Chapter (5): Jacobians – Velocities and Static Forces.

* Differentiation of Position Vectors:

${}^{B}Q$: position vector representing point Q in frame {B}

The velocity of point Q is expressed by:

$${}^{B}V_Q = \frac{{}^{B}d}{dt} Q$$

– The rate of change of the position of point Q with respect to {B}

=> Sometimes a point is fixed wrt {B} but is moving wrt another frame

$$\boxed{{}^{A}({}^{B}V_Q) = {}^{A}\frac{d}{dt}{}^{B}Q}$$

– The derivative of the position vector of point Q wrt frame {B}, expressed in frame {A}.

Note: ${}^{B}({}^{B}V_Q) = {}^{B}V_Q$ (for simplicity)

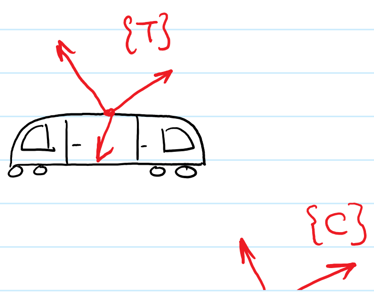
$${}^{A}({}^{B}V_Q) = \underbrace{{}^{A}_{B}R\,{}^{B}V_Q}_{\text{we will use this notation.}}$$

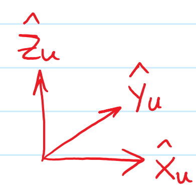
Note: If {U} is a universal fixed frame

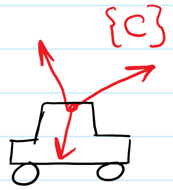
$${}^{U}V_{corg} = v_c$$

$${}^{A}({}^{U}V_{corg}) = {}^{A}v_c = {}^{A}_{U}R\, v_c$$

Example:







* Train is travelling at 100 mph in the direction of $\hat{X}_u$
* Car is travelling at 30 mph in the direction of $\hat{X}_u$

$^{u}_{T}R$, $^{u}_{c}R$ Knowns

Find ${}^{u}\frac{d}{dt}{}^{u}P_{corg}$, ${}^{c}({}^{u}V_{Torg})$, ${}^{c}({}^{T}V_{corg})$

$$v_T = {}^{u}V_{Torg} = 100\hat{X}_u$$

$$v_c = {}^{u}V_{corg} = 30\hat{X}_u$$

(*) ${}^{u}\frac{d}{dt}{}^{u}P_{corg} = {}^{u}({}^{u}V_{corg}) = {}^{u}V_{corg} = v_c = 30\hat{X}_u$ mph.

(*) ${}^{c}({}^{u}V_{Torg}) = {}^{c}v_T = {}^{c}_{u}R\, v_T = {}^{u}_{c}R^{T}(100\hat{X}_u)$ mph.

(*) ${}^{c}({}^{T}V_{corg}) = {}^{c}_{T}R\,{}^{T}V_{corg} = {}^{c}_{u}R\,{}^{u}_{T}R\,{}^{T}V_{corg}$

$$= {}^{u}_{c}R^{T}\,{}^{u}_{T}R\,{}^{T}V_{corg}$$

We know from dynamics that $v_B = v_A + {}^{A}v_B$

$$v_c = v_T + {}^{T}v_c$$

$$30\hat{X}_u = 100\hat{X}_u + {}^{T}v_c$$

$${}^{T}v_c = {}^{T}V_{corg} = -70\hat{X}_u$$

$\therefore {}^{c}({}^{T}V_{corg}) = {}^{u}_{c}R^{T}\,{}^{u}_{T}R(-70\hat{X}_u)$.

* The angular Velocity Vector

${}^{A}\Omega_B$: Vector representing the angular velocity for the rotation of frame {B} wrt frame {A}.

