Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

1

Watch this lecture and download the slides

http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

More Online Courses at: http://www.jarrar.info

Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

2

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

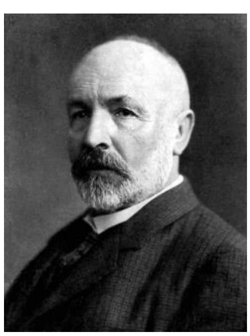
6.1 Basics of Sets

In this lecture:

- ❑ **Part 1: Basic Concepts and Notations**
- ❑ Part 2: Subsets, proper subsets, and Set Equalities
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Empty Sets
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: Power Sets & Cartesian Products

3

History



Georg Cantor
1845 – 1918
Born in Saint Petersburg, Russia
Moved to Germany 1856
PhD: University of Berlin 1867
Work: University of Halle

Set theory is the branch of mathematical logic that studies sets, which informally are collections of objects.

Initiated by Georg Cantor in 1870s

4

Basic Concepts and Notations

Cantor suggested a set as a:

"collection into a whole ***M*** of definite and separate objects of our intuition or our thought".

$\boldsymbol{M} = \{$ Ali, Adam, Sara $\}$

Each object is called an elements (or member of) of ***M***.

Ali $\in$ ***M*** (Ali belongs to *M*)

Rami $\notin$ ***M*** (Rami does not belong to *M*)

,

Basic Concepts and Notations

The order of elements is irrelevant

{Ali, Adam, Sara} = {Adam, Sara, Ali}

Redundancy is not allowed

{Ali, Adam, Adam, Sara}

A set can be an element inside another set

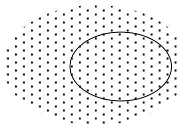
{1, {1}{ has two elements

Notation of elements

{Ali} ≠ Ali different elements

,

Defining Sets by a Property



$$A = \{x \in S \mid P(x)\}$$

The set of all "x is dummy" — Property

Examples:

The set of all integers that are more than -2 and less than 5

$\{x \in \mathbf{Z} \mid -2 < x < 5\}$

The set of all persons who born in Palestine

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine})\}$

The set of all persons who born in Palestine and Love Homus

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine}) \land \textit{Love}(x, \text{Homus})\}$

7

Set Versus Element

In Set theory → **Set** vs. **Element** ← Mathematical Set

In JAVA → Class vs. Object

- The **extension** of a set is its elements.
- The **order** of elements is irrelevant
- In set theory: an element itself might be a set.

8

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

6.1 Basics of Sets

In this lecture:

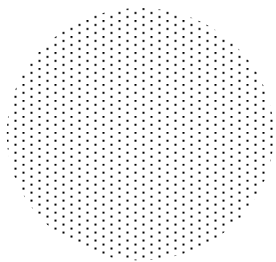
- ❑ Part 1: Basic Concepts and Notations
- ❑ **Part 2: Subsets, proper subsets, and Set Equalities**
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Empty Sets
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: Power Sets & Cartesian Products

Subsets

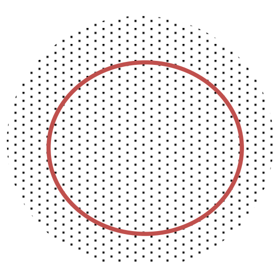
المجموعة الجزئية

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

Subsets Versus JAVA SubClasses

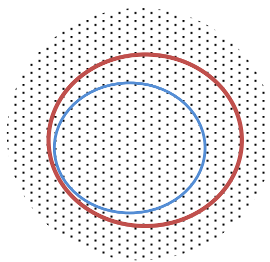


Subsets Versus JAVA SubClasses



$\mathit{Animal} = \{x \in \text{LivingOrganism} \mid \mathit{CanMove}(x)\}$

Subsets Versus JAVA SubClasses



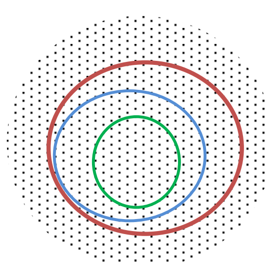
$Animal = \{x \in \text{LivingOrganism} \mid CanMove(x)\}$

$Human = \{x \in \text{Animal} \mid \text{Has}Mind(x) \wedge \text{Utter}(x)\}$

Human ⊂ Animal

13

Subsets Versus JAVA SubClasses



$Animal = \{x \in \text{LivingOrganism} \mid CanMove(x)\}$

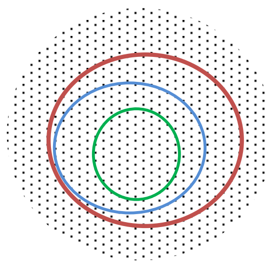
$Human = \{x \in \text{Animal} \mid \text{Has}Mind(x) \wedge \text{Utter}(x)\}$

$Woman = \{x \in \text{Human} \mid \text{GivesBirth}(x)\}$

Woman ⊂ Human ⊂ Animal

14

Subsets Versus JAVA SubClasses

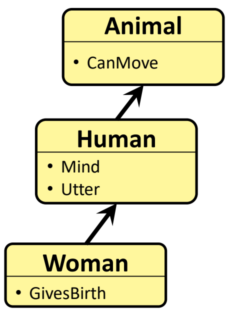


Animal = {x ∈ LivingOrganism | *CanMove*(x)}

Human = {x ∈ Animal | Has*Mind*(x) ∧ Utter (x)}

Woman = {x ∈ Human| GivesBirth(x)}

Woman ⊂ Human ⊂ Animal



Every subclass inherits the properties of its super class, thus:

- Human is a living organism that can move, has mind and utter.
- Woman is a living organism that can move, has mind and utter, and able to give birth.

