Solution – Assignment 4:

Problem 1:

$$ {}^0_3T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & L_1C_1 + L_2C_1C_2 \\ S_1C_{23} & -S_1S_{23} & -C_1 & L_1S_1 + L_2S_1C_2 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and:

$$ {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad {}^0_4T = {}^0_3T\,{}^3_4T $$

we could then find ${}^0J(\underline{\theta})$ quite easily by differentiating ${}^0P_{YORG}$. Finally, ${}^4J(\underline{\theta})$ can be calculated as ${}^4_0R\,{}^0J(\underline{\theta})$. This might be tedious, so lets try "standard" velocity propagation as done in the text:

$$ {}^1W_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} {}^1V_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2W_2 = {}^2_1R\,{}^1W_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} $$

$$ {}^2W_2 = \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} {}^2V_2 = {}^2_1R({}^1V_1 + {}^1W_1 \times {}^1P_2) $$

$$ {}^2V_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_1\dot{\theta}_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix} $$

$$ {}^3W_3 = {}^3_2R\,{}^2W_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_2\dot\theta_1 \\ C_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} $$

$$ {}^3W_3 = \begin{bmatrix} S_{23}\dot\theta_1 \\ C_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} \quad {}^3V_3 = {}^3_2R({}^2V_2 + {}^2W_2 \times {}^2P_3) $$

$$ {}^3V_3 = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ -L_1\dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ L_2\dot\theta_2 \\ -L_2C_2\dot\theta_1 \end{bmatrix} \right) $$

$$ {}^3V_3 = \begin{bmatrix} S_3L_2\dot\theta_2 \\ C_3L_2\dot\theta_2 \\ -L_1\dot\theta_1 - L_2C_2\dot\theta_1 \end{bmatrix} \quad {}^4W_4 = {}^3W_3 $$

$$ {}^4V_4 = {}^4_3R({}^3V_3 + {}^3W_3 \times {}^3P_y) = {}^3V_3 + {}^3W_3 \times {}^3P_4 $$

$$ = \begin{bmatrix} S_2L_2\dot\theta_2 \\ C_3L_2\dot\theta_2 \\ -L_1\dot\theta_1 - L_2C_2\dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ L_3(\dot\theta_2 + \dot\theta_3 \\ -L_3C_{23}\dot\theta_1 \end{bmatrix} $$

$$ = \begin{bmatrix} S_3L_2\dot\theta_2 \\ C_3L_2\dot\theta_2 - L_3(\dot\theta_2 + \dot\theta_3) \\ -L_1\dot\theta_1 - L_2C_2\dot\theta_1 - L_3C_{23}\dot\theta_1 \end{bmatrix} $$

$$ \therefore {}^4J(\underline{\theta}) = \begin{bmatrix} 0 & S_3L_2 & 0 \\ 0 & C_3L_2 + L_3 & L_3 \\ (-L_1 - L_2C_2 - L_3C_{23}) & 0 & 0 \end{bmatrix} $$

<u>Problem 2:</u>

The mapping which potentially can be singular is: $Y = J(\theta)\underline{\dot\theta}$ for the "position domain", and $\underline{\tau} = J^T(\theta)F$ for the "force domain". Now since transposition has nothing to do with the rank of a (square) matrix, its clear that the singularities of $J(\underline{\theta})$ are the same as those of $J^T(\underline{\theta})$.

Problem 3:

$$\underline{\tau} = {}^3J^T\, {}^3\underline{F} \quad \therefore {}^3\underline{F} = {}^3J^{-T}\,\underline{\tau}$$

$$^3J = \begin{bmatrix} L_1S_2 & 0 \\ L_1C_2 + L_2 & L_2 \end{bmatrix}$$

$$^3J^T = \begin{bmatrix} L_1S_2 & L_1C_2 + L_2 \\ 0 & L_2 \end{bmatrix}$$

so,

$$^3J^{-T} = \frac{1}{L_1L_2S_2}\begin{bmatrix} L_2 & -L_1C_2 - L_2 \\ 0 & L_1S_2 \end{bmatrix}$$

Problem 4:

From (5.103):

$$^B\nu = \begin{bmatrix} {}^B_AR & -{}^B_AR\,{}^AP X \\ 0 & {}^B_AR \end{bmatrix} {}^A\nu$$

$${}^B_AR\,{}^APX = \begin{bmatrix} 0.86 & 0.5 & 0 \\ -0.5 & 0.86 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -5 & 0 \\ 5 & 0 & -10 \\ 0 & 10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & -4.3 & -5.0 \\ 4.3 & 2.5 & -8.6 \\ 0 & 10 & 0 \end{bmatrix}$$

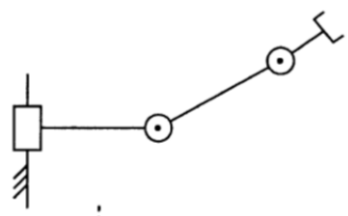
$$^B\nu = \begin{bmatrix} 0.86 & 0.5 & 0 & -2.5 & 4.3 & 5.0 \\ -0.5 & 0.86 & 0 & -4.3 & -2.5 & 8.6 \\ 0 & 0 & 1 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0.86 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0.86 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -3 \\ 1.41 \\ 1.41 \\ 0 \end{bmatrix}$$

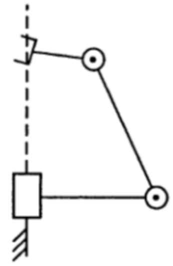
$$^B\nu = [3.52 \ -7.80 \ -17.1 \ 1.91 \ 0.51 \ 0]^T$$

Problem 5:

"Workspace boundary" any angle set: $\{\theta_1, \theta_2, 0\}$

"Workspace interior" any angle set such that: $L_1 + L_2C_2 + L_3C_{23} = 0$ (θ_1 is arbitrary)





Problem 6:

$$\underline{\tau} = {}^{O}J^{T}(\underline{\theta})\,{}^{O}\underline{F}$$

$$\underline{\tau} = \begin{bmatrix} -L_1S_1 - L_2S_{12} & L_1C_1 + L_2C_{12} \\ -L_2S_{12} & L_2C_{12} \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\tau_1 = -10S_1L_1 - 10L_2S_{12}$$

$$\tau_2 = -10L_2S_{12}$$