(b_{21}) \\ a_{11}(b_{12}) + a_{12}(b_{22}) & a_{21}(b_{12}) + a_{22}(b_{22}) \\ a_{11}(b_{13}) + a_{12}(b_{23}) & a_{21}(b_{13}) + a_{22}(b_{23}) \end{bmatrix}$$

Answer : $([A]\,[B])^T = [B]^T\,[A]^T$

A.8 $[T] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}$

$$[C] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \qquad [C]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$|\,[T]\,| = C^2 + S^2 = 1$$

$$[T]^{-1} = \frac{[C]^T}{|[T]|} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

and

$$[T]^T = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}$$

$$\therefore \quad [T]^T = [T]^{-1} \text{ and } T \text{ is an orthogonal matrix}$$

A.9 Show $\{X\}^T [A] \{X\}$ is symmetric. Given

$$\{X\} = \begin{bmatrix} x & y \\ 1 & x \end{bmatrix}, [A] = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\{X\}^T = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix}$$

$$\{X\}^T [A] \{X\} = \begin{bmatrix} x & 1 \\ y & x \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax+b & bx+c \\ ay+bx & by+cx \end{bmatrix} \begin{bmatrix} x & y \\ 1 & x \end{bmatrix}$$

$$= \begin{bmatrix} ax^2+bx+bx+c & axy+by+bx^2+cx \\ axy+bx^2+by+cx & ay^2+bxy+bxy+cx^2 \end{bmatrix}$$

as the 1–2 term = 2–1 term $\{X\}^T [A] \{X\}$ is symmetric.

A.10 Evaluate $[K] = \int_0^L [B]^T \; E\,[B]\,dx$, $[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$

$$[K] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx$$

$$[K] = \int_0^L \begin{bmatrix} \frac{1}{L^2} & \frac{-1}{L^2} \\ \frac{-1}{L^2} & \frac{1}{L^2} \end{bmatrix} E\,dx$$

$$[K] = E \begin{bmatrix} \frac{1}{L} & \frac{1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{bmatrix} = \frac{E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(Should multiply by A to get actual $[K]$ for a bar)

A.11 The following integral represents the strain energy in a bar

$$U = \frac{A}{2} \int_0^L \{d\}^T [B]^T [D] [B] \{d\}\,dx$$

where $\{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$ $\quad [B] = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}$ $\quad [D] = E$

Show that $\frac{dU}{d\{d\}}$ yields $[k]\,\{d\}$, where $[k]$ is the bar stiffness matrix given by

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{d\}^T = [d_1\ d_2] \qquad [B]^T = \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}$$

$$U = \frac{A}{2}\int_0^L \{d\}^T\ [B]^T\ [D]\ [B]\ \{d\}\ dx$$

$$U = \frac{AL}{2}\ \{d\}^T [B]^T [D]^T [B]\ \{d\} = \frac{AL}{2}\ [d_1 \quad d_2] \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AL}{2}\begin{bmatrix} \frac{d_2 - d_1}{L} \end{bmatrix} [E] \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\begin{bmatrix} \frac{d_2 - d_1}{L} \end{bmatrix} \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{AEL}{2}\begin{bmatrix} \frac{d_1 - d_2}{L^2} & \frac{-d_1 + d_2}{L^2} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$U = \frac{AEL}{2}\left[\frac{d_1^2 - d_1 d_2 - d_1 d_2 + d_2^2}{L^2}\right] = \frac{AE}{2L}\ [\, d_1^2 - 2d_1 d_2 + d_2^2\,]$$

$$\frac{dU}{d\{d\}} = \begin{Bmatrix} \frac{\partial U}{\partial d_1} \\ \frac{\partial U}{\partial d_2} \end{Bmatrix} = \begin{Bmatrix} \frac{AE}{2L}(2d_1 - 2d_2) \\ \frac{AE}{2L}(2d_2 - 2d_1) \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_2 - d_1 \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} d_1 - d_2 \\ -d_1 + d_2 \end{Bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\frac{dU}{d\{d\}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = [k]\ \{d\} \text{ knowing that } [k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Thus $\qquad \dfrac{dU}{d\{d\}} = [k]\ \{d\}$