15

Distinction between ∈ and ⊇

Which of the following are true statements?

$2 \in \{1, 2, 3\}$

$\{2\} \in \{1, 2, 3\}$

$2 \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{\{1\}, \{2\}\}$

$\{2\} \in \{\{1\}, \{2\}\}$

16

Distinction between ∈ and ⊇

Which of the following are true statements?

✓ $2 \in \{1, 2, 3\}$

✗ $\{2\} \in \{1, 2, 3\}$

✗ $2 \subseteq \{1, 2, 3\}$

✓ $\{2\} \subseteq \{1, 2, 3\}$

✗ $\{2\} \subseteq \{\{1\}, \{2\}\}$

✓ $\{2\} \in \{\{1\}, \{2\}\}$

,

17

Subsets Notations

Not Subset:

$$A \nsubseteq B \Leftrightarrow \exists x \,.\, x \in A \;\; and \;\; x \notin B$$

Notations:

$A = B$		A equals B
$A \subset B$	$B \supset A$	A is subset of B
$A \subseteq B$	$B \supseteq A$	A is subset or equal of B
$A \not\subset B$	$B \not\supset A$	A is not a subset of B
$A \nsubseteq B$	$B \nsupseteq A$	A is not a subset or equal of B
$A \subsetneq B$	$B \supsetneq A$	A is a subset but not equal of B

Examples: Person ⊇ Man, $Z \supseteq Z^{+}$, $R \not\subset Z$

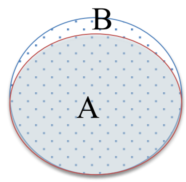
,

18

Proper Subsets

Definition

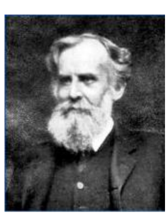
Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.



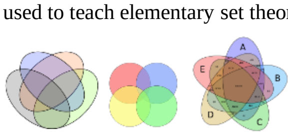
Examples of proper subsets:

{-2, -3, 4} ⊂ {-2, -2.5, -3, -3.5, -4}
{1, 2, 3, 4, …} ⊂ {0, 1, 2, 3, 4, ...}

Venn Diagrams

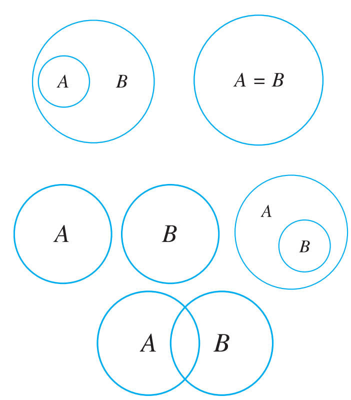


John Venn, British
(1834-1923)
Represented sets as diagrams in1881.
used to teach elementary set theory,



$A \subseteq B$

$A \not\subseteq B$



Z: integers (صحيحة)
Q: rational numbers (نسبية)
R: real numbers(حقيقية)

Proving and Disproving Subset Relations

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$

$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Prove that $A \subseteq B$.

Suppose x is a particular but arbitrarily chosen element of A.

Show that $x \in B$, means show that $x = 3\cdot(integer)$.

$x = 6r + 12$

$= 3\cdot(2r + 4)$.

Let $s = 2r + 4$.

Also, $3s = 3(2r + 4)$

$= 6r + 12$

$= x$

Therefore, x is an element of B.

,

21

Set Equality

Definition

Given sets A and B, A **equals** B, written $\mathbf{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Example: Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$

$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$

Is $A = B$?

Yes. To prove this, both subset relations $\mathbf{A \subseteq B}$ and $\mathbf{B \subseteq A}$ must be proved.

Part 1, Proof That $A \subseteq B$:

…..

Part 2, Proof That $B \subseteq A$:

…..

