Mechanical Vibrations

Some Figures Courtesy Addison Wesley

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- Multi degree-of-freedom systems
  - Free vibrations
  - Forced vibrations

Mechanical vibrations

- Defined as oscillatory motion of bodies in response to disturbance.
- Oscillations occur due to the presence of a restoring force
- Vibrations are everywhere:
    - Human body: eardrums, vocal cords, walking and running
    - Vehicles: residual imbalance of engines, locomotive wheels
    - Rotating machinery: Turbines, pumps, fans, reciprocating machines
    - Musical instruments
- Excessive vibrations can have detrimental effects:
    - Noise
    - Loosening of fasteners
    - Tool chatter
    - Fatigue failure
    - Discomfort
- When vibration frequency coincides with natural frequency, resonance occurs.

Mechanical vibrations

- Aeolian, wind-induced or vortex-induced vibration of the Tacoma Narrows bridge on 7 November 1940 caused it to resonate resulting in catastrophic failure.

Tacoma Narrows Bridge Collapse Video

Mechanical vibrations

- Millennium Bridge, London: Pedestrians, in reaction to lateral motion of the bridge, altered their gait and started behaving in concert to induce the structure to resonate further (forced periodic excitation):

Video link

Fundamentals

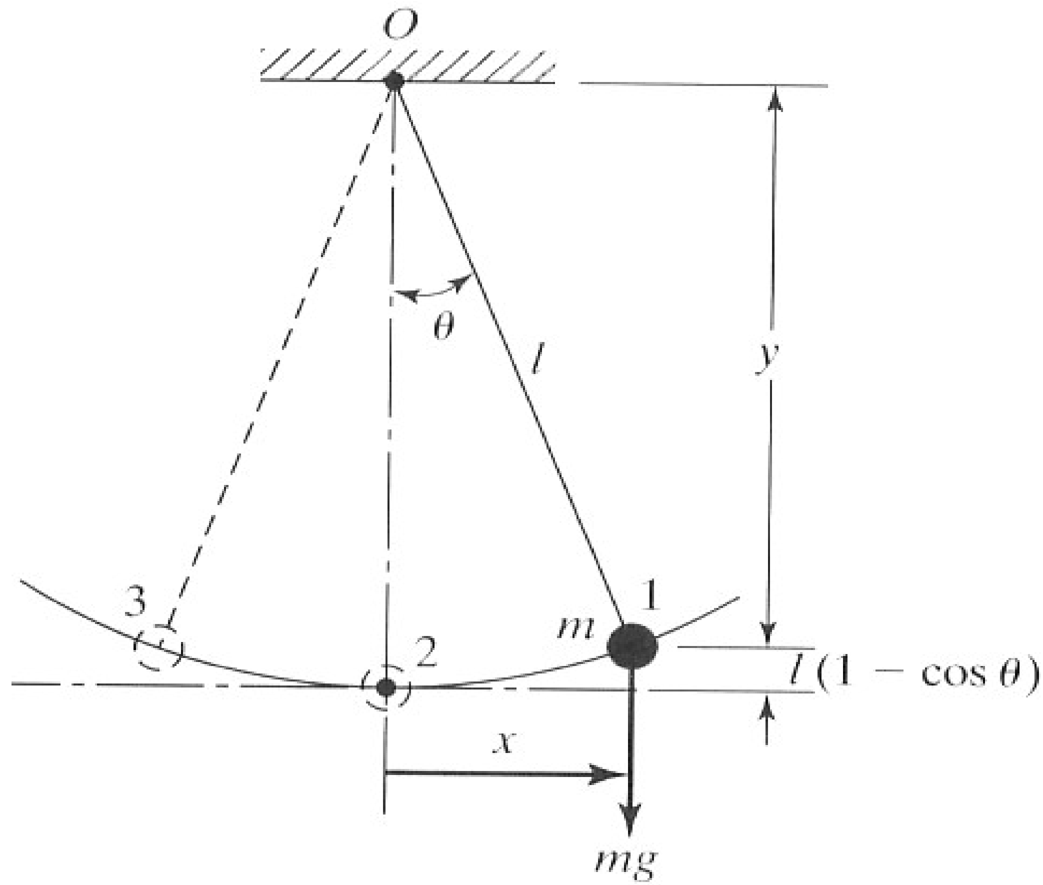
- In simple terms, a vibratory system involves the transfer of potential energy to kinetic energy and vice-versa in alternating fashion.
- When there is a mechanism for dissipating energy (damping) the oscillation gradually diminishes.
- In general, a vibratory system consists of three basic components:
  - A means of storing potential energy (spring, gravity)
  - A means of storing kinetic energy (mass, inertial component)
  - A means to dissipate vibrational energy (damper)

Fundamentals

- This can be observed with a pendulum:
- At position 1: the kinetic energy is zero and the potential energy is

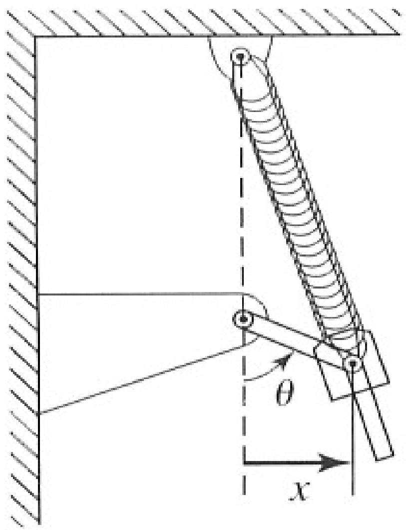
$$mgl(1 - \cos\theta)$$

- At position 2: the kinetic energy is at its maximum
- At position 3: the kinetic energy is again zero and the potential energy at its maximum.
- In this case the oscillation will eventually stop due to aerodynamic drag and pivot friction → HEAT

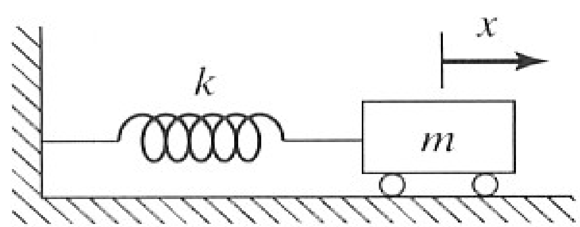


Degrees of Freedom

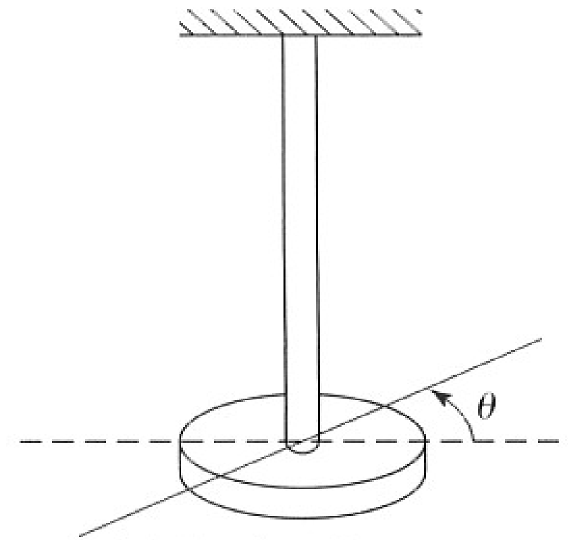
- The number of degrees of freedom : number of independent coordinates required to completely determine the motion of all parts of the system at any time.
- Examples of single degree of freedom systems:



(a) Slider-crank-spring mechanism



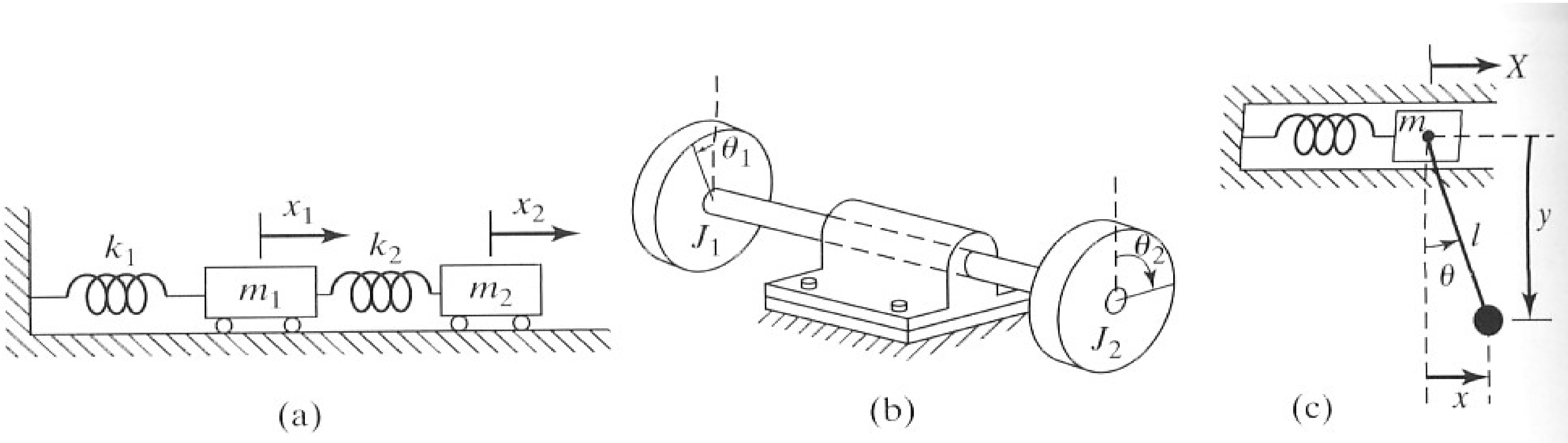
(b) Spring-mass system



(c) Torsional system

Degrees of Freedom

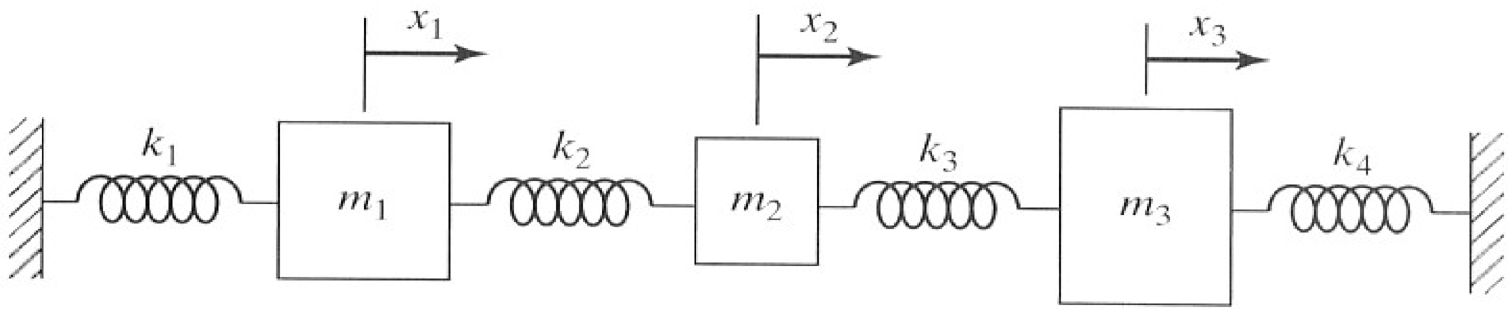
- Examples of two degree of freedom systems:



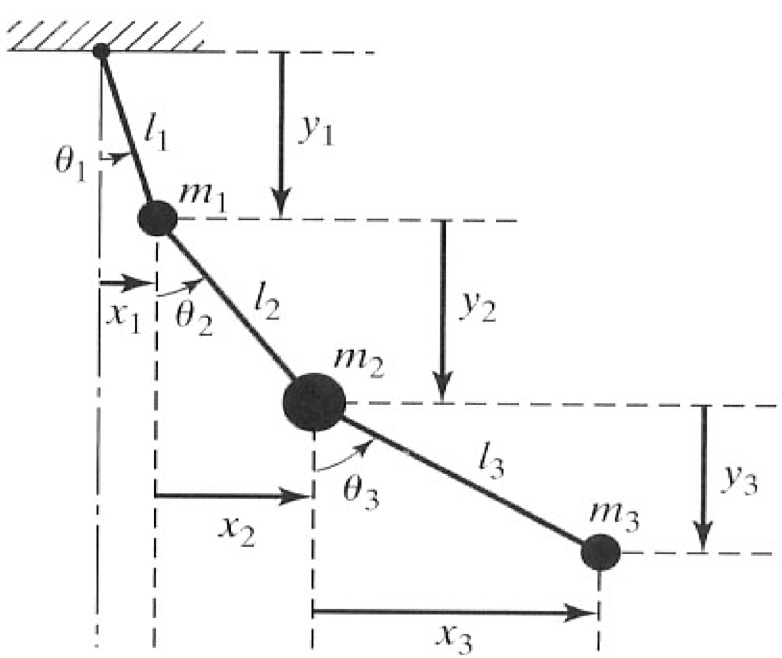
(a) (b) (c)

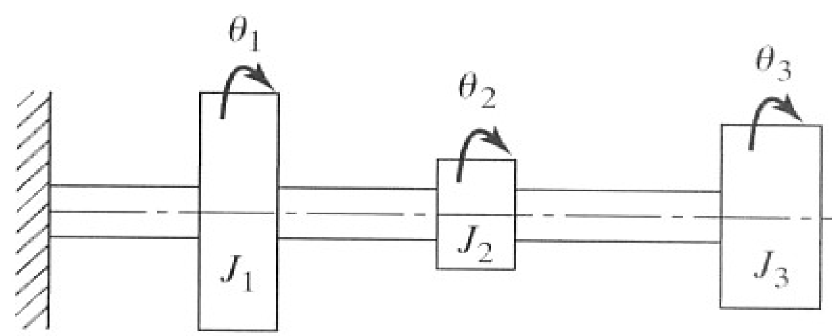
Degrees of Freedom

- Examples of three degree of freedom systems:



(a)



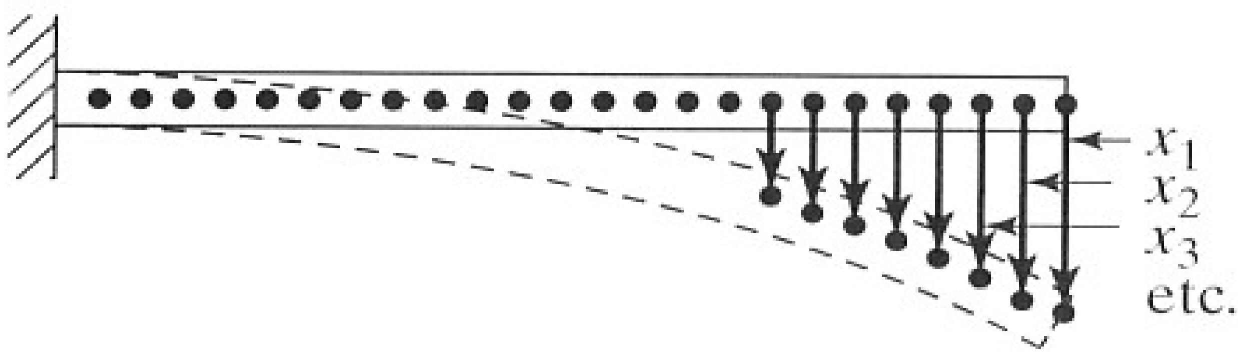


(c)

Discrete and continuous systems

- Many practical systems small and large or structures can be describe with a finite number of DoF. These are referred to as *discrete* or *lumped* parameter systems
- Some large structures (especially with continuous elastic elements) have an infinite number of DoF These are referred to as *continuous* or *distributed* systems.

- In most cases, for practical reasons, continuous systems are approximated as discrete systems with sufficiently large numbers lumped masses, springs and dampers. This equates to a large number of degrees of freedom which affords better accuracy.



Classification of Vibration

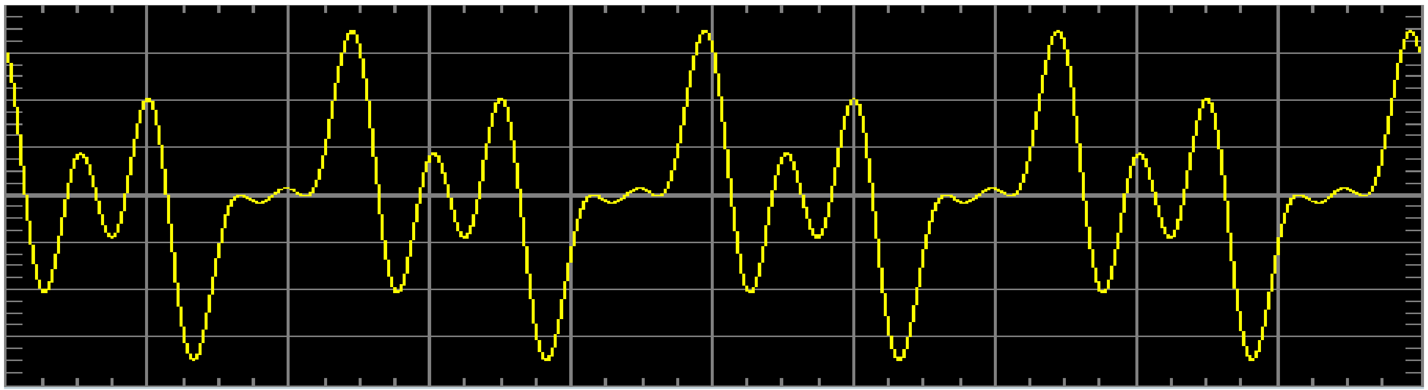
- **Free and Forced vibrations**
  - ***Free vibration***: Initial disturbance, system left to vibrate without influence of external forces.
  - ***Forced vibration***: Vibrating system is stimulated by external forces. If *excitation* frequency coincides with *natural* frequency, resonance occurs.

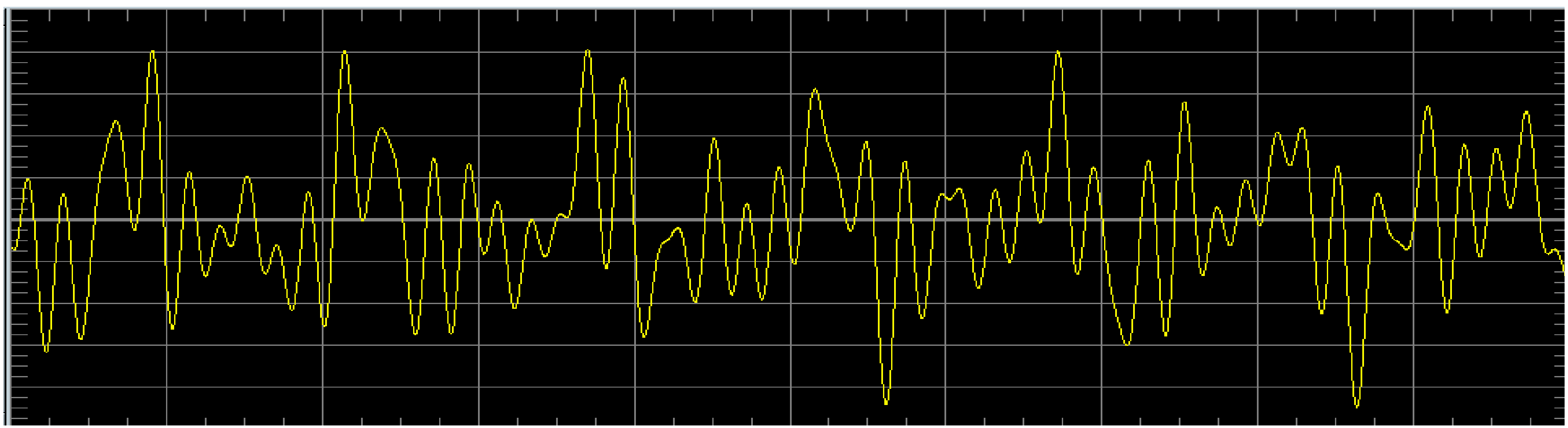
- **Undamped and damped vibration**
  - ***Undamped vibration***: No dissipation of energy. In many cases, damping is (negligibly) small (steel 1 – 1.5%). However small, damping has critical importance when analysing systems at or near resonance.
  - ***Damped vibration***: Dissipation of energy occurs - vibration amplitude decays.

- **Linear and nonlinear vibration**
  - ***Linear vibration***: Elements (mass, spring, damper) behave linearly. Superposition holds - double excitation level = double response level, mathematical solutions well defined.
  - ***Nonlinear vibration***: One or more element behave in nonlinear fashion (examples). Superposition does not hold, and analysis technique not clearly defined.

Classification of Vibration

- **Deterministic and Random vibrations**
  - ***Deterministic vibration***: Can be described by implicit mathematical function as a function of time.
  - ***Random vibration***: Cannot be predicted. Process can be described by statistical means.

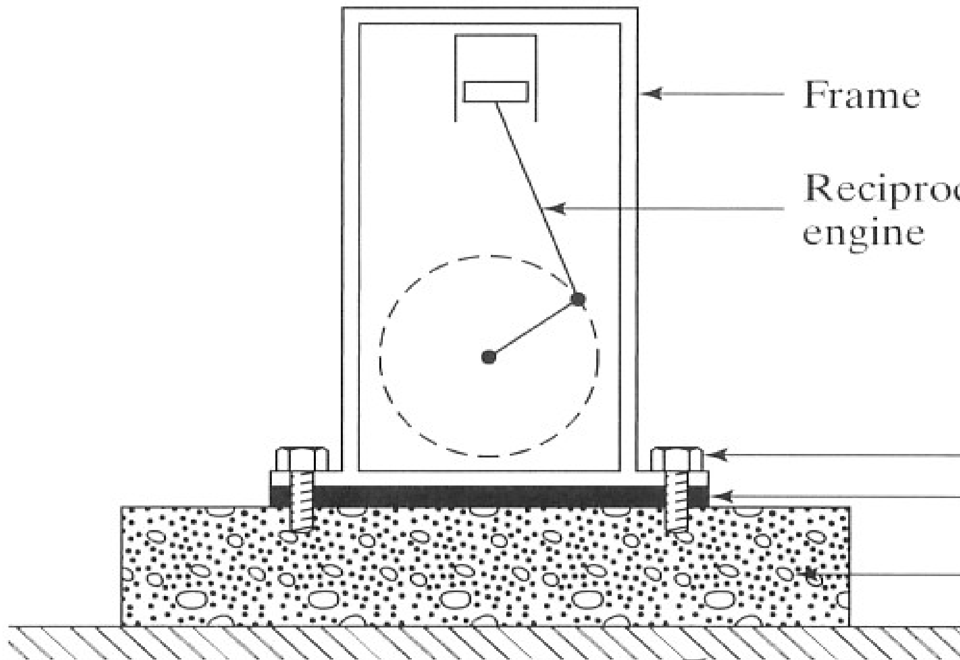




Vibration Analysis

- Input (excitation) and output (response) are wrt time
- Response depend on ***initial*** conditions and external forces
- Most practical systems very complex – (mathematical) modelling requires simplification
- Procedure:
  - → Mathematical modelling
  - → Derivation / statement of governing equations
  - → Solving of equations for specific boundary conditions and external forces
  - → Interpretation of solution(s)

Vibration Analysis



Example (1.3 Ed.3)

Spring Elements

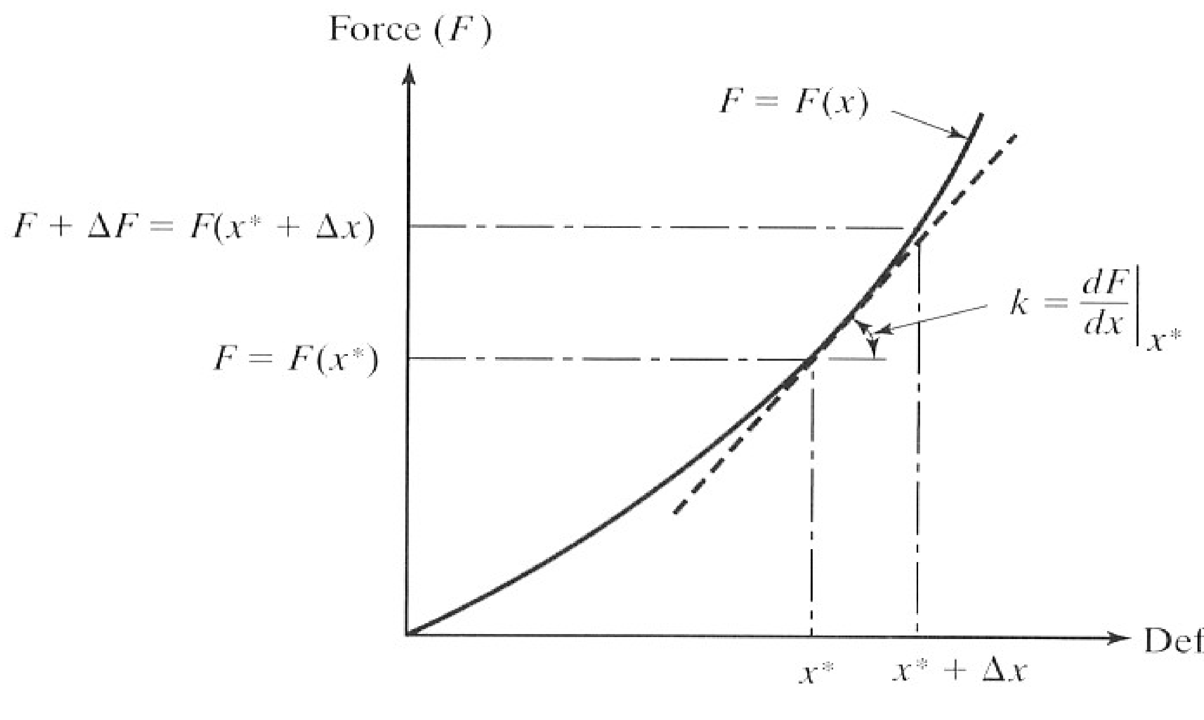
- Pure spring element considered to have negligible mass and damping
- Force proportional to spring deflection (relative motion between ends):

$$F = k\Delta x$$

- For linear springs, the potential energy stored is:

$$U = \tfrac{1}{2}k\left(\Delta x\right)^2$$

- Actual springs sometimes behave in nonlinear fashion
- Important to recognize the presence and significance (magnitude) of nonlinearity
- Desirable to generate linear estimate



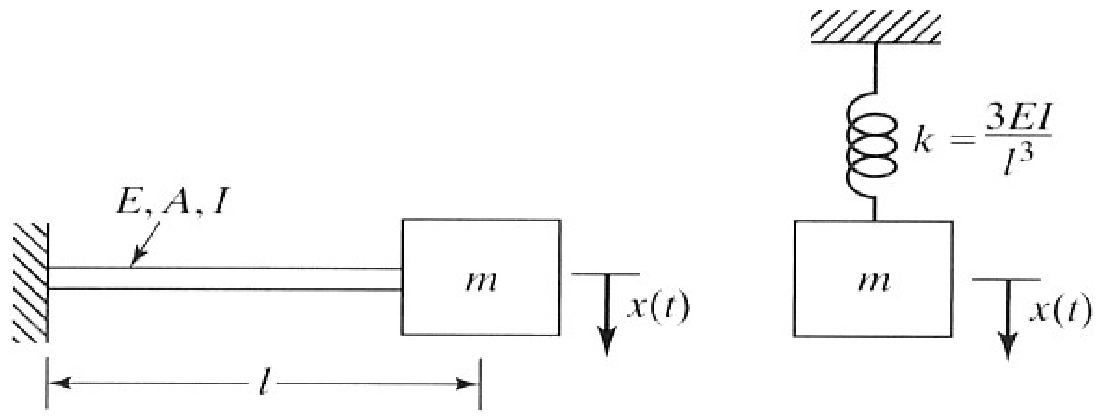
Spring Elements

- Equivalent spring constant.
  - Eg: cantilever beam: Mass of beam assumed negligible cf lumped mass
  - Deflection at free end:

$$\delta = \frac{mgl^3}{3EI}$$

  - Stiffness (Force/defln):

$$k = \frac{mg}{\delta} = \frac{3EI}{l^3}$$



(a) Actual system
(b) Single degree of freedom model

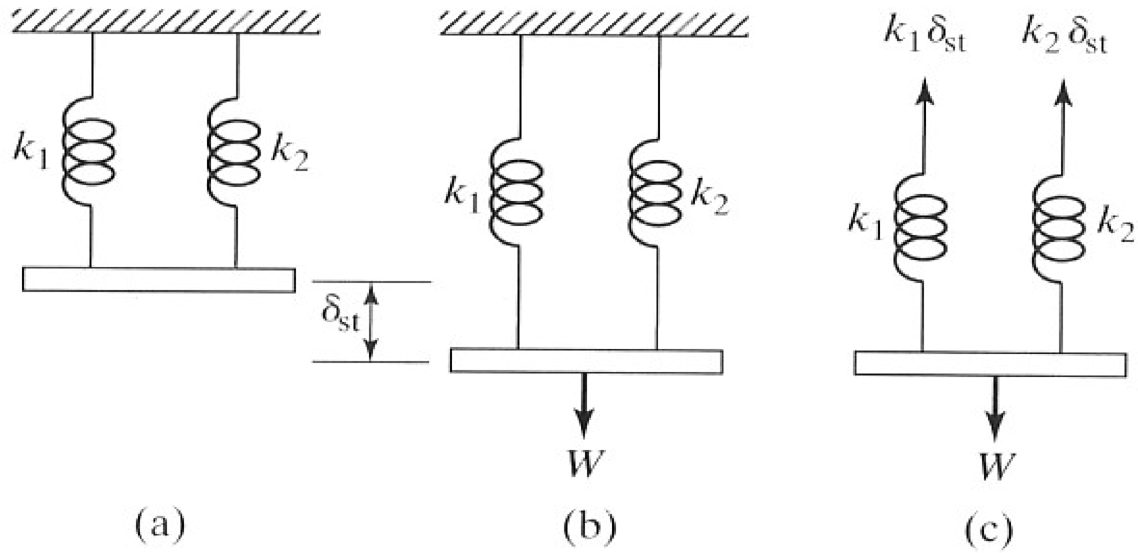
- This procedure can be applied for various geometries and boundary conditions. (see appendix)

Spring Elements

- Equivalent spring constant.
  - Springs in parallel:

$$w = mg = k_1\delta + k_2\delta$$

$$w = mg = k_{eq}\delta$$



- where

$$k_{eq} = k_1 + k_2$$

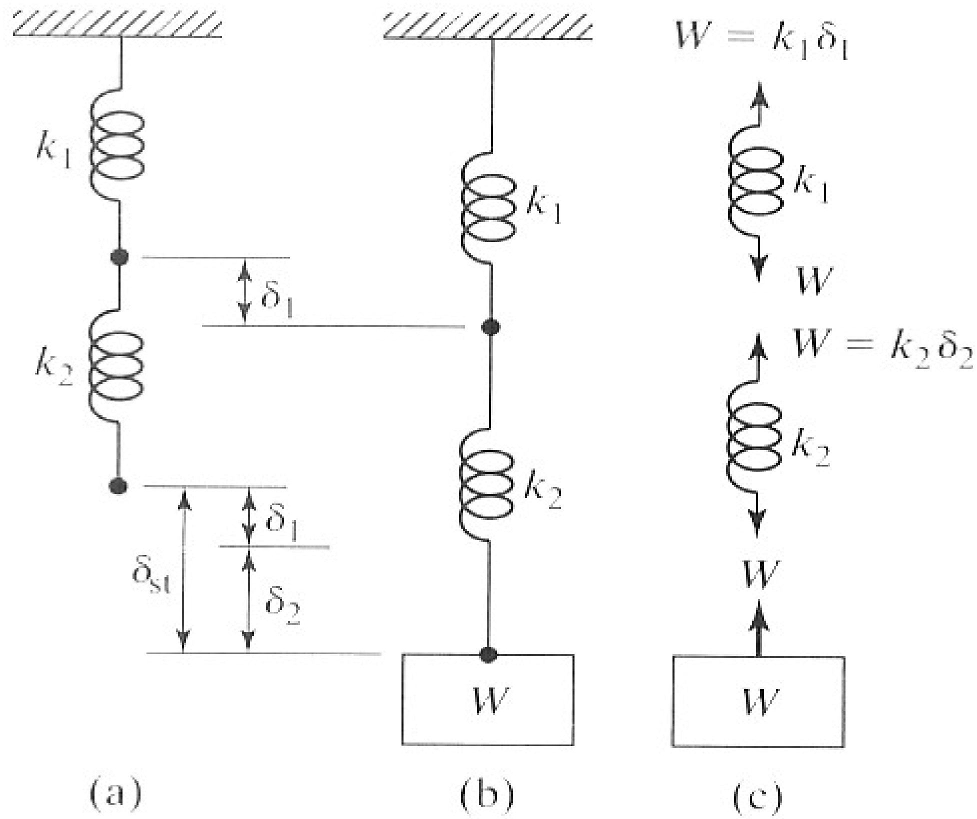
- In general, for n springs in parallel:

$$k_{eq} = \sum_{i=1}^{i=n} k_i$$

Spring Elements

- Equivalent spring constant.
  - **Springs in series:**

$$\delta_t = \delta_1 + \delta_2$$



- Both springs are subjected to the same force:

$$mg = k_1\delta_1 = k_2\delta_2$$

$$mg = k_{eq}\delta_t$$

- Combining the above equations:

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_t$$

$$\delta_1 = \frac{k_{eq}\delta_t}{k_1} \quad and \quad \delta_2 = \frac{k_{eq}\delta_t}{k_2}$$

Spring Elements

- **Springs in series (cont'd):**

  - Substituting into first eqn:

$$\delta_t = \frac{k_{eq}\delta_t}{k_1} + \frac{k_{eq}\delta_t}{k_2}$$

  - Dividing by $k_{eq}\delta_t$ throughout:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

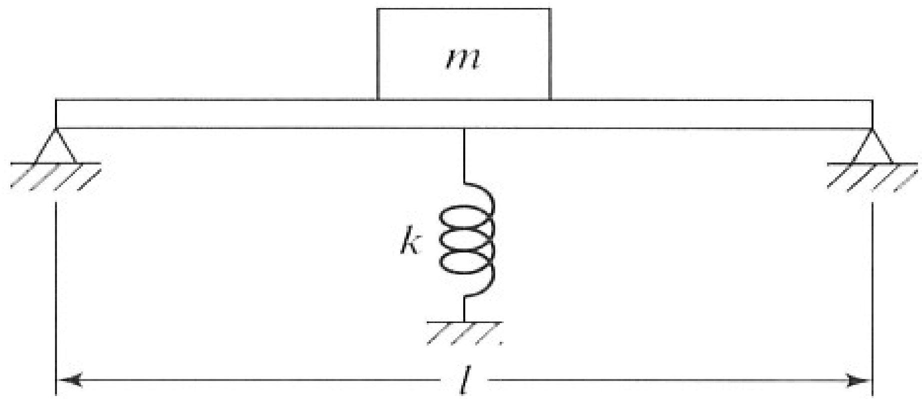
  - For n springs in series:

$$\frac{1}{k_{eq}} = \sum_{i=1}^{i=n}\left[\frac{1}{k_i}\right]$$

Spring Elements

- Equivalent spring constant.
  - When springs are connected to rigid components such as pulleys and gears, the energy equivalence principle must be used.

- Example:

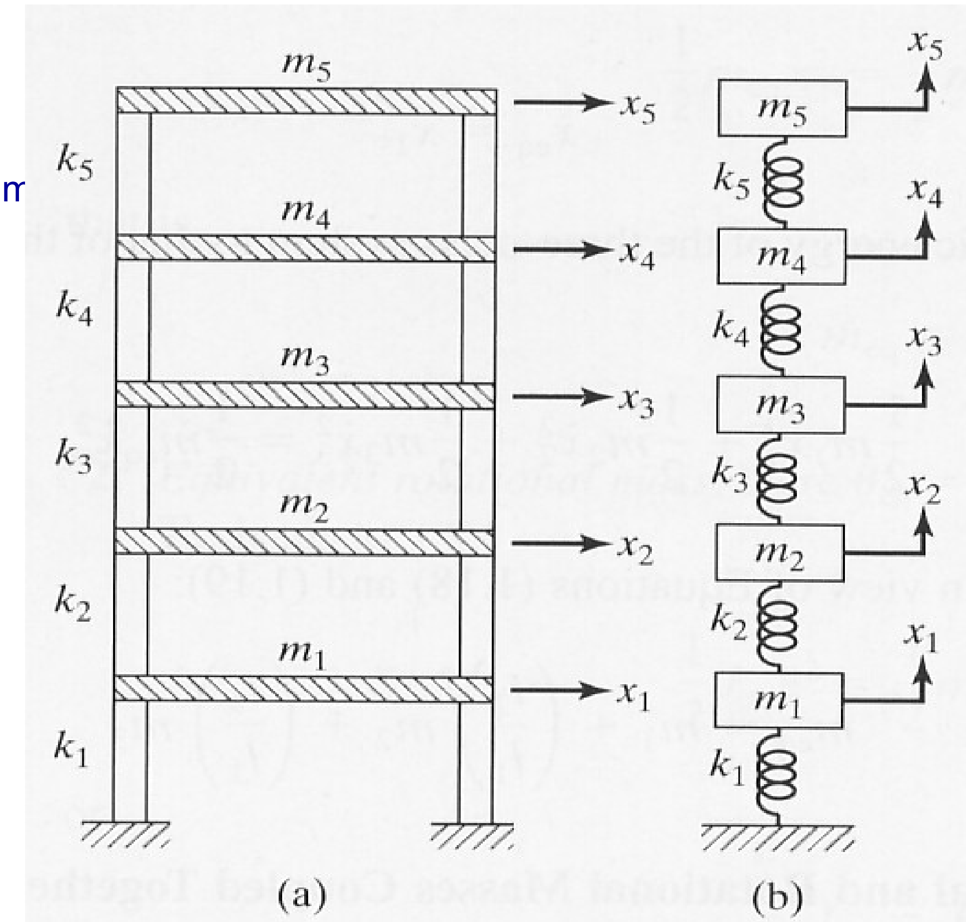


Example (1.10 Ed.3)

Mass / Inertia Elements

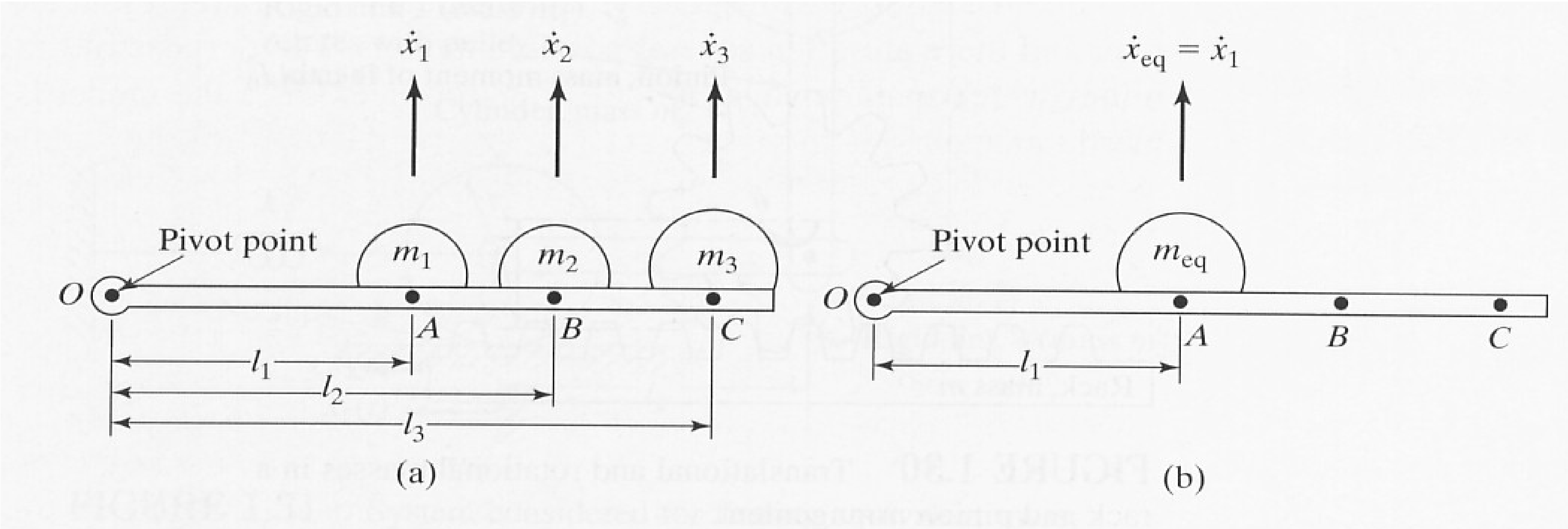
- Mass or inertia element assumed rigid (lumped mass)
- Its energy (kinetic) is proportional to velocity.
- Force ∝ mass * acceleration
- Work = force * displacement
- Work done on mass is stored as Kinetic Energy

- Modelling with lumped mass elements. Example: assum frame mass is negligible cf mass of floors.



Mass / Inertia Elements

- Equivalent mass - example:



- The velocities of the mass elements can be written as:

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1 \quad and \quad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1$$

- To determine the equivalent mass at position l_1:

$$\dot{x}_{eq} = \dot{x}_1$$

Mass / Inertia Elements

- Equivalent mass – example (cont'd)

  - Equating the kinetic energies:

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{eq}\dot{x}_{eq}^2$$

  - Substituting for the velocity terms:

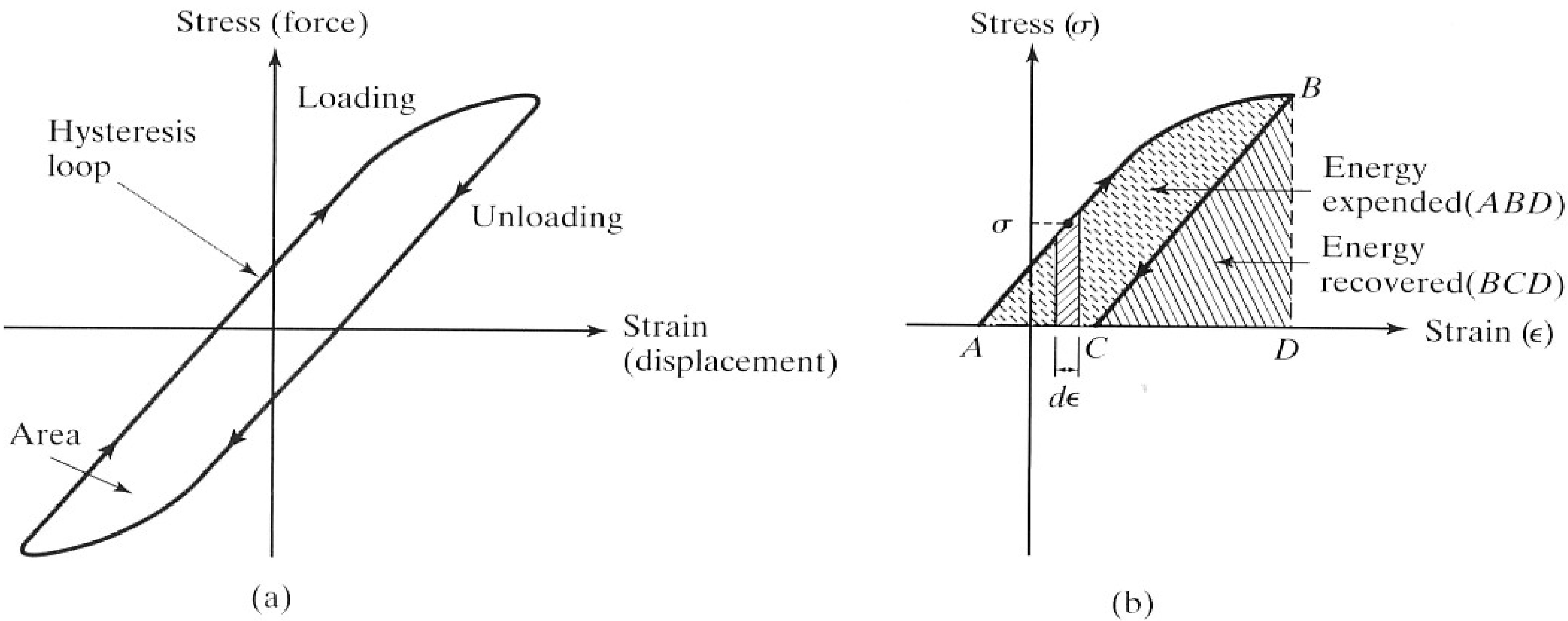
$$m_{eq} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$$

Damping Elements

- Absorbs energy from vibratory system $\rightarrow$ vibration amplitude decays.
- Damping element considered to have no mass or elasticity
- Real damping systems very complex, damping modelled as:
  - **Viscous damping:**
    - Based on viscous fluid flowing through gap or orifice.
    - Eg: film between sliding surfaces, flow b/w piston & cylinder, flow thru orifice, film around journal bearing.
    - Damping force $\propto$ relative velocity between ends
  - **Coulomb (dry Friction) damping:**
    - Based on friction between unlubricated surfaces
    - Damping force is constant and opposite the direction of motion

Damping Elements

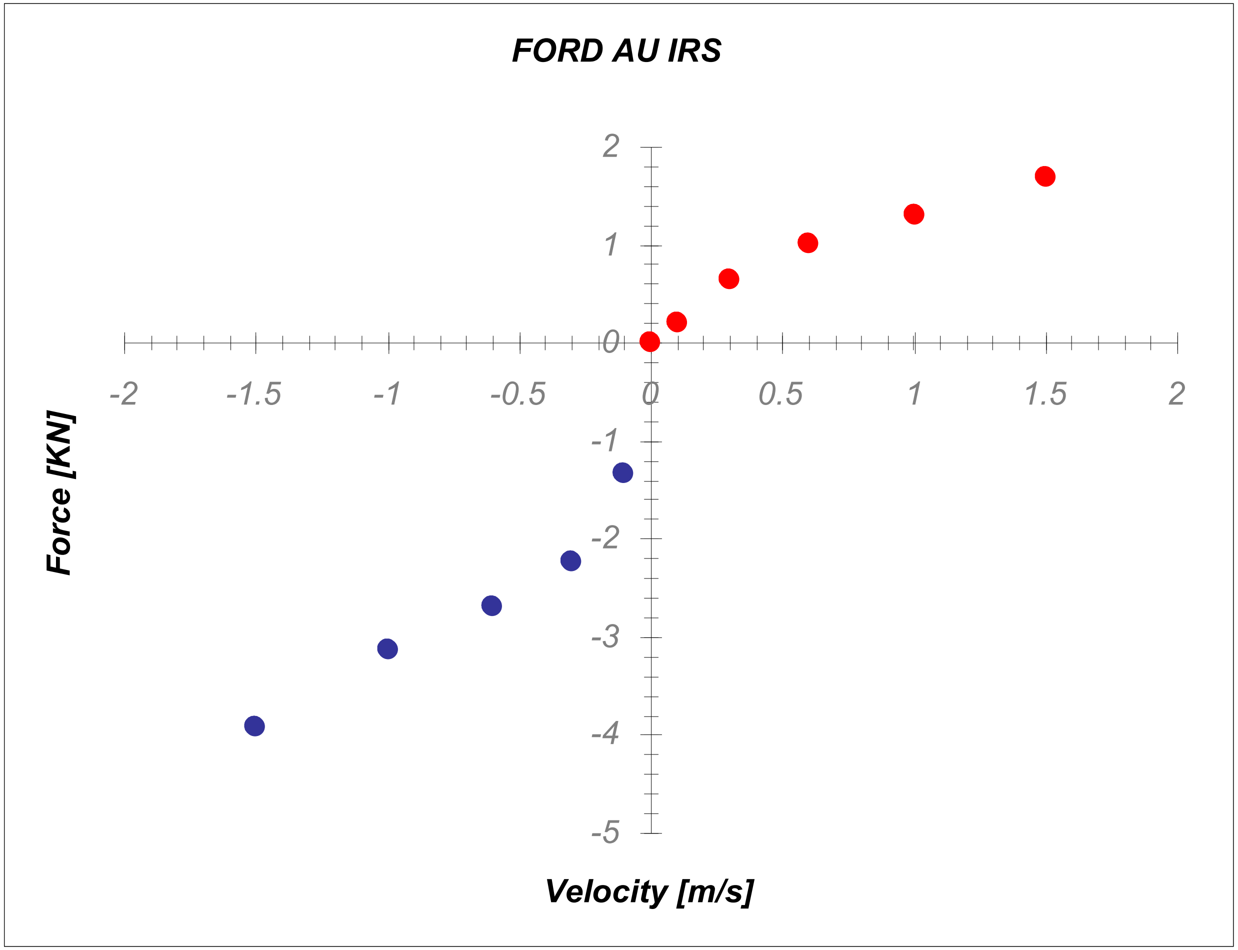
- **Hysteretic (material or solid) damping:**
  - Based on plastic deformation of materials (energy loss due to slippage b/w grains)
  - Energy lost due to hysteresis loop in force-deflection (stress-strain) curve of element when load is applied:



Damping Elements

- **Equivalent damping element:**
  - Combinations of damping elements can be replace by equivalent damper using same procedures as for spring and mass/inertia elements.

Damping Elements



Harmonic Motion

- Harmonic motion: simplest form of periodic motion (deterministic).
- Pure sinusoidal (co-sinusoidal) motion
- Eg: Scotch-yoke mechanism rotating with angular velocity ω - simple harmonic motion:
- The motion of mass m is described by:

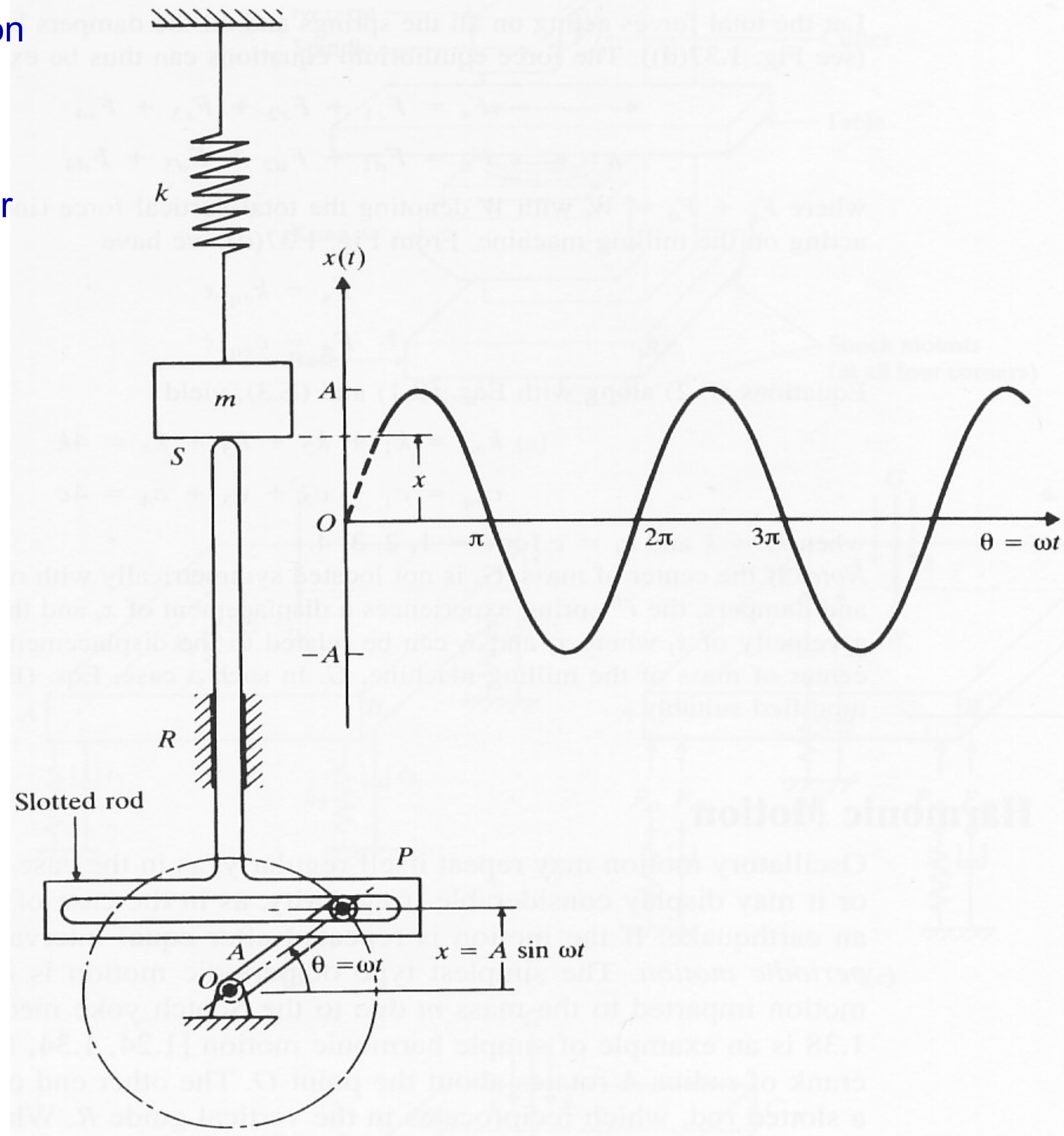
$$x = A\sin(\theta) = A\sin(\omega t)$$

- Its velocity and acceleration are:

$$\frac{dx}{dt} = \omega A\cos(\omega t)$$

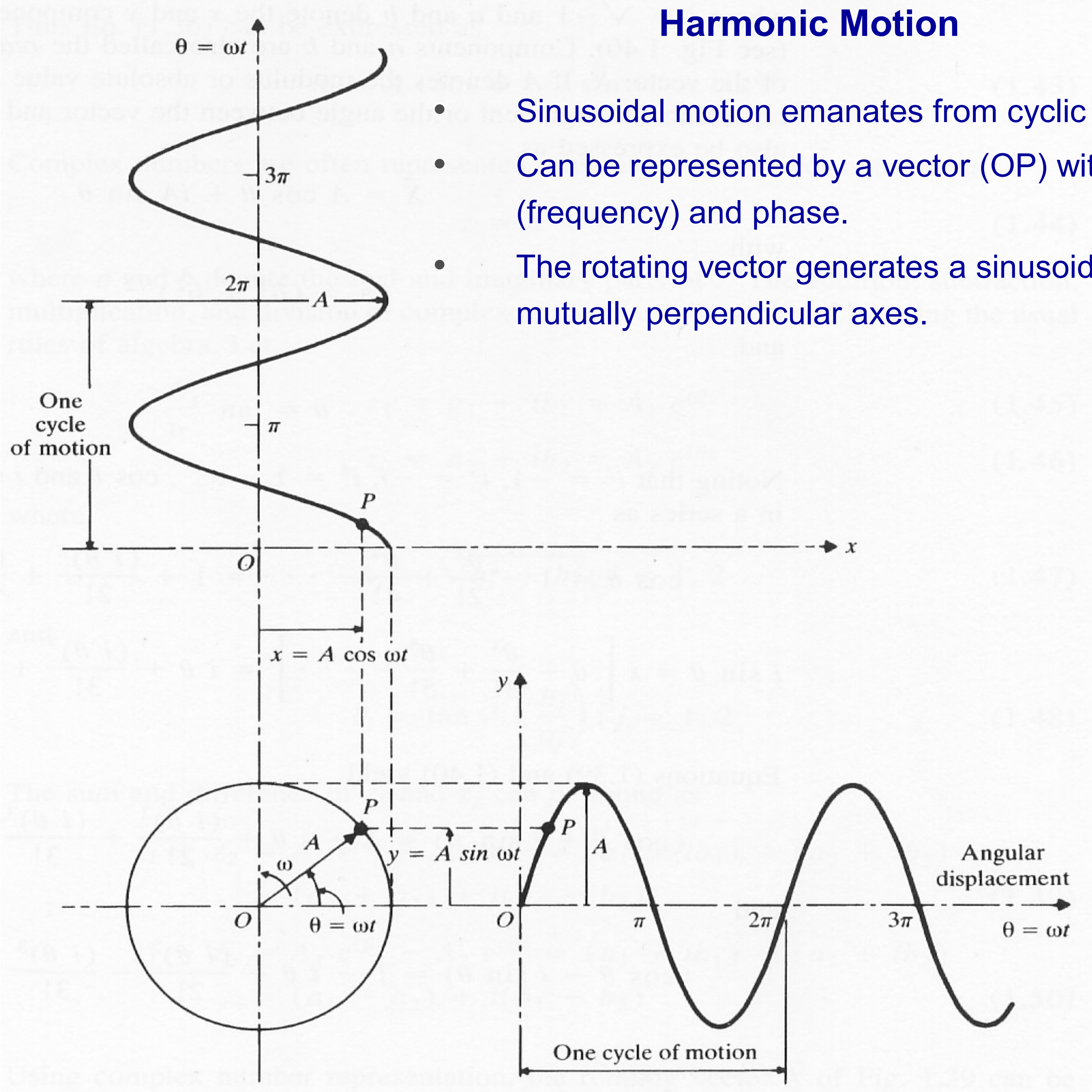
and

$$\frac{d^2x}{dt^2} = -\omega^2 A\sin(\omega t) = -\omega^2 x$$



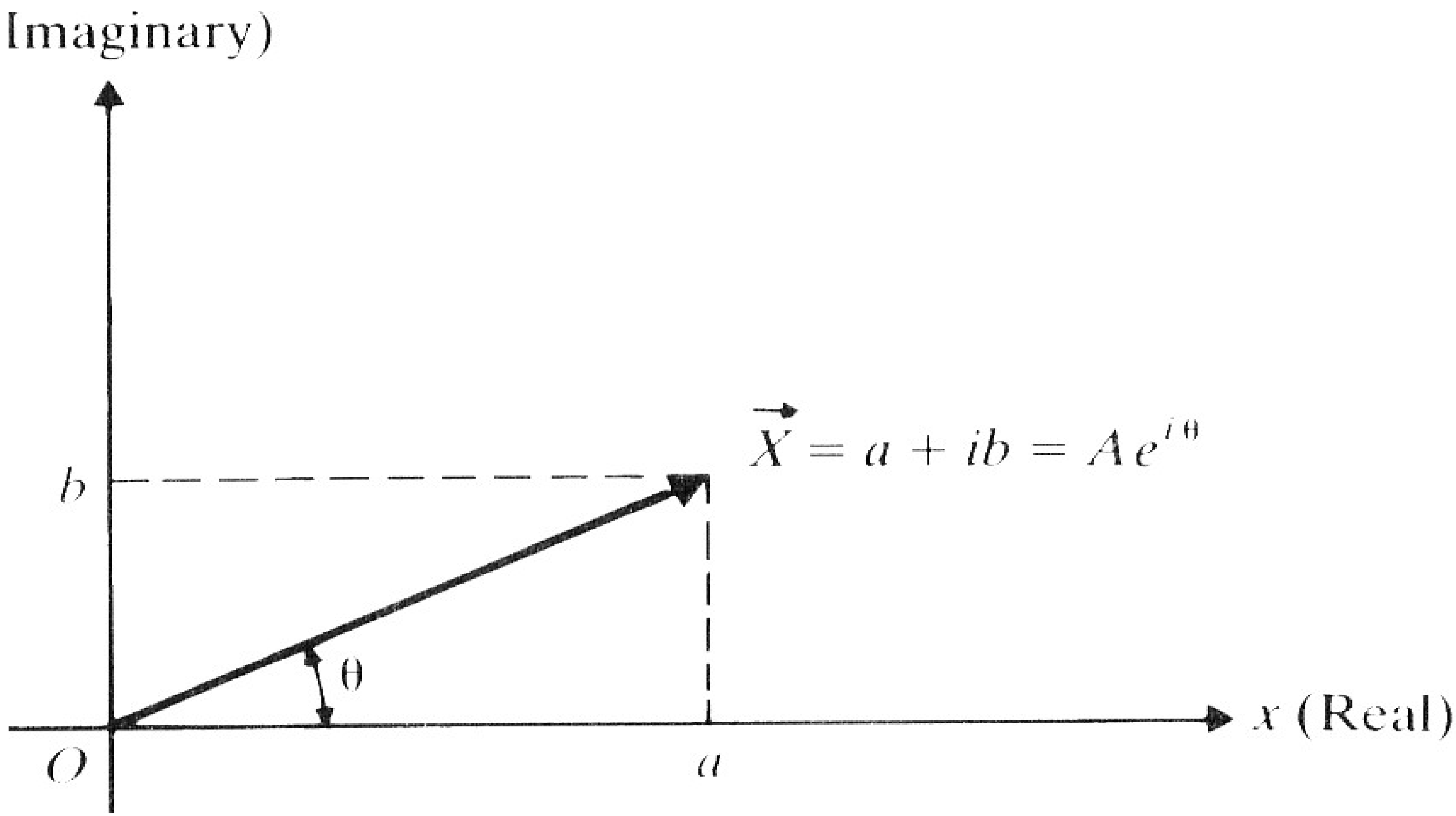
Harmonic Motion

- Sinusoidal motion emanates from cyclic motion
- Can be represented by a vector (OP) with a magnitude, angular velocity (frequency) and phase.
- The rotating vector generates a sinusoidal and a co-sinusoidal components along mutually perpendicular axes.