Appendix B

B.1 By Cramer's Rule

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 12 \end{Bmatrix}$$

$$x_1 = \frac{\left|\{d\}^{(1)}\right|}{|[a]|} = \frac{\begin{vmatrix} 5 & 3 \\ 12 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{-5-36}{-13} = 3.15$$

$$x_2 = \frac{\left|\{d\}^{(2)}\right|}{|[a]|} = \frac{\begin{vmatrix} 1 & 5 \\ 4 & 12 \end{vmatrix}}{-13} = \frac{12-20}{-13} = 0.62$$

B.2 By Inverse method

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 12 \end{Bmatrix}$$

$$C_{11} = (-1)^{1+1}|-1| = -1,\ C_{12} = (-1)^{1+2}|4| = -4$$

$$C_{21} = (-1)^{2+1}|3| = -3,\ C_{22} = (-1)^{2+2}|1| = 1$$

$$[C] = \begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}, \qquad [C]^T = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$$

$$|[a]| = a_{11}\, C_{11} + a_{12}\, C_{12} = 1(-1) + 3(-4) = -13$$

$$[a]^{-1} = \frac{\left|[C]^T\right|}{|[a]|} = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -4 & -1 \end{bmatrix}$$

$$\therefore \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = -\frac{1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{Bmatrix} 5 \\ 12 \end{Bmatrix} = -\frac{1}{13}\begin{Bmatrix} -5-36 \\ -20+12 \end{Bmatrix}$$

$$x_1 = \frac{-41}{-13} = 3.15, \qquad x_2 = \frac{-8}{-13} = 0.62$$

B.3 By Gauss elimination

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ -2 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ -3 \end{Bmatrix}$$

Multliply row 1 by $-\left(\frac{-2}{1}\right)$ and add to row 3

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ 0 & -9 & -8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ 5 \end{Bmatrix}$$

Multliply row 2 by $-\left(\frac{-9}{3}\right)$ and add to row 3

$$\begin{bmatrix} 1 & -4 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -1 \\ 2 \end{Bmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

By (3)

$$\therefore \quad 4x_3 = 2$$

$$x_3 = \frac{1}{2}$$

By (2)

$$3x_2 + 4x_3 = -1$$

$$x_2 = \frac{-1-2}{3} = -1$$

By (1)

$$x_1 - 4(-1) - 5\left(\frac{1}{2}\right) = 4$$

$$x_1 = \frac{5}{2}$$

B.4 $\quad 2x_1 + x_2 - 3x_3 = 11$

$$4x_1 - 2x_2 + 3x_3 = 8$$

$$-2x_1 + 2x_2 - x_3 = -6$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 11 \\ 4 & -2 & 3 & 8 \\ -2 & 2 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & -4 & 9 & -14 \\ 0 & 3 & -4 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & \frac{-9}{4} & \frac{7}{2} \\ 0 & 0 & \frac{11}{4} & \frac{-11}{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & \frac{-9}{4} & \frac{7}{2} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\therefore \quad x_1 = 3, \quad x_2 = -1, \quad x_3 = -2$$

