

VECTOR MECHANICS FOR ENGINEERS:
STATICS

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Distributed Forces:
Moments of Inertia

Vector Mechanics for Engineers: Dynamics

Contents

[Introduction](#)

[Moments of Inertia of an Area](#)

[Moment of Inertia of an Area by
Integration](#)

[Polar Moment of Inertia](#)

[Radius of Gyration of an Area](#)

[Sample Problem 9.1](#)

[Sample Problem 9.2](#)

[Parallel Axis Theorem](#)

[Moments of Inertia of Composite
Areas](#)

[Sample Problem 9.4](#)

[Sample Problem 9.5](#)

[Product of Inertia](#)

[Principal Axes and Principal Moments
of Inertia](#)

[Sample Problem 9.6](#)

[Sample Problem 9.7](#)

[Mohr's Circle for Moments and Products
of Inertia](#)

[Sample Problem 9.8](#)

[Moment of Inertia of a Mass](#)

[Parallel Axis Theorem](#)

[Moment of Inertia of Thin Plates](#)

[Moment of Inertia of a 3D Body by
Integration](#)

[Moment of Inertia of Common Geometric
Shapes](#)

[Sample Problem 9.12](#)

[Moment of Inertia With Respect to an
Arbitrary Axis](#)

[Ellipsoid of Inertia. Principle Axes of
Axes of Inertia of a Mass](#)



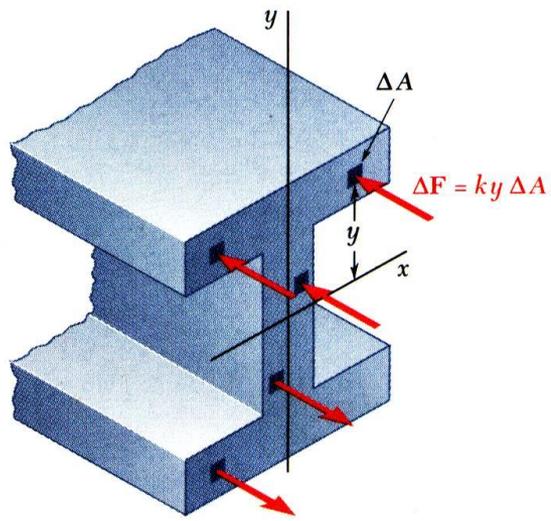
Introduction

- Previously considered distributed forces which were proportional to the area or volume over which they act.
 - The resultant was obtained by summing or integrating over the areas or volumes.
 - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.
- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
 - It will be shown that the magnitude of the resultant depends on the first moment of the force distribution with respect to the axis.
 - The point of application of the resultant depends on the second moment of the distribution with respect to the axis.
- Current chapter will present methods for computing the moments and products of inertia for areas and masses.



Vector Mechanics for Engineers: Dynamics

Moment of Inertia of an Area

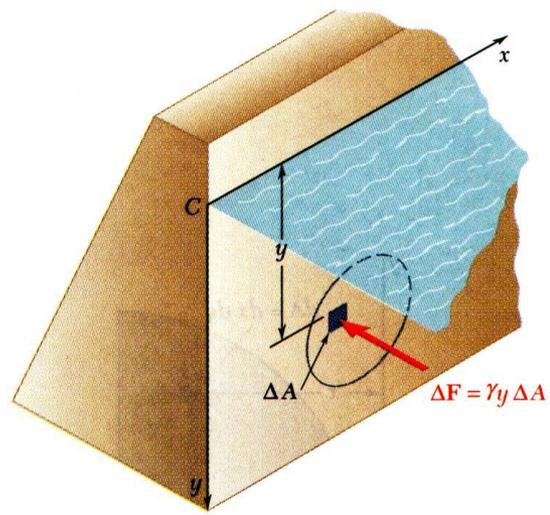


- Consider distributed forces $\Delta \vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.
- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta \vec{F} = ky\Delta A$$

$$R = k \int y dA = 0 \quad \int y dA = Q_x = \text{first moment}$$

$$M = k \int y^2 dA \quad \int y^2 dA = \text{second moment}$$



- Example: Consider the net hydrostatic force on a submerged circular gate.

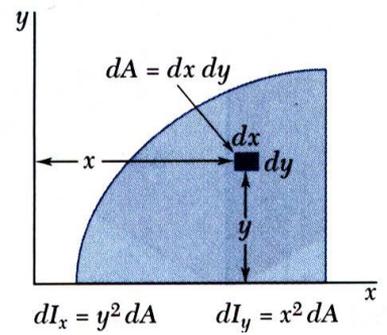
$$\Delta F = p\Delta A = \gamma y\Delta A$$

$$R = \gamma \int y dA$$

$$M_x = \gamma \int y^2 dA$$

Vector Mechanics for Engineers: Dynamics

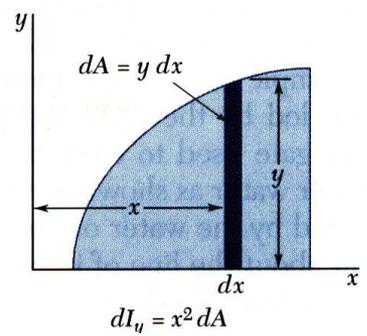
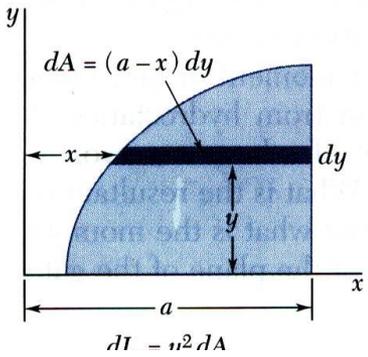
Moment of Inertia of an Area by Integration



- *Second moments or moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \quad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

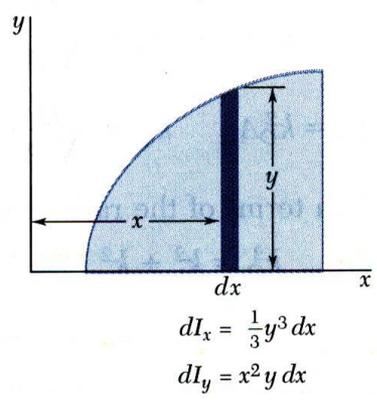
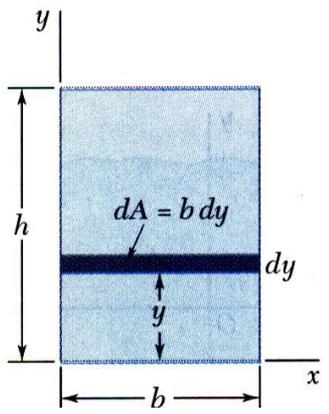


- For a rectangular area,

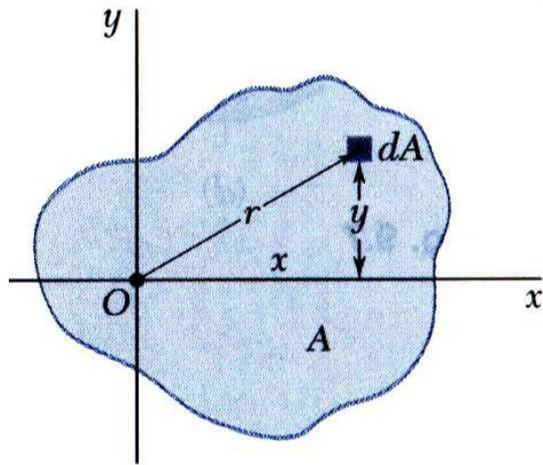
$$I_x = \int y^2 dA = \int_0^h y^2 b dy = \frac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \frac{1}{3} y^3 dx \quad dI_y = x^2 dA = x^2 y dx$$



Polar Moment of Inertia



- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = \int r^2 dA$$

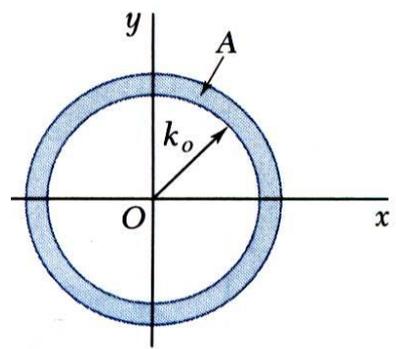
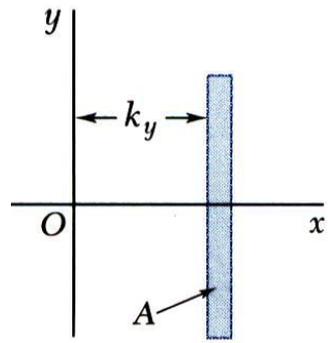
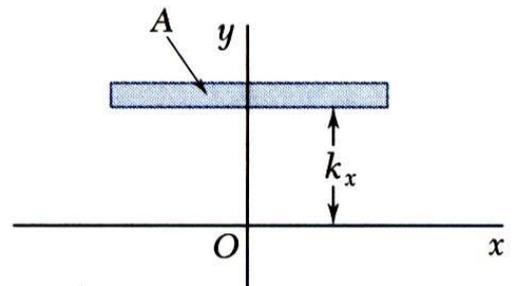
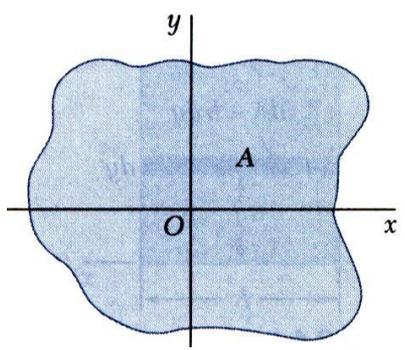
- The polar moment of inertia is related to the rectangular moments of inertia,

$$\begin{aligned} J_0 &= \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA \\ &= I_y + I_x \end{aligned}$$



Vector Mechanics for Engineers: Dynamics

Radius of Gyration of an Area



- Consider area A with moment of inertia I_x . Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x .

$$I_x = k_x^2 A \quad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ radius of gyration with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \quad k_y = \sqrt{\frac{I_y}{A}}$$

