Chapter 3
Manipulator Kinematics

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Kinematics:

- Kinematics: the science of motion that treats the subject without regard to the forces that cause it. (Position, Velocity, Acceleration, Jerk …)

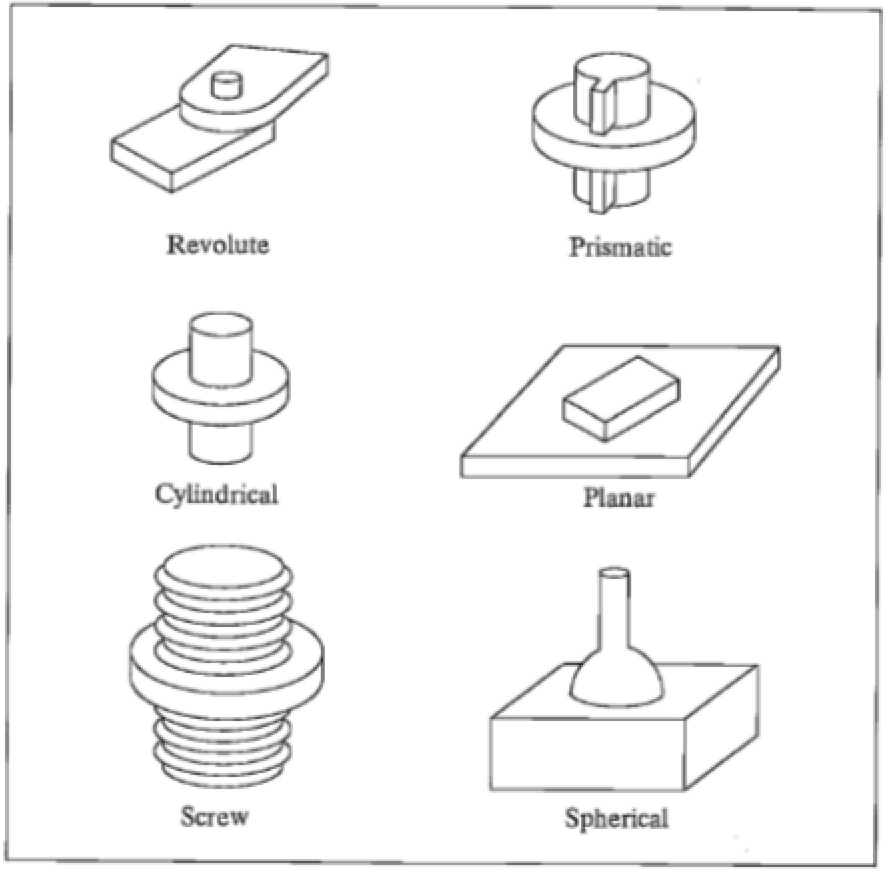
- In this Chapter we will study the position and orientation of the manipulator links in static situations.

- The approach is to attach frames to the various parts and describe the relationships between these frames.

- Main Goal: Compute the manipulator's end-effector relative to the base of the manipulator as a function of the joint variables.

Link Descriptions:

- Manipulator: a set of links connected in a chain by joints.
- Joint: Connection between 2 links allowing restricted relative motion between the neighboring links.



- We will only consider

  joints with one DOF:

  - Revolute Joints

  - Prismatic Joints

Link Descriptions:

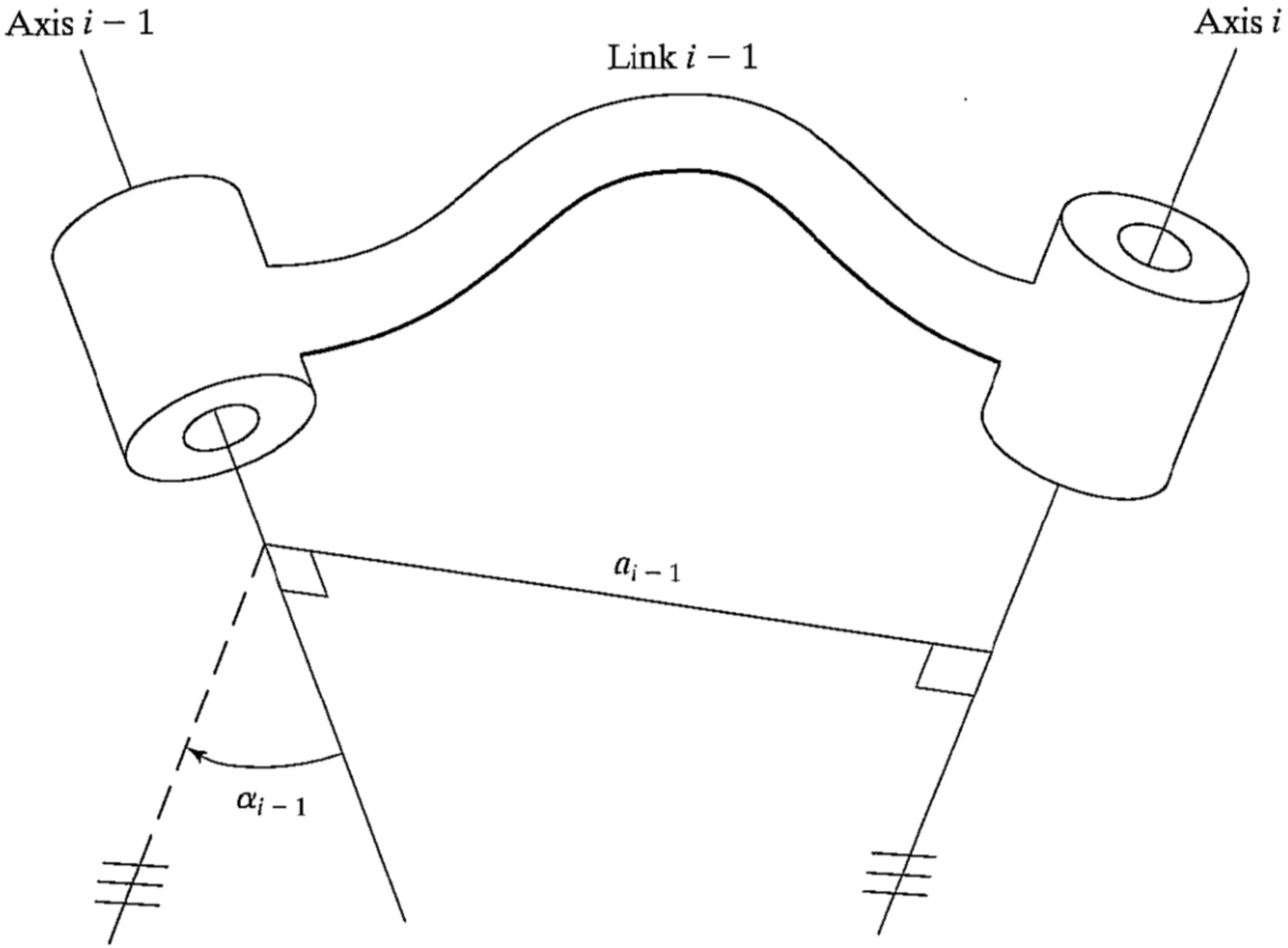
- The links are numbered starting from the fixed base → 0
- The first moving body is numbered Link 1, and so on, out to the free end of the arm, which is Link n.
- Typical manipulators have 5 or 6 joints.

