Number Theory
and Proof Methods

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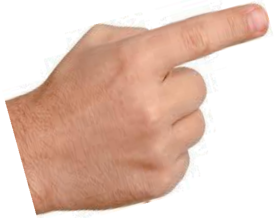
4.1 Introduction

4.2 Rational Numbers



4.3 Divisibility

4.4 Quotient-Remainder Theorem

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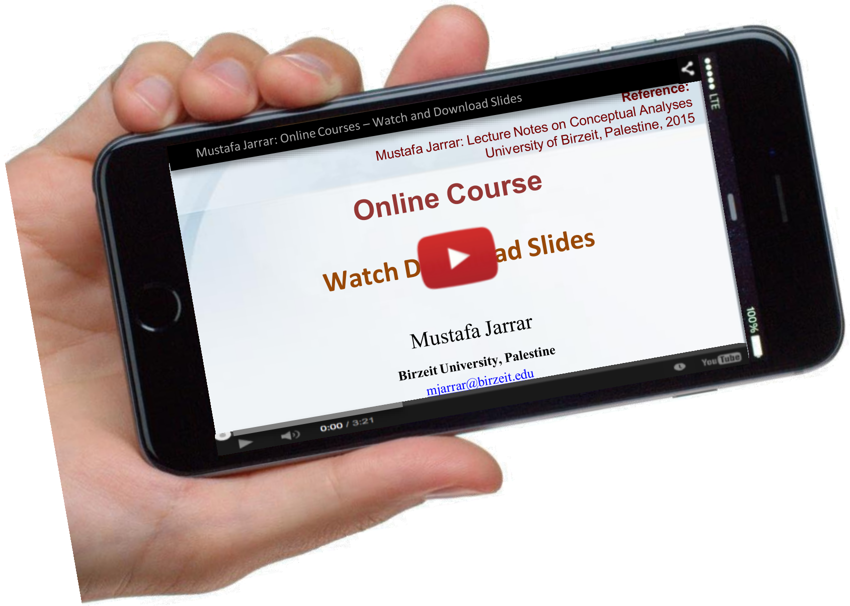
Watch this lecture and download the slides



http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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Acknowledgement:
This lecture is based on, but not limited to, chapter 3 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Number Theory

4.3 Divisibility

In this lecture:

- Part 1: **What is Divisibility;**
- Part 2: Proving Properties of Divisibility;
- Part 3: The Unique Factorization Theorem

Keywords: Number Theory, Prove, Disapprove, Direct Proofs, divisibility, factorization

What is Divisibility?

• Definition

If n and d are integers and $d \neq 0$ then

n is **divisible by** d if, and only if, n equals d times some integer.

Instead of "n is divisible by d," we can say that

n **is a multiple of** d, or
d **is a factor of** n, or
d **is a divisor of** n, or
d **divides** n.

The notation **d | n** is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$:

$$d \mid n \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = dk.$$

Examples

- ✓ Is 21 divisible by 3?
- ✓ Does 5 divide 40?
- ✓ Does $7 \mid 42$?
- ✓ Is 32 a multiple of -16?
- ✓ Is 6 a factor of 54?
- ✓ Is 7 a factor of -7?
- ✓ If k is any integer, does k divide **0?**

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Positive Divisor of a Positive Integer

Theorem 4.3.1 A Positive Divisor of a Positive Integer

For all integers a and b, if a and b are positive and a divides b, then $a \leq b$.

Proof:

$$b = a.k$$

Thus $1 \leq k$

$a.1 \leq k \, . \, a$ multiply both sides with a.

Thus $a \leq k \, . \, a = b$

Thus $a \leq b$

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Divisibility of Algebraic Expressions

If *a* and *b* are integers, is *3a* + *3b* divisible by 3?

$3a + 3b = 3(a + b)$ and $a + b$ is an integer because it is a sum of two integers.

If *k* and *m* are integers, is *10km* divisible by *5?*

10k m = 5 · (2k m) and 2k m is an integer because it is a product of three integers.

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Not divisible

For all integers n and d, $\quad d \nmid n \quad \Leftrightarrow \quad \dfrac{n}{d}$ is not an integer.

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Prime Numbers and Divisibility

An alternative way to define a prime number is to say that:

an integer $n > 1$ is prime if, and only if, its only positive integer divisors are 1 and itself.

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Transitivity of Divisibility

Theorem 4.3.3 Transitivity of Divisibility

For all integers a, b, and c, if a divides b and b divides c, then a divides c.

Proof:

Starting Point: Suppose a, b, and c are particular but arbitrarily chosen integers such that $a \mid b$ and $b \mid c$.

We need to show: $a \mid c$.

since $a \mid b$, $b = ar$ for some integer r.
And since $b \mid c$, $c = bs$ for some integer s.
Hence, $c = bs = (ar)s$
But $(ar)s = a(rs)$ by the associative law
Hence $c = a(rs)$.
As rs is an integer, then $a \mid c$.

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Divisibility by a Prime

Theorem 4.3.4 Divisibility by a Prime

Any integer $n > 1$ is divisible by a prime number.

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Counterexamples and Divisibility

Checking a Proposed Divisibility Property

Is it true or false that for

all integers *a* and *b*, if *a* | *b* and *b*|*a* then *a* = *b*?

Counterexample: Let $a = 2$ and $b = -2$. Then
$a \mid b$ since $2 \mid (-2)$ and $b \mid a$ since $(-2) \mid 2$, but $a \neq b$ since $2 \neq -2$.
Therefore, the proposed divisibility property is false.

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Important Theory

The Unique Factorization Theorem

By a German mathematician (Carl Friedrich Gauss) in 1801.

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The Unique Factorization Theorem

أي رقم اكبر من واحد إما ان يكون عدد أولي او حاصل ضرب أعداد أولية

Any integer greater than 1 either is prime or can be written as a product of prime numbers in a way that is unique except,

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2$$

Theorem 4.3.5 Unique Factorization of Integers Theorem (Fundamental Theorem of Arithmetic)

Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_k^{e_k},$$

and any other expression for n as a product of prime numbers is identical to this except, perhaps, for the order in which the factors are written.

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The Standard factored Form

• Definition

Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; $e_1, e_2, \ldots, e_k$ are positive integers; and $p_1 < p_2 < \cdots < p_k$.

Example: Write 3,300 in standard factored form.

$$\begin{aligned} 3{,}300 &= 100 \,.\, 33 \\ &= 4 \,.\, 25 \,.\, 3 \,.\, 11 \\ &= 2 \,.\, 2 \,.\, 5 \,.\, 5 \,.\, 3 \,.\, 11 \\ &= 2^2 \,.\, 3^1 \,.\, 5^2 \,.\, 11^1. \end{aligned}$$

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Using Unique Factorization to Solve a Problem

Suppose m is an integer such that

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot m = 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10$$

Does $17 \mid m$?

Solution:

Since 17 a prime in the left, it should be also in the right side.
Since we cannot produce 17 form (8,7,6,5,4,3 or 2) it should be a prime factor of *m*

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