B.5 $x_1 = 2y_1 - y_2 \qquad z_1 = - x_1 - x_2$

$x_2 = y_1 - y_2 \qquad z_2 = 2x_1 + x_2$

(a) $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$

$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

(b) $z_1 = -(2y_1 - y_2) - (y_1 - y_2)$

$z_1 = -3y_1 + 2y_2$

$z_2 = 2(2y_1 - y_2) + (y_1 - y_2)$

$z_2 = 5y_1 - 3y_2$

$\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$

(c) $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

$\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = 9 - 10 = -1$

$[C] = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}$

$[C]^T = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$

B.6 $\{X\}^T = (1, 1, 1, 1, 1)$

First iteration

$$2x_1 - x_2 = -1 \Rightarrow x_1 = \frac{1}{2}(x_2 - 1)$$

$$= \frac{1}{2}(1 - 1) = 0$$

$$6x_2 = x_1 + x_3 + 4 \Rightarrow x_2 = \frac{1}{6}(x_1 + x_3 + 4)$$

$$= \frac{1}{6}(0 + 1 + 4)$$

$$= 0.833$$

$$4x_3 = 2x_2 + x_4 + 4 \Rightarrow x_3 = \frac{1}{4}(2x_2 + x_4 + 4)$$

$$x_3 = \frac{1}{4}(2(0.833) + 1 + 4)$$

$$= 1.667$$

$$4x_4 = x_3 + x_5 + 6 \Rightarrow x_4 = \frac{1}{4}(x_3 + x_5 + 6)$$

$$x_4 = \frac{1}{4}(1.667 + 1 + 6)$$

$$= 2.167$$

$$2x_5 = x_4 - 2 \Rightarrow x_5 = \frac{1}{2}(x_4 - 2)$$

$$= \frac{1}{2}(2.167 - 2)$$

$$= 0.083$$

2nd iteration

$$x_1 = \frac{1}{2}(x_2 - 1) = \frac{1}{2}(0.833 - 1) = -0.084$$

$$x_2 = \frac{1}{6}(x_1 + x_3 + 4) = \frac{1}{6}(-0.084 + 1.667 + 4)$$

$$= 0.93$$

$$x_3 = \frac{1}{4}(2x_2 + x_4 + 4) = \frac{1}{4}(2(0.93) + 2.167 + 4)$$

$$= 2.007$$

$$x_4 = \frac{1}{4}(x_3 + x_5 + 6) = \frac{1}{4}(2.007 + 0.083 + 6)$$

$$= 2.023$$

$$x_5 = \frac{1}{2}(x_4 - 2) = \frac{1}{2}(2.023 - 2)$$

$$= 0.011$$

3rd iteration	4th iteration	5th iteration
$x_1 = -0.035$	$x_1 = -0.003$	$x_1 = 0$
$x_2 = 0.995$	$x_2 = 1.000$	$x_2 = 1$
$x_3 = 2.003$	$x_3 = 2.001$	$x_3 = 2$
$x_4 = 2.004$	$x_4 = 2.001$	$x_4 = 2$
$x_5 = 0.002$	$x_5 = 0.000$	$x_5 = 0$

B.7 By Gauss-Seidel

Initial guess $x_1 = 1$ $x_2 = 1$

1st iteration (Reorder equations so a_{ii} largest)

$$4x_1 - x_2 = 12 \Rightarrow x_1 = \frac{1}{4}(12 + x_2)$$

$$= \frac{1}{4}(12 + 1) = 3.25$$

$$x_1 + 3x_2 = 5 \Rightarrow x_2 = \frac{1}{3}(5 - x_1)$$

$$= \frac{1}{3}(5 - 3.25) = 0.583$$

2nd iteration

$$x_1 = 3.146$$
$$x_2 = 0.618$$

3rd iteration

$$x_1 = 3.155$$
$$x_2 = 0.615$$

4th iteration

$$x_1 = 3.154$$
$$x_2 = 0.615$$

5th iteration

$$x_1 = 3.154$$
$$x_2 = 0.615$$

B.8

(a) $2x_1 - 4x_2 = 2$

$$-9x_1 + 12x_2 = -6$$

$$|[a]| \neq 0$$

$$\begin{vmatrix} 2 & -4 \\ -9 & 12 \end{vmatrix} = 24 + 36 = 60 \neq 0$$

$\therefore$ unique solution

(b) $10x_1 + x_2 = 0$

$$5x_1 + \frac{1}{2}x_2 = 3$$

$$\begin{vmatrix} 10 & 1 \\ 5 & \frac{1}{2} \end{vmatrix} = 0$$

Non existent solution

(c) $2x_1 + x_2 + x_3 = 6$

$$3x_1 + x_2 - x_3 = 4$$
$$5x_1 + 2x_2 + 2x_3 = 8$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & -1 \\ 5 & 2 & 2 \end{vmatrix} \neq 0$$