- Note:

In order to describe the position and orientation of the end-effector we need at least 6 joints.

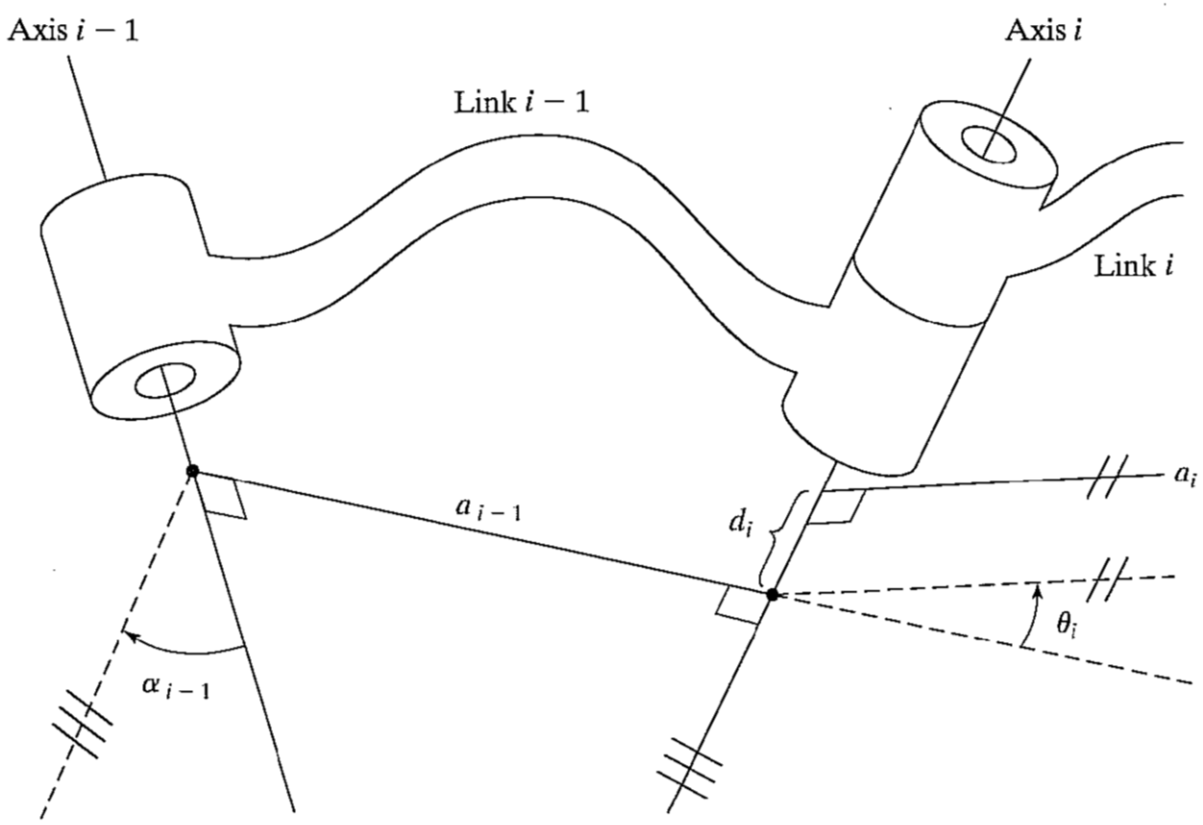
Link Components:

- Joint Axis i: a vector direction about which link i rotates relative to link i – 1.
- Link Length a_{i-1} : a line along the length of the link that is perpendicular to both joint axes.
- Link Twist α_{i-1}: the angle measured from axis $i-1$ to axis i in the right hand sense



Neighboring Links:

- Neighboring links have a common joint axis between them.
- Link Offset d_i : the vertical distance measured along the axis of joint i between the points of intersection of a_{i-1} and a_i with the joint axis i.
- Joint Angle θ_i: the angle that describes the amount of rotation of link i with respect to link $i-1$. It is measured from a_{i-1} to a_i.