Harmonic Motion

- Often convenient to represent sinusoidal and co-sinusoidal components (mutually perpendicular) in complex number format
- Where a and b denote the sinusoidal (x) and co-sinusoidal (y) components
- a and b = real and imaginary parted of vector **X**



Harmonic Motion

Definition of terms:

- **Cycle**: motion of body from equilibrium position → extreme position → equilibrium position → extreme position in other direction → equilibrium position .
- **Amplitude**: Maximum value of motion from equilibrium. (Peak – Peak = 2 x amplitude)
- **Period**: Time taken to complete one cycle

$$\tau = \frac{2\pi}{\omega}$$

ω = circular frequency

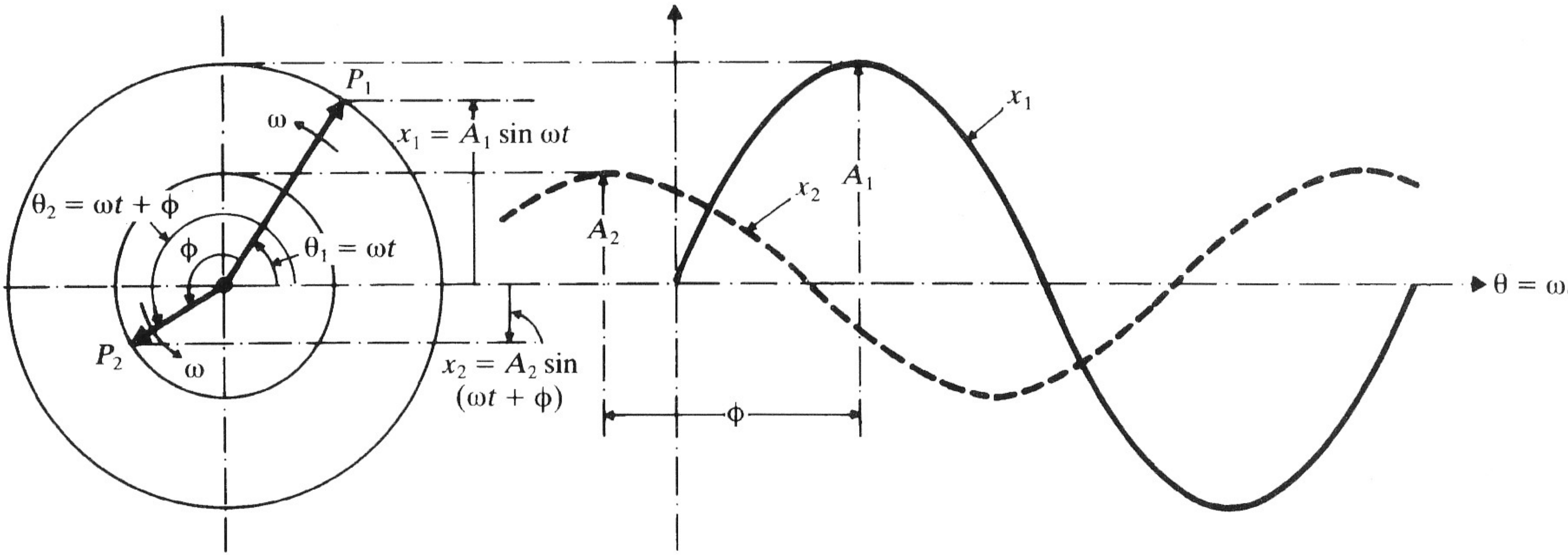
- **Frequency**: number of cycles per unit time.

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

ω : radians/s f Hertz (cycles /s)

Harmonic Motion

- **Phase angle**: the difference in angle (lead or lag) by which two harmonic motions of the same frequency reach their corresponding value (maxima, minima, zero up-cross, zero down-cross)



Harmonic Motion

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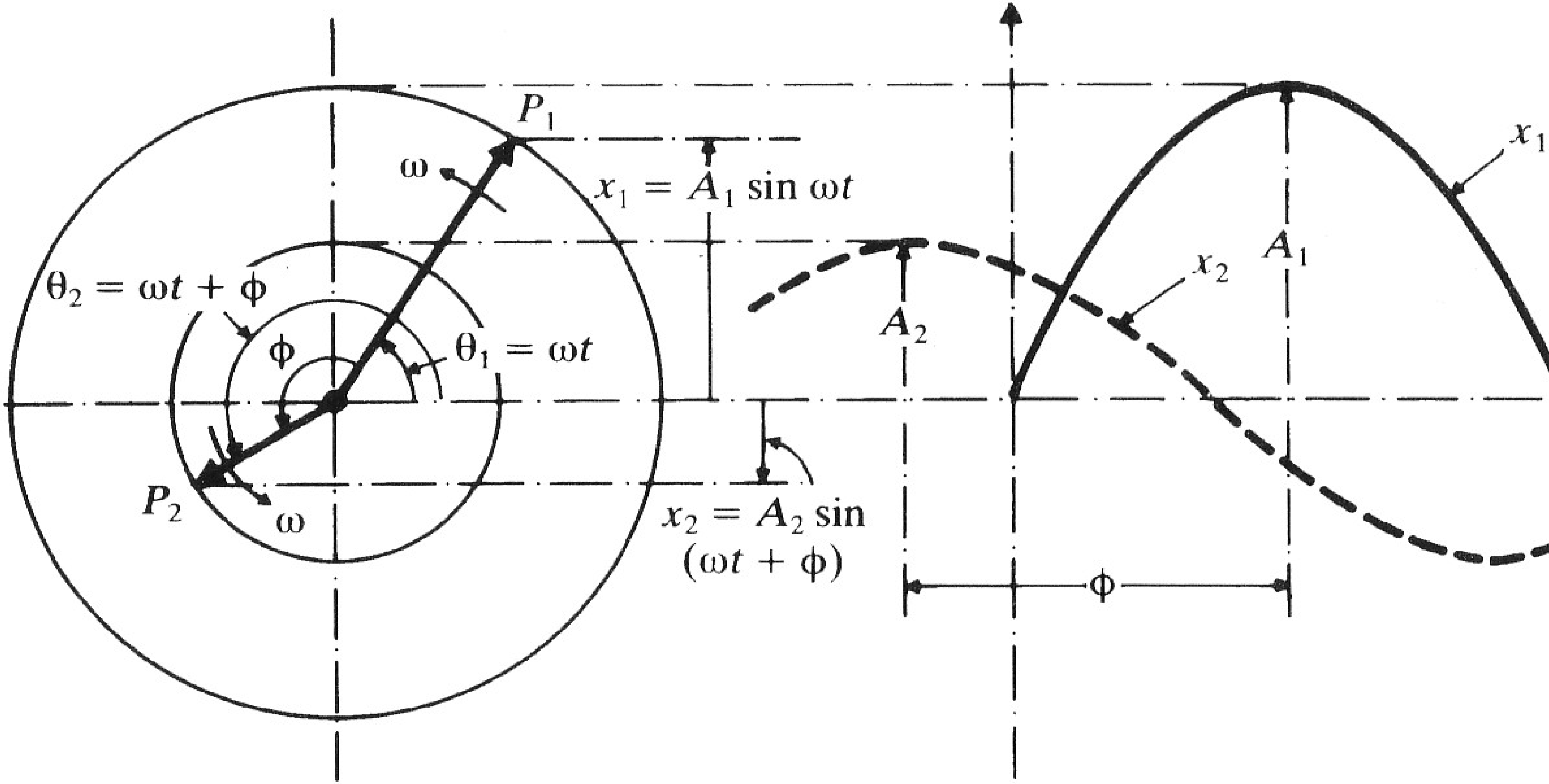


Harmonic Motion

- **Natural frequency:** the frequency at which a system vibrates without external forces after an initial disturbance. The number of natural frequencies always matches the number of DoF.
- **Beats:** the effect produced by adding two harmonic motions with similar (close) frequencies.

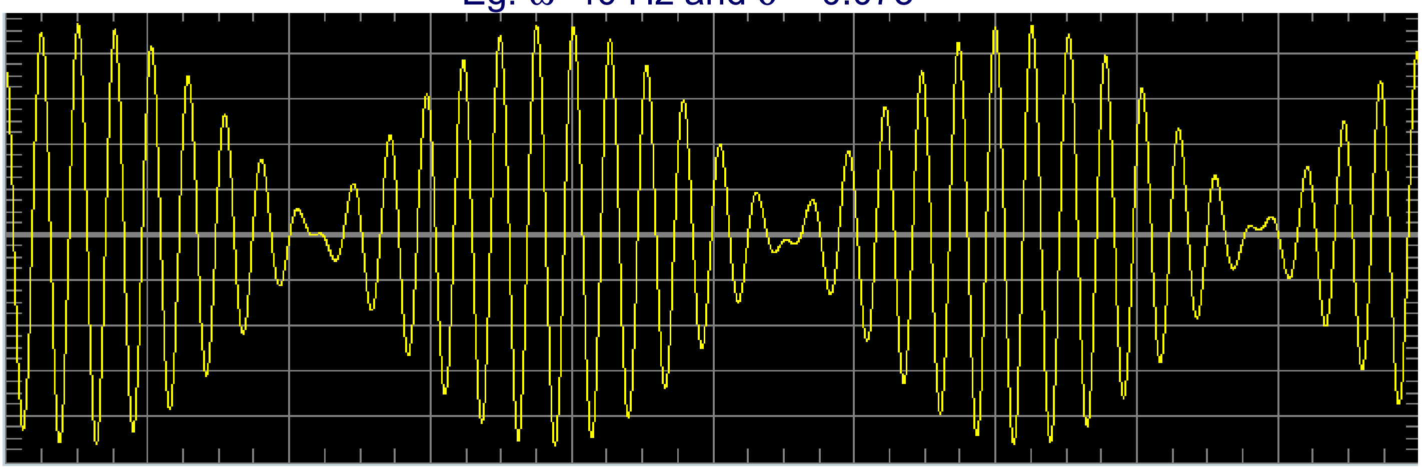
$$x_1 = A\sin(\omega t) \qquad x_2 = A\sin(\omega t + \delta\omega t)$$

$$x_t = x_1 + x_2 = A\,[\sin(\omega t) + \sin(\omega t + \delta\omega t)]$$

$$\text{Since} \quad \sin M + \sin N = 2\sin\frac{M+N}{2}\cos\frac{M-N}{2}$$

$$x_t = 2A\sin\left(\omega t + \frac{\delta\omega t}{2}\right)\cos\left(\frac{\delta\omega t}{2}\right)$$

Eg: ω=40 Hz and δ= -0.075



- In mechanical vibratory systems, beats occur when the (harmonic) excitation (forcing) frequency is close to the natural frequency.

Harmonic Motion

- **Octave:** doubling of any quantity. Used mainly for frequency.
- **Octave band (frequency):** maximum is double of minimum. Eg: 64 – 128 Hz, 1000 – 2000 Hz.
- **Decibel:** defined as 10 x log(power ratio)

$$dB = 10Log\left(\frac{P}{P_0}\right)$$

In electrical systems (as in mechanical vibratory systems) power is proportional to the value squared hence:

$$dB = 20Log\left(\frac{X}{X_0}\right)$$

Harmonic (Fourier) Analysis

- Many vibratory systems not harmonic but often periodic
- Any periodic function can be represented by the Fourier series – infinite sum of sinusoids and co-sinusoids.

$$x(t) = \frac{a_o}{2} + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \dots\dots\dots$$

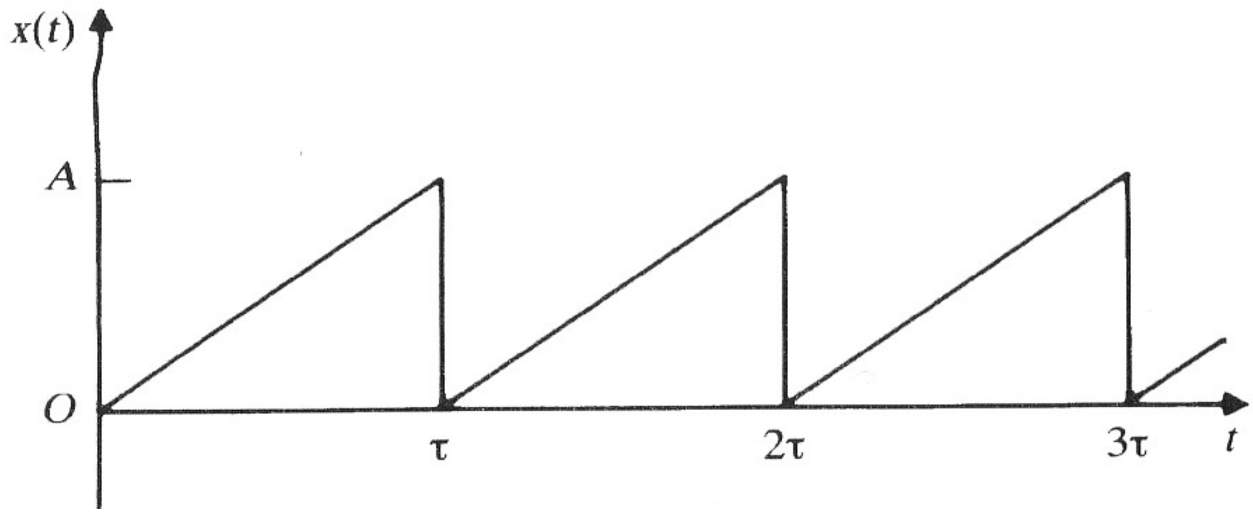
$$+ b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \dots\dots\dots$$

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- To obtain a_n and b_n the series is multiplied by *cos(nωt)* and *sin(nωt)* respectively and integrated over one period.

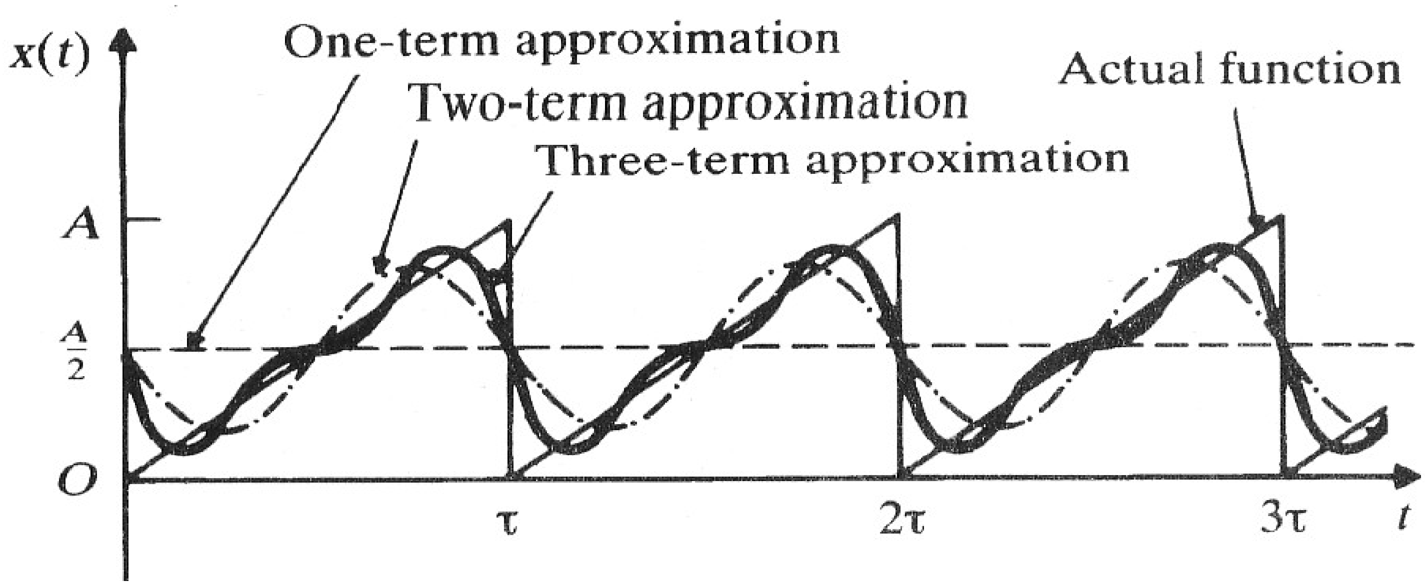
Harmonic (Fourier) Analysis

- Example:



Harmonic (Fourier) Analysis

- Example:



Harmonic (Fourier) Analysis

- As for simple harmonic motion, Fourier series can be expressed with complex numbers:

$$e^{i\omega t} = cos(\omega t) + i\,sin(\omega t)$$

$$e^{-i\omega t} = cos(\omega t) - i\,sin(\omega t)$$

$$cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

- The Fourier series:

$$x(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n\, cos(n\omega t) + b_n\, sin(n\omega t)]$$

Can be written as:

$$x(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left\{ a_n \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) + b_n \left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) \right\}$$

Harmonic (Fourier) Analysis

- Defining the complex Fourier coefficients

$$c_n = \frac{a_n - ib_n}{2} \quad and \quad c_{n-1} = \frac{a_n + ib_n}{2}$$

- The (complex) Fourier series is simplified to:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\omega t}$$

Harmonic (Fourier) Analysis

$$x(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

- The Fourier series is made-up of **harmonics.**
- Their amplitudes and phases are defined as:

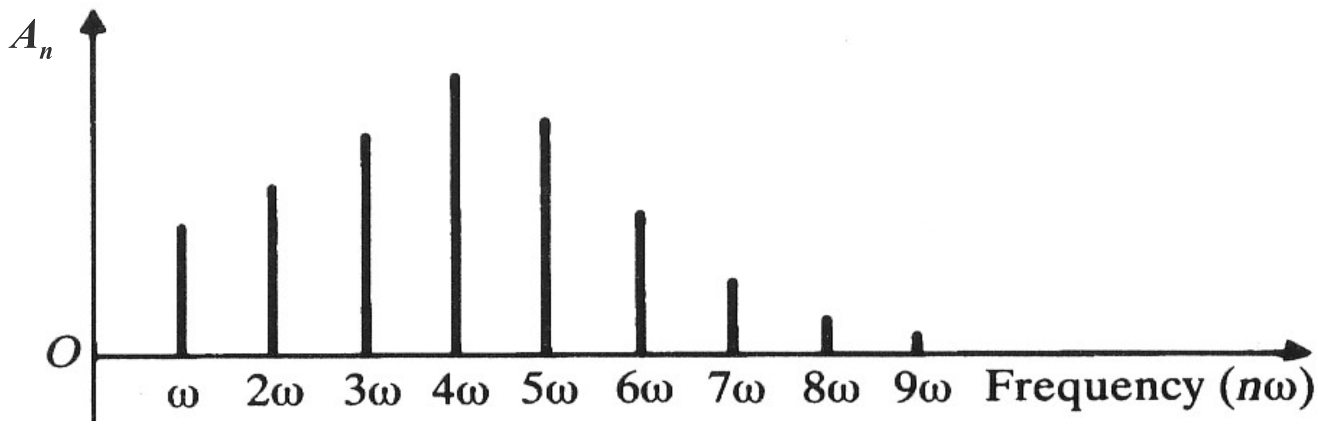
$$A_n = \sqrt{(a_n^2 + b_n^2)}$$

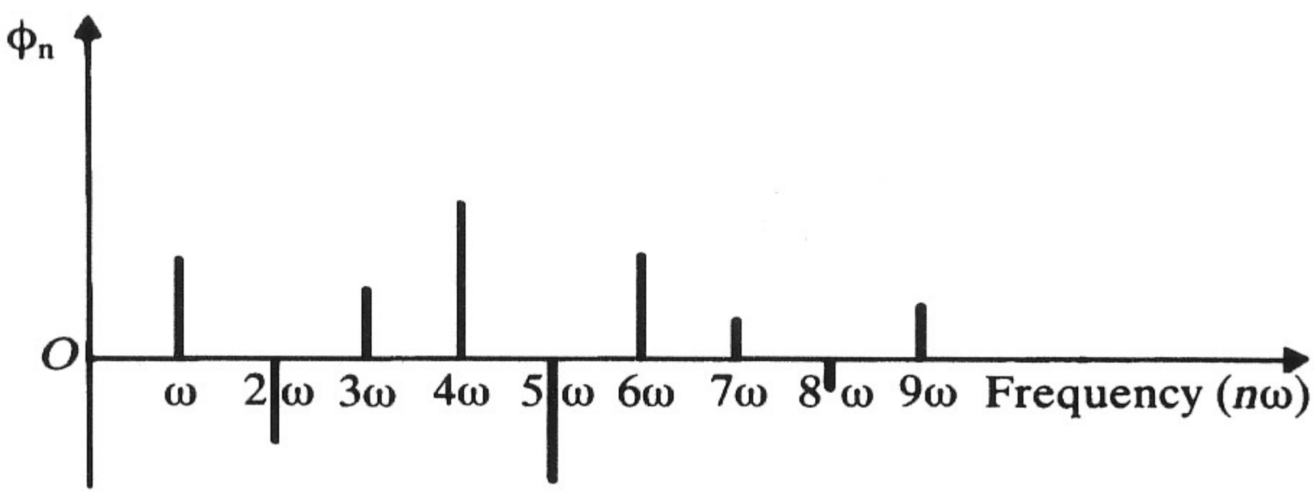
$$\phi_n = a\tan\left(\frac{b_n}{a_n}\right)$$

harmonics

Harmonic (Fourier) Analysis

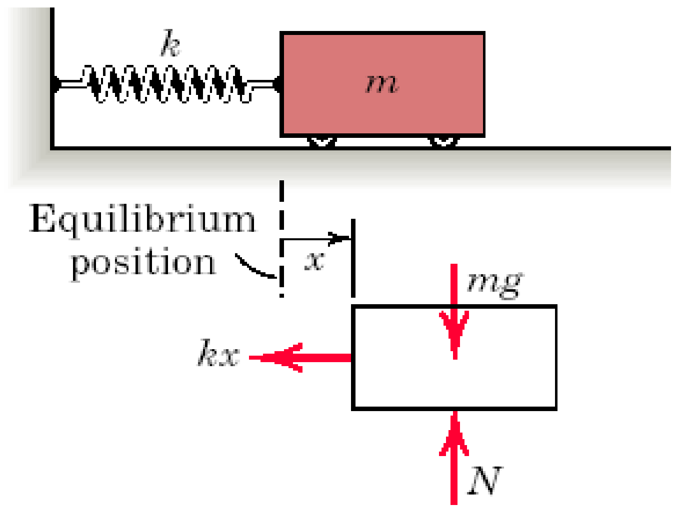
- The amplitudes (magnitudes) and phases of the harmonics can be plotted as a function of frequency to form the ***frequency spectrum*** of ***spectral diagram:***





Free undamped vibration single DoF

- Recall: Free vibrations $\rightarrow$ system given initial disturbance and oscillates free of external forces.
- Undamped: no decay of vibration amplitude
- Single DoF:
    - mass treated as rigid, limped (particle)
    - Elasticity idealised by single spring
    - only one natural frequency.



- The equation of motion can be derived using
    - Newton's second law of motion
    - D'Alembert's Principle,
    - The principle of virtual displacements and,
    - The principle of conservation of energy.

Free undamped vibration single DoF

- Using Newton's second law of motion to develop the **equation of motion.**
  1. Select suitable coordinates
  2. Establish (static) equilibrium position
  3. Draw free-body-diagram of mass
  4. Use FBD to apply Newton's second law of motion:

     "*Rate of change of momentum = applied force*"

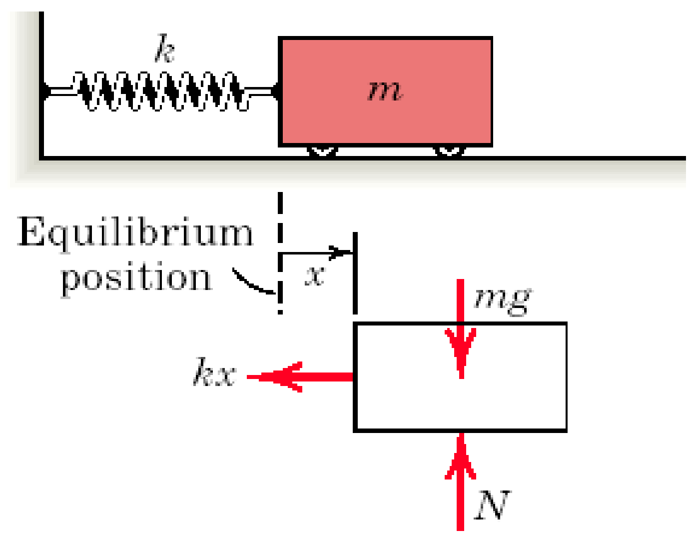
$$F(t) = \frac{d}{dt}\left(m\frac{dx(t)}{dt} \right)$$

As m is constant

$$F(t) = m\frac{d^2x(t)}{dt^2} = m\ddot{x}$$

For rotational motion

$$M(t) = J\ddot{\theta}$$



For the free, undamped single DoF system

$$F(t) = -kx = m\ddot{x}$$

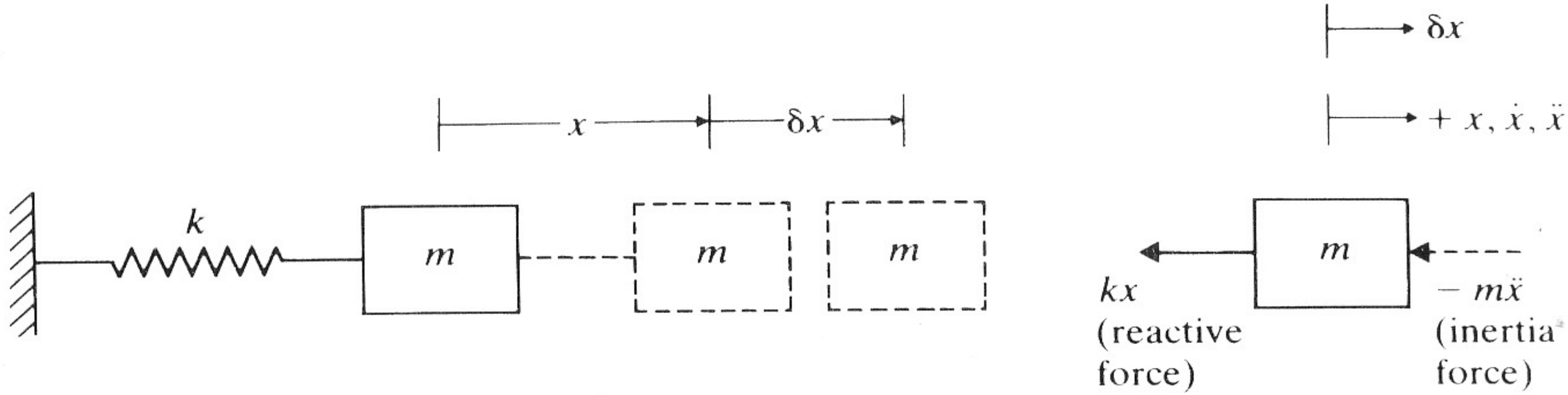
$$or$$

$$m\ddot{x} + kx = 0$$

Free undamped vibration single DoF

Principle of virtual displacements:

- “When a system in equilibrium under the influence of forces is given a virtual displacement. The total work done by the virtual forces = 0”
- Displacement is imaginary, infinitesimal, instantaneous and compatible with the system



(a) Mass under a displacement x

(b) Free-body diagram

- When a virtual displacement *dx* is applied, the sum of work done by the spring force and the inertia force are set to zero:

$$-(kx)\delta x \;-(m\ddot{x})\delta x = 0$$

- Since *dx* ≠ 0 the equation of motion is written as:

$$kx \;+ m\ddot{x} = 0$$

Free undamped vibration single DoF

Principle of conservation of energy:

- No energy is lost due to friction or other energy-dissipating mechanisms.
- If no work is done by external forces, the system total energy = constant
- For mechanical vibratory systems:

$$KE + PE = constant$$

$$or$$

$$\frac{d}{dt}\left(KE + PE \right) = 0$$

- Since

$$KE = \frac{1}{2}m\dot{x}^2 \qquad and \qquad PE = \frac{1}{2}kx^2$$

$$then$$

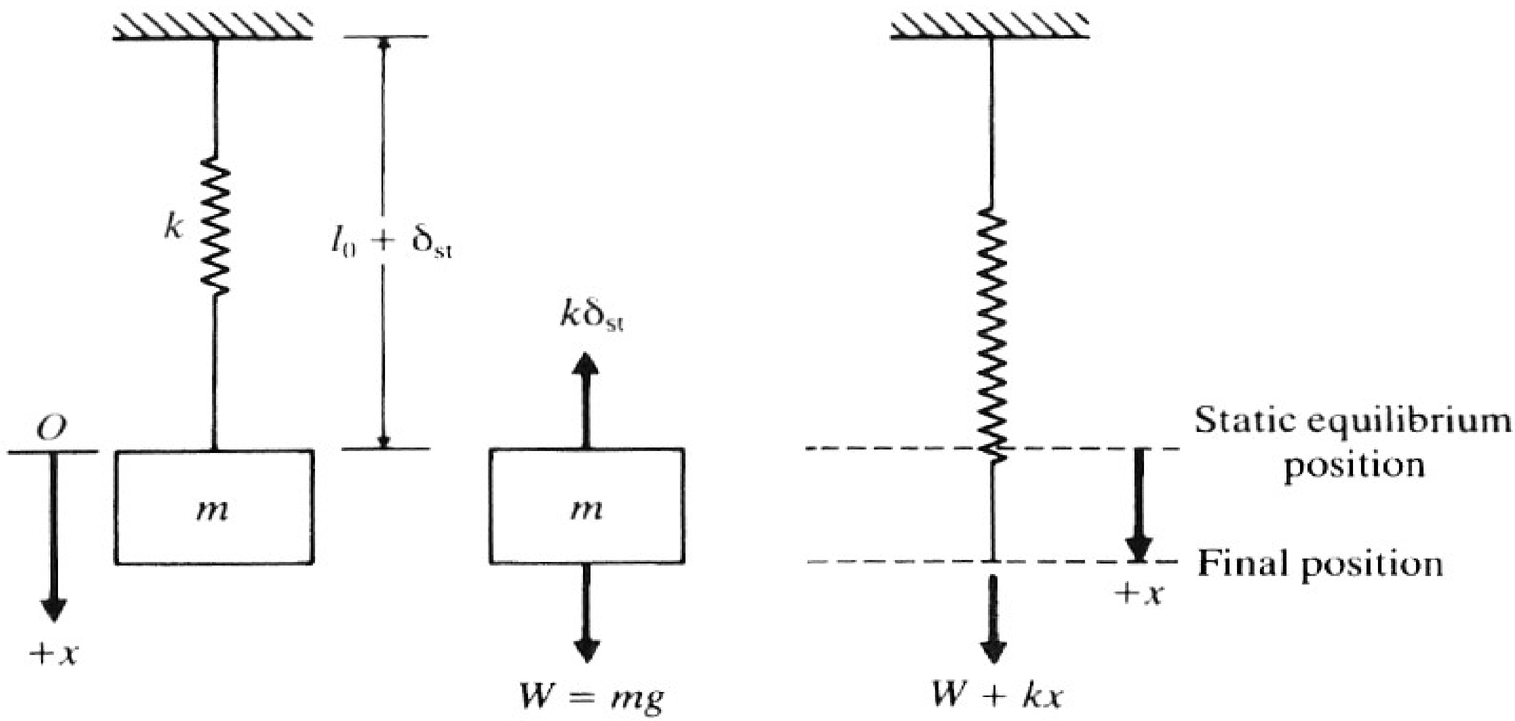
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \right) = 0$$

$$or$$

$$m\ddot{x} + kx = 0$$

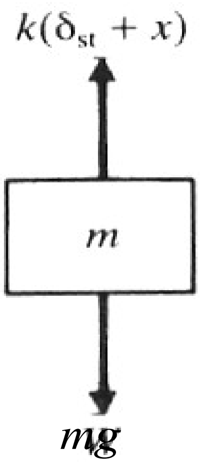
Free undamped vibration single DoF

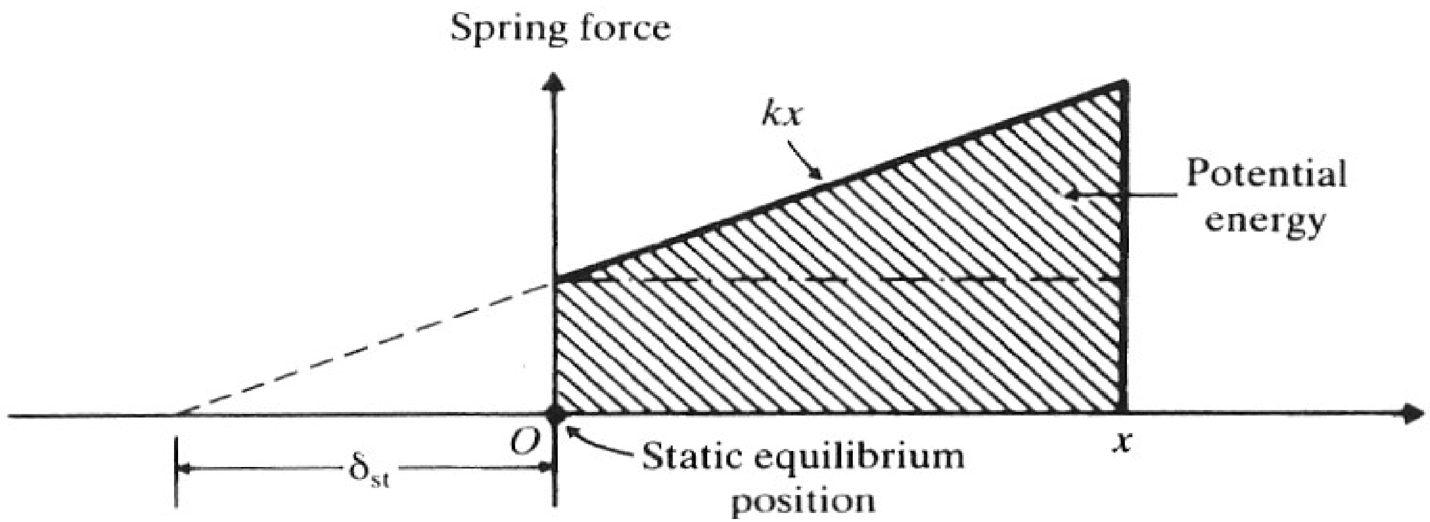
Vertical mass-spring system:



Free undamped vibration single DoF

Vertical mass-spring system:





- From the free body diagram:, using Newton's second law of motion:

$$m\ddot{x} = -k(x + \delta_{st}) + mg$$
$$since \quad k\delta_{st} = mg$$
$$m\ddot{x} + kx = 0$$

- Note that this is the same as the eqn. of motion for the horizontal mass-spring system

∀ ∴ if x is measured from the static equilibrium position, gravity (weight) can be ignored

- This can be also derived by the other three alternative methods.

Free undamped vibration single DoF

- **The solution to the differential eqn. of motion.**
- As we anticipate oscillatory motion, we may propose a solution in the form:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

or

$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}$$

alternatively, if we let $s = \pm i\omega_n$

$$x(t) = Ce^{\pm st}$$

- By substituting for x(t) in the eqn. of motion:

$$C(ms^2 + k) = 0$$

since $c \neq 0$,

$$ms^2 + k = 0 \qquad \neg \text{ Characteristic equation}$$

and

$$s = \pm i\omega_n = \pm\sqrt{\frac{k}{m}} \qquad \neg \text{ roots = eigenvalues}$$

or

$$\omega_n = \sqrt{\frac{k}{m}}$$

Free undamped vibration single DoF

- **The solution to the differential eqn. of motion.**
- Applying the initial conditions to the general solution: $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

$$x_{(t=0)} = A = x_0 \quad \textit{initial displacement}$$

$$\dot{x}_{(t=0)} = B\omega_n = \dot{x}_0 \quad \textit{initial velocity}$$

- The solution becomes:

$$x(t) = x_0\cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n}\sin(\omega_n t)$$

$$\textit{if we let} \quad A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} \quad \textit{and} \quad \phi = a\tan\left(\frac{x_0\omega_n}{\dot{x}_o}\right) \quad \textit{then}$$

$$x(t) = A_0\sin(\omega_n t + \phi)$$

- This describes motion of harmonic oscillator:
  - Symmetric about equilibrium position
  - Thru equilibrium: velocity is maximum & acceleration is zero
  - At peaks and valleys, velocity is zero and acceleration is maximum
  - ∀ $\omega_n = \sqrt{(k/m)}$ is the natural frequency

Free undamped vibration single DoF

- Note: for vertical systems, the natural frequency can be written as:

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$since \quad k = \frac{mg}{\delta_{st}}$$

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad or \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Free undamped vibration single DoF

- **Torsional vibration.**
- Approach same as for translational system. Laboratory exercise.

Free undamped vibration single DoF

- **Compound pendulum.**
- Given an initial angular displacement or velocity, system will oscillate due to gravitational acceleration.
- Assume rigid body → single DoF

Restoring torque:

$$mgd \sin\theta$$

$\therefore$ *Equation of motion :*

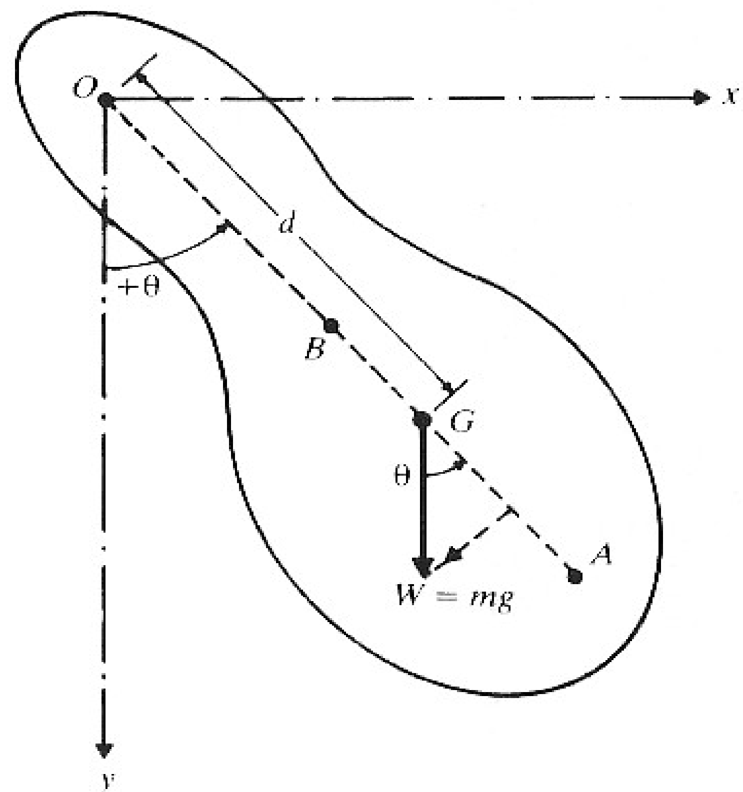
$$J_o\ddot{\theta} + mgd \sin\theta = 0 \quad \neg \text{ nonlinear } 2^{nd} \text{ order ODE}$$

Linearity is approximated if $\sin\theta \approx \theta$ *Therefore :*

$$J_o\ddot{\theta} + mgd\theta = 0$$

Natural frequency :

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$



Free undamped vibration single DoF

Natural frequency :

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$

since for a simple pendulum

$$\omega_n = \sqrt{\frac{g}{l}}$$

Then, $l = \frac{J_o}{md}$ *and since* $J_o = mk_o^2$ *then*

$$\omega_n = \sqrt{\frac{gd}{k_o^2}} \quad \text{and} \quad l = \frac{k_o^2}{d}$$

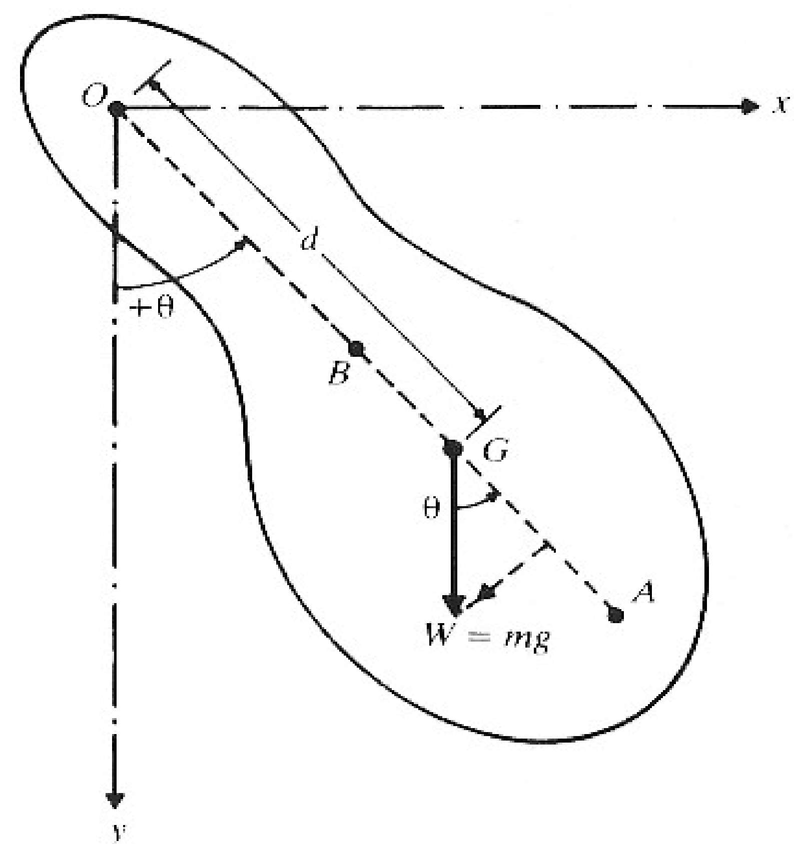
Applying the parallel axis theorem $k_o^2 = k_G^2 + d^2$

$$l = \frac{k_G^2}{d} + d$$

Let $l = GA + d = OA$

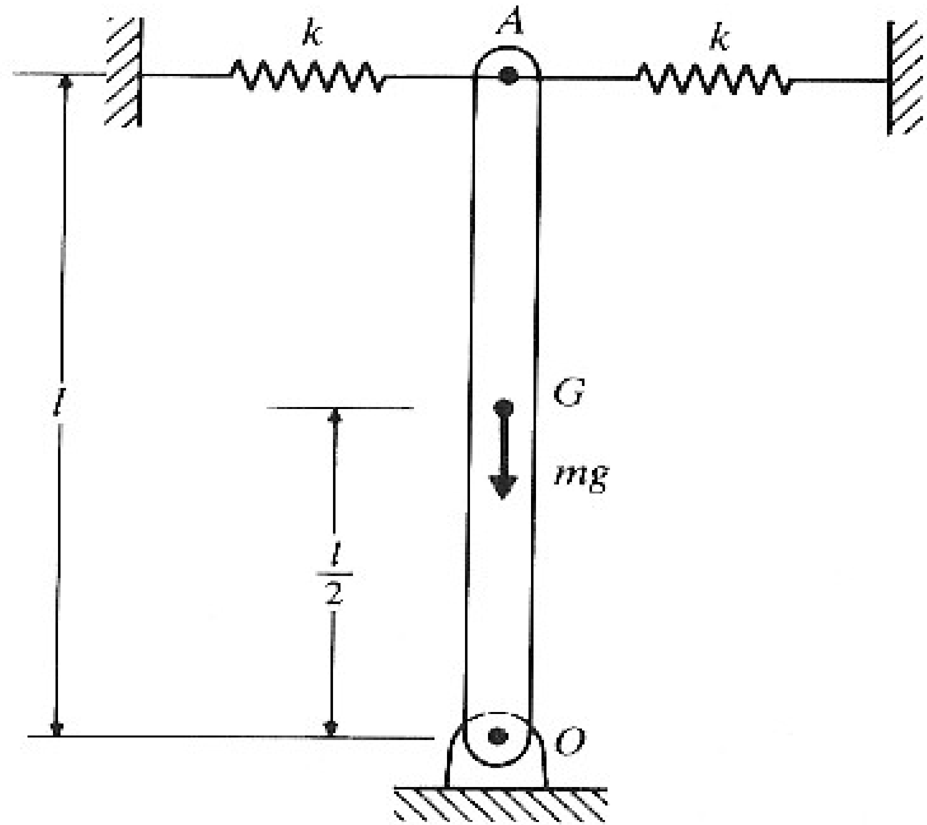
$$\omega_n = \sqrt{\frac{g}{k_o^2 / d}} = \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{OA}}$$

The location A $\left(GA = \frac{k_G^2}{d} \right)$ *is the "centre of percussion"*



Free undamped vibration single DoF

- **Stability.**
- Some systems may have inherent instability



Free undamped vibration single DoF

- **Stability.**
- Some systems may have inherent instability
- When the bar is deflected by θ,

The spring force is :

$$2kl\sin\theta$$

The gravitational force thru G is :

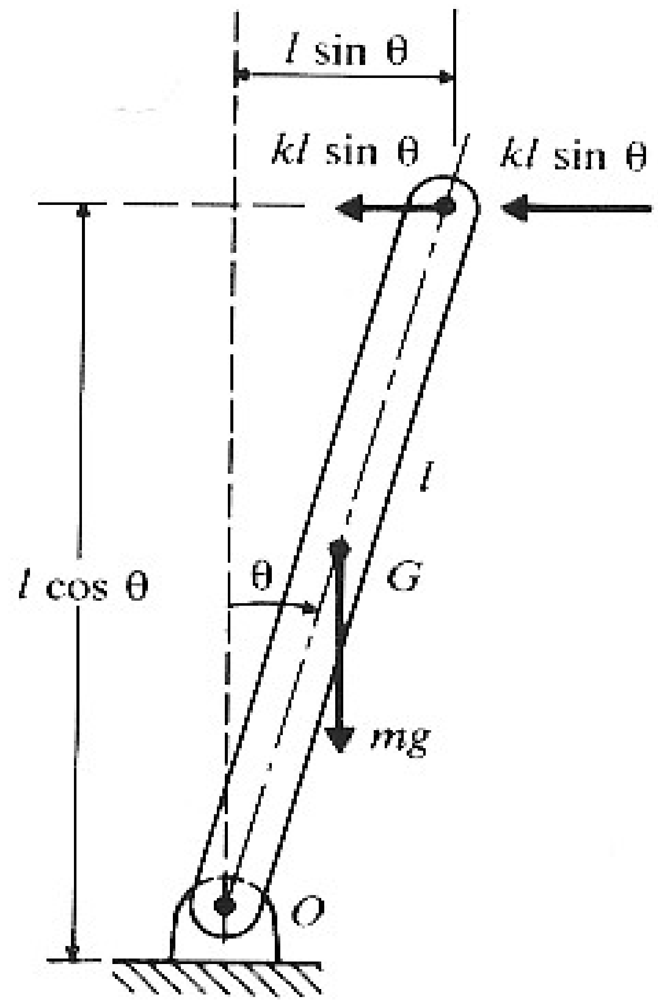
$$mg$$

The inertial moment about O due to the angular acceleration $\ddot{\theta}$ is :

$$J_o\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta}$$

The eqn. of motion is written as :

$$\frac{ml^2}{3}\ddot{\theta} + (2kl\sin\theta)\,l\cos\theta - mg\frac{l}{2}\sin\theta = 0$$



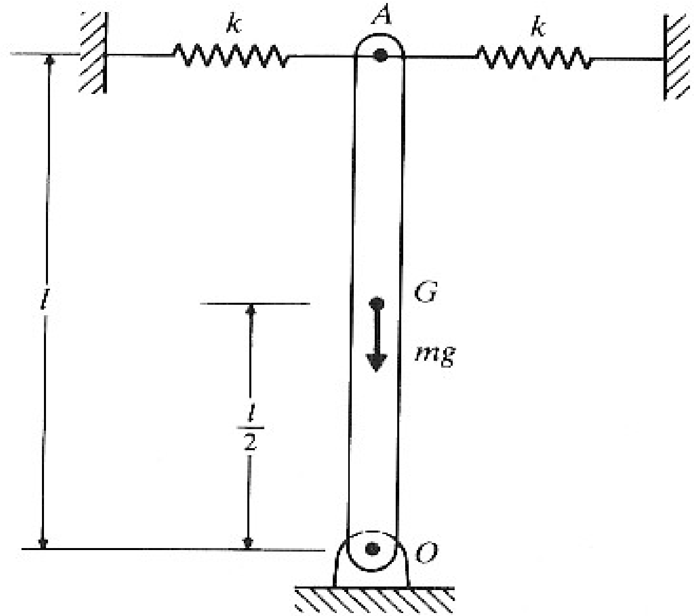
Free undamped vibration single DoF

For small oscillations, $\sin\theta = \theta$ *and* $\cos\theta = 1$. *Therefore*

$$\frac{ml^2}{3}\theta + 2kl^2\theta - \frac{mgl}{2}\theta = 0$$

or

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$



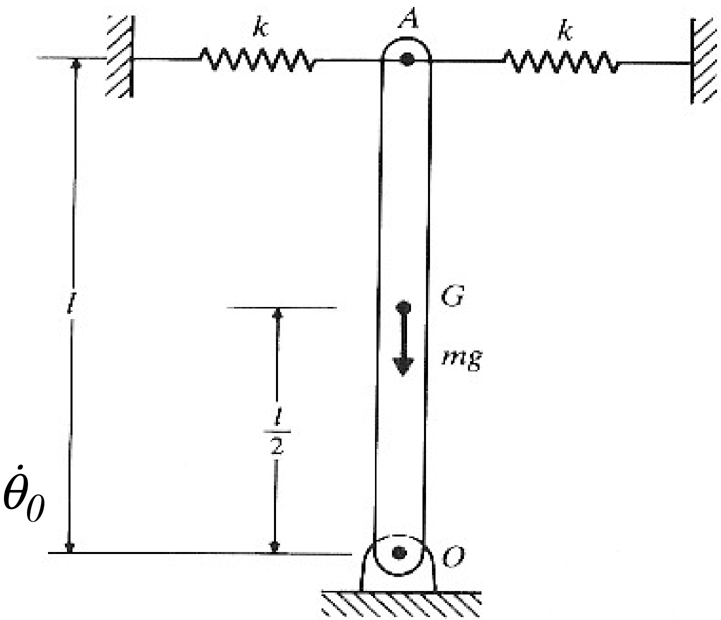
The solution to the eqn. of motion depends of the sign of ( )

(1) If ( ) >0, the resulting motion is oscillatory (simple harmonic) with a natural frequency

$$\omega_n = \sqrt{\left(\frac{12kl^2 - 3mgl}{2ml^2}\right)}$$

Free undamped vibration single DoF

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2} \right) \theta = 0$$



(2) If ( ) =0, the eqn. of motion reduces to:

$$\ddot{\theta} = 0$$

The solution is obtained by integrating twice yielding :

$$\theta(t) = C_1 t + C_2$$

Applying initial conditions $\theta(t=0) = \theta_0$ *and* $\dot{\theta}(t=0) = \dot{\theta}_0$

$$\theta(t) = \dot{\theta}_0 t + \theta_0$$

Which shows a linear increase of angular displ. at constant velocity.