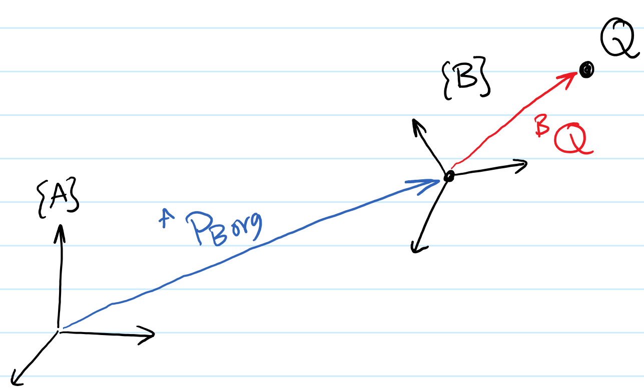
${}^{C}({}^{A}\Omega_B)$: Same as above but the vector is expressed wrt frame {C}

${}^{U}\Omega_C = \omega_C$

${}^{A}\omega_C$: Angular velocity of {C} expressed in {A}
{C} is rotating about {U}.

* Linear & Rotational Velocities of Rigid Bodies:

1) Motion is pure translation

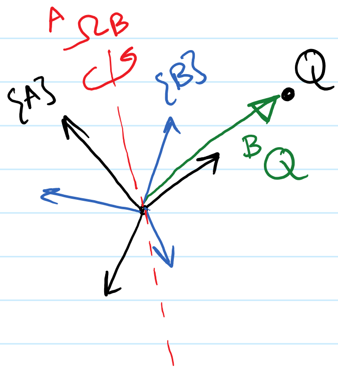


Pure Translation → ${}^{A}_{B}R$ is constant
${}^{A}P_{Borg}$ changes.

$${}^{A}V_Q = {}^{A}\frac{d}{dt}\,{}^{A}Q = {}^{A}\frac{d}{dt}\left({}^{A}P_{Borg} + {}^{B}V_Q\right)$$

$$\boxed{{}^{A}V_Q = {}^{A}V_{Borg} + {}^{A}_{B}R\,{}^{B}V_Q}$$

2) Motion is pure Rotation



^A_BR is changing

$^AP_{Borg}$ is constant $= 0$.

$$^AV_Q = {}^A\Omega_B \times {}^AQ \qquad \vec{v} = \vec{\omega} \times \vec{r} \text{ (from dynamics)}$$

if Q is changing wrt $\{B\}$ then:

$$^AV_Q = {}^A({}^BV_Q) + {}^A\Omega_B \times {}^AQ$$

$$^AV_Q = {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BQ$$

3) Motion is a Combination of Rotation & Translation

$$^AV_Q = {}^AV_{Borg} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BQ$$

* A Property of the derivative of the orthonormal Matrix

Why didn't we include the derivative of ^A_BR ?!

$R\,R^T = I_n$ for a $n \times n$ orthonormal matrix R.

derive the equation:

$R\dot{R}' + \dot{R}R' = O_n$ $\quad O_n$: $n\times n$ matrix of zeros.

from Algebra

$$R\dot{R}^T + (R\dot{R}^T)^T = O_n$$

let $S = R\dot{R}^T$

$$S + S^T = O_n \Rightarrow S \text{ is a skew-symmetric matrix.}$$

$$S = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}_{3\times 3}$$

Back to the case of pure rotation. ${}^A_BR$ changes with time

$${}^AP = {}^A_BR\,{}^BP$$

$${}^AV_P = {}^A_B\dot{R}\,{}^BP$$

${}^BP$ is constant
(P and $\{B\}$ rotate together)

$$= {}^A_B\dot{R}\,{}^B_AR\,{}^AP$$

$$= {}^A_B\dot{R}\,{}^A_BR^T\,{}^AP$$

$$\therefore \boxed{{}^AV_P = {}^A_BS\,{}^AP}$$

Define $S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_x & -\Omega_y & 0 \end{bmatrix}$

$${}^AV_P = \begin{bmatrix} 0 & -{}^A\Omega_{Bz} & {}^A\Omega_{By} \\ {}^A\Omega_{Bz} & 0 & -{}^A\Omega_{Bx} \end{bmatrix}\begin{bmatrix} {}^AP_x \\ {}^AP_y \end{bmatrix}$$