Link Parameters:

- Any robot can be described kinematically by giving the values of four quantities for each link:

→ Link Length a_{i-1}
→ Link Twist α_{i-1}
→ Link Offset d_i
→ Joint Angle θ_i

- These parameters are called the **Denavit-Hartenberg** Parameters (**DH parameters**)

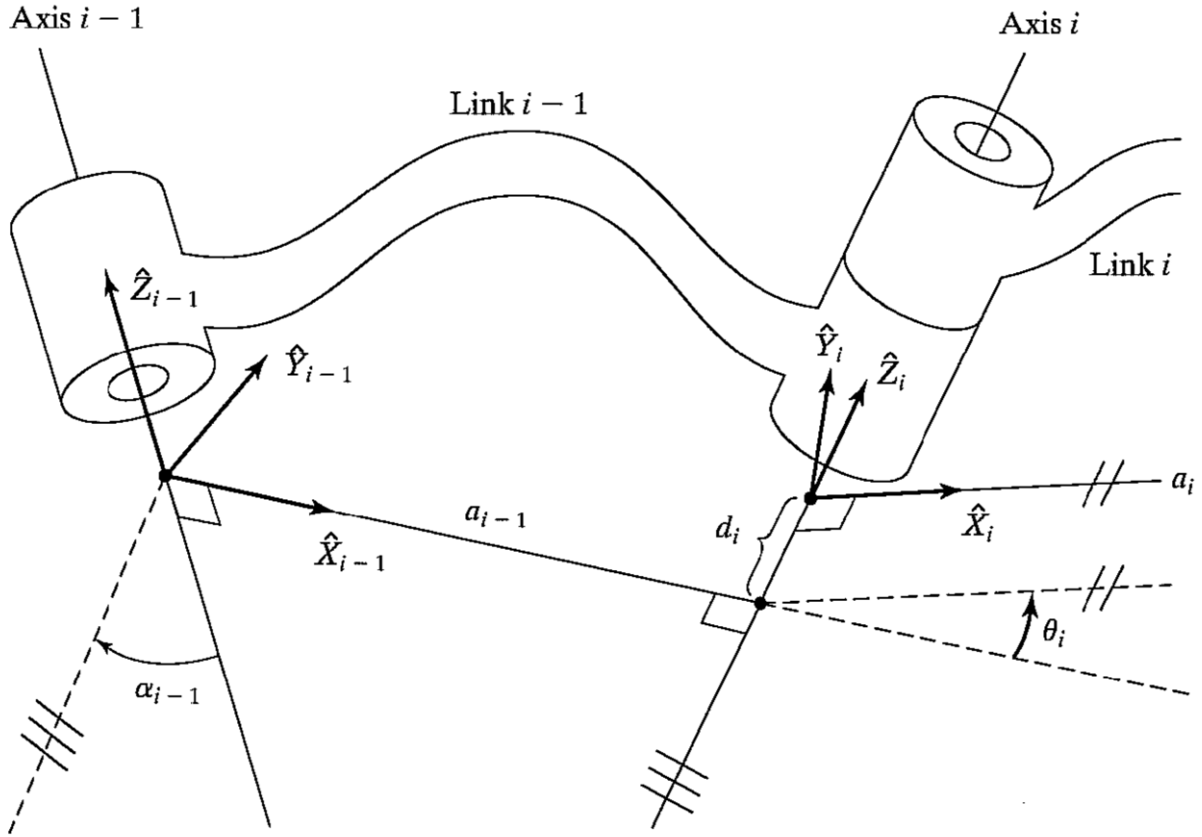
- For a revolute joint, all parameters are constants, except for θ_i → Joint Variable
- For a prismatic joint, all parameters are constants except for d_i → Joint Variable

- Note that a_i and α_i depend on joint axes i and $i+1$, thus at both ends of the chain:

$a_0 = a_n = 0$
$\alpha_0 = \alpha_n = 0$

Convention for Attaching Frames to Links

- Frame i is rigidly attached to link i.
- $\hat{Z}_i$ is coincident with the joint axis i.
- The origin of Frame i is placed at the point of intersection between a_i and the joint axis i.
- $\hat{X}_i$ points along a_i in the direction from joint i to $i+1$
- If $a_i = 0$, $\hat{X}_i$ is normal to the plane including $\hat{Z}_i$ and $\hat{Z}_{i+1}$.



First and Last Links in the Chain:

- Frame {0} is arbitrary, usually we define it to be coincident with frame {1} when $\theta_1 = 0$.
- This ensures that:

→ $a_0 = 0$
→ $\alpha_0 = 0$
→ $d_1 = 0$ if joint 1 is revolute
→ $\theta_1 = 0$ if joint 1 is prismatic

- In Frame {n} $\hat{X}_n$ is chosen so that it aligns with $\hat{X}_{n-1}$ when $\theta_n = 0$, and the origin of Frame {n} is chosen so that $d_n = 0$