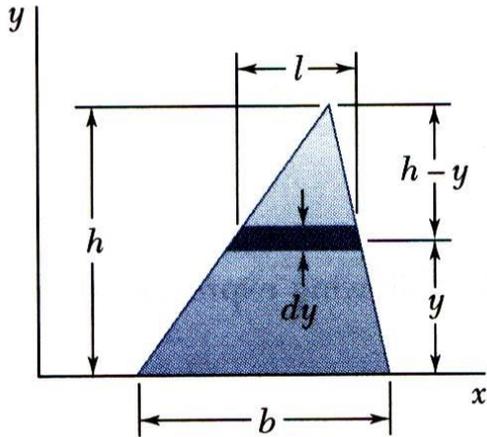
$$J_O = k_O^2 A \quad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.1



SOLUTION:

- A differential strip parallel to the x axis is chosen for dA .

$$dI_x = y^2 dA \quad dA = l dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h - y}{h} \quad l = b \frac{h - y}{h} \quad dA = b \frac{h - y}{h} dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

$$I_x = \int y^2 dA = \int_0^h y^2 b \frac{h - y}{h} dy = \frac{b}{h} \int_0^h (hy^2 - y^3) dy$$

$$= \frac{b}{h} \left[h \frac{y^3}{3} - \frac{y^4}{4} \right]_0^h$$

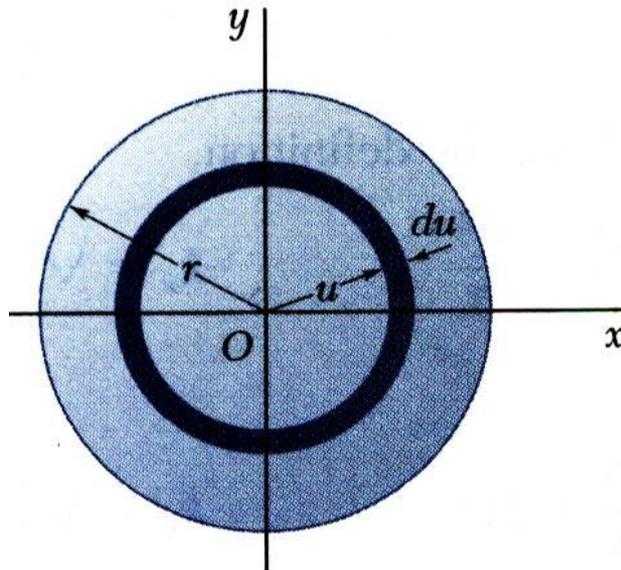
$$I_x = \frac{bh^3}{12}$$

Determine the moment of inertia of a triangle with respect to its base.



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Sample Problem 9.2



SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \quad dA = 2\pi u du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u du) = 2\pi \int_0^r u^3 du$$

$$J_O = \frac{\pi}{2} r^4$$

- Determine the centroidal polar moment of inertia of a circular area by direct integration.
- Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.

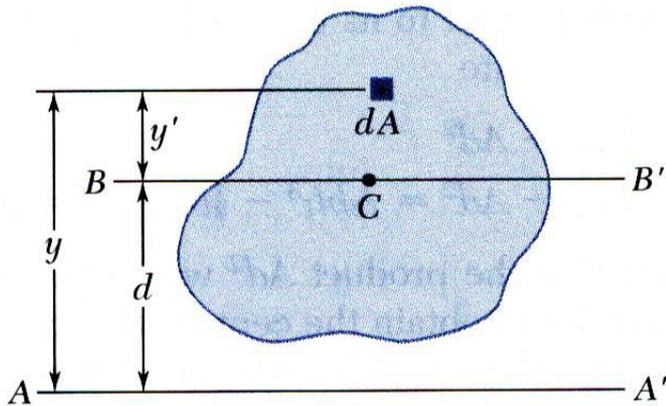
- From symmetry, $I_x = I_y$

$$J_O = I_x + I_y = 2I_x \quad \frac{\pi}{2} r^4 = 2I_x$$

$$I_{diameter} = I_x = \frac{\pi}{4} r^4$$

Vector Mechanics for Engineers: Dynamics

Parallel Axis Theorem



- Consider moment of inertia I of an area A with respect to the axis AA'

$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a *centroidal axis*.

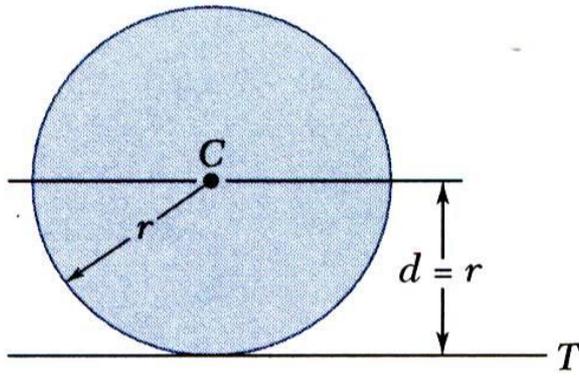
$$\begin{aligned} I &= \int y^2 dA = \int (y' + d)^2 dA \\ &= \int y'^2 dA + 2d \int y' dA + d^2 \int dA \end{aligned}$$

$$I = \bar{I} + Ad^2 \quad \textit{parallel axis theorem}$$



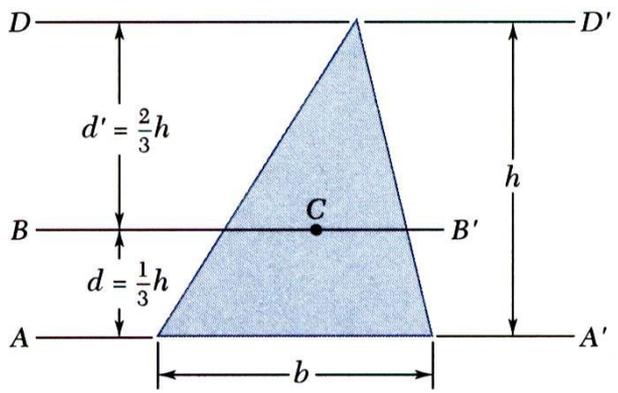
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Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \frac{1}{4}\pi r^4 + (\pi r^2)r^2 = \frac{5}{4}\pi r^4$$



- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$

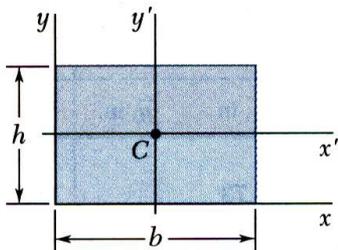
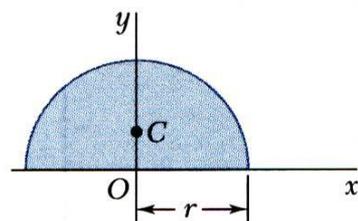
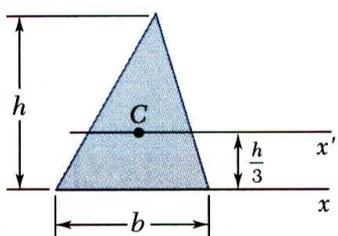
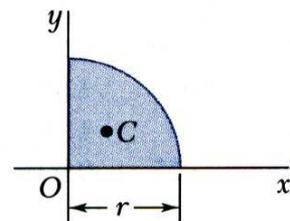
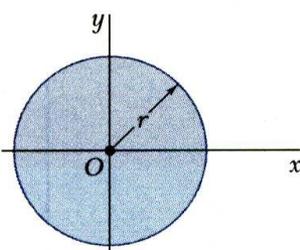
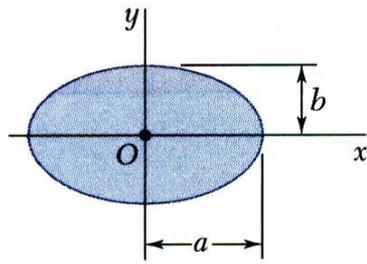
$$I_{BB'} = I_{AA'} - Ad^2 = \frac{1}{12}bh^3 - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2 = \frac{1}{36}bh^3$$



Vector Mechanics for Engineers: Dynamics

Moments of Inertia of Composite Areas

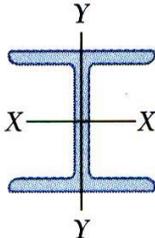
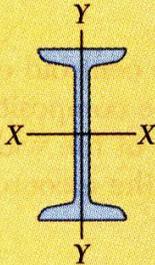
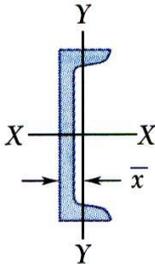
- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, \dots , with respect to the same axis.

<p>Rectangle</p>		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	<p>Semicircle</p>		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
<p>Triangle</p>		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	<p>Quarter circle</p>		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
<p>Circle</p>		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	<p>Ellipse</p>		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

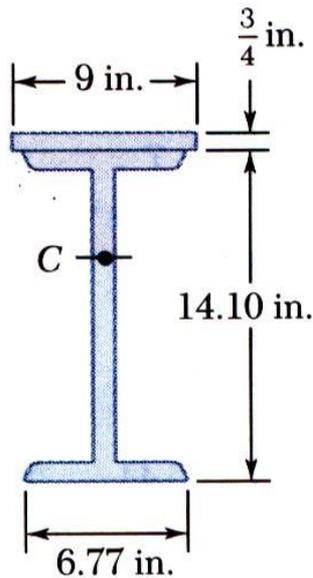


Vector Mechanics for Engineers: Dynamics

Moments of Inertia of Composite Areas

	Designation	Area mm ²	Depth mm	Width mm	Axis X-X			Axis Y-Y			
					\bar{I}_x 10 ⁶ mm ⁴	\bar{k}_x mm	\bar{y} mm	\bar{I}_y 10 ⁶ mm ⁴	\bar{k}_y mm	\bar{x} mm	
W Shapes (Wide-Flange Shapes)		W460 × 113†	14400	463	280	554	196.3		63.3	66.3	
		W410 × 85	10800	417	181	316	170.7		17.94	40.6	
		W360 × 57	7230	358	172	160.2	149.4		11.11	39.4	
		W200 × 46.1	5890	203	203	45.8	88.1		15.44	51.3	
S Shapes (American Standard Shapes)		S460 × 81.4†	10390	457	152	335	179.6		8.66	29.0	
		S310 × 47.3	6032	305	127	90.7	122.7		3.90	25.4	
		S250 × 37.8	4806	254	118	51.6	103.4		2.83	24.2	
		S150 × 18.6	2362	152	84	9.2	62.2		0.758	17.91	
C Shapes (American Standard Channels)		C310 × 30.8†	3929	305	74	53.7	117.1		1.615	20.29	17.73
		C250 × 22.8	2897	254	65	28.1	98.3		0.949	18.11	16.10
		C200 × 17.1	2181	203	57	13.57	79.0		0.549	15.88	14.50
		C150 × 12.2	1548	152	48	5.45	59.4		0.288	13.64	13.00

Sample Problem 9.4



The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

Determine the moment of inertia and radius of gyration with respect to an axis which is parallel to the plate and passes through the centroid of the section.