$$
{}^{A}V_P = \begin{bmatrix} 0 & -{}^{A}\Omega_{Bz} & {}^{A}\Omega_{By} \\ {}^{A}\Omega_{Bz} & 0 & -{}^{A}\Omega_{Bx} \\ {}^{A}\Omega_{Bx} & -{}^{A}\Omega_{By} & 0 \end{bmatrix} \begin{bmatrix} {}^{A}P_x \\ {}^{A}P_y \\ {}^{A}P_z \end{bmatrix}
$$

$$
= \begin{bmatrix} -{}^{A}\Omega_{Bz}\,{}^{A}P_y + {}^{A}\Omega_{By}\,{}^{A}P_z \\ {}^{A}\Omega_{Bz}\,{}^{A}P_x - {}^{A}\Omega_{Bx}\,{}^{A}P_z \\ {}^{A}\Omega_{Bx}\,P_x - {}^{A}\Omega_{By}\,P_y \end{bmatrix} = \begin{bmatrix} {}^{A}\Omega_{Bx} \\ {}^{A}\Omega_{By} \\ {}^{A}\Omega_{Bz} \end{bmatrix} \times {}^{A}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}
$$

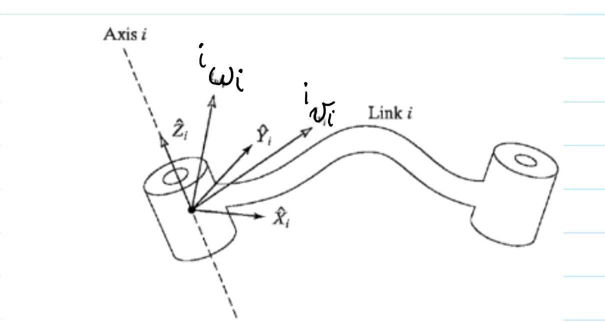
$$
{}^{A}V_P = {}^{A}\Omega_B \times {}^{A}P
$$

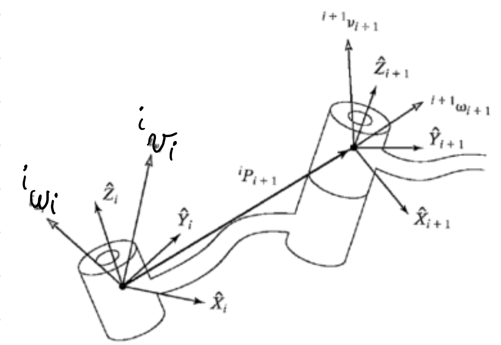
$\Rightarrow$ which is the result we obtained before.

⊛ Motion of the links of a Robot

- → frame $\{0\}$ is our reference frame
- → v_i is the linear velocity of the origin of $\{i\}$
- → ω_i is the angular velocity of the origin of $\{i\}$

⊛ Velocity propagation from link to link:





$$
{}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}
$$

$$
{}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}
$$

To express the previous 2 equations wrt $\{i+1\}$, multiply by ${}^{i+1}_{i}R$

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{\ i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}}$$

$$\boxed{{}^{i+1}v_{i+1} = {}^{i+1}_{\ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)}$$

For revolute Joints

In case of a prismatic joint:

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{\ i}R\,{}^{i}\omega_i$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{\ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

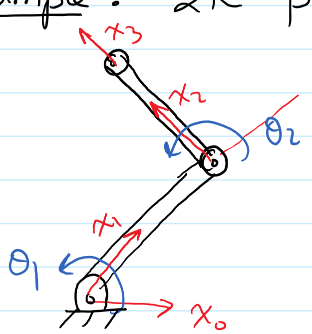
$\Rightarrow$ Applying these equations from link to link we can obtain:

${}^{N}\omega_N$ & ${}^{N}v_N$

$\Rightarrow$ Then $\quad {}^{0}v_N = {}^{0}_{N}R\,{}^{N}v_N$

$\qquad\qquad {}^{0}\omega_N = {}^{0}_{N}R\,{}^{0}\omega_N$

<u>Example</u>: 2R planar Robot $\rightarrow$ find ${}^{0}v_3$ as a function of $\dot{\theta}_1$, $\dot{\theta}_2$