And if $\dot{\theta}_0 = 0$ *the bar remains in static equilibrium at* $\theta(t) = \theta_0$

Free undamped vibration single DoF

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$

(3) If ( ) < 0, we define:

$$\alpha = -\left(\frac{12kl^2 - 3mgl}{2ml^2}\right) = \left(\frac{3mgl - 12kl^2}{2ml^2}\right)$$

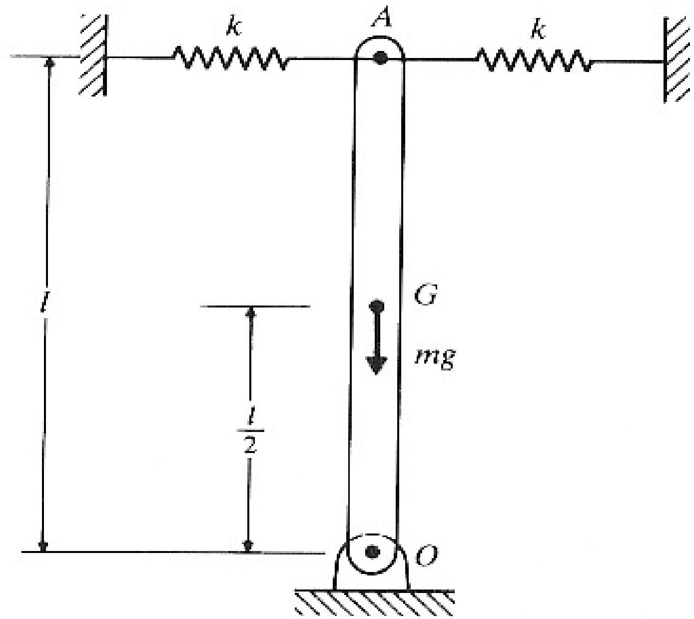
The solution of the eq. of motion is :

$$\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t}$$

Applying initial conditions $\theta(t=0) = \theta_0$ *and* $\dot{\theta}(t=0) = \dot{\theta}_0$

$$\theta(t) = \frac{1}{2\alpha}\left[\left(\alpha\theta_0 + \dot{\theta}_0\right)e^{\alpha t} + \left(\alpha\theta_0 - \dot{\theta}_0\right)e^{-\alpha t}\right]$$

which shows that $\theta(t)$ *increases exponentially with time and is therefore unstable because the restoring moment (springs) is less than the non-restoring moment due to gravity.*



Free undamped vibration single DoF

- **Rayleigh's Energy method to determine natural frequency**
- Recall: Principle of conservation of energy:

$$T_1 + U_1 = T_2 + U_2$$

- Where T_1 and U_1 represent the energy components at the time when the kinetic energy is at its maximum ($\therefore$ U_1=0) and T_2 and U_2 the energy components at the time when the potential energy is at its maximum ($\therefore$ T_2=0)

$$T_1 + 0 = 0 + U_2$$

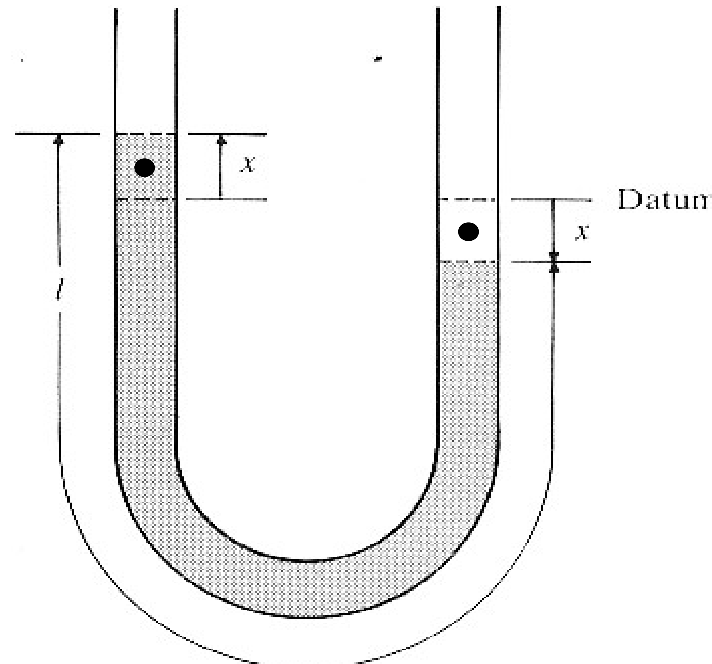
- For harmonic motion

$$T_{max} = U_{max}$$

Free undamped vibration single DoF

- **Rayleigh's Energy method to determine natural frequency: Application example:**

- Find minimum length of mercury u-tube manometer tube so that f_n of fluid column < 2 Hz.
- Determine U_{max} and T_{max}:
- Umax = potential energy of raised fluid column + potential energy of depressed fluid column.



$$U = mg\left.\frac{x}{2}\right|_{raised} + mg\left.\frac{x}{2}\right|_{depressed}$$

$$= (Ax\gamma)\left.\frac{x}{2}\right|_{raised} + (Ax\gamma)\left.\frac{x}{2}\right|_{depressed}$$

$$= A\gamma x^2$$

A : *cross sectional area and* γ : *specific weight of mercury*

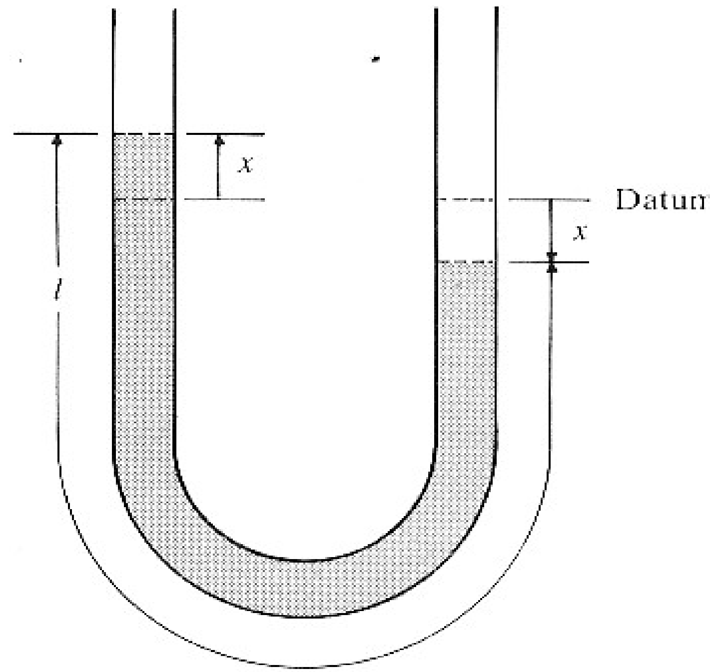
- Kinetic energy:

$$T = \frac{1}{2}(mass\ of\ mercury\ col)\,vel^2$$

$$= \frac{1}{2}\left(\frac{Al\gamma}{g}\right)\dot{x}^2$$

Free undamped vibration single DoF

- **Rayleigh's Energy method to determine natural frequency: Application example:**



- If we assume harmonic motion:

$$x(t) = X\cos(2\pi f_n t) \quad \text{where } X \text{ is the max. displacement}$$

$$\dot{x}(t) = 2\pi f_n X \sin(2\pi f_n t) \quad \text{where } 2\pi f_n X \text{ is the max. velocity}$$

- Substituting for the maximum displacement and velocity:

$$U_{max} = A\gamma X^2 \quad \text{and} \quad T_{max} = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

$$U_{max} = T_{max} \quad \therefore \quad A\gamma X^2 = \frac{1}{2}\left(\frac{Al\gamma}{g}\right)(2\pi f_n)^2 X^2$$

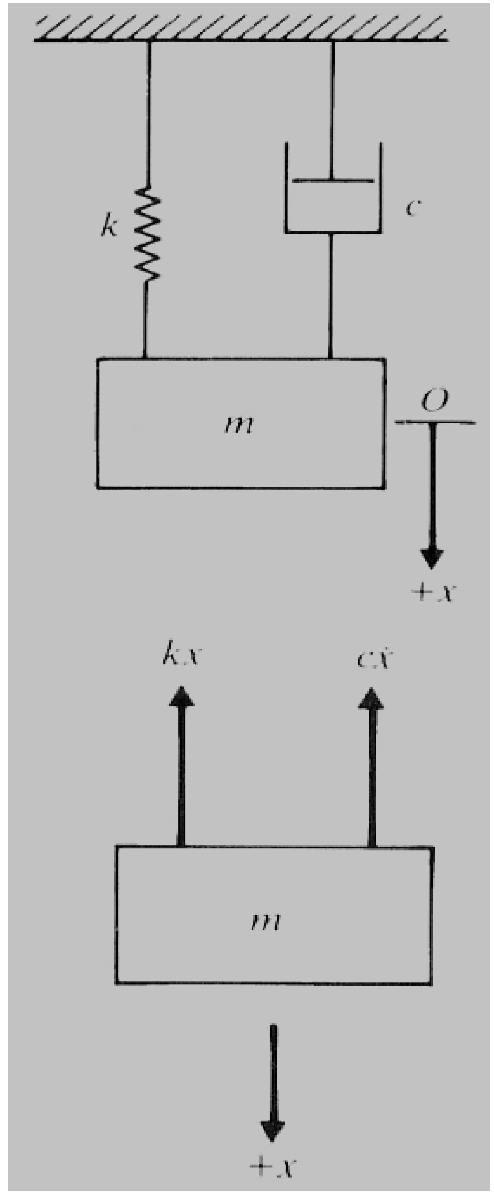
$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{2g}{l}\right)}$$

- Minimum length of column:

$$f_n = \frac{1}{2\pi}\sqrt{\left(\frac{2g}{l}\right)} \le 1.5\ Hz$$

$$l \ge 0.221\ m$$

Free single DoF vibration + viscous damping



- Recall: viscous damping force ∝ velocity:

$$F = -c\dot{x} \qquad c = damping\ constant\ or\ coefficient\ [Ns/m]$$

Applying Newton's second law of motion to obtain the eqn. of motion :

$$m\ddot{x} = -c\dot{x} - kx \qquad or \qquad m\ddot{x} + c\dot{x} + kx = 0$$

If the solution is assumed to take the form :

$$x(t) = Ce^{st} \qquad where\ s = \pm i\omega_n$$

$$then: \quad \dot{x}(t) = sCe^{st} \quad and \quad \ddot{x}(t) = s^2Ce^{st}$$

Substituting for $x, \dot{x}$ and $\ddot{x}$ in the eqn. of motion

$$ms^2 + cs + k = 0$$

The root of the characteristic eqn. are :

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

The two solutions are :

$$x_1(t) = C_1e^{s_1t} \qquad and \qquad x_2(t) = C_2e^{s_2t}$$

Free single DoF vibration + viscous damping

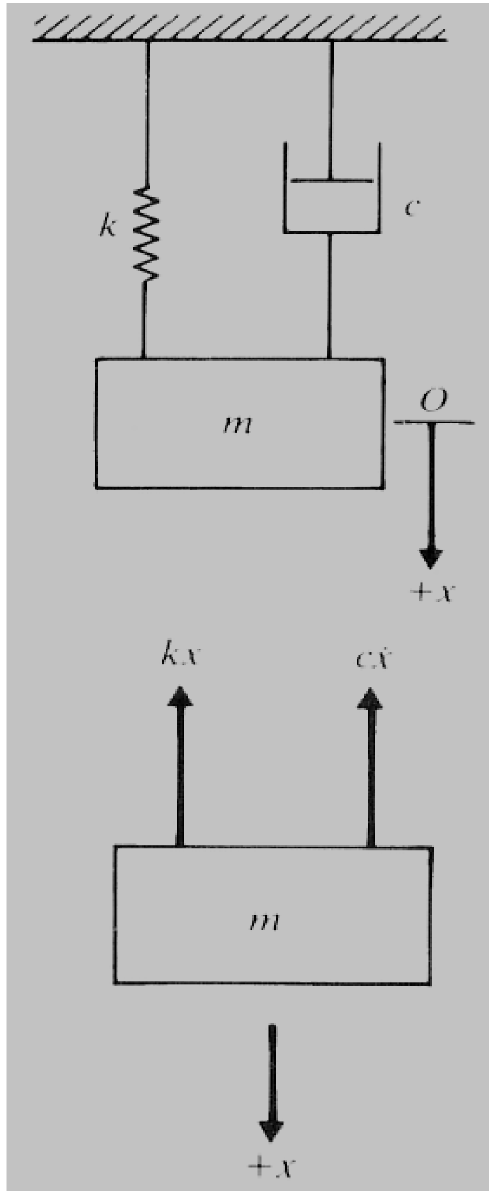
- The general solution to the Eqn. Of motion is:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

or

$$x(t) = C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right\}t}$$

where C_1 and C_2 are arbitrary constants determined from the initial conditions.



Free single DoF vibration + viscous damping

- **Critical damping (c_c):** value of c for which the radical in the general solution is zero:

$$\left(\frac{c_c}{2m}\right)^2-\left(\frac{k}{m}\right)=0 \qquad or \qquad c_c=2m\sqrt{\frac{k}{m}}=2m\omega_n=2\sqrt{km}$$

- **Damping ratio (ζ):** damping coefficient : critical damping coefficient.

$$\zeta=\frac{c}{c_c} \qquad or \qquad \frac{c}{2m}=\frac{c}{c_c}\frac{c_c}{2m}=\zeta\omega_n$$

The roots can be re-written :

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}=\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega_n$$

And the solution becomes :

$$x(t)=C_1e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+C_2e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

- The response x(t) depends on the roots s_1 and s_2 → the behaviour of the system is dependent on the damping ratio **ζ.**

Free single DoF vibration + viscous damping

$$x(t)=C_1e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+C_2e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

- When ζ <1, the system is underdamped. (ζ^2-1) is negative and the roots can be written as:

$$s_1=\left(-\zeta+i\sqrt{1-\zeta^2}\right)\omega_n \quad and \quad s_2=\left(-\zeta-i\sqrt{1-\zeta^2}\right)\omega_n$$

And the solution becomes :

$$x(t)=C_1e^{\left(-\zeta+i\sqrt{1-\zeta^2}\right)\omega_n t}+C_2e^{\left(-\zeta-i\sqrt{1-\zeta^2}\right)\omega_n t}$$

$$x(t)=e^{-\zeta\omega_n t}\left\{C_1e^{\left(i\sqrt{1-\zeta^2}\right)\omega_n t}+C_2e^{\left(-i\sqrt{1-\zeta^2}\right)\omega_n t}\right\}$$

$$x(t)=e^{-\zeta\omega_n t}\left\{(C_1+C_2)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+i(C_1-C_2)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

$$x(t)=e^{-\zeta\omega_n t}\left\{C_1'\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+C_2'\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

$$x(t)=Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right) \quad or \quad x(t)=X_0e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t-\phi_o\right)$$

Where C'_1, C'_2; X, ϕ and X_o, ϕ_o are arbitrary constant determined from initial conditions.

Free single DoF vibration + viscous damping

$$x(t)=e^{-\zeta\omega_n t}\left\{C_1' \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+C_2' \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- For the initial conditions:

$$x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$$

Then

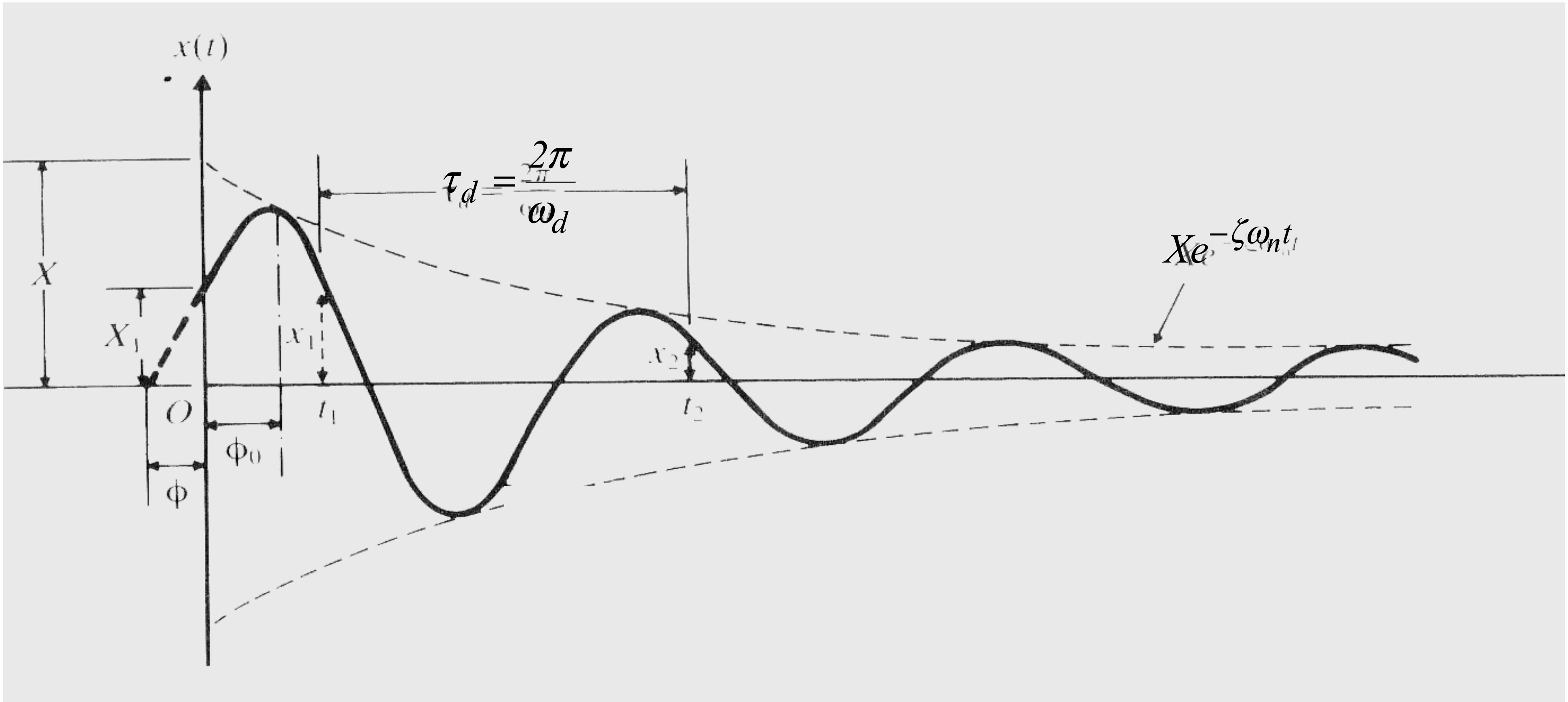
$$C_1'=x_0 \qquad and \qquad C_2'=\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

Therefore the solution becomes

$$x(t)=e^{-\zeta\omega_n t}\left\{x_0 \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- This represents a decaying (damped) harmonic motion with angular frequency $\sqrt{(1-\zeta^2)}\omega_n$ also known as the damped natural frequency. The factor $e^{-(\,)}$ causes the exponential decay.

Free single DoF vibration + viscous damping



Exponentially decaying harmonic – free SDoF vibration with viscous damping .

Underdamped oscillatory motion and has important engineering applications.

Free single DoF vibration + viscous damping

$$x(t) = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \quad or \quad x(t) = X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_o\right)$$

The constants (X, ϕ) and (X_0, ϕ_0) representing the magnitude and phase become :

$$X = X_0 = \sqrt{\left(C_1'\right)^2 + \left(C_2'\right)^2}$$

$$\phi = a\tan\left(\frac{C_1'}{C_2'}\right) \quad and \quad \phi_0 = a\tan\left(-\frac{C_2'}{C_1'}\right)$$

Free single DoF vibration + viscous damping

- When $\zeta = 1$, $c=c_c$, system is critically damped and the two roots to the eqn. of motion become:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

and solution is

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

Applying the initial conditions $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$ *yields*

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

The solution becomes :

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0 \right) t \right] e^{-\omega_n t}$$

- As $t \to \infty$, the exponential term diminished toward zero and depicts ***aperiodic*** motion

Free single DoF vibration + viscous damping

- When $\zeta > 1$, $c>c_c$, system is overdamped and the two roots to the eqn. of motion are real and negative:

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

with $s_2 = s_1$ *and the initial conditions* $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$

the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

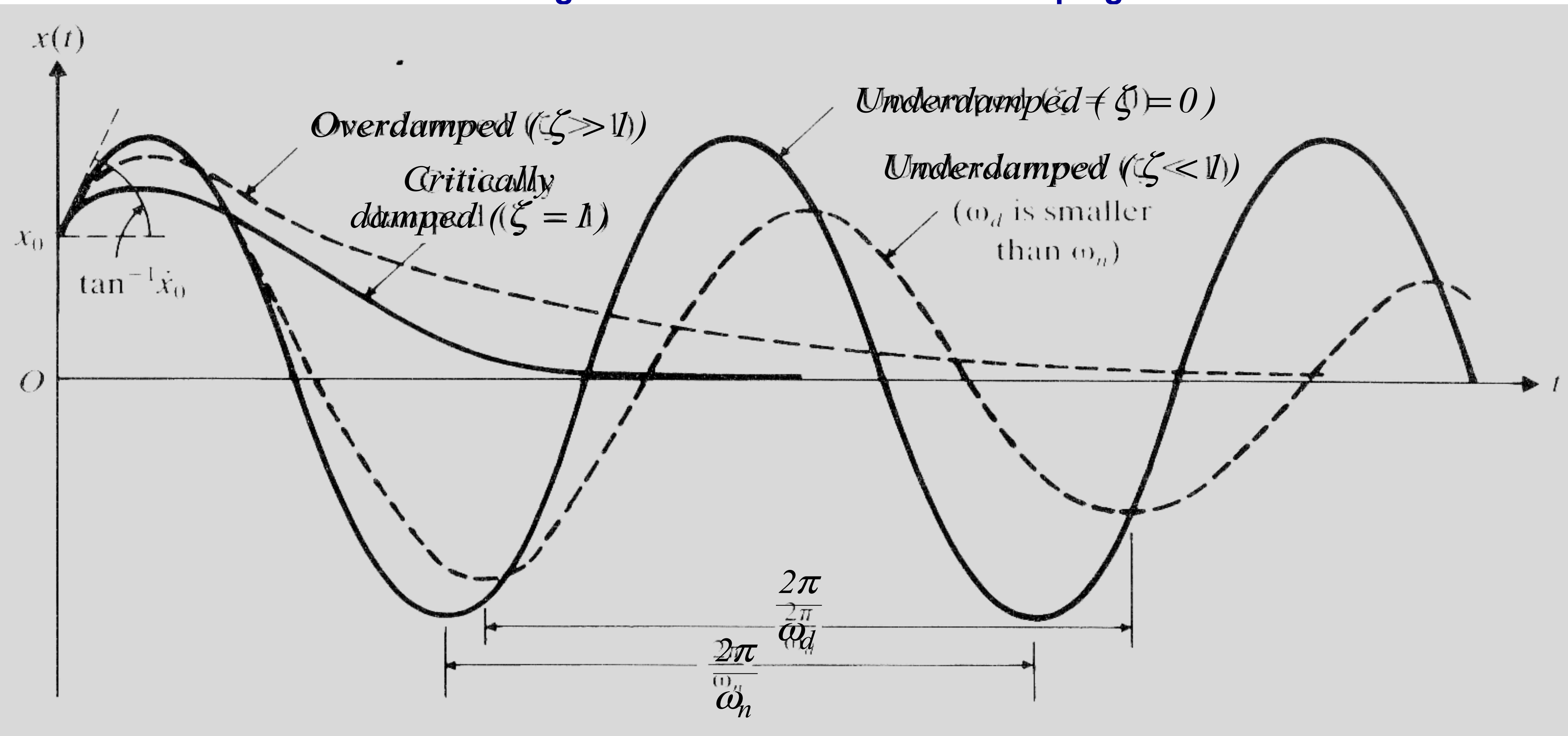
where

$$C_1 = \frac{x_0\omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Which shows ***aperiodic*** motion which diminishes exponentially with time.

Free single DoF vibration + viscous damping



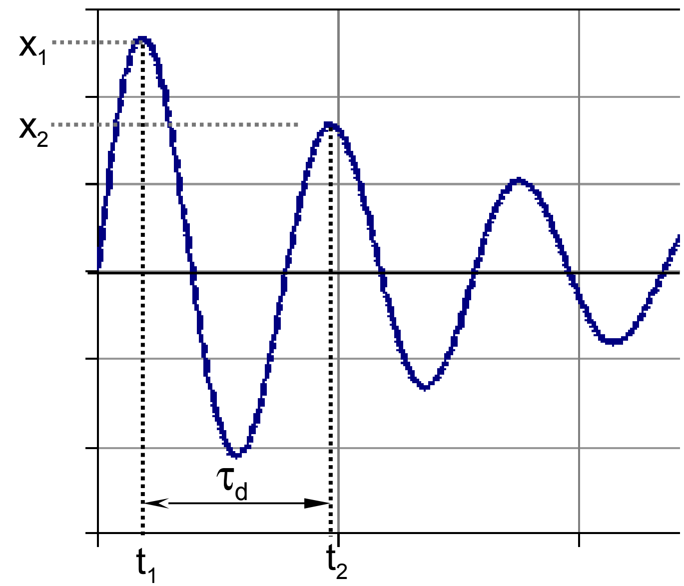
Critically damped systems have lowest required damping for aperiodic motion and mass returns to equilibrium position in shortest possible time.

Free single DoF vibration + viscous damping

Example

Free single DoF vibration + viscous damping

- **Logarithmic decrement:** Natural logarithm of ratio of two successive peaks (or troughs) in an exponentially decaying harmonic response.
- Represents the rate of decay
- Used to determine damping constant from experimental data.
- Using the solution for underdamped systems:



$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)}$$

$$\text{Let} \quad t_2 = t_1 + \tau_d = t_1 + \frac{2\pi}{\omega_d} \quad \text{then}$$

$$\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$$

and

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d}$$

Applying the natural ln on both sides,

the log arithmic decrement δ is obtained :

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\omega_d}$$

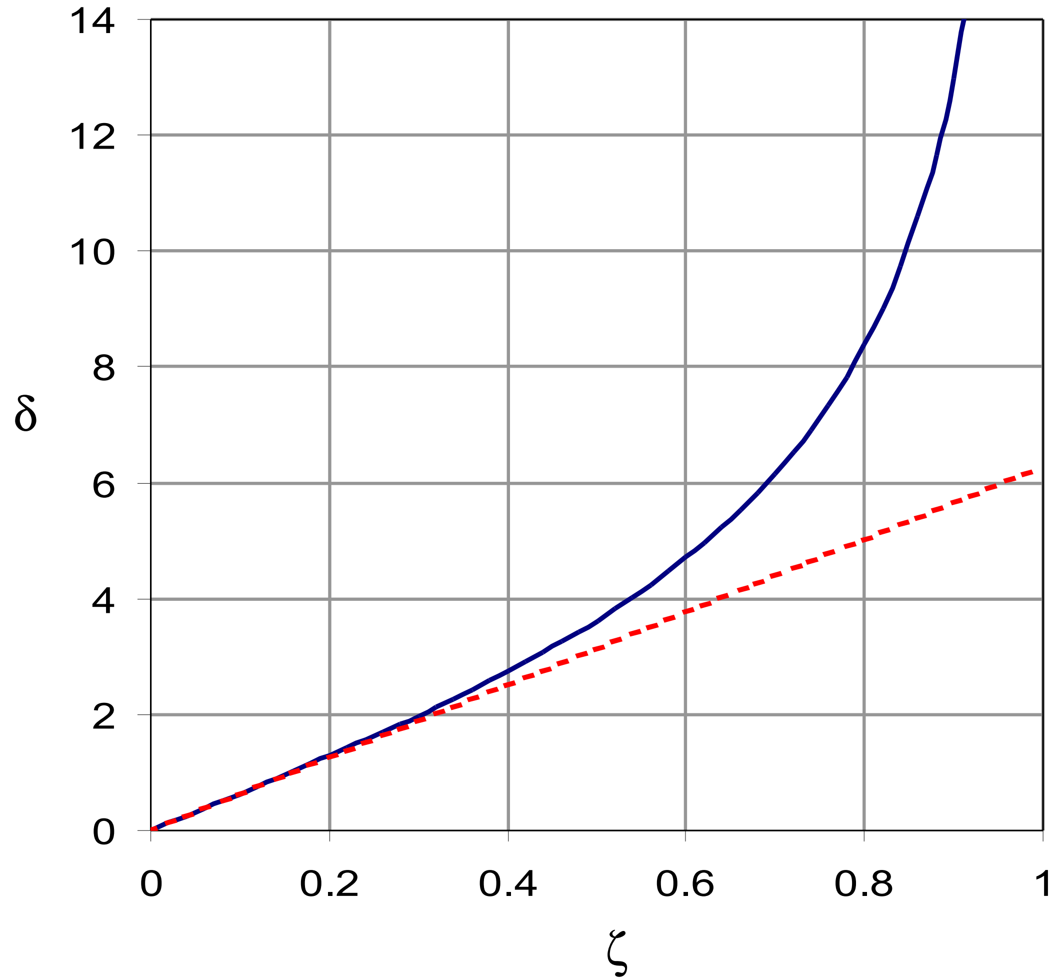
Free single DoF vibration + viscous damping

- **Logarithmic decrement:**

For low damping (ζ << 1)

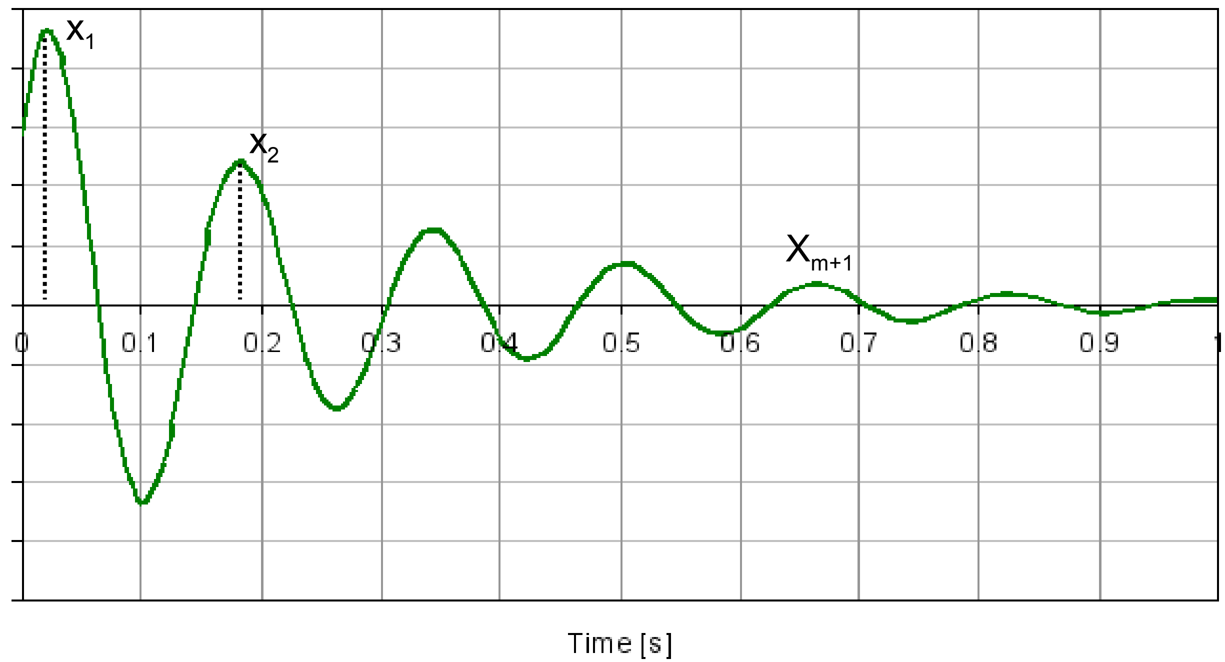
$$\delta = ln\left(\frac{x_1}{x_2}\right) = 2\pi\zeta$$

Valid for ζ < .3



Free single DoF vibration + viscous damping

- **Logarithmic decrement after n cycles:**



- Since the period of oscillation is constant:

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}}$$

Since $\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d}$ *then*

$$\frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n\tau_d}\right)^m = e^{m\zeta\omega_n\tau_d}$$

The log arithmic decrement can therefore be obtained from a number m of successive decaying oscillations

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$$

Free single DoF vibration + Coulomb damping

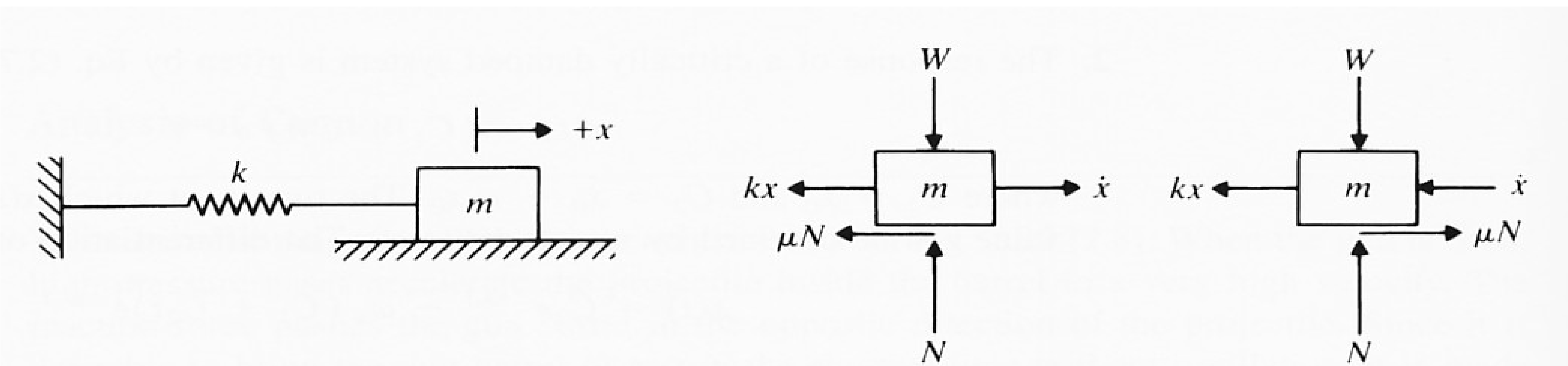
- Coulomb or dry friction dampers are simple and convenient
- Occurs when components slide / rub
- Force proportional to normal force:

$$F = \mu N$$

$$F = \mu m g \qquad \textit{for free-standing systems}$$

$$\textit{where } \mu \textit{ is the coefficient of friction.}$$

- Force acts in opposite direction to velocity and is independent of displacement and velocity.
- Consider SDOF system with dry friction:



Case 1. Case 2.

Free single DoF vibration + Coulomb damping

- Case 1: Mass moves from left to right. x = positive and x’ is positive or x = negative and x’ is positive.
- The eqn. of motion is:

$$m\ddot{x} = -kx - \mu N \quad or \quad m\ddot{x} + kx = -\mu N \quad \neg \; 2^{nd} \text{ order homogeneous DE}$$

For which the general solution is :

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) - \frac{\mu N}{k} \qquad (1)$$

where the frequency of vibration ω_n *is* $\sqrt{\frac{k}{m}}$ *and* A_1 *and* A_2 *are constants dependent on the initial conditions of this portion of the cycle.*

- Case 2: Mass moves from right to left. x = positive and x’ is negative or x = negative and x’ is negative.
- The eqn. of motion is:

$$m\ddot{x} = -kx + \mu N \quad or \quad m\ddot{x} + kx = \mu N$$

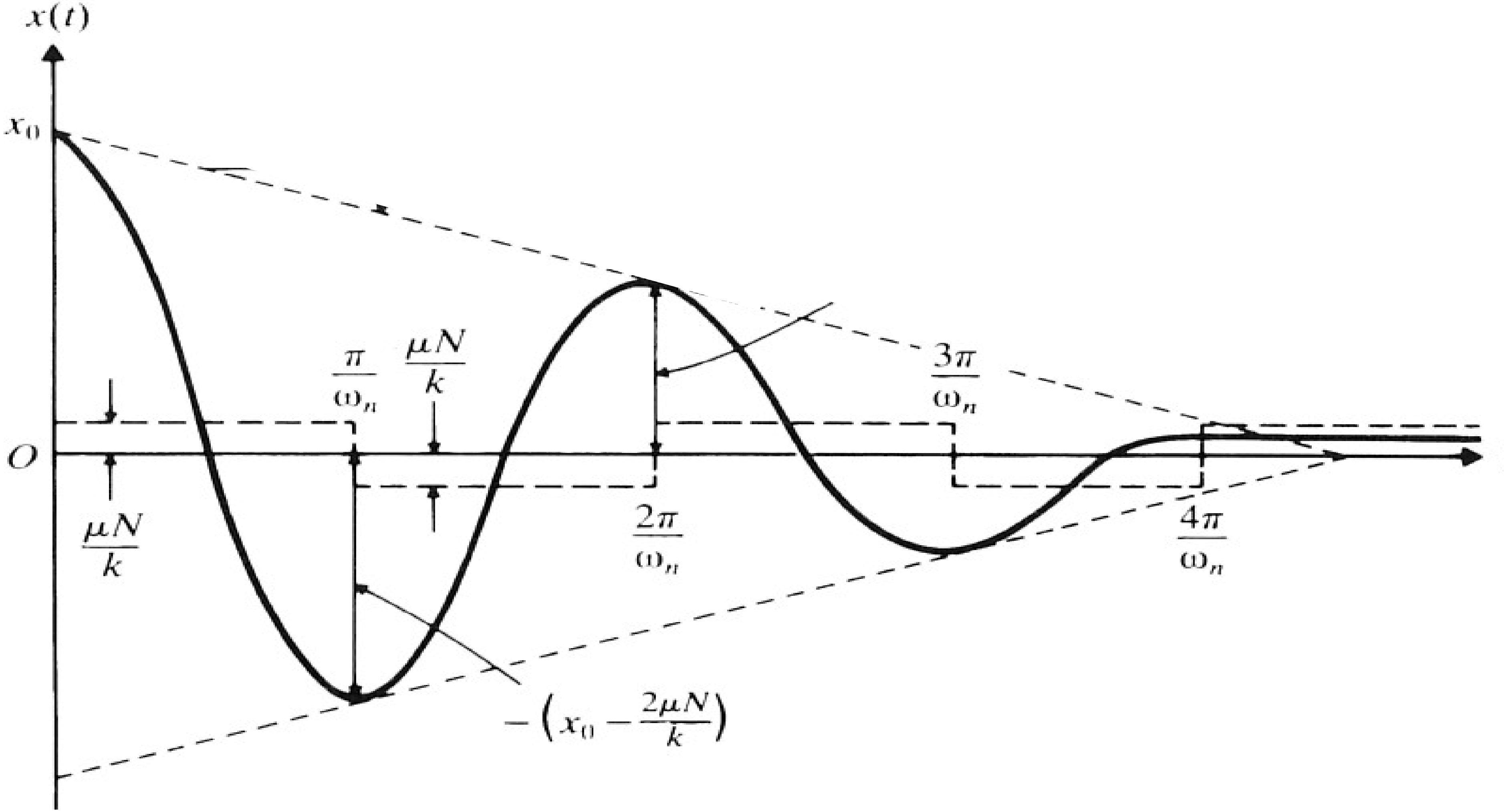
For which the general solution is :

$$x(t) = A_3 \cos(\omega_n t) + A_4 \sin(\omega_n t) + \frac{\mu N}{k} \qquad (2)$$

where the frequency of vibration ω_n *is again* $\sqrt{\frac{k}{m}}$ *and* A_3 *and* A_4 *are constants dependent on the initial conditions of this portion of the cycle.*

Free single DoF vibration + Coulomb damping

- The term μN/k [m] is a constant representing the virtual displacement of the spring k under force μN. The equilibrium position oscillates between +μN/k and -μN/k 1 for each harmonic half cycle of motion.



Free single DoF vibration + Coulomb damping

- To find a more specific solution to the eqn. of motion we apply the simple initial conditions:

$$x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$$

The motion starts from the extreme right (ie. velocity is zero)

Substituting into

$$x(t)=A_3\cos(\omega_n t)+A_4\sin(\omega_n t)+\frac{\mu N}{k} \qquad (2)$$

and

$$\dot{x}(t)=-A_3\omega_n\sin(\omega_n t)+A_4\omega_n\cos(\omega_n t)+0$$

gives

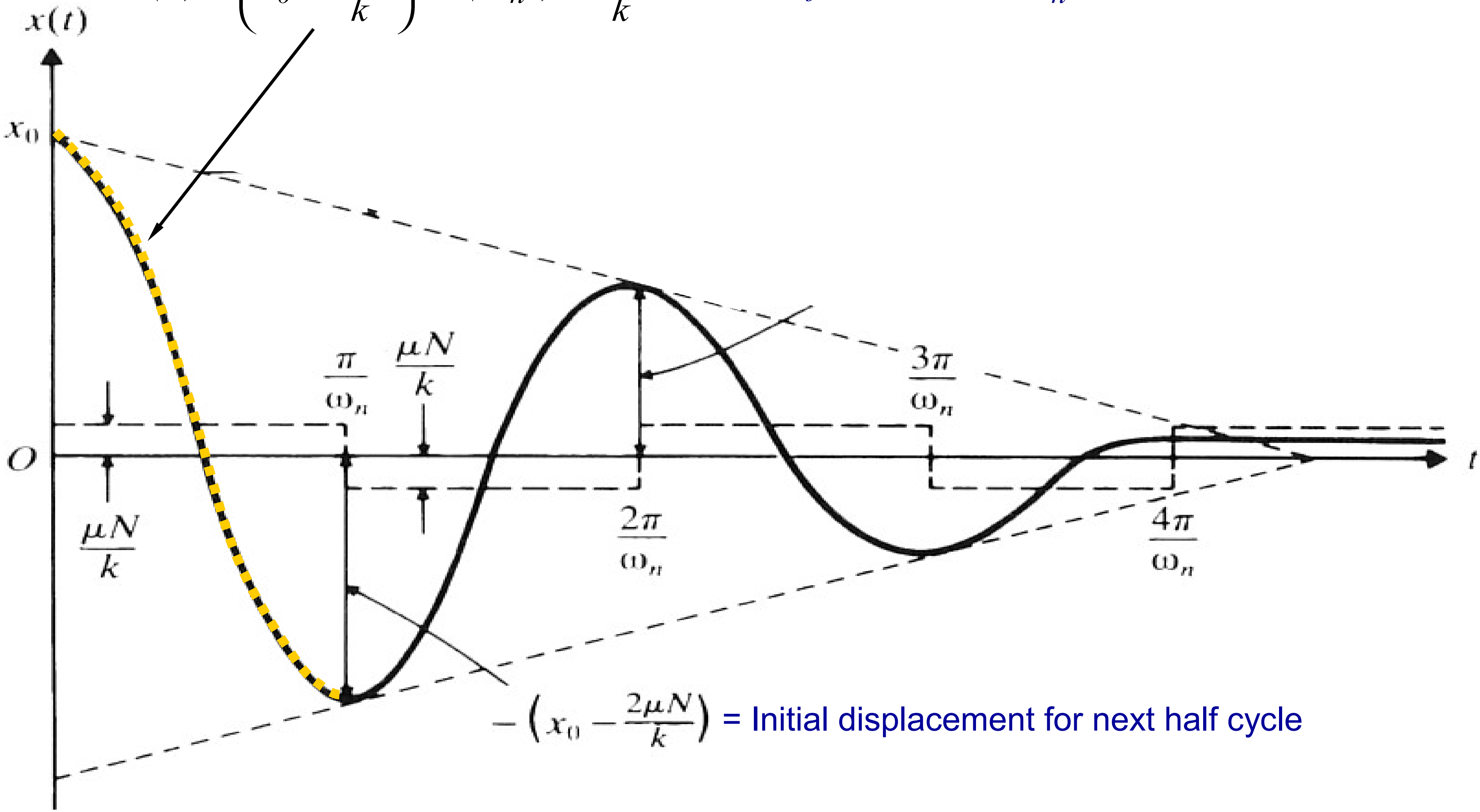
$$A_3=x_0-\frac{\mu N}{k} \quad and \quad A_4=0$$

Eqn.(2) becomes

$$x(t)=\left(x_0-\frac{\mu N}{k}\right)\cos(\omega_n t)+\frac{\mu N}{k} \qquad (2a) \qquad valid\ for\ 0\le t\le\pi/\omega_n$$

Free single DoF vibration + Coulomb damping

$$x(t) = \left(x_0 - \frac{\mu N}{k} \right) \cos(\omega_n t) + \frac{\mu N}{k} \qquad \text{valid for } 0 \le t \le \pi / \omega_n$$



Free single DoF vibration + Coulomb damping

- The displacement at π/ω_n becomes the initial displacement for the next half cycle, x_1.

$$-x_1 = x\left(t = \frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k}\right)\cos(\pi) + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

and the initial velocity $\dot{x}(t=0)$ *is* $= \dot{x}\left(t = \frac{\pi}{\omega_n}\right)$ *in eqn (2a)*

Substituting these initial conditions into eqn.(1)

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) - \frac{\mu N}{k} \qquad (1)$$

and its derivative

$$\dot{x}(t) = -\omega_n A_1 \sin(\omega_n t) + \omega_n A_2 \cos(\omega_n t)$$

gives

$$A_1 = x_0 - \frac{3\mu N}{k} \qquad and \qquad A_2 = 0$$

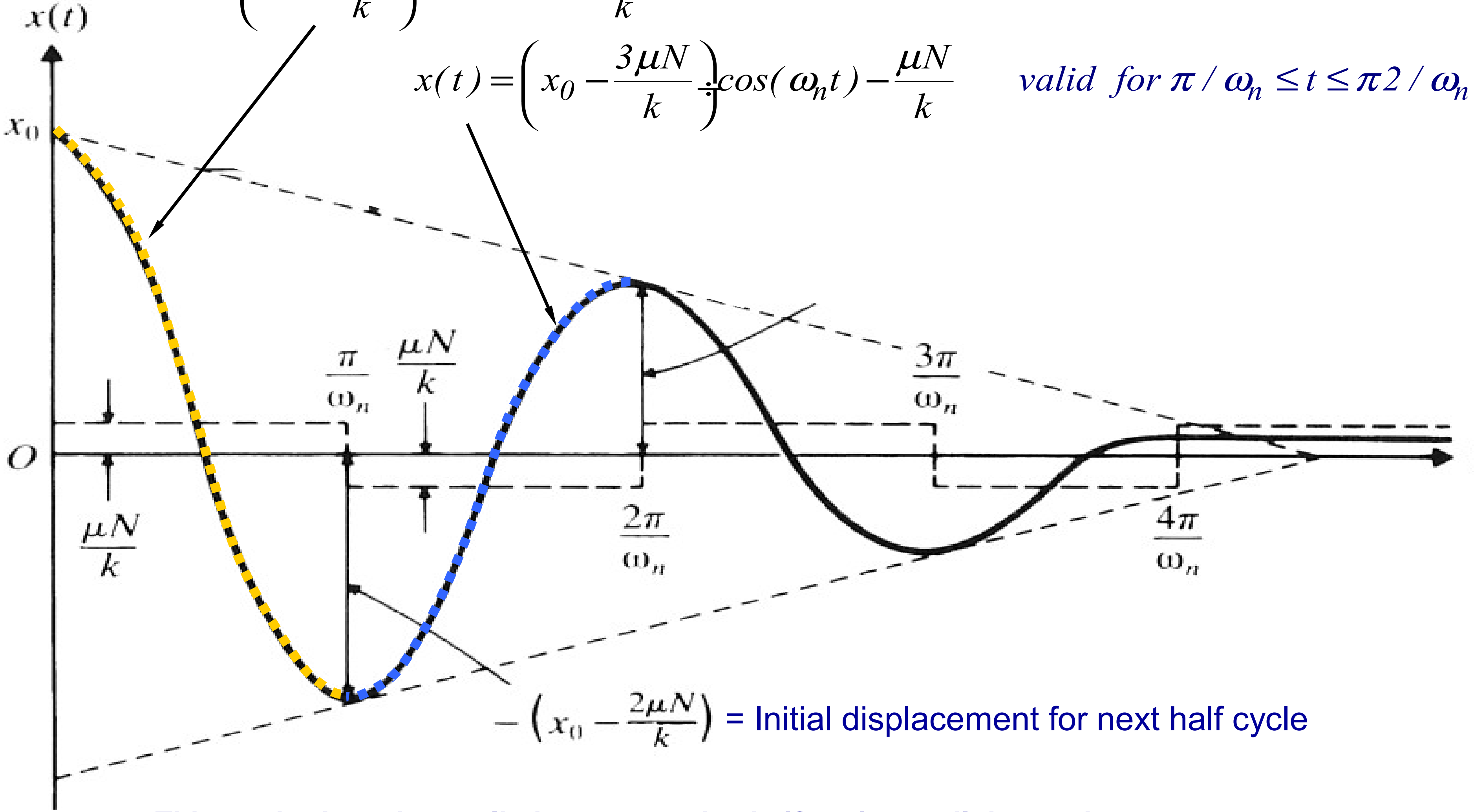
such that eqn.(1) becomes :

$$x(t) = \left(x_0 - \frac{3\mu N}{k}\right)\cos(\omega_n t) - \frac{\mu N}{k} \qquad (1a) \quad valid\ for\ \pi/\omega_n \le t \le \pi 2/\omega_n$$

Free single DoF vibration + Coulomb damping

$$x(t)=\left(x_0-\frac{\mu N}{k}\right)\cos(\omega_n t)+\frac{\mu N}{k} \qquad \text{valid for } 0\le t\le \pi/\omega_n$$

$$x(t)=\left(x_0-\frac{3\mu N}{k}\right)\cos(\omega_n t)-\frac{\mu N}{k} \qquad \text{valid for } \pi/\omega_n \le t\le \pi 2/\omega_n$$



This method can be applied to successive half cycles until the motion stops.

Free single DoF vibration + Coulomb damping

- During each half period π/ω_n the reduction in magnitude (peak height) is $2\mu N/k$
- Any two succesive peaks are related by:

$$x_m = x_{m-1} - \left(\frac{4\mu N}{k}\right)$$

- The motion will stop when $x_n < \mu N/k$
- The total number of half vibration cycles, *r*, is obtained from:

$$x_0 - r\left(\frac{2\mu N}{k}\right) \leq \left(\frac{\mu N}{k}\right)$$

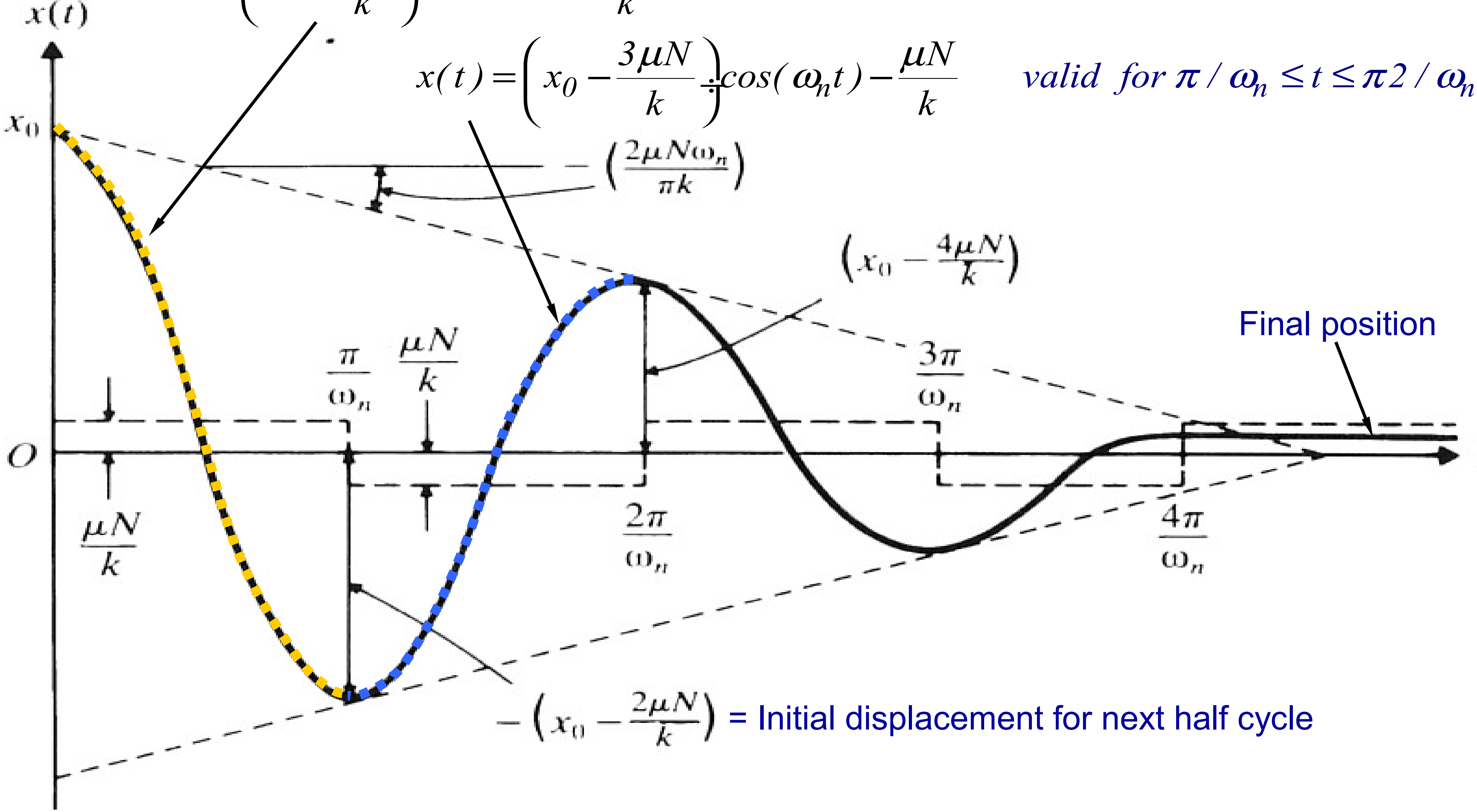
or

$$r \geq \left\{\frac{x_0 - \frac{\mu N}{k}}{\left(\frac{2\mu N}{k}\right)}\right\}$$

Free single DoF vibration + Coulomb damping

$$x(t)=\left(x_0-\frac{\mu N}{k}\right)\cos(\omega_n t)+\frac{\mu N}{k} \qquad \textit{valid for } 0\le t\le \pi/\omega_n$$

$$x(t)=\left(x_0-\frac{3\mu N}{k}\right)\cos(\omega_n t)-\frac{\mu N}{k} \qquad \textit{valid for } \pi/\omega_n\le t\le \pi 2/\omega_n$$



Free single DoF vibration + Coulomb damping

- Important features of Coulomb damping:
  1. The equation of motion is nonlinear (cf. linear for viscous damping)
  2. Coulomb damping **does not** alter the system's natural frequency (cf. damped natural frequency for viscous damping).
  3. The motion is always periodic (cf. overdamped for viscous systems)
  4. Amplitude reduces linearly (cf. exponential decay for viscous systems)
  5. System eventually comes to rest – number of vibration cycles finite (cf. sustained vibration with viscous damping)
  6. The final position is the permanent displacement (not equilibrium) equivalent to the friction force (cf. approaches zero for viscous systems)

Forced (harmonically excited) single DoF vibration

- External energy supplied to system as applied force or imposed motion (displacement, velocity or acceleration)
- This section deals only with ***harmonic excitation*** which results in ***harmonic response*** (cf. steady-state or transient response from non-harmonic excitation).
- Harmonic forcing function takes the form:

$$F(t) = F_0\, e^{i(\omega t + \phi)} \qquad or \qquad F(t) = F_0 \cos(\omega t + \phi) \qquad or \qquad F(t) = F_0 \sin(\omega t + \phi)$$

- Where F_0 is the amplitude, ω the frequency and ϕ the phase angle.
- The response of a linear system subjected to harmonic excitation is also harmonic.
- The response amplitude depends on the ratio of the excitation frequency to the natural frequency.
- Some "common" harmonic forcing functions are:
  - Rotating machine / element with (large) residual imbalance
  - Regular shedding of vortices caused by laminar flow across slender structures (VIV) – ie: chimneys, bridges, overhead cables, mooring cables, tethers, pylons…
  - Vehicle travelling on pavement corrugations or sinusoidal surfaces
  - Structures excited by regular (very narrow banded) ocean / water waves

Forced (harmonically excited) single DoF vibration

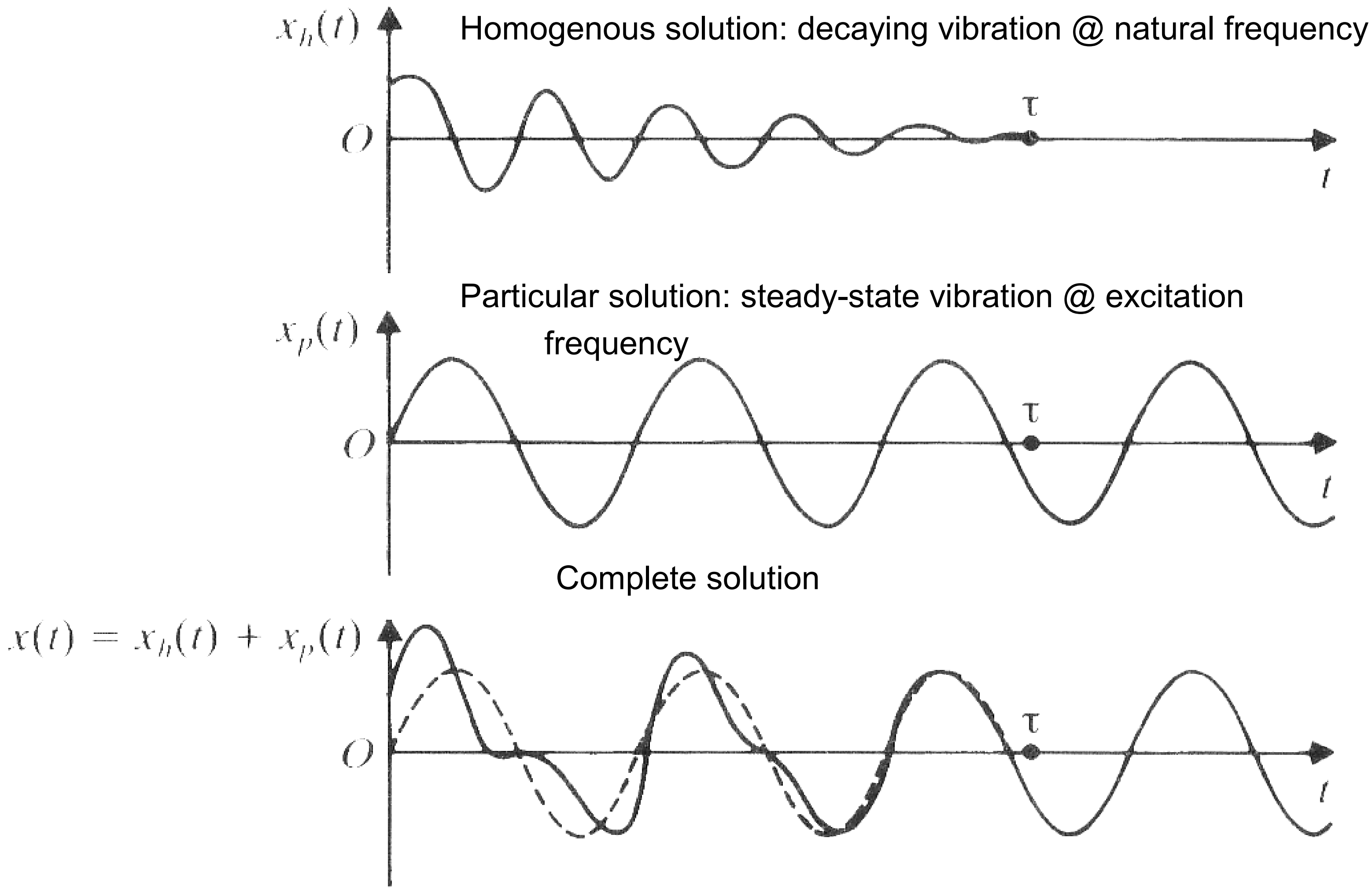
- Equation of motion when a force is applied to a viscously damped SDOF system is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad \neg \quad non\,hom\,ogeneous\ differential\ eqn.$$

- The general solution to a nonhomogeneous DE is the sum if the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$.
- The homogeneous solution represents the solution to the free SDOF which is known to decay over time for all conditions (underdamped, critically damped and overdamped).
- The general solution therefore reduces to the particular solution $x_p(t)$ which represents the steady-state vibration which exists as long as the forcing function is applied.