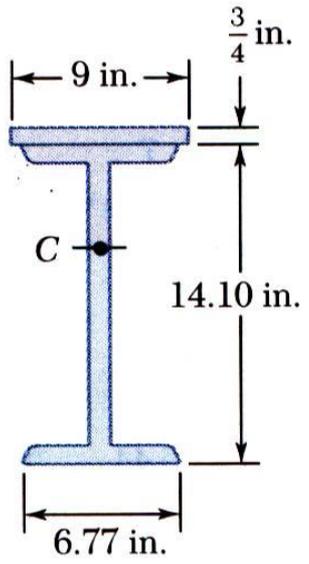
SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.
- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.
- Calculate the radius of gyration from the moment of inertia of the composite section.



Vector Mechanics for Engineers: Dynamics

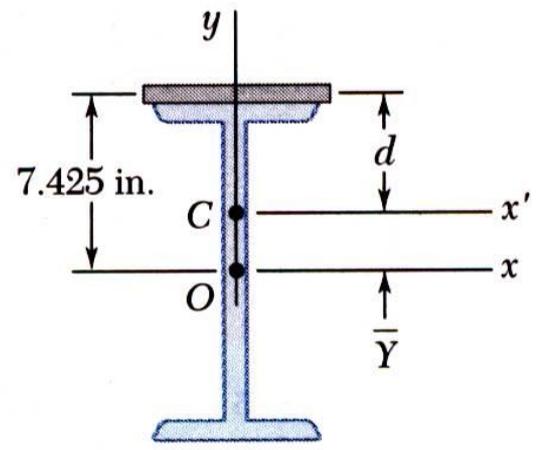
Sample Problem 9.4



SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.

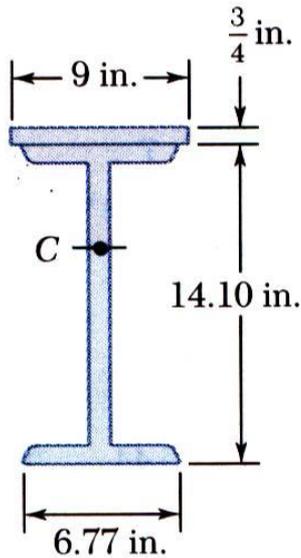
Section	A, in^2	$\bar{y}, \text{in.}$	$\bar{y}A, \text{in}^3$
Plate	6.75	7.425	50.12
Beam Section	11.20	0	0
	$\sum A = 17.95$		$\sum \bar{y}A = 50.12$



$$\bar{Y} \sum A = \sum \bar{y}A \quad \bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{50.12 \text{ in}^3}{17.95 \text{ in}^2} = 2.792 \text{ in.}$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.4



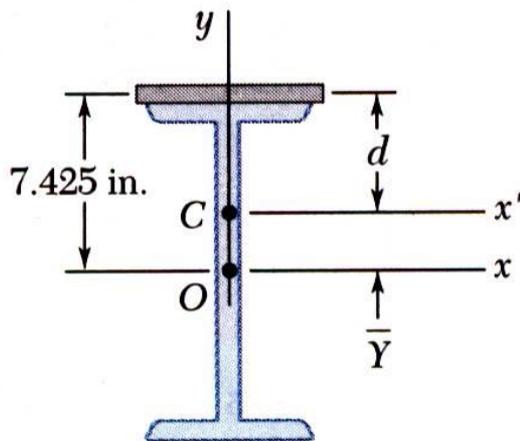
- Apply the parallel axis theorem to determine moments of inertia of beam section and plate with respect to composite section centroidal axis.

$$I_{x', \text{beam section}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2 = 472.3 \text{ in}^4$$

$$I_{x', \text{plate}} = \bar{I}_x + Ad^2 = \frac{1}{12}(9)\left(\frac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2 = 145.2 \text{ in}^4$$

$$I_{x'} = I_{x', \text{beam section}} + I_{x', \text{plate}} = 472.3 + 145.2$$

$$I_{x'} = 618 \text{ in}^4$$



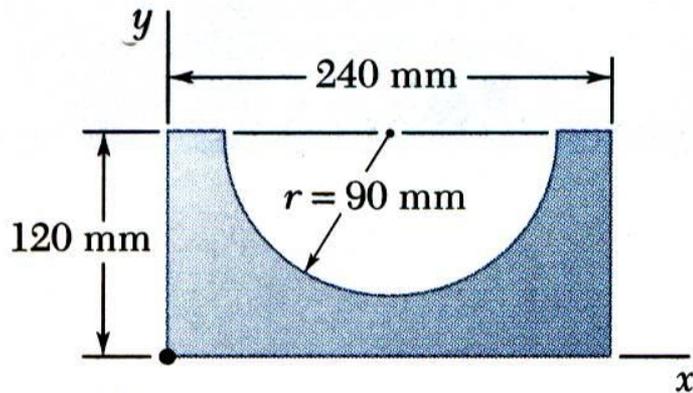
- Calculate the radius of gyration from the moment of inertia of the composite section.

$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \frac{617.5 \text{ in}^4}{17.95 \text{ in}^2}$$

$$k_{x'} = 5.87 \text{ in.}$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.5



SOLUTION:

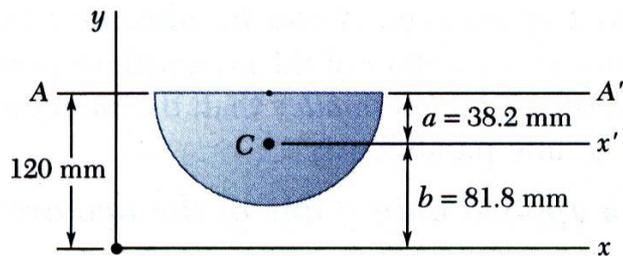
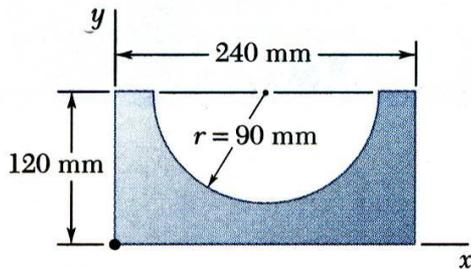
- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

Determine the moment of inertia of the shaded area with respect to the x axis.



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.5



$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2 \text{ mm}$$

$$b = 120 - a = 81.8 \text{ mm}$$

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(90)^2 = 12.72 \times 10^3 \text{ mm}^2$$

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.

Rectangle:

$$I_x = \frac{1}{3}bh^3 = \frac{1}{3}(240)(120)^3 = 138.2 \times 10^6 \text{ mm}^4$$

Half-circle:

moment of inertia with respect to AA' ,

$$I_{AA'} = \frac{1}{8}\pi r^4 = \frac{1}{8}\pi(90)^4 = 25.76 \times 10^6 \text{ mm}^4$$

moment of inertia with respect to x' ,

$$\begin{aligned} \bar{I}_{x'} &= I_{AA'} - Aa^2 = (25.76 \times 10^6) - (12.72 \times 10^3) \\ &= 7.20 \times 10^6 \text{ mm}^4 \end{aligned}$$

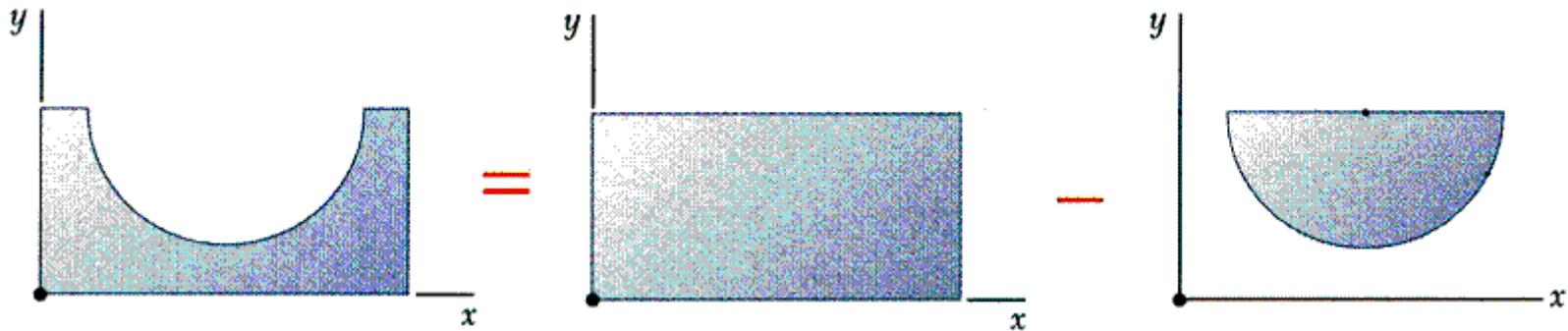
moment of inertia with respect to x ,

$$\begin{aligned} I_x &= \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + (12.72 \times 10^3)(81.8)^2 \\ &= 92.3 \times 10^6 \text{ mm}^4 \end{aligned}$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.5

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

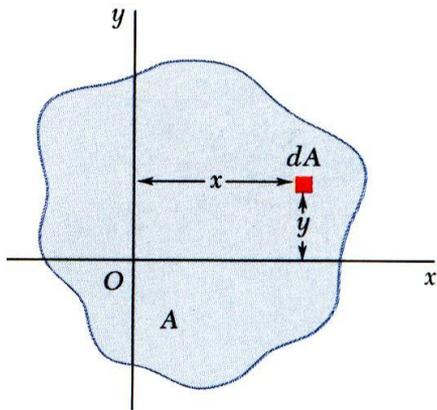


$$I_x = 138.2 \times 10^6 \text{ mm}^4 - 92.3 \times 10^6 \text{ mm}^4$$

$$I_x = 45.9 \times 10^6 \text{ mm}^4$$

Vector Mechanics for Engineers: Dynamics

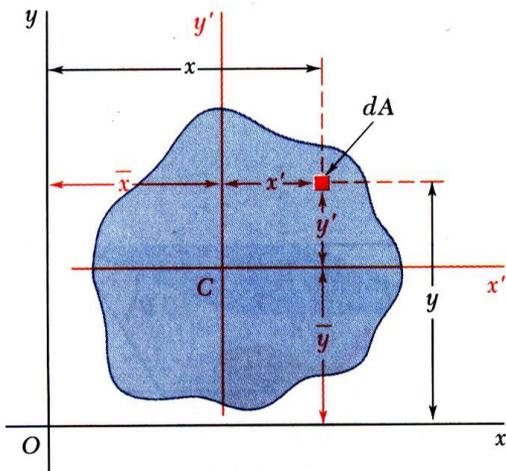
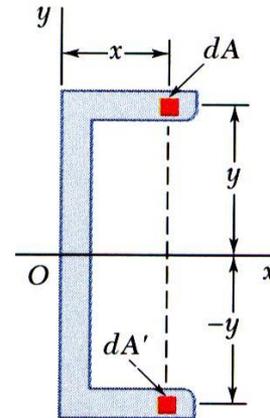
Product of Inertia



- *Product of Inertia:*

$$I_{xy} = \int xy \, dA$$

- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.



