

CHAPTER

5

*VECTOR MECHANICS FOR ENGINEERS:*  
**STATICS**

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Lecture Notes:  
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Distributed Forces:  
Centroids and Centers  
of Gravity

# Vector Mechanics for Engineers: Statics

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# Vector Mechanics for Engineers: Statics

## Introduction

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.
- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

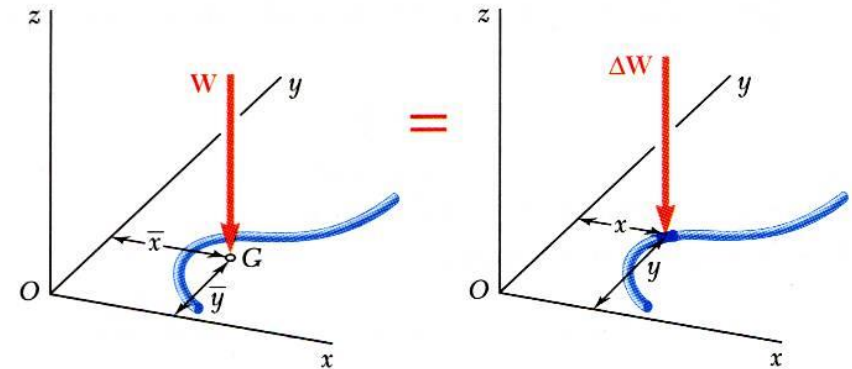
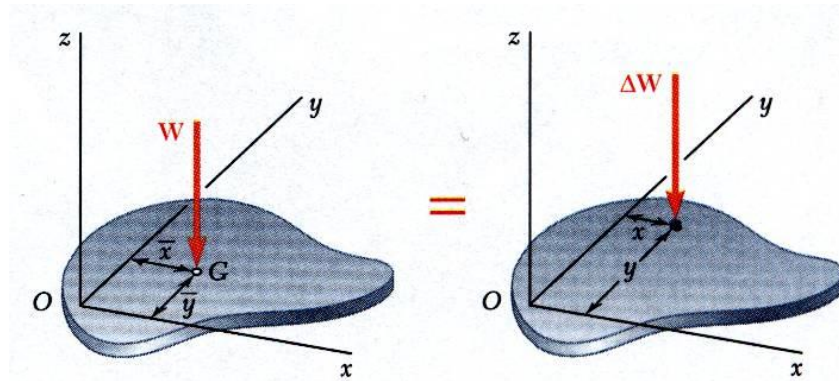


# Vector Mechanics for Engineers: Statics

## Center of Gravity of a 2D Body

- Center of gravity of a plate

- Center of gravity of a wire



$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$

$$= \int x dW$$

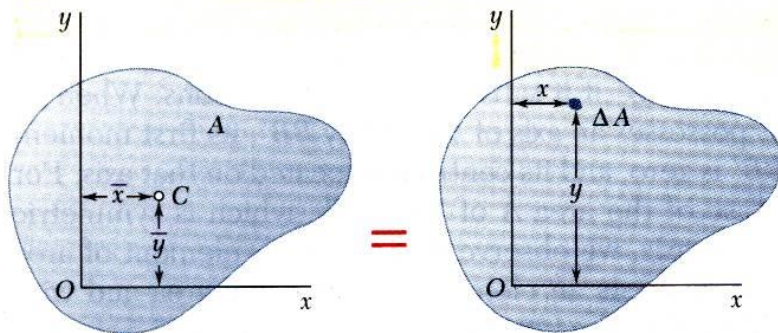
$$\sum M_x \quad \bar{y}W = \sum y\Delta W$$

$$= \int y dW$$

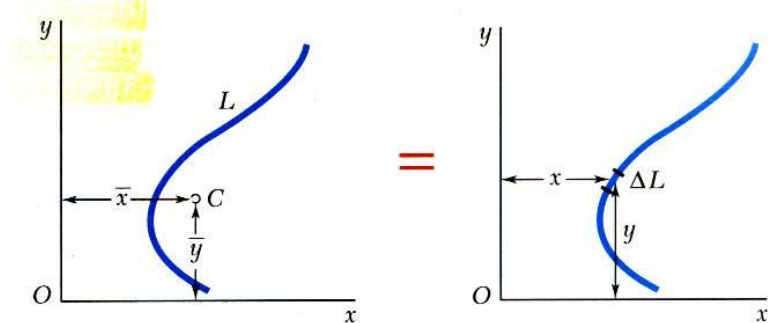
# Vector Mechanics for Engineers: Statics

## Centroids and First Moments of Areas and Lines

- Centroid of an area



- Centroid of a line



$$\bar{x}W = \int x dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t) dA$$

$$\bar{x}A = \int x dA = Q_y$$

= first moment with respect to y

$$\bar{y}A = \int y dA = Q_x$$

= first moment with respect to x

$$\bar{x}W = \int x dW$$

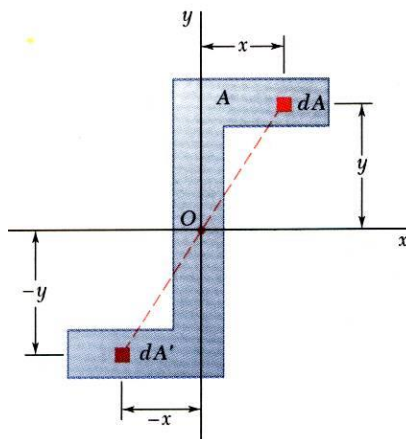
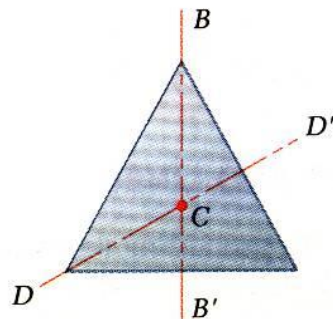
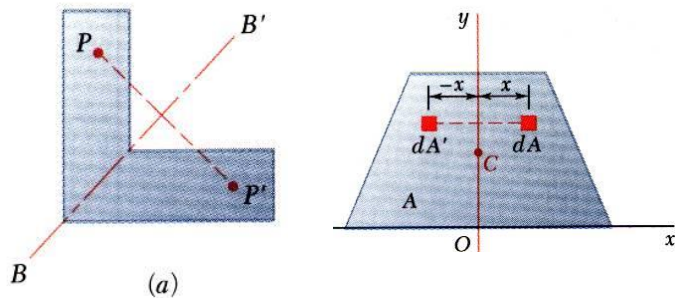
$$\bar{x}(\gamma L a) = \int x(\gamma a) dL$$

$$\bar{x}L = \int x dL$$

$$\bar{y}L = \int y dL$$

# Vector Mechanics for Engineers: Statics

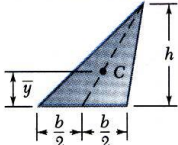
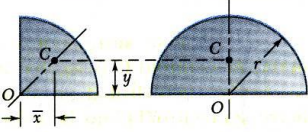
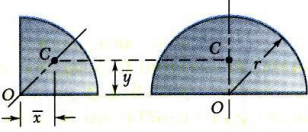
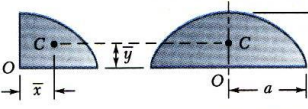
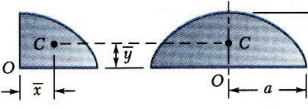
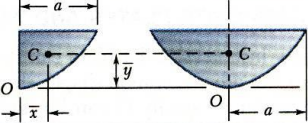
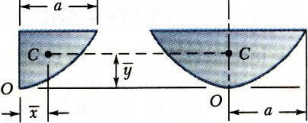
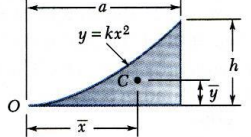
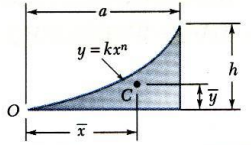
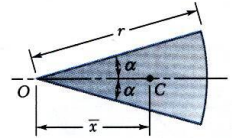
## First Moments of Areas and Lines



- An area is symmetric with respect to an axis  $BB'$  if for every point  $P$  there exists a point  $P'$  such that  $PP'$  is perpendicular to  $BB'$  and is divided into two equal parts by  $BB'$ .
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center  $O$  if for every element  $dA$  at  $(x, y)$  there exists an area  $dA'$  of equal area at  $(-x, -y)$ .
- The centroid of the area coincides with the center of symmetry.