Summary of DH-parameters

→ a_i is the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$

→ α_i is the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$

→ d_i is the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$

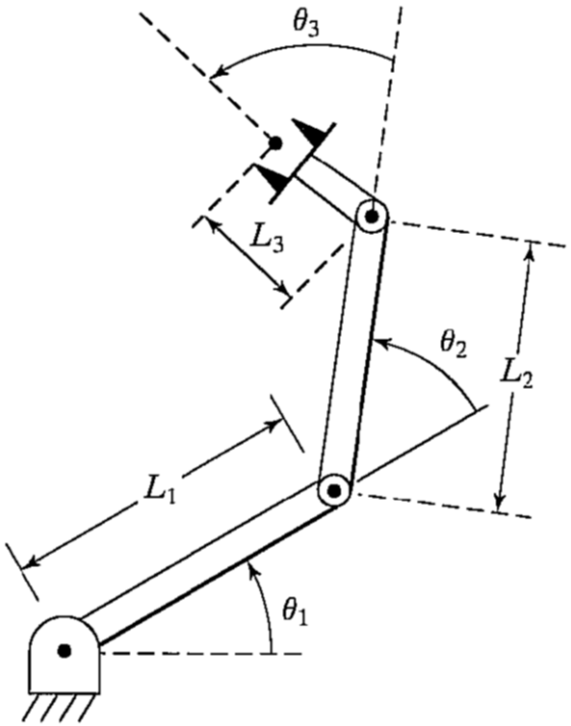
→ θ_i is the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$

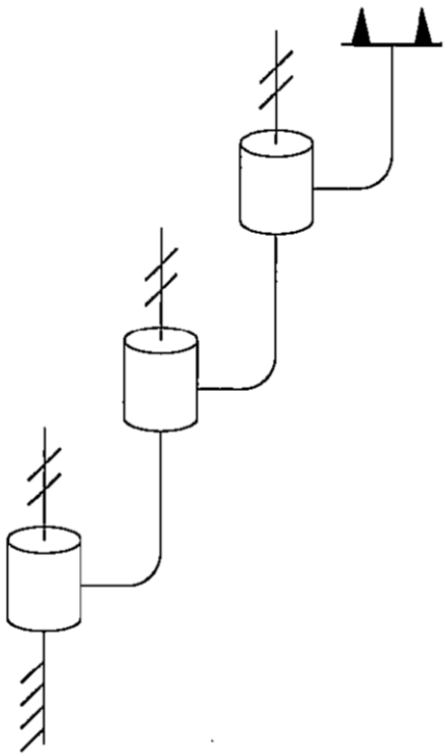
- Usually a_i is a positive value (distance)
- The rest of the parameters can be positive or negative.

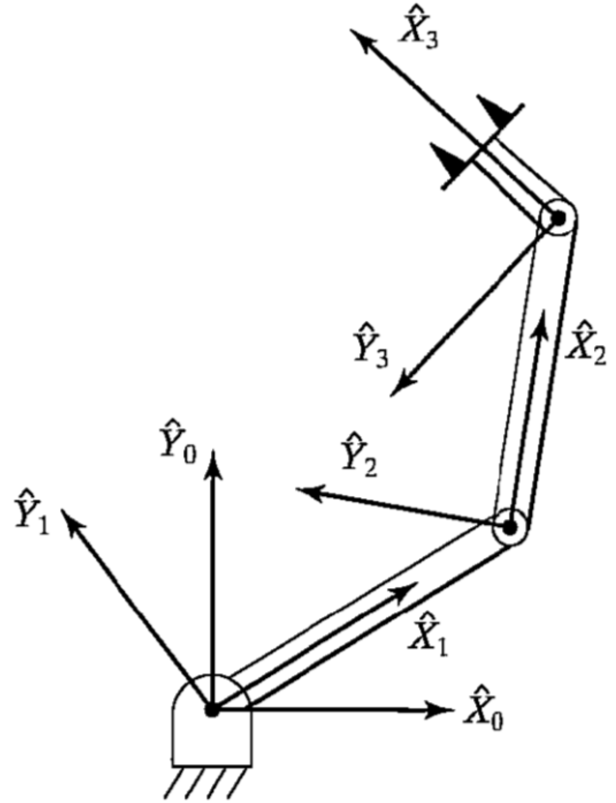
Summary of Link Frame Attachment Procedure

1. Identify the joint axes and imagine infinite lines along them. For steps 2-5 consider 2 of these neighboring lines (at axes i and $i+1$)
2. Identify the common perpendicular between them. At the point where the common perpendicular meets the i^{th} axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the i^{th} joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete the right-hand coordinate system.
6. Assign {0} to match {1} when the first joint variable is 0. For {n}, choose an origin location and $\hat{X}_n$ direction freely, but generally so as to cause as many link parameters as possible to become zero.

