Lecture Notes on **Discrete Mathematics**.
Birzeit University, Palestine, 2021

Number Theory
and Proof Methods

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&

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4.1 Introduction

4.2 Rational Numbers

4.3 Divisibility

4.4 Quotient-Remainder Theorem

Watch this lecture and download the slides

http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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Acknowledgement:
This lecture is based on, but not limited to, chapter 3 in “Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)”.

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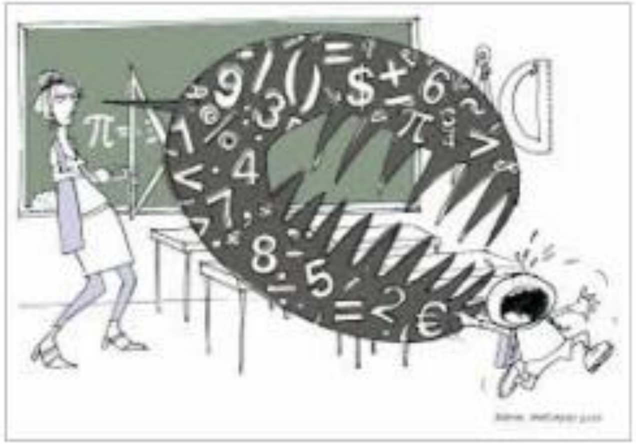
4.1 Introduction to Number Theory & Proofs Methods

In this lecture:

- Part 1: **Why Number theory for programmers**
- Part 2: Odd-Even & Prime-Composite Numbers
- Part 3: How to prove statements;
- Part 4: Disprove by counterexample;
- Part 5: Direct proofs

Keywords: Number Theory, Prove, Disapprove, Direct Proofs, Odd, even, Prime, Composite

Why Number Theory for Programmers?



- How to learn to be precise in thinking and in programing?
- Mistakes and bugs in programs: e.g., medical applications, military applications, …
- We use numbers everywhere in programs especially in loops and conditions.
- Studying number theory (properties of numbers) is very helpful, especially **how to prove and disapprove**
- For example: (dis/)approve the following properties:
  - ❖ The product of any two even integers is even?
  - ❖ The sum/difference of any two odd integers is even?
  - ❖ The product of any two odd integers is odd?

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Odd and Even Numbers

• Definitions

An integer n is **even** if, and only if, n equals twice some integer. An integer n is **odd** if, and only if, n equals twice some integer plus 1.

Symbolically, if n is an integer, then

$$n \text{ is even} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k.$$
$$n \text{ is odd} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k + 1.$$

Examples

Is 0 even?

Is −301 odd?

If a and b are integers, is $6a^2b$ even?

Ifa and b are integers, is $10a + 8b + 1$ odd?

Is every integer either even or odd?

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Examples

Is 0 even? ✓
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Is every integer either even or odd? ✓

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Prime and Composite Numbers

• Definition

An integer n is **prime** if, and only if, $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n. An integer n is **composite** if, and only if, $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

In symbols:

n is prime $\Leftrightarrow$ $\forall$ positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

n is composite $\Leftrightarrow$ $\exists$ positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

Example

Is 1 prime? ✗

Is it true that every integer greater than 1 is either prime or composite? ✓

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How to (dis)approve statements

Before (dis)approving, write a math statements as a Universal or an Existential Statement:

	Proving	Disapproving
$\exists x \in D \ . \ Q(x)$	One example	Negate then direct proof
$\forall x \in D \ . \ Q(x)$	Direct proof	Counter example

This chapter: Direct proofs with numbers

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Disproof by Counterexample

$$\forall a,b \in \mathbf{R} \;.\; a^2 = b^2 \;\rightarrow\; a = b.$$

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Disproof by Counterexample

$$\forall a,b \in \mathbf{R}\ .\ a^2 = b^2 \rightarrow a = b.$$

Counterexample:

Let a = 1 and b = -1. Then $a^2 = 1^2 = 1$ and $b^2 = (-1)^2 = 1$, and so $a^2 = b^2$. But $a \neq b$ since $1 \neq -1$.

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Proving Universal Statements

The Method of Exhaustion

The majority of mathematical statements to be proved are universal.

$$\forall x \in D \,.\, P(x) \rightarrow Q(x)$$

One way to prove such statements is called T**he Method of Exhaustion**,
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Example

Use the method of exhaustion to prove the following:

$\forall n \in Z$, if n is even and $4 \leq n \leq 26$, then n can be written as a sum of two prime numbers.

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Example

Use the method of exhaustion to prove the following:

$\forall n \in Z$, if n is even and $4 \leq n \leq 26$, then n can be written as a sum of two prime numbers.

$4 = 2 + 2$	$6 = 3 + 3$	$8 = 3 + 5$	$10 = 5 + 5$
$12=5+7$	$14=11+3$	$16=5+11$	$18=7+11$
$20=7+13$	$22=5+17$	$24=5+19$	$26=7+19$

➔ **This method** is obviously impractical, as we cannot check all possibilities.

Direct Proofs

Method of Generalizing from the Generic Particular

To show that every element of a set satisfies a certain property, suppose x is a *particular* but *arbitrarily chosen* element of the set, and show that x satisfies the property.

Method of Direct Proof

1. Express the statement to be proved in the form "$\forall x \in D$, if $P(x)$ then $Q(x)$." (This step is often done mentally.)
2. Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true. (This step is often abbreviated "Suppose $x \in D$ and $P(x)$.")
3. Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules for logical inference.

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Example

Prove that the sum of any two even integers is even.

Formal Restatement:

Starting Point:

We need to Show:

[This is what we needed to show.]

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Example

Prove that the sum of any two even integers is even.

Formal Restatement: $\forall m,n \in \mathbf{Z}\ .\ \text{Even}(m) \wedge \text{Even}(n) \to \text{Even}(m+n)$

Starting Point: Suppose m and n are even *[particular but arbitrarily chosen]*

We need to Show: $m+n$ is even

$m = 2k$
$n = 2j$
$m+n = 2k + 2j = 2(k+j)$
$(k+j)$ is integer
Thus: $2(k+j)$ is even

[This is what we needed to show.]

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In the next sections

we will practice proving more examples