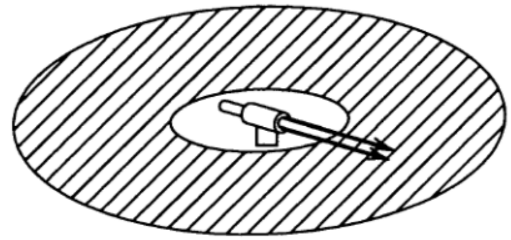
Solution – Assignment 3:

Problem 1:

A flat disk of zero thickness with inner radius equal to minimum extension, and outer radius equal to maximum extension.



Problem 2:

${}^S_T T$ is given, so compute:

$${}^B_W T = {}^B_S T \, {}^S_T T \, {}^W_T T^{-1}$$

Now ${}^B_W T = {}^0_3 T$ which we write out as:

$${}^0_3 T = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the solution of exercise 3 from chapter 3 we have:

$${}^0_3 T = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & C_1(C_2 L_2 + L_1) \\ S_1 C_{23} & -S_1 S_{23} & -C_1 & S_1(C_2 L_2 + L_1) \\ S_{23} & C_{23} & 0 & S_2 L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equate elements $(1,3)$: $S_1 = R_{13}$

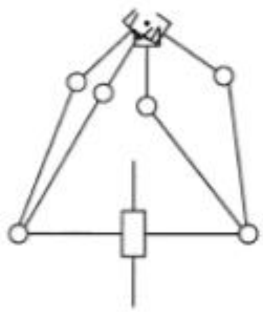
Equate elements $(2,3)$: $-C_1 = R_{23}$

$\therefore \quad \boxed{\theta_1 = \text{atan2}(R_{13}, -R_{23})}$

If both $R_{13} = 0$ and $R_{23} = 0$ the goal is unattainable.

Equate elements $(1,4)$: $P_x = C_1(C_2 L_2 + L_1)$

Equate elements $(2,4)$: $P_y = S_1(C_2 L_2 + L_1)$



If $C_1 \neq 0$ then $C_2 = \frac{1}{L_2}\left(\frac{P_x}{C_1} - L_1\right)$

$$\text{Else } C_2 = \frac{1}{L_2}\left(\frac{P_y}{S_1} - L_1\right)$$

Equate Elements (3,4): $P_z = S_2 L_2$

so, $\boxed{\theta_2 = \operatorname{atan2}\left(\frac{P_z}{L_2}, C_2\right)}$

Equate elements (3, 1): $S_{23} = R_{31}$

Equate elements (3, 2): $C_{23} = R_{32}$

so, $\boxed{\boldsymbol{\theta_3 = \operatorname{atan2}(R_{31}, R_{32}) - \theta_2}}$

If both R_{31} and R_{32} are zero, the goal is unattainable.

A second interpretation of the problem is that only a desired position is given (no orientation). In this there may be up to four solutions:

Assume ${}^3P_{\text{tool}} = L_3 \hat{X}_3$, then

$${}^0P_{\text{tool}} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} L_1C_1 + L_2C_1C_2 + L_3C_1C_{23} \\ L_1S_1 + L_2S_1C_2 + L_3S_1C_{23} \\ L_2S_2 + L_3S_{23} \end{bmatrix}$$

First,

$$S_1 = \frac{P_y}{L_1 + L_2C_2 + L_3C_{23}} \quad C_1 = \frac{P_x}{L_1 + L_2C_2 + L_3C_{23}}$$

so, $\boxed{\boldsymbol{\theta_1 = \operatorname{atan2}(P_y, P_x) \text{ or } \operatorname{atan2}(-P_y, -P_x)}}$

Since the sign of the "$L_1 + L_2C_2 + L_3C_{23}$" term may be + or −.

Next, define:

$$\alpha = \begin{cases} \frac{P_x}{C_1} - L_1 & \text{if } C_1 \neq 0 \\ \frac{P_y}{S_1} - L_1 & \text{if } S_1 \neq 0 \end{cases}$$

And we have:

$$L_2C_2 + L_3C_{23} = \alpha$$

$$L_2S_2 + L_3S_{23} = P_z$$

Square and add these two equations to get:

$$L_2^2 + L_3^2 + 2L_2L_3C_3 = \alpha^2 + P_z^2$$

$$C_3 = \frac{1}{2L_2L_3}(\alpha^2 + P_z^2 - L_2^2 - L_3^2)$$

$S_3 = \pm\sqrt{1 - C_3^2}$; $\boxed{\theta_3 = \operatorname{atan2}(S_3, C_3)}$

Finally,

$$L_3 C_{23} = \alpha - L_2 C_2$$

$$L_3 S_{23} = P_z - L_2 S_2$$

so, $\boxed{\theta_2 = \mathbf{atan2}(P_z - L_2 S_2, \alpha - L_2 C_2) - \theta_3}$

Problem 3:

Since $S\alpha_1 = 0$, easy to use Pieper's method:

$z = K_4$ (see pages 129–131 of text)

or

$$^oP_{4z} = C\alpha_1 F_3 + C\alpha_1 d_2$$

$$^oP_{4z} = F_3$$

$^oP_{4z} = A_3 S\alpha_2 S_3 + D_3 C\alpha_2$ (other terms are zero)

$$1.707 = \sqrt{2}\frac{\sqrt{2}}{2} S_3 + \sqrt{2}\frac{\sqrt{2}}{2}$$

$$1.707 = S_3 + 1$$

$$S_3 = 1.707 - 1 = 0.707$$

since $-180 < \theta_3 < 180$, there are 2 sols:

$\boxed{\theta_3 = 45^\circ \text{ or } \theta_3 = 135^\circ}$