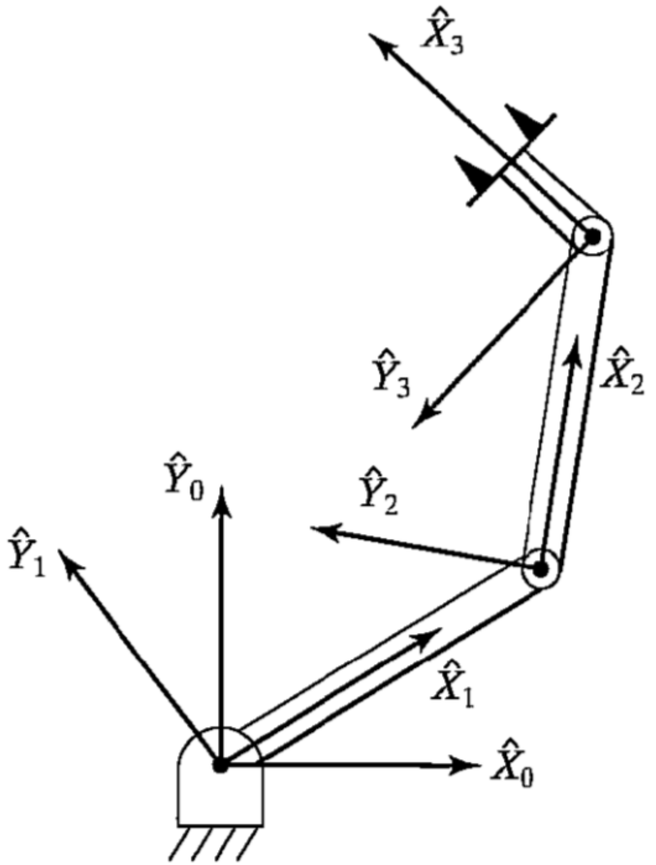
Example: RRR (3R) Robot





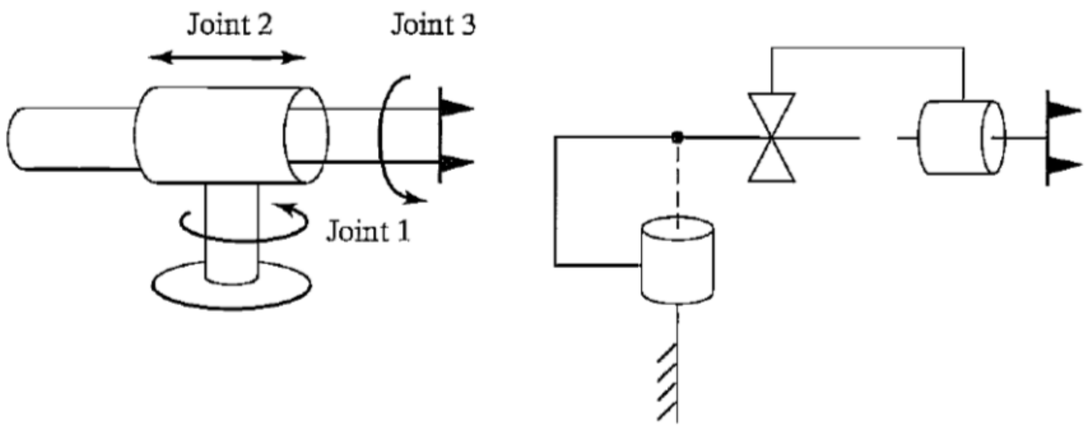


Example: RRR (3R) Robot

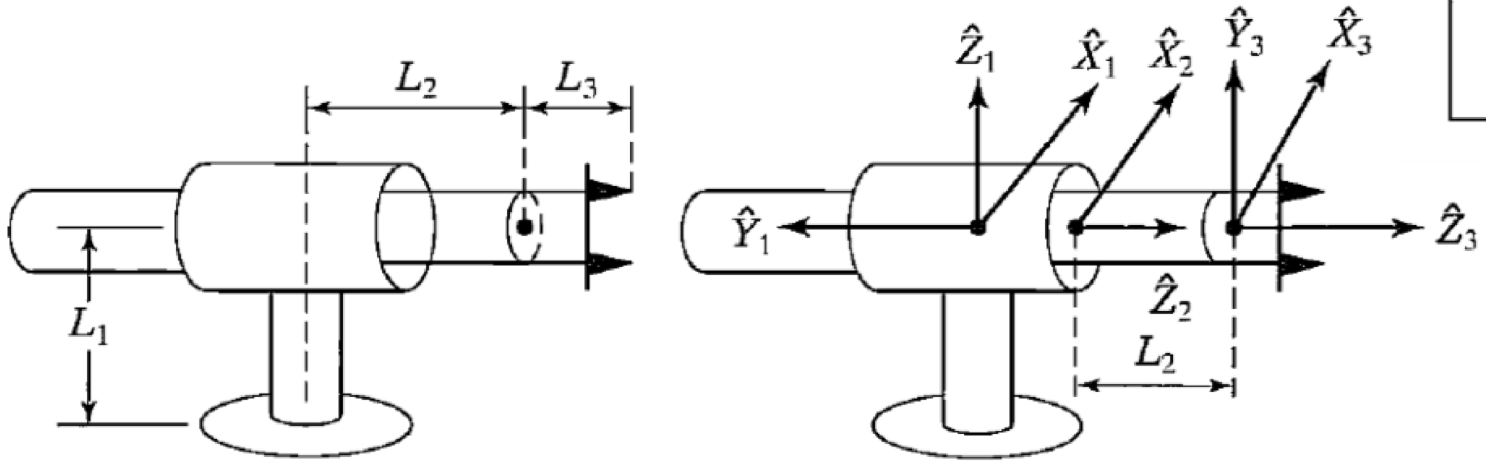


i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Example: RPR Robot



(a) (b)



