Chapter (4): Inverse Kinematics.

Saturday, February 25, 2017 1:55 PM

Transcendental Equations: Equations containing transcendental functions (not algebraic functions) ex: $\sin(x)$, $\cos(x)$, $\ln(x)$ ...

Algebraic Equations – Example:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

(Closed form solution / No need for numerical solutions)

Transcendental Equations – Example:

$$\boxed{a\cos\theta + b\sin\theta = c}$$

(Numerical Solution needed)

⇒ There are 3 methods that can be used to find a closed form solution for the above equation:

1) Quadratic Equation

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$$

↳ Substitute in the equation

$$a\sqrt{1-\sin^2\theta} + b\sin\theta = c$$

$$\left(a\sqrt{1-\sin^2\theta}\right)^2 = (c - b\sin\theta)^2$$

$$a^2 - a^2\sin^2\theta = c^2 - 2cb\sin\theta + b^2\sin^2\theta$$

Rearrange to get: $(a^2-b^2)\sin^2\theta - 2bc\sin\theta + (c^2-a^2) = 0$

This has a closed form solution

$$\boxed{\sin\theta = \frac{2bc \mp \sqrt{4b^2c^2 - 4(a^2-b^2)(c^2-a^2)}}{(a^2-b^2)}}$$

$$\boxed{\cos\theta = \sqrt{1-\sin^2\theta}}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)}$$

Matlab: atan2(sin(t), cos(t))

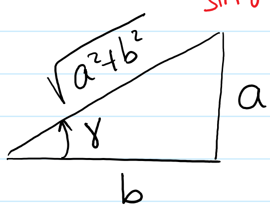
$$\boxed{\theta = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)}$$

Matlab: atan2(sin(t), cos(t))

2) Introducing an angle γ

Divide the equation by $\sqrt{a^2+b^2}$

$$\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\sin\gamma}\cos\theta + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\cos\gamma}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$



$$\sin\gamma\cos\theta + \cos\gamma\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

$$\sin(\gamma+\theta) = \frac{c}{\sqrt{a^2+b^2}}$$

Thus $$\boxed{\theta = \sin^{-1}\frac{c}{\sqrt{a^2+b^2}} - \sin^{-1}\frac{a}{\sqrt{a^2+b^2}}}$$

3) Variable Substitution

let $u = \tan\frac{\theta}{2}$

Recall: $\cos(2A) = 2\cos^2 A - 1$

Then $\rightarrow$ $\cos(A) = 2\cos^2\frac{A}{2} - 1$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$$

$$u^2 = \tan^2\frac{\theta}{2} = \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{1-\cos^2\theta/2}{\cos^2\theta/2}$$

$$u^2 = \frac{1-\left(\frac{1+\cos\theta}{2}\right)}{\frac{1+\cos\theta}{2}} = \frac{2-1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$u^2(1+\cos\theta) = 1-\cos\theta$$

$$(u^2+1)\cos\theta = 1-u^2 \Longrightarrow \begin{cases} \cos\theta = \dfrac{1-u^2}{1+u^2} \\ \sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{2u}{1+u^2} \end{cases}$$

Substitute in the equation

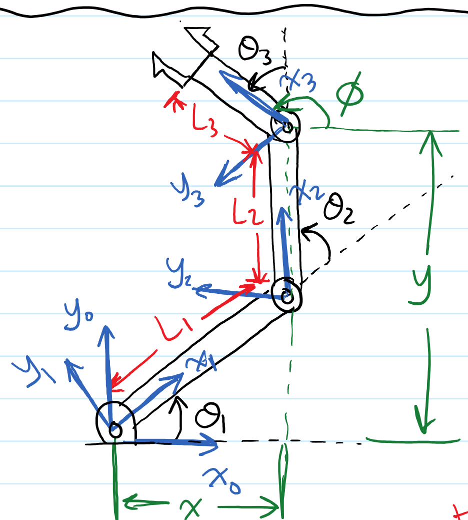
$$a\frac{(1-u^2)}{(1+u^2)} + b\frac{(1-u^2)}{(1+u^2)} = c$$

Rearrange to get $(a+c)u^2 - 2bu + (c-a) = 0$

Solve for $\boxed{u = \dfrac{2b \mp \sqrt{4b^2-4(c^2-a^2)}}{2(a+c)} = \tan\dfrac{\theta}{2}}$

Solve for θ

(*) Inverse Kinematics of a RRR planar Robot



Inverse Kinematics problem:

Knowns: x, y, ϕ

unknowns: $\theta_1, \theta_2, \theta_3$

rot about z

trans. along x & y

$$^0_3T = \begin{bmatrix} c\phi & -s\phi & 0 & x \\ s\phi & c\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

These 2 matrices are equal.