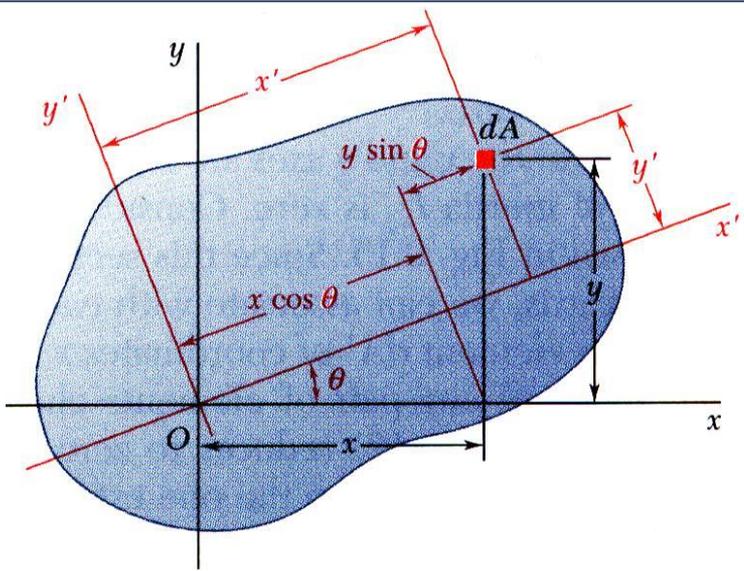
- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$



Vector Mechanics for Engineers: Dynamics

Principal Axes and Principal Moments of Inertia



- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

Given $I_x = \int y^2 dA$ $I_y = \int x^2 dA$
 $I_{xy} = \int xy dA$

we wish to determine moments and product of inertia with respect to new axes x' and y' .

Note: $x' = x \cos \theta + y \sin \theta$
 $y' = y \cos \theta - x \sin \theta$

- The equations for $I_{x'}$ and $I_{y'}$ are the parametric equations for a circle,

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

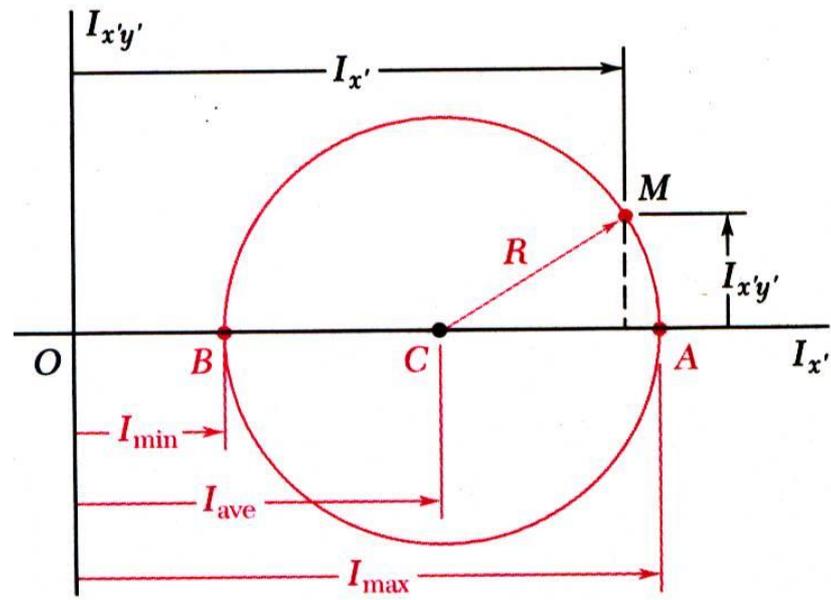
$$I_{ave} = \frac{I_x + I_y}{2} \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- The equations for $I_{y'}$ and $I_{x'y'}$ lead to the same circle.



Vector Mechanics for Engineers: Dynamics

Principal Axes and Principal Moments of Inertia



- At the points A and B , $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively
 $I_{\max, \min} = I_{ave} \pm R$

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

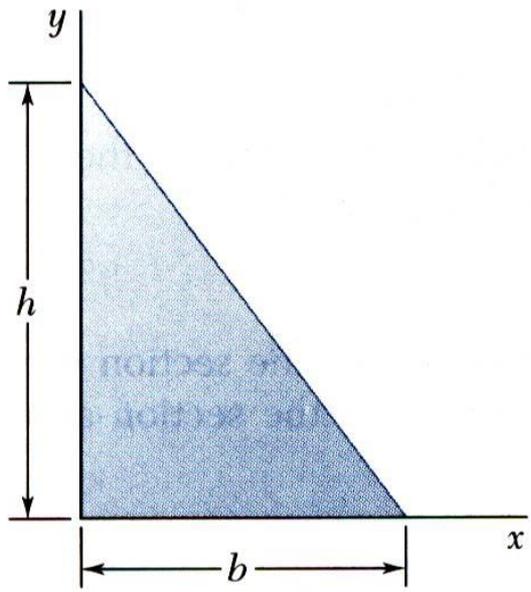
- The equation for θ_m defines two angles, 90° apart which correspond to the *principal axes* of the area about O .
- I_{\max} and I_{\min} are the *principal moments of inertia* of the area about O .

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.6



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

Determine the product of inertia of the right triangle (*a*) with respect to the *x* and *y* axes and (*b*) with respect to centroidal axes parallel to the *x* and *y* axes.



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.6

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \quad dA = y dx = h\left(1 - \frac{x}{b}\right) dx$$

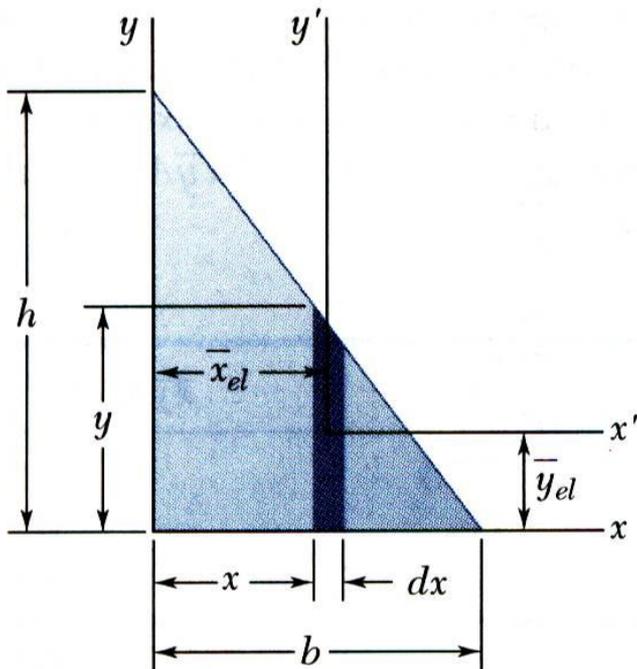
$$\bar{x}_{el} = x \quad \bar{y}_{el} = \frac{1}{2} y = \frac{1}{2} h\left(1 - \frac{x}{b}\right)$$

Integrating dI_{xy} from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el} \bar{y}_{el} dA = \int_0^b x \left(\frac{1}{2}\right) h^2 \left(1 - \frac{x}{b}\right)^2 dx$$

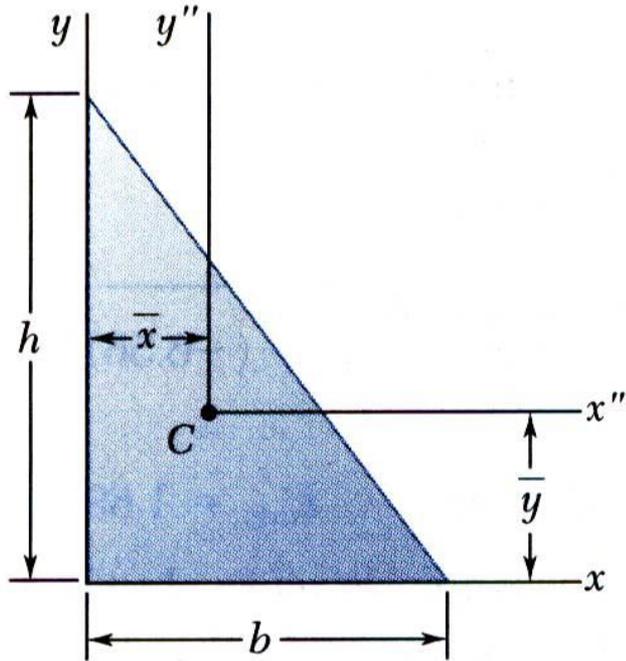
$$= h^2 \int_0^b \left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right) dx = h^2 \left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2} \right]_0^b$$

$$I_{xy} = \frac{1}{24} b^2 h^2$$



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.6



- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \frac{1}{3}b \quad \bar{y} = \frac{1}{3}h$$

With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

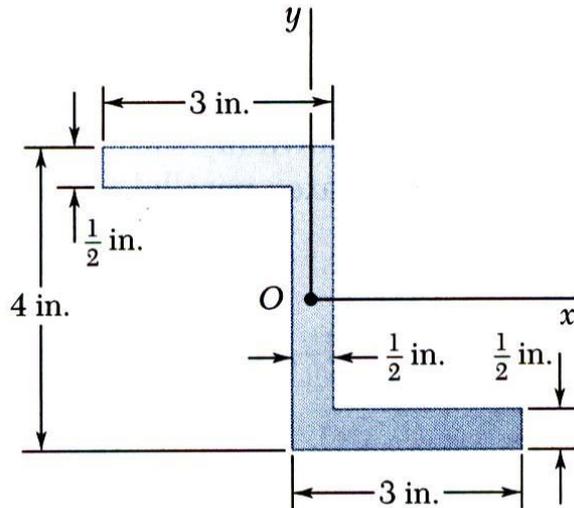
$$\bar{I}_{x''y''} = \frac{1}{24}b^2h^2 - \left(\frac{1}{3}b\right)\left(\frac{1}{3}h\right)\left(\frac{1}{2}bh\right)$$

$$\bar{I}_{x''y''} = -\frac{1}{72}b^2h^2$$



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.7



SOLUTION:

- Compute the product of inertia with respect to the xy axes by dividing the section into three rectangles and applying the parallel axis theorem to each.
- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9.27).