Forced (harmonically excited) damped single DoF vibration

- Example of solution to harmonically excited damped SDOF system:



Forced (harmonically excited) single DoF vibration – undamped.

- Let the forcing function acting on the mass of an undamped SDOF system be:

$$F(t) = F_0 \cos(\omega t)$$

- The eqn. of motion reduces to:

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

- Where the homogeneous solution is:

$$x_h(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

$$\text{where } \omega_n = \sqrt{k/m}$$

- As the excitation is harmonic, the particular solution is also harmonic with the same frequency:

$$x_p(t) = X \cos(\omega t)$$

- Substituting $x_p(t)$ in the eqn. of motion and solving for X gives:

$$X = \frac{F_0}{k - m\omega^2}$$

- The complete solution becomes

$$x(t) = x_h(t) + x_p(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \frac{F_0}{k - m\omega^2} \cos(\omega t)$$

Forced (harmonically excited) single DoF vibration – undamped.

- Applying the initial conditions $x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$ gives:

$$C_1 = x_0 - \frac{F_0}{k-m\omega^2} \qquad and \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

- The complete solution becomes:

$$x(t) = \left(x_0 - \frac{F_0}{k-m\omega^2} \right) cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} \right) sin(\omega_n t) + \frac{F_0}{k-m\omega^2} cos(\omega t)$$

- The maximum amplitude of the steady-state solution can be written as:

$$\frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n} \right)^2} \qquad where\ \delta_{st} = \frac{F_0}{k}$$

- X/δ_{st} is the ratio of the dynamic to the static amplitude and is known as the ***amplification factor*** or ***amplification ratio*** and is dependent on the frequency ratio $r = \omega/\omega_n$.

Forced (harmonically excited) single DoF vibration – undamped.

- When $\omega/\omega_n < 1$ the denominator of the steady-state amplitude is positive and the amplification factor increases as ω approaches the natural frequency ω_n. The response is ***in-phase*** with the excitation.

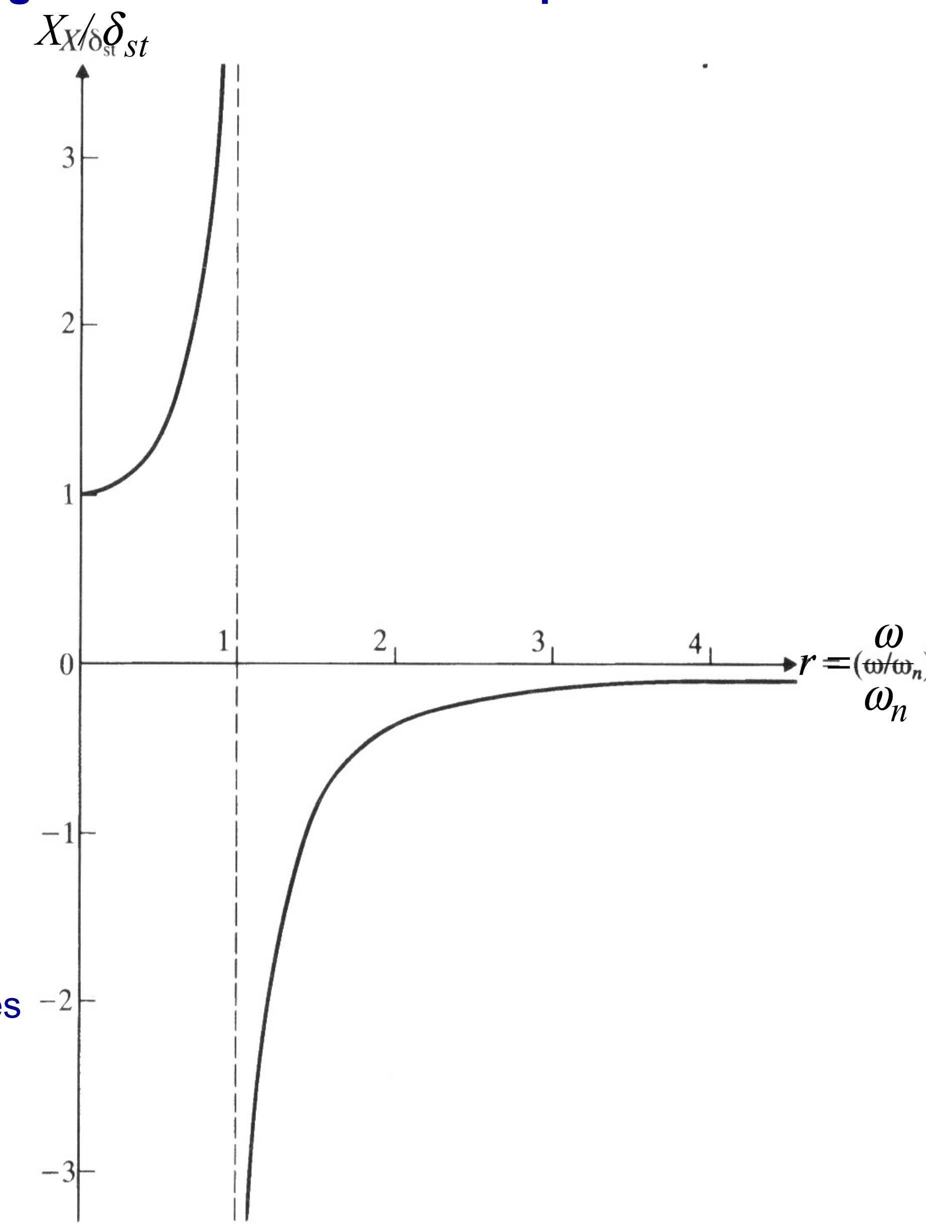
- When $\omega/\omega_n > 1$ the denominator of the steady-state amplitude is negative an the amplification factor is redefined as:

$$\frac{X}{\delta_{st}} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$

and the steady – state response becomes :

$$x_p(t) = -X\cos(\omega t)$$

which shows that the response is out-of-phase with the excitation and decreases ($\rightarrow$ zero) as ω increases ($\rightarrow \infty$)



Forced (harmonically excited) single DoF vibration – undamped.

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$$\frac{X}{\delta_{st}} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$

$$\textit{and the steady-state response becomes:}$$

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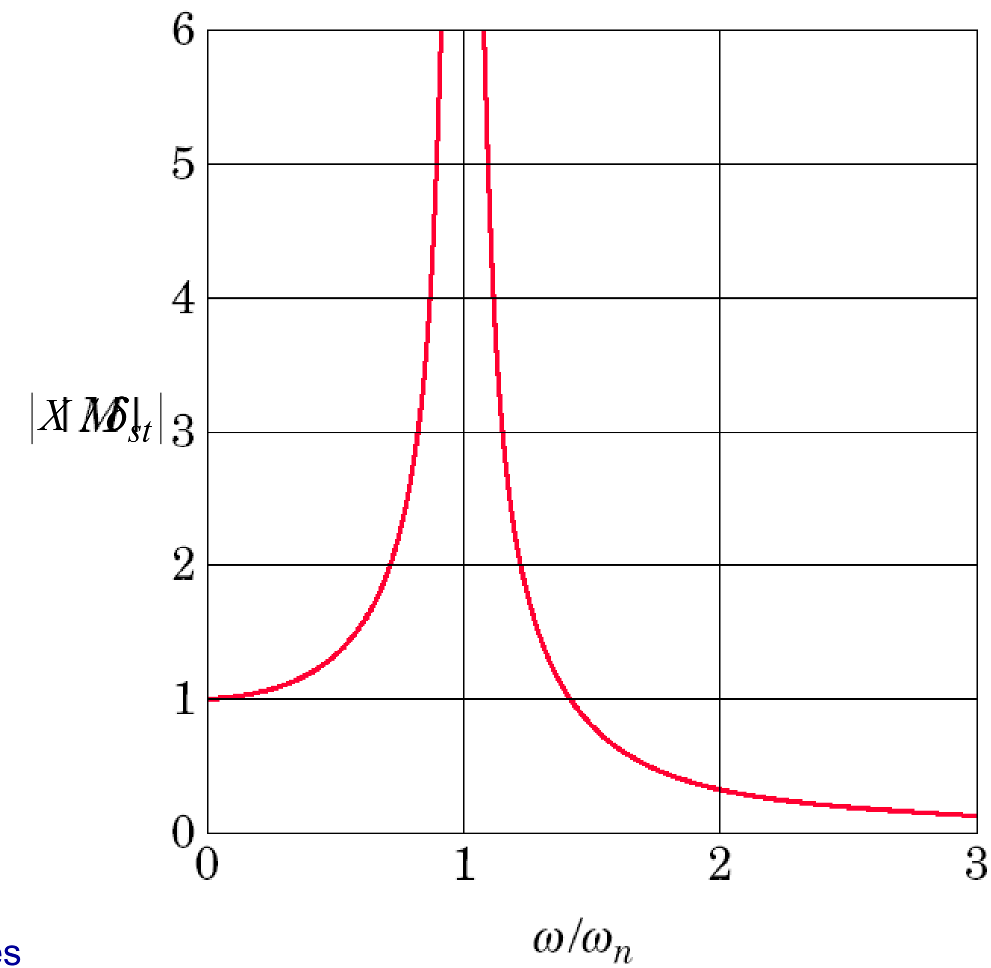
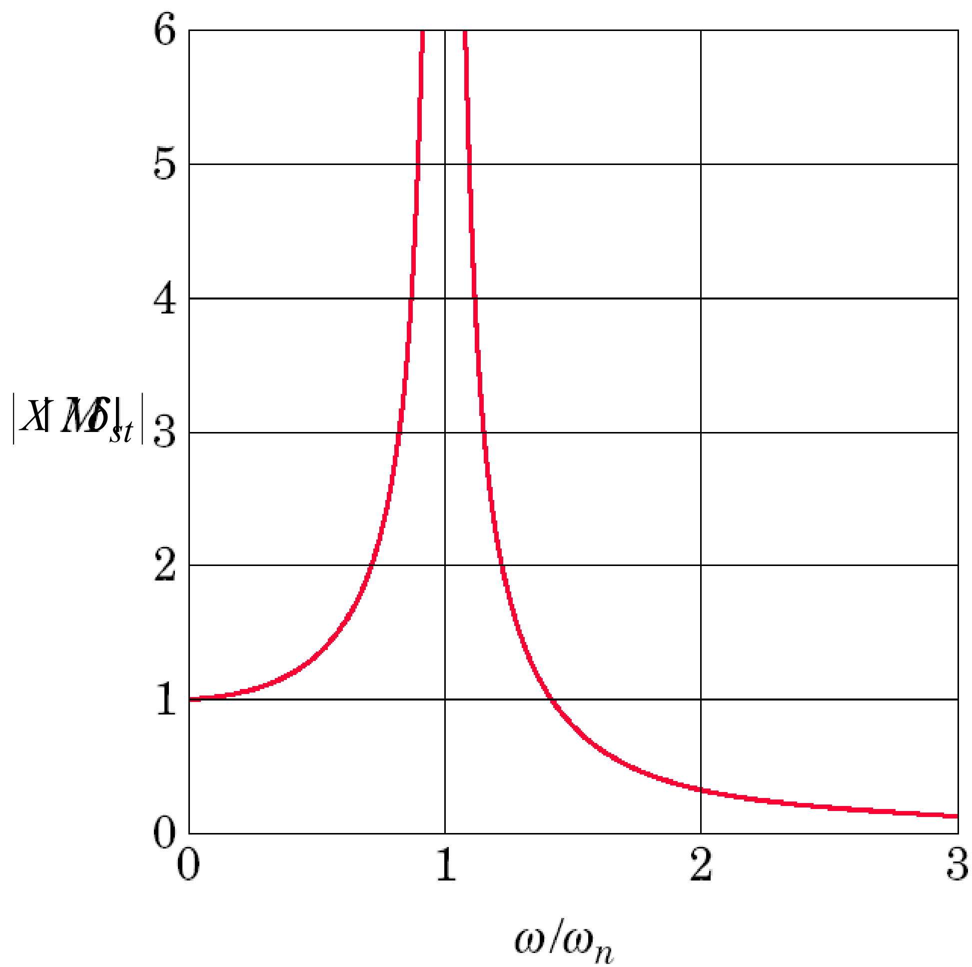


Forced (harmonically excited) single DoF vibration – undamped.

- When ω/ω_n = 1 the denominator of the steady-state amplitude is zero an the response becomes infinitely large. This condition when $\omega=\omega_n$ is known as resonance.



Forced (harmonically excited) single DoF vibration – undamped.

- The complete solution

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin(\omega_n t) + \frac{F_0}{k - m\omega^2} \cos(\omega t)$$

can be written as:

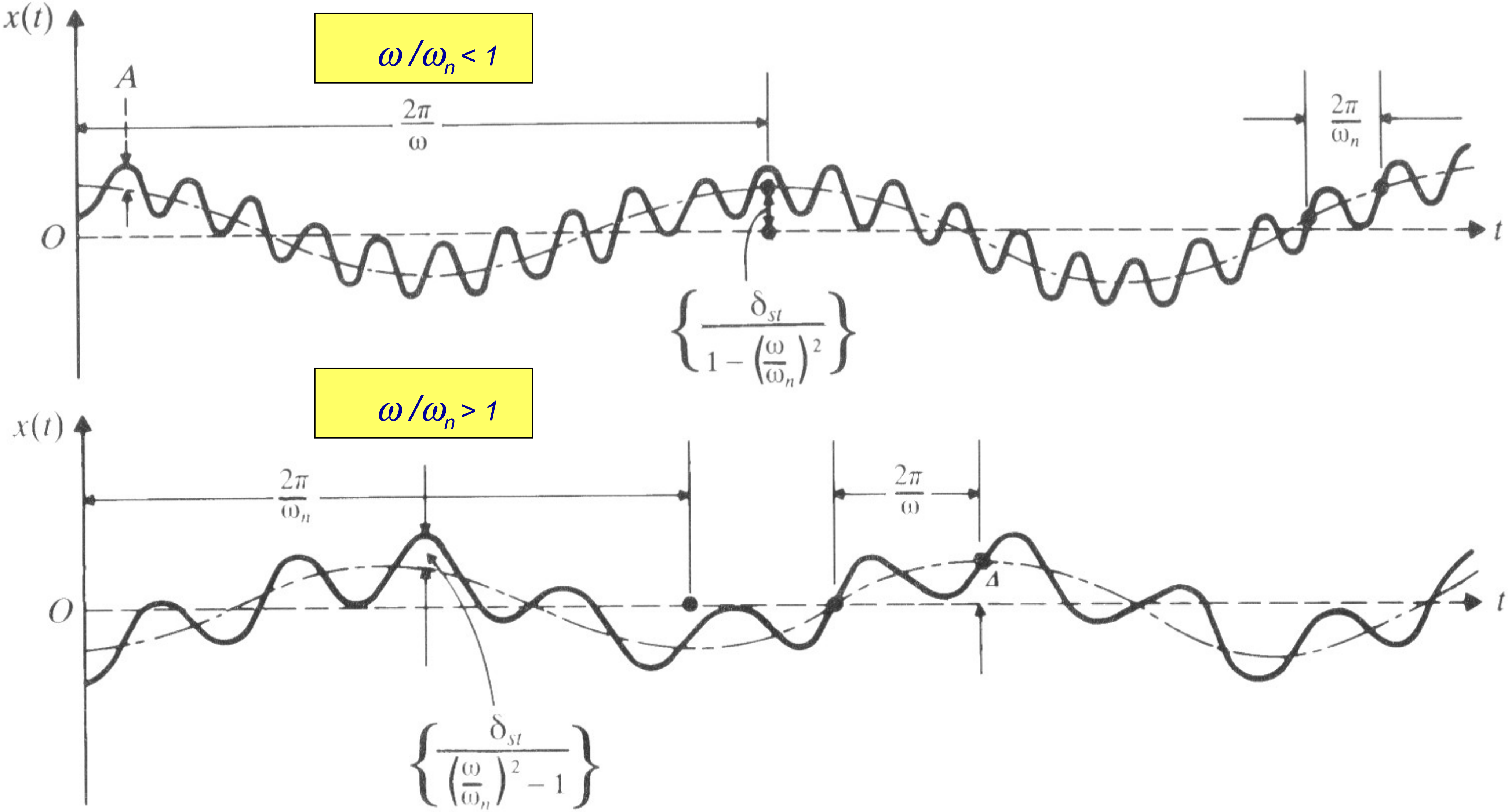
$$x(t) = A\cos(\omega_n t + \phi) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \cos(\omega t) \qquad for\ \omega / \omega_n < 1$$

$$x(t) = A\cos(\omega_n t + \phi) - \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \cos(\omega t) \qquad for\ \omega / \omega_n > 1$$

where A and ϕ are functions of x_0 and $\dot{x}_0$ as before.

- The complete solution is a sum of two cosines with frequencies corresponding to the natural and forcing (excitation) frequencies.

Forced (harmonically excited) single DoF vibration – undamped.



Forced (harmonically excited) single DoF vibration – undamped.

- When the excitation frequency ω is close but not exactly equal to the natural frequency ω_n beating may occur.
- Letting the initial conditions $x_0 = x'_0 = 0$, the complete solution:

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} \right) sin(\omega_n t) + \frac{F_0}{k - m\omega^2} cos(\omega t)$$

reduces to :

$$x(t) = \frac{(F_0 / m)}{(\omega_n^2 - \omega^2)} [cos(\omega_n t) - cos(\omega t)] = \frac{(F_0 / m)}{(\omega_n^2 - \omega^2)} \left[2\,sin\left\{ \left(\frac{\omega + \omega_n}{2} \right) t \right\} \times sin\left\{ \left(\frac{\omega - \omega_n}{2} \right) t \right\} \right]$$

If we let the excitation frequency be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon$$

where ε is a small positive number. Then

$$\omega_n \approx \omega \quad and \quad \omega_n + \omega = 2\omega$$

therefore :

$$(\omega_n - \omega)(\omega_n + \omega) = \omega_n^2 - \omega^2 = 4\varepsilon\omega$$

Substituting for $\omega_n - \omega$, $\omega_n + \omega$ and $\omega_n^2 - \omega^2$ in the complete solution yields :

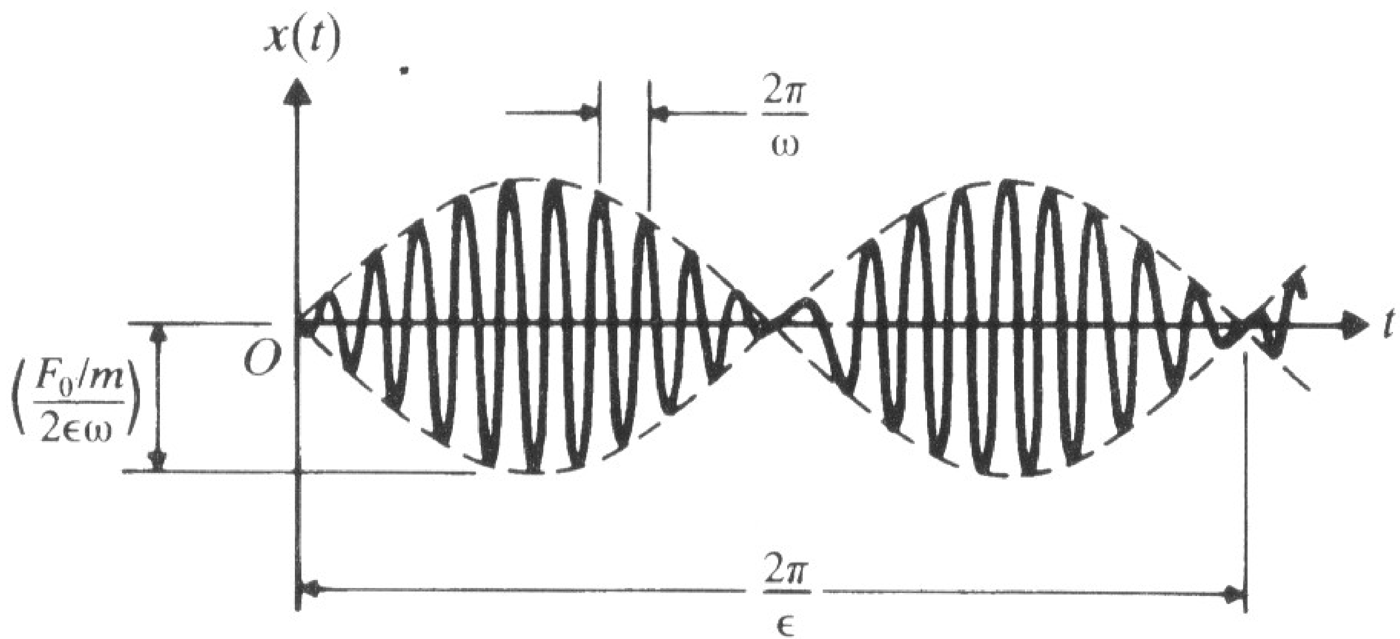
$$x(t) = \frac{(F_0 / m)}{(2\varepsilon\omega)} sin(\varepsilon t) \times sin(\omega t)$$

Forced (harmonically excited) single DoF vibration – undamped.

$$x(t) = \frac{(F_0 / m)}{(2\varepsilon\omega)} \sin(\varepsilon t) \times \sin(\omega t)$$

- Since ε is small, sin(ε t) has a long period. The solution can then be considered as harmonic motion with a principal frequency ω an a variable amplitude equal to

$$X(t) = \frac{(F_0 / m)}{(2\varepsilon\omega)} \sin(\varepsilon t)$$



Forced (harmonically excited) single DoF vibration – Damped.

- **Steady-state Solution**
- If the forcing function is harmonic:

$$F(t) = F_0 \cos(\omega t)$$

- The equation of motion of a SDOF system with viscous damping is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

- The steady-state response is given by the particular solution which is also expected to be harmonic:

$$x_p(t) = X \cos(\omega t - \phi)$$

where the amplitude X and the phase angle ϕ are to be determined

Forced (harmonically excited) single DoF vibration – Damped.

- Substituting x_p into the steady-state eqn. of motion yields:

$$X\left[\left(k-m\omega^2\right)cos(\omega t-\phi)-c\omega\,sin(\omega t-\phi)\right]=F_0\,cos(\omega t)$$

applying the trigonometric relationships :

$$cos(\omega t-\phi)=cos(\omega t)cos(\phi)+sin(\omega t)sin(\phi)$$

$$sin(\omega t-\phi)=sin(\omega t)cos(\phi)-cos(\omega t)sin(\phi)$$

we obtain :

$$X\left[\left(k-m\omega^2\right)cos(\phi)+c\omega\,sin(\phi)\right]=F_0$$

$$X\left[\left(k-m\omega^2\right)sin(\phi)-c\omega\,cos(\phi)\right]=0$$

which gives :

$$X=\frac{F_0}{\left[\left(k-m\omega^2\right)^2-(c\omega)^2\right]^{1/2}}\qquad and\qquad \phi=a\,tan\left(\frac{c\omega}{k-m\omega^2}\right)$$

for the particular solution

$$x_p(t)=X\,cos(\omega t-\phi)$$

Forced (harmonically excited) single DoF vibration – Damped.

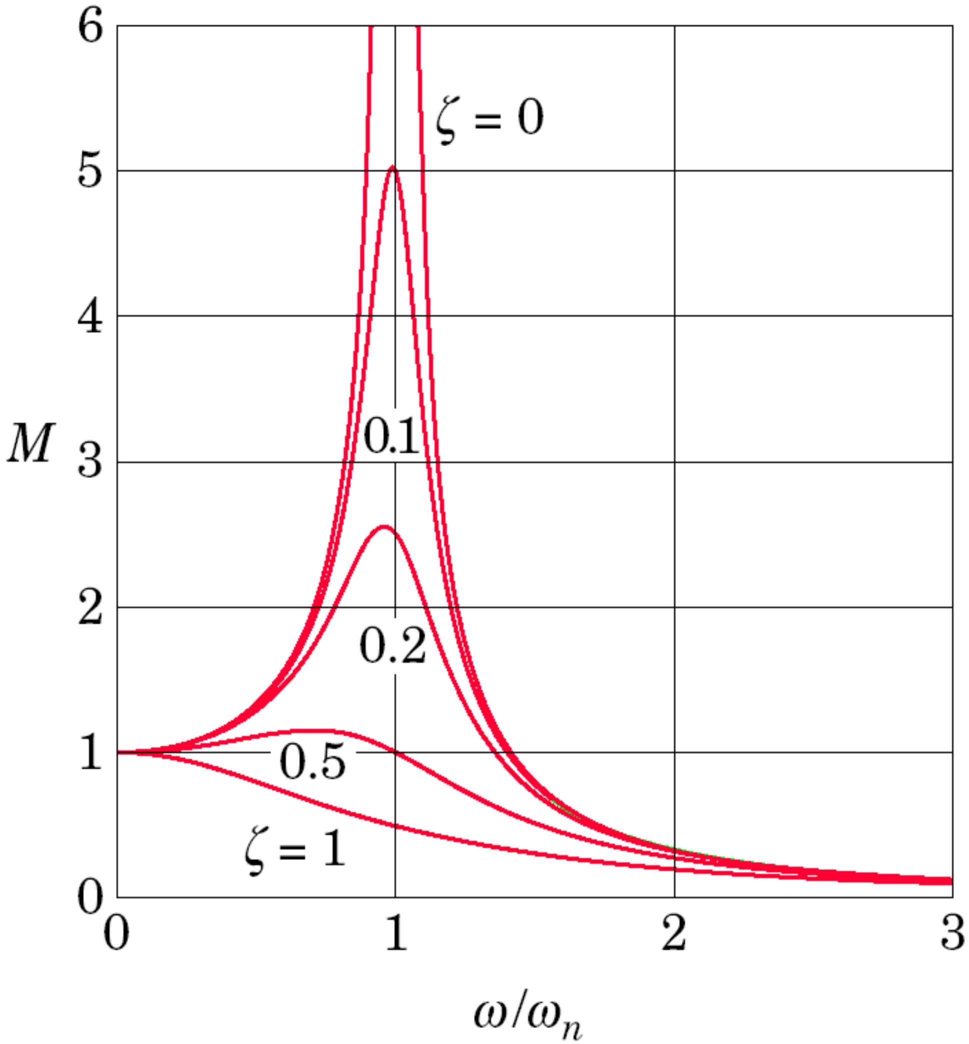
- Alternatively, the amplitude and phase can be written in terms of the frequency ratio $r = \omega/\omega_n$ and the damping coefficient ζ:

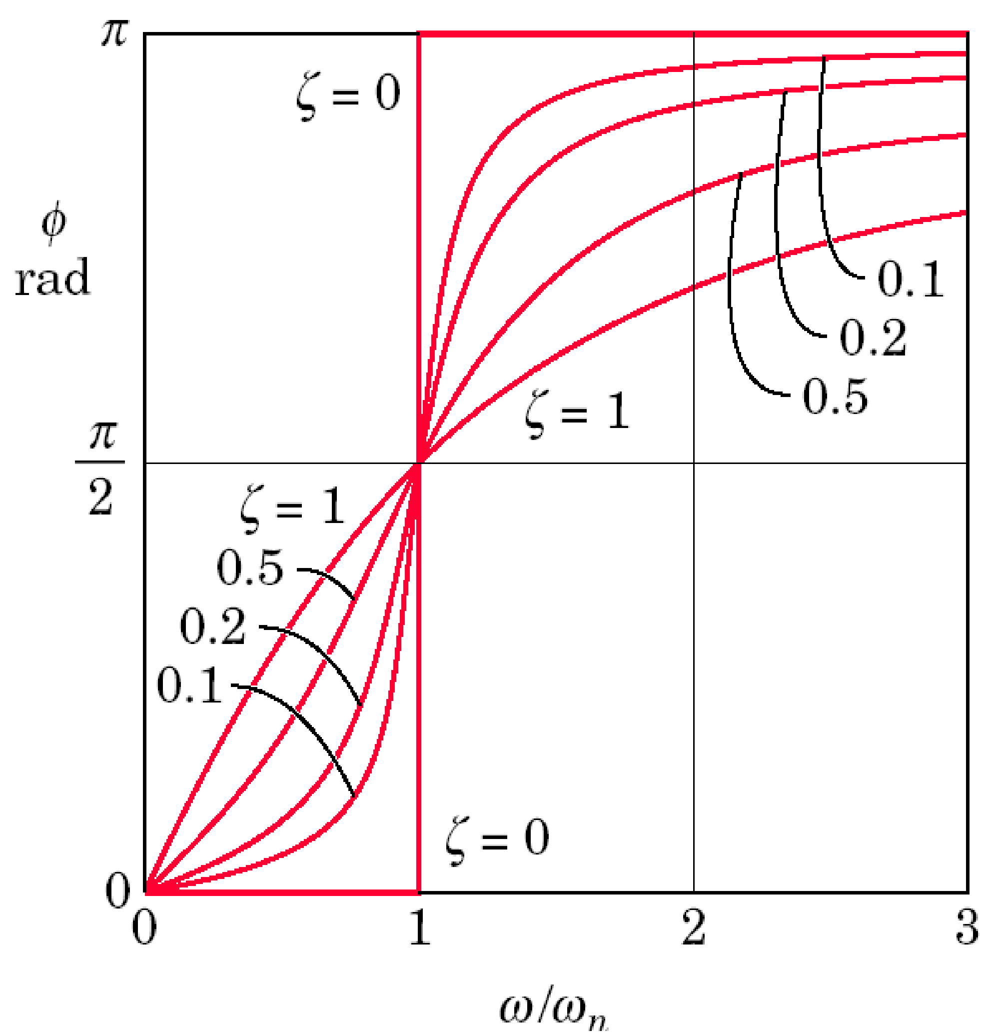
$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\left\{\left[1-r^2\right]^2 + \left[2\zeta r\right]^2\right\}^{1/2}}$$

$$\phi = a\tan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$

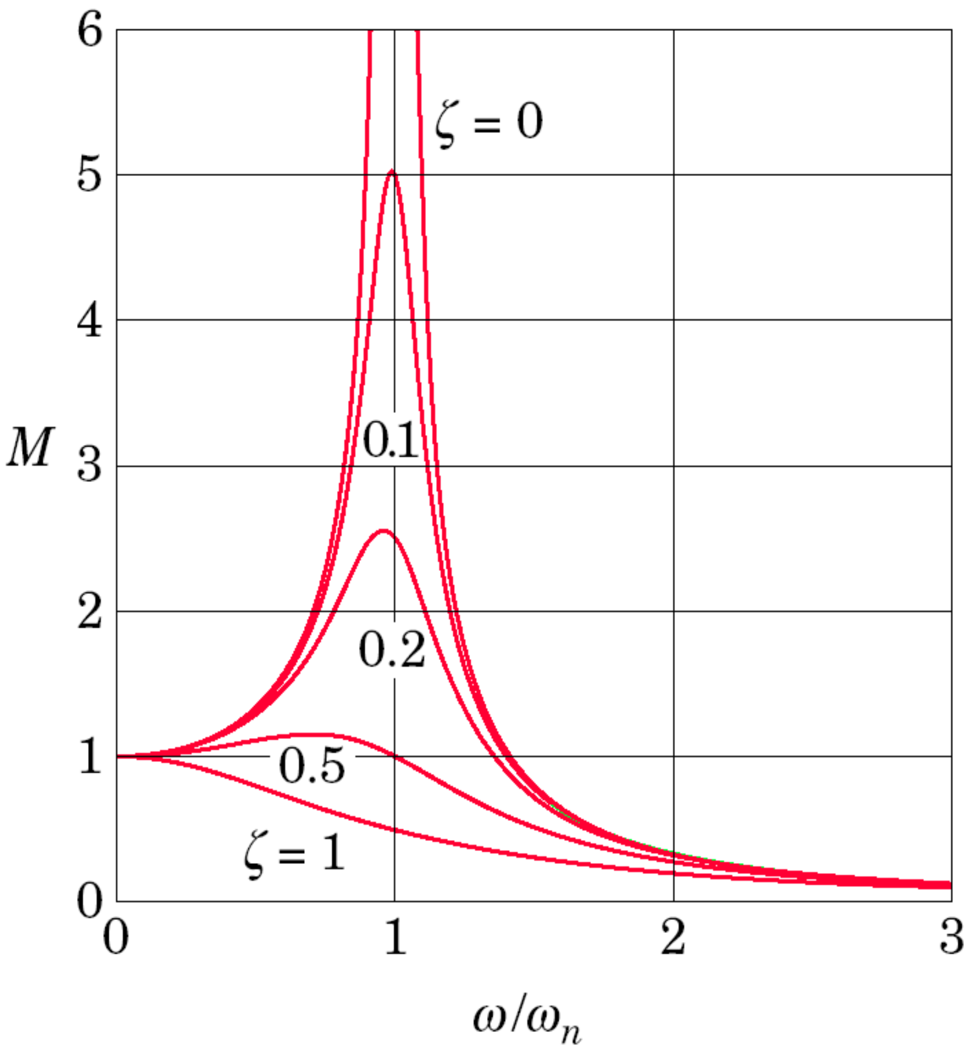
Forced (harmonically excited) single DoF vibration – Damped.

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-r^2\right]^2+\left[2\zeta r\right]^2\right\}^{1/2}} \qquad \phi = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$



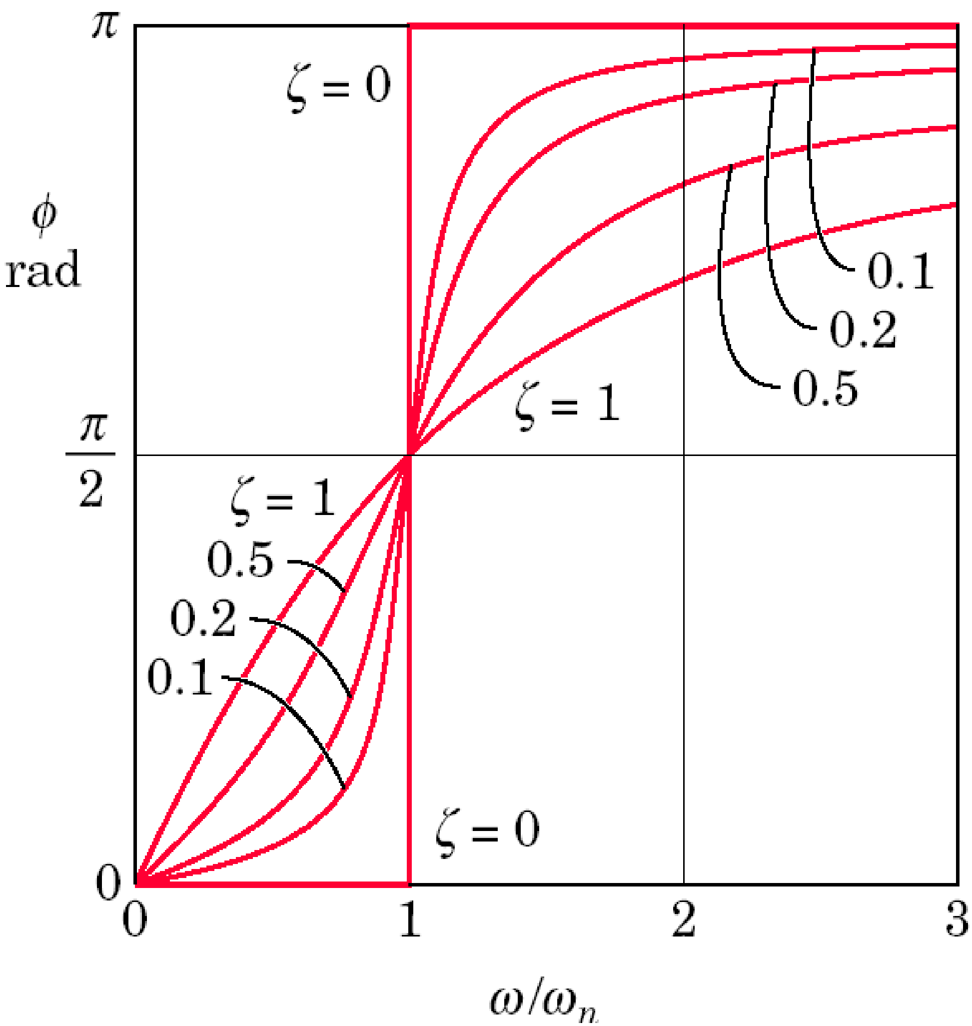


Forced (harmonically excited) single DoF vibration – Damped.



- The magnification ratio at all frequencies is reduced with increased damping.
- The effect of damping on the magnification ratio is greatest at or near resonance.
- The magnification ratio approaches 1 as the frequency ratio approaches 0 (DC)
- The magnification ratio approaches 0 as the frequency ratio approaches ∞
- For $0 < \zeta < 1/\sqrt{2}$ the magnification ratio maximum occurs at $r = \sqrt{(1 - 2\zeta^2)}$ or $\omega = \omega_n \sqrt{(1 - 2\zeta^2)}$ which is lower than both the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n \sqrt{(1 - \zeta^2)}$
- When $r = \sqrt{(1 - 2\zeta^2)}$ $M_{max} = 1/[2\zeta \sqrt{(1 - \zeta^2)}]$ → *if* M_{max} can be measured, the damping ratio can be determined.
- When $\zeta = 1/\sqrt{2}$ $dM/dr = 0$ *at* $r = 0$.
- When $\zeta > 1/\sqrt{2}$ M decreases monotonically with increasing frequency.

Forced (harmonically excited) single DoF vibration – Damped.



- For undamped systems (*ζ = 0)* the phase angle is 0° (response in phase with excitation) for r<1 and 180° (response out of phase with excitation) for r>1.
- For damped systems (*ζ > 0)* when *r < 1* the phase angle is less than 90° and response lags the excitation and when *r >1* the phase angle is greater than 90° and the response leads the excitation (approaches 180° for large frequency ratios..
- For damped systems (*ζ > 0)* when *r =1* the phase lag is always 90°.

Forced (harmonically excited) single DoF vibration – Damped.

- **Complete Solution**
- The complete solution is the sum of the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$:

$$x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi)$$

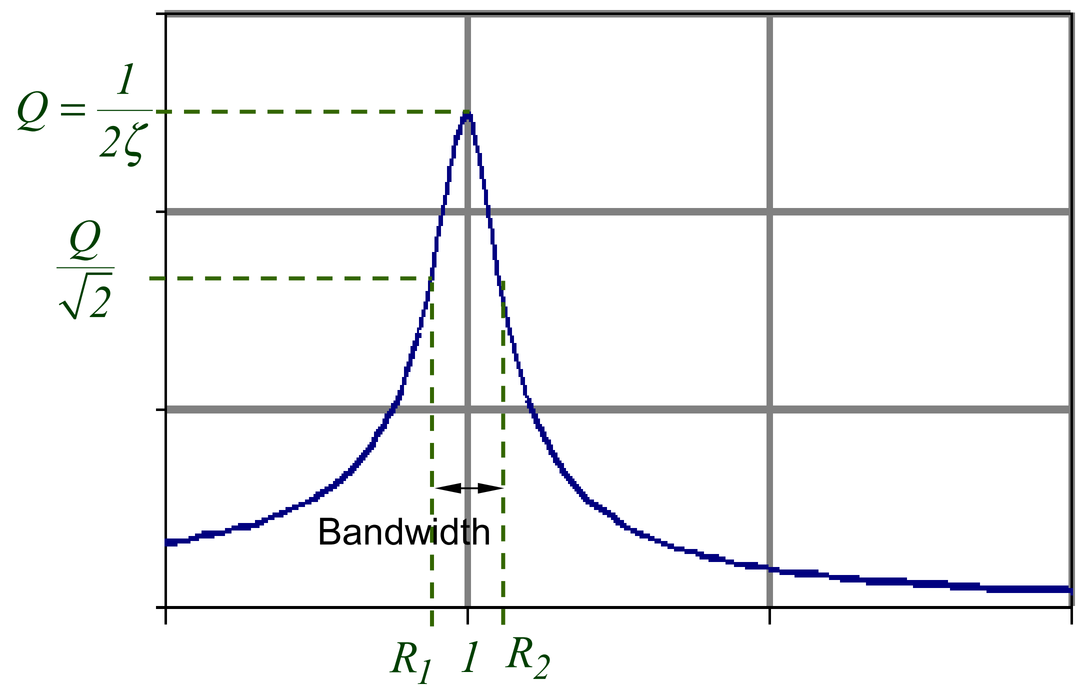
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, *X and* ϕ *are given as before, and* X_0 *and* ϕ_0 *are determined from the initial conditions*

Forced (harmonically excited) single DoF vibration – Damped.

- **Quality Factor & Bandwidth**
- When damping is small ($\zeta < 0.05$) the peak magnification ratio corresponds with resonance ($\omega = \omega_n$).
- The value of the magnification ratio (**Q**uality factor or **Q** factor) becomes:

$$Q = \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{2\zeta}$$

- The points where the magnification ratio falls below $Q/\sqrt{2}$, are called the half power points R_1 and R_2. (Power is proportional to amplitude squared: *Power = Fv = cv² = c(dx/dt)²*)
- The Quality factor Q can be used to estimate the equivalent viscous damping of systems.
- The difference between the half power frequencies is called the ***bandwidth***.



Forced (harmonically excited) single DoF vibration – Damped.

- The values of the half power frequencies are determined as follows:

$$\left(\frac{X}{\delta_{st}}\right) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

In terms of the frequency ratio r :

$$r^4 - r^2(2-4\zeta^2)+(1-8\zeta^2)=0$$

Which, when solved gives :

$$r_1^2 = 1-2\zeta^2-2\zeta\sqrt{1+\zeta^2} \qquad and \qquad r_2^2 = 1-2\zeta^2+2\zeta\sqrt{1+\zeta^2}$$

When ζ is small, ζ^2 is negligible and the solutions can be reduced to :

$$r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1-2\zeta \qquad and \qquad r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1+2\zeta$$

$$\therefore \qquad \omega_2^2-\omega_1^2 = \left(R_2^2-R_1^2\right)\omega_n^2 \approx 4\zeta\omega_n^2$$

Forced (harmonically excited) single DoF vibration – Damped.

Since $\dfrac{\omega_2+\omega_1}{2}=\omega_n$ *and* $\omega_2^2-\omega_1^2=(\omega_2+\omega_1)(\omega_2-\omega_1)$,

the bandwidth $\Delta\omega=\omega_2-\omega_1$ *can be written as :*

$$\Delta\omega=\frac{\omega_2^2-\omega_1^2}{\omega_2+\omega_1}\;;\;\frac{4\zeta\omega_n^2}{2\omega_n}\;;\;2\zeta\omega_n$$

The qualily factor Q can then be expressed in terms of the natural frequency and bandwidth :

$$Q\;;\;\frac{1}{2\zeta}\;;\;\frac{\omega_n}{\Delta\omega}$$

Forced (harmonically excited) single DoF vibration – Damped.

- **Complex notation.**
- Recall that a harmonic function may expressed as follows:

$$F(t) = F_0 \cos(\omega t + \phi) \quad = \quad F_0 \sin(\omega t + \phi) \quad = \quad F_0\, e^{i(\omega t + \phi)}$$

- If the harmonic forcing function is expressed in complex form:

$$F = F_0 e^{i\omega t}$$

- The equation of motion for a damped SDOF system becomes:

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

- The actual excitation function is real and is represented by the real part of the complex function. Consequently, the steady-state response is also real and is represented by the real part of the complex particular solution which takes the form:

$$x_p(t) = Xe^{i\omega t}$$

Therefore :

$$\dot{x}_p(t) = i\omega Xe^{i\omega t} \quad and \quad \ddot{x}_p(t) = -\omega Xe^{i\omega t}$$

- Substituting in the eqn. of motion gives:

$$-m\omega^2 Xe^{i\omega t} + ic\omega Xe^{i\omega t} + kXe^{i\omega t} = F_0 e^{i\omega t}$$

Forced (harmonically excited) single DoF vibration – Damped.

- The response amplitude becomes:

$$X = \frac{F_0}{\left[\left(k - m\omega^2\right) + ic\omega\right]} \quad \neg \quad X / F_0 \text{ is called the RECEPTANCE (Dynamic compliance)}$$

multiplying the numerator & denominator on the RHS by $\left(k - m\omega^2\right) - ic\omega$
and separating real and imaginary components :

$$X = F_0 \left[\frac{k - m\omega^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2} - i\frac{c\omega}{\left(k - m\omega^2\right)^2 + c^2\omega^2} \right]$$

applying the complex relationships : $x + iy = Ae^{i\phi}$ *where* $A = \sqrt{x^2 + y^2}$ *and* $\phi = a\tan\left(\frac{y}{x}\right)$

The magnitude of the response can be written as :

$$X = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{-i\phi} \qquad \text{where} \quad \phi = a\tan\left(\frac{c\omega}{k - m\omega^2}\right)$$

And the steady – state solution becomes :

$$x_p(t) = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{i(\omega t - \phi)}$$

Forced (harmonically excited) single DoF vibration – Damped.

- As before the response amplitude:

$$X = \frac{F_0}{\left[\left(k - m\omega^2\right) + ic\omega\right]}$$

can be written in terms of the frequency ratio r and the damping ratio ζ :

$$\frac{kX}{F_0} = \frac{1}{1 - r^2 + i2\zeta r} \equiv H(i\omega) \neg \text{ Complex Frequency Response Function (FRF)}$$

The magnitude of $H(i\omega)$ is given by :

$$\left|H(i\omega)\right| = \left|\frac{kX}{F_0}\right| = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}} \quad \text{which is the same as the magnification ratio } M \text{ :}$$

It can be shown that the complex FRF and its magnitude are related by :

$$H(i\omega) = \left|H(i\omega)\right| e^{-i\phi} \quad \text{where } e^{-i\phi} = \cos\phi + i\sin\phi \quad \text{and} \quad \phi = a\tan\left(\frac{2\zeta r}{1 - r^2}\right)$$

The steady-state response can therefore be expressed as :

$$x_p(t) = \frac{F_0}{k}\left|H(i\omega)\right| e^{i(\omega t - \phi)}$$

- Measurements of the magnitude FRF can be used to experimentally determine the values of m, c and k.

Forced (harmonically excited) single DoF vibration – Damped.

- When the excitation function is described by: $F(t) = F_0 \cos(\omega t)$
- The steady-state response is given by the real part of the solution:

$$x_p(t) = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{i(\omega t - \phi)} = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} \cos(\omega t - \phi)$$

$$= Re\left[\frac{F_0}{k} H(i\omega) e^{i\omega t}\right]$$

$$= Re\left[\frac{F_0}{k} \left|H(i\omega)\right| e^{i(\omega t - \phi)}\right]$$

- Conversely, when the excitation function is described by: $F(t) = F_0 \sin(\omega t)$
- The steady-state response is given by the imaginary part of the solution:

$$x_p(t) = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} e^{i(\omega t - \phi)} = \frac{F_0}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} \sin(\omega t - \phi)$$

$$= Im\left[\frac{F_0}{k} H(i\omega) e^{i\omega t}\right]$$

$$= Im\left[\frac{F_0}{k} \left|H(i\omega)\right| e^{i(\omega t - \phi)}\right]$$

Forced (harmonically excited) single DoF vibration – Damped.

- **Complex Vector Notation of Harmonic Motion:**
- Harmonic excitation and response can be represented in the complex plane

Steady – state displacement :

$$x_p(t) = \frac{F_0}{k}\left|H(i\omega)\right|e^{i(\omega t-\phi)}$$

Steady – state velocity :

$$\dot{x}_p(t) = i\omega\frac{F_0}{k}\left|H(i\omega)\right|e^{i(\omega t-\phi)} = i\omega\, x_p(t)$$

Steady – state acceleration :

$$\ddot{x}_p(t) = (i\omega)^2\frac{F_0}{k}\left|H(i\omega)\right|e^{i(\omega t-\phi)} = -\omega^2\, x_p(t)$$

Since i and -1 respectively can be written as :

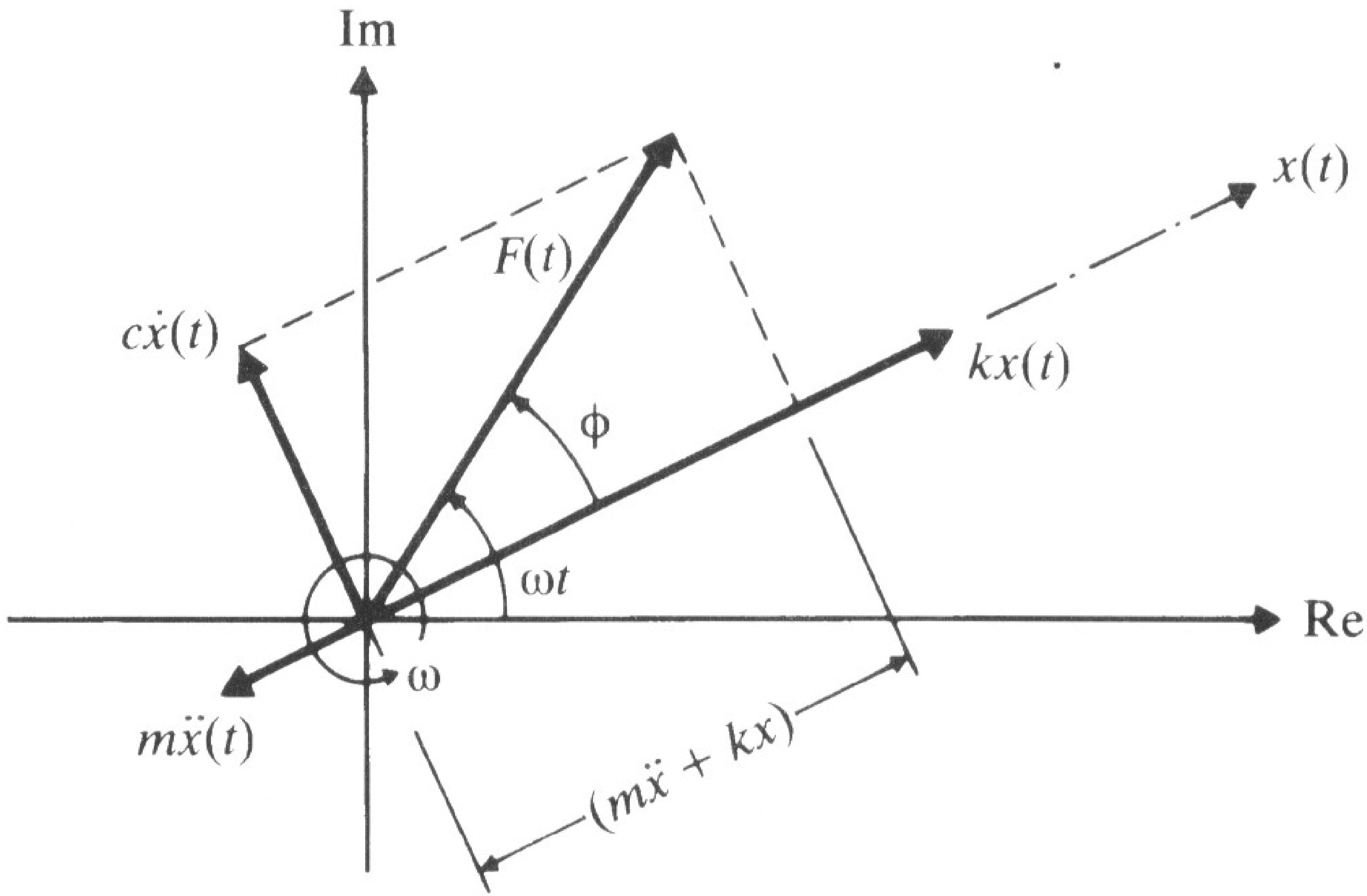
$$i = cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}} \quad and \quad -1 = cos(\pi) + i\,sin(\pi) = e^{i\pi}$$

- It can be seen that:
  - The velocity leads the displacement by 90° and is multiplied by ω.
  - The acceleration leads the displacement by 180° and is multiplied by ω^2.

Forced (harmonically excited) single DoF vibration – Damped.

- **Complex Vector Notation of Harmonic Motion:**

$$x_p(t) = \frac{F_0}{k}\left|H(i\omega)\right|e^{i(\omega t-\phi)} \qquad \dot{x}_p(t) = i\omega\, x_p(t) \qquad \ddot{x}_p(t) = -\omega^2\, x_p(t)$$



Forced (harmonically excited) single DoF vibration – Damped.

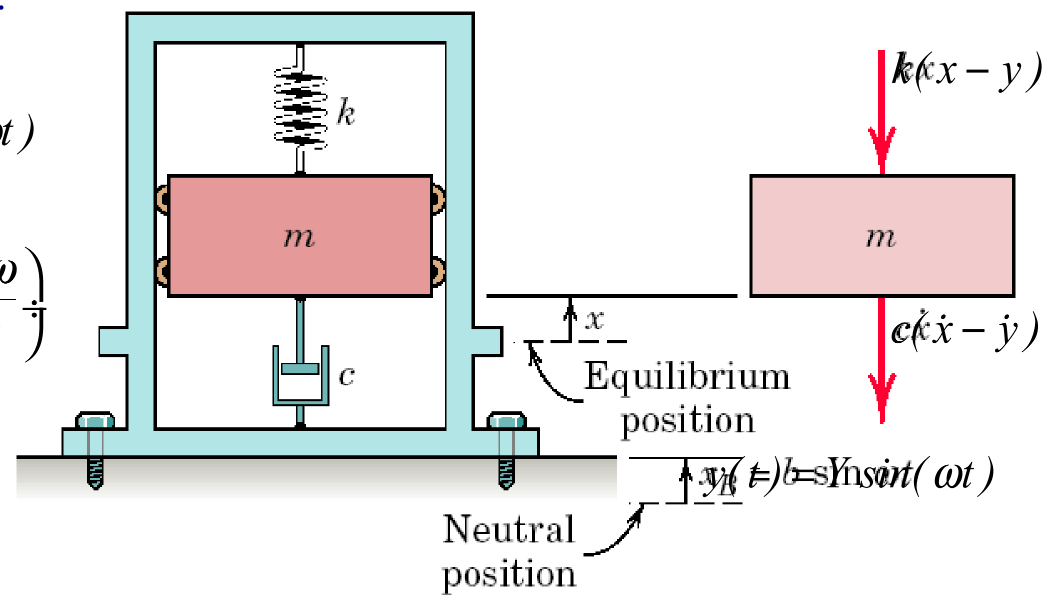
- **Response due to base motion (harmonic)**
- In this case, the excitation is provided by the imposed harmonic motion of the supporting base.
- The displacement of the base about a neutral position is denoted by y(t) and the response of the mass from its static equilibrium position by x(t).
- At any time, the length of the spring is x – y and the relative velocity between the two ends of the damper is x' – y'.
- The equation of motion is:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

If $y(t) = Y\sin(\omega t)$ *the eqn. of motion becomes :*

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx &= c\dot{y} + ky \\ &= c\omega Y\cos(\omega t) + kY\sin(\omega t) \\ &= A\sin(\omega t - \alpha) \end{aligned}$$

$$where \quad A = Y\sqrt{k^2 + (c\omega)^2} \quad and \quad \alpha = a\tan\left(-\frac{c\omega}{k}\right)$$



- The applied displacement has the same effect of applying a harmonic force of magnitude A to the mass.

Forced (harmonically excited) single DoF vibration – Damped.