(*) Check matlab code ... RRR ... (*)

$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

are equal.

(*) Check matlab code forkineRRR.m (*)

$$ {}^{0}_{3}T = \begin{bmatrix} C_{1+2+3} & -S_{1+2+3} & 0 & L_1 C_1 + L_2 C_{1+2} \\ S_{1+2+3} & C_{1+2+3} & 0 & L_1 S_1 + L_2 S_{1+2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Note: DH-parameters are:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

We can obtain 4 different equations:

$$ \left.\begin{aligned} C_{1+2+3} &= C\phi \quad \text{—(1)} \\ S_{1+2+3} &= S\phi \quad \text{—(2)} \end{aligned}\right\} \quad \phi = \theta_1 + \theta_2 + \theta_3 $$

$$ L_1 C_1 + L_2 C_{1+2} = x \quad \text{—(3)} $$

$$ L_1 S_1 + L_2 S_{1+2} = y \quad \text{—(4)} $$

Square equations (3) & (4):

$$ x^2 = L_1^2 C_1^2 + 2L_1 L_2 C_1 C_{1+2} + L_2^2 C_{1+2}^2 \quad \text{—(3)} $$

$$ y^2 = L_1^2 S_1^2 + 2L_1 L_2 S_1 S_{1+2} + L_2^2 S_{1+2}^2 \quad \text{—(4)} $$

Add equations (3) & (4):

$$ x^2 + y^2 = L_1^2 + L_2^2 + 2L_1 L_2 (C_1 C_{1+2} + S_1 S_{1+2}) $$

$\cos(\theta_1 + \theta_2 - \theta_1) = \cos\theta_2$

$$x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2\, c_2$$

Rearrange to get:

$$\boxed{\cos\theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}}$$

$$\boxed{\sin\theta_2 = \mp\sqrt{1-\cos^2\theta}}\quad \text{(2 solutions)}$$

$$\boxed{\theta_2 = \tan^{-1}\left(\frac{\sin\theta_2}{\cos\theta_2}\right)}$$

To find θ_1:

$$L_1 c_1 + L_2 c_{1+2} = x \quad \text{— (3)}$$

$$L_1 c_1 + L_2(c_1c_2 - s_1s_2) = x$$

$$\boxed{(L_1 + L_2c_2)\, c_1 - (L_2 s_2)\, s_1 = x}$$

$$L_1 s_1 + L_2 s_{1+2} = y$$

$$L_1 s_1 + L_2(s_1c_2 + s_2c_1) = y$$

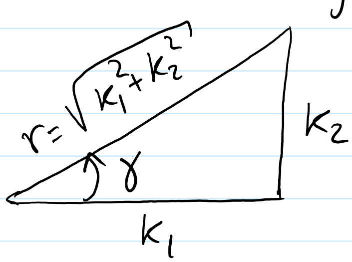
$$\boxed{(L_1 + L_2c_2)\, s_1 + (L_2s_2)\, c_1 = y}$$

let $k_1 = L_1 + L_2c_2$ (one value)

$k_2 = L_2 s_2$ (two values)

$$\boxed{\begin{aligned} k_1c_1 - k_2s_1 &= x \\ k_1s_1 + k_2c_1 &= y \end{aligned}}$$

→ To solve this system of equations introduce an angle γ



$$\left.\begin{aligned} k_1 &= r\cos\gamma \\ k_2 &= r\sin\gamma \end{aligned}\right\}\ \text{substitute in the above 2 equations}$$

$$r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1 = x$$

$$r\cos\gamma \sin\theta_1 + r\sin\gamma\cos\theta_1 = y$$

$$\left.\begin{aligned} \cos(\gamma+\theta_1) &= \frac{x}{r} \\ \sin(\gamma+\theta_1) &= \frac{y}{r} \end{aligned}\right\}$$ Solve for γ → Solve for θ_1