,

22

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

6.1 Basics of Sets

In this lecture:

- Part 1: Basic Concepts and Notations
- Part 1: Subsets, proper subsets, and Set Equalities
- Part 3: **Set Operations** (Union, Intersection, Difference, Complement)
- Part 4: Empty Sets
- Part 5: Partitions of Sets
- Part 6: Power Sets & Cartesian Products

Operations on Sets

• Definition

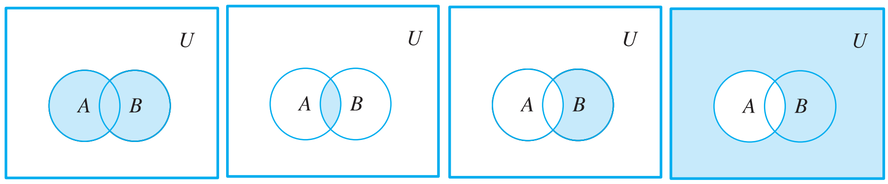
Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Operations on Sets



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Distinction between $\cap$ and $\wedge$

Between sets → $\cap$ and $\wedge$ ← Between predicate and propositions

$\cup$ and $\vee$

Indexed Collection of Sets

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0,\ A_1,\ A_2,\ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

,

27

Finding Unions and Intersections of More than Two Sets

For each positive integer *i*, let $A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$

A_1: set of all real numbers between -1 and 1
A_2: set of all real numbers between -1/2 and 1/2
A_3: set of all real numbers between - 1/3 and 1/3

Find $A_1 \cup A_2 \cup A_3$ $= (-1,1)$, because $\left(-\frac{1}{2}, \frac{1}{2}\right)\left(-\frac{1}{3}, \frac{1}{3}\right)$ included

Find $A_1 \cap A_2 \cap A_3$ $= \left(-\frac{1}{3}, \frac{1}{3}\right)$, because (-1,1)$\left(-\frac{1}{2}, \frac{1}{2}\right)$ are included

Find $\bigcup_{i=1}^{\infty} A_i = (-1,1)$ Find $\bigcap_{i=1}^{\infty} A_i = 0$

,

28

Formalizing Statements in Set Theory

- **All smart students**
- **Students who are not Smart**
- **There are no smart students from Palestine**
- **There are no smart students from Palestine among the winners**

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

6.1 Basics of Sets

In this lecture:

- ❑ Part 1: Basic Concepts and Notations
- ❑ Part 1: Subsets, proper subsets, and Set Equalities
- ❑ Part 3: Operations on Sets
- ❑ Part 4: **Empty Sets**
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: Power Sets & Cartesian Products

The Empty Set

The empty set is not the same thing as nothing; rather, it is a set with nothing inside it and a set is always something. This issue can be overcome by **viewing a set as a bag—an empty bag undoubtedly still exists.**

Describe the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

$\forall A: \emptyset \subseteq A$

$\forall A: A \times \cup\emptyset = \cup\emptyset$

$\forall A: A \cup \cup\emptyset \subseteq A$

$\forall A: A \times \cup\emptyset \Rightarrow A = \cup\emptyset$

$\forall A: A \cap \cup\emptyset \subseteq \cup\emptyset$

While the empty set is a standard and widely accepted mathematical concept, it remains an ontological curiosity, **whose meaning and usefulness are debated by philosophers and logicians**.

,

31

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

6.1 Basics of Sets

In this lecture:

- ❑ Part 1: Basic Concepts and Notations
- ❑ Part 1: Subsets, proper subsets, and Set Equalities
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Empty Sets
- ❑ Part 5: **Partitions of Sets**
- ❑ Part 6: Power Sets & Cartesian Products

,

32

Disjoint Sets

• **Definition**

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Man ∩ Woman = ∅

• **Definition**

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

,

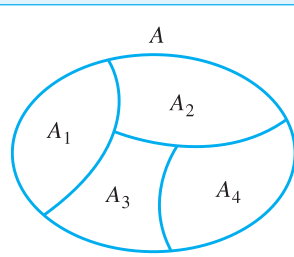
33

Partitions of Sets

• **Definition**

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.



Man ∩ Woman = ∅

Person = Man ∪ Woman

,

34

Example

Let **Z** be the set of all integers and let

$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \quad \text{for some integer } k\}$,
$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}$,
$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of Z?

Yes. By the quotient-remainder theorem, every integer *n* can be represented in exactly one of the three forms

$$n=3k \quad \text{or} \quad n=3k+1 \quad \text{or} \quad n=3k+2$$

It also implies that every integer is in one of the sets T_0, T_1, or T_2.
So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$.

,

35

Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

6.1 Basics of Sets

In this lecture:

- ❑ Part 1: Basic Concepts and Notations
- ❑ Part 1: Subsets, proper subsets, and Set Equalities
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Empty Sets
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: **Power Sets & Cartesian Products**

,

36

Power Sets

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$**,** is the set of all subsets of A.

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$

$$= \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

,

37

n-tuples

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered** n**-tuple,** $(x_1, x_2, \ldots, x_n)$**,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Order *n*-tuples:

Is (1,2) =(2,1)?

Is $(3, (-2)^2, 1/3) = (\sqrt{9}, 4, \frac{3}{9})$?

,

38

Cartesian Products

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example: Let $A_1 = \{x, y\}$, $A_2 = \{1,2,3\}$, and $A_3 = \{a,b\}$.

$A_1 \times A_2 =$

$= \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$

Example

Let A = {Ali, Ahmad},
B = {AI, Dmath, DB},
C = {Pass, Fail}

Find (A × B) × C =

Find A × B × C =