- The steady-state response of the mass is given by the particular solution $x_p(t)$:

$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{1/2}} \sin(\omega t - \phi_1 - \alpha)$$

where $\alpha = a\tan\left(-\frac{c\omega}{k}\right)$ *and* $\phi_1 = a\tan\left(\frac{c\omega}{k - m\omega^2}\right)$

The solution can be simplified to :

$$x_p(t) = X \sin(\omega t - \phi)$$

where

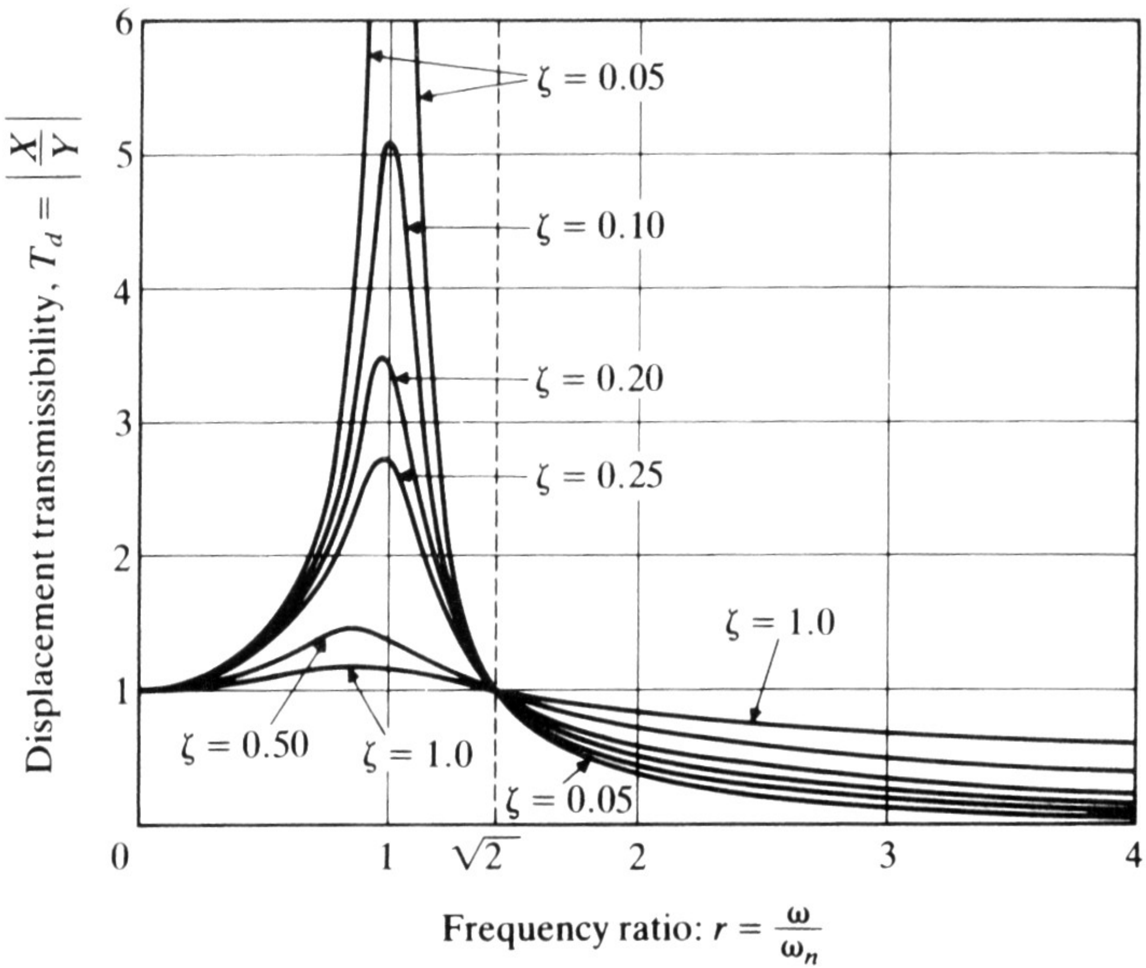
$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{\left(k - m\omega^2\right)^2 + c^2\omega^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{\left(1 - r^2\right)^2 + (2\zeta r)^2}\right]^{1/2} \leftarrow \textit{Displacement Transmissibility}$$

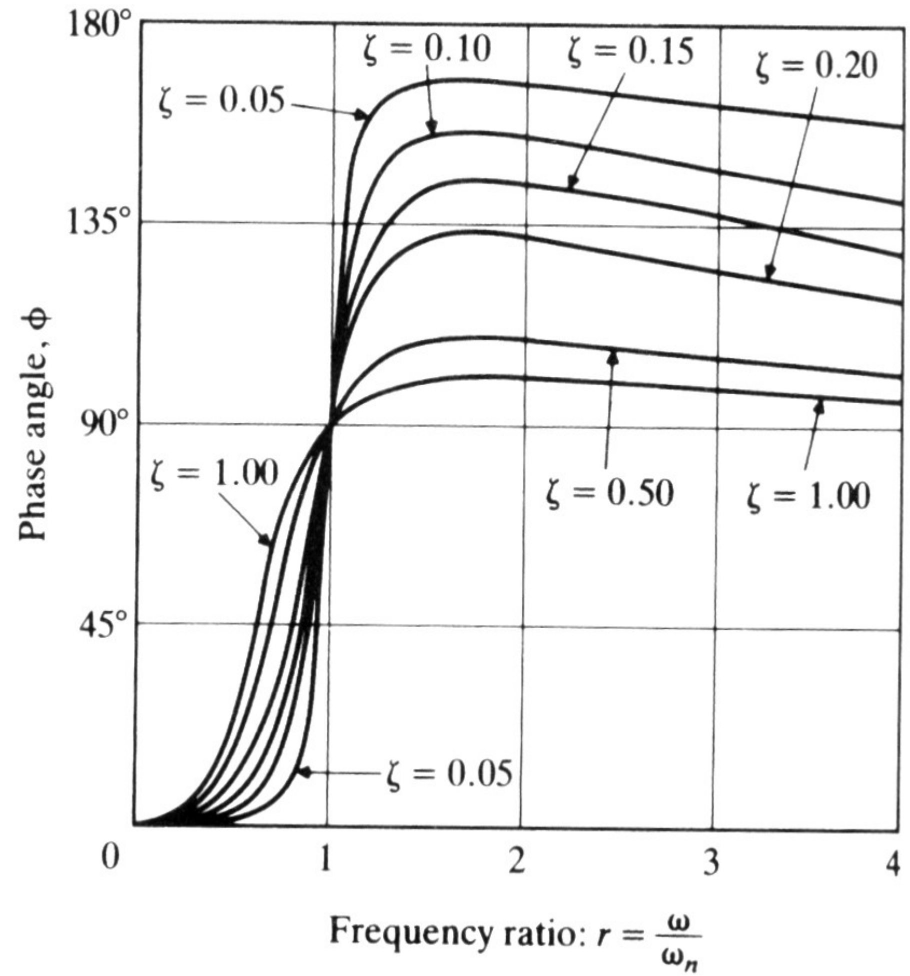
and

$$\phi = a\tan\left(\frac{mc\omega^3}{k\left(k - m\omega^2\right) + (c\omega)^2}\right) = a\tan\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

Forced (harmonically excited) single DoF vibration – Damped.

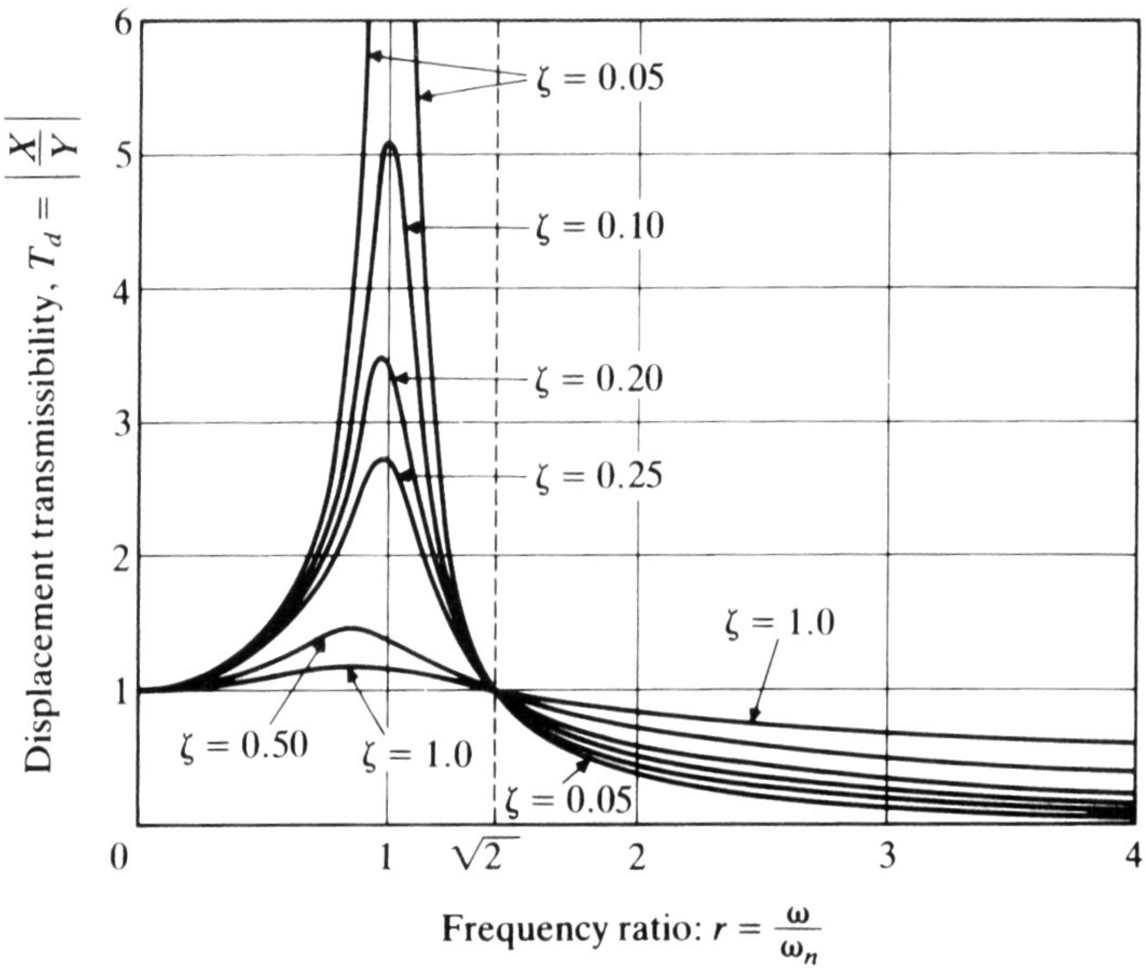
$$\frac{X}{Y}=\left[\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+(2\zeta r)^2}\right]^{1/2} \quad \text{and} \quad \phi = a\tan\left(\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)$$





Forced (harmonically excited) single DoF vibration – Damped.

- **Characteristics of the displacement transmissibility:**
- The transmissibility is 1 when r = 0 (DC) and close to 1 when r is small.
- For undamped systems (ζ = 0), $T_d \rightarrow \infty$ at resonance (r = 1)
- For all damping values $T_d<1$ for $r>\sqrt{2}$ and $T_d = 1$ for $r = \sqrt{2}$
- For $r<\sqrt{2}$ T_d is inversely proportional to ζ
- For $r>\sqrt{2}$ T_d is proportional to ζ



Forced (harmonically excited) single DoF vibration – Damped.

- **Transmitted Force**
- The force transmitted to the base/support is caused by the reaction of the spring and damper:

$$F = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x}$$

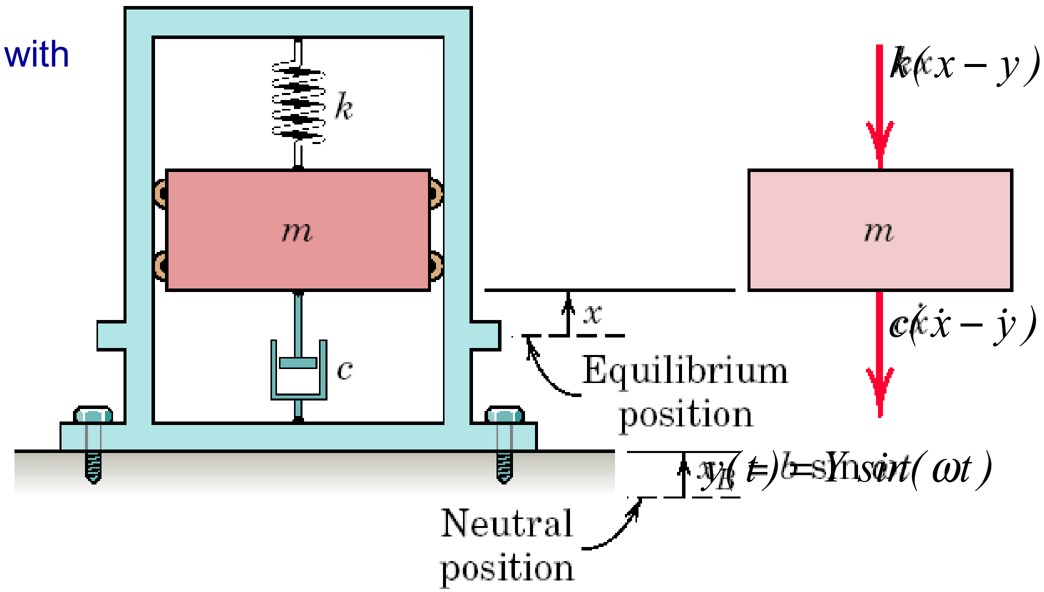
Since the steady – state (particular) solution is $x_p(t) = X\sin(\omega t - \phi)$ *, F can be written as :*

$$F = m\omega^2 X\sin(\omega t - \phi) = F_T \sin(\omega t - \phi)$$

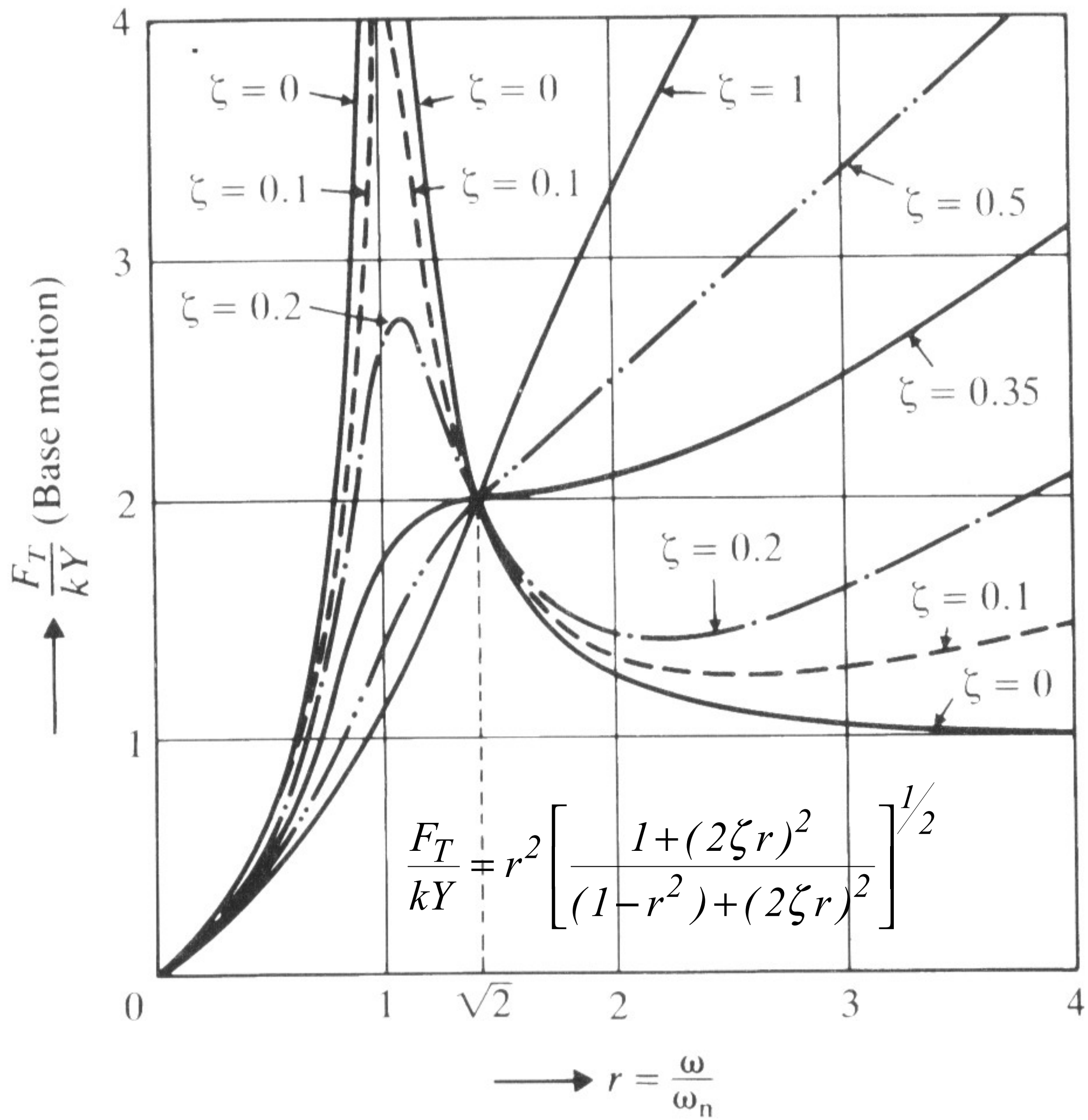
- Where F_T is the amplitude of the transmitted force and is given by:

$$\frac{F_T}{kY} = r^2\left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2} \quad \neg \quad \textit{Force Transmissibility}$$

- Note that the transmitted force is always in–phase with the motion of the mass x(t):



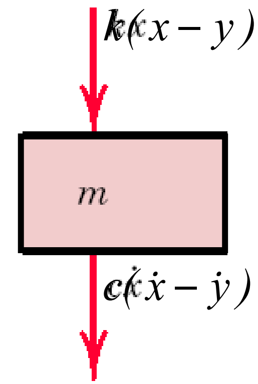
Forced (harmonically excited) single DoF vibration – Damped.



$$\frac{F_T}{kY} = r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)+(2\zeta r)^2}\right]^{1/2}$$

Forced (harmonically excited) single DoF vibration – Damped.

- **Relative Motion**
- If z = x – y represents the motion of the mass relative to the base, the eqn. of motion:



$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

can be written as :

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \sin(\omega t)$$

The (steady – state) solution of which is :

$$z(t) = \frac{m\omega^2 Y \sin(\omega t - \phi_1)}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^{1/2}} = Z \sin(\omega t - \phi_1)$$

where the amplitude Z is given by :

$$Z = \frac{m\omega^2 Y}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^{1/2}} = Y \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]^{1/2}}$$

and the phase ϕ_1 is given by :

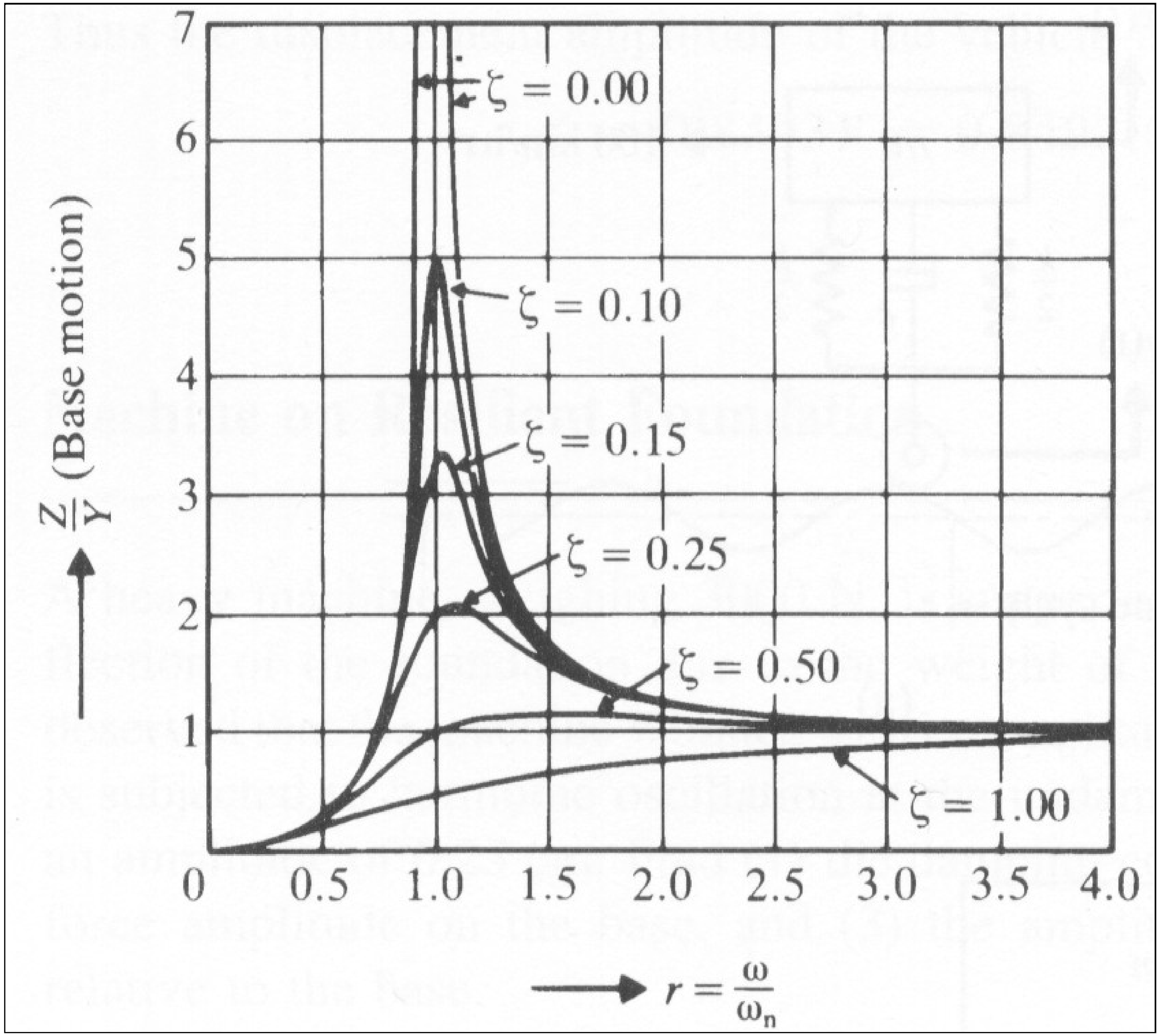
$$\phi_1 = a\tan\left(\frac{c\omega}{k - m\omega^2}\right) = a\tan\left(\frac{2\zeta r}{1 - r^2}\right)$$

Forced (harmonically excited) single DoF vibration – Damped.

- **Relative Motion**

$$\frac{Z}{Y} = \frac{r^2}{\left[\left(1-r^2\right)^2 + \left(2\zeta r\right)^2\right]^{1/2}}$$

$$\phi_1 = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$



Forced (harmonically excited) single DoF vibration – Damped.

- **Rotating Imbalance Excitation**
- With the horizontal components cancelled the vertical component of the excitation is:

$$F(t) = me\omega^2 \sin(\omega t)$$

The eqn. of motion is :

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin(\omega t)$$

and the steady-state solution becomes :

$$x_p(t) = X\sin(\omega t - \phi) = Im\left[\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 |H(i\omega)| e^{i(\omega t - \phi)}\right]$$

The response amplitude and phase are given by :

$$X = \frac{me\omega^2}{\left[\left(k - M\omega^2\right)^2 + (c\omega)^2\right]^{1/2}} = \frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 |H(i\omega)| \quad or \quad \frac{MX}{me} = \frac{r^2}{\left[\left(1 - r^2\right)^2 + (2\zeta r)^2\right]^{1/2}} = r^2 |H(i\omega)|$$

$$\phi = a\tan\left(\frac{c\omega}{k - M\omega^2}\right) = a\tan\left(\frac{2\zeta r}{1 - r^2}\right)$$

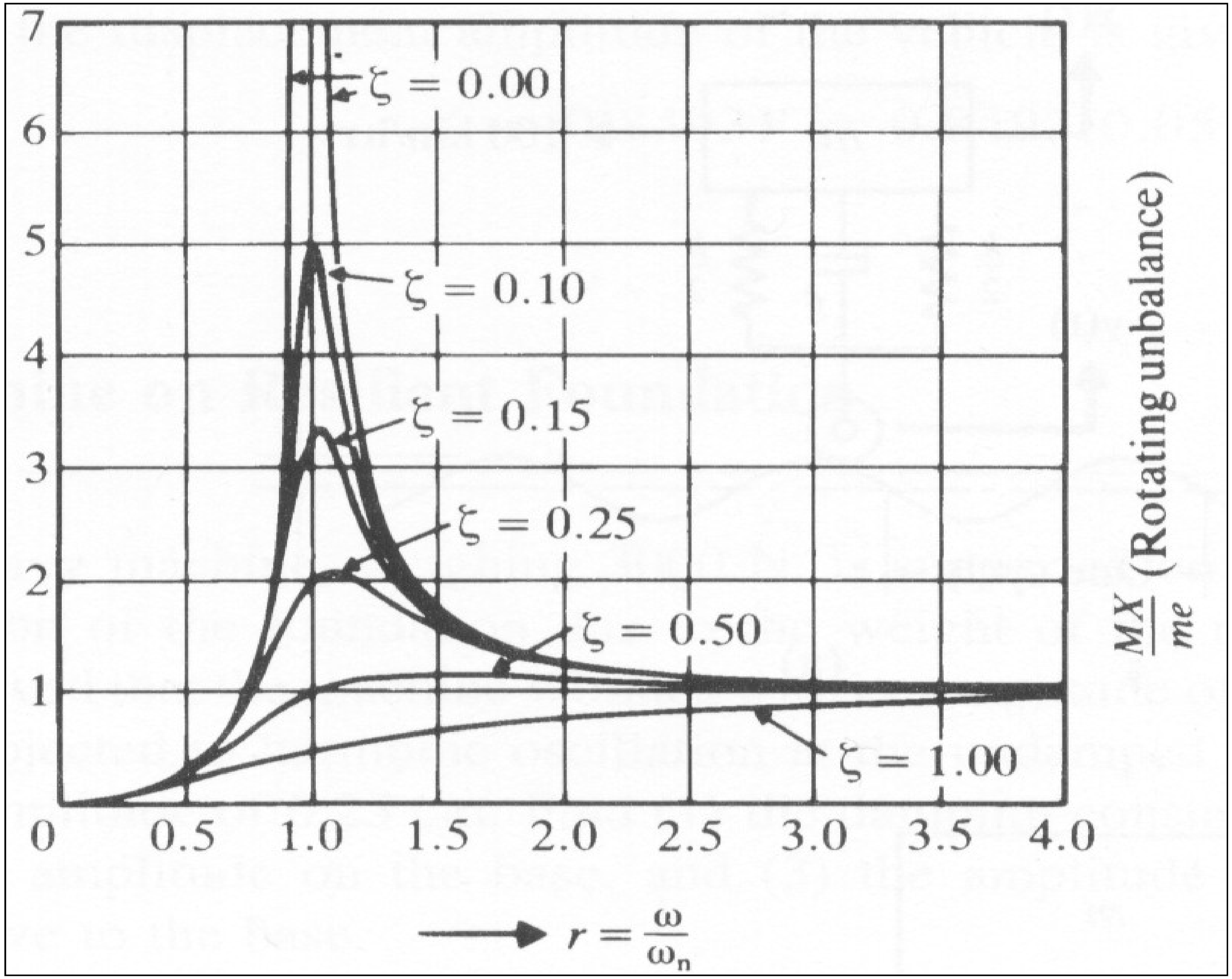
Forced (harmonically excited) single DoF vibration – Damped.

- **Rotating Imbalance Excitation**

$$\frac{MX}{me} = \frac{r^2}{\left[\left(1-r^2\right)^2 + \left(2\zeta r\right)^2\right]^{1/2}}$$

$$= r^2 \left|H(i\omega)\right|$$

$$\phi = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$



Forced (harmonically excited) single DoF vibration – Damped.

- **Forced Vibration with Coulomb Damping**
- The equation of motion for a SDOF with Coulomb damping subjected to a harmonic force is:

$$M\ddot{x} + kx \pm \mu N = F_0 \sin(\omega t)$$

- Solution complicated.
- If μN is large cf F_0, motion of mass m is discontinuous
- If $\mu N << F_0$ motion of mass m will approximate harmonic motion
- When $\mu N << F_0$ an approximate solution to eqn. of motion may be used to determine equivalent viscous damping ratio.
- This is achieved by equating dissipated energy for both cases.
- For Coulomb damping, the energy dissipated during a cycle of amplitude X is:

$$\Delta W = 4(\mu N X) \quad - \quad 4 \text{ quarter cycles}$$

- For viscous damping, the energy dissipated during a cycle of amplitude X is:

$$\Delta W = \int_{t=0}^{2\pi/\omega} Fv\,dt = \int_{t=0}^{2\pi/\omega} c_{eq}\left(\frac{dx}{dt}\right)^2 dt = \int_{t=0}^{2\pi} c_{eq} X^2 \omega \cos^2(\omega t)\; d(\omega t)$$

$$= \pi c_{eq} \omega X^2$$

Forced (harmonically excited) single DoF vibration – Damped.

- Equating the dissipated energies:

$$c_{eq} = \frac{4\mu N}{\pi \omega X^2}$$

- And the equivalent damping ratio is defined as:

$$\zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2m\omega_n} = \frac{4\mu N}{2m\omega_n \pi \omega X} = \frac{2\mu N}{\pi m \omega_n \omega X}$$

- The amplitude X and the phase ϕ of the response becomes:

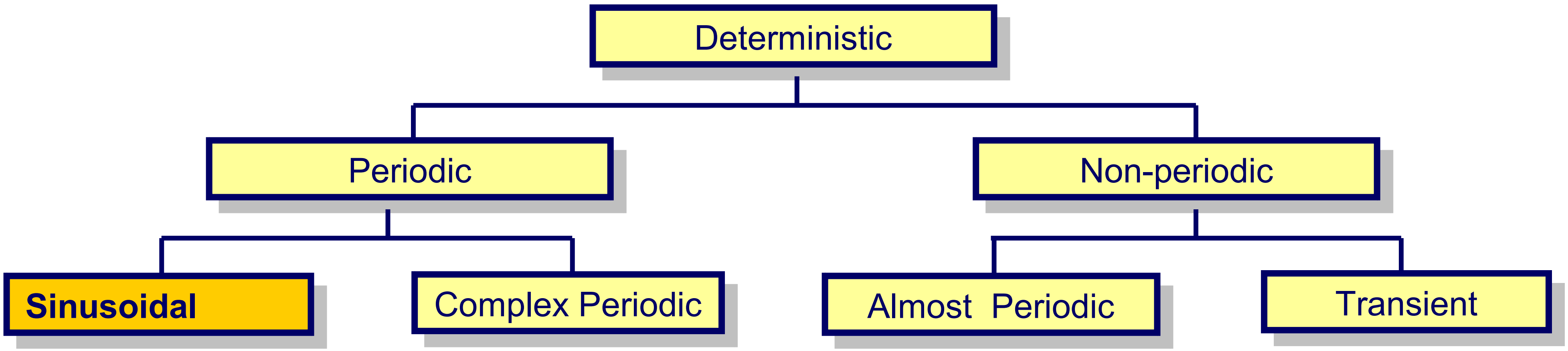
$$X = \frac{F_0}{k}\left[\frac{1-\left(\frac{4\mu N}{\pi F_0}\right)^2}{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2}\right]^{1/2} \qquad \phi = a\tan\left[\frac{\pm 1 - \frac{4\mu N}{\pi F_0}}{\left\{1-\left(\frac{4\mu N}{\pi F_0}\right)^2\right\}^{1/2}}\right]$$

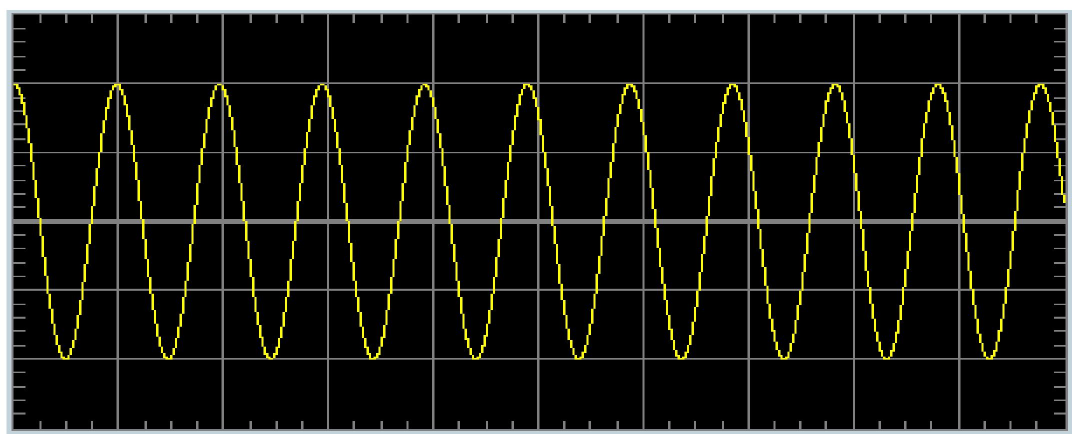
- These approximations are only valid for $\mu N << F_0$

SDoF systems – General forcing functions

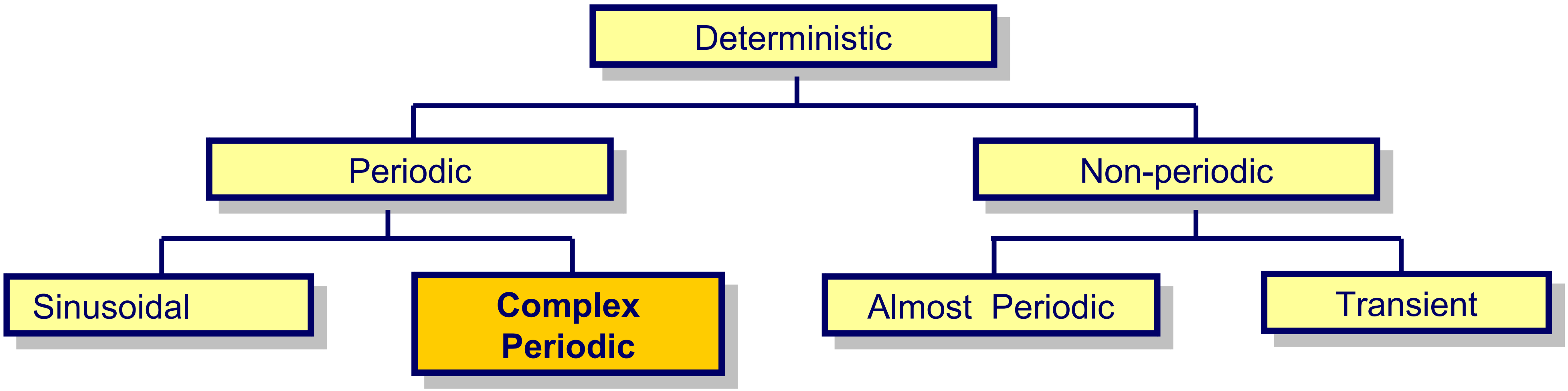
- Methods to solve response due to general (nonharmonic) forcing functions.
- General forcing function may be periodic (nonharmonic) or aperiodic.
- Aperiodic forcing functions may be finite or infinite
- When the duration of a transient forcing function << natural period of system, forcing function called SHOCK.
- When forcing function is periodic (not harmonic), it can be described with a series (sum) of harmonic or Fourier components.

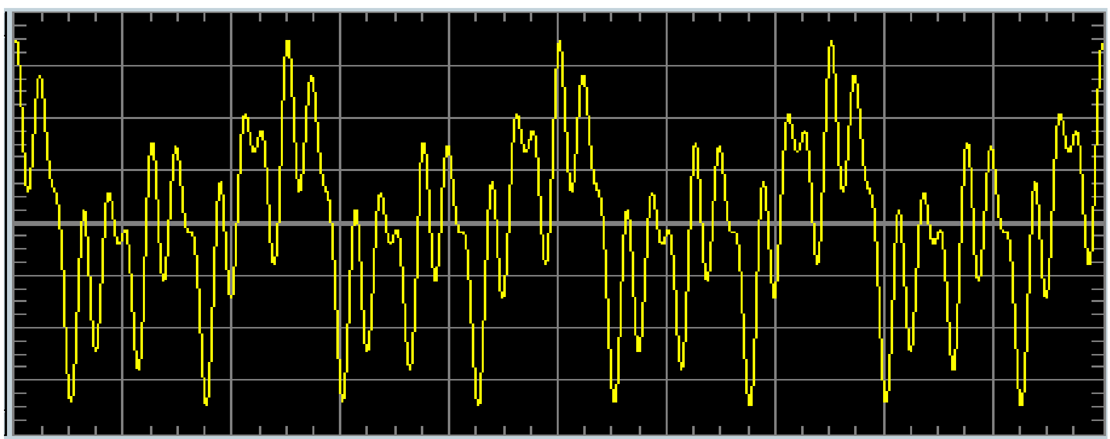
Types of deterministic forcing functions.





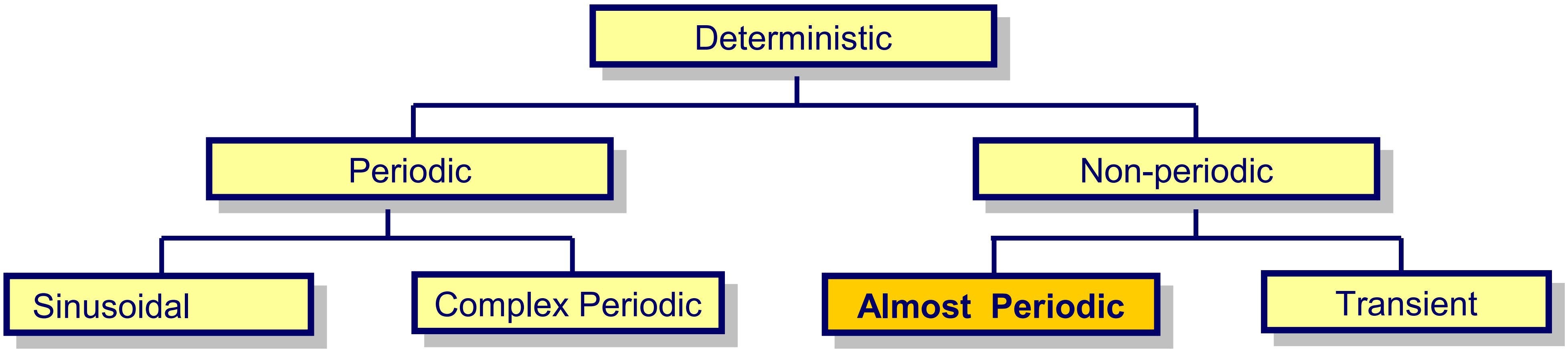
Types of deterministic forcing functions.

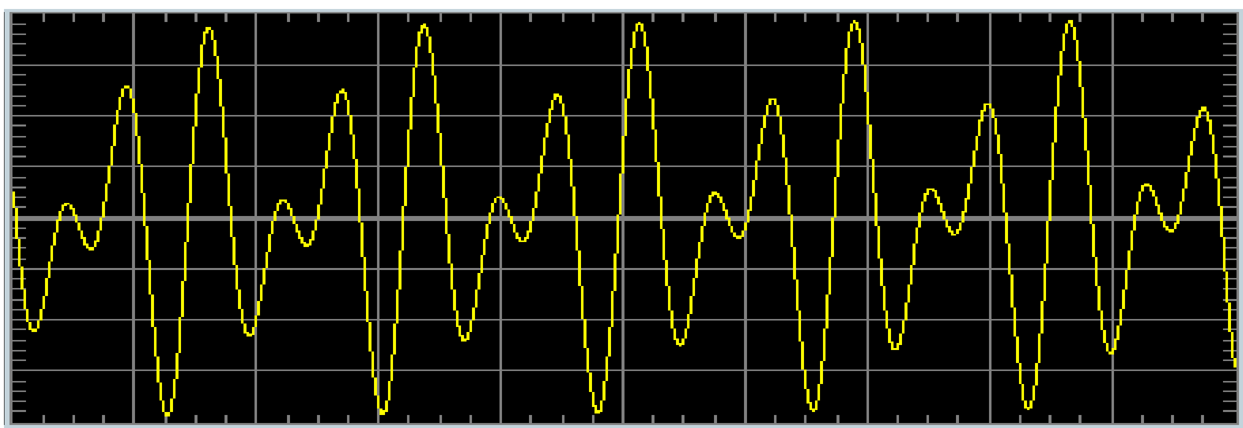




Can be defined mathematically. Waveform contains harmonics which are multiples if the fundamental frequency (show spectrum) Signal factory.vee

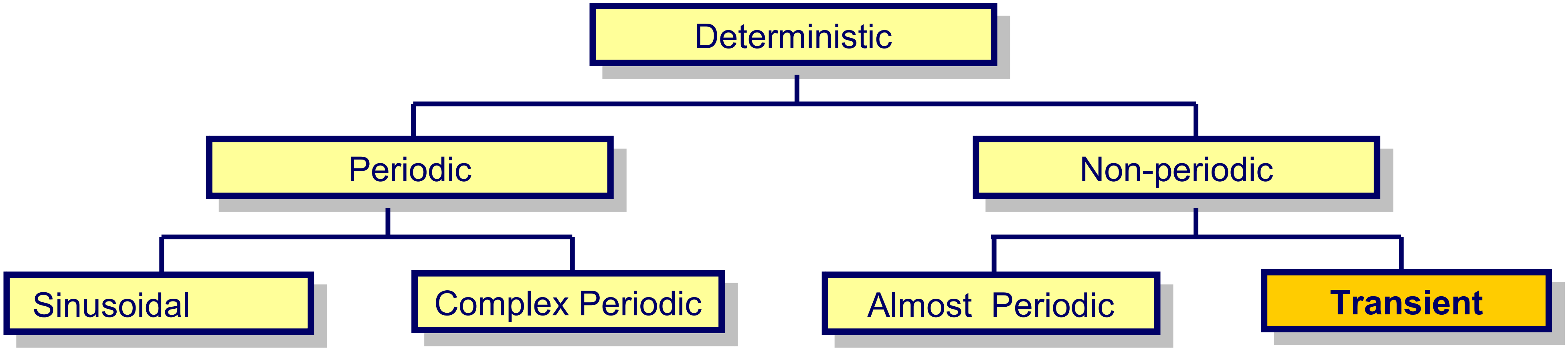
Types of deterministic forcing functions.

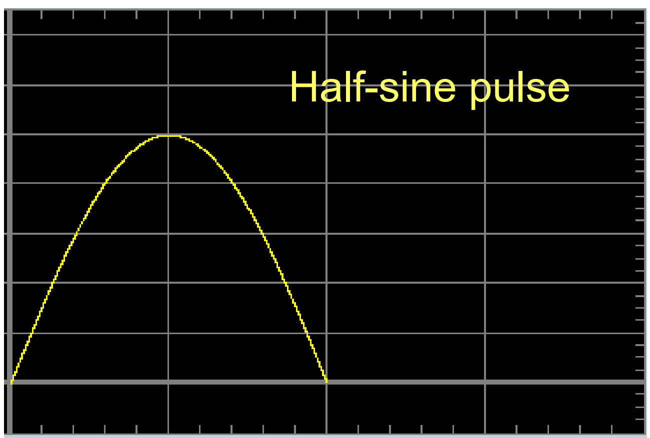


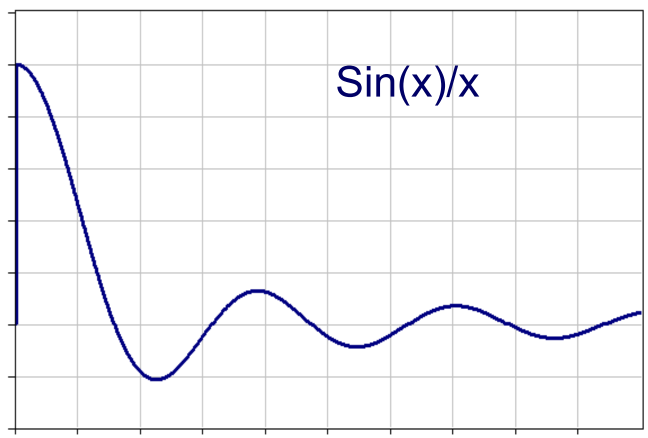


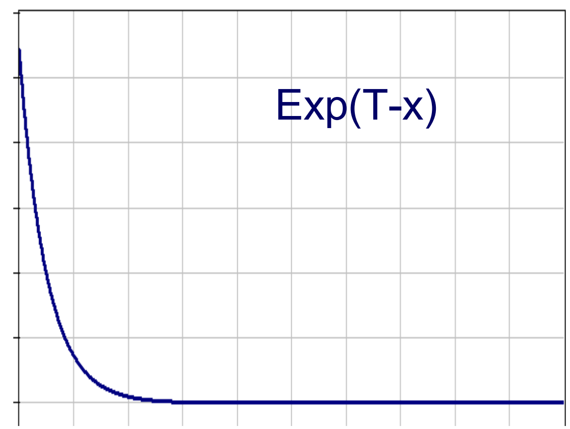
Contains sine wave of arbitrary frequencies which frequency ratios are not rational numbers (show spectrum) Signal factory.vee

Types of deterministic forcing functions.









All other deterministic data that can be described by a suitable function

SDoF systems – General forcing functions - Periodic

- For periodic forcing functions, the response of system is obtained by using the **principle of superposition**:
- The total response consists of sum of response functions due to individual harmonic functions in forcing function.
- The periodic forcing function (period $\tau = 2\pi/\omega$) can be expressed as a Fourier series:

$$F(t) = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j=1}^{\infty} b_j \sin(j\omega t)$$

where

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t)\cos(j\omega t)\ dt \qquad for \quad j = 0, 1, 2.....$$

$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t)\sin(j\omega t)\ dt, \qquad for \quad j = 1, 2, 3.....$$

- The eqn. of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_o}{2} + \sum_{j=1}^{\infty} a_j \cos(j\omega t) + \sum_{j=1}^{\infty} b_j \sin(j\omega t)$$

- The RHS is a constant + a sum of harmonic functions.

SDoF systems – General forcing functions - Periodic

- Using the principle of superposition, the steady-state solution is the sum of the steady-state solution for the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_o}{2} \qquad (1)$$

$$m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\infty} a_j \cos(j\omega t) \qquad (2)$$

$$m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\infty} b_j \sin(j\omega t) \qquad (3)$$

- The steady-state solutions of (1), (2) and (3) are

$$x_p(t) = \frac{a_o}{2k}$$

$$x_p(t) = \frac{a_j / k}{\sqrt{\left(1 - j^2 r^2\right)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j)$$

$$x_p(t) = \frac{b_j / k}{\sqrt{\left(1 - j^2 r^2\right)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j)$$

SDoF systems – General forcing functions - Periodic

- The entire steady-state solution is given by:

$$x_p(t)=\frac{a_o}{2k}+\sum_{j=1}^{\infty}\frac{a_j/k}{\sqrt{\left(1-j^2r^2\right)^2+(2\zeta jr)^2}}cos(j\omega t-\phi_j)+\sum_{j=1}^{\infty}\frac{b_j/k}{\sqrt{\left(1-j^2r^2\right)^2+(2\zeta jr)^2}}sin(j\omega t-\phi_j)$$

where

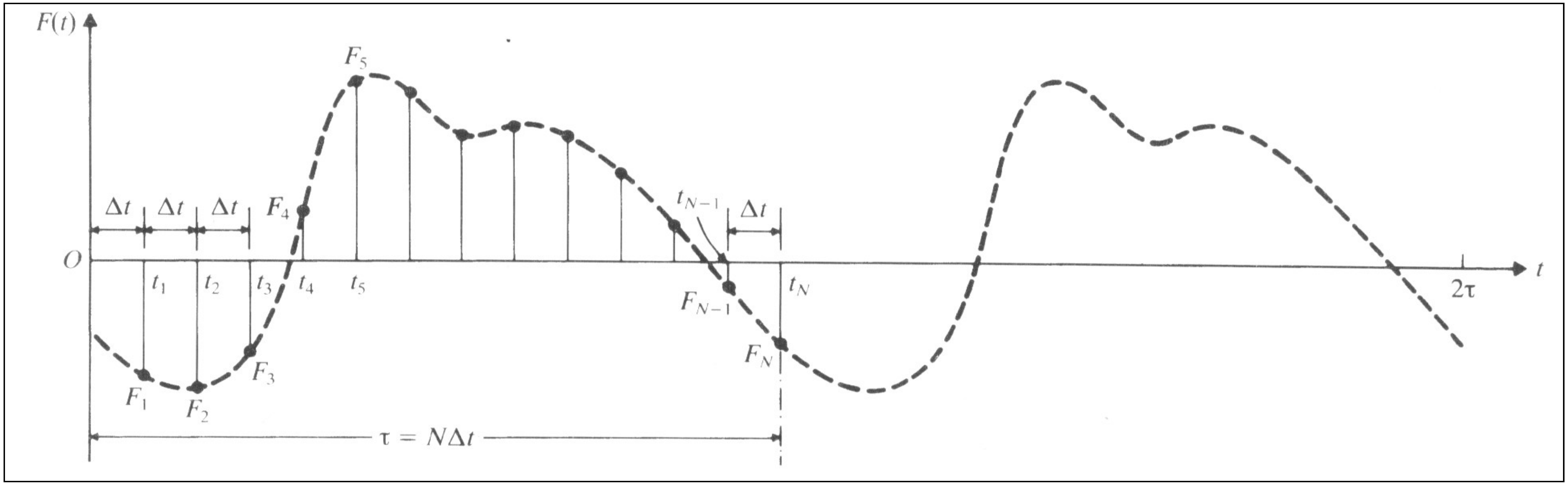
$$\phi_j=a\,tan\left(\frac{2\zeta jr}{1-j^2r^2}\right) \quad and \quad r=\frac{\omega}{\omega_n}$$

- The response amplitude and phase for each harmonic (j^{th} term) depend on *j*.
- When $r = 1$ the response amplitude is relatively high for any value *j* (more so when both *j* and ζ are small)
- As j becomes larger (higher harmonics) the amplitude response becomes smaller $\rightarrow$ the first few terms are usually needed to generate a reasonably accurate response.
- **Complete Solution**
- The complete solution is obtained by including the transient part of the solution which is dependent on the initial conditions.
- This requires setting the complete solution and its derivative to the specified initial displacement and velocity which produces a complicated expression for the transient part of the solution.

Example: Triangular forcing function. Vee & Excel

SDoF systems – General forcing functions - Periodic

- Situation sometimes arises when the periodic forcing function is given (obtained) experimentally (eg: wave, wind , seismic, topography..) and represented by discrete measurement data.



- When the (measured) data cannot be readily described by a mathematical function
- The discrete measurement data can be integrated numerically to obtain the Fourier coefficients.

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} F_i \qquad a_j = \frac{2}{N}\sum_{i=1}^{N} F_i \cos\left(\frac{2j\pi t_i}{\tau}\right) \quad \text{and} \quad b_j = \frac{2}{N}\sum_{i=1}^{N} F_i \sin\left(\frac{2j\pi t_i}{\tau}\right) \quad \text{for} \quad j = 1, 2.....$$

- The Fourier coefficients can then be used to find the solution with the excitation frequency taken as the lowest frequency component of the data.

$$\omega = \frac{2\pi}{\tau}$$

SDoF systems – General forcing functions – Nonperiodic

- When the forcing function is arbitrary and nonperiodic (aperiodic) it cannot be represented with a Fourier series
- Alternative methods for determining the response must be used:
  - Representation of the excitation function with a ***Convolution integral***
  - Using ***Laplace Transformations***
  - Approximating F(t) with a suitable ***interpolation method*** then using a numerical procedure
  - ***Numerical integration*** of the equations of motion.

SDoF systems – General forcing functions – Nonperiodic

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SDoF systems – General forcing functions – Nonperiodic

- **Convolution integral**
- Consider one of the simplest nonperiodic exciting force: Impulsive force: which has a large magnitude F which acts for a very short time Δt.
- An impulse can be measured by the resulting change in momentum:

$$Im\ pulse = F\,\Delta t = m\dot{x}_2 - m\dot{x}_1$$

where $\dot{x}_1$ *and* $\dot{x}_2$ *represent the velocity of the lumped mass before and after the impulse.*

- The magnitude of the impulse *F*Δt is represented by

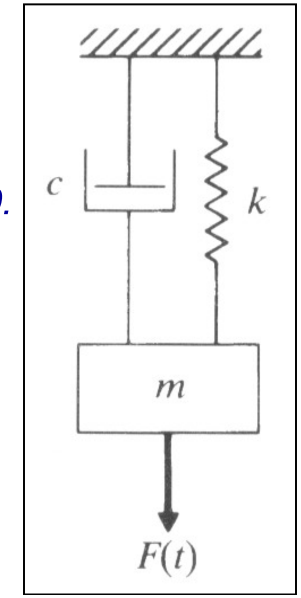
$$\underset{\sim}{F} = \int_t^{t+\Delta t} F\,dt$$

and a unit impulse is defined as

$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = Fdt = 1$$

- For *Fdt* to have a finite value, *F* approaches infinity as *Δt* nears zero.

SDoF systems – General forcing functions – Nonperiodic



- **Convolution integral – Impulse response**
- Consider a (viscously) damped SDoF (mass-spring-damper system) subjected to an impulse at *t=0.*
- For an underdamped system, the eqn. of motion is:

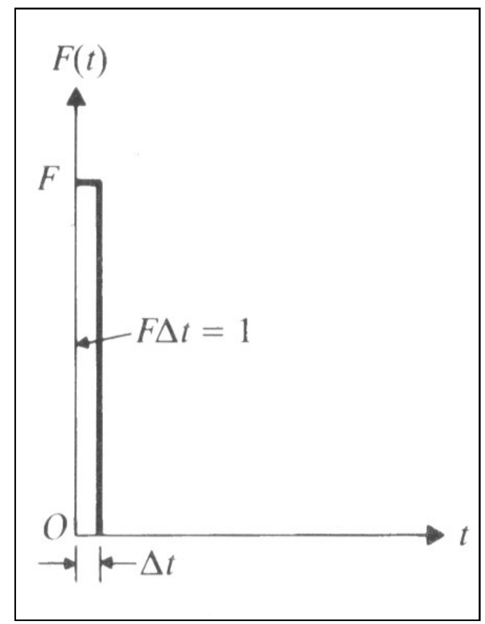
$$m\ddot{x} + c\dot{x} + kx = 0$$

- And its solution:

$$x(t) = e^{-\zeta\omega_n t}\left\{ x_0 \cos\left(\omega_d t\right) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\left(\omega_d t\right) \right\}$$

where

$$\zeta = \frac{c}{2m\omega_n} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad \omega_n = \sqrt{\frac{k}{m}}$$



- If, prior to the impulse load being applied, the mass is at rest, then:

$$x(t<0) = 0 \;\; and \;\; \dot{x}(t<0) = 0 \quad or \quad x(t=0^-) = 0 \;\; and \;\; \dot{x}(t=0^-) = 0$$

- The impulse-momentum equation gives:

$$\underset{\sim}{f} = 1 = m\dot{x}(t=0) - m\dot{x}(t=0^-) = m\dot{x}_0$$

- And the initial conditions are given by:

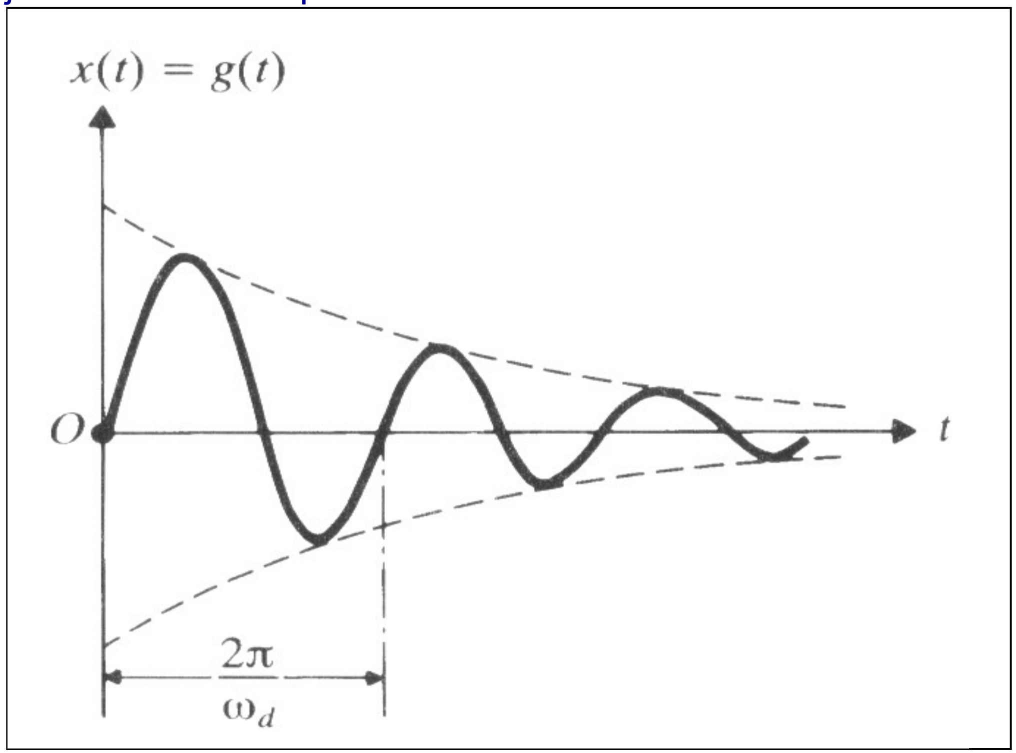
$$x(t=0) = x_0 = 0 \qquad and \qquad \dot{x}(t=0) = \dot{x}_0 = \frac{1}{m}$$

SDoF systems – General forcing functions – Nonperiodic

- **Convolution integral – Impulse response**
- The solution reduces to:

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} sin\left(\omega_d t\right)$$

- *g(t)* is the ***impulse response function*** an represents the response of a viscously damped single degree of freedom system subjected to a unit impulse.