$${}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{1}\omega_1 = {}^{1}_{0}R\,{}^{0}\omega_0 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^{2}\omega_2 = {}^{1}_{2}R\,{}^{1}\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^{3}\omega_3 = {}^{1}_{3}R\,{}^{2}\omega_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^{0}v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{1}v_1 = {}^{1}_{0}R\left({}^{0}v_0 + {}^{0}\omega_0 \times {}^{0}P_1\right) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{2}v_2 = {}^{2}_{1}R\left({}^{1}v_1 + {}^{1}\omega_1 \times {}^{1}P_2\right) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \dot{\theta}_1 l_1 s_2 \\ \dot{\theta}_1 l_1 c_2 \\ 0 \end{bmatrix} $$

$$ {}^{3}v_3 = {}^{3}_{2}R\left({}^{2}v_2 + {}^{2}\omega_2 \times {}^{2}P_3\right) = I \left(\begin{bmatrix} \dot{\theta}_1 l_1 s_2 \\ \dot{\theta}_1 l_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \right) $$

$$ = \begin{bmatrix} \dot{\theta}_1 l_1 s_2 \\ \dot{\theta}_1 l_1 c_2 + (\dot{\theta}_1 + \dot{\theta}_2) l_2 \\ 0 \end{bmatrix} $$

$$ {}^0_3R = {}^0_1R \, {}^1_2R \, {}^2_3R = \begin{bmatrix} C_{1+2} & -S_{1+2} & 0 \\ S_{1+2} & C_{1+2} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0v_3 = {}^0_3R \, {}^3v_3 = \begin{bmatrix} -l_1 S_1 \dot{\theta}_1 - l_2 S_{1+2}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 C_1 \dot{\theta}_1 + l_2 C_{1+2}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} $$

* Jacobians

In robotics, we use the Jacobian matrix to relate joint velocities to Cartesian velocities.

$$ v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} $$

⇒ a vector that represents the velocity of the end effector in the cartesian space.

\# of rows = # of DOF of the end-effector.

$$ \dot{\Theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} $$

⇒ A vector that represents the velocities of joints in joint space.

\# of rows = # of joints in the manipulator.

$$ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} & \cdots & \\ \vdots & \ddots & & \vdots \\ \frac{\partial \omega_z}{\partial \dot{\theta}_1} & \cdots & & \frac{\partial \omega_z}{\partial \dot{\theta}_6} \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_6 \end{bmatrix} $$

* Usually this equation is expressed wrt a certain frame.

$$ \boxed{{}^0v = {}^0J(\Theta)\,\dot{\Theta}} $$

$$\boxed{{}^0\nu = {}^0J(\Theta)\,\dot{\Theta}}$$

↳ the Jacobian depend on Θ which represents a given position of the manipulator.

This means that the relationship is instantaneous and of course changes with time.

Example: For the previous 2R planar robot

$$ {}^3v_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} = \underbrace{\begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}}_{{}^3J(\Theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ {}^0v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{1+2}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{1+2}(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix} = \underbrace{\begin{bmatrix} -l_1 s_1 - l_2 s_{1+2} & -l_2 s_{1+2} \\ l_1 c_1 + l_2 c_{1+2} & l_2 c_{1+2} \end{bmatrix}}_{{}^0J(\Theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

* You can add another row to represent ω_z

$$ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{1+2} & -l_2 s_{1+2} \\ l_1 c_1 + l_2 c_{1+2} & l_2 c_{1+2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} $$

* If a Jacobian is expressed wrt a frame and it is required to express it wrt a different frame:

$$ {}^A\nu = \begin{bmatrix} {}^Av \\ {}^A\omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_BR & 0 \\ \hline 0 & {}^A_BR \end{array}\right] \begin{bmatrix} {}^Bv \\ {}^B\omega \end{bmatrix} $$

$$ \begin{bmatrix} {}^Av \\ {}^A\omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_BR & 0 \\ \hline 0 & {}^A_BR \end{array}\right] {}^BJ(\Theta)\,\dot{\Theta} $$

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & 0 \\ \hline 0 & {}^{A}_{B}R \end{array}\right] {}^{B}J(\Theta)\,\dot{\Theta}$$

$$= {}^{A}J(\Theta)\,\dot{\Theta}$$

Then:

$$\boxed{{}^{A}J(\Theta) = \left[\begin{array}{c|c} {}^{A}_{B}R & 0 \\ \hline 0 & {}^{A}_{B}R \end{array}\right] {}^{B}J(\Theta)}$$

(*) Jacobian Inverse & Singularities

If we have a desired linear and angular velocities of the end-effector in cartesian space, then we can use this to calculate the required joint angular velocities:

$$\dot{\Theta} = \underbrace{J^{-1}(\Theta)}\, v$$

↳ for certain values of Θ, $J(\Theta)$ is singular so the inverse does not exist.