For the section shown, the moments of inertia with respect to the x and y axes are $I_x = 10.38 \text{ in}^4$ and $I_y = 6.97 \text{ in}^4$.

Determine (a) the orientation of the principal axes of the section about O , and (b) the values of the principal moments of inertia about O .

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.7

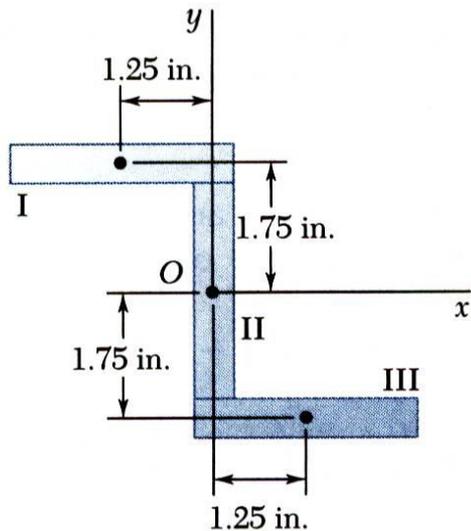
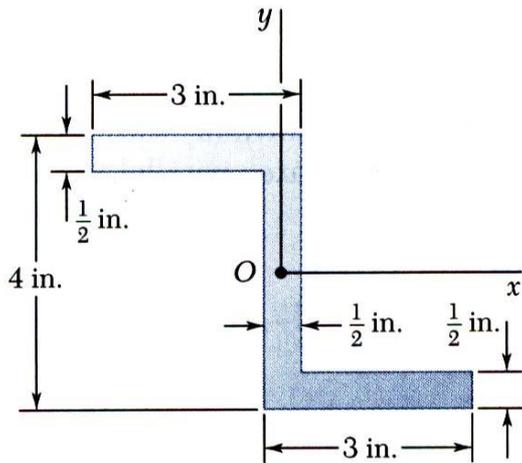
SOLUTION:

- Compute the product of inertia with respect to the xy axes by dividing the section into three rectangles.

Apply the parallel axis theorem to each rectangle,

$$I_{xy} = \sum (\bar{I}_{x'y'} + \bar{x}\bar{y}A)$$

Note that the product of inertia with respect to centroidal axes parallel to the xy axes is zero for each rectangle.

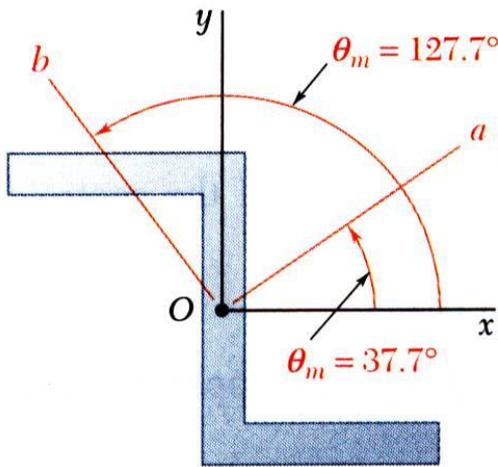


Rectangle	Area, in ²	\bar{x} , in.	\bar{y} , in.	$\bar{x}\bar{y}A$, in ⁴
<i>I</i>	1.5	-1.25	+1.75	-3.28
<i>II</i>	1.5	0	0	0
<i>III</i>	1.5	+1.25	-1.75	-3.28
				$\sum \bar{x}\bar{y}A = -6.56$

$$I_{xy} = \sum \bar{x}\bar{y}A = -6.56 \text{ in}^4$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.7



- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9.27).

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$2\theta_m = 75.4^\circ \text{ and } 255.4^\circ$$

$$\theta_m = 37.7^\circ \text{ and } \theta_m = 127.7^\circ$$

$$I_x = 10.38 \text{ in}^4$$

$$I_y = 6.97 \text{ in}^4$$

$$I_{xy} = -6.56 \text{ in}^4$$

$$\begin{aligned} I_{\max, \min} &= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \\ &= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2} \end{aligned}$$

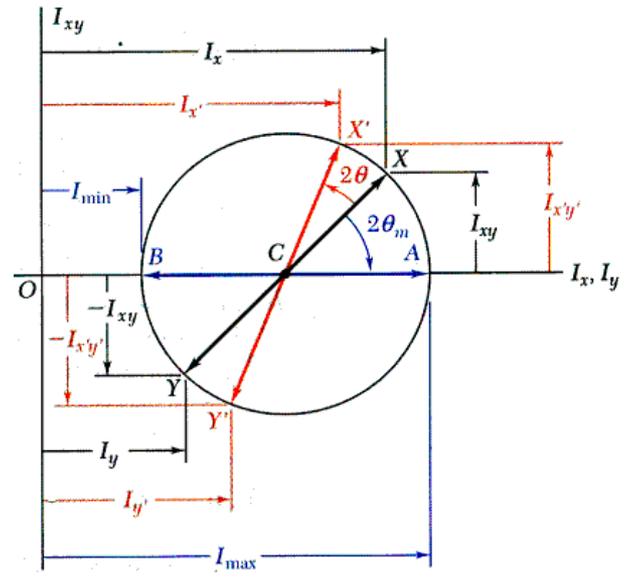
$$I_a = I_{\max} = 15.45 \text{ in}^4$$

$$I_b = I_{\min} = 1.897 \text{ in}^4$$



Vector Mechanics for Engineers: Dynamics

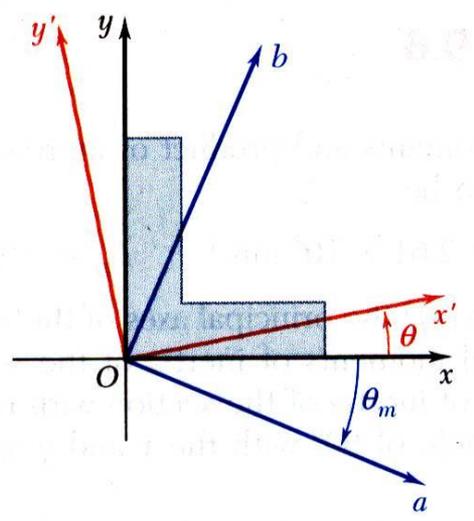
Mohr's Circle for Moments and Products of Inertia



- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle*,

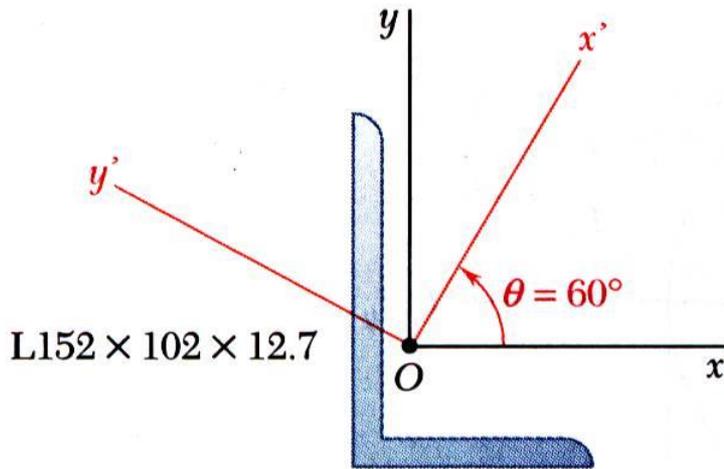
$$I_{ave} = \frac{I_x + I_y}{2} \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes and principal moments and products of inertia.



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.8



SOLUTION:

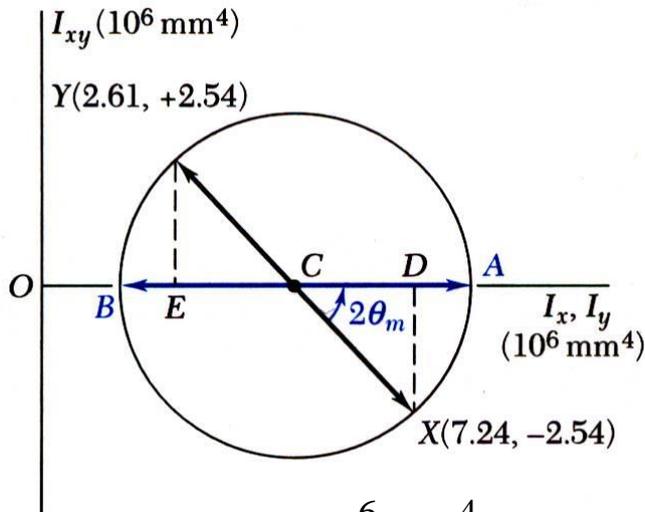
- Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

The moments and product of inertia with respect to the x and y axes are $I_x = 7.24 \times 10^6 \text{ mm}^4$, $I_y = 2.61 \times 10^6 \text{ mm}^4$, and $I_{xy} = -2.54 \times 10^6 \text{ mm}^4$.

Using Mohr's circle, determine (a) the principal axes about O , (b) the values of the principal moments about O , and (c) the values of the moments and product of inertia about the x' and y' axes

Vector Mechanics for Engineers: Dynamics

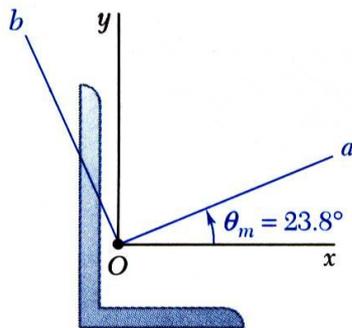
Sample Problem 9.8



$$I_x = 7.24 \times 10^6 \text{ mm}^4$$

$$I_y = 2.61 \times 10^6 \text{ mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \text{ mm}^4$$



SOLUTION:

- Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \frac{1}{2}(I_x + I_y) = 4.925 \times 10^6 \text{ mm}^4$$

$$CD = \frac{1}{2}(I_x - I_y) = 2.315 \times 10^6 \text{ mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{ mm}^4$$

- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

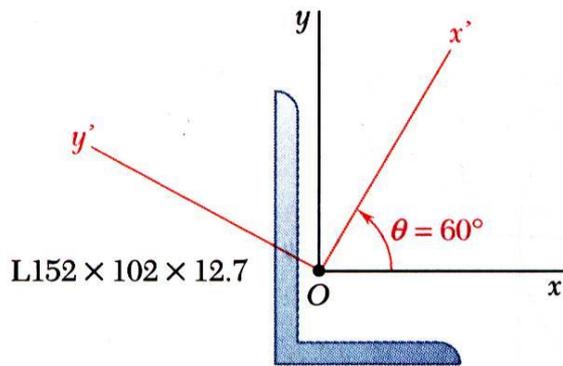
$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6^\circ \quad \theta_m = 23.8^\circ$$

$$I_{\max} = OA = I_{ave} + R \quad I_{\max} = 8.36 \times 10^6 \text{ mm}^4$$

$$I_{\min} = OB = I_{ave} - R \quad I_{\min} = 1.49 \times 10^6 \text{ mm}^4$$

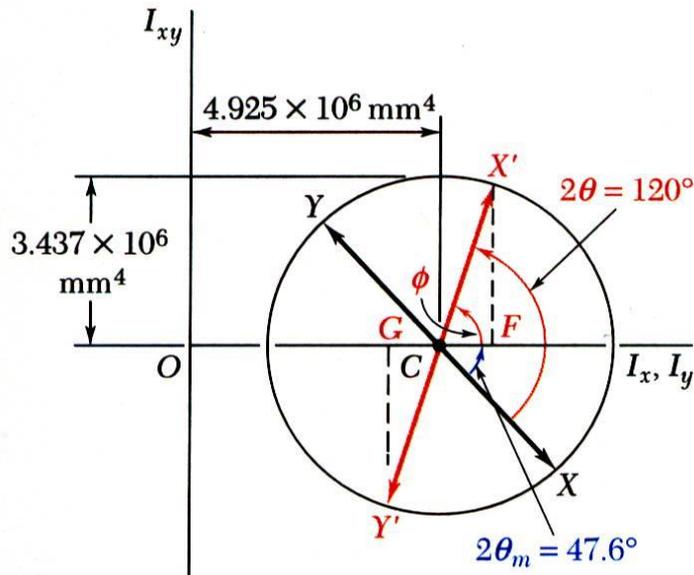
Vector Mechanics for Engineers: Dynamics

Sample Problem 9.8



- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

The points X' and Y' corresponding to the x' and y' axes are obtained by rotating CX and CY counterclockwise through an angle $\Theta = 2(60^\circ) = 120^\circ$. The angle that CX' forms with the x' axes is $\phi = 120^\circ - 47.6^\circ = 72.4^\circ$.



$$I_{x'} = OF = OC + CX' \cos \phi = I_{ave} + R \cos 72.4^\circ$$

$$I_{x'} = 5.96 \times 10^6 \text{ mm}^4$$

$$I_{y'} = OG = OC - CY' \cos \phi = I_{ave} - R \cos 72.4^\circ$$

$$I_{y'} = 3.89 \times 10^6 \text{ mm}^4$$

$$I_{x'y'} = FX' = CY' \sin \phi = R \sin 72.4^\circ$$

$$I_{x'y'} = 3.28 \times 10^6 \text{ mm}^4$$

$$OC = I_{ave} = 4.925 \times 10^6 \text{ mm}^4$$

$$R = 3.437 \times 10^6 \text{ mm}^4$$

Vector Mechanics for Engineers: Dynamics

Moment of Inertia of a Mass

- Angular acceleration about the axis AA' of the small mass Δm due to the application of a couple is proportional to $r^2 \Delta m$.

$$r^2 \Delta m = \text{moment of inertia of the mass } \Delta m \text{ with respect to the axis } AA'$$

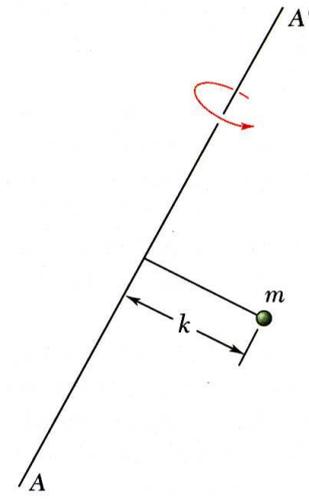
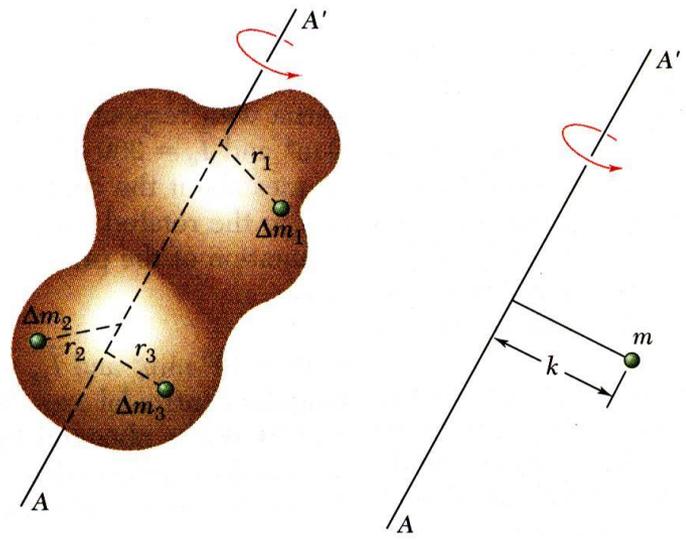
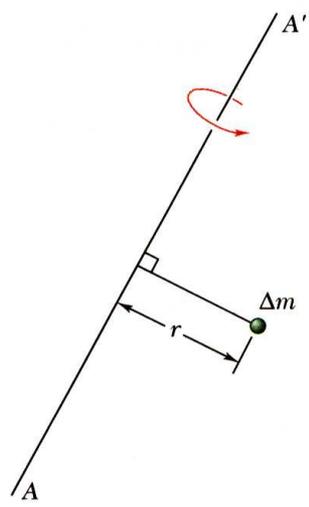
- For a body of mass m the resistance to rotation about the axis AA' is

$$I = r_1^2 \Delta m + r_2^2 \Delta m + r_3^2 \Delta m + \dots$$

$$= \int r^2 dm = \text{mass moment of inertia}$$

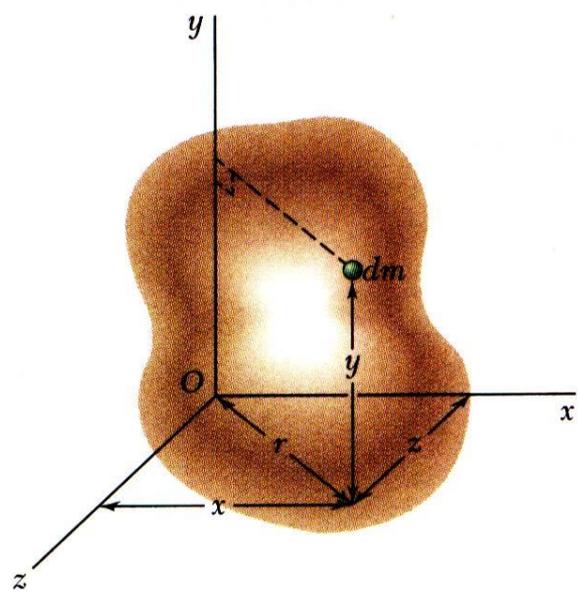
- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \quad k = \sqrt{\frac{I}{m}}$$



Vector Mechanics for Engineers: Dynamics

Moment of Inertia of a Mass



- Moment of inertia with respect to the y coordinate axis is

$$I_y = \int r^2 dm = \int (z^2 + x^2) dm$$

- Similarly, for the moment of inertia with respect to the x and z axes,

$$I_x = \int (y^2 + z^2) dm$$

$$I_z = \int (x^2 + y^2) dm$$

- In SI units,

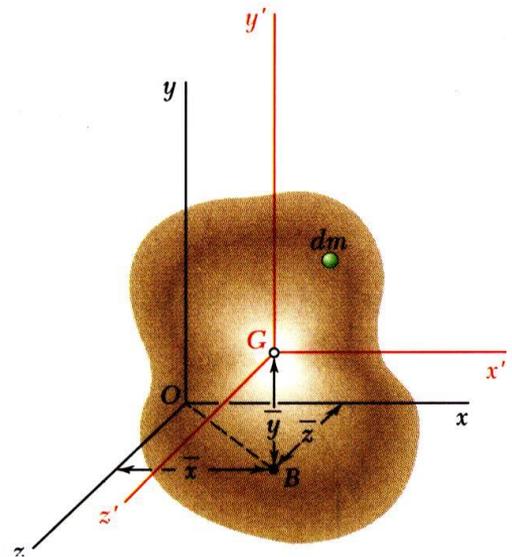
$$I = \int r^2 dm = (\text{kg} \cdot \text{m}^2)$$

In U.S. customary units,

$$I = (\text{slug} \cdot \text{ft}^2) = \left(\frac{\text{lb} \cdot \text{s}^2}{\text{ft}} \text{ft}^2 \right) = (\text{lb} \cdot \text{ft} \cdot \text{s}^2)$$



Parallel Axis Theorem



- For the rectangular axes with origin at O and parallel centroidal axes,

$$I_x = \int (y^2 + z^2) dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2] dm$$

$$= \int (y'^2 + z'^2) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + (\bar{y}^2 + \bar{z}^2) \int dm$$

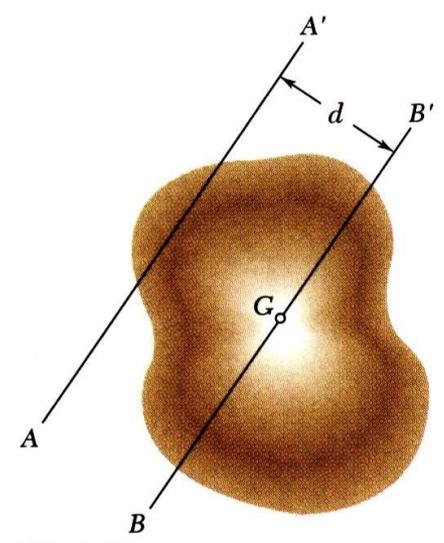
$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)$$

$$I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2)$$

$$I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2)$$

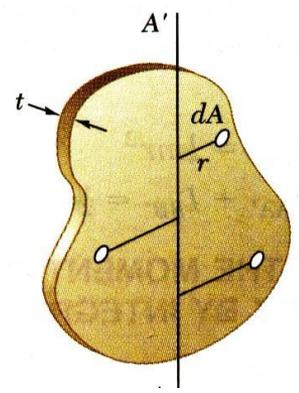
- Generalizing for any axis AA' and a parallel centroidal axis,

$$I = \bar{I} + md^2$$



Vector Mechanics for Engineers: Dynamics

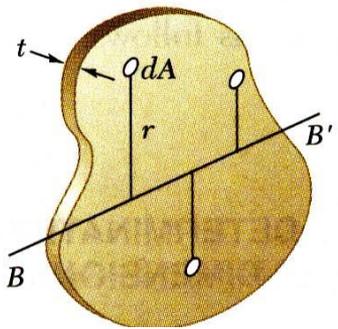
Moments of Inertia of Thin Plates



- For a thin plate of uniform thickness t and homogeneous material of density ρ , the mass moment of inertia with respect to axis AA' contained in the plate is

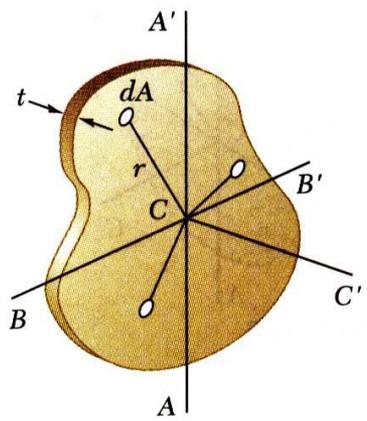
$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$

$$= \rho t I_{AA',area}$$



- Similarly, for perpendicular axis BB' which is also contained in the plate,

$$I_{BB'} = \rho t I_{BB',area}$$



- For the axis CC' which is perpendicular to the plate,

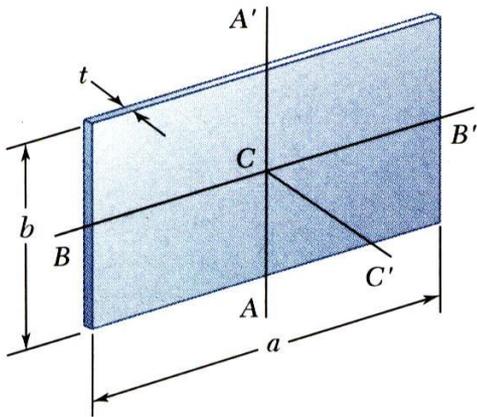
$$I_{CC'} = \rho t J_{C,area} = \rho t (I_{AA',area} + I_{BB',area})$$

$$= I_{AA'} + I_{BB'}$$



Vector Mechanics for Engineers: Dynamics

Moments of Inertia of Thin Plates

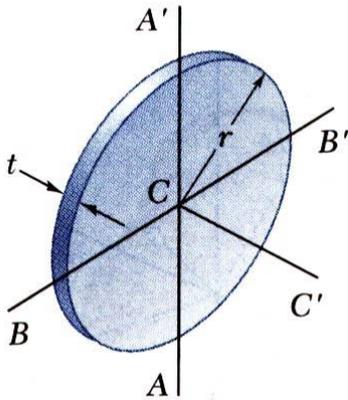


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t I_{AA',area} = \rho t \left(\frac{1}{12} a^3 b \right) = \frac{1}{12} m a^2$$

$$I_{BB'} = \rho t I_{BB',area} = \rho t \left(\frac{1}{12} a b^3 \right) = \frac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \frac{1}{12} m (a^2 + b^2)$$



- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t I_{AA',area} = \rho t \left(\frac{1}{4} \pi r^4 \right) = \frac{1}{4} m r^2$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{2} m r^2$$



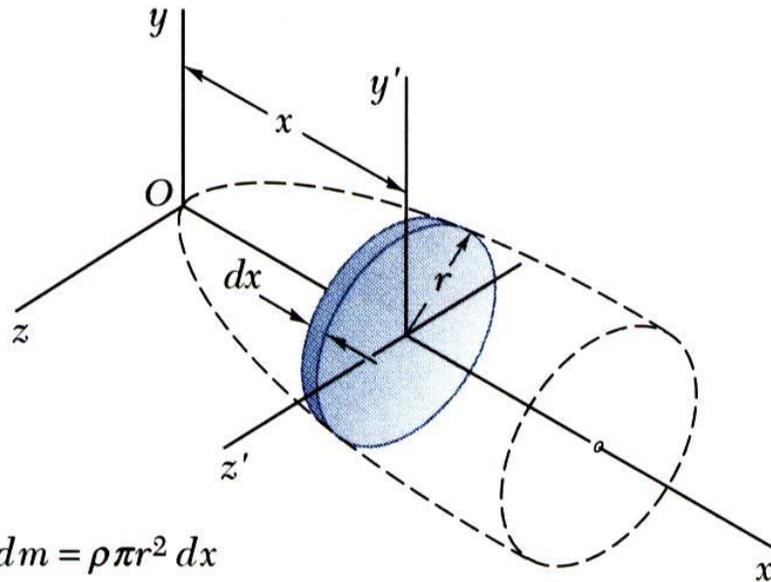
Vector Mechanics for Engineers: Dynamics

Moments of Inertia of a 3D Body by Integration

- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

$$I = \rho \int r^2 dV$$

- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for dm .



$$dm = \rho \pi r^2 dx$$

$$dI_x = \frac{1}{2} r^2 dm$$

$$dI_y = dI_{y'} + x^2 dm = \left(\frac{1}{4} r^2 + x^2 \right) dm$$

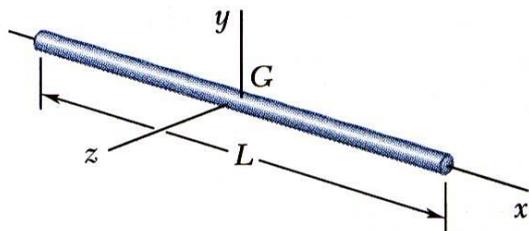
$$dI_z = dI_{z'} + x^2 dm = \left(\frac{1}{4} r^2 + x^2 \right) dm$$

- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

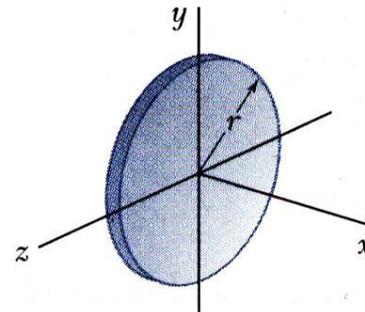


Vector Mechanics for Engineers: Dynamics

Moments of Inertia of Common Geometric Shapes

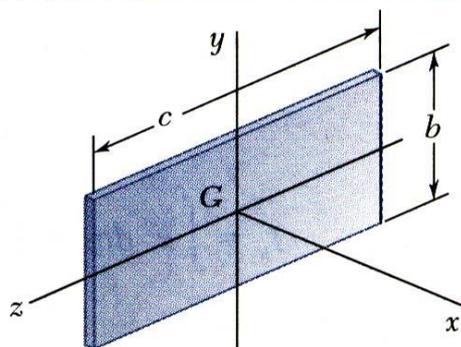


$$I_y = I_z = \frac{1}{12} mL^2$$



$$I_x = \frac{1}{2} mr^2$$

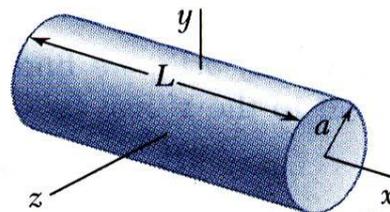
$$I_y = I_z = \frac{1}{4} mr^2$$



$$I_x = \frac{1}{12} m(b^2 + c^2)$$

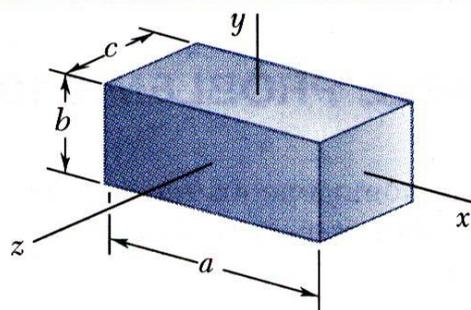
$$I_y = \frac{1}{12} mc^2$$

$$I_z = \frac{1}{12} mb^2$$



$$I_x = \frac{1}{2} ma^2$$

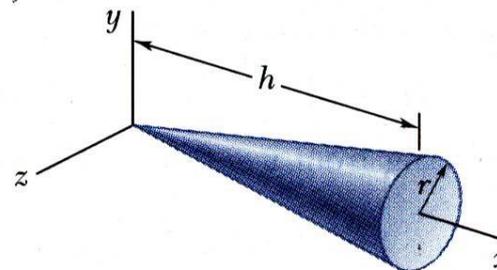
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$



$$I_x = \frac{1}{12} m(b^2 + c^2)$$

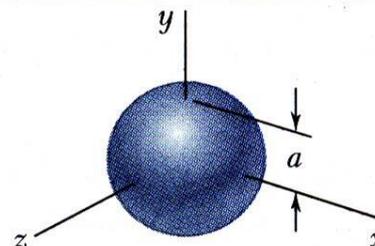
$$I_y = \frac{1}{12} m(c^2 + a^2)$$

$$I_z = \frac{1}{12} m(a^2 + b^2)$$



$$I_x = \frac{3}{10} ma^2$$

$$I_y = I_z = \frac{3}{5} m\left(\frac{1}{4}a^2 + h^2\right)$$

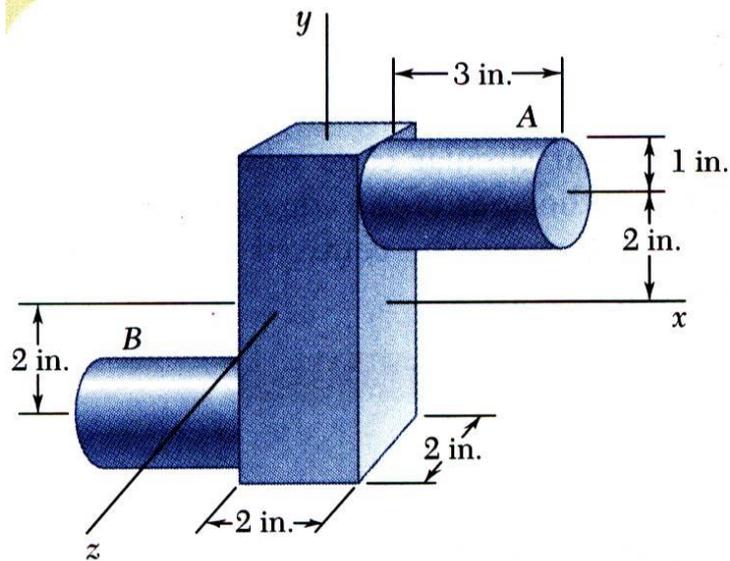


$$I_x = I_y = I_z = \frac{2}{5} ma^2$$



Vector Mechanics for Engineers: Dynamics

Sample Problem 9.12



SOLUTION:

- With the forging divided into a prism and two cylinders, compute the mass and moments of inertia of each component with respect to the xyz axes using the parallel axis theorem.
- Add the moments of inertia from the components to determine the total moments of inertia for the forging.

Determine the moments of inertia of the steel forging with respect to the xyz coordinate axes, knowing that the specific weight of steel is 490 lb/ft^3 .

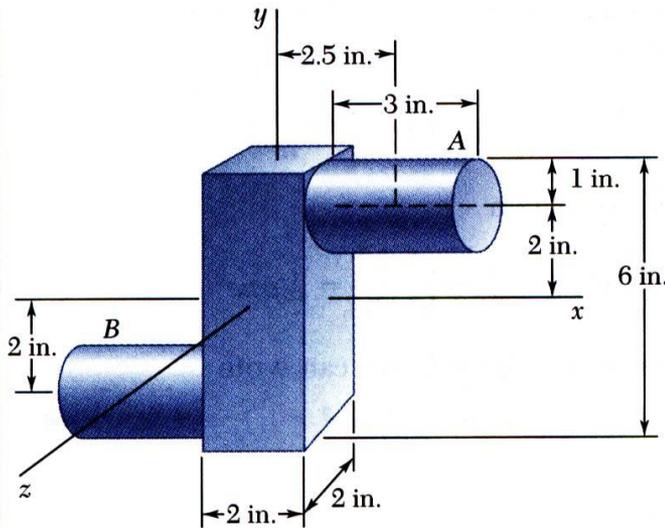


Vector Mechanics for Engineers: Dynamics

Sample Problem 9.12

SOLUTION:

- Compute the moments of inertia of each component with respect to the xyz axes.



cylinders ($a = 1\text{ in.}$, $L = 3\text{ in.}$, $\bar{x} = 2.5\text{ in.}$, $\bar{y} = 2\text{ in.}$):

$$\begin{aligned} I_x &= \frac{1}{2}ma^2 + m\bar{y}^2 \\ &= \frac{1}{2}(0.0829)\left(\frac{1}{12}\right)^2 + (0.0829)\left(\frac{2}{12}\right)^2 \\ &= 2.59 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \end{aligned}$$

$$\begin{aligned} I_y &= \frac{1}{12}m[3a^2 + L^2] + m\bar{x}^2 \\ &= \frac{1}{12}(0.0829)\left[3\left(\frac{1}{12}\right)^2 + \left(\frac{3}{12}\right)^2\right] + (0.0829)\left(\frac{2.5}{12}\right)^2 \\ &= 4.17 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \end{aligned}$$

each cylinder :

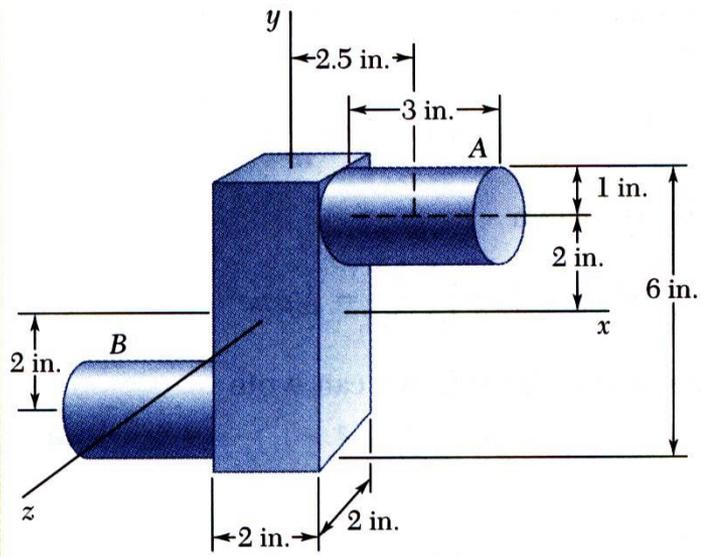
$$m = \frac{\gamma V}{g} = \frac{(490 \text{ lb/ft}^3)(\pi \times 1^2 \times 3) \text{ in}^3}{(1728 \text{ in}^3/\text{ft}^3)(32.2 \text{ ft/s}^2)}$$

$$m = 0.0829 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$\begin{aligned} I_y &= \frac{1}{12}m[3a^2 + L^2] + m[\bar{x}^2 + \bar{y}^2] \\ &= \frac{1}{12}(0.0829)\left[3\left(\frac{1}{12}\right)^2 + \left(\frac{3}{12}\right)^2\right] + (0.0829)\left[\left(\frac{2.5}{12}\right)^2 + \left(\frac{2}{12}\right)^2\right] \\ &= 6.48 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \end{aligned}$$

Vector Mechanics for Engineers: Dynamics

Sample Problem 9.12



prism ($a = 2$ in., $b = 6$ in., $c = 2$ in.):

$$I_x = I_z = \frac{1}{12} m [b^2 + c^2] = \frac{1}{12} (0.211) \left[\left(\frac{6}{12}\right)^2 + \left(\frac{2}{12}\right)^2 \right]$$

$$= 4.88 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = \frac{1}{12} m [c^2 + a^2] = \frac{1}{12} (0.211) \left[\left(\frac{2}{12}\right)^2 + \left(\frac{2}{12}\right)^2 \right]$$

$$= 0.977 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

- Add the moments of inertia from the components to determine the total moments of inertia.

prism :

$$m = \frac{\gamma V}{g} = \frac{(490 \text{ lb/ft}^3)(2 \times 2 \times 6) \text{ in}^3}{(1728 \text{ in}^3/\text{ft}^3)(32.2 \text{ ft/s}^2)}$$

$$m = 0.211 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$I_x = 10.06 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_y = 0.977 \times 10^{-3} + 2(4.17 \times 10^{-3})$$

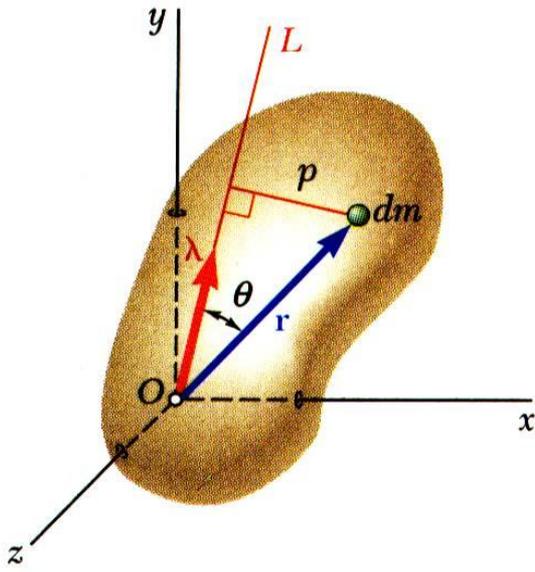
$$I_y = 9.32 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$I_z = 4.88 \times 10^{-3} + 2(6.48 \times 10^{-3})$$

$$I_z = 17.84 \times 10^{-3} \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Vector Mechanics for Engineers: Dynamics

Moment of Inertia With Respect to an Arbitrary Axis



- I_{OL} = moment of inertia with respect to axis OL

$$I_{OL} = \int p^2 dm = \int |\vec{\lambda} \times \vec{r}|^2 dm$$

- Expressing $\vec{\lambda}$ and \vec{r} in terms of the vector components and expanding yields

$$I_{OL} = I_x \lambda_x^2 + I_y \lambda_y^2 + I_z \lambda_z^2 - 2I_{xy} \lambda_x \lambda_y - 2I_{yz} \lambda_y \lambda_z - 2I_{zx} \lambda_z \lambda_x$$

- The definition of the mass products of inertia of a mass is an extension of the definition of product of inertia of an area

$$I_{xy} = \int xy dm = \bar{I}_{x'y'} + m\bar{x}\bar{y}$$

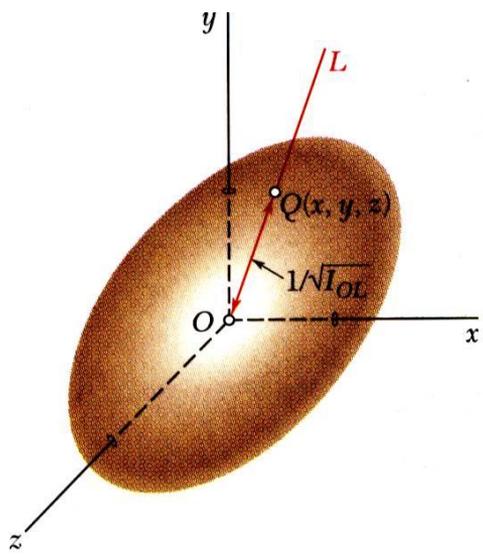
$$I_{yz} = \int yz dm = \bar{I}_{y'z'} + m\bar{y}\bar{z}$$

$$I_{zx} = \int zx dm = \bar{I}_{z'x'} + m\bar{z}\bar{x}$$



Vector Mechanics for Engineers: Dynamics

Ellipsoid of Inertia. Principal Axes of Inertia of a Mass



- Assume the moment of inertia of a body has been computed for a large number of axes OL and that point Q is plotted on each axis at a distance $OQ = 1/\sqrt{I_{OL}}$
- The locus of points Q forms a surface known as the *ellipsoid of inertia* which defines the moment of inertia of the body for any axis through O .
- x', y', z' axes may be chosen which are the *principal axes of inertia* for which the products of inertia are zero and the moments of inertia are the *principal moments of inertia*.