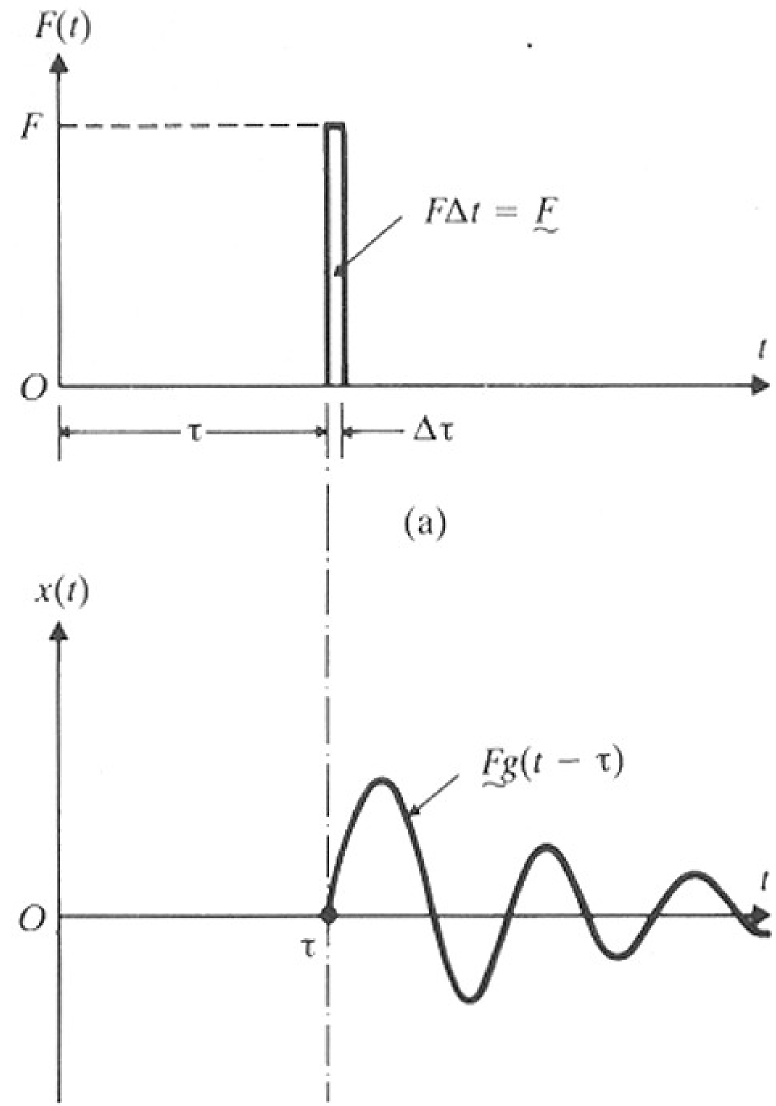
SDoF systems – General forcing functions – Nonperiodic

- **Convolution integral – Impulse response**
- If the magnitude of the impulse is $\underset{\sim}{F}$ instead of unity, the initial velocity $x'_0 = F/m$ and the response becomes:

$$x(t) = \frac{\underset{\sim}{F} e^{-\zeta\omega_n t}}{m\omega_d} \sin\left(\omega_d t\right) = \underset{\sim}{F}\, g(t)$$

- If the impulse is applied to a stationary system at an arbitrary time $t = \tau$ the response is

$$x(t) = \underset{\sim}{F}\, g(t-\tau)$$



(a)
(b)

SDoF systems – General forcing functions – Nonperiodic

- **Convolution integral – Arbitrary exciting force**
- If we consider the arbitrary force to comprise of a series of impulses of varying magnitudes such that at time τ, the force $F(\tau)$ acts on the system for a short period $\Delta\tau$.
- The impulse acting at $t = \tau$ is given by $F(\tau)\Delta\tau$.
- At any time t the elapsed time is $t - \tau$
- The system response at t due to the impulse is

$$x(t) = \tilde{F}\, g(t-\tau) = F(\tau)\Delta\tau\, g(t-\tau)$$

- The total response at time t is determined by summing the responses caused by the impulses acting al all times τ:

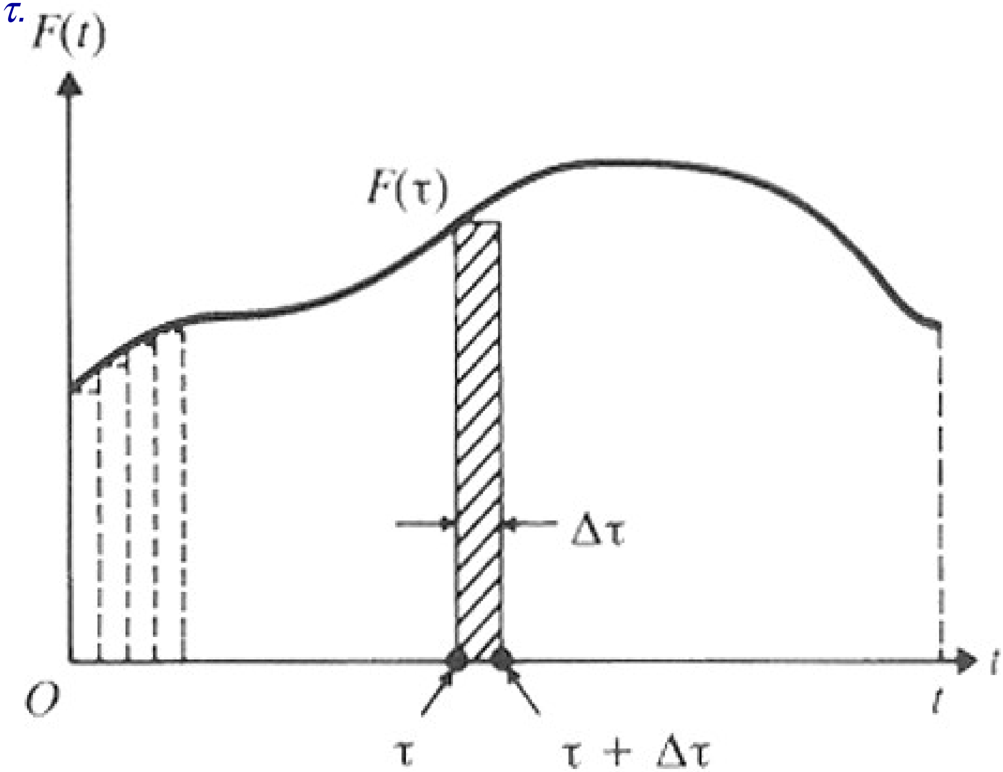
$$x(t) = \sum F(\tau)\, g(t-\tau)\, \Delta\tau$$

Making $\Delta\tau \to 0$ the response can be expressed as :

$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$$

Substituting the impulse response function $g(t-\tau)$:

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin\left[\omega_d\left(t-\tau\right)\right] d\tau \quad \neg \quad \text{Convolution or Duhamel integral}$$



- This solution does not account for initial conditions.
- Can be integrated explicitly or numerically depending on $F(t)$

SDoF systems – General forcing functions – Nonperiodic

- **Convolution integral – Arbitrary exciting force**
- In the case where the excitation is provided by an arbitrary imposed motion of the base, *y(t),* the relative displacement is given by:

$$z(t) = \frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta \omega_n (t-\tau)} \sin\left[\omega_d \left(t-\tau\right)\right] d\tau$$

Example: Step load

SDoF systems – General forcing functions – Nonperiodic

- When the forcing function is arbitrary and nonperiodic (aperiodic) it cannot be represented with a Fourier series
- Alternative methods for determining the response must be used:
  - Representation of the excitation function with a ***Convolution integral***
  - Using ***Laplace Transformations***
  - Approximating F(t) with a suitable ***interpolation method*** then using a numerical procedure
  - ***Numerical integration*** of the equations of motion.

SDoF systems – General forcing functions – Nonperiodic

- **Laplace Transformation**
- Efficient method to generate solution of linear differential equations
- Converts differential equations into algebraic equations to facilitate solving
- Can be applied to discontinuous functions
- Can be used for any type of excitation including periodic & harmonic
- Automatically accounts for initial conditions
- The Laplace transform of *x(t)* is given by:

$$\bar{x}(s) = \mathcal{L}\,x(t) = \int_0^\infty e^{-st}\,x(t)\,dt$$

- Where *s* the subsidiary variable and is usually complex.

- To use Laplace Transform:
  1. Write the equation of motion
  2. Compute or look-up the Laplace transform of each term using known initial conditions
  3. Solve the transformed (algebraic) equation of motion
  4. Use the inverse Laplace transform to obtain the response (solution)

SDoF systems – General forcing functions – Nonperiodic

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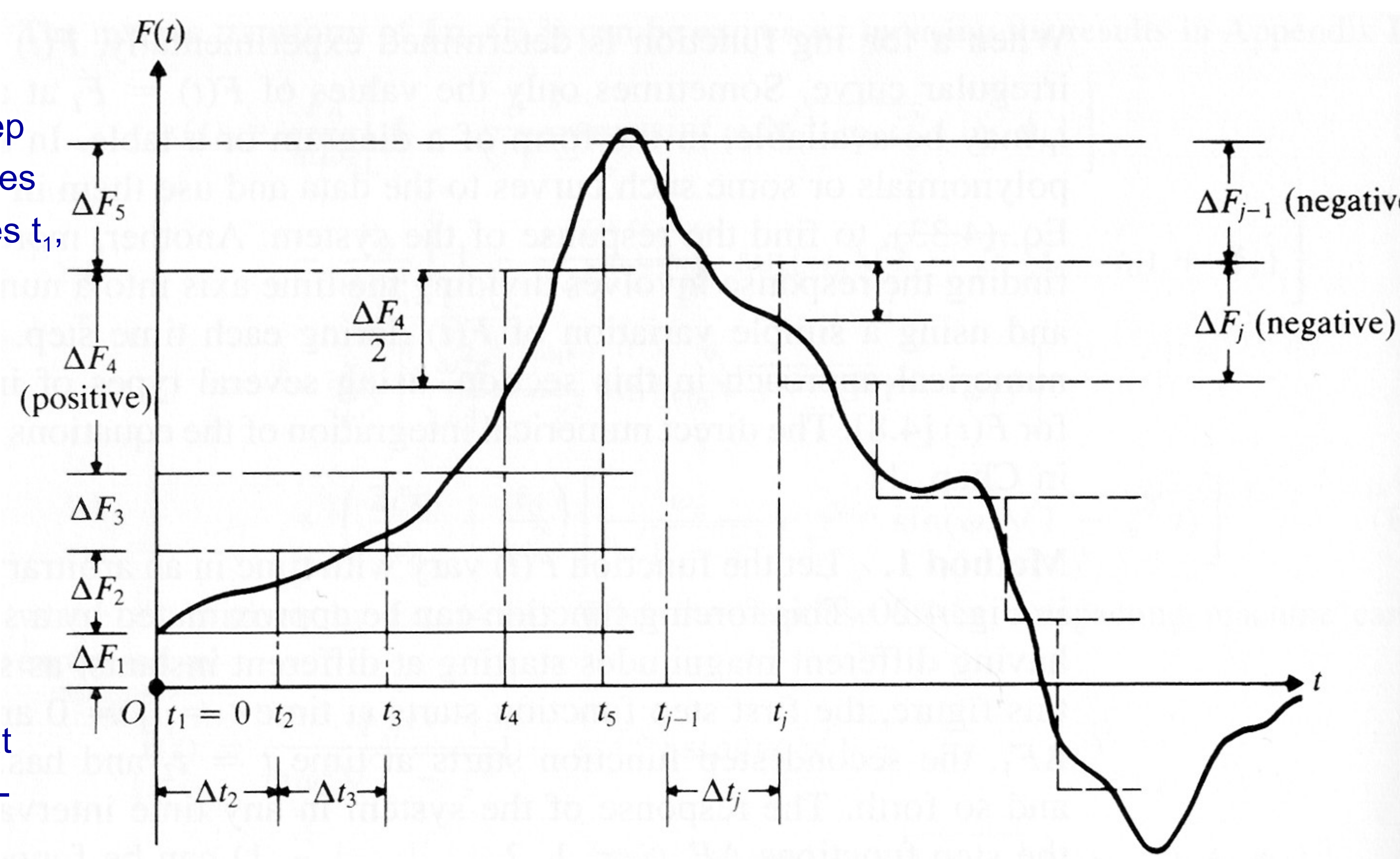
SDoF systems – General forcing functions – Nonperiodic

- **Numerical Methods (interpolation)**
- Used when the nonperiodic forcing function cannot be described mathematically
- It may be possible to “fit” a mathematical approximation (say polynomial) to data then use the convolution integral
- Often more practical to represent the digitised data with a series of incremental functions:
- **Step functions**
- The arbitrary function is represented by a series of step functions of varying magnitudes ΔF_1, ΔF_2, ΔF_3… and start times t_1, t_2, t_3….
- Note that the polarity of ΔF changes with the slope of the function
- Smaller intervals yield better accuracy.
- The approximation is also improved by choosing the subsequent start times so that F(t) intersects the step at mid-height of the step.



SDoF systems – General forcing functions – Nonperiodic

- **Numerical Methods (interpolation) - Step functions**
- The system response due to a step excitation ΔF_i for any time interval $t_{i-1} < t < t_i$ $(i = 1, 2, 3 \ldots.j\text{-}1)$ can be determined from the previous example:

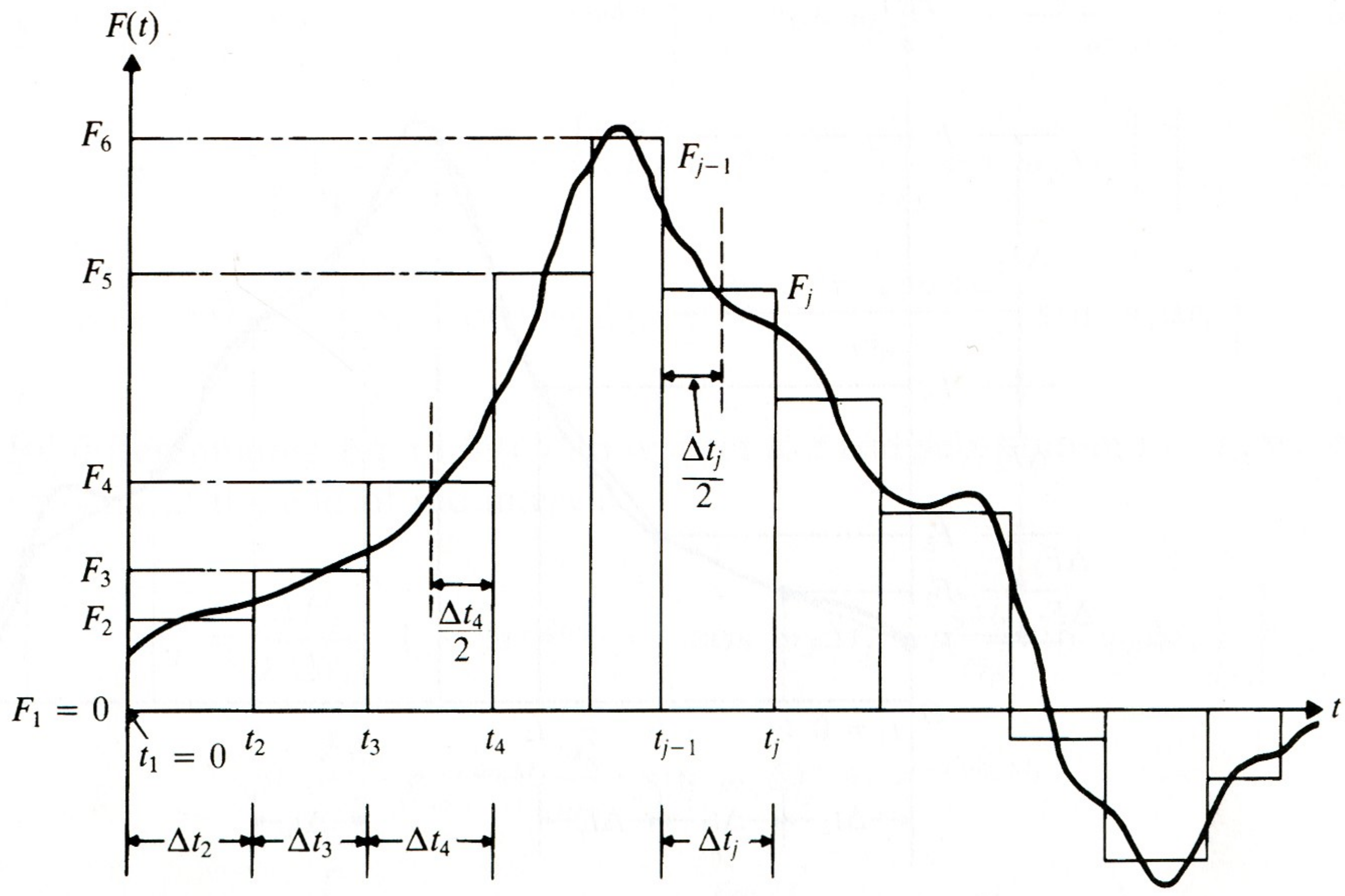
$$x(t) = \frac{1}{k}\sum_{i=1}^{j-1} \Delta F_i \left[1 - e^{-\zeta\omega_n(t-t_i)} \left\{ cos\left(\omega_d\left(t-t_i\right)\right) + \frac{\zeta\omega_n}{\omega_d} sin\left(\omega_d\left(t-t_i\right)\right) \right\} \right]$$

- When $t = t_j$ the response is:

$$x(t) = \frac{1}{k}\sum_{i=1}^{j-1} \Delta F_i \left[1 - e^{-\zeta\omega_n(t_j-t_i)} \left\{ cos\left(\omega_d\left(t_j-t_i\right)\right) + \frac{\zeta\omega_n}{\omega_d} sin\left(\omega_d\left(t_j-t_i\right)\right) \right\} \right]$$

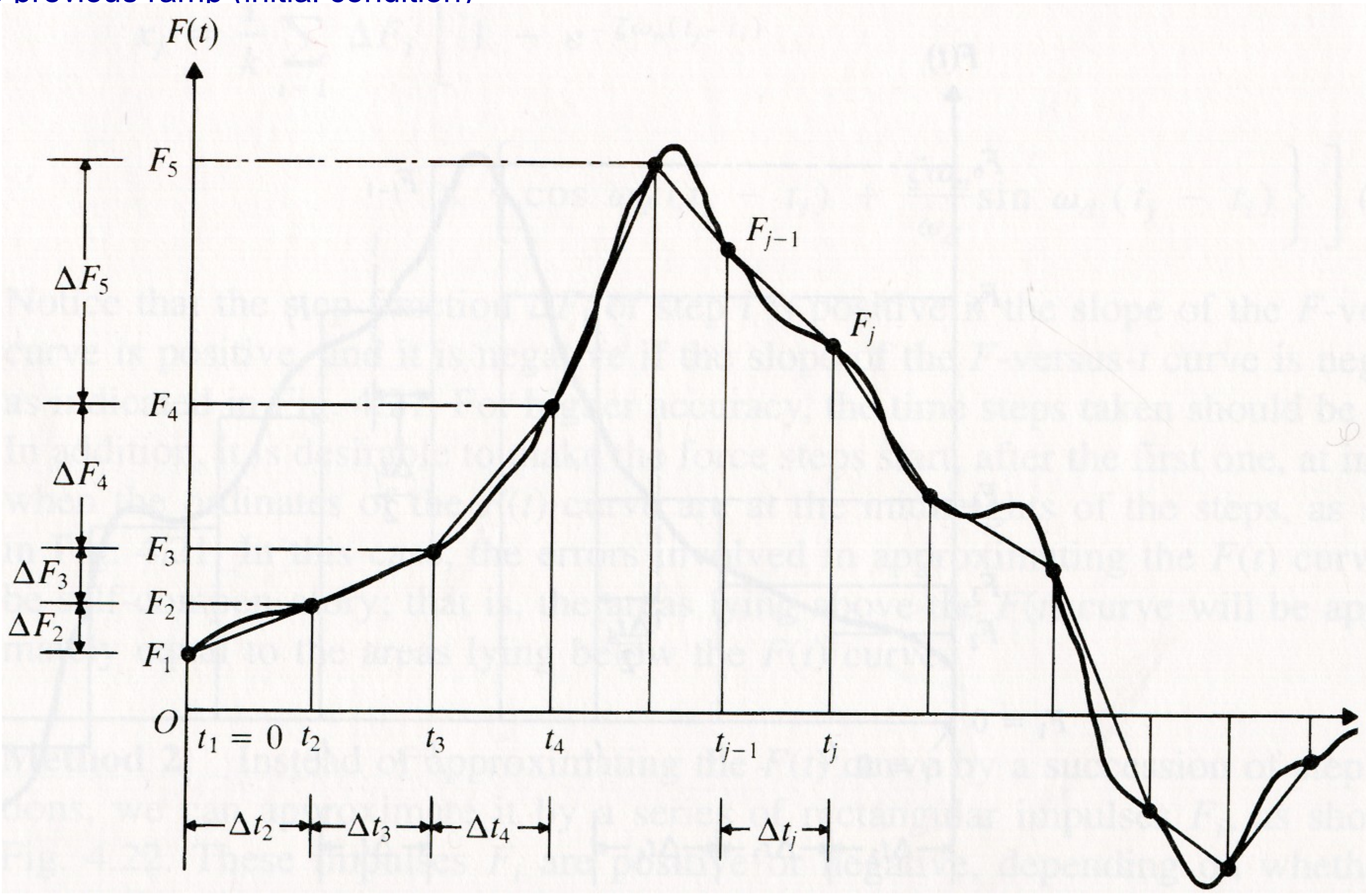
SDoF systems – General forcing functions – Nonperiodic

- **Numerical Methods (interpolation) - Rectangular impulses**
- The arbitrary function is represented by a series of rectangular impulses F_i the polarity of which depends on the polarity of F(t) at that instant.
- The response of the system in any time interval $t_{i-1} < t < t_i$ is obtained by adding the response caused by F_j (applied over Δt_j to the response at $t = t_i$ which represent the initial condition:



SDoF systems – General forcing functions – Nonperiodic

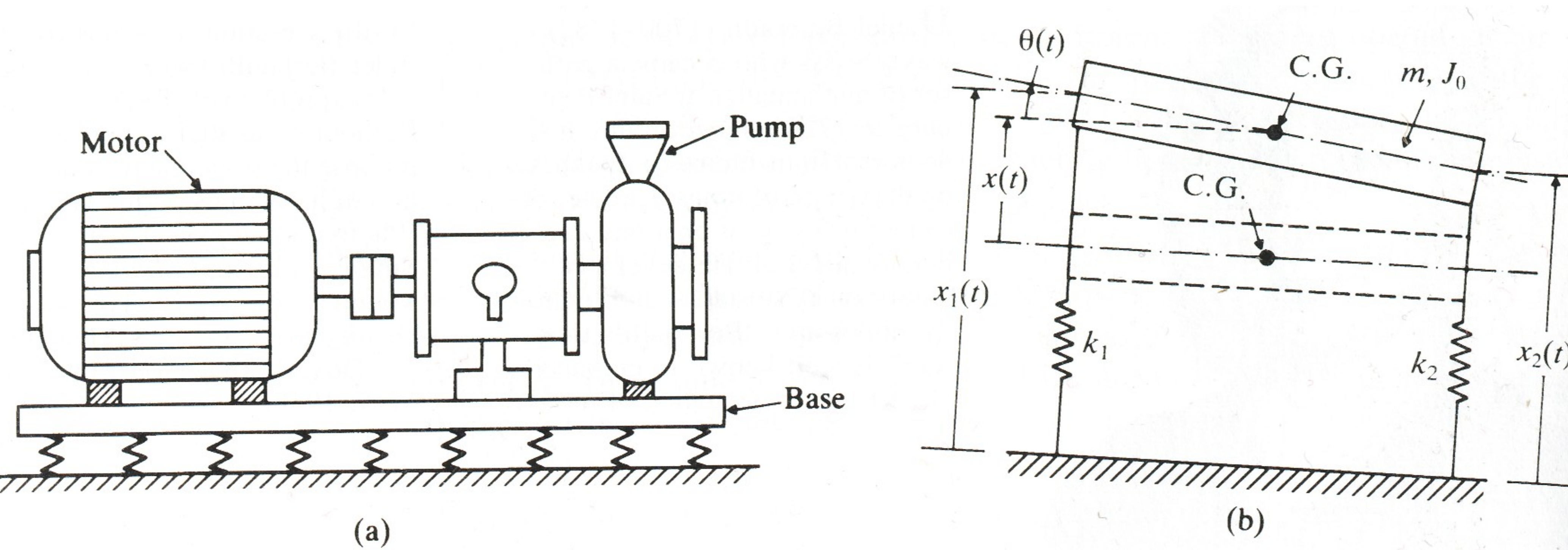
- **Numerical Methods (interpolation) – Ramps (linear) approximation**
- The arbitrary function is represented by a series of linear functions and the response of the system in any time interval $t_{i-1} < t < t_i$ is obtained by adding the response caused by the linear (ramp) during a specified interval to the response due to the previous ramp (initial condition)



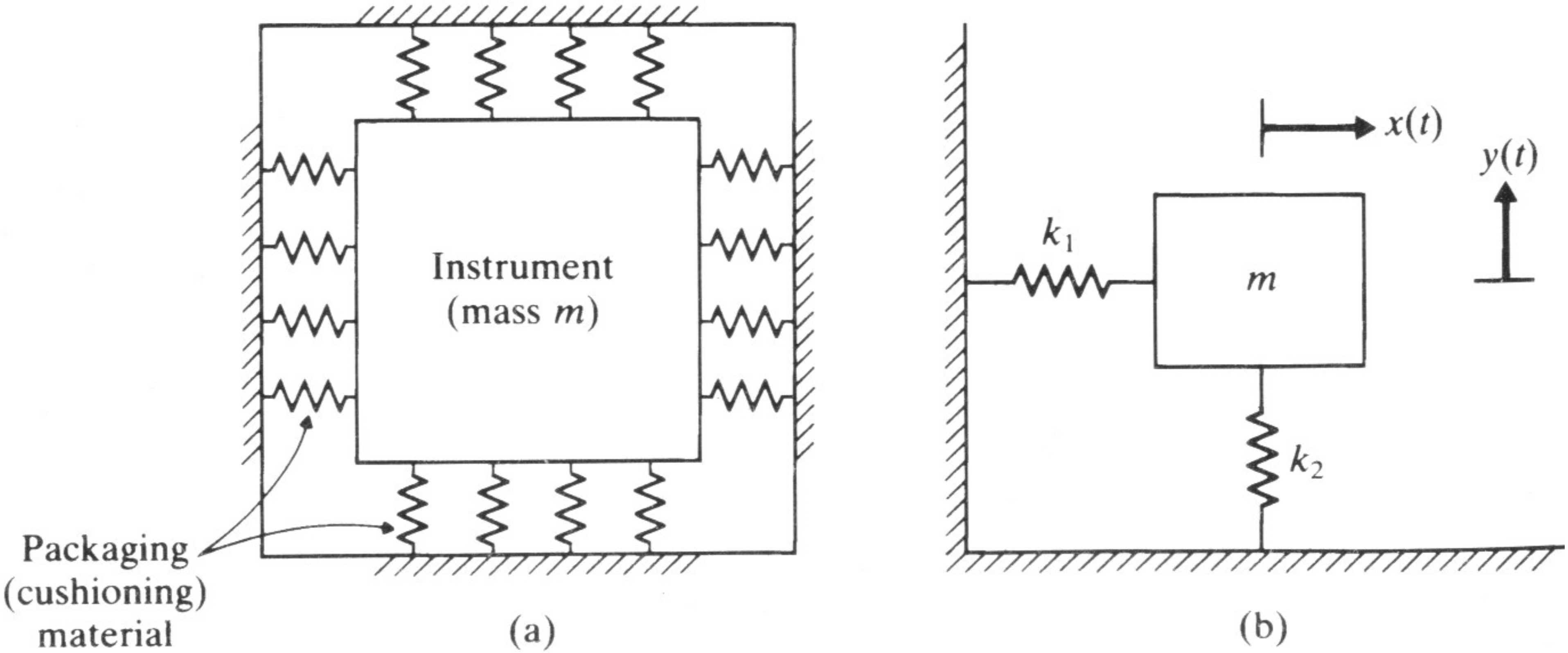
SDoF systems – General forcing functions – Nonperiodic

- **(Shock) Response Spectrum**
- Shows the variation in maximum response of a damped SDOF due to a particular transient (shock) excitation.
- The Shock Response Spectrum (SRS) is plotted for a range of natural frequencies usually at fractional octave intervals.
- The SRS is used to determine the effect of a particular (shock) excitation function on damped SDoF systems.
- Given the nature of real shocks, the SRS is usually computed using numerical means.

- **Two degree of freedom systems:**

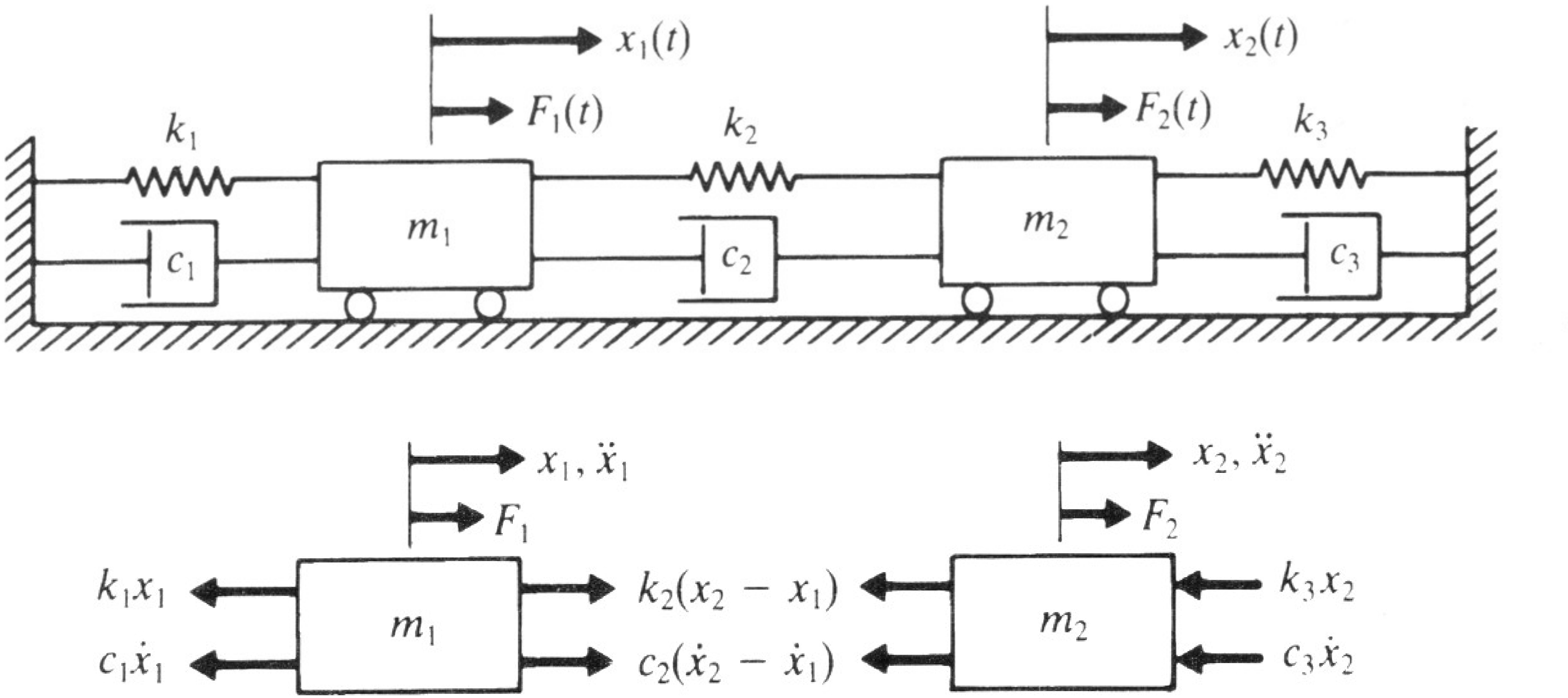


- **Two degree of freedom systems:**



- No. of DoF of system = No. of mass elements x number of motion types for each mass
- For each degree of freedom there exists an equation of motion – usually **coupled** differential equations.
- Coupled means that the motion in one coordinate system depends on the other
- If harmonic solution is assumed, the equations produce two natural frequencies and the amplitudes of the two degrees of freedom are related by the *natural, principal or normal* mode of vibration.
- Under an arbitrary initial disturbance, the system will vibrate freely such that the two normal modes are superimposed.
- Under sustained harmonic excitation, the system will vibrate at the excitation frequency. Resonance occurs if the excitation frequency corresponds to one of the natural frequencies of the system

- **Equations of motion**
- Consider a viscously damped system:
- Motion of system described by position $x_1(t)$ and $x_2(t)$ of masses m_1 and m_2
- The free-body diagram is used to develop the equations of motion using Newton's second law

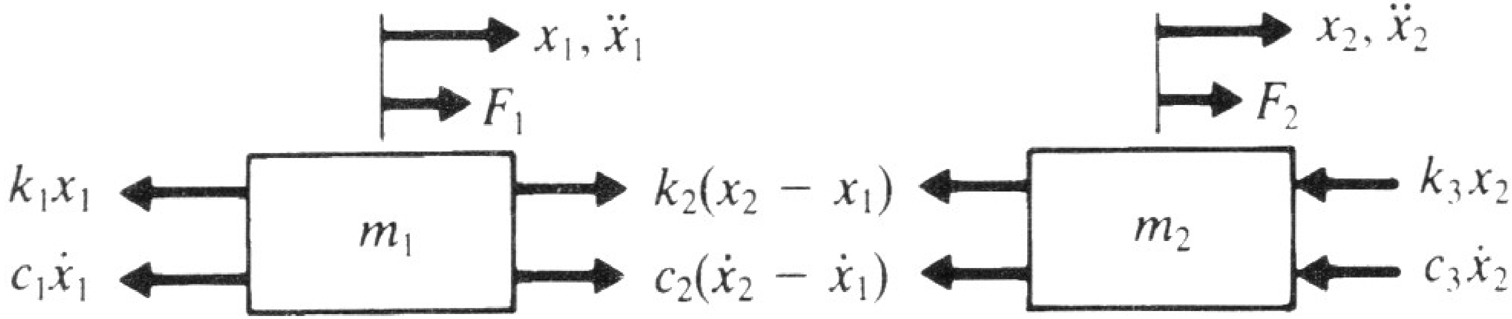


Spring k_1 under tension for $+x_1$

Spring k_2 under tension for $+(x_2 - x_1)$

Spring k_3 under compression for $+x_2$

- **Equations of motion**



$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1$$
$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + c_3\dot{x}_2 + k_3x_2 = F_2$$

or

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

- The differential equations of motion for mass m_1 and mass m_2 are **coupled**.
- The motion of each mass is influenced by the motion of the other.

- **Equations of motion**

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=F_1$$
$$m_2\ddot{x}_2-c_2\dot{x}_1+(c_2+c_3)\dot{x}_2-k_2x_1+(k_2+k_3)x_2=F_2$$

- The coupled differential eqns. of motion can be written in matrix form:

$$[m]\,\ddot{\vec{x}}(t)+[c]\,\dot{\vec{x}}(t)+[k]\,\vec{x}(t)=\vec{F}(t)$$

where $[m]$, $[c]$ and $[k]$ are the mass, damping and stiffness matrices respectively and are given by:

$$[m]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [c]=\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix} \qquad [k]=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

$\vec{x}(t)$, $\dot{\vec{x}}(t)$, $\ddot{\vec{x}}(t)$ and $\vec{F}(t)$ are the displacement, velocity, acceleration and force vectors respectively and are given by :

$$\vec{x}(t)=\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \quad \dot{\vec{x}}(t)=\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \quad \ddot{\vec{x}}(t)=\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} \quad and \quad \vec{F}(t)=\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

- Note: the mass, damping and stiffness matrices are all square and symmetric $[m] = [m]^T$ and consist of the mass, damping and stiffness constants.

- **Free vibrations of undamped systems**
- The eqns. of motion for a free and undamped TDoF system become:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

- Let us assume that the resulting motion of each mass is harmonic: For simplicity, we will also assume that the response frequencies and phase will be the same:

$$x_1(t) = X_1 \cos(\omega t + \phi) \quad \text{and} \quad x_2(t) = X_2 \cos(\omega t + \phi)$$

- Substituting the assumed solutions into the eqns. of motion:

$$\left[\left\{-m_1\omega^2 + (k_1 + k_2)\right\} X_1 - k_2X_2\right]\cos(\omega t + \phi) = 0$$
$$\left[-k_2X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\} X_2\right]\cos(\omega t + \phi) = 0$$

As these equations must be zero for all values of t, the cosine terms cannot be zero. Therefore:

$$\left\{-m_1\omega^2 + (k_1 + k_2)\right\} X_1 - k_2X_2 = 0$$
$$-k_2X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\} X_2 = 0$$

- Represent two simultaneous algebraic equations with a trivial solution when X_1 and X_2 are both zero – no vibration.

- **Free vibrations of undamped systems**
- Written in matrix form it can be seen that the solution exists when the determinant of the mass / stiffness matrix is zero:

$$\begin{bmatrix} \left\{-m_1\omega^2 + (k_1 + k_2)\right\} & -k_2 \\ -k_2 & \left\{-m_2\omega^2 + (k_2 + k_2)\right\} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

or

$$m_1 m_2 \omega^4 - \left\{(k_1 + k_2) m_2 + (k_2 + k_3) m_1\right\} \omega^2 + (k_1 + k_2)(k_2 + k_2) - k_2^2 = 0$$

- The solution to the ***characteristic equation*** yields the natural frequencies of the system.
- The roots of the characteristic equation are:

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2) m_2 + (k_2 + k_3) m_1}{m_1 m_2}\right\}$$
$$\pm \frac{1}{2}\left[\left\{\frac{(k_1 + k_2) m_2 + (k_2 + k_3) m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

- This shows that the homogenous solution is harmonic with natural frequencies ω_1 and ω_2

- **Free vibrations of undamped systems**
- Because the system is coupled, the constants X_1 and X_2 are a function of both natural frequencies ω_1 and ω_2
- Let the values of X_1 and X_2 corresponding to ω_1 be $X_1^{(1)}$ and $X_2^{(1)}$ and *those* corresponding to ω_2 be $X_1^{(2)}$ and $X_2^{(2)}$
- Since the simultaneous algebraic equations are homogeneous only the ***amplitude ratios*** $r_1 = (X_2^{(1)}/X_1^{(1)})$ and $r_2 = (X_2^{(2)}/X_1^{(2)})$ can be determined.
- Substituting ω_1 and ω_2 gives:

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2+k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + (k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2+k_3)}$$

$$\left\{-m_1\omega^2 + (k_1+k_2)\right\} X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \left\{-m_2\omega^2 + (k_2+k_3)\right\} X_2 = 0$$

- The normal modes of vibration corresponding to the natural frequencies ω_1 and ω_2 can be expressed in vector form known as the ***modal vectors:***

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix} \quad and \quad \vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

- The modal vectors describe the ***relative amplitude*** of vibration of each mass for each of the natural frequencies.

- **Free vibrations of undamped systems**
- The motion (free vibration) of each mass is given by:

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} \rightarrow First\ mod\ e$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} \rightarrow First\ mod\ e$$

- The constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 *and* ϕ_2 are determined from the initial conditions.

- **Free vibrations of undamped systems**
- Two initial conditions for each mass need to be specified (second order D.E.s)
- The system can be made to vibrate freely in either mode *(i = 1, 2)* by applying the appropriate initial conditions

$$x_1(t=0) = X_1^{(i)} \qquad \dot{x}_1(t=0) = 0$$

$$x_2(t=0) = r_1 X_1^{(i)} \qquad \dot{x}_2(t=0) = 0$$

- Any other combination of initial conditions will result in the excitation of both modes
- Two initial conditions for each mass need to be specified (second order D.E.s)
- The resulting motion is obtained by superposition of the normal modes:

$$\vec{x}(t) = \vec{x}^{(1)}(t) + \vec{x}^{(2)}(t)$$

or

$$\vec{x}_1(t) = \vec{x}_1^{(1)}(t) + \vec{x}_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$\vec{x}_2(t) = \vec{x}_2^{(1)}(t) + \vec{x}_2^{(2)}(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

- If the initial conditions are:

$$x_1(t=0) = x_1(0) \qquad \dot{x}_1(t=0) = \dot{x}_1(0)$$

$$x_2(t=0) = x_2(0) \qquad \dot{x}_2(t=0) = \dot{x}_2(0)$$

- The constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 *and* ϕ_2 can be by substituting the initial conditions in the combined motion eqns.

- **Free vibrations of undamped systems**

$$x_1(t) = X_1^{(1)} cos(\omega_1 t + \phi_1) + X_1^{(2)} cos(\omega_2 t + \phi_2)$$

$$\vec{x}_2(t) = r_1 X_1^{(1)} cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} cos(\omega_2 t + \phi_2)$$

substituting the initial conditions:

$$x_1(0) = X_1^{(1)} cos(\phi_1) + X_1^{(2)} cos(\phi_2)$$

$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} sin(\phi_1) - \omega_2 X_1^{(2)} sin(\phi_2)$$

$$x_2(0) = r_1 X_1^{(1)} cos(\phi_1) + r_2 X_1^{(2)} cos(\phi_2)$$

$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)} sin(\phi_1) - \omega_2 r_2 X_1^{(2)} sin(\phi_2)$$

The following unknowns can be identified:

$$x_1(0) = \boldsymbol{X_1^{(1)} cos(\varphi_1)} + \boldsymbol{X_1^{(2)} cos(\varphi_2)}$$

$$\dot{x}_1(0) = -\omega_1 \boldsymbol{X_1^{(1)} sin(\varphi_1)} - \omega_2 \boldsymbol{X_1^{(2)} sin(\varphi_2)}$$

$$x_2(0) = r_1 \boldsymbol{X_1^{(1)} cos(\varphi_1)} + r_2 \boldsymbol{X_1^{(2)} cos(\varphi_2)}$$

$$\dot{x}_2(0) = -\omega_1 r_1 \boldsymbol{X_1^{(1)} sin(\varphi_1)} - \omega_2 r_2 \boldsymbol{X_1^{(2)} sin(\varphi_2)}$$

- **Free vibrations of undamped systems**
- Solving for the identified constants yields:

$$X_1^{(1)} \cos(\phi_1) = \left\{ \frac{r_2 x_1(0) - x_2(0)}{r_2 - r_1} \right\} \qquad X_1^{(2)} \cos(\phi_2) = \left\{ \frac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1} \right\}$$

$$X_1^{(1)} \sin(\phi_1) = \left\{ \frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1 \left(r_2 - r_1 \right)} \right\} \qquad X_1^{(2)} \sin(\phi_2) = \left\{ \frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2 \left(r_2 - r_1 \right)} \right\}$$

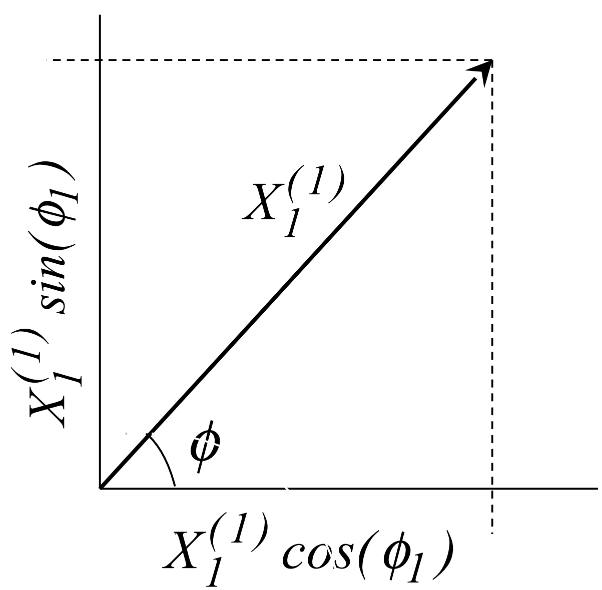
Therefore:

$$X_1^{(1)} = \sqrt{\left\{ X_1^{(1)} \cos(\phi_1) \right\}^2 + \left\{ X_1^{(1)} \sin(\phi_1) \right\}^2}$$

$$X_1^{(2)} = \sqrt{\left\{ X_1^{(2)} \cos(\phi_2) \right\}^2 + \left\{ X_1^{(2)} \sin(\phi_2) \right\}^2}$$

$$\phi_1 = a\tan \left\{ \frac{X_1^{(1)} \sin(\phi_1)}{X_1^{(1)} \cos(\phi_1)} \right\}$$

$$\phi_2 = a\tan \left\{ \frac{X_1^{(2)} \sin(\phi_2)}{X_1^{(2)} \cos(\phi_2)} \right\}$$



- **Free vibrations of undamped systems**
- In terms of the amplitude ratios r_i and natural frequencies ω_i:

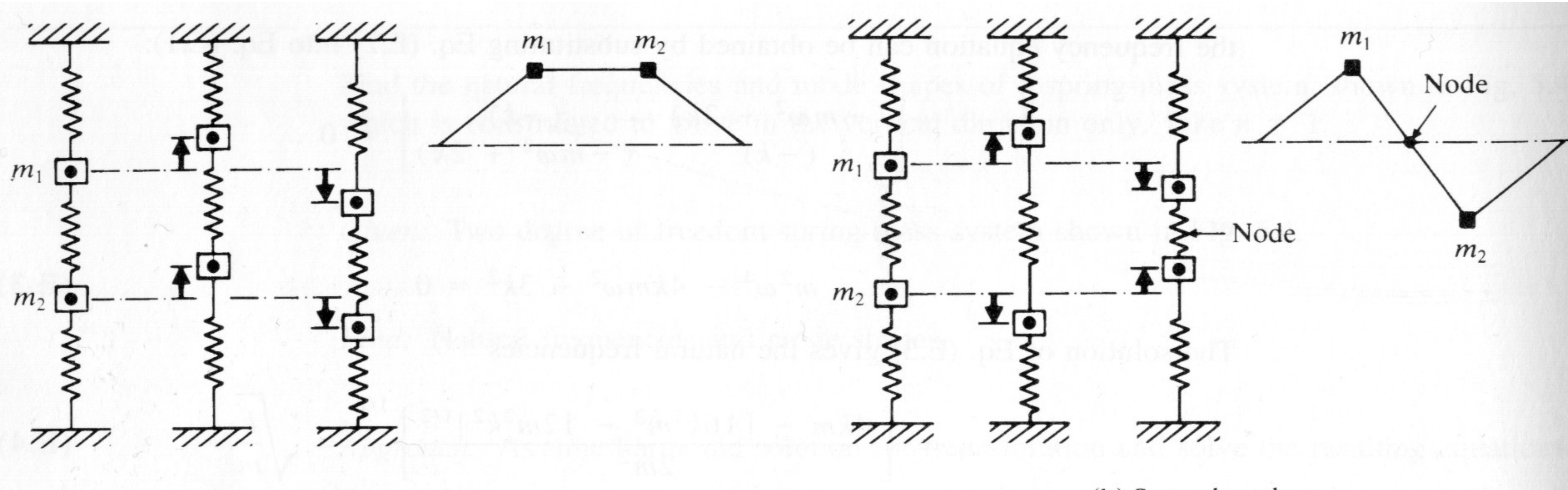
$$X_1^{(1)} = \frac{1}{(r_2 - r_1)}\sqrt{\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2\dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2}}$$

$$X_1^{(2)} = \frac{1}{(r_2 - r_1)}\sqrt{\{-r_1 x_1(0) - x_2(0)\}^2 + \frac{\{r_1\dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_2^2}}$$

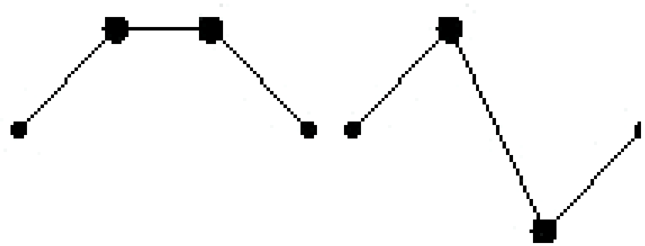
$$\phi_1 = a\tan\left\{\frac{-r_2\dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[r_2 x_1(0) - x_2(0)]}\right\}$$

$$\phi_2 = a\tan\left\{\frac{r_1\dot{x}_1(0) + \dot{x}_2(0)}{\omega_2[-r_1 x_1(0) - x_2(0)]}\right\}$$

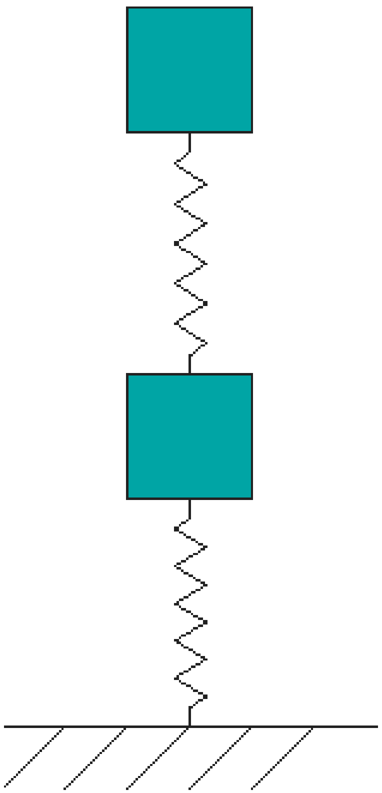
- **Free vibrations of undamped system**
- Example:



(a) First mode (b) Second mode



- **Free vibrations of undamped system**
- Example:

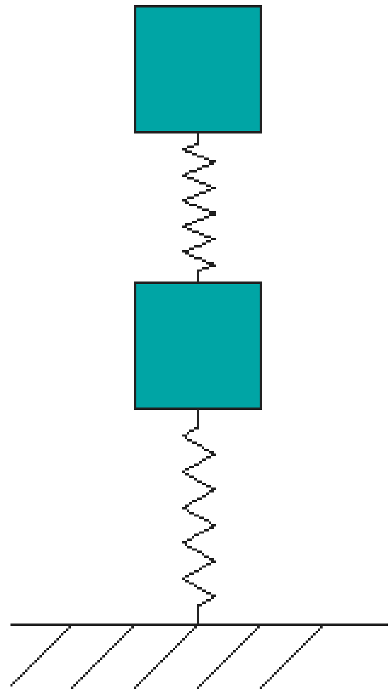


Masses: 0.71 kg each

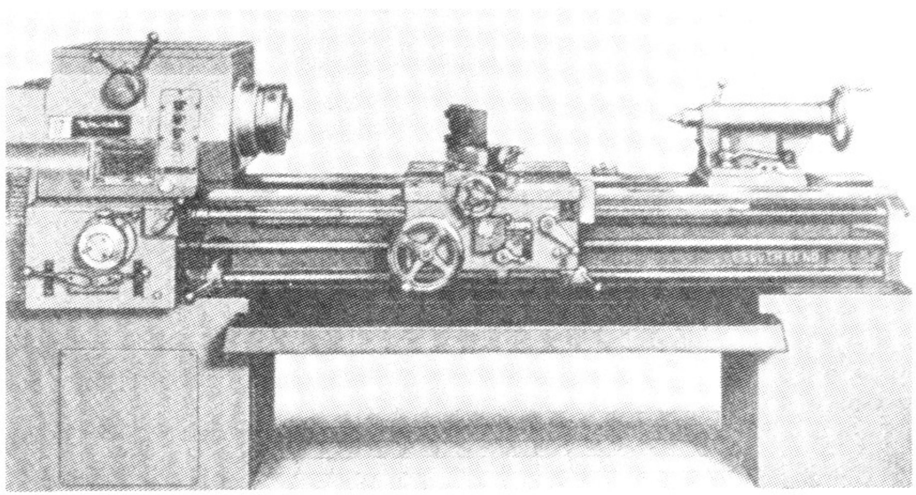
Middle spring: 175 N/m

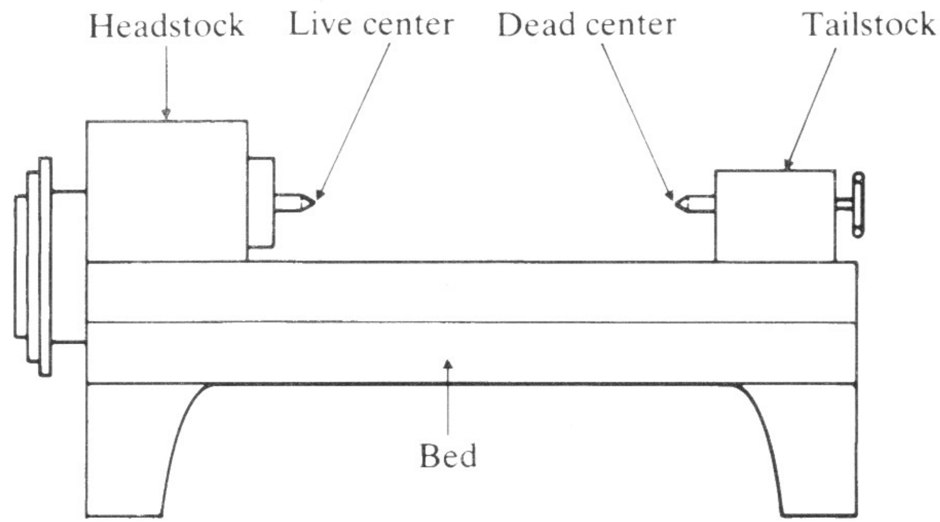
Bottom spring: 350 N/m

Animations courtesy Tom Irvine (Vibrationdata)



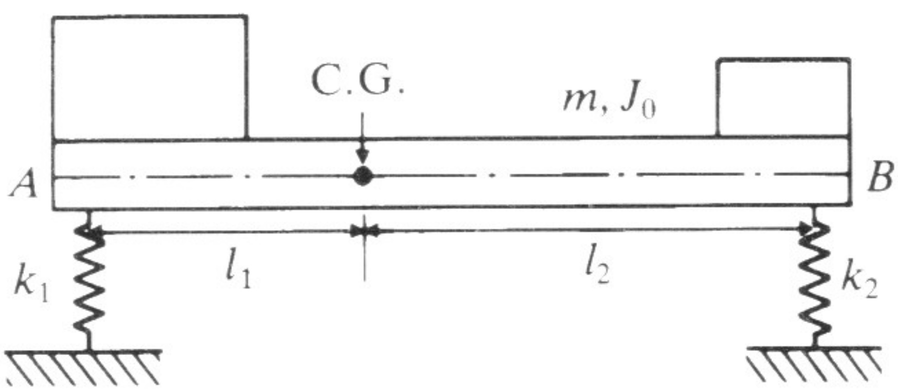
- **Coordinate Coupling**
- Whenever possible, the coordinates are chosen so that they are independent based from the equilibrium position.
- In some cases, another pair of coordinates may be used – ***generalised coordinates***

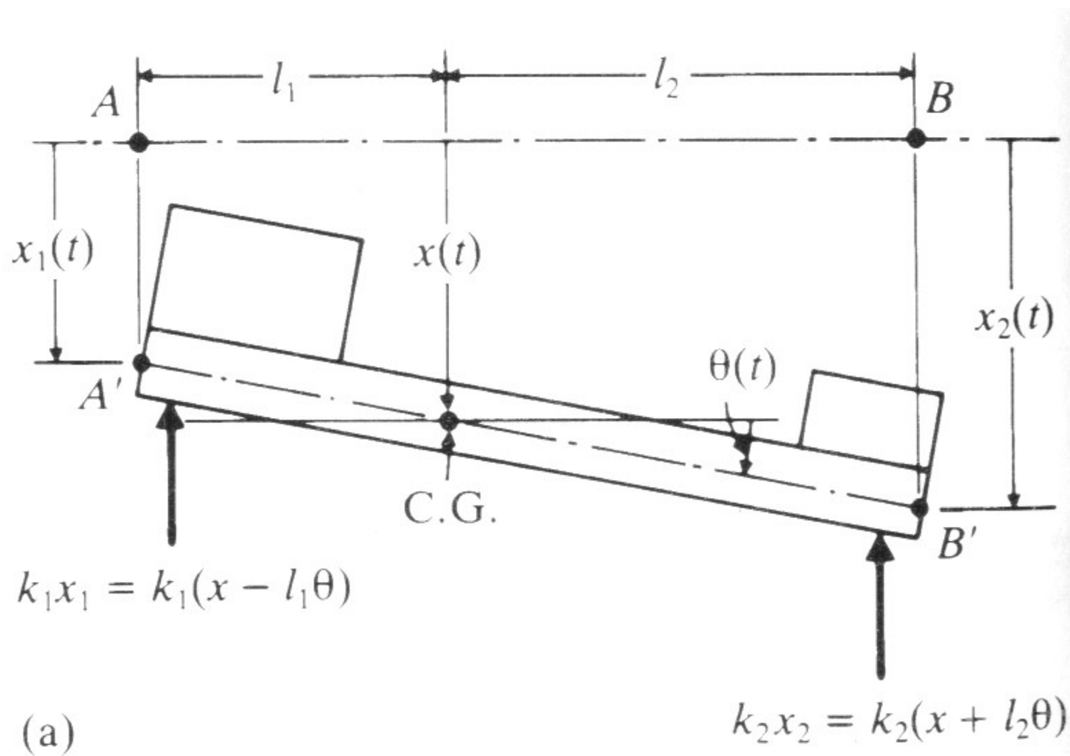




- The lathe can be simplified to be represented by a 2DoF with the bed considered as a rigid body with two lumped masses representing the headstock and tailstock assemblies. The supports are represented by two springs.
- The following set of coordinates can be used to describe the system:

- **Coordinate Coupling**
- (1): the deflection at each extremity of the lathe $x_1(t)$ and $x_2(t)$
- (2): the deflection at the centre of gravity $x(t)$ and the rotation $\theta(t)$
- (3): the deflection at extremity A $x_1(t)$ and the rotation $\theta(t)$



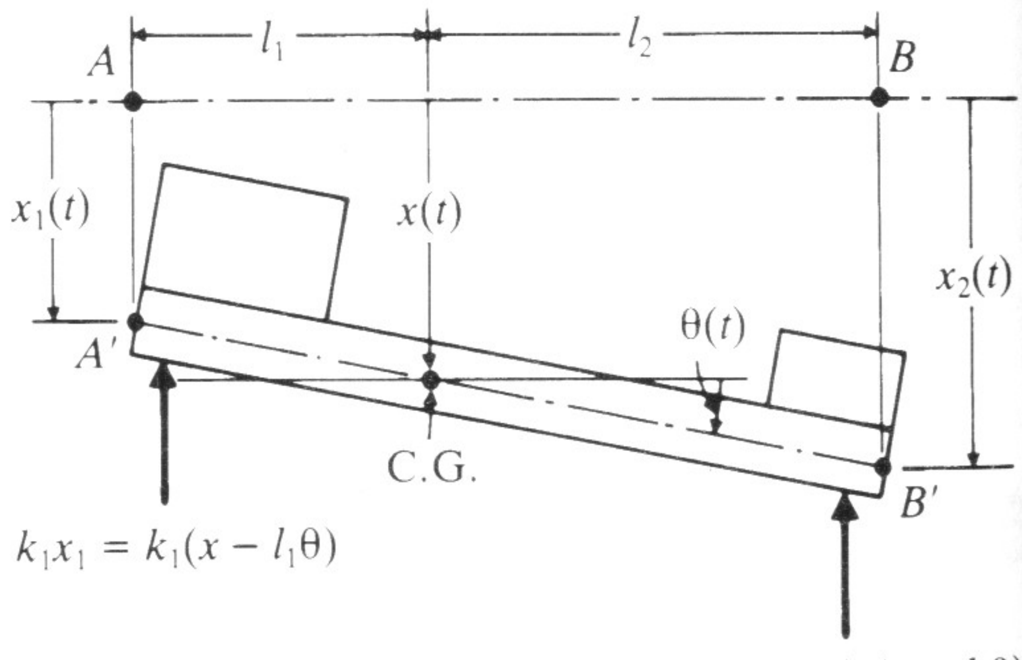


- **Coordinate Coupling**
- Equations of motion using *x(t)* and *θ(t)*
- Using the FBD, in the vertical direction and about the C.G. respectively:
- $m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \quad and \quad J_o\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2$

in matrix form:

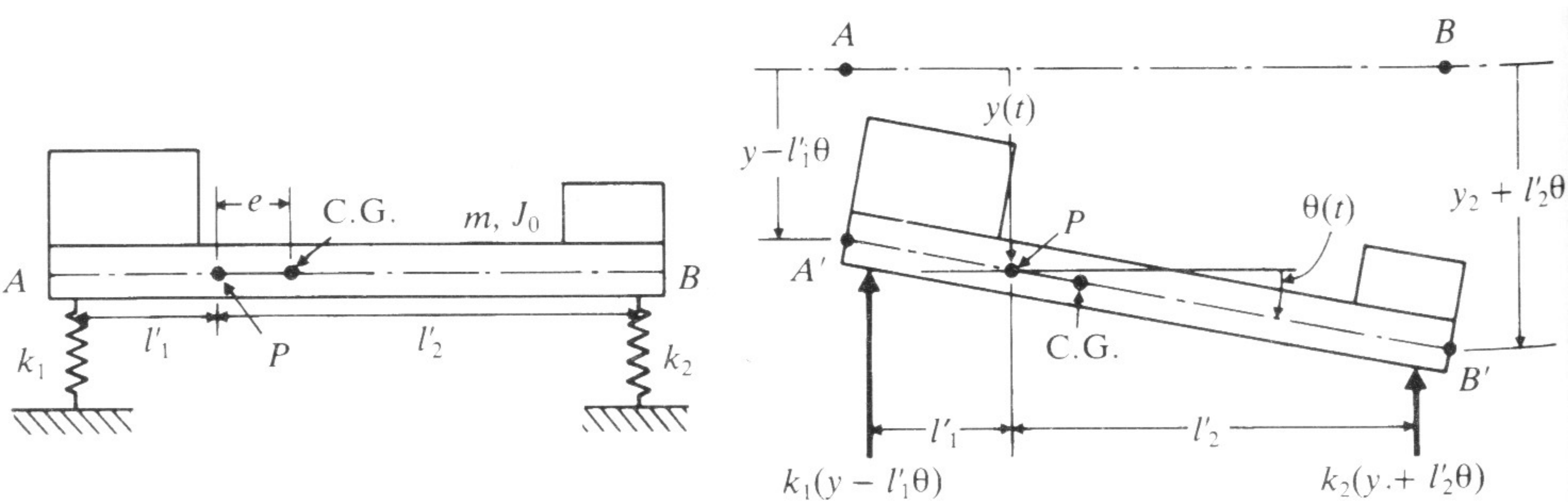
$$\begin{bmatrix} m & 0 \\ 0 & J_o \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & (k_1l_1^2 + k_2l_2^2) \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- As each eqn. contains both *x* and θ the system is coupled – ***Elastic*** or ***static coupling***
- Whenever a displacement or torque is applied thru the C.G. the resulting motion will contain **both** translation and rotation.
- The system is uncoupled (eqns. independent) **only** when $k_1l_1 = k_2l_2$
- Only then can pure translation or rotation be generated by a displacement or torque thru the C.G.



