Chapter (2): Spatial Descriptions and Transformations.

Monday, February 20, 2017 10:13 AM

* There is always a universal Coordinate System (CS) to which everything is referenced.

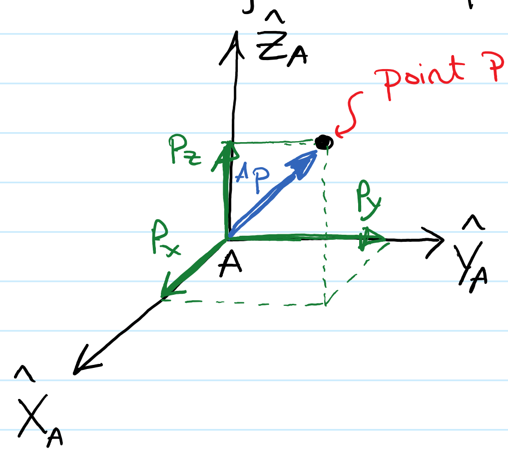
* Description of a position:

Any point in the universe can be located using a 3x1 position vector

P is described in CS {A}

$$^{A}P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

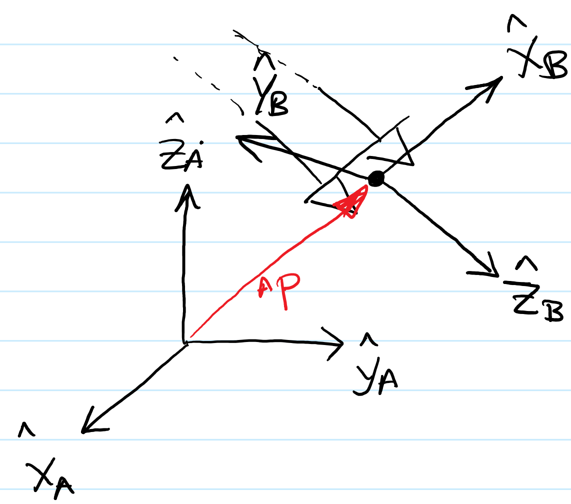
The result of projecting the vector P on x, y, z of frame {A} respectively.



* Description of orientation:

To describe the orientation of a body:

1) Attach a CS to the body
2) Describe this CS wrt a reference CS.



To describe {B} wrt {A}:

Define: The rotation matrix:

Define : the rotation matrix :

$$\underbrace{{}^{A}_{B}R}_{3\times3} = \left[\ \underbrace{{}^{A}\hat{X}_B}_{3\times1}\quad \underbrace{{}^{A}\hat{Y}_B}_{3\times1}\quad \underbrace{{}^{A}\hat{Z}_B}_{3\times1}\ \right]$$

projection of $\hat{X}_B$ on $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$

Projection of $\hat{Y}_B$ on $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$

Projection of $\hat{Z}_B$ on $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$

* Projection is defined using the dot product :

$${}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

Notice that the rows of ${}^{A}_{B}R$ are the descriptions of $\{A\}$ wrt $\{B\}$

$${}^{A}_{B}R = \begin{bmatrix} {}^{B}\hat{X}_A^{T} \\ {}^{B}\hat{Y}_A^{T} \\ {}^{B}\hat{Z}_A^{T} \end{bmatrix} = {}^{B}_{A}R^{T}$$

projecting $\{B\}$ on $\{A\}$ using ${}^{A}_{B}R$ then projecting $\{A\}$ on $\{B\}$ using ${}^{B}_{A}R$ results in no change of the orientation

$${}^{A}_{B}R \times {}^{B}_{A}R = \underbrace{I_3}_{\text{identity matrix } (3\times3)}$$

Since ${}^{B}_{A}R = {}^{A}_{B}R^{T} \implies {}^{A}_{B}R \times {}^{A}_{B}R^{T} = I_3$

$$\implies \boxed{{}^{A}_{B}R^{T} = {}^{A}_{B}R^{-1}}$$

Note that ${}^{A}_{B}R$ has orthonormal columns (their magnitude is 1 and they are orthogonal)

From linear algebra $\Rightarrow$ if a matrix is orthonormal

$$R^T = R^{-1}$$

* Description of a frame:

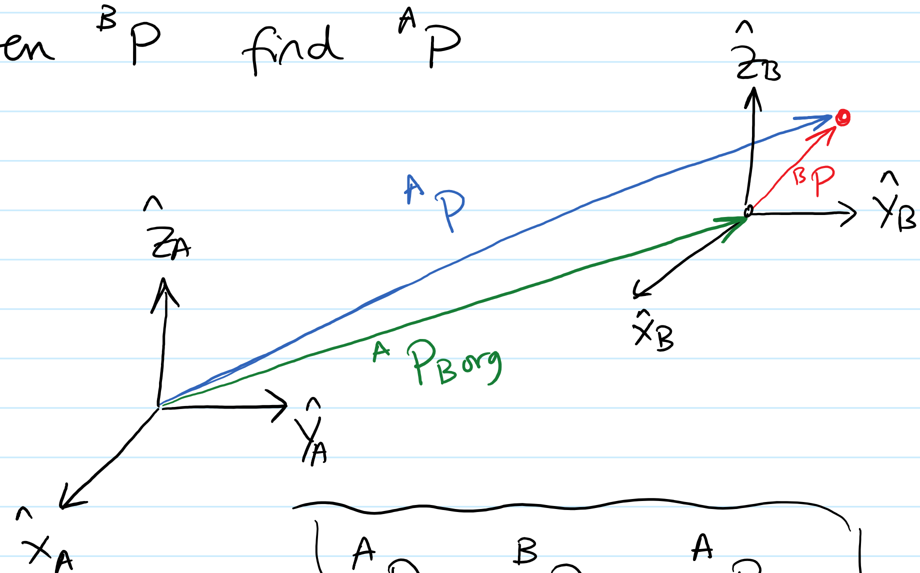
A frame is a set of four vectors giving position & orientation information: (Rotation matrix + position vector)

$$\{B\} = \{{}^{A}_{B}R,\ {}^{A}P_{Borg}\}$$

Mapping: Changing descriptions from one frame to another.

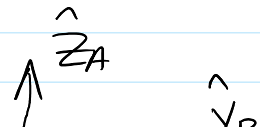
(1) Mapping involving translated frames:

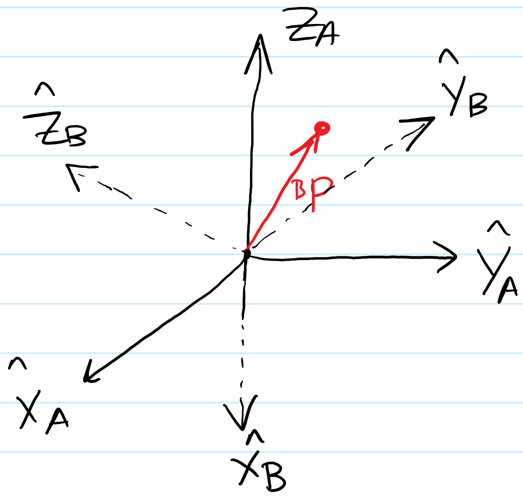
Given ${}^{B}P$ find ${}^{A}P$



$$\boxed{{}^{A}P = {}^{B}P + {}^{A}P_{Borg}}$$

(2) Mapping involving rotated frames



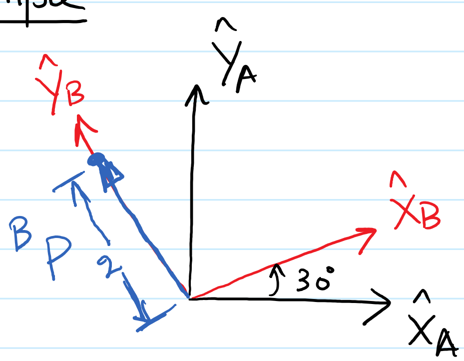


$$ {}^A_B R = \left[{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \right] $$

$$ \boxed{{}^A P = {}^A_B R \, {}^B P} $$

$$ {}^A P = \begin{bmatrix} {}^A\hat{X}_B \cdot {}^B P \\ {}^A\hat{Y}_B \cdot {}^B P \\ {}^A\hat{Z}_B \cdot {}^B P \end{bmatrix} $$

<u>example</u>



Find ${}^A P$:

By inspection :

$$ {}^A P = \begin{bmatrix} -2 \sin 30 \\ 2 \cos 30 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix} $$

<u>Using mapping</u>

$$ {}^A_B R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} \cos 30 & -\cos 60 & 0 \\ \cos 60 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix} $$

$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}P = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow {}^{A}P = {}^{A}_{B}R\,{}^{B}P = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}$$

(3) Mapping involving general frames:

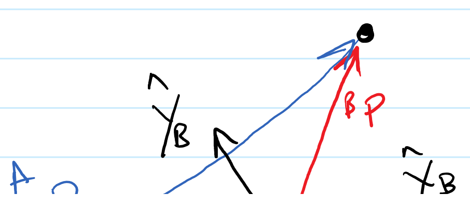
$$\boxed{\underset{(3\times1)}{{}^{A}P} = \underset{(3\times3)}{{}^{A}_{B}R}\ \underset{(3\times1)}{{}^{B}P} + \underset{(3\times1)}{{}^{A}P_{Borg}} = \underbrace{{}^{A}_{B}T}_{\text{Transformation matrix}}\ {}^{B}P}$$

$${}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

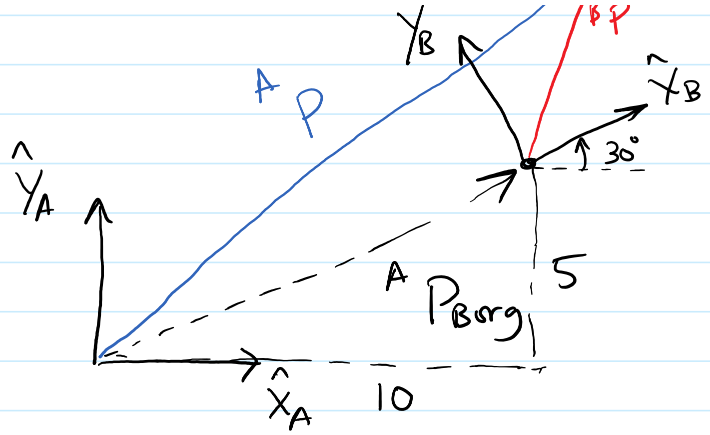
$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{Borg} \\ 1 \end{bmatrix}$$

<u>example</u>



$${}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 \end{bmatrix}$$

find $^A P$

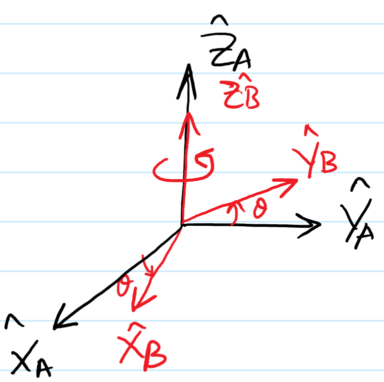
$$^A_B R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad ^A P_{Borg} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} ^A P \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}$$

* How do we easily find $^A_B R$ without performing the dot product. ?

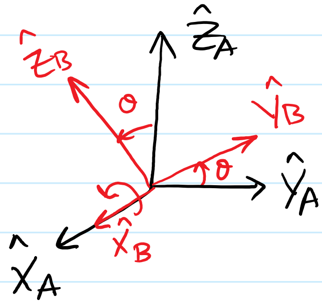
$\Rightarrow$ Three Cases : Any rotation is a combination of these cases.

(1) Rotation about z-axis :



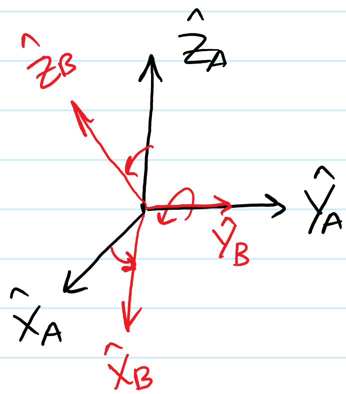
$$^A_B R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) Rotation about x-axis



$${}^A_B R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

(3) Rotation about y-axis



$${}^A_B R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

* Compound Transformations :

$\{C\}$ is known wrt $\{B\}$ $\Rightarrow {}^B_C T$ ${}^B P = {}^B_C T \, {}^C P$

$\{B\}$ is known wrt $\{A\}$ $\Rightarrow {}^A_B T$ ${}^A P = {}^A_B T \, {}^B P$

Find ${}^A P$ given ${}^C P$

$$\boxed{{}^A P = {}^A_B T \, {}^B_C T \, {}^C P}$$

* There are 2 different conventions for describing orientations.

