Propositional Logic

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 2 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

What is Discrete Math?

- Discrete mathematics is the branch of mathematics dealing with objects that can assume only distinct, separated values
- It is a branch of mathematics concerned with the study of objects that can be represented finitely (or countable)
- Discrete mathematics is the mathematical language of computer science

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What is Discrete Math?

- Concepts and notations from discrete mathematics are useful in studying and describing objects and problems in all branches of computer science, such as algorithms, artificial intelligence, programming languages, security and cryptography, automated theorem proving, and software development in general
- Mathematical reasoning is interesting but also a great way to increase creative mathematical thinking. It helps in your personal life as much as your output as a software developer

Propositional Logic

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What is Logic?

- Logic is concerned with the methods of reasoning
- Logic provides rules and techniques to determine whether a given argument is valid
- Logic is the language used for most formal specification languages
- Programs can be described with mathematics, and Propositional logic can be used to reason about their correctness (design and analysis of algorithms)

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Proposition

- The basic building blocks of logic—propositions
- A **proposition** is a statement (that is, a sentence that declares a fact) that is either true or false, but not both

- Examples:

Amman is the capital of Jordan.
1 + 1 = 2.
2 + 2 = 3.

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Proposition

- Consider the following examples, are they propositions?

- What time is it?

- Read this carefully.

- $x + 1 = 2$.

- $x + y = z$.

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Negation

• Definition

If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

p	$\sim p$
T	F
F	T

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Conjunction

• Definition

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

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Disjunction

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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Truth Table for Exclusive Or

$$p \oplus q \quad \text{or} \quad p \text{ XOR } q$$

$$(p \vee q) \wedge \sim(p \wedge q)$$

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Truth Table for Exclusive Or

$$p \oplus q \quad \text{or} \quad p \text{ XOR } q$$

$$(p \vee q) \wedge \sim(p \wedge q)$$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

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Logical Equivalence

• Definition

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

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Logical Equivalence

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

p and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

Negations of And and Or: De Morgan's Laws

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q.$$

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Negations of And and Or: De Morgan's Laws

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q.$$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ always have the same truth values, so they are logically equivalent

Tautologies and Contradictions

• Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

all T's so $p \vee \sim p$ is a tautology

all F's so $p \wedge \sim p$ is a contradiction

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

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Simplifying Statement Forms

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

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Simplifying Statement Forms

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

$\sim(\sim p \wedge q) \wedge (p \vee q)$	$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	by De Morgan's laws
	$\equiv (p \vee \sim q) \wedge (p \vee q)$	by the double negative law
	$\equiv p \vee (\sim q \wedge q)$	by the distributive law
	$\equiv p \vee (q \wedge \sim q)$	by the commutative law for $\wedge$
	$\equiv p \vee \mathbf{c}$	by the negation law
	$\equiv p$	by the identity law.

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