Singularities of the mechanism:

Points at which $J^{-1}(\Theta)$ cannot be found:

(1) At the boundaries of the manipulator's workspace

(2) At some points within the workspace

Workspace boundaries singularity

Workspace interior singularity

When 2 or more links are alligned.

fully stretched out / fully folded in

For the previous 2R planar Robot:

$$J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

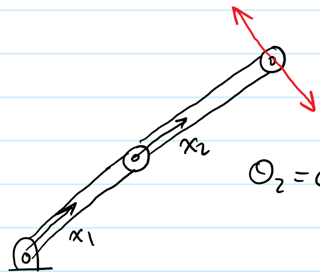
to find points causing singularities:

$$J(\theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

To find points causing singularities:

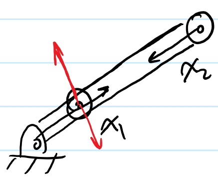
$$\det(J(\theta)) = l_1 l_2 s_2 - 0 = 0$$

A singularity occurs if $\sin\theta_2 = 0 \Rightarrow \theta_2 = 0$, $\theta_2 = 180$



$\theta_2 = 0$

In this case, motion is only possible in one direction (perpindicular to the arm) $\rightarrow$ the robot lost a degree of freedom.



$\theta_2 = 180°$ Same thing happens here

Both cases are considered Work-space boundary Singularities.

<u>Example</u> if the same 2R planar Robot has an end-effector that travels at 1 m/s in the X direction, Show that the joint rates are reasonable as long as we are far from singularities, and tend to have huge values close to a singularity.

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = {}^0J^{-1}(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_{1+2} & -l_2 s_{1+2} \\ l_1 c_1 + l_2 c_{1+2} & l_2 c_{1+2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\det({}^0J(\theta)) = -l_1 s_1 l_2 c_{1+2} - \cancel{l_2 s_{1+2} l_2 c_{1+2}} + l_2 s_{1+2} l_1 c_1 + \cancel{l_2 s_{1+2} l_2 c_{1+2}} = l_1 l_2 (c_1 s_{1+2} - s_1 c_{1+2})$$

$$= l_1 l_2 S_2$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{l_1 l_2 S_2} \begin{bmatrix} l_2 C_{1+2} & l_2 S_{1+2} \\ -l_1 C_1 - l_2 C_{1+2} & -l_1 S_1 - l_2 S_{1+2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{\theta}_1 &= \frac{l_2 C_{1+2}}{l_1 l_2 S_2} \\ \dot{\theta}_2 &= \frac{-l_1 C_1 - l_2 C_{1+2}}{l_1 l_2 S_2} \end{aligned}\right\}$$

as $\theta_2 \to 0^\circ / 180^\circ$

$S_2 \to 0$

thus $\dot{\theta}_1, \dot{\theta}_2 \to \infty$

* For the PUMA 560 A singularity occurs @ $\theta_5 = 0$ in this case z_4 & z_6 are aligned $\to$ $\theta_4 = \theta_6$. Thus the robot loses a degree of freedom.

$\to$ See Matlab Code : Singularity.m

* Manipulability : A quantitative measure of how far the manipulator is from a singularity.

$$w = \sqrt{\det(J(\theta) J^T(\theta))}$$

The higher w is $\to$ the furthest we are from singularities.

See code : manipulability.m

Notes on the Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial \theta_1} & \frac{\partial \dot{x}}{\partial \theta_2} & \cdots & \frac{\partial \dot{x}}{\partial \theta_6} \\ \vdots & \ddots & & \\ & & & \frac{\partial w_z}{\partial \theta_6} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}$$

Check codes: meaning_of_Jacobian.m

exercises_on_J.m.