(a)

- **Coordinate Coupling**
- (1): the deflection *y(t)* at point P located at distance *e* to the left of the C.G. and the rotation *θ(t)*



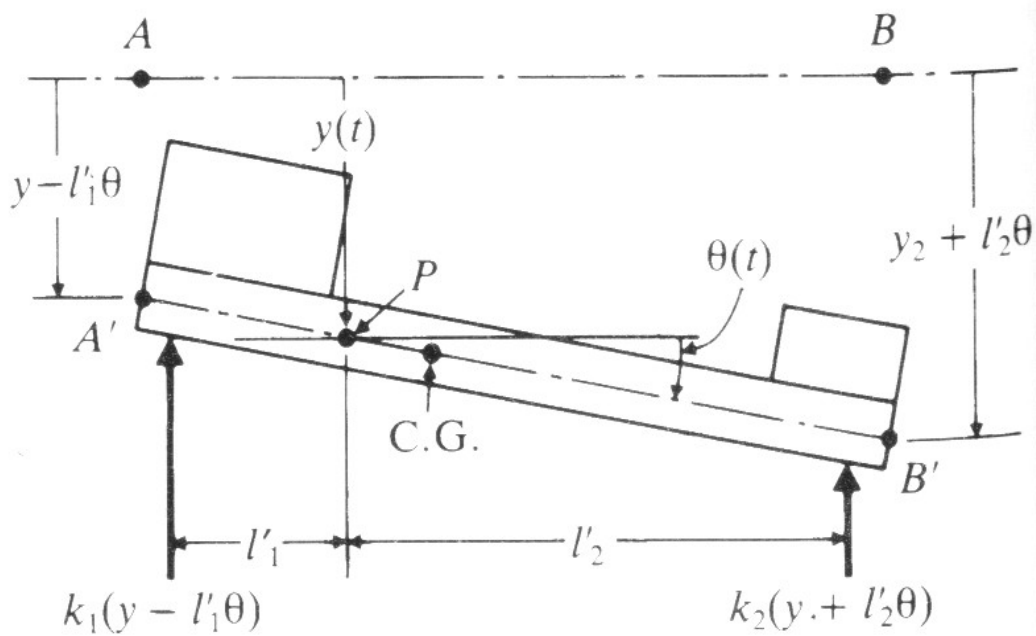
- **Coordinate Coupling**
- Using the FBD, the translational and rotational equations of motion are:

$$m\ddot{y} = -k_1(y - l_1'\theta) - k_2(y - l_2'\theta) - me\ddot{\theta} \qquad and \qquad J_p\ddot{\theta} = k_1(y - l_1'\theta)l_1' - k_2(y - l_2'\theta)l_2' - me\ddot{y}$$

in matrix form:

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & \left(k_2 l_2' - k_1 l_1'\right) \\ \left(k_2 l_2' - k_1 l_1'\right) & \left(k_1 l_1'^2 + k_2 l_2'^2\right) \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- As each eqn. contains both *y, y”, θ and θ’’* the system is coupled with both ***elastic*** (***static) and mass (dynamic) coupling***
- When $k_1 l'_1 = k_2 l'_2$, the system is **dynamically** coupled **only** → the inertial force my’’ produced by vertical motion will induce a rotational motion (my’’e) and vice verca.

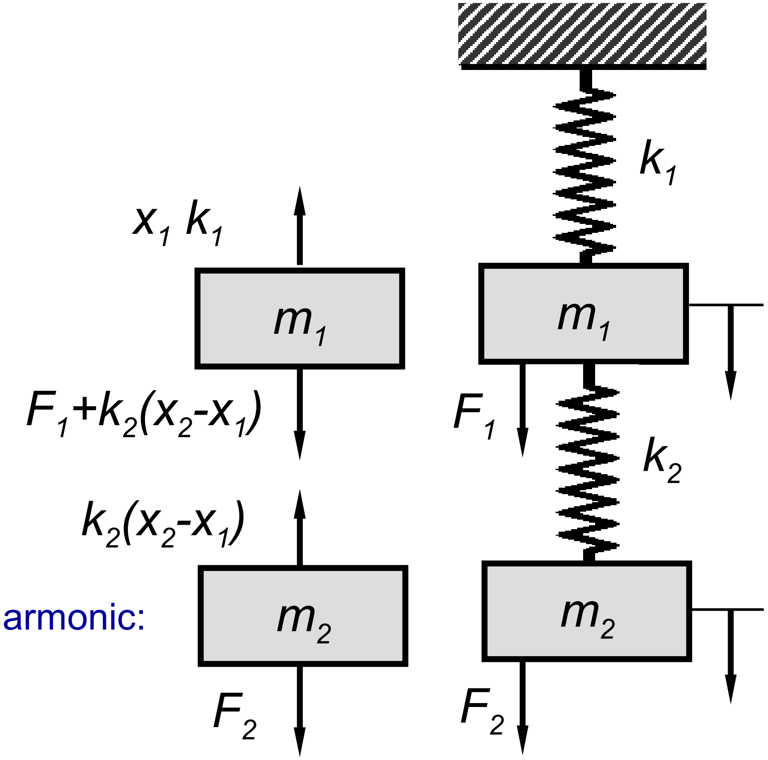


- **Coordinate Coupling**
- General case for viscously damped 2DoF:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- System has elastic (static) coupling if the stiffness matrix is not diagonal
- System has damping or velocity (dynamic) coupling if the damping matrix is not diagonal
- System has mass or inertial (dynamic) coupling if the mass matrix is not diagonal

- The system behaviour does not depend on the choice of coordinates!
- There exists a set of coordinates which will produce (statically and dynamically) uncoupled equations of motions → ***principal*** or ***natural*** coordinates. These uncoupled equations can be solved independently.



- **Harmonically forced vibrations – undamped**
- The harmonic excitation forces are:

$$F_1(t) = F_1 \sin(\omega_f t) \quad and \quad F_2(t) = F_2 \sin(\omega_f t)$$

where ω_f is the forcing frequency.

- Applying Newton's 2nd law gives the eqns. of motion:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1 \sin(\omega_f t)$$
$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = F_2 \sin(\omega_f t)$$

- Assuming that the solutions will take the form of the excitation – harmonic:

$$x_1 = X_1 \sin(\omega_f t) \quad and \quad x_2 = X_2 \sin(\omega_f t)$$

- Substituting for x_1 and x_2 in the eqns. of motion:

$$(-m_1\omega_f^2 + k_1 + k_2)X_1 \sin(\omega_f t) - k_2 X_2 \sin(\omega_f t) = F_1 \sin(\omega_f t)$$
$$(-m_2\omega_f^2 + k_2)X_2 \sin(\omega_f t) - k_2 X_1 \sin(\omega_f t) = F_2 \sin(\omega_f t)$$

- **Harmonically forced vibrations – undamped**

Dividing throughout by $\sin(\omega_f t)$ and putting in matrix form :

$$\begin{bmatrix} \left(k_1 + k_2 - m_1\omega_f^2\right) & -k_2 \\ -k_2 & \left(k_2 - m_2\omega_f^2\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

or

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \rightarrow \quad d_{11}X_1 + d_{12}X_2 = F_1 \quad \text{and} \quad d_{21}X_1 + d_{22}X_2 = F_2$$

The response amplitudes X_1 and X_2 can be determined using Cramer's rule:

$$X_1 = \frac{\begin{vmatrix} F_1 & d_{12} \\ F_2 & d_{22} \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{22}F_1 - d_{12}F_2}{d_{11}d_{22} - d_{21}d_{12}} \quad \text{and} \quad X_2 = \frac{\begin{vmatrix} d_{11} & F_1 \\ d_{21} & F_2 \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{11}F_2 - d_{21}F_1}{d_{11}d_{22} - d_{21}d_{12}}$$

- Note: the determinant (characteristic equation) can be equated to zero ($d_{11}d_{22} - d_{21}d_{12} = 0$) to define the system natural frequencies.
- Under forced excitation, when $d_{11}d_{22} - d_{21}d_{12} = 0$ the response amplitudes X_1 and $X_2 \rightarrow \infty$
- This defines resonance conditions (excitation frequency corresponds to either natural frequencies)
- Note: Due to coupling both masses will exhibit resonance when the excitation force is applied to only one mass:

- **Harmonically forced vibrations – undamped absorber**
- A mass-spring assembly added to a single degree of freedom with a natural frequency ω_n tuned to the forcing frequency ω_f will act as a vibration absorber and reduce the vibration of the main mass to zero.
- Undamped vibration absorbers are designed so that the natural frequencies of the resulting system are displaced away from the excitation frequency.
- The equations of motion of the main mass m_1 and the auxiliary mass m_2 are:

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_0 \sin(\omega t)$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

Rearranging

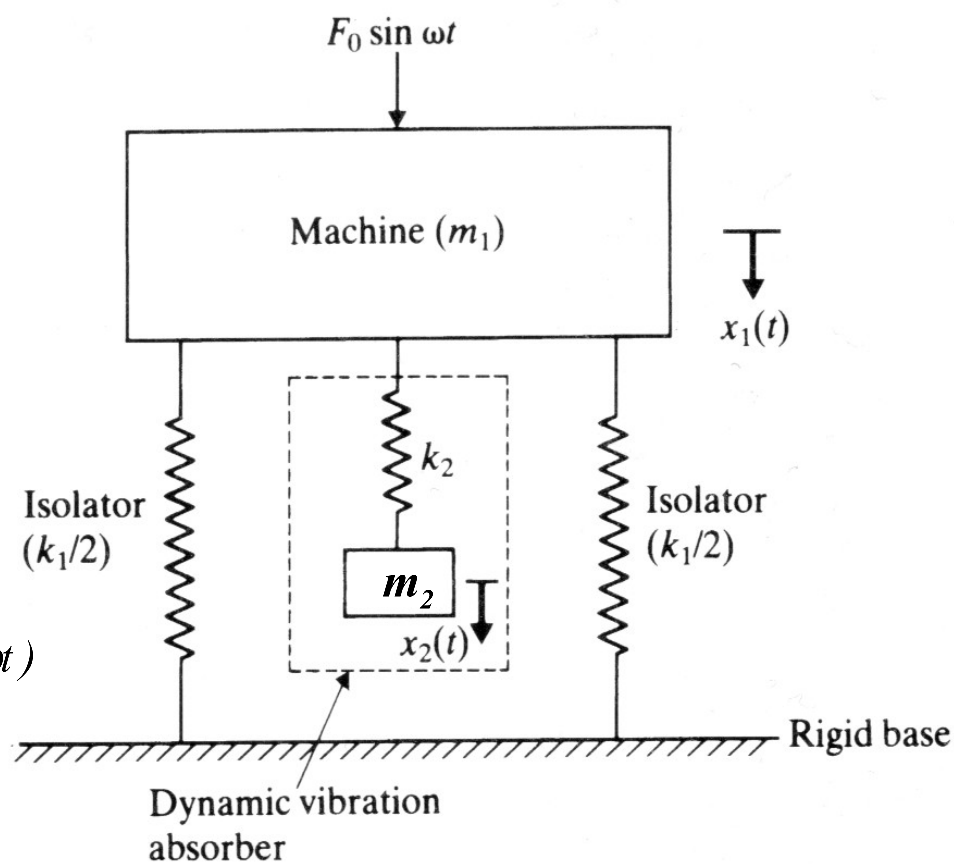
$$m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = F_0 \sin(\omega t)$$
$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

Assuming harmonic solutions

$$x_j(t) = X_j \sin(\omega t) \qquad j = 1, 2$$

And substituting into the eqns. of motion:

$$\left[-\omega^2 m_1 X_1 + (k_1 + k_2)X_1 - k_2X_2\right]\sin(\omega t) = F_0 \sin(\omega t)$$
$$-\omega^2 m_2 X_2 + k_2X_2 - k_2X_1 = 0$$



- **Harmonically forced vibrations – undamped absorber**

In matrix form :

$$\begin{bmatrix} -\omega^2 m_1 + (k_1 + k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Using Cramer's rule to determine the response amplitudes X_1 and X_2 :

$$X_1 = \frac{\begin{vmatrix} F_1 & d_{12} \\ F_2 & d_{22} \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{22}F_1 - d_{12}F_2}{d_{11}d_{22} - d_{21}d_{12}} \quad \text{and} \quad X_2 = \frac{\begin{vmatrix} d_{11} & F_1 \\ d_{21} & F_2 \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{d_{11}F_2 - d_{21}F_1}{d_{11}d_{22} - d_{21}d_{12}}$$

Or

$$X_1 = \frac{\left(k_2 - \omega^2 m_2\right) F_0}{\left(k_1 + k_2 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - k_2^2} \quad \text{and} \quad X_2 = \frac{k_2 F_0}{\left(k_1 + k_2 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - k_2^2}$$

- In order to minimise the amplitude of mass 1, the numerator of X_1 should be equated to zero which produces:

$$\omega^2 = \frac{k_2}{m_2}$$

- **Harmonically forced vibrations – undamped absorber**

If the original machine was operating near resonance :

$$\omega^2 \; ; \; \omega_1^2 = \frac{k_1}{m_1}$$

If the absorber is designed so that its natural frequency corresponds to the forcing frequency :

$$\omega^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1}$$

The amplitude of the machine (m_1) at its original resonant frequency will be zero.

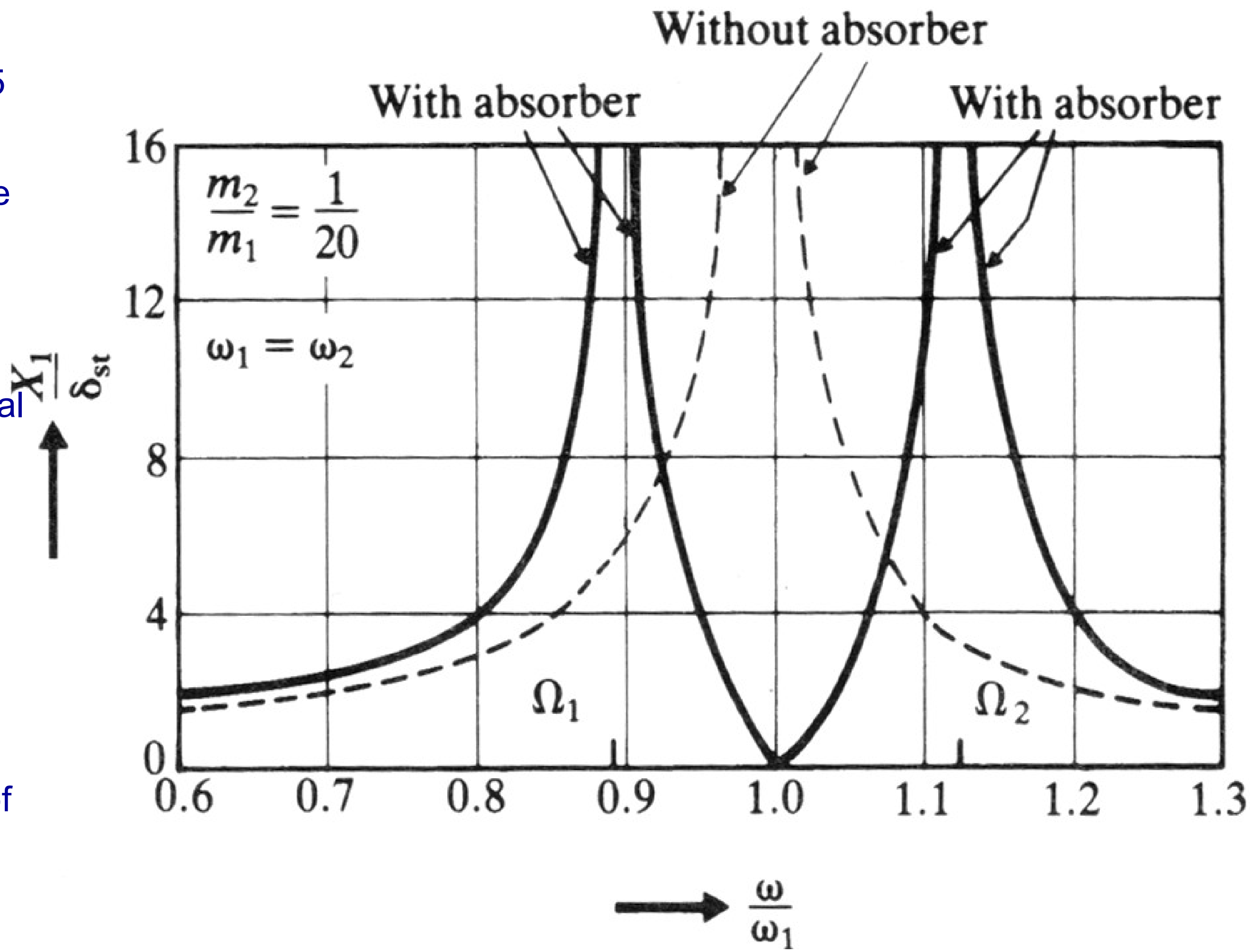
Since

$$\delta_{st} = \frac{F_0}{k_1}, \quad \omega_1 = \sqrt{\frac{k_1}{m_1}} \quad and \quad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

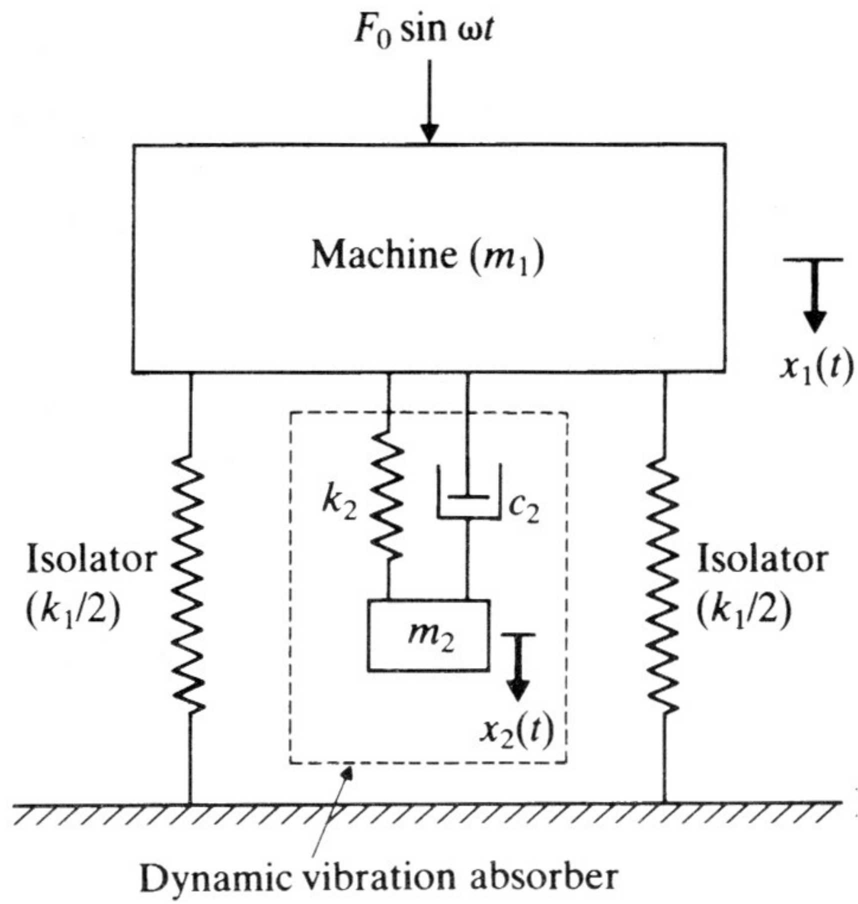
The dynamic response (magnification factor) of the main mass and the auxiliary mass (absorber) are :

$$\frac{X_1}{\delta_{st}} = \frac{1-\left(\frac{\omega}{\omega_2}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}} \quad \text{and} \quad \frac{X_2}{\delta_{st}} = \frac{1}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}}$$

- **Harmonically forced vibrations – undamped absorber**
- The size of the auxiliary mass m_2 is governed by the allowable deflection X_2.
- These systems can be quite effective over a reasonable frequency band ± 5 %.
- The new system has an added degree of freedom hence two resonance peaks.
- The system will pass thru the first resonance during startup, it is essential that the run-up time is minimised.
- Otherwise, introduce damping to prevent large vibrations of m_1 if the excitation frequency is likely to vary.
- At $\omega = \omega_1$ $X_1 = 0$ and $X_2 = -k_1 \delta_{st}/k_2 = -F_0/k_2$ which shows that the force exerted by the absorber mass is out of phase with (counteracts) the exciting force which causes X_1 to reduce to zero.



- **Harmonically forced vibrations – damped absorber**



- Introducing a viscous damper produces the following eqns. of motion:

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) = F_0 \sin(\omega t)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = 0$$

Assuming harmonic solutions in the form :

$$x_j(t) = X_j e^{i\omega t} \qquad j = 1, 2$$

Yields the steady-state amplitudes:

$$X_1 = \frac{F_0\left(k_2 - \omega^2 m_2 + ic_2\omega\right)}{\left[\left(k_1 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - m_2 k_2 \omega^2\right] + ic_2\omega\left(k_1 - \omega^2 m_1 - \omega^2 m_2\right)}$$

$$X_2 = \frac{X_1\left(k_2 + ic_2\omega\right)}{\left(k_2 - \omega^2 m_2 + ic_2\omega\right)}$$

- **Harmonically forced vibrations – damped absorber**

Using the following definitions :

Mass ratio :	$\mu = m_2 / m_1$
Static deflection :	$\delta_{st} = F_0 / k_1$
Square absorber natural frequency :	$\omega_a^2 = k_2 / m_2$
Square main mass natural frequency :	$\omega_n^2 = k_1 / m_1$
Natural frequency ratio :	$f = \omega_a / \omega_n$
Forced frequency ratio :	$g = \omega / \omega_n$
Critical damping constant :	$c_c = 2 m_2 \omega / \omega_n$
Damping ratio :	$\zeta = c_2 / c_c$

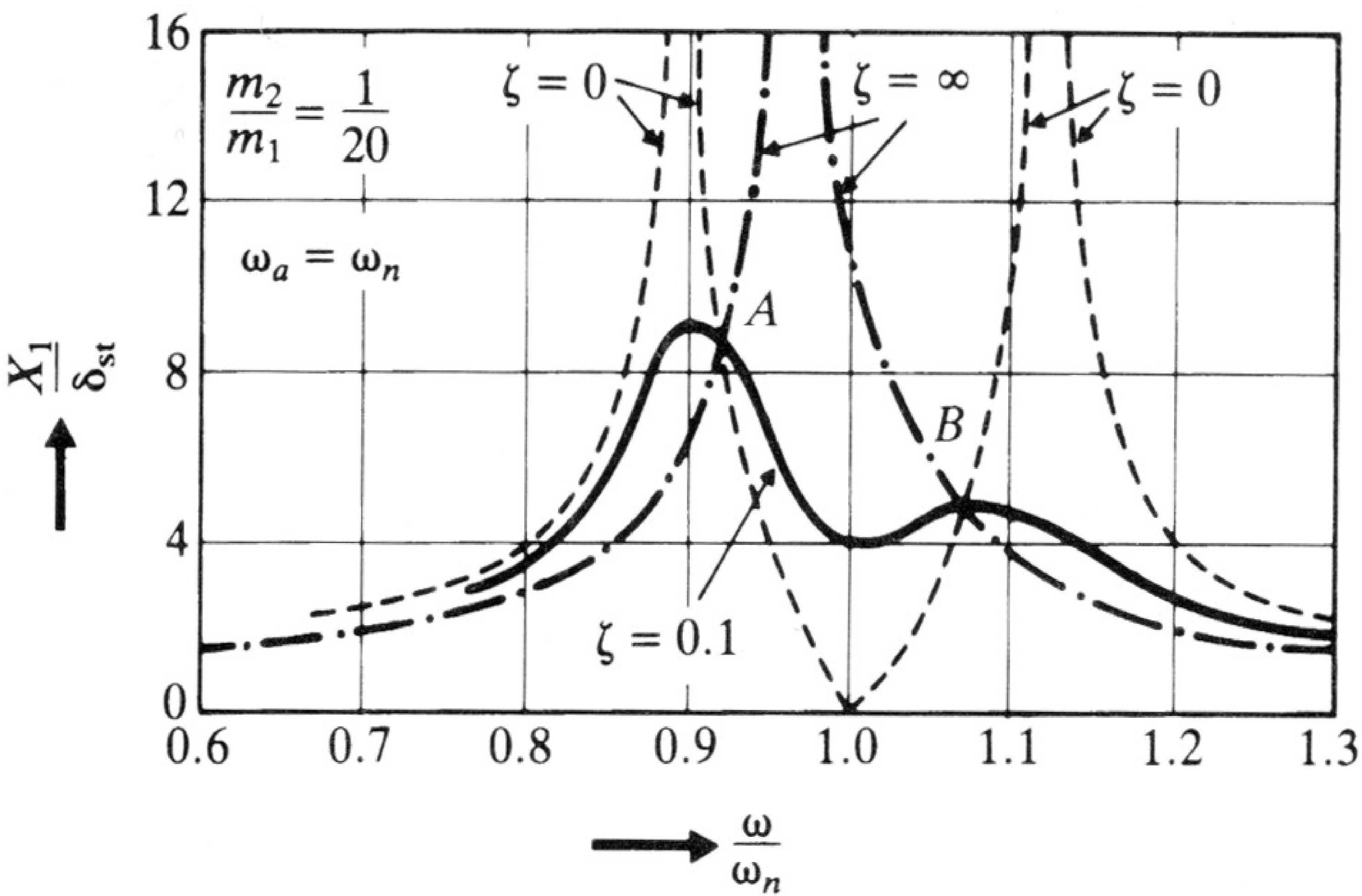
The magnitude ratios can be written as :

$$\frac{X_1}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + \left(g^2 - f^2\right)^2}{(2\zeta g)^2\left(g^2 - 1 + \mu g^2\right)^2 + \left\{\mu f^2 g^2 - \left(g^2 - 1\right)\left(g^2 - f^2\right)\right\}^2}}$$

$$\frac{X_2}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + f^4}{(2\zeta g)^2\left(g^2 - 1 + \mu g^2\right)^2 + \left\{\mu f^2 g^2 - \left(g^2 - 1\right)\left(g^2 - f^2\right)\right\}^2}}$$

- **Harmonically forced vibrations – damped absorber**

$$\frac{X_1}{\delta_{st}} = \sqrt{\frac{(2\zeta g)^2 + \left(g^2 - f^2\right)^2}{(2\zeta g)^2\left(g^2 - 1 + \mu g^2\right)^2 + \left\{\mu f^2 g^2 - \left(g^2 - 1\right)\left(g^2 - f^2\right)\right\}^2}}$$



- **Harmonically forced vibrations – damped absorber**

- When damping is infinite, the two masses are rigidly coupled and the system behaves as an undamped single DoF system with mass $m_1 + m_2$ and stiffness k_1
- X_1 approaches ∞ when $\zeta = 0$ and $\zeta = \infty$
- The amplitude of the absorber mass is always greater that that of the main mass. Allow for large vibration amplitudes and consider fatigue issues for design of absorber springs.
- X_1 will have a minimum
- All damping values produce curves which intersect at **A** and **B**
- The frequencies of A and B can be located by substituting the extreme conditions $\zeta = 0$ and $\zeta = \infty$ into the magnitude ratio equation.
- It has been shown that vibration absorbers operate optimally when the ordinates of A and B are equal for which:

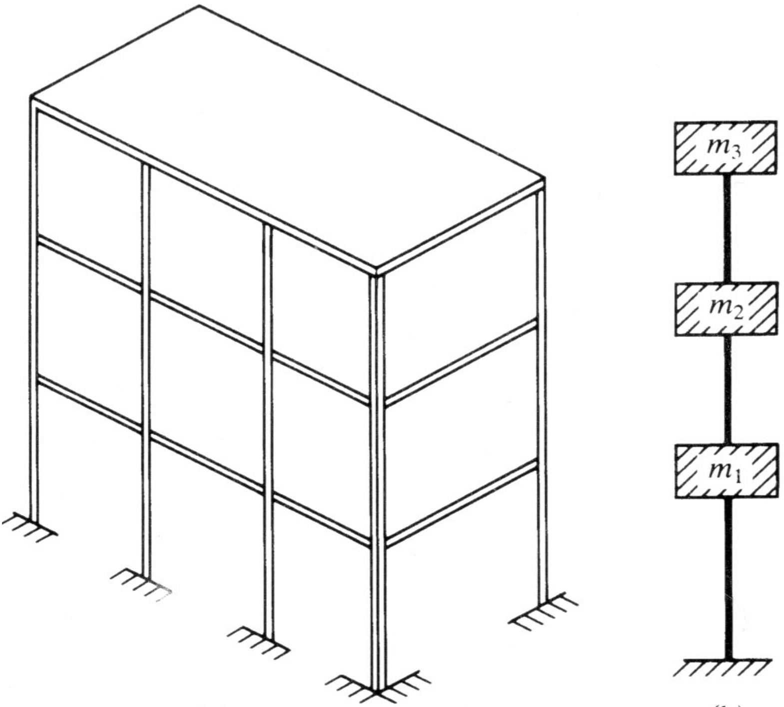
$$f = \omega_a / \omega_n = \frac{1}{(1+\mu)} = \frac{1}{(1+m_2/m_1)}$$

- Such systems are known as ***tuned vibration absorbers***.

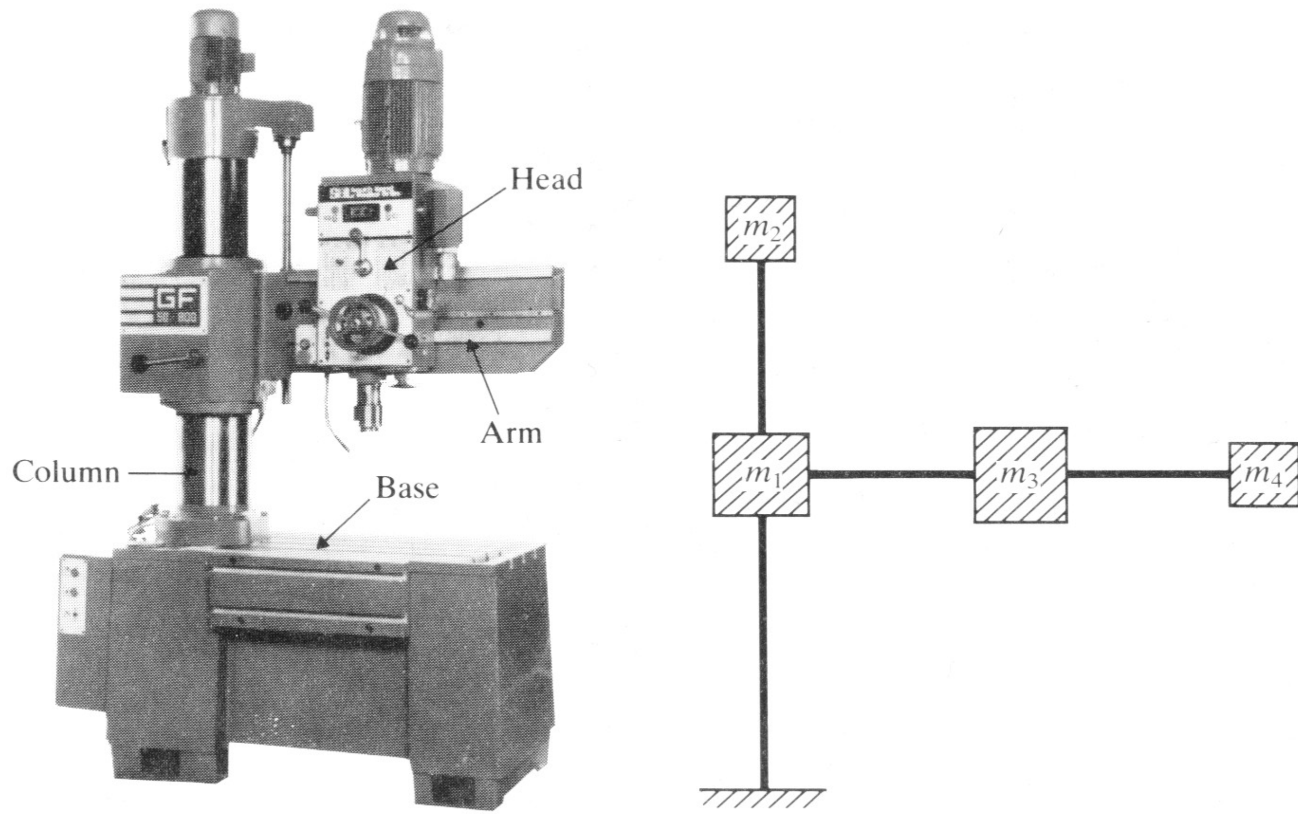
- Vibration analysis of continuous systems require solution to partial differential equations which do not always exist
- Analysis of multi DoF systems requires solution of a collection of ordinary differential equations.
- Continuous systems are often approximated by MDoF systems.
- Previous principles apply:
  - One eqn. of motion for each degree of freedom
  - One generalised coordinate for each degree of freedom
  - The number of natural frequencies and mode shapes are equal to the number of DoFs
  - The natural frequencies are determined by equating the determinant to zero (solution to characteristic equations becomes more complex as number of DoF increases)
- Eqns. of motion obtained from Newton's second law, influence coefficients or Lagrange's equations.

- Modelling continuous systems as MDoF systems:
  - ***Finite element models:***
    - The geometry of a distributed mass system is replaced by a large number of small structural elements (m,c,k)
    - A simple solution is assumed for each element
    - Inter-element compatibility and equilibrium is used to approximate the solution
  - ***Lumped-mass or discrete-mass models:***
    - The (distributed) mass or inertia of the system is replaced by a finite number of rigid bodies (lumped mass)
    - These lumped mass are connected by mass-less spring and damping elements.
    - Linear or angular coordinates are used to describe the motion of each lumped mass element
    - Better accuracy is usually achieved when more lumped masses are used

- **Lumped-mass or discrete-mass models:**



(a) (b)



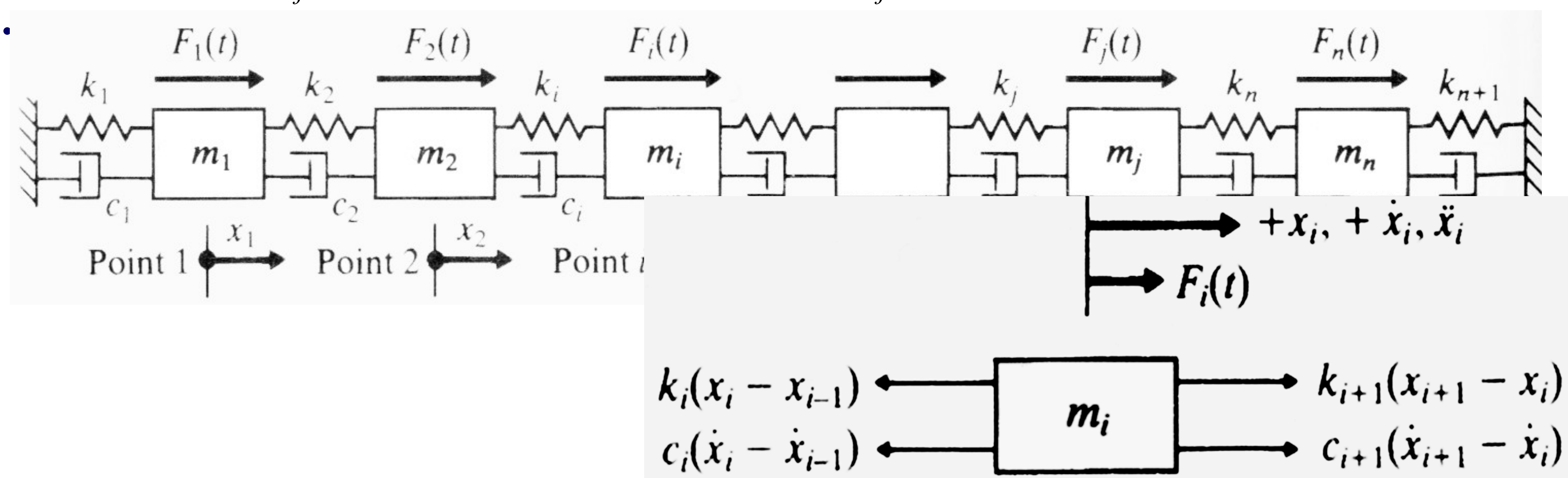
(a) (b)

- **Equations of Motion – Newton's second law.**

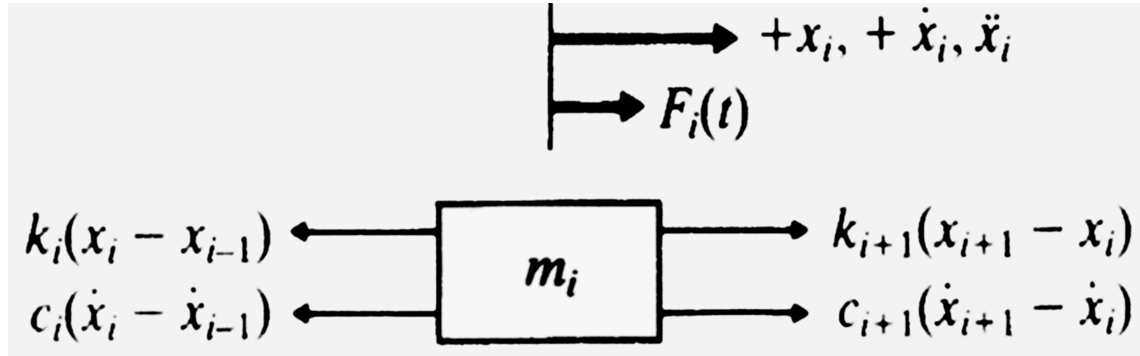
1. Define suitable coordinates to describe the position of each lumped mass in the model
2. Establish the static equilibrium of the system and determine the displacement of each lumped mass wrt to their respective static equilibrium position.
3. Draw the free-body diagram for each lumped mass in the model. Indicate the spring, damping and external forces on each mass element when a positive displacement and velocity is applied to each mass element.
4. Generate the equation of motion for each mass element by applying Newton's second law of motion with reference to the free-body diagrams:

$$m_i\ddot{x}_i = \sum_j F_{ij} \quad (\text{for mass } m_i) \quad \text{and} \quad J_i\ddot{\theta}_i = \sum_j M_{ij} \quad (\text{for rigid body of inertia } J)$$

-



- **Equations of Motion – Newton's second law.**



$$m_i\ddot{x}_i = -k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1}) + c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) + F_i \qquad for\ i = 1,2,3...,n-1$$

Rearranging:

$$m_i\ddot{\boldsymbol{x}}_{\boldsymbol{i}} - c_i\dot{\boldsymbol{x}}_{\boldsymbol{i-1}} + (c_i + c_{i+1})\dot{\boldsymbol{x}}_{\boldsymbol{i}} - c_{i+1}\dot{\boldsymbol{x}}_{\boldsymbol{i+1}} - k_i\boldsymbol{x}_{\boldsymbol{i-1}} + (k_i + k_{i+1})\boldsymbol{x}_{\boldsymbol{i}} - k_{i+1}\boldsymbol{x}_{\boldsymbol{i+1}} = \boldsymbol{F}_{\boldsymbol{i}} \qquad for\ i = 1,2,3...,n-1$$

- Note that the system has both stiffness and damping coupling
- The equations of motion of masses m_1 *and* m_n at the extremities of the system are obtained by setting $i = 1$ & $x_{i-1} = 0$ and $i = n$ & $x_{n+1} = 0$

$$m_1\ddot{\boldsymbol{x}}_{\boldsymbol{1}} + (c_1 + c_2)\dot{\boldsymbol{x}}_{\boldsymbol{1}} - c_2\dot{\boldsymbol{x}}_{\boldsymbol{2}} + (k_1 + k_2)\boldsymbol{x}_{\boldsymbol{1}} - k_2\boldsymbol{x}_{\boldsymbol{2}} = \boldsymbol{F}_{\boldsymbol{1}}$$

$$m_n\ddot{\boldsymbol{x}}_{\boldsymbol{n}} - c_n\dot{\boldsymbol{x}}_{\boldsymbol{n-1}} + (c_n + c_{n+1})\dot{\boldsymbol{x}}_{\boldsymbol{n}} - k_n\boldsymbol{x}_{\boldsymbol{n-1}} + (k_n + k_{n+1})\boldsymbol{x}_{\boldsymbol{n}} = \boldsymbol{F}_{\boldsymbol{n}}$$

- In matrix form:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

- **Equations of Motion – Newton's second law.**
- Where the mass matrix [*m*], the damping matrix [*c*] and the stiffness matrix [*k*] are given by:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & m_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & m_3 & \dots & 0 & 0 \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ 0 & 0 & 0 & \dots & 0 & m_n \end{bmatrix}$$

$$[c] = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & \dots & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & \dots & 0 & 0 \\ 0 & -c_3 & (c_3 + c_4) & \dots & 0 & 0 \\ . & & & & . & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & 0 & 0 & \dots & -c_n & (c_n + c_{n+1}) \end{bmatrix}$$

- **Equations of Motion – Newton's second law.**

$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & \dots & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & \dots & 0 & 0 \\ . & & & & . & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & 0 & 0 & \dots & -k_n & (k_n+k_{n+1}) \end{bmatrix}$$

- And the displacement. Velocity, acceleration and excitation force vectors are given by:

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{Bmatrix} \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ x_2(t) \\ . \\ . \\ . \\ \dot{x}_n(t) \end{Bmatrix} \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ . \\ . \\ . \\ \ddot{x}_n(t) \end{Bmatrix} \quad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ . \\ . \\ . \\ F_n(t) \end{Bmatrix}$$

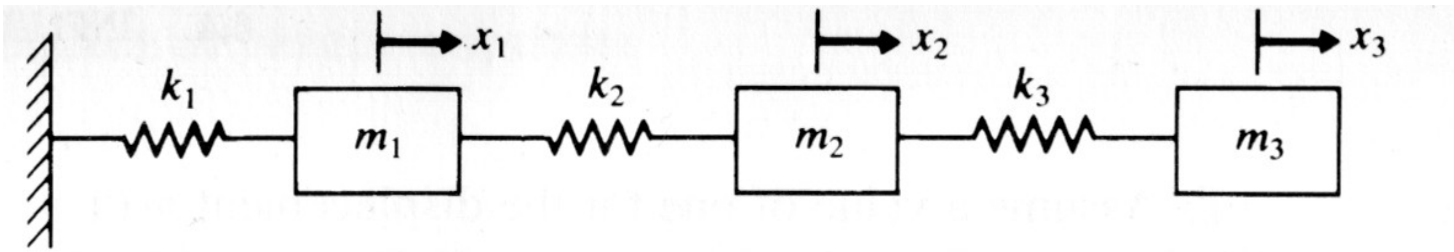
- **Equations of Motion – Newton's second law.**
- In general terms:

$$[m]=\begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ m_{n1} & m_{n1} & m_{n3} & \dots & m_{nn} \end{bmatrix} \quad [c]=\begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{bmatrix} \quad [k]=\begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{bmatrix}$$

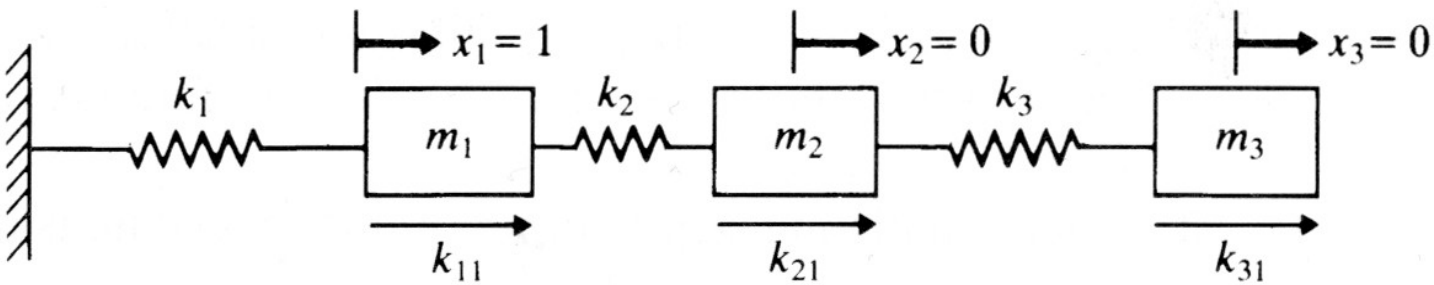
- **Influence coefficients.**
- It is sometimes practical to express the eqns. of motion of MDoF systems in terms of ***influence coefficients***
- The elements of the stiffness matrix are known as the ***stiffness*** influence coefficients and relate the force at a point in the system with the displacement applied at another point in the system.
- The stiffness influence coefficient k_{ij} is defined as the force at point *i* due to a unit displacement at point *j* when all other points, except *j*, are fixed.
- The total force at *i* is the sum of the forces due to all applied displacements.:

$$F_i = \sum_{j=1}^{n} k_{ij} x_j \qquad i = 1,2,3...n \qquad \text{or} \qquad \vec{F} = [k]\,\vec{x} \qquad \text{where} \qquad [k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{bmatrix}$$

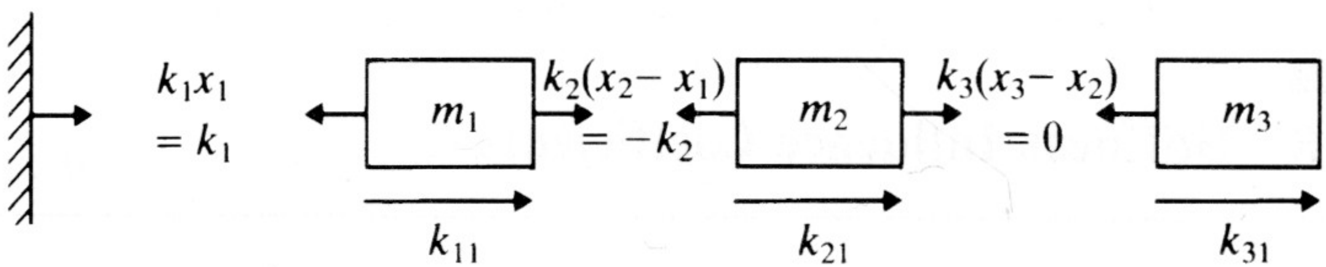
- **Influence coefficients – stiffness.**
- Example:



- Use static equilibrium to determine the stiffness influence coefficients.
- Step 1: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

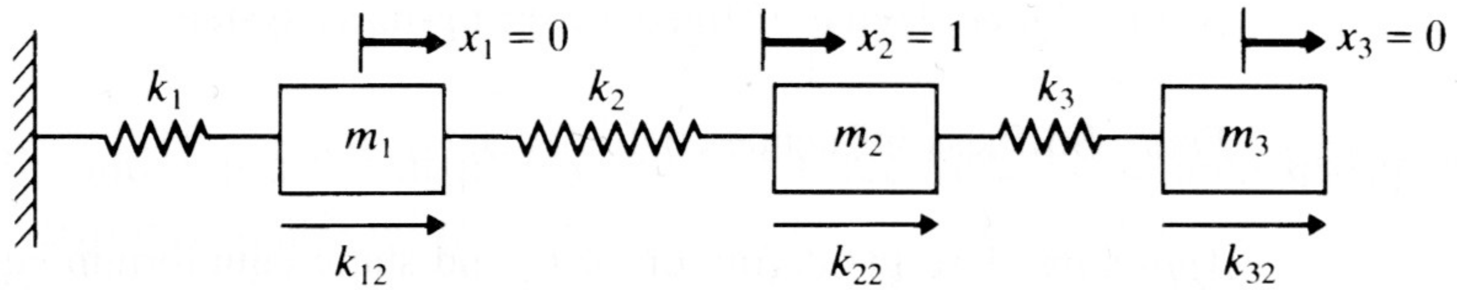


- For which the free–body diagram is:

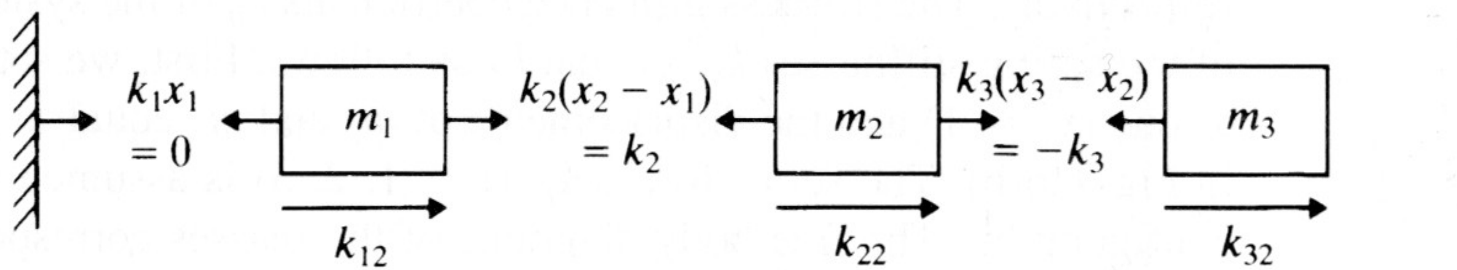


$$k_{11} = F_1 / x_{11} = F_1 = k_1 + k_2$$
$$k_{21} = -k_2$$
$$k_{31} = 0$$

- **Influence coefficients – stiffness.**
- Step 2: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$.



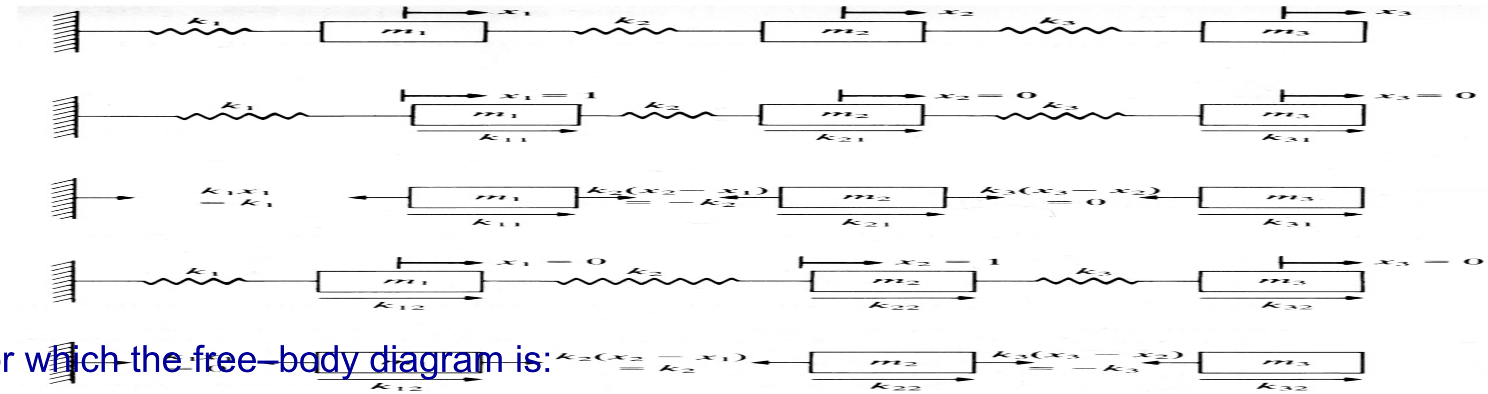
- For which the free–body diagram is:



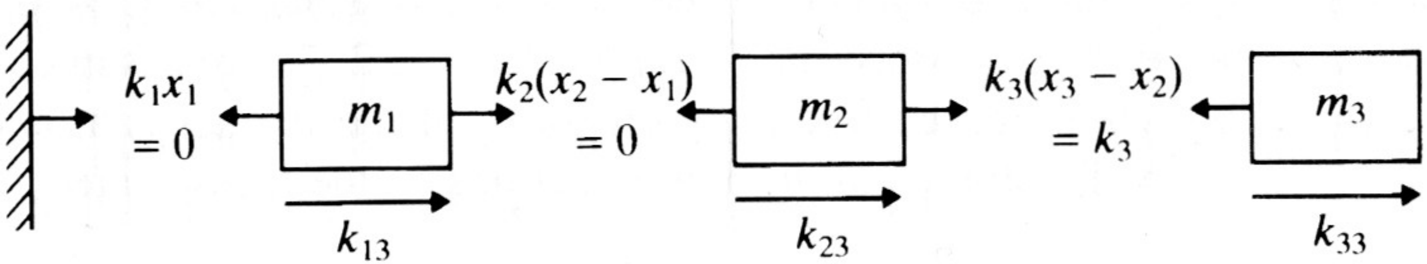
$$k_{12} = -k_2$$
$$k_{22} = k_2 + k_3$$
$$k_{32} = -k_3$$

- **Influence coefficients – stiffness.**

- Step 3: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$.