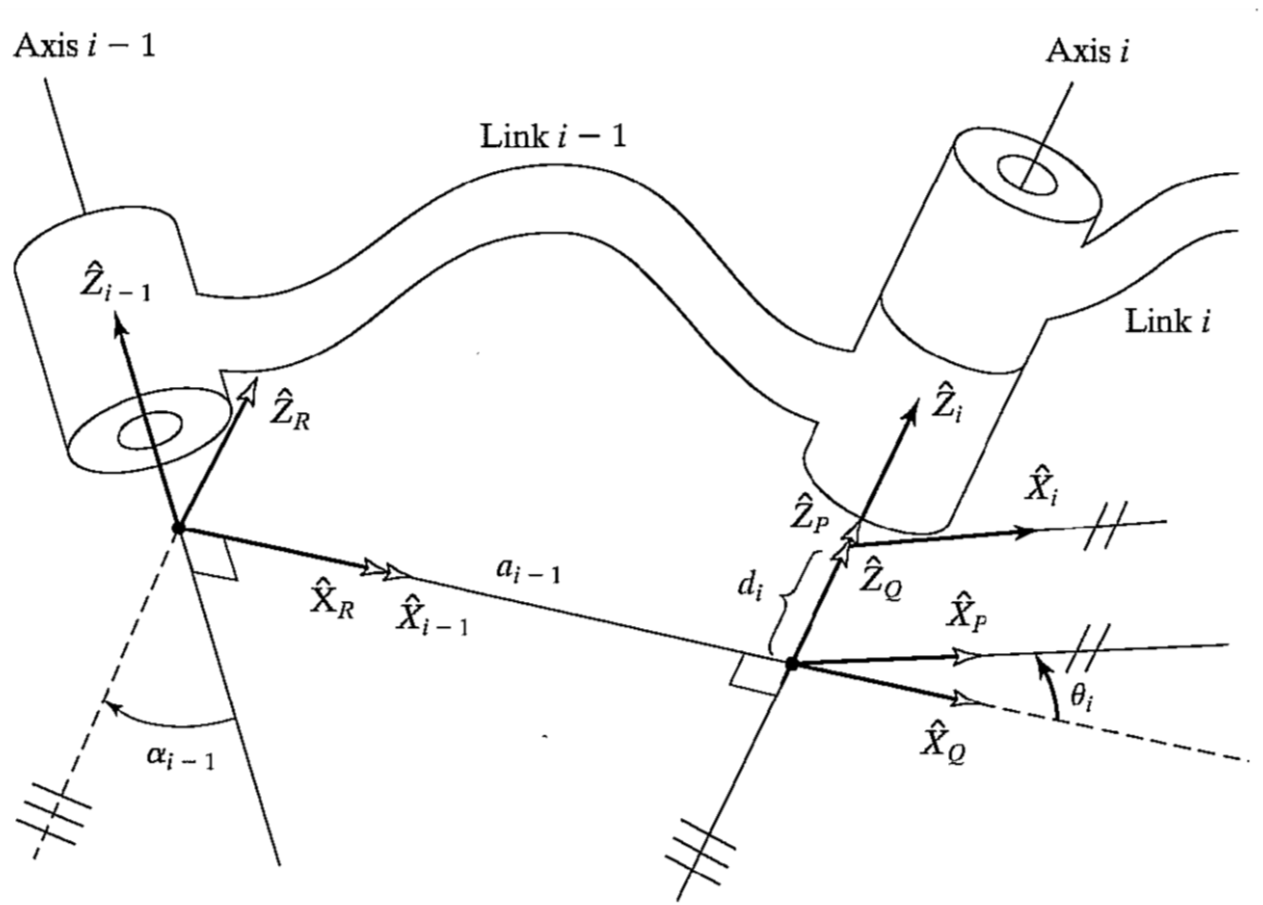
(a) (b)

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0
3	0	0	L_2	θ_3

Derivation of the Transformation Matrix

To find the transformation matrix ${}_{i-1}^{\;\;i}T$ we should define 3 intermediate frames between $\{i-1\}$ and $\{i\}$:

1. $\{R\}$ differs from $\{i-1\}$ by a rotation about $\hat{X}_{i-1}$ with an angle α_{i-1}
2. $\{Q\}$ differs from $\{R\}$ by a translation along $\hat{X}_R$ with a distance a_{i-1}
3. $\{P\}$ differs from $\{Q\}$ by a rotation about $\hat{Z}_Q$ with an angle θ_i
4. $\{i\}$ differs from $\{P\}$ by a translation along $\hat{Z}_P$ with a distance d_i