# Vector Mechanics for Engineers: Statics

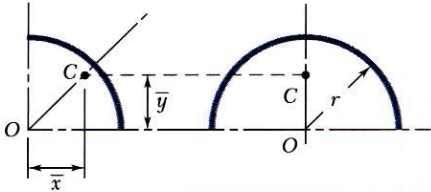
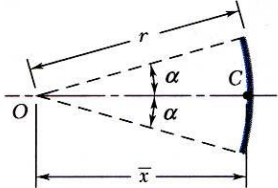
## Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	$\alpha r^2$



# Vector Mechanics for Engineers: Statics

## Centroids of Common Shapes of Lines

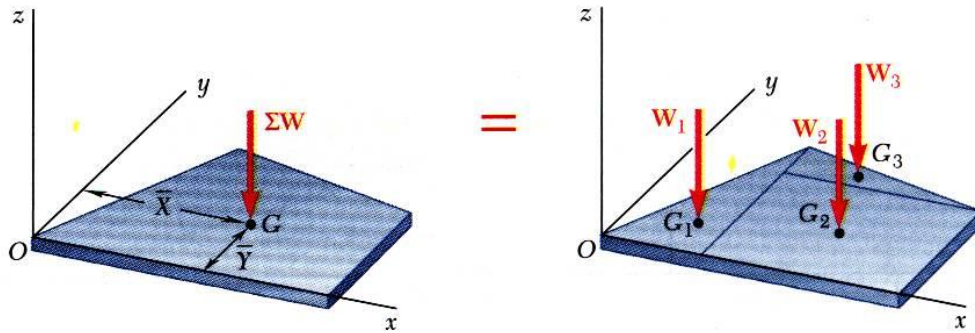
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	$\pi r$
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$





# Vector Mechanics for Engineers: Statics

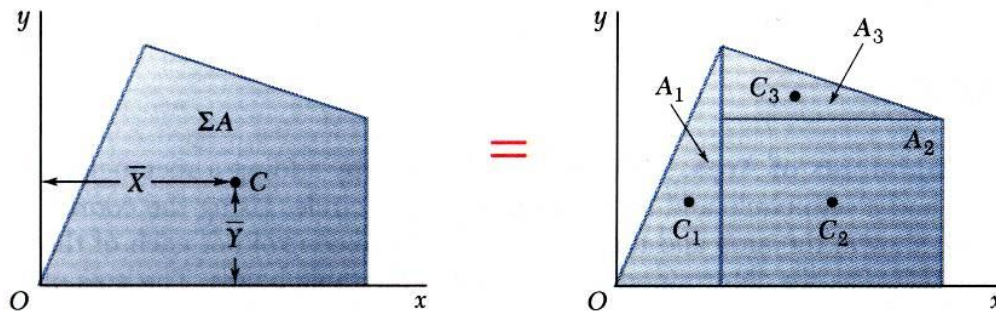
## Composite Plates and Areas



- Composite plates

$$\bar{X} \sum W = \sum \bar{x} W$$

$$\bar{Y} \sum W = \sum \bar{y} W$$



- Composite area

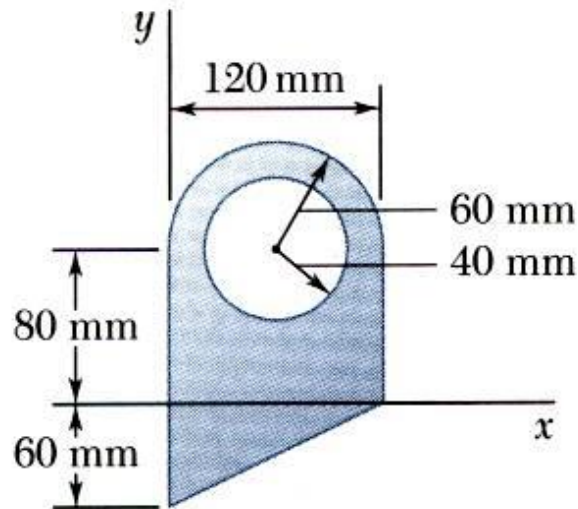
$$\bar{X} \sum A = \sum \bar{x} A$$

$$\bar{Y} \sum A = \sum \bar{y} A$$



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.1



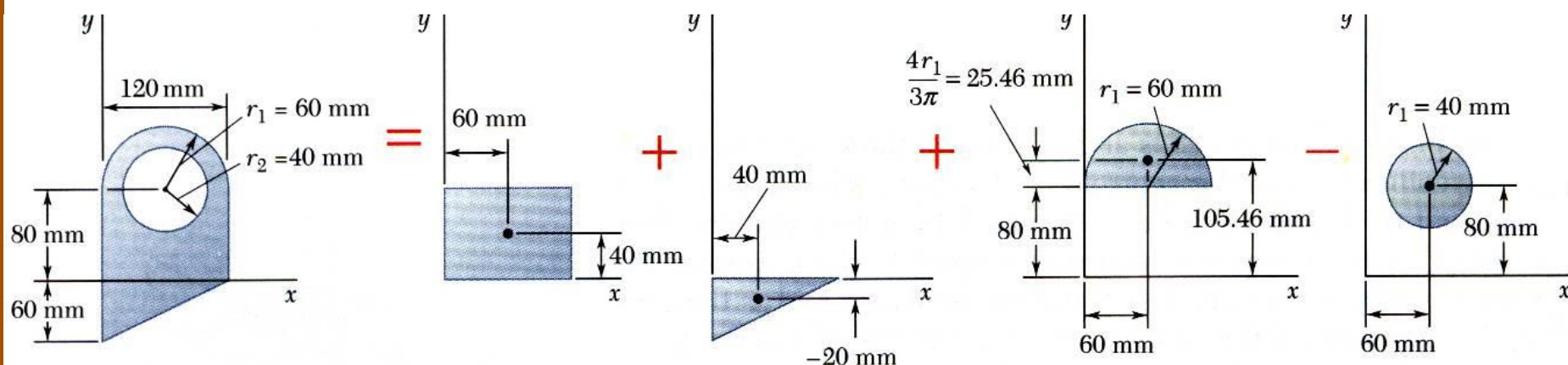
### SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

For the plane area shown, determine the first moments with respect to the  $x$  and  $y$  axes and the location of the centroid.

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.1



Component	A, mm <sup>2</sup>	$\bar{x}$ , mm	$\bar{y}$ , mm	$\bar{x}A$ , mm <sup>3</sup>	$\bar{y}A$ , mm <sup>3</sup>
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	$-72 \times 10^3$
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	$-301.6 \times 10^3$	$-402.2 \times 10^3$
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

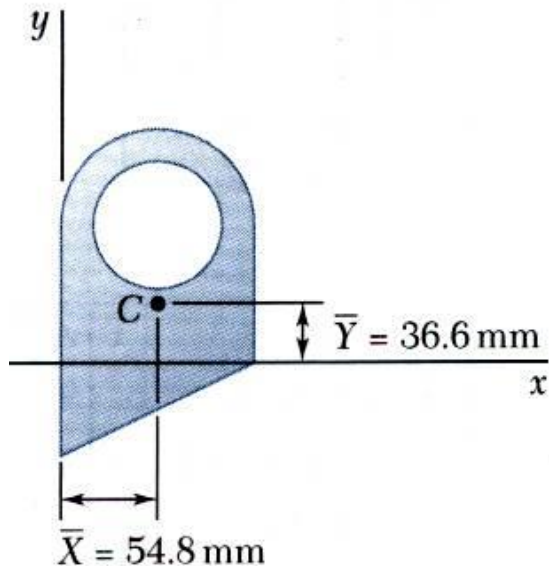
$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$

$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\bar{X} = 54.8 \text{ mm}$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\bar{Y} = 36.6 \text{ mm}$$

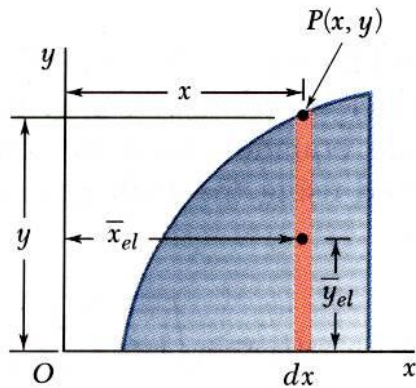
# Vector Mechanics for Engineers: Statics

## Determination of Centroids by Integration

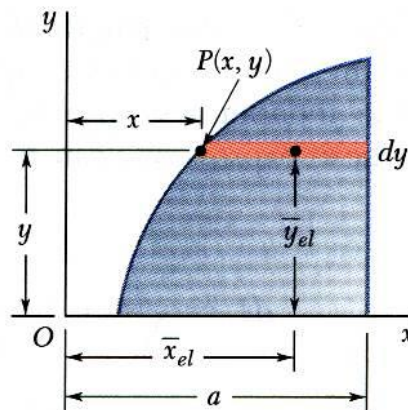
$$\bar{x}A = \int x dA = \iint x dx dy = \int \bar{x}_{el} dA$$

$$\bar{y}A = \int y dA = \iint y dx dy = \int \bar{y}_{el} dA$$

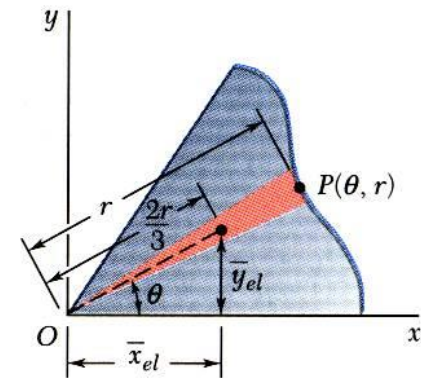
- Double integration to find the first moment may be avoided by defining  $dA$  as a thin rectangle or strip.



$$\begin{aligned}\bar{x}A &= \int \bar{x}_{el} dA \\ &= \int x(y dx) \\ \bar{y}A &= \int \bar{y}_{el} dA \\ &= \int \frac{y}{2}(y dx)\end{aligned}$$



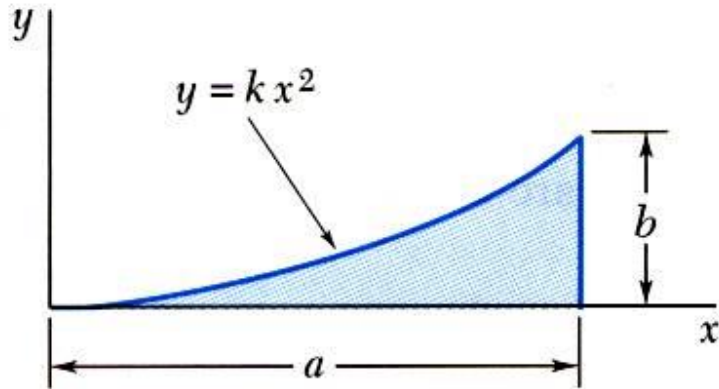
$$\begin{aligned}\bar{x}A &= \int \bar{x}_{el} dA \\ &= \int \frac{a+x}{2}[(a-x)dx] \\ \bar{y}A &= \int \bar{y}_{el} dA \\ &= \int y[(a-x)dx]\end{aligned}$$



$$\begin{aligned}\bar{x}A &= \int \bar{x}_{el} dA \\ &= \int \frac{2r}{3} \cos \theta \left( \frac{1}{2} r^2 d\theta \right) \\ \bar{y}A &= \int \bar{y}_{el} dA \\ &= \int \frac{2r}{3} \sin \theta \left( \frac{1}{2} r^2 d\theta \right)\end{aligned}$$

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.4



### SOLUTION:

- Determine the constant  $k$ .
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Determine by direct integration the location of the centroid of a parabolic spandrel.



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.4

### SOLUTION:

- Determine the constant  $k$ .

$$y = k x^2$$

$$b = k a^2 \Rightarrow k = \frac{b}{a^2}$$

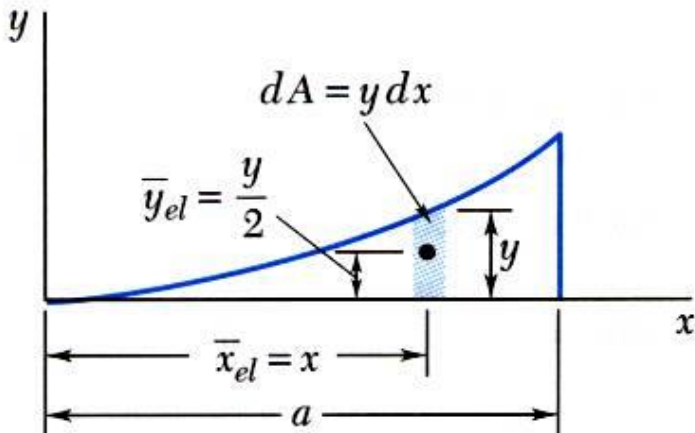
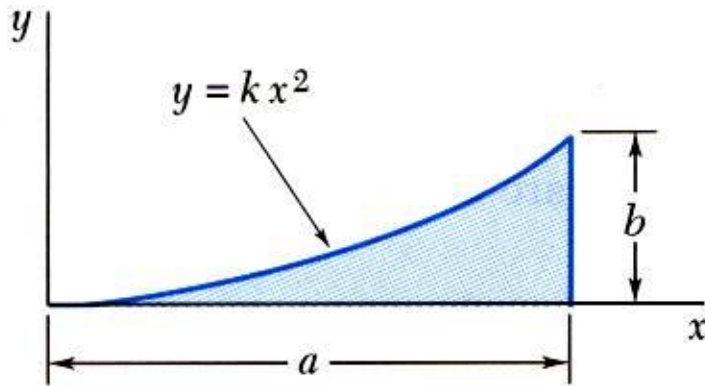
$$y = \frac{b}{a^2} x^2 \quad \text{or} \quad x = \frac{a}{b^{1/2}} y^{1/2}$$

- Evaluate the total area.

$$A = \int dA$$

$$= \int y dx = \int_0^a \frac{b}{a^2} x^2 dx = \left[ \frac{b}{a^2} \frac{x^3}{3} \right]_0^a$$

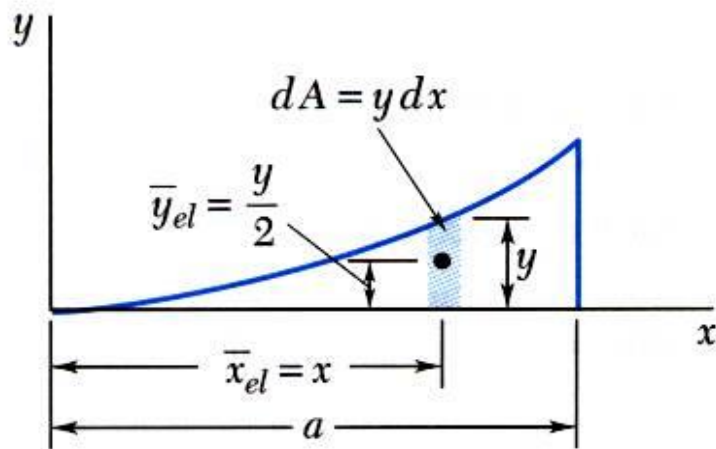
$$= \frac{ab}{3}$$



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.4

- Using vertical strips, perform a single integration to find the first moments.



$$Q_y = \int \bar{x}_{el} dA = \int xy dx = \int_0^a x \left( \frac{b}{a^2} x^2 \right) dx$$

$$= \left[ \frac{b}{a^2} \frac{x^4}{4} \right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y dx = \int_0^a \frac{1}{2} \left( \frac{b}{a^2} x^2 \right)^2 dx$$

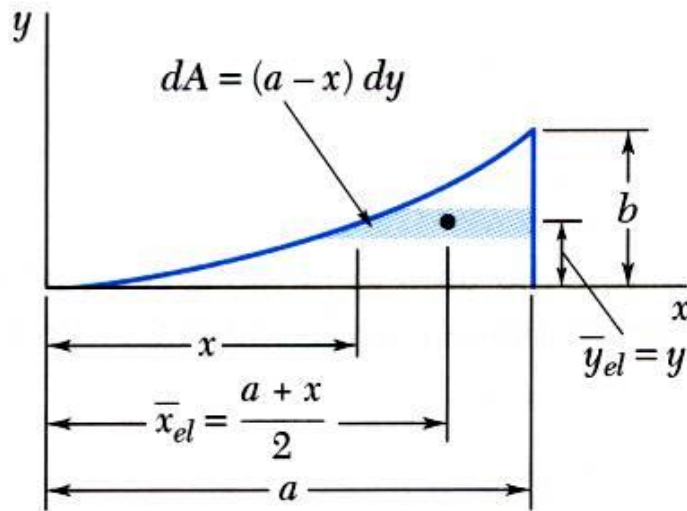
$$= \left[ \frac{b^2}{2a^4} \frac{x^5}{5} \right]_0^a = \frac{ab^2}{10}$$



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.4

- Or, using horizontal strips, perform a single integration to find the first moments.



$$Q_y = \int \bar{x}_{el} dA = \int \frac{a+x}{2} (a-x) dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

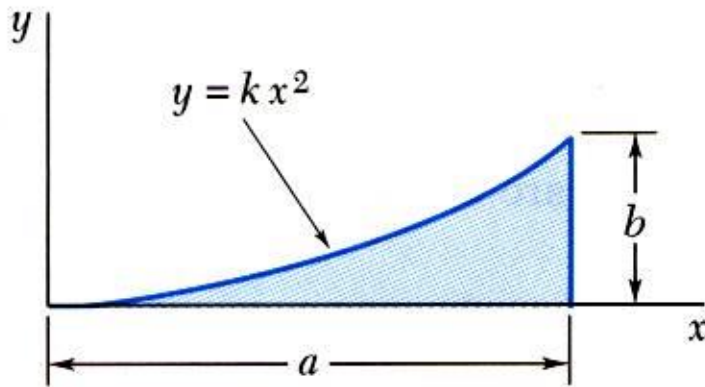
$$= \frac{1}{2} \int_0^b \left( a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int y(a-x) dy = \int y \left( a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left( ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.4



- Evaluate the centroid coordinates.

$$\bar{x}A = Q_y$$

$$\bar{x} \frac{ab}{3} = \frac{a^2b}{4}$$

$$\bar{x} = \frac{3}{4}a$$

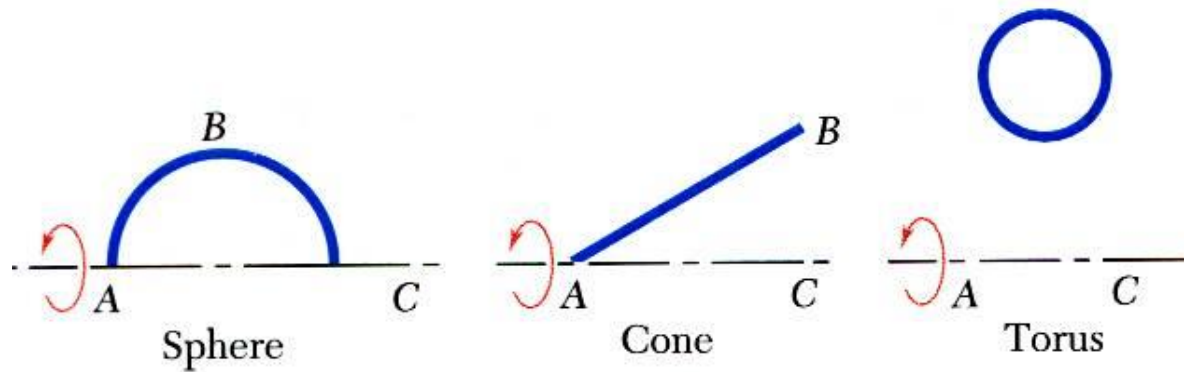
$$\bar{y}A = Q_x$$

$$\bar{y} \frac{ab}{3} = \frac{ab^2}{10}$$

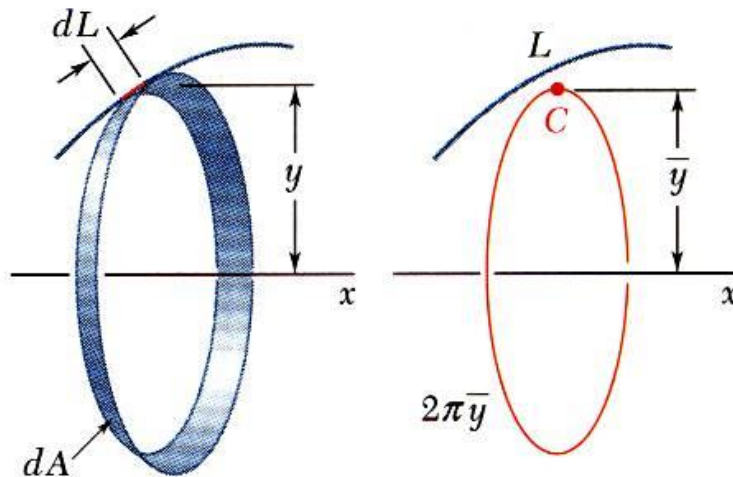
$$\bar{y} = \frac{3}{10}b$$

# Vector Mechanics for Engineers: Statics

## Theorems of Pappus-Guldinus



- Surface of revolution is generated by rotating a plane curve about a fixed axis.

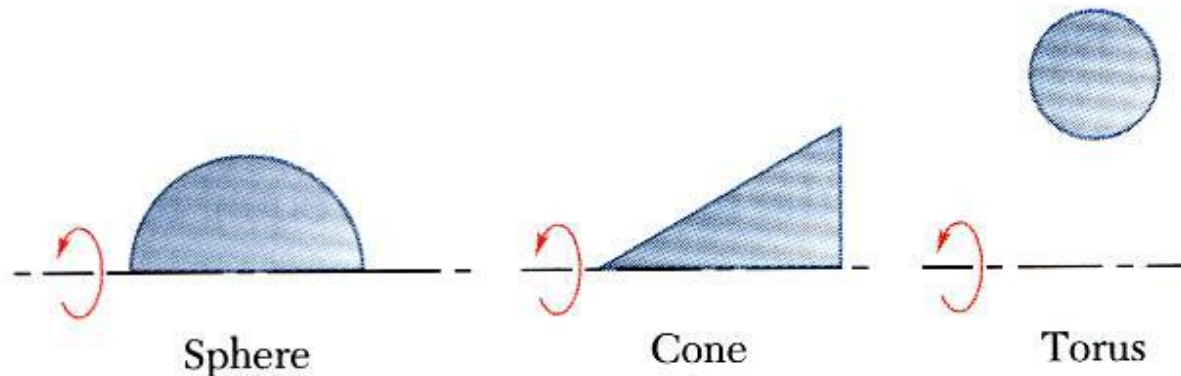


- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

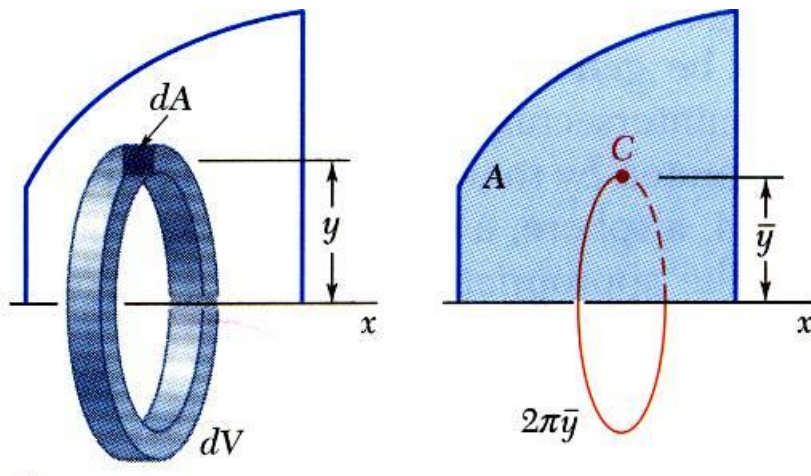
$$A = 2\pi \bar{y} L$$

# Vector Mechanics for Engineers: Statics

## Theorems of Pappus-Guldinus



- Body of revolution is generated by rotating a plane area about a fixed axis.

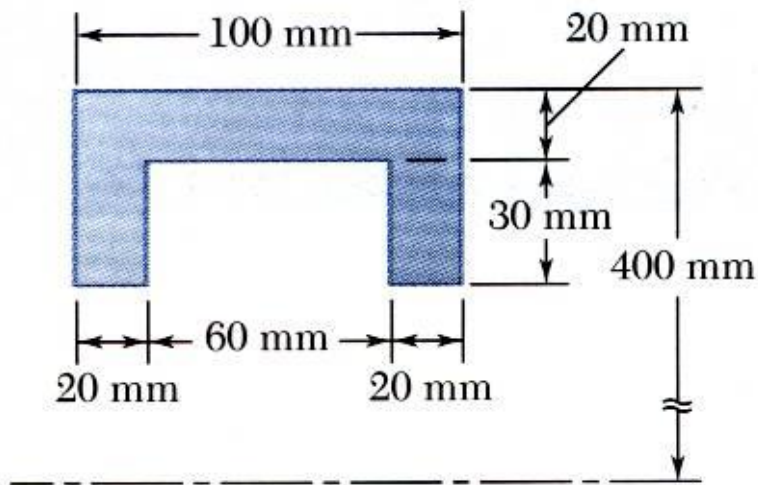


- Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

$$V = 2\pi \bar{y} A$$

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.7



### SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and acceleration.

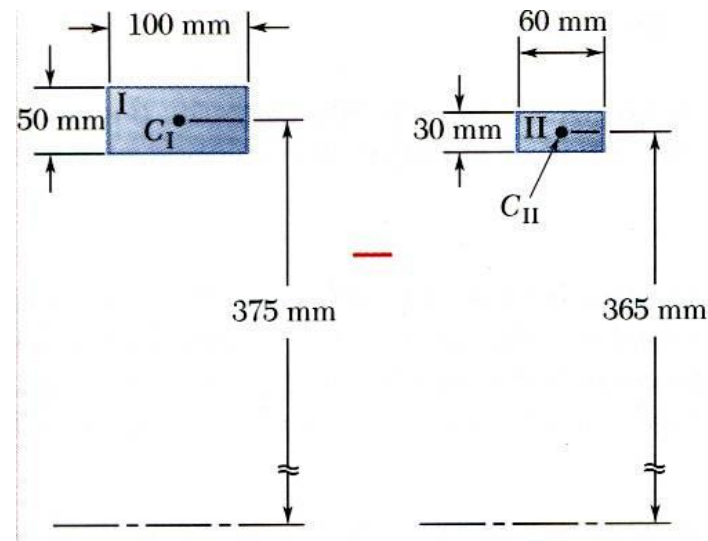
The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is  $\rho = 7.85 \times 10^3 \text{ kg/m}^3$  determine the mass and weight of the rim.

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.7

### SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and acceleration.



	Area, mm <sup>2</sup>	$\bar{y}$ , mm	Distance Traveled by $C$ , mm	Volume, mm <sup>3</sup>
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	-1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim = $7.65 \times 10^6$

$$m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3) (7.65 \times 10^6 \text{ mm}^3) (10^{-9} \text{ m}^3/\text{mm}^3)$$

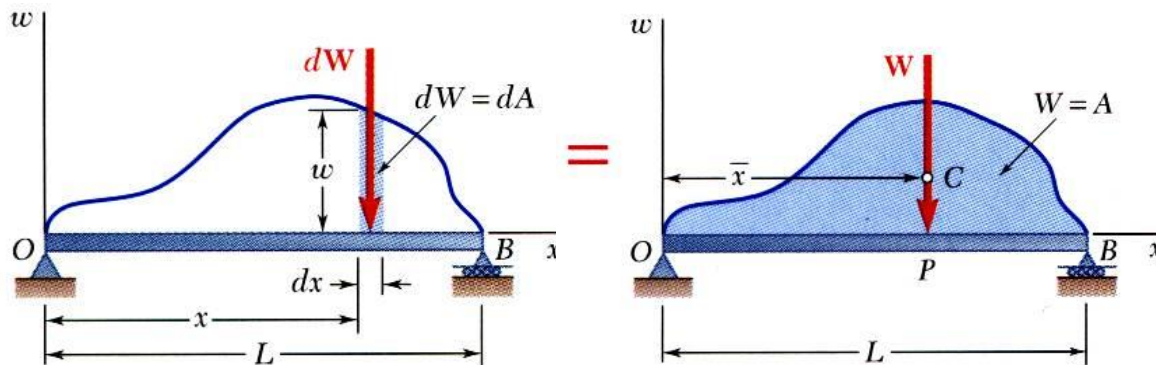
$$m = 60.0 \text{ kg}$$

$$W = mg = (60.0 \text{ kg})(9.81 \text{ m/s}^2)$$

$$W = 589 \text{ N}$$

# Vector Mechanics for Engineers: Statics

## Distributed Loads on Beams



$$W = \int_0^L w dx = \int dA = A$$

- A distributed load is represented by plotting the load per unit length,  $w$  (N/m). The total load is equal to the area under the load curve.

$$(OP)W = \int x dW$$

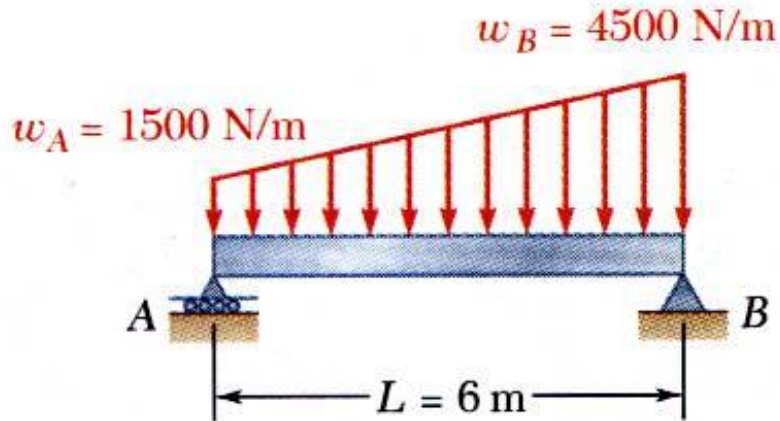
$$(OP)A = \int_0^L x dA = \bar{x}A$$

- A distributed load can be replaced by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.9



### SOLUTION:

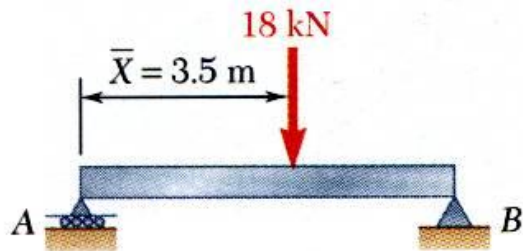
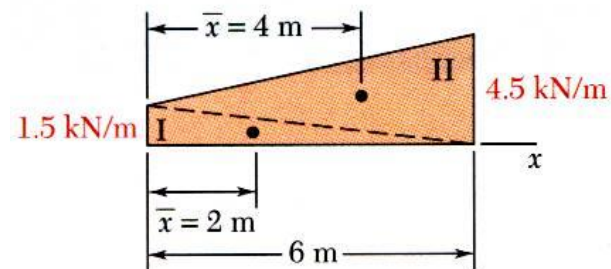
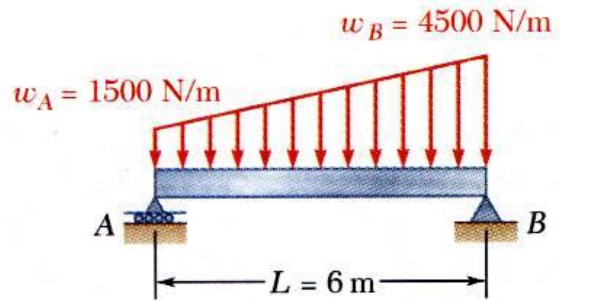
- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.



# Vector Mechanics for Engineers: Statics

## Sample Problem 5.9



### SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$F = 18.0 \text{ kN}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

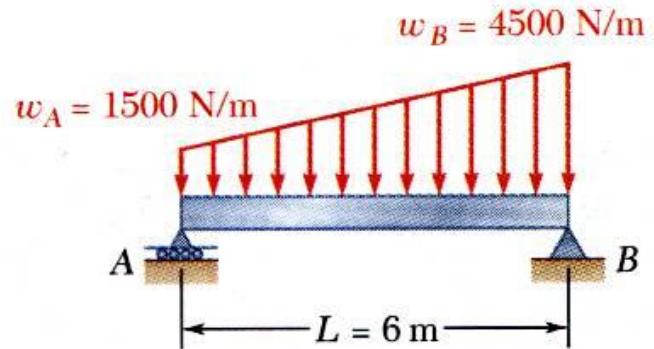
$$\bar{X} = \frac{63 \text{ kN} \cdot \text{m}}{18 \text{ kN}}$$

$$\bar{X} = 3.5 \text{ m}$$

Component	A, kN	$\bar{x}$ , m	$\bar{x}A$ , kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

# Vector Mechanics for Engineers: Statics

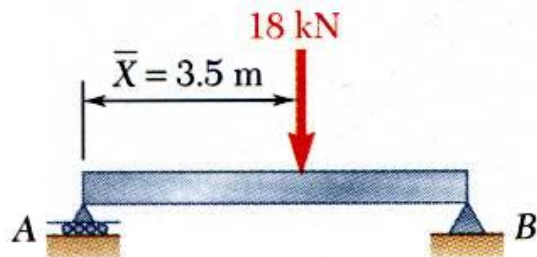
## Sample Problem 5.9



- Determine the support reactions by summing moments about the beam ends.

$$\sum M_A = 0: \quad B_y(6 \text{ m}) - (18 \text{ kN})(3.5 \text{ m}) = 0$$

$$B_y = 10.5 \text{ kN}$$

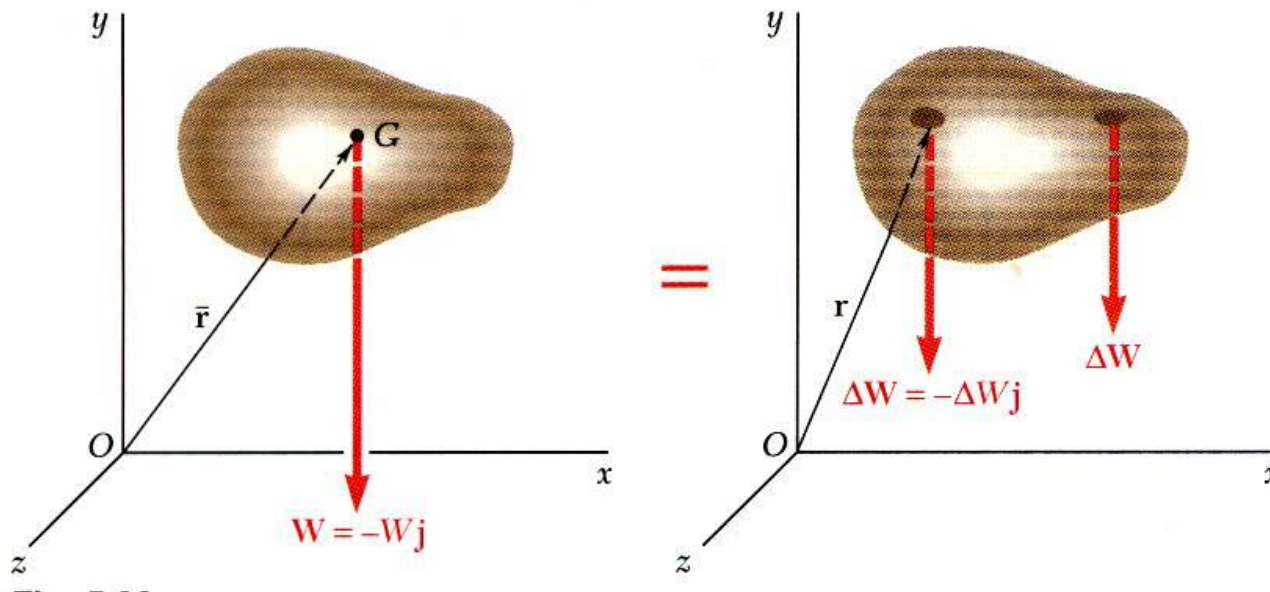


$$\sum M_B = 0: \quad -A_y(6 \text{ m}) + (18 \text{ kN})(6 \text{ m} - 3.5 \text{ m}) = 0$$

$$A_y = 7.5 \text{ kN}$$

# Vector Mechanics for Engineers: Statics

## Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity  $G$

$$-W\vec{j} = \sum (-\Delta W\vec{j})$$

$$\vec{r}_G \times (-W\vec{j}) = \sum [\vec{r} \times (-\Delta W\vec{j})]$$

$$\vec{r}_G W \times (-\vec{j}) = (\sum \vec{r} \Delta W) \times (-\vec{j})$$

$$W = \int dW \quad \vec{r}_G W = \int \vec{r} dW$$

- Results are independent of body orientation,

$$\bar{x}W = \int x dW \quad \bar{y}W = \int y dW \quad \bar{z}W = \int z dW$$

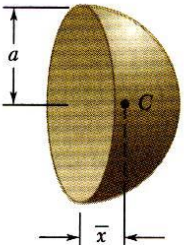
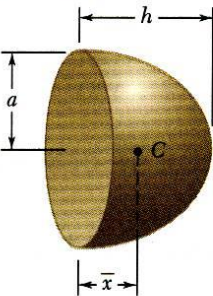
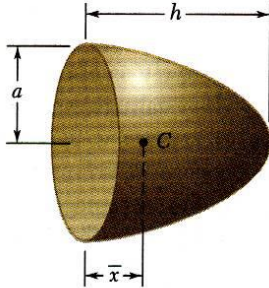
- For homogeneous bodies,

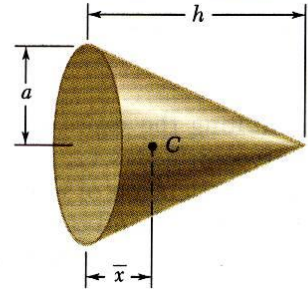
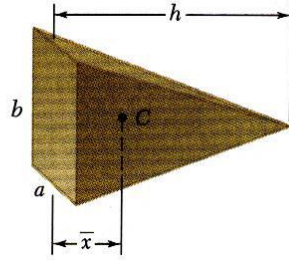
$$W = \gamma V \quad \text{and} \quad dW = \gamma dV$$

$$\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV$$

# Vector Mechanics for Engineers: Statics

## Centroids of Common 3D Shapes

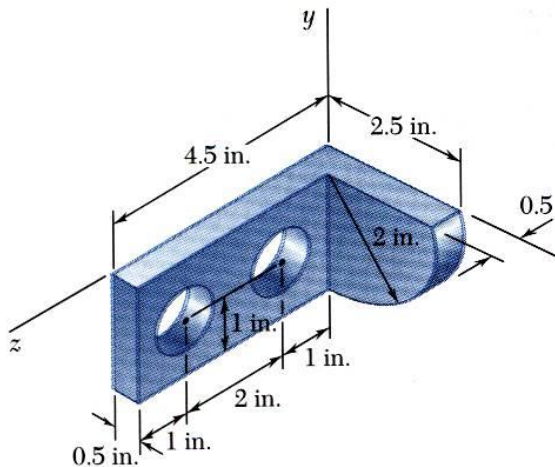
Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$

Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$



# Vector Mechanics for Engineers: Statics

## Composite 3D Bodies

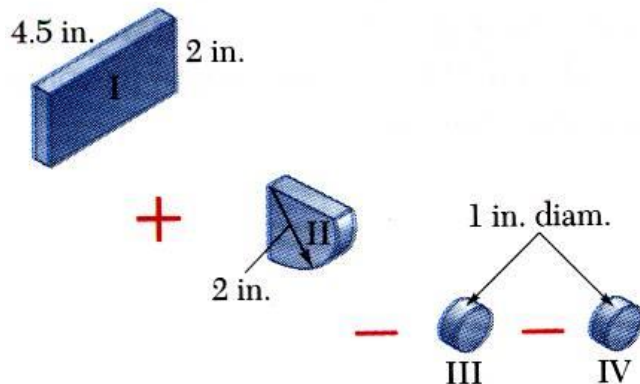


- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X} \sum W = \sum \bar{x} W \quad \bar{Y} \sum W = \sum \bar{y} W \quad \bar{Z} \sum W = \sum \bar{z} W$$

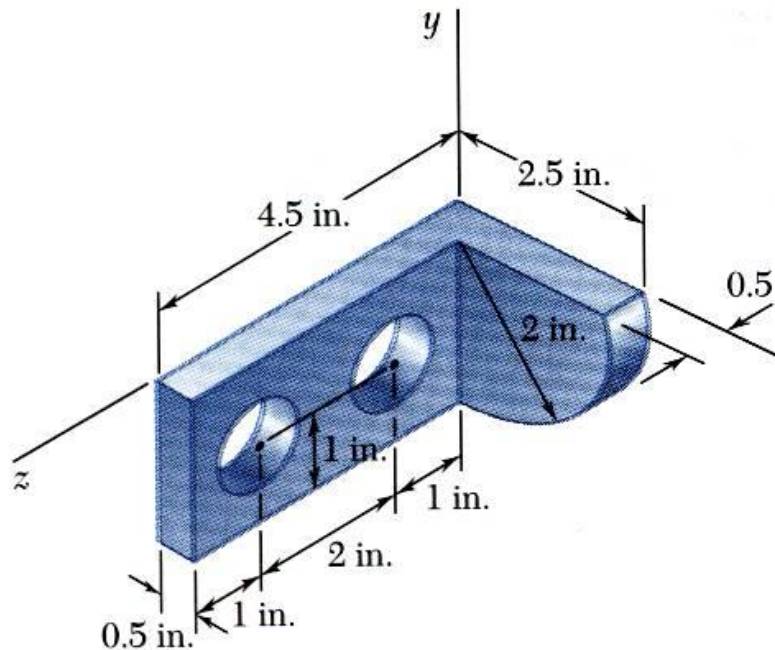
- For homogeneous bodies,

$$\bar{X} \sum V = \sum \bar{x} V \quad \bar{Y} \sum V = \sum \bar{y} V \quad \bar{Z} \sum V = \sum \bar{z} V$$



# Vector Mechanics for Engineers: Statics

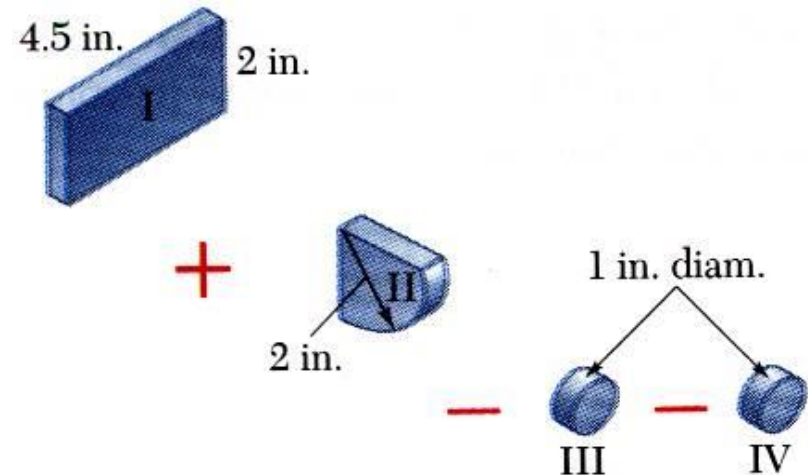
## Sample Problem 5.12



Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.

### SOLUTION:

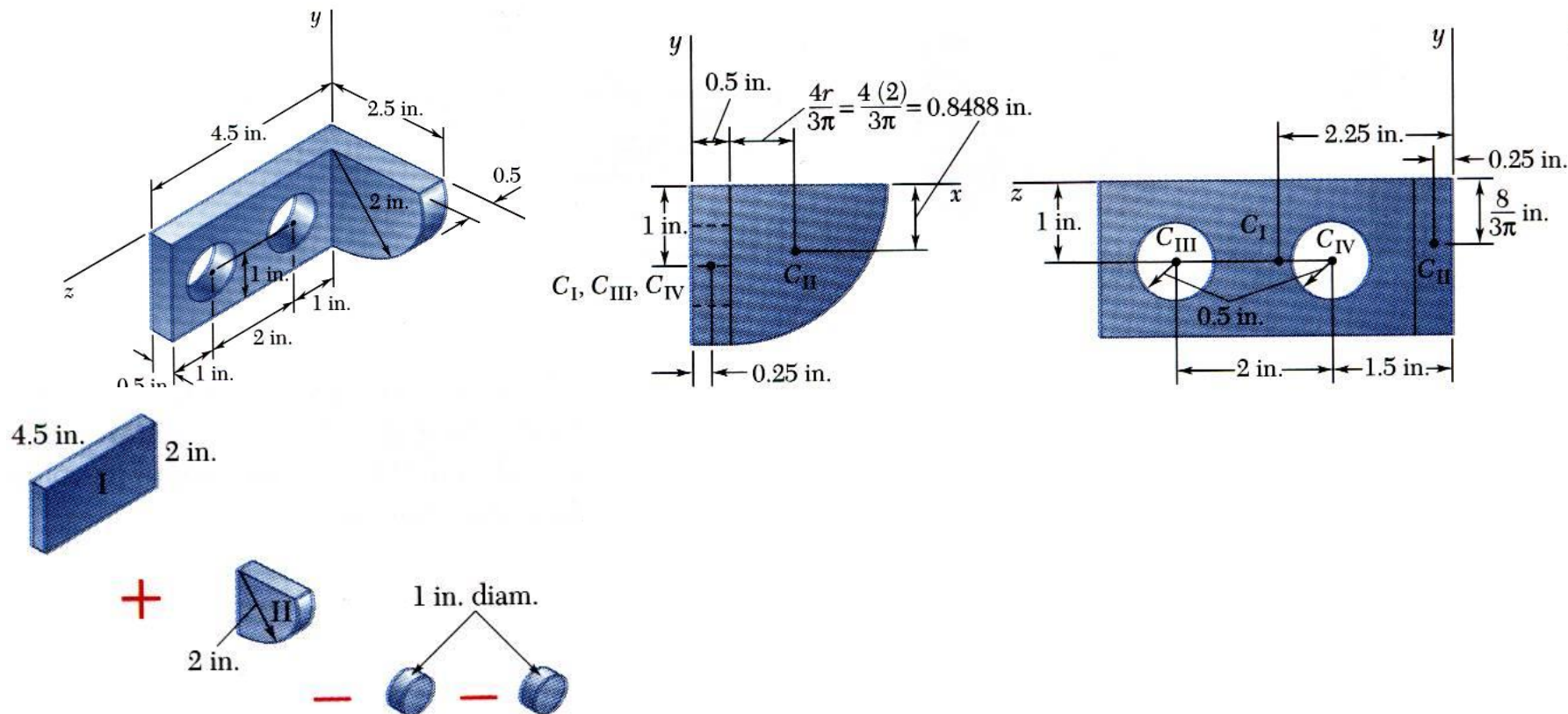
- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.





# Vector Mechanics for Engineers: Statics

## Sample Problem 5.12

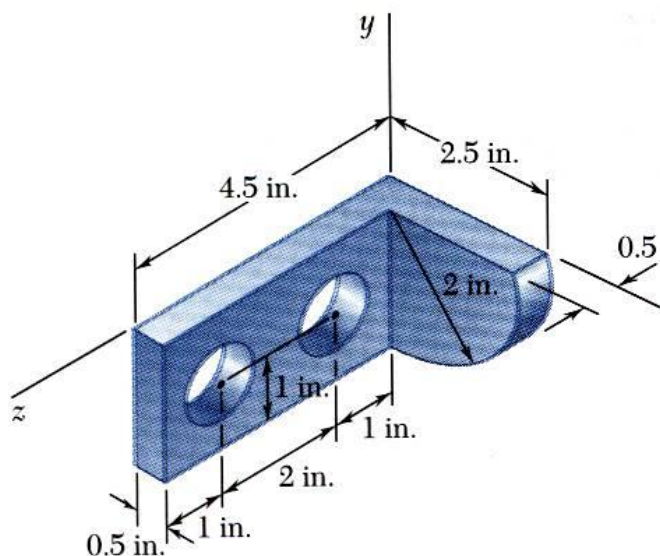


	$V, \text{in}^3$	$\bar{x}, \text{in.}$	$\bar{y}, \text{in.}$	$\bar{z}, \text{in.}$	$\bar{x}V, \text{in}^4$	$\bar{y}V, \text{in}^4$	$\bar{z}V, \text{in}^4$
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$

# Vector Mechanics for Engineers: Statics

## Sample Problem 5.12

	$V, \text{in}^3$	$\bar{x}, \text{in.}$	$\bar{y}, \text{in.}$	$\bar{z}, \text{in.}$	$\bar{x}V, \text{in}^4$	$\bar{y}V, \text{in}^4$	$\bar{z}V, \text{in}^4$
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
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IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$



$$\bar{X} = \Sigma \bar{x}V / \Sigma V = (3.08 \text{ in}^4) / (5.286 \text{ in}^3)$$

$$\bar{X} = 0.577 \text{ in.}$$

$$\bar{Y} = \Sigma \bar{y}V / \Sigma V = (-5.047 \text{ in}^4) / (5.286 \text{ in}^3)$$

$$\bar{Y} = 0.577 \text{ in.}$$

$$\bar{Z} = \Sigma \bar{z}V / \Sigma V = (1.618 \text{ in}^4) / (5.286 \text{ in}^3)$$

$$\bar{Z} = 0.577 \text{ in.}$$