- For which the free–body diagram is:



$$k_{13} = 0$$
$$k_{23} = -k_3$$
$$k_{33} = k_3$$

- **Influence coefficients – stiffness.**
- The system stiffness matrix is:

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

- The calculation of *n* stiffness influence coefficients require the solution of *n* simultaneous equations.
- Thus the computation of stiffness influence coefficients for a system with n degrees of freedom may require a significant effort (up to n^2 computations)

Multi-level building example

- **Influence coefficients - flexibility.**
- It is sometimes easier to define the system in terms of the ***flexibility influence coefficients***
- The flexibility influence coefficients relates the displacement at a point in the system with the force applied at another point in the system.
- The flexibility influence coefficient a_{ij} is defined as the deflection at point *i* due to a unit force point *j* with no other forces acting on the system.
- For a linear system:

$$x_{ij} = a_{ij} F_j$$

- When several forces act at various points in the system, F_j *for j = 1, 2, 3….n,* the total deflection at point *i* is the sum of the deflections caused by each individual applied force:

$$x_i = \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_{ij} F_j \qquad i = 1,2,3.....n \qquad \text{in matrix form:} \qquad \vec{x} = [a]\,\vec{F}$$

where $\vec{x}$ and $\vec{F}$ are the displacement and force vectors and $[a]$ is the flexibility matrix:

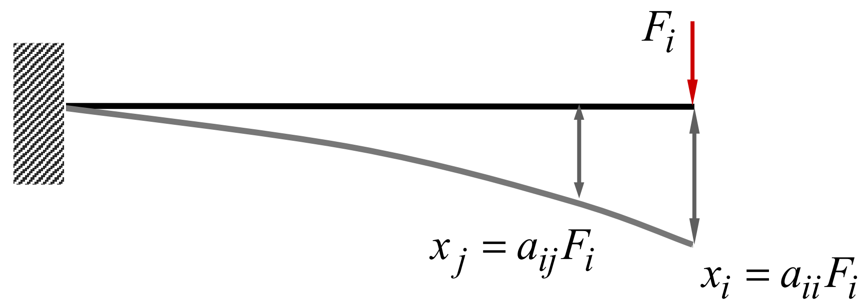
$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

- **Influence coefficients - flexibility.**
- Not unexpected that the flexibility matrix is related to the stiffness matrix.

$$[a]^{-1}\vec{x} = [a]\vec{F}[a]^{-1}$$

$$\vec{F} = [a]^{-1}\vec{x} = [k]\vec{x}$$

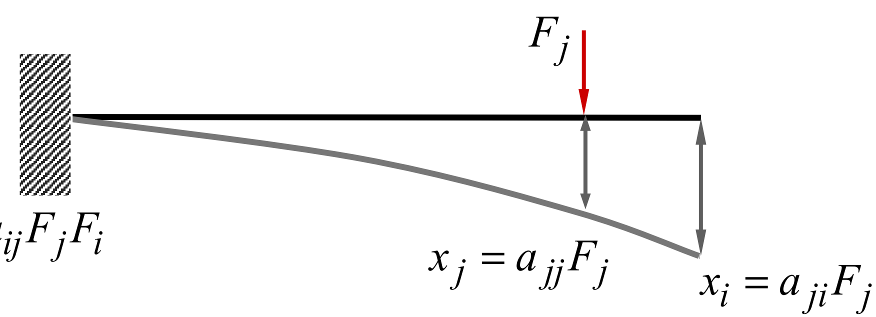
$$[a]^{-1} = [k]$$



- **Reciprocity theorem:** For a linear system : $a_{ij} = a_{ji}$
- Consider the work done by forces F_i andf F_j

Case 1: $W_i = \frac{1}{2}F_i x_i = \frac{1}{2}a_{ii}F_i^2$

Case 2 : $W_j = \frac{1}{2}F_j x_j = \frac{1}{2}a_{jj}F_j^2$



When F_i and F_j are applied sequentially the total work is:

$$W_{ij} = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + x_j F_i = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + a_{ij}F_j F_i$$

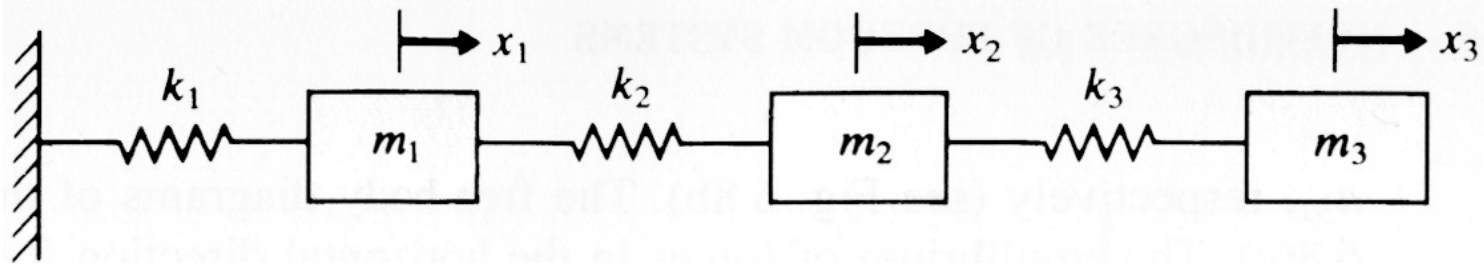
and when F_j is applied before F_i the total work is:

$$W_{ji} = \frac{1}{2}a_{jj}F_j^2 + \frac{1}{2}a_{ii}F_i^2 + x_i F_j = \frac{1}{2}a_{ii}F_i^2 + \frac{1}{2}a_{jj}F_j^2 + a_{ji}F_i F_j$$

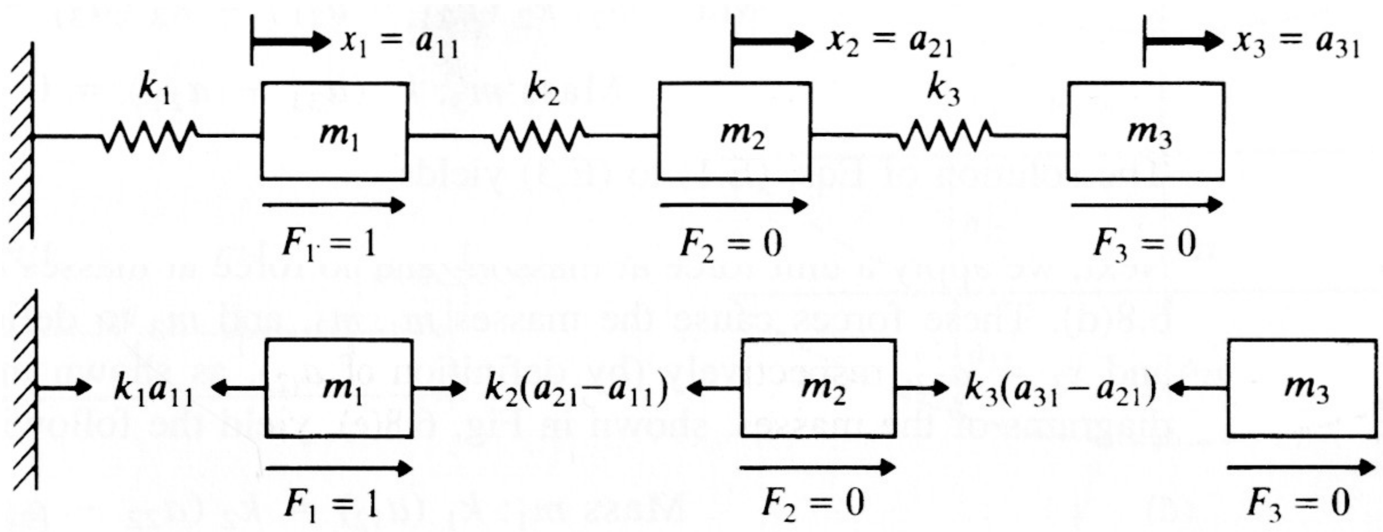
Since the total work done is not dependent on the sequence of applied force :

$W_{ij} = W_{ji}$ hence $a_{ij} = a_{ji}$

- **Influence coefficients - flexibility.**
- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 1: Apply a unit load at point 1 only and calculate the deflections of each mass due to the unit load at 1.



$$a_{11} = x_{11} / F_1 = x_{11}$$

Mass 1:

$$k_1 a_{11} = k_2(a_{21} - a_{11}) + F_1$$

$$k_1 a_{11} = k_2(a_{21} - a_{11}) + 1$$

Mass 2 :

$$k_2(a_{21} - a_{11}) = k_3(a_{31} - a_{21})$$

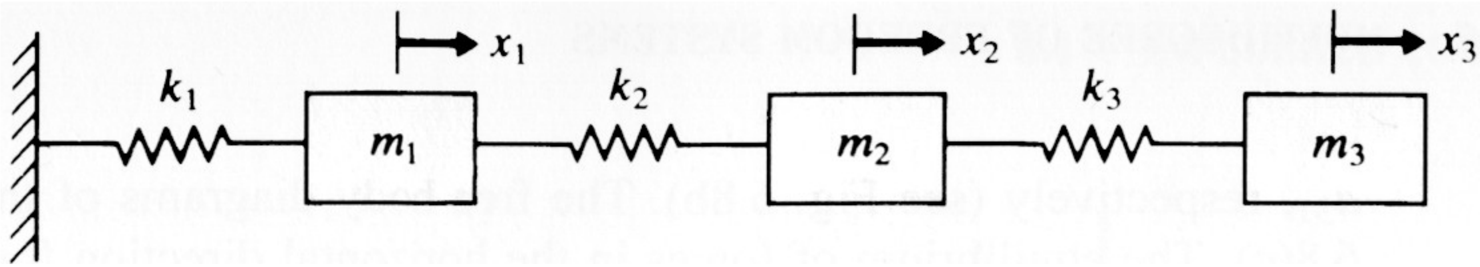
Mass 3 :

$$k_3(a_{31} - a_{21}) = 0$$

Solving :

$$a_{11} = \frac{1}{k_1},\ a_{21} = \frac{1}{k_1},\ a_{31} = \frac{1}{k_1},$$

- **Influence coefficients - flexibility.**
- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 2: Apply a unit load at point 2 only and calculate the deflections of each mass due to the unit load at 2.

Mass 1:

$$k_1 a_{12} = k_2 (a_{22} - a_1)$$

Mass 2 :

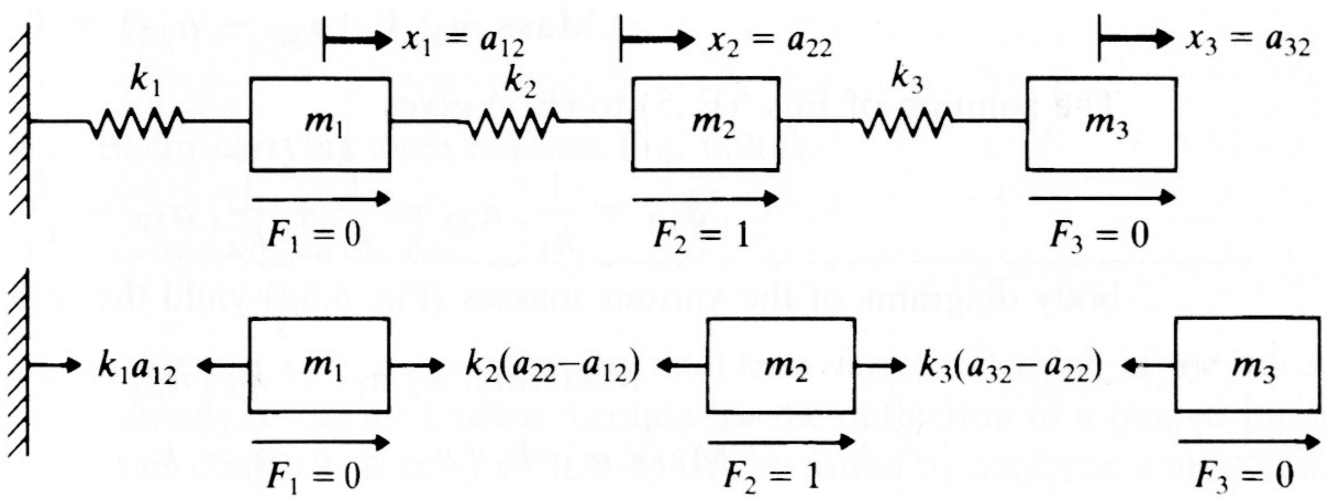
$$k_2 (a_{22} - a_{13}) = k_3 (a_{32} - a_{22}) + 1$$

Mass 3 :

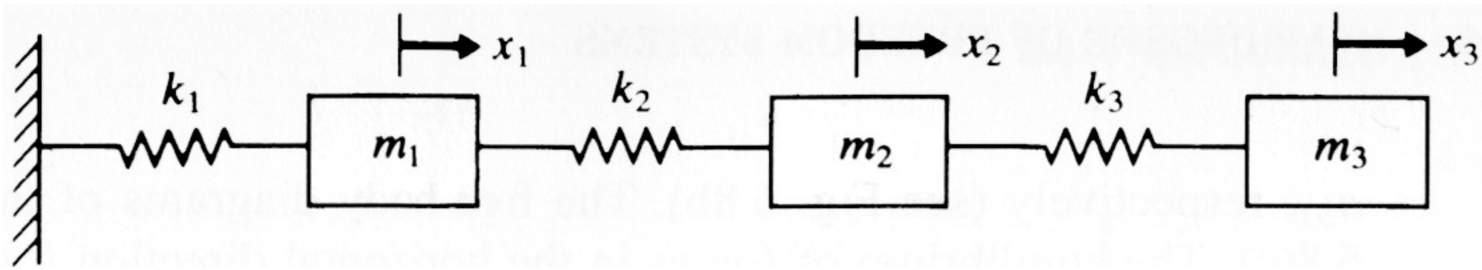
$$k_3 (a_{32} - a_{22}) = 0$$

Solving :

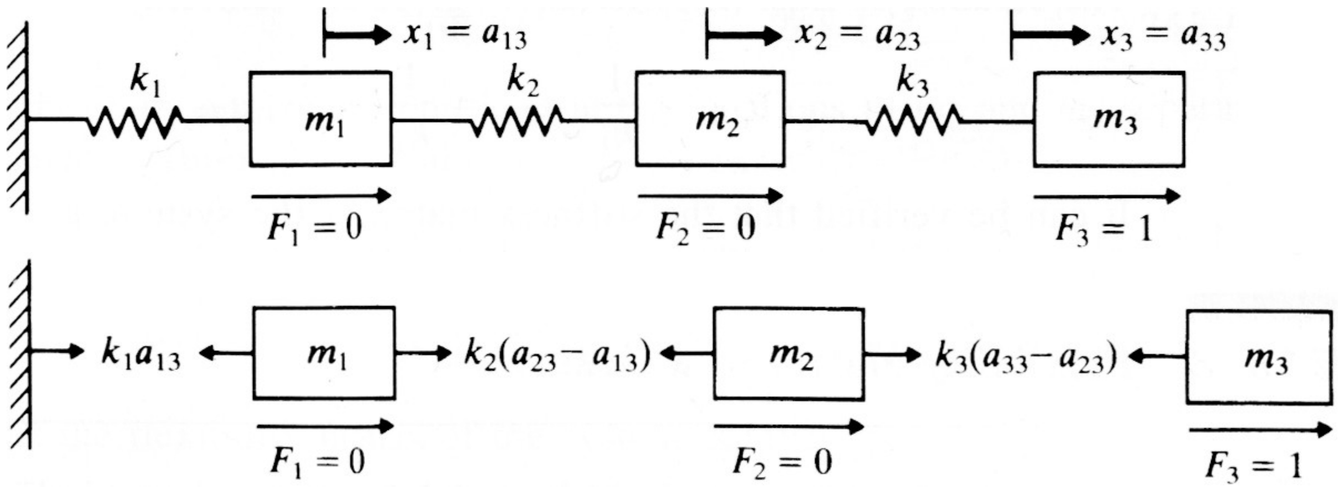
$$a_{12} = \frac{1}{k_1}, \; a_{22} = \frac{1}{k_1} + \frac{1}{k_2}, \; a_{32} = \frac{1}{k_1} + \frac{1}{k_2}$$



- **Influence coefficients - flexibility.**
- Example: Use static equilibrium to determine the flexibility matrix of the system.



- Step 3: Apply a unit load at point 3 only and calculate the deflections of each mass due to the unit load at 3.



Mass 1:

$$k_1 a_{13} = k_2 (a_{23} - a_3)$$

Mass 2 :

$$k_2 (a_{23} - a_{13}) = k_3 (a_{33} - a_{23})$$

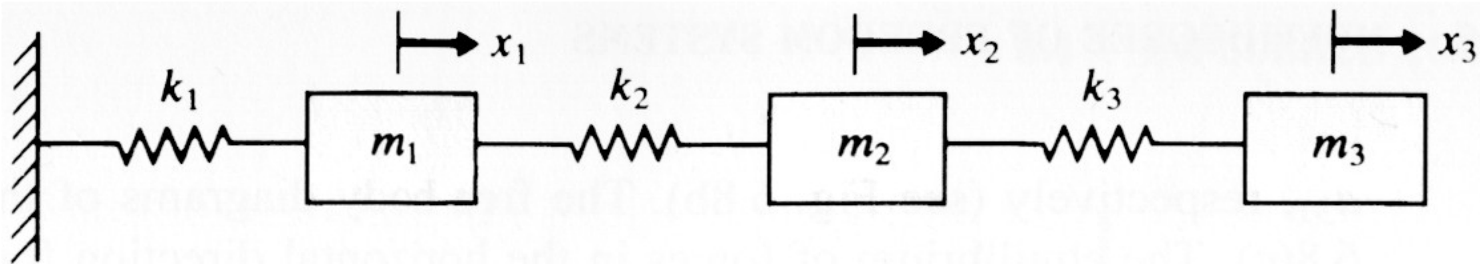
Mass 3 :

$$k_3 (a_{33} - a_{23}) = 1$$

Solving :

$$a_{13} = \frac{1}{k_1}, \quad a_{23} = \frac{1}{k_1} + \frac{1}{k_2}, \quad a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

- **Influence coefficients - flexibility.**
- Example: Use static equilibrium to determine the flexibility matrix of the system.



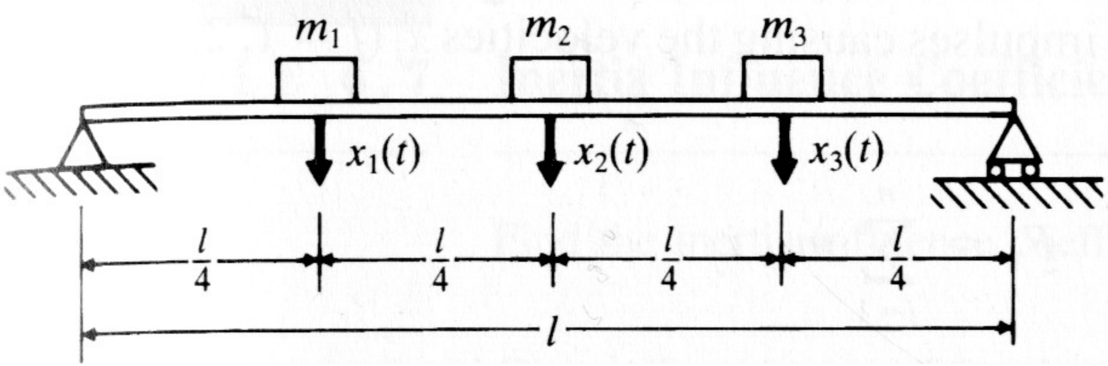
- The flexibility matrix of the system is:

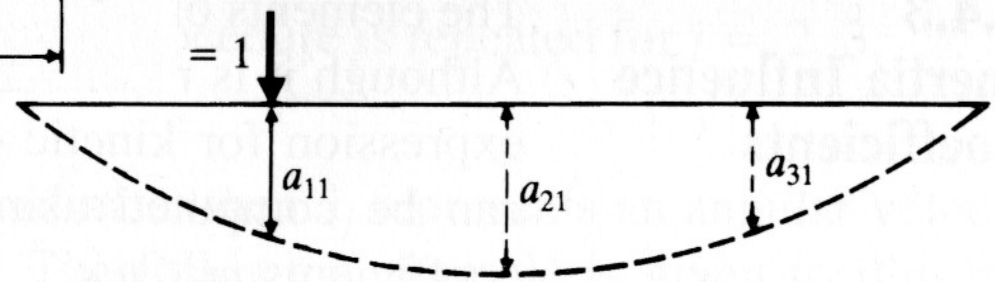
$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & k_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2) \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2 + 1/k_3) \end{bmatrix}$$

- It can be verified that the inverse of this flexibility matrix is the system stiffness matrix:

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

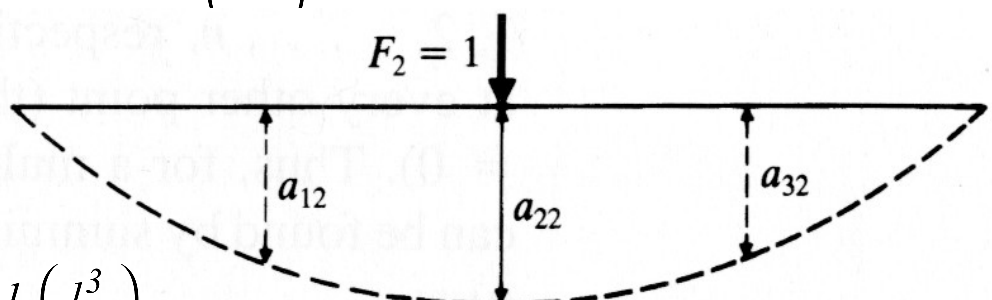
- **Influence coefficients - flexibility.**
- Example: Use static equilibrium to determine the flexibility matrix of the system.





- Step 1: Apply a unit load at point 1 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 1.

$$a_{11} = x_{11} / F_1 = x_{11} = \frac{9}{768}\left(\frac{l^3}{EI}\right) \qquad a_{12} = \frac{11}{768}\left(\frac{l^3}{EI}\right) \qquad a_{13} = \frac{7}{768}\left(\frac{l^3}{EI}\right)$$

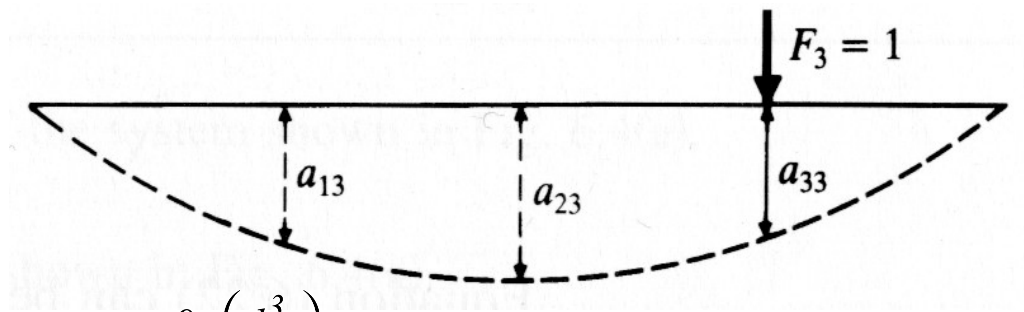


- Step 2: Apply a unit load at point 2 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 2.

$$a_{21} = a_{12} = \frac{11}{768}\left(\frac{l^3}{EI}\right) \qquad a_{22} = \frac{1}{48}\left(\frac{l^3}{EI}\right) \qquad a_{23} = \frac{11}{768}\left(\frac{l^3}{EI}\right)$$

- **Influence coefficients - flexibility.**

- Step 3: Apply a unit load at point 3 only and calculate the deflections at points 1, 2 and 3 due to the unit load at 3.



$$a_{31} = a_{13} = \frac{7}{768}\left(\frac{l^3}{EI}\right) \qquad a_{32} = a_{23} = \frac{11}{48}\left(\frac{l^3}{EI}\right) \qquad a_{33} = \frac{9}{768}\left(\frac{l^3}{EI}\right)$$

- The system flexibility matrix is:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & k_{32} & a_{33} \end{bmatrix} = \frac{l^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

- **Influence coefficients - inertia.**
- The elements of the mass matrix are referred to as the ***inertia influence coefficients***.
- The inertia influence coefficients of a system can be determined by applying the impulse-momentum equations.
- The inertia influence coefficients m_{1j}, m_{2j}, m_{3k}.....m_{nj} are defined as the impulses applied at points *1, 2,3…n* to produce a unit velocity at point *j* and zero velocity at every other point in the system.
- The total impulse at point *i* is:

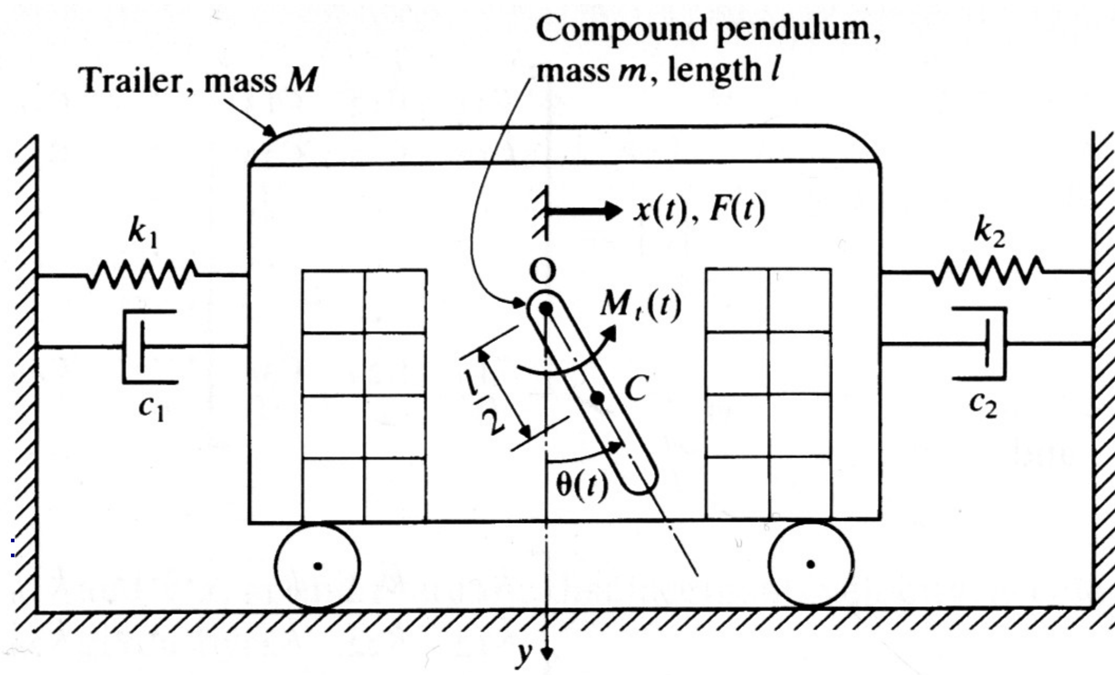
$$\underset{\sim}{F}_i = \sum_{j=1}^{n} m_{ij}\dot{x}_j \qquad i = 1,2,3...n \qquad \text{or} \qquad \underset{\sim}{\vec{F}} = [m]\vec{\dot{x}} \qquad \text{where} \qquad [m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ m_{n1} & k_{n2} & k_{n3} & \cdots & m_{nn} \end{bmatrix}$$

and $\underset{\sim}{\vec{F}}$ and $\vec{\dot{x}}$ are the impulse and velocity vectors.

- The inertia influence coefficients of linear systems are symmetrical:

$$m_{ij} = m_{ji}$$

- **Influence coefficients - inertia.**
- Example: Determine the inertia influence coefficients (mass matrix) of the 2DoF system:
- Step 1: Apply impulses F_1 *(trailer)* along *x(t)* and F_2 *(pendulum)* along *θ(t)* which will result in a unit velocity along *x (x' = 1)* and zero velocity along *θ (θ' = 0).*



Applying the linear impulse - momentum eqn :

$$F_1 = m_{11}\dot{x}_1 = m_{11}$$

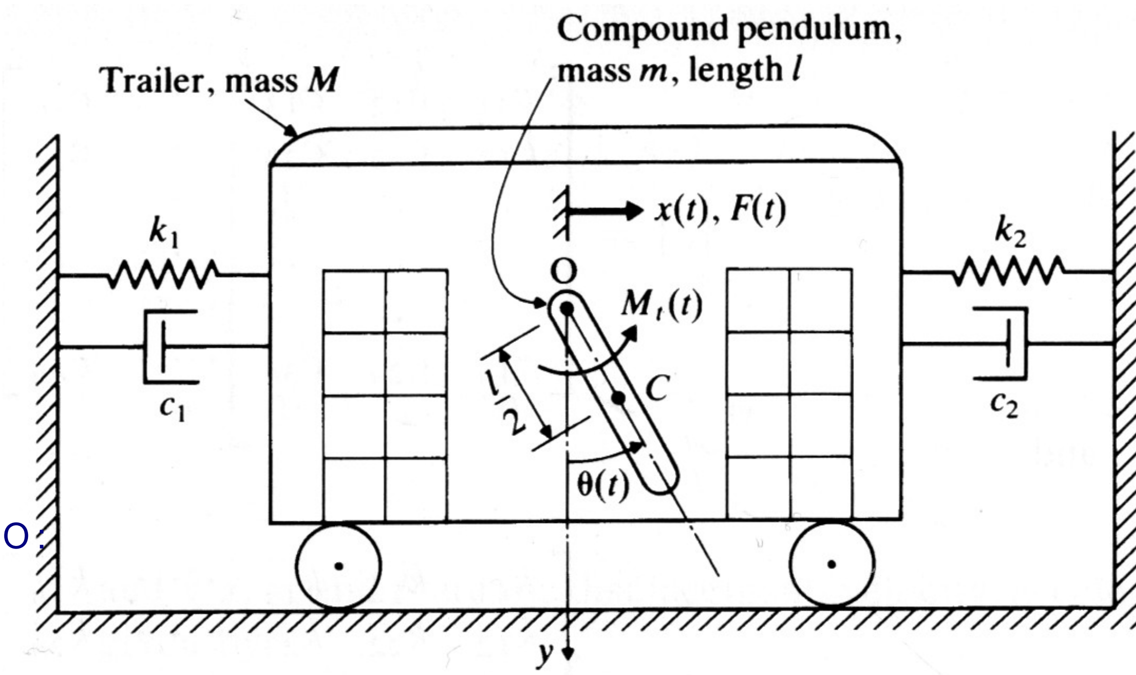
$$m_{11} = (M+m)\dot{x} + \frac{l}{2}m\dot{\theta} = (M+m)$$

Applying the angular impulse - momentum eqn about O :

$$F_2 = m_{21}\dot{x}_1 = m_{21}$$

$$m_{21} = \frac{l}{2}m\dot{x} + \left(\frac{ml^2}{3}\right)\dot{\theta} == \frac{l}{2}m$$

- **Influence coefficients - inertia.**
- Example: Determine the inertia influence coefficients (mass matrix) of the 2DoF system:
- Step 2: Apply impulses F_1 *(trailer)* along *x(t)* and F_2 *(pendulum)* along *θ(t)* which will result in zero velocity along *x (x' = 0)* and a unit velocity along *θ (θ' = 1).*



Applying the linear impulse - momentum eqn :

$$F_1 = m_{12}\dot{x}_2 = m_{12}$$

$$m_{12} = (M + m)\dot{x} + \frac{l}{2}m\dot{\theta} = \frac{l}{2}m$$

Applying the angular impulse - momentum eqn about O :

$$F_2 = m_{22}\dot{x}_2 = m_{22}$$

$$m_{22} = \frac{l}{2}m\dot{x} + \left(\frac{ml^2}{3}\right)\dot{\theta} = \frac{ml^2}{3}$$

The mass or inertia matrix of the system is therefore :

$$[m] = \begin{bmatrix} (M+m) & \frac{ml}{2} \\ \frac{ml}{2} & \frac{ml^2}{3} \end{bmatrix}$$

- **Eigenvalues and Eigenvectors**
- The solution to the eqn. of motion of a free undamped MDoF system

$$[m]\ddot{\vec{x}} + [k]\vec{x} = 0$$

- defines the (steady-state) harmonic vibration of the system due to an initial disturbance (initial conditions).
- The solution is established by assuming a solution in the form:

$$x_i(t) = X_i T(t) \qquad i = 1,2,3....n$$

where X_i is a constant and T is a function of time.

The amplitude ratio of any two coordinates $\left\{\dfrac{x_i(t)}{x_j(t)}\right\}$ is independent of time.

Which signify that the motion (vibration) of all the degrees of freedom are synchronised - mode shape is fixed

and is written as :

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{Bmatrix}$$

- **Eigenvalues and Eigenvectors**
- Substituting the assumed solution into the eqn. of motion gives:

$$[m]\vec{X}\ddot{T}(t)+[k]\vec{X}T(t)=\vec{0}$$

in scalar form:

$$\left(\sum_{j=1}^{n} m_{ij}X_j\right)\ddot{T}(t)+\left(\sum_{j=1}^{n} k_{ij}X_j\right)T(t)=0 \qquad i=1,2,3.....,n$$

which gives:

$$-\frac{\ddot{T}(t)}{T(t)}=\frac{\sum_{j=1}^{n} k_{ij}X_j}{\sum_{j=1}^{n} m_{ij}X_j} \qquad i=1,2,3.....,n$$

$$-\frac{\ddot{T}(t)}{T(t)}=\frac{\sum_{j=1}^{n} k_{ij}X_j}{\sum_{j=1}^{n} m_{ij}X_j}=\omega^2$$

or :

$$\ddot{T}(t)+\omega^2 T(t)=0$$

- **Eigenvalues and Eigenvectors**

  Then :

$$\sum_{j=1}^{n}\left(k_{ij}-\omega^2 m_{ij}\right)X_j=0 \qquad i=1,2,3.....,n$$

or in matrix form:

$$\left[\,[k]-\omega^2[m]\,\right]\vec{X}=\vec{0} \qquad (a)$$

as found previously, the solution to the above can be written as :

$$T(t)=C_1\cos(\omega t+\phi)$$

- This solution reveals that the degrees of freedom can vibrate harmonically at the same frequency ω and phase angle ϕ as long as the frequency satisfies eqn. (a) which represents a set on *n* linear homogeneous equations.
- For non-trivial solutions, the determinant of the coefficient matrix must be zero which gives the *characteristic equation*:

$$\left|k_{ij}-\omega^2 m_{ij}\right|=\left|[k]-\omega^2[m]\right|=0$$

- This is known as the eigenvalue problem, where ω^2 is the eigenvalue and ω the natural frequency of the system.
- Expansion of the characteristic equation gives an n^{th} order polynomial in terms of ω^2 the solution of which produces *n* real and positive roots when the mass and stiffness matrices are symmetric and positive.
- The *n* natural frequencies are in ascending order $\omega_1 \le \omega_2 \le \omega_3 \le \ldots \le \omega_n$ with ω_1 being the fundamental natural frequency.

- **Eigenvalues and Eigenvectors**

If we let :

$$\lambda = \frac{1}{\omega^2}$$

Equation (a) becomes:

$$\left[\ \lambda[k]-[m]\right]\vec{X}=\vec{0}$$

and multiplying both sides by $[k]^{-1}$ gives :

$$\left[\lambda[I]-[D]\right]\vec{X}=\vec{0}$$

or

$$\lambda[I]\vec{X}=[D]\vec{X}$$

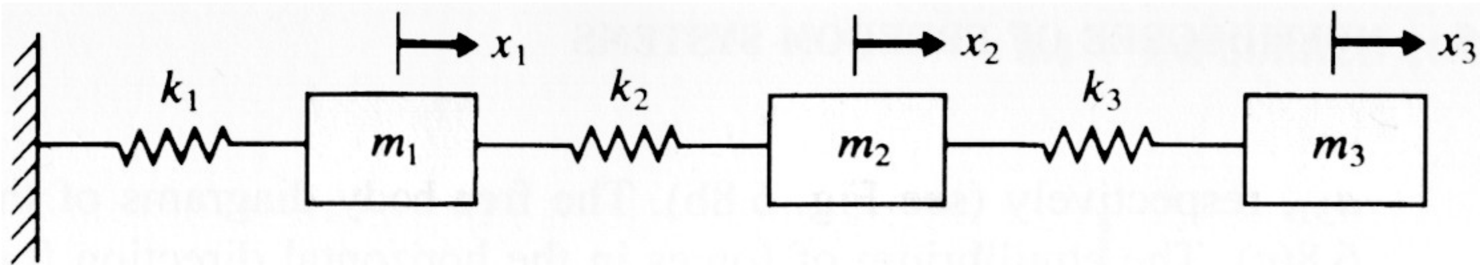
where $[D]=[k]^{-1}[m]$ is the ***dynamical matrix***.

for a non-trivial solution the determinant of the characteristic eqn. must be zero:

$$\left|\lambda[I]-[D]\right|=0$$

- Expanding gives an n^{th} degree polynomial in terms of λ
- This form lends itself to obtaining solutions by numerical (computer) methods to determine the roots of a polynomial equation.

- **Eigenvalues and Eigenvectors**
- Example: Find the natural frequencies and mode shapes of the system when $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$.



- The dynamical matrix is given by:

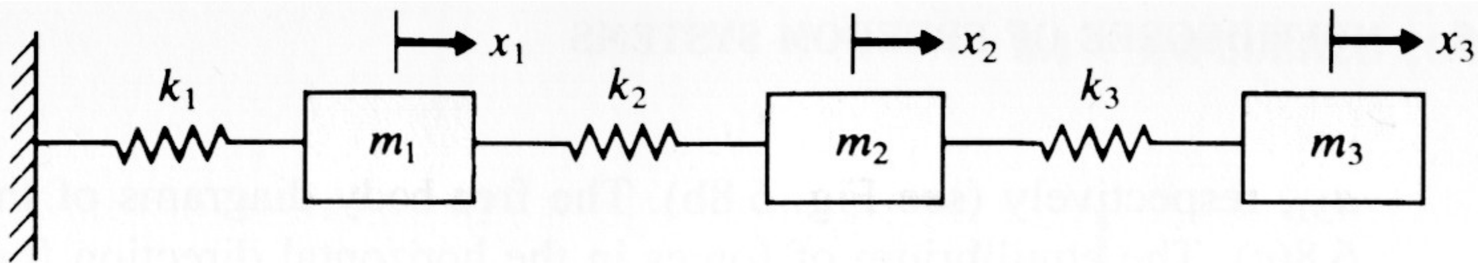
$$[D] = [k]^{-1}[m] \equiv [a][m]$$

- And the flexibility and mass matrix were determined previously:

$$[a] = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2) \\ 1/k_1 & (1/k_1 + 1/k_2) & (1/k_1 + 1/k_2 + 1/k_3) \end{bmatrix} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad therefore: \qquad [D] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

- **Eigenvalues and Eigenvectors**
- Example: Find the natural frequencies and mode shapes of the system when $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$.



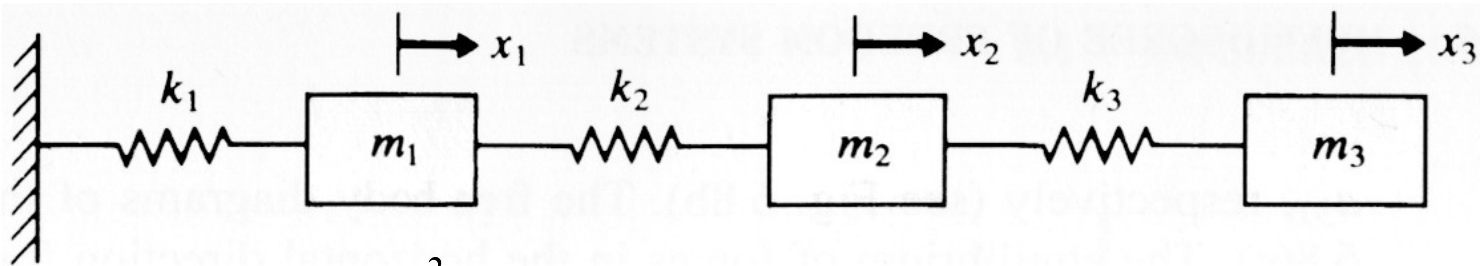
- Equating the C.E. determinant to zero:

$$\left|\lambda[I]-[D]\right| = \left|\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - \frac{m}{k}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix}\right| = 0 \qquad \left(\lambda = \frac{1}{\omega^2}\right)$$

subtracting and dividing throughout by λ:

$$= \begin{vmatrix}\left(1-\frac{m}{k\lambda}\right) & \left(-\frac{m}{k\lambda}\right) & \left(-\frac{m}{k\lambda}\right)\\ \left(-\frac{m}{k\lambda}\right) & \left(1-\frac{2m}{k\lambda}\right) & \left(-\frac{2m}{k\lambda}\right)\\ \left(-\frac{m}{k\lambda}\right) & \left(-\frac{2m}{k\lambda}\right) & \left(1-\frac{3m}{k\lambda}\right)\end{vmatrix} = 0$$

- **Eigenvalues and Eigenvectors**
- Example: Find the natural frequencies and mode shapes of the system when $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$.



If $\alpha = \frac{m}{k\lambda} = \frac{m\omega^2}{k}$

$$\begin{vmatrix} 1-\alpha & -\alpha & -\alpha \\ -\alpha & 1-2\alpha & -2\alpha \\ -\alpha & -2\alpha & 1-3\alpha \end{vmatrix} = \alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0$$

whose roots (eigenvalues) are:

$$\alpha_1 = \frac{m\omega_1^2}{k} = 0.198 \qquad \omega_1 = 0.445\sqrt{\frac{k}{m}}$$

$$\alpha_2 = \frac{m\omega_2^2}{k} = 1.555 \qquad \omega_2 = 1.247\sqrt{\frac{k}{m}}$$

$$\alpha_3 = \frac{m\omega_3^2}{k} = 3.249 \qquad \omega_3 = 1.803\sqrt{\frac{k}{m}}$$

- **Eigenvalues and Eigenvectors**

The mode shapes are determined by calculating the eigenvectors :

$$\left[\lambda_i[I]-[D]\right]\vec{X}^{(i)}=\vec{0}$$ (i denotes the ith mode shape)

First mode : substituting $\lambda_1 = \dfrac{1}{\omega_1^2} = 5.049\dfrac{m}{k}$ gives :

$$\left[5.049\frac{m}{k}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{m}{k}\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}\right]\begin{Bmatrix}X_1^{(1)}\\X_2^{(1)}\\X_3^{(1)}\end{Bmatrix}=\begin{bmatrix}4.049&-1&-1\\-1&3.049&-2\\-1&-2&2.049\end{bmatrix}\begin{Bmatrix}X_1^{(1)}\\X_2^{(1)}\\X_3^{(1)}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

From the first and second rows :

$$X_2^{(1)}+X_3^{(1)}=4.049X_1^{(1)} \qquad and \qquad 3.049X_2^{(1)}-2X_3^{(1)}=X_1^{(1)}$$

Solving for $X_2^{(1)}$ and $X_3^{(1)}$ in terms $X_1^{(1)}$:

$$X_2^{(1)}=1.802X_1^{(1)} \qquad and \qquad X_3^{(1)}=2.247X_1^{(1)}$$

Therefore the first mode shape is : $\vec{X}^{(1)}=X_1^{(1)}=\begin{Bmatrix}1\\1.802\\2.247\end{Bmatrix}$

- **Eigenvalues and Eigenvectors**

Second mode : substituting $\lambda_2 = \frac{1}{\omega_2^2} = 0.643\frac{m}{k}$ gives :

$$\left[0.643\frac{m}{k}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \frac{m}{k}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix}\right]\begin{Bmatrix}X_1^{(2)}\\ X_2^{(2)}\\ X_3^{(2)}\end{Bmatrix} = \begin{bmatrix}-0.357 & -1 & -1\\ -1 & -1.357 & -2\\ -1 & -2 & -2.357\end{bmatrix}\begin{Bmatrix}X_1^{(2)}\\ X_2^{(2)}\\ X_3^{(2)}\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

From the first and second rows :

$$-X_2^{(2)} - X_3^{(2)} = 0.357X_1^{(2)} \quad \text{and} \quad -1.357X_2^{(2)} - 2X_3^{(2)} = X_1^{(2)}$$

Solving for $X_2^{(2)}$ and $X_3^{(2)}$ in terms $X_1^{(2)}$:

$$X_2^{(2)} = 0.445X_1^{(2)} \quad \text{and} \quad X_3^{(2)} = -0.802X_1^{(2)}$$

Therefore the second mode shape is : $\vec{X}^{(2)} = X_1^{(2)} = \begin{Bmatrix}1\\ 0.445\\ -0.802\end{Bmatrix}$

- **Eigenvalues and Eigenvectors**

Third mode : substituting $\lambda_3 = \frac{1}{\omega_3^2} = 0.308\frac{m}{k}$ gives :

$$\left[0.308\frac{m}{k}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{m}{k}\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}\right]\begin{Bmatrix}X_1^{(3)}\\X_2^{(3)}\\X_3^{(3)}\end{Bmatrix} = \begin{bmatrix}-0.692&-1&-1\\-1&-1.692&-2\\-1&-2&-2.692\end{bmatrix}\begin{Bmatrix}X_1^{(3)}\\X_2^{(3)}\\X_3^{(3)}\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

From the first and second rows :

$$-X_2^{(3)} - X_3^{(3)} = 0.692X_1^{(3)} \quad \text{and} \quad -1.692X_2^{(3)} - 2X_3^{(3)} = X_1^{(3)}$$

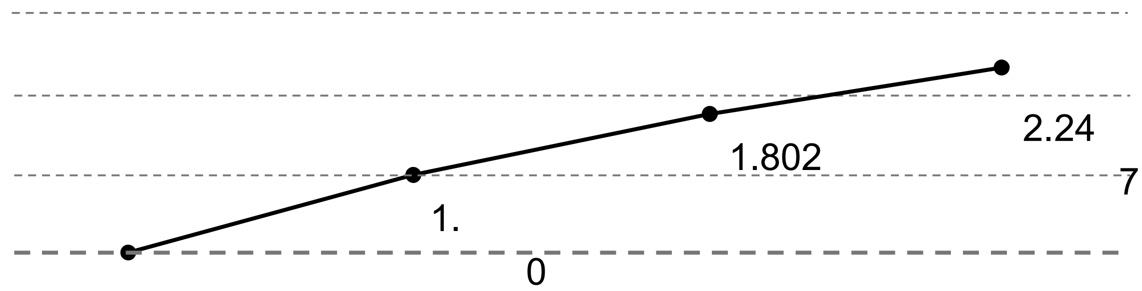
Solving for $X_2^{(3)}$ and $X_3^{(3)}$ in terms $X_1^{(3)}$:

$$X_2^{(3)} = -1.247X_1^{(3)} \quad \text{and} \quad X_3^{(3)} = 0.554X_1^{(3)}$$

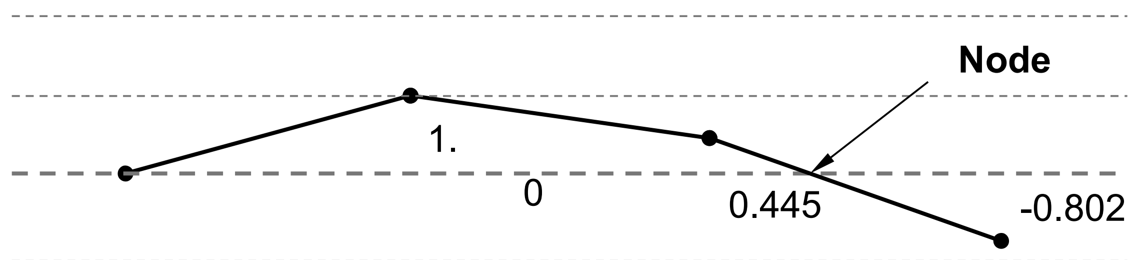
Therefore the third mode shape is : $\vec{X}^{(3)} = X_1^{(3)} = \begin{Bmatrix}1\\-1.247\\0.554\end{Bmatrix}$

- **Eigenvalues and Eigenvectors**

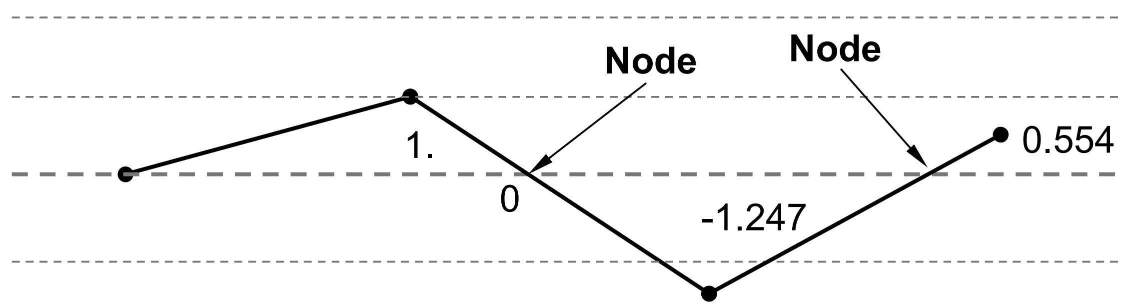
$$X_1^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}$$



$$X_1^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix}$$

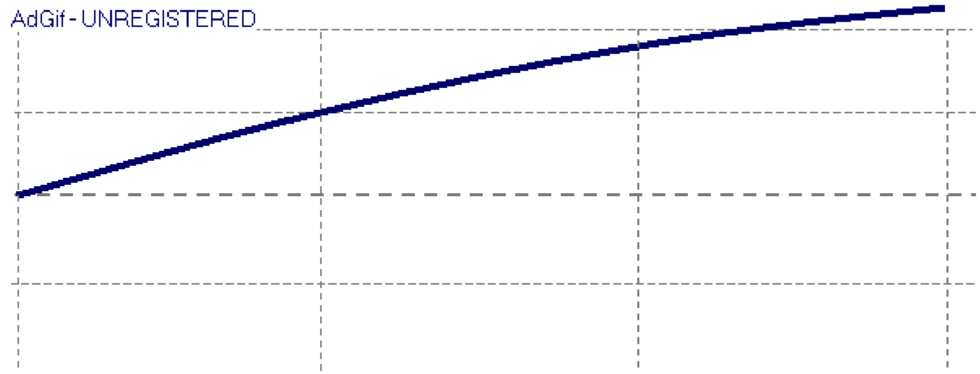


$$X_1^{(3)} = \begin{Bmatrix} 1 \\ -1.247 \\ 0.554 \end{Bmatrix}$$



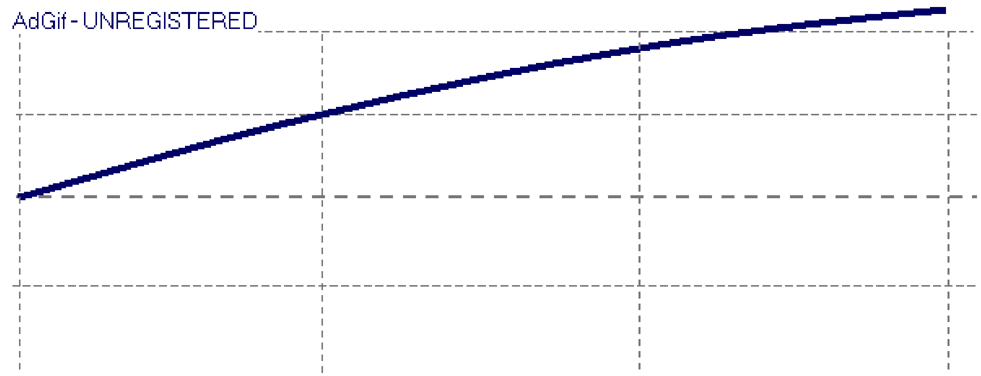
- **Eigenvalues and Eigenvectors**

$$Mode\ \#1\ \omega_n = 0.45\sqrt{\frac{k}{m}} \quad X_1^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}$$

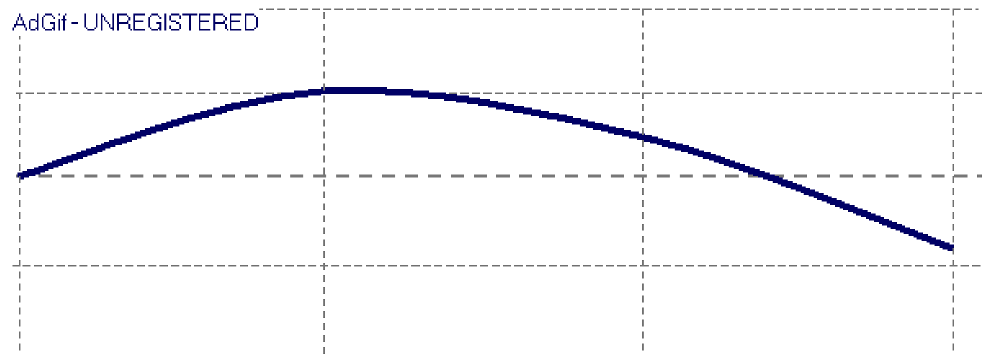


- **Eigenvalues and Eigenvectors**

$$Mode\ \#1\ \omega_n = 0.45\sqrt{\frac{k}{m}}\quad X_1^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}$$

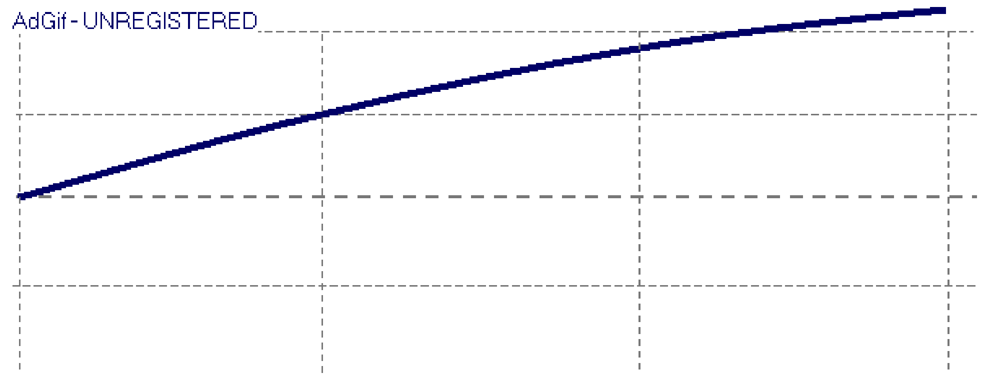


$$Mode\ \#2\ \omega_n = 1.25\sqrt{\frac{k}{m}}\quad X_1^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix}$$

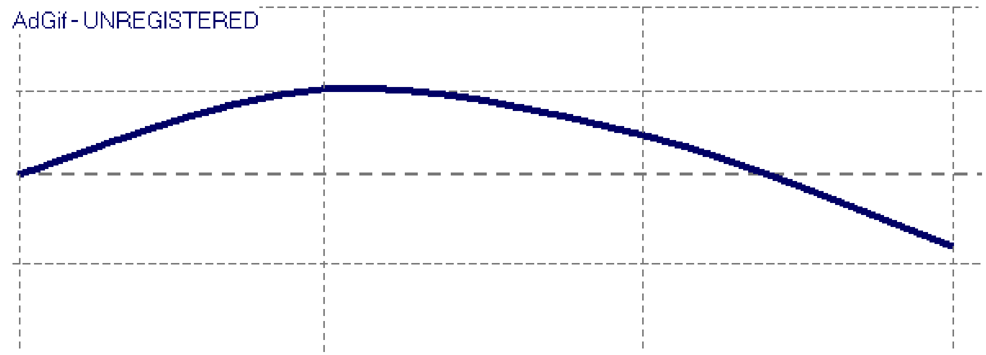


- **Eigenvalues and Eigenvectors**

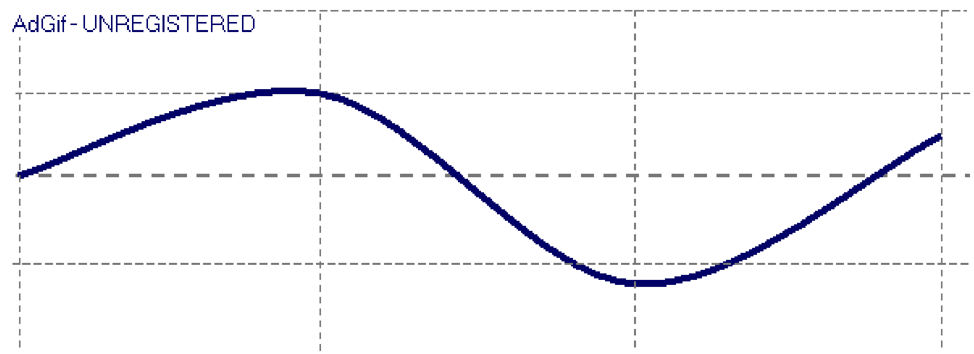
$$Mode\ \#1\ \omega_n = 0.45\sqrt{\frac{k}{m}}\quad X_1^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}$$



$$Mode\ \#2\ \omega_n = 1.25\sqrt{\frac{k}{m}}\quad X_1^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix}$$



$$Mode\ \#3\ \omega_n = 1.80\sqrt{\frac{k}{m}}\quad X_1^{(3)} = \begin{Bmatrix} 1 \\ -1.247 \\ 0.554 \end{Bmatrix}$$



Mechanical Vibrations

Good luck for the exam!