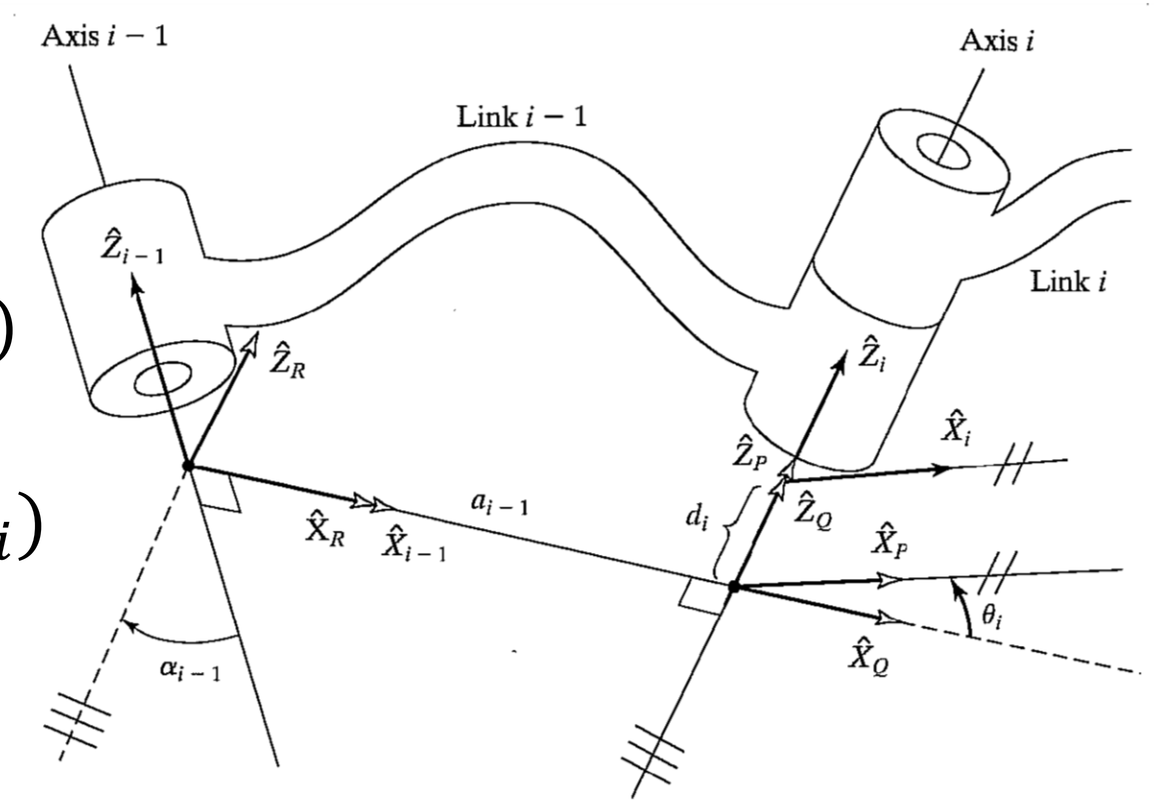
Derivation of the Transformation Matrix

- Note that rotations and translations are happening with respect to the new axes (Euler angles)

$$^{i-1}_{i}T = {}^{i-1}_{R}T \; ^{R}_{Q}T \; ^{Q}_{P}T \; ^{P}_{i}T$$

$$^{i-1}_{i}T = R_x(\alpha_{i-1})D_x(a_{i-1})R_z(\theta_i)D_z(d_i)$$

$$^{i-1}_{i}T = screw_x(\alpha_{i-1}, a_{i-1})screw_z(\theta_i, d_i)$$



- $screw_x(\alpha_{i-1}, a_{i-1})$ represents a transformation matrix that includes a rotation about the x-axis with an angle α_{i-1} and a translation along the x-axis with a distance a_{i-1}

Derivation of the Transformation Matrix

$$ {}^{i-1}_{i}T = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1}s\theta_i & c\alpha_{i-1}c\theta_i & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\alpha_{i-1}s\theta_i & s\alpha_{i-1}c\theta_i & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example: RPR Robot

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0
3	0	0	L_2	θ_3

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

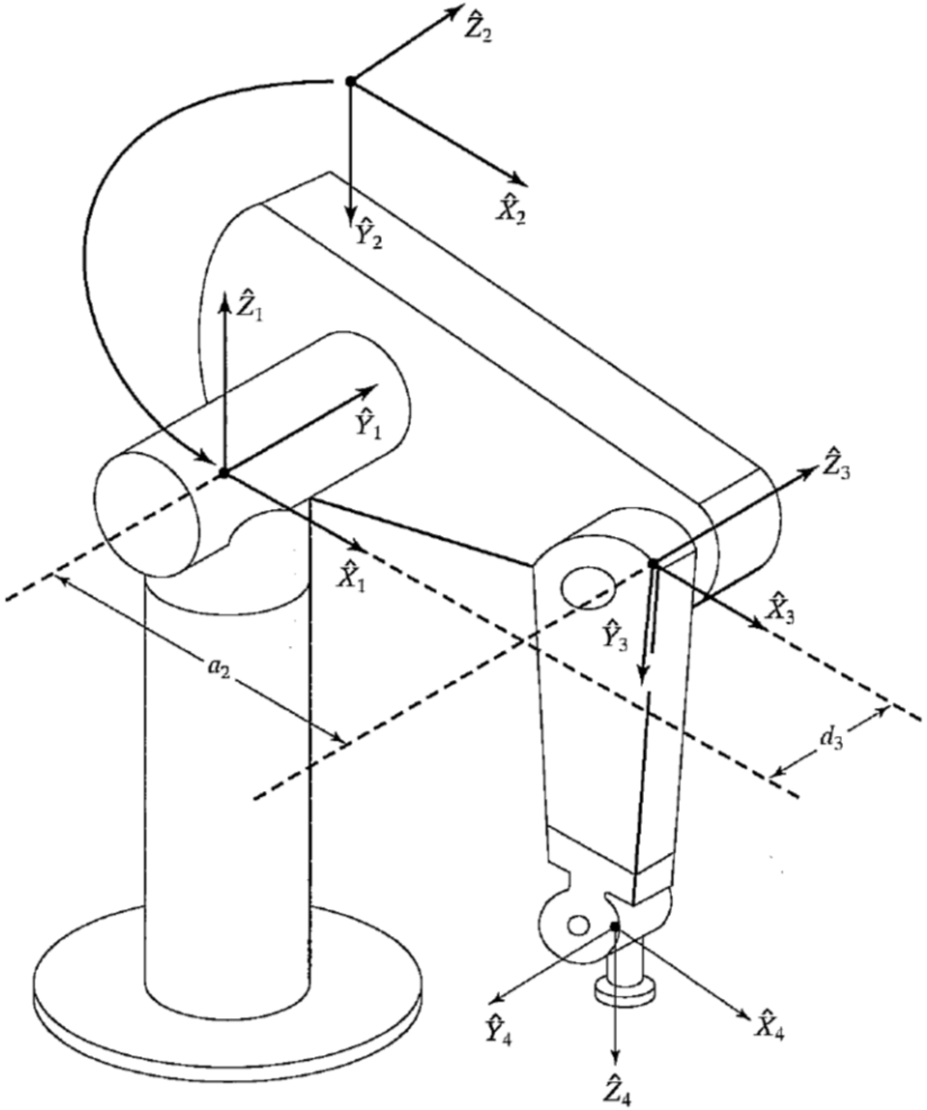
Concatenating Link Transformations:

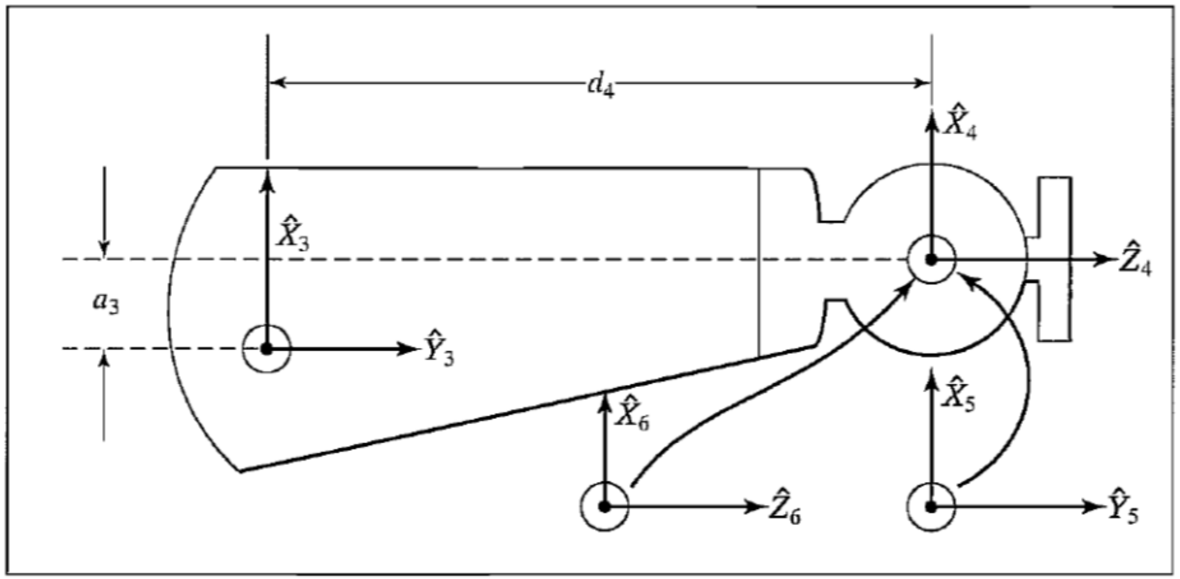
- After obtaining the Transformation Matrix of each 2 adjacent joints, we can get the total Transformation Matrix using:

$$ {}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \ldots {}^{N-1}_{N}T $$

- The transformation is a function of all joint variables.
- If the joint variables are know for a certain instant, the Cartesian position and orientation of the last link can be calculated using the Transformation Matrix shown above.

Example: PUMA 560





Example: PUMA 560

i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

$$ {}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

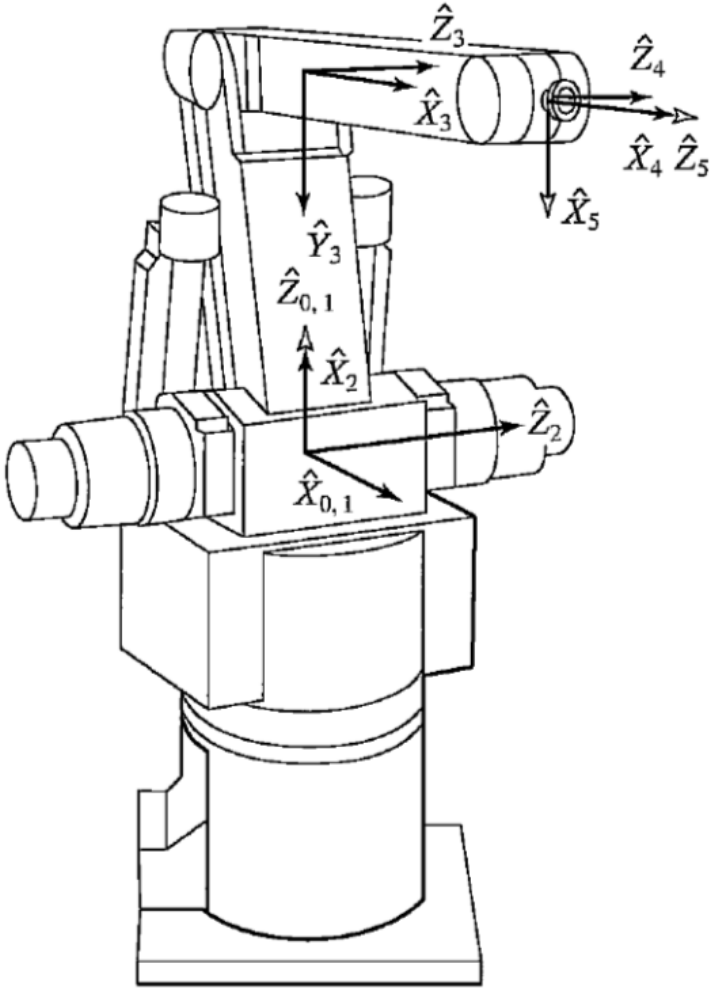
$$ {}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Example: Yasukawa L-3



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0	L_2	0	θ_3
4	0	L_3	0	θ_4
5	90°	0	0	θ_5

Example: Yasukawa L-3

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & l_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & l_3 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Frames with Standard Names:

- The base frame {B}: is located at a nonmoving part at the base of the manipulator.
- The station frame {S}: is located at a task-relevant point. It is universal and the actions of the robot are performed relative to it. {S} is always specified with respect to {B}.
- The wrist frame {W}: is fixed to the last link of the manipulator and moves with it. It is usually specified with respect to {B}.
- The tool frame {T} is fixed to the end of the tool that the robot is holding. It is always specified with respect to {W}
- The goal frame {G}: it describes the end point of the motion. {T} should coincide with {G} to know that the manipulator did its job.

Frames with Standard Names: