Lecture Notes on **Discrete Mathematics**.
Birzeit University, Palestine, 2015
Modified in July 2021

First Order Logic

Mustafa Jarrar

&

Radi Jarrar

1. **Predicates and Quantified Statements I**
2. Predicates and Quantified Statements II
3. *Statements with Multiple Quantifiers*

Watch this lecture and download the slides

http://jarrar--courses.blogspot.com/2014/03/discrete--mathematics--course.html

More Lectures Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on, but not limited to, chapter 3 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

In this Lecture

We will learn

- **Part 1: What is a predicate, and Predicate Logic**
- Part 2: Universal and Existential Quantifiers: ∀, ∃
- Part 3: Formalize and Verablize Statements;
- Part 4: Different Writings of Quantified Statements;
- Part 5: Tarski's World (Simple Example)

Keywords: Predicates, FOL, First Order Logic, Universal Quantifier, Existential Quantifier, Tarski's World.

First Order Logic

is also called:

- The Logic of Quantified Statements
- Predicate Logic
- First--Order Predicate Calculus
- Lower Predicate Calculus
- Quantification theory

,

First Order Logic

A **proposition** is basically a sentence that has a truth value that can either be true or false, but it needs to be assigned any of the two values and not be ambiguous. Propositional logic is used to analyze a statement or group of statements.

Predicates can be seen as properties or additional information to express the subject of the sentence.

A quantified predicate is a proposition, that is, when you assign values to a predicate with variables it can be made a proposition.

,

What is First Order Logic?

Propositional Logic

- Set of propositional symbols (e.g., Ahmed, Student, P, Q)
- No binding of variables (joined together by logical operators to form sentences)

$\neg P$ Negation

$P \wedge Q$ Conjunction P

$\vee\ Q$ Disjunction P

$\rightarrow Q$ Implication P

$\leftrightarrow Q$ Equivalence

We regard the world as *Propositions*

First Order Logic

P(x..y), Q(t,..s) Predicates

(Allows quantification over variables)

$\neg P$ Negation

$P \wedge Q$ Conjunction P

$\vee\ Q$ Disjunction P

$\rightarrow Q$ Implication P

$\leftrightarrow Q$ Equivalence

$\forall$ Universal quantification

$\exists$ *Existential* quantification

We regard the world as *Quantified Predicates*

,

What is Predicate?

- **Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

$$P(x_1, x_2 \; , \ldots , x_n)$$

Examples

Person(Amjad)

University(BZU)

StudyAt(Amjad, BZU)

,

Arity of Predicates

Arity is the number of arguments or operands taken by a function or relation in logic, mathematics, and computer science

$$P(x_1, x_2, \ldots, x_n)$$

Predicate Name

Arguments

Examples:

Unary Predicates: Person(Amjad), University(BZU)

Binary Predicates: StudyAt(Amjad, BZU)

Ternary Predicates StudyAt(Amjad,BZU, CS)

Quaternary Predicate: StudyAt(Amjad,BZU, CS, 2015)

n-ary Predicate: StudyAt(Amjad,BZU, CS, 2015, BA, ...)

Truth of Predicates

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

$$\{x \in Organization \mid University(x)\}$$

The set of all organizations that are universities.

$$\{x \in Person \mid student(x)\}$$

The set of all persons that are students.

,

In this Lecture

We will learn

- Part 1: What is a predicate, and Predicate Logic
- **Part 2: Universal and Existential Quantifiers: ∀, ∃**
- Part 3: Formalize and Verablize Statements;
- Part 4: Different Writings of Quantified Statements;
- Part 5: Tarski’s World (Simple Example)

The Universal Quantifier: ∀

Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

$$\forall P \in \text{Palestinian} \;.\; \text{Likes}(p, \text{Zatar})$$

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Let $D = \{1, 2, 3, 4, 5\}$. $\forall x \in D, x^2 \geq x.$

,

The Existential Quantifier: ∃

• Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

$$\exists p \in \text{Person} \,.\, \text{Likes}(p, \text{Zatar})$$

$$\exists m \in \mathbf{Z}^{+} \text{ such that } m^2 = m.$$

Let $E = \{5, 6, 7, 8\}$ $\quad \exists m \in E$ such that $m^2 = m$.

,

In this Lecture

We will learn

- Part 1: What is a predicate, and Predicate Logic
- Part 2: Universal and Existential Quantifiers: $\forall$, $\exists$


- **Part 3: Formalize and Verablize Statements;**
- Part 4: Different Writings of Quantified Statements;
- Part 5: Tarski’s World (Simple Example)

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R}, x^2 \geq 0.$

$\forall x \in \mathbf{R}, x^2 \neq -1.$

$\exists m \in \mathbf{Z}^+$ such that $m^2 = m.$

$\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4.$

,

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R}, x^2 \geq 0.$

All real numbers have non-negative squared value
OR Every real number has a non-negative squared value
OR The square of any real number has a non-negative value

$\forall x \in \mathbf{R}, x^2 \neq -1.$

$\exists m \in \mathbf{Z}^+ \text{such that } m^2 = m.$

$\forall x \in \mathbf{R}, \text{if } x > 2 \text{ then } x^2 > 4.$

,

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R}, x^2 \geq 0.$

All real numbers have non-negative squared value

OR Every real number has a non-negative squared value

OR The square of any real number has a non-negative value

$\forall x \in \mathbf{R}, x^2 \neq -1.$

All real numbers have squares that are not equal to -1

OR No real value has a square equals to -1

$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$

$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$

,

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R}, x^2 \geq 0.$

All real numbers have non-negative squared value

OR Every real number has a non-negative squared value

OR The square of any real number has a non-negative value

$\forall x \in \mathbf{R}, x^2 \neq -1.$

All real numbers have squares that are not equal to -1

OR No real value has a square equals to -1

$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$

There is a positive integer whose square equals to itself

OR We can find at least one positive integer equal to its own

OR some positive integer equals to its own square

$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$

,

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R}, x^2 \geq 0.$

All real numbers have non-negative squared value

OR Every real number has a non-negative squared value

OR The square of any real number has a non-negative value

$\forall x \in \mathbf{R}, x^2 \neq -1.$

All real numbers have squares that are not equal to -1

OR No real value has a square equals to -1

$\exists m \in \mathbf{Z}^+ \text{such that } m^2 = m.$

There is a positive integer whose square equals to itself

OR We can find at least one positive integer equal to its own

OR some positive integer equals to its own square

$\forall x \in \mathbf{R}, \text{if } x > 2 \text{ then } x^2 > 4.$

If a real number is greater than 2 then its square is greater than 4

OR The square of any real number is greater than 4

OR the squares of all real numbers greater than 2 are greater than 4

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

No dogs have wings

Some programs are structured

If a real number is an integer, then it is a rational number

All bytes have eight bits

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall \text{ triangles } t, t \text{ has three sides}$

$OR\ \forall t \in T, t \text{ has three sides}$

No dogs have wings

Some programs are structured

If a real number is an integer, then it is a rational number

All bytes have eight bits

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall\ triangles\ t, t\ has\ three\ sides$

$OR\ \forall t \in T, t\ has\ three\ sides$

No dogs have wings

$\forall\ dogs\ d, d\ has\ no\ wings$

$OR\ \forall d \in D, d\ does\ not\ have\ wings$

Some programs are structured

If a real number is an integer, then it is a rational number

All bytes have eight bits

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall\ triangles\ t, t\ has\ three\ sides$

$OR\ \forall t \in T, t\ has\ three\ sides$

No dogs have wings

$\forall\ dogs\ d, d\ has\ no\ wings$

$OR\ \forall d \in D, d\ does\ not\ have\ wings$

Some programs are structured

$\exists a\ program\ p\ such\ that\ p\ is\ structured$

OR ∃p ∈ P such that p is structured

If a real number is an integer, then it is a rational number

All bytes have eight bits

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall\, triangles\ t, t\ has\ three\ sides$

$OR\ \forall t \in T, t\ has\ three\ sides$

No dogs have wings

$\forall\, dogs\ d, d\ has\ no\ wings$

$OR\ \forall d \in D, d\ does\ not\ have\ wings$

Some programs are structured

$\exists a\ program\ p\ such\ that\ p\ is\ structured$

OR ∃p ∈ P such that p is structured

If a real number is an integer, then it is a rational number

$\forall\, real\ numbers\ x, if\ \ x\ is\ an\ integer, then\ x\ is\ a\ rational\ number$

$OR\ \forall x \in R, if x \in Z\ then\ x \in Q$

All bytes have eight bits

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall \text{ triangles } t, t \text{ has three sides}$

$OR\ \forall t \in T, t \text{ has three sides}$

No dogs have wings

$\forall \text{ dogs } d, d \text{ has no wings}$

$OR\ \forall d \in D, d \text{ does not have wings}$

Some programs are structured

$\exists a \text{ program } p \text{ such that } p \text{ is structured}$

OR ∃p ∈ P such that p is structured

If a real number is an integer, then it is a rational number

$\forall \text{ real numbers } x, \text{if } x \text{ is an integer, then } x \text{ is a rational number}$

$OR\ \forall x \in R, \text{if } x \in Z \text{ then } x \in Q$

All bytes have eight bits

$\forall x, \text{if } x \text{ is a byte then } x \text{ has eight bits}$

No fire trucks are green

,

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall\, triangles\ t, t\ has\ three\ sides$

$OR\ \forall t \in T, t\ has\ three\ sides$

No dogs have wings

$\forall\, dogs\ d, d\ has\ no\ wings$

$OR\ \forall d \in D, d\ does\ not\ have\ wings$

Some programs are structured

$\exists a\ program\ p\ such\ that\ p\ is\ structured$

OR ∃p ∈ P such that p is structured

If a real number is an integer, then it is a rational number

$\forall\, real\ numbers\ x, if\ \ x\ is\ an\ integer, then\ x\ is\ a\ rational\ number$

$OR\ \forall x \in R, if x \in Z\ then\ x \in Q$

All bytes have eight bits

$\forall\, x, if\ \ x\ is\ a\ byte\ then\ x\ has\ eight\ bits$

No fire trucks are green

$\forall\, x, if\ \ x\ is\ a\ firetruck\ then\ x\ is\ not\ green$

,

In this Lecture

We will learn

- Part 1: What is a predicate , and Predicate Logic
- Part 2: Universal and Existential Quantifiers: ∀ , ∃
- Part 3: Formalize and Verablize Statements;
- **Part 4: Different Writings of Quantified Statements;**
- Part 5: Tarski's World (Simple Example)

,

Different Writings

$\forall\ x \in Square$. Rectangle (x)

$\forall x$. If x is as square then x is a rectangle

$\forall$ *Squares* x . *x* *is a* a rectangle

Although the book uses this notation but it's not recommended as predicates are not clear.

$\forall$ p$\in$Palestinian . Likes(p, Zatar)

$\forall$ P . Palestinian(p) $\wedge$ Likes(p, Zatar)

$\exists$p$\in$Person . Likes(p, Zatar)

$\exists$p. Person(p) $\wedge$ Likes(p, Zatar)

,

Quantifications might be Implicit

Formalize the following:

If a number is an integer, then it is a rational number.

If a person was born in Hebron then s/he is Khalili

People like Hommos are smart

,

Quantifications might be Implicit

Formalize the following:

If a number is an integer, then it is a rational number.

$\forall n \cdot Integer(n) \rightarrow Rational(n)$

If a person was born in Palestine then s/he is Palestinian

People like Hommos are smart

,

Quantifications might be Implicit

Formalize the following:

If a number is an integer, then it is a rational number.

$\forall n \cdot Integer(n) \rightarrow Rational(n)$

If a person was born in Palestine then s/he is Palestinian

$\forall x \in Person \cdot BornInPalestine(x) \rightarrow Palestinian(x)$

$\forall x \in Person \cdot BornIn(x, Palestine) \rightarrow Palestinian(x)$

People like Hommos are smart

,

Quantifications might be Implicit

Formalize the following:

If a number is an integer, then it is a rational number.

$\forall n \cdot Integer(n) \rightarrow Rational(n)$

If a person was born in Palestine then s/he is Palestinian

$\forall x \in Person \cdot BornInPalestine(x) \rightarrow Palestinian(x)$

$\forall x \in Person \cdot BornIn(x, Palestine) \rightarrow Palestinian(x)$

People like Hommos are smart

$\forall x \in Person \cdot Like(x, Homos) \rightarrow Smart(x)$

$\forall x \in Person \cdot LikeHomos(x) \rightarrow Smart(x)$

,

In this Lecture

We will learn

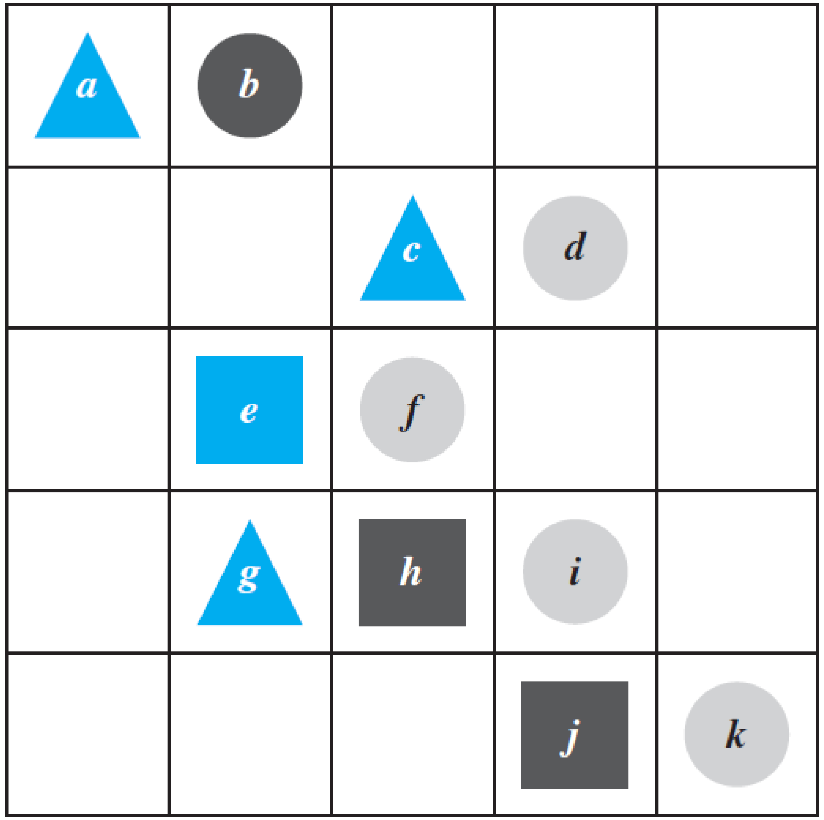
- Part 1: What is a predicate, and Predicate Logic
- Part 2: Universal and Existential Quantifiers: ∀, ∃
- Part 3: Formalize and Verablize Statements;
- Part 4: Different Writings of Quantified Statements;

- **Part 5: Tarski’s World (Simple Example)**

,

Tarski's World Example

The following statements are true or false?



$$\forall t \,.\, \text{Triangle}(t) \rightarrow \text{Blue}(t).$$

$$\forall x \,.\, \text{Blue}(x) \rightarrow \text{Triangle}(x).$$

$$\exists y \,.\, \text{Square}(y) \wedge \text{RightOf}(d, y).$$

$$\exists z \,.\, \text{Square}(z) \wedge \text{Gray}(z).$$

,