(θ_1 will have 2 solutions, one for every θ_2)

To find θ_3:

$$\phi = \theta_1 + \theta_2 + \theta_3$$

(θ_3 will have 2 solutions, one for every θ_1, θ_2 set)

Summary

$$\begin{array}{ccc} & {}^0_3T & \\ \swarrow & & \searrow \\ \theta_2 & & \theta_2 \\ \downarrow & & \downarrow \\ \theta_1 & & \theta_1 \\ \downarrow & & \downarrow \\ \theta_3 & & \theta_3 \end{array}$$

$$\cos\theta_2 = \frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}$$

$$\sin\theta_2 = \sqrt{1-\cos^2\theta_2}$$

$$K_1 = L_1 + L_2\cos\theta_2$$

$$K_2 = L_2\sin\theta_2$$

$$\gamma = \tan^{-1}(K_2, K_1)$$

$$\theta_1 + \gamma = \tan^{-1}(y, x)$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

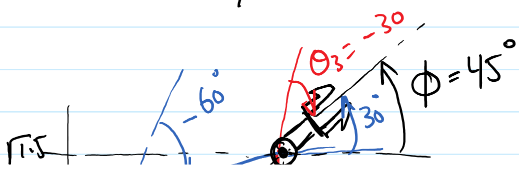
Example a RRR-Robot has $L_1 = L_2 = 1$ m

find the joint angles to achieve:

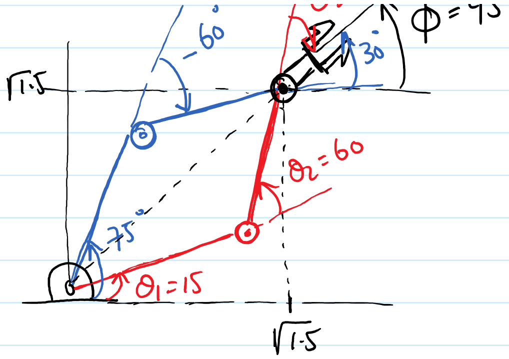
$x = \sqrt{1.5}$ m

$y = \sqrt{1.5}$ m

$\phi = 45^\circ$



$$\cos\theta_2 = 1.5 + 1.5 - 1 - 1 = \frac{1}{2} \Rightarrow \theta_2 = 60^\circ$$



$$\cos\theta_2 = \frac{1.5 + 1.5 - 1 - 1}{2(1)(1)} = \frac{1}{2} \Rightarrow \theta_2 = 60°$$

$$\sin\theta_2 = \mp\sqrt{1 - 0.5^2} = \mp\frac{\sqrt{3}}{2} \Rightarrow \theta_2 = -60°$$

$$K_1 = 1 + 0.5 = 1.5$$

$$K_2 = \mp\frac{\sqrt{3}}{2}$$

$$\tan\gamma = \mp\frac{\sqrt{3}/2}{3/2} = \mp\frac{1}{\sqrt{3}}$$

for $\theta_2 = 60 \rightarrow \gamma = 30$

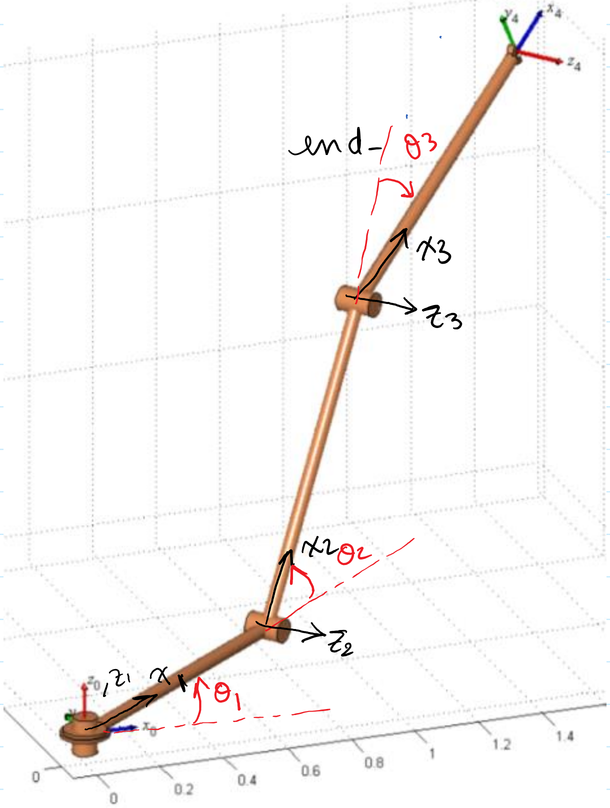
$\theta_2 = -60 \Rightarrow \gamma = -30$

$$\theta_1 + \gamma = \tan^{-1} y/x = 45°$$

$$\theta_2 = 60° \rightarrow \theta_1 = 15° \rightarrow \theta_3 = -30$$

$$\theta_2 = -60° \rightarrow \theta_1 = 75° \rightarrow \theta_3 = 30°$$

(*) Inverse Kinematics of a 3R spatial Robot – graphical solution

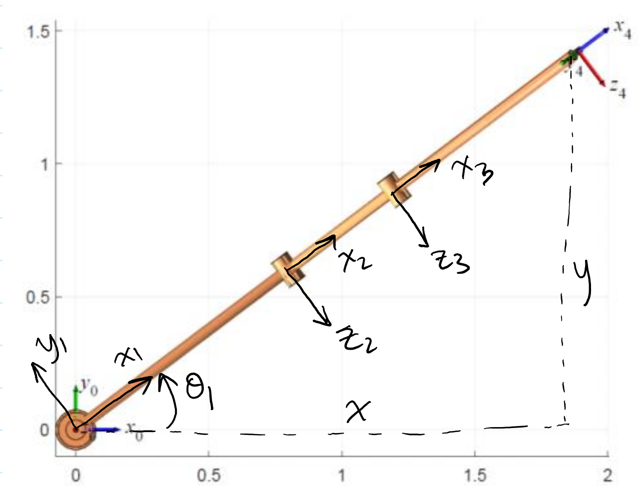


DH- parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i	nown
1	0	0	0	θ_1	
2	$\pi/2$	L_1	0	θ_2	
3	0	L_2	0	θ_3	
4	0	L_3	0	0	

O_4 which is fixed to the end effector, has a known orientation & position with respect to the base frame $\{0\}$

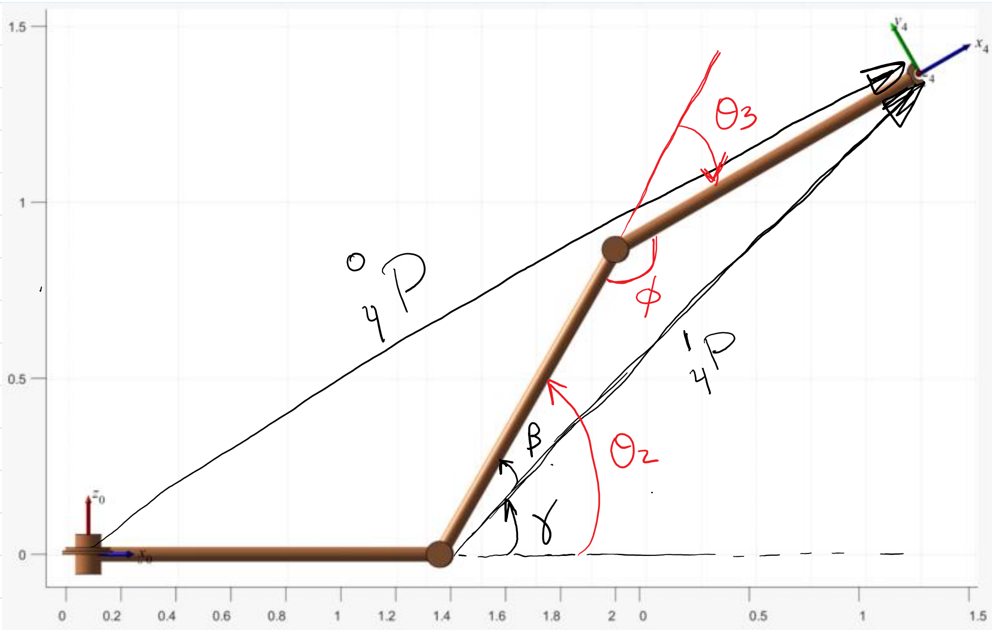
To take a clean look, check the top view.



x: horizontal position of the e.e. wrt $\{0\}$ $\rightarrow$ known
y: vertical position of the e.e. wrt $\{0\}$ $\rightarrow$ known

$$\boxed{\theta_1 = \tan^{-1} \frac{y}{x}}$$

The following view shows θ_2, θ_3 clearly.



$${}^1_4P = \begin{bmatrix} x - L_1 \\ 0 \\ z \end{bmatrix}$$

z: height of e.e. wrt $\{0\} \to$ known

$$\circledast \quad {}^0_4P = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

$$\gamma = \tan^{-1}\left(\frac{z}{x-L_1}\right)$$

$$L_3^2 = L_2^2 + \left(z^2 + (x-L_1)^2\right) - 2L_2\sqrt{z^2+(x-L_1)^2}\cos\beta \Rightarrow \text{find } \beta$$

$$\boxed{\theta_2 = \gamma + \beta}$$

$$z^2 + (x-L_1)^2 = L_2^2 + L_3^2 - 2L_2L_3\cos\phi \Rightarrow \text{find } \phi$$

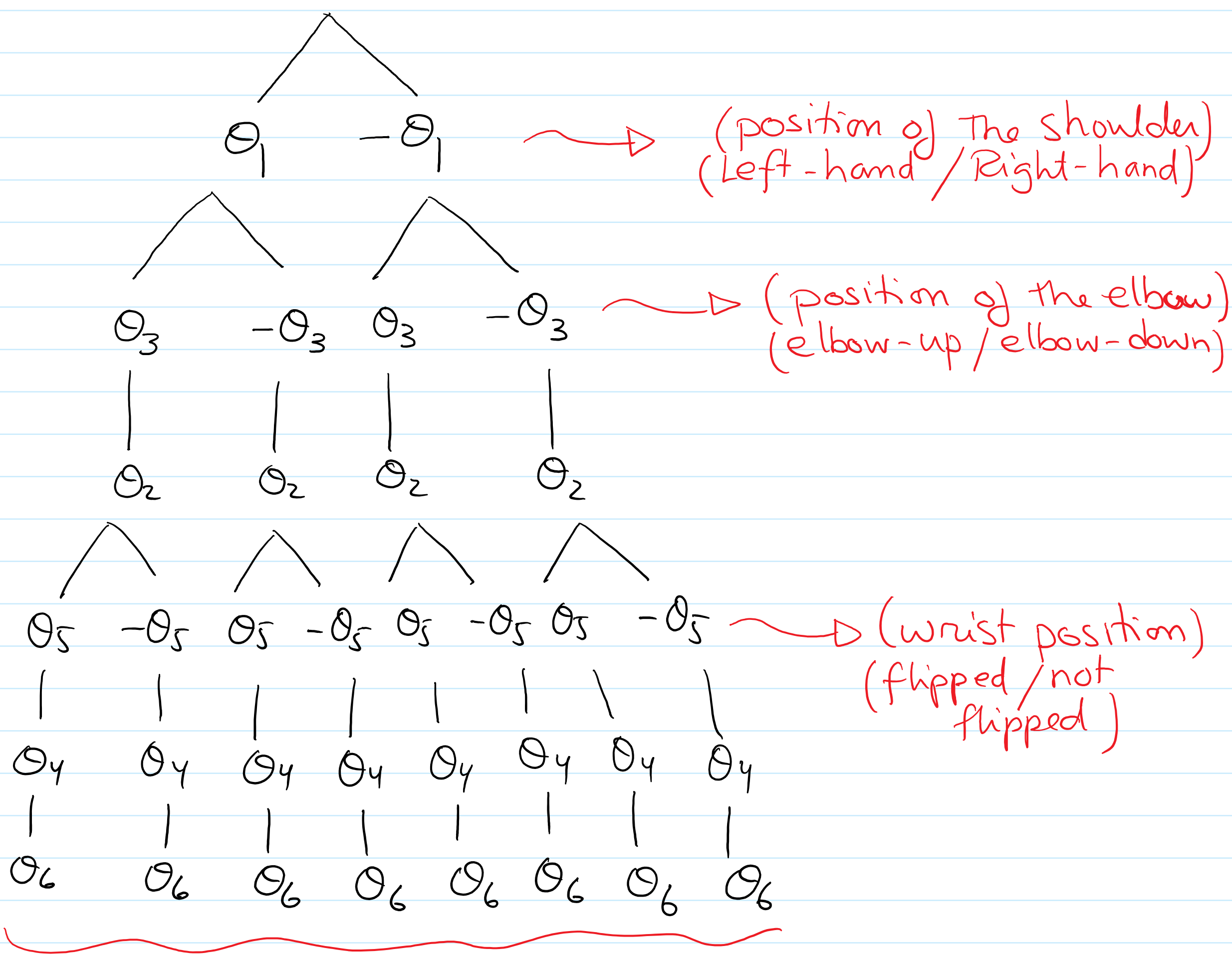
$$\boxed{\theta_3 = 180^\circ - \phi}$$

⊛ Inverse Kinematics of the PUMA 560 manipulator – MATLAB (Robotic Toolbox)

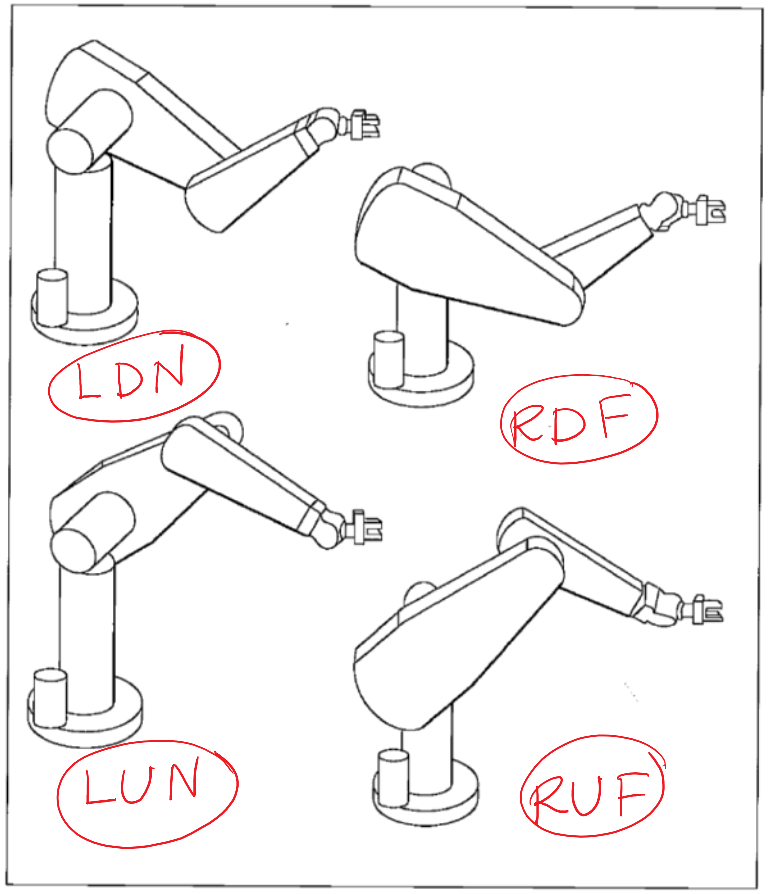
<u>Recall</u> The number of solutions of the inverse kinematic problem depend on :

(1) The number of joints (DOF)
(2) The link parameters ($a_i, \alpha_i, d_i, \theta_i$)
(3) Joint variable limitations.

For the PUMA 560 $\longrightarrow$ 8 solutions exist for the IK problem



8 - different Solutions.



* Check Matlab code eightSol.m