(1) X-Y-Z fixed angles:

Start with frame {B} coincident with frame {A}
Rotate {B} about $\hat{X}_A$ by an angle γ
Rotate {B} about $\hat{Y}_A$ by an angle β
Rotate {B} about $\hat{Z}_A$ by an angle α
} roll, pitch, yaw angles.

The equivalent Rotation Matrix

$$ {}^{A}_{B}R_{xyz}(\gamma, \beta, \alpha) = R_z(\alpha)\, R_y(\beta)\, R_x(\gamma) $$

$$ = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} $$

$$ = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha S\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix} $$

(2) Z-Y-X Euler Angles

Start with frame {B} coincident with frame {A}
Rotate {B} about $\hat{Z}_B$ by an angle α
Rotate {B} about $\hat{Y}_B$ by an angle β
Rotate {B} about $\hat{X}_B$ by an angle γ

The equivalent Rotation matrix

$$ {}^{A}_{B}R_{x'y'z'} = R_z(\alpha)\, R_y(\beta)\, R_x(\gamma) $$

* See Attached matlab animation;

* Inverting a Transform

$${}^{A}P = {}^{A}_{B}T \, {}^{B}P$$

$${}^{B}P = {}^{A}_{B}T^{-1} \, P^{A}$$

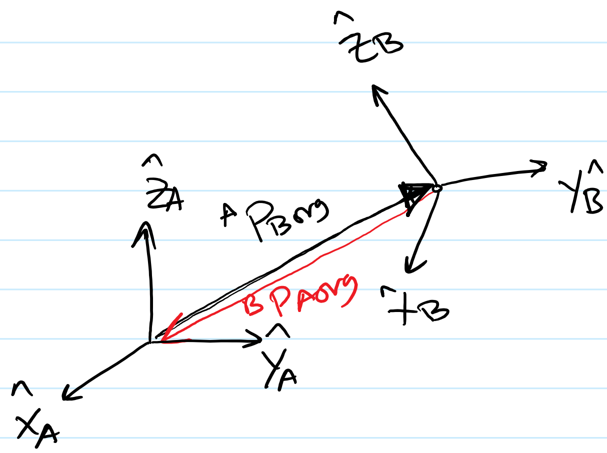
difficult to compute.

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{B}_{A}R & & {}^{B}P_{Aorg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$${}^{B}_{A}R = {}^{A}_{B}R^{T}$$

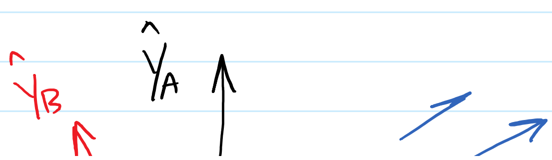
$${}^{B}P_{Aorg} = -{}^{A}_{B}R^{T} \, {}^{A}P_{Borg}$$

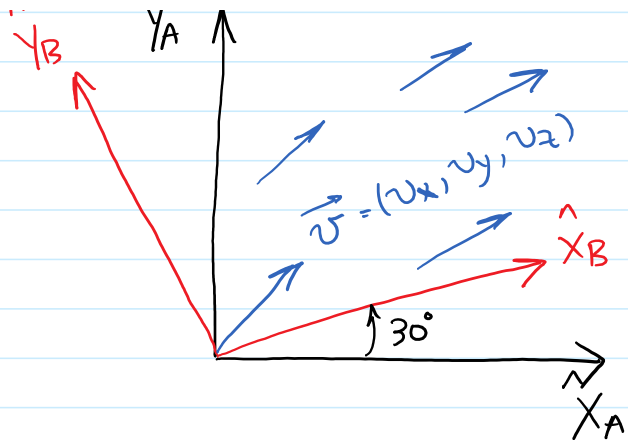
$${}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \, {}^{A}P_{Borg} \\ \hline 0 \; 0 \; 0 & 1 \end{array}\right]$$



$${}^{B}P_{Aorg} = -\underbrace{{}^{A}_{B}R^{T}}_{{}^{B}_{A}R} \, {}^{A}P_{Borg}$$

* The concept of a free vector (velocity)





The starting point of a free vector is not critical and does not affect the analysis of the vector.
Example: velocity vector

⟹ For Free Vectors, we only apply Rotation when we do transformations & not translations.

* Displacement & Force are NOT free vectors.