* Static Forces in Manipulators:

→ Solve for the joint torques that are needed to keep the manipulator in static equilibrium while holding something or acting on something.

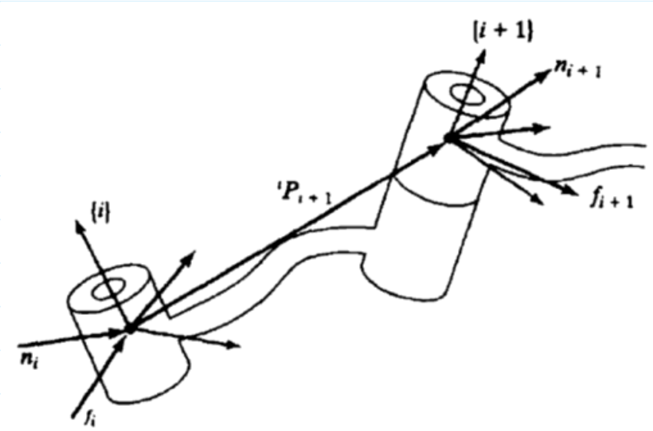
Steps: (1) Lock all the joints so that the manipulator becomes a structure
(2) Write a force-moment balance relationship in terms of link frames.
(3) Compute static torques that need to act on the joint axes to keep the manipulator in static equilibrium.

* In this Chapter we will not consider weights of links, this will be discussed in Chapter 6.

Notation:

f_i : force excerted on link i by link $i-1$

n_i : torque/moment excerted on link i by link $i-1$



Summing forces & Setting them to zero:

$${}^{i}f_i - {}^{i}f_{i+1} = 0$$

Summing torques about the origin of frame i:

$${}^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0$$

So:

$${}^{i}f_i = {}^{i}f_{i+1} \quad \text{--- (1)}$$

$${}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} \quad \text{--- (2)}$$

These two equations can be written as:

$$\boxed{\begin{aligned} {}^{i}f_i &= {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} \\ {}^{i}n_i &= {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i \end{aligned}}$$

* What Torques are needed in the joints?

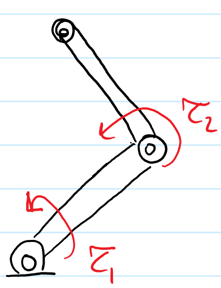
All components of the force & moment vectors are resisted by the structure of the mechanism except for the torque about the joint axes.

$$\tau_i = {}^{i}n_i \cdot {}^{i}\hat{Z}_i = {}^{i}n_i^{T}\,{}^{i}\hat{Z}_i \quad \text{(revolute joint)}$$

$$\tau_i = {}^{i}f_i \cdot {}^{i}\hat{Z}_i = {}^{i}f_i^{T}\,{}^{i}\hat{Z}_i \quad \text{(prismatic joint)}$$

↳ Note that we are using the symbol τ even for the linear actuator

Example 2R planar Robot.



+ve direction of τ_i is in the direction of increasing θ_i.

$${}^{3}F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} \quad \text{acts on joint 3}$$

find τ_1, τ_2

find τ_1, τ_2

$$^3f_3 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} C_2 f_x - S_2 f_y \\ S_2 f_x + C_2 f_y \\ 0 \end{bmatrix}$$

$$^3n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2$$

$$= 0 + \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}$$

$$^1n_1 = {}^1_2R\,{}^2n_2 + {}^1_2P \times {}^1f_1$$

$$= \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} C_2 f_x - S_2 f_y \\ S_2 f_x + C_2 f_y \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ l_1 S_2 f_x + l_1 C_2 f_y + l_2 f_y \end{bmatrix}$$

Therefore $\tau_1 = l_1 S_2 f_x + l_1 C_2 f_y + l_2 f_y$

$\tau_2 = l_2 f_y$

In matrix form $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underbrace{\begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix}} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$

this matrix is
the transpose of the Jacobian
we found in the velocity example

* In general

$$\tau = J^T(\theta)\,\tilde{\mathcal{F}}$$

where $\tilde{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}_{6\times1}$