$\therefore$ unique solution

(d) $x_1 + x_2 + x_3 = 4$

$$2x_1 + 2x_2 + 2x_3 = 2$$
$$3x_1 + 3x_2 + 3x_3 = 3$$
$$|[a]| = 0$$

$\therefore$ Non unique

all lines are parallel to each other

B.9 1^{st} Figure $n_d = 2$, $m = 3$

$$n_b = n_d\,(m + 1) = 2\,(3 + 1) = 8$$

2^{nd} Figure $n_d = 2$, $m = 5$

$$n_b = 2(5 + 1) = 12$$

Appendix D

D.1 Using table D–1

(a) Load case 1

$$P = 10 \text{ kip} \quad L = 20 \text{ ft}$$

$$f_{1y} = f_{2y} = \frac{-10}{2} = -5 \text{ kip}$$

$$m_1 = -m_2 = \frac{-10\,(20)}{8} = -25 \text{ kip}\cdot\text{ft}$$

(b) Load case 3

$$P = 5 \text{ kip} \quad L = 20 \text{ ft} \quad \alpha = \frac{1}{4}$$

$$f_{1y} = f_{2y} = -5 \text{ kip}$$

$$m_1 = -m_2 = -\frac{1}{4}\left(1-\frac{1}{4}\right)(5)\,(20)$$

$$= -18.75 \text{ kip}\cdot\text{ft}$$

(c) Load case 4

$$w = 1000 \ \frac{\text{lb}}{\text{ft}} \quad L = 30 \text{ ft}$$

$$f_{1y} = f_{2y} = \frac{-(1000)\,(30)}{2} = -15000 \text{ lb}$$

$$m_1 = -m_2 = \frac{-(1000)\,(30)^2}{12} = -75000 \text{ lb}\cdot\text{ft}$$

(d) Load cases 1 and 7

$$P = 5 \text{ kip}, \quad L = 20 \text{ ft} \quad w = 2 \ \frac{\text{kip}}{\text{ft}}$$

$$f_{1y} = \frac{-5}{2} - \frac{13(20)(2)}{32} = -18.75 \text{ kip}$$

$$f_{2y} = \frac{-5}{2} - \frac{3(20)(2)}{32} = -6.25 \text{ kip}$$

$$m_1 = \frac{-5(20)}{8} - \frac{11(2)(20)^2}{192} = -12.5 - 45.83$$

$$= -58.33 \text{ kip}\cdot\text{ft}$$

$$m_2 = \frac{5(20)}{8} - \frac{5(2)(20)^2}{192} = 12.5 + 20.83$$

$$= 33.33 \text{ kip}\cdot\text{ft}$$

(e) Load case 5

Note: Switch nodes 1 and 2

$$w = 2000\ \frac{\text{lb}}{\text{ft}} \qquad L = 20\ \text{ft}$$

$$f_{1y} = \frac{-3(2000)20}{20} = -\,6000\ \text{lb}$$

$$f_{2y} = \frac{-7(2000)(20)}{20} = -\,14{,}000\ \text{lb}$$

$$m_1 = \frac{(2000)(20)^2}{30} = -\,26{,}667\ \text{lb}\cdot\text{ft}$$

$$m_2 = \frac{(2000)(20)^2}{20} = 40{,}000\ \text{lb}\cdot\text{ft}$$

(f) Load case 2

$$P = 5\ \text{kN}, \quad L = 7\ \text{m}, \quad a = 5\ \text{m}, \quad b = 2\ \text{m}$$

$$f_{2y} = \frac{-5(2)^2\,[7+2(5)]}{7^3} = -\,0.99\ \text{kN}$$

$$f_{1y} = \frac{-5(5)^2\,[7+2(2)]}{7^3} = -\,4.01\ \text{kN}$$

$$m_2 = \frac{-5(5)\,(2)^2}{7^2} = -\,2.04\ \text{kN}\cdot\text{m}$$

$$m_1 = \frac{5(5)^2\,(2)}{7^2} = 5.10\ \text{kN}\cdot\text{m}$$

(g) Load case 6

$$w = 4\ \frac{\text{kN}}{\text{m}},\ L = 6\ \text{m}$$

$$f_{1y} = f_{2y} = \frac{-4(6)}{4} = -\,6\ \text{kN}$$

$$m_1 = -\,m_2 = \frac{-5(4)(6)^2}{96} = -\,7.5\ \text{kN}\cdot\text{m}$$

(h) Load case 4

$$w = 5\ \frac{\text{kN}}{\text{m}},\ L = 4\ \text{m}$$

$$f_{1y} = f_{2y} = \frac{-5(4)}{2} = -\,10\ \text{kN}$$

$$m_1 = -\,m_2 = \frac{-5(4)^2}{12} = -\,6.67\ \text{kN}\cdot\text{m}$$