Set Theory

6.1. Basics of Set Theory

6.2 Properties of Sets and Element Argument

6.3 Algebraic Proofs

6.4 Boolean Algebras

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ **Part 1: Set Relations and Identities**
- ❑ Part 2: Proving Set Identities (Element Argument)
- ❑ Part 3: Examples of proving Set Identities

Set Relations

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

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Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
2. $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
3. $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x,y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

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Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

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Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

➔ We will prove some of these theories in the lecture, please prove others at home

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ Part 1: Set Relations and Identities
- ❑ Part 2: **Proving Set Identities (Element Argument)**
- ❑ Part 3: Examples of proving Set Identities

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Proving Set Identities

Proving That Sets Are Equal

e.g., prove: *HumanMale* = *Man*

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

But: How to prove that $X \subseteq Y$?

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The Element Argument Method

For Proving a set is a subset of another

e.g., prove: *HumanMale* ⊆ *Man*

i.e., Prove that every element in *HumanMale* is an element in *Man*

The Element Argument Method:

Let sets X and Y be Given, To Prove that $\boldsymbol{X \subseteq Y}$**:**

Step 1. Suppose that x is a particular but arbitrarily chosen element if X.

Step 2. Show that x is an element of Y.

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The Element Argument Method

In details

Example: Prove that: A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

That is:

Prove: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

That is, show $\forall x$, if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$. [Show $x \in (A \cup B) \cap (A \cup C)$.]

…

Thus $x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Prove: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

That is, show $\forall x$, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$. [Show $x \in A \cup (B \cap C)$.]

…

Thus $x \in A \cup (B \cap C)$.

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ Part 1: Set Relations and Identities
- ❑ Part 2: Proving Set Identities (Element Argument)
- ❑ Part 3: **Examples of proving Set Identities**

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Proving: **A Distributive Law for Sets**

Theorem 6.2.2(3)(a) A Distributive Law for Sets
For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Suppose $x \in A \cup (B \cap C)$.
$x \in A$ or $x \in B \cap C$. (by def. of union)
Case 1 ($x \in A$): then
$x \in A \cup B$ (by def. of union) and
$x \in A \cup C$ (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)
Case 2 ($x \in B \cap C$): then
$x \in B$ and $x \in C$ (def. of intersection)
As $x \in B$, $x \in A \cup B$ (by def. of union)
As $x \in C$, $x \in A \cup C$, (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)

In both cases, $x \in (A \cup B) \cap (A \cup C)$.
Thus: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
by definition of subset

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Suppose $x \in (A \cup B) \cap (A \cup C)$.
$x \in A \cup B$ and $x \in A \cup C$. (def. of intersection)
Case 1 ($x \in A$): then
$x \in A \cup (B \cap C)$ (by def. of union)
Case 2 ($x \notin A$): then
$x \in B$ and $x \in C$, (def. of intersection)
Then, $x \in B \cap C$ (def. of intersection)
$\therefore x \in A \cup (B \cap C)$

In both cases $x \in A \cup (B \cap C)$.
Thus: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
by definition of subset,

Conclusion: Since both subset relations have been proved, it follows by definition of set equality that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proving: **A De Morgan's Law for Sets**

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$

Same As: proving whether: the people who are not students or employees is the same as the people who are either students nor employees.

$(A \cup B)^c \subseteq A^c \cap B^c$

Suppose $x \in (A \cup B)^c$. *[We must show that $x \in A^c \cap B^c$.]* By definition of complement,

$$x \notin A \cup B.$$

But to say that $x \notin A \cup B$ means that

it is false that (x is in A or x is in B).

By De Morgan's laws of logic, this implies that

x is not in A and x is not in B,

which can be written $x \notin A$ and $x \notin B$.

Hence $x \in A^c$ and $x \in B^c$ by definition of complement. It follows, by definition of intersection, that $x \in A^c \cap B^c$ *[as was to be shown]*. So $(A \cup B)^c \subseteq A^c \cap B^c$ by definition of subset.

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$A^c \cap B^c \subseteq (A \cup B)^c$

Suppose $x \in A^c \cap B^c$. *[We must show that $x \in (A \cup B)^c$.]* By definition of intersection, $x \in A^c$ and $x \in B^c$, and by definition of complement,

$$x \notin A \quad \text{and} \quad x \notin B.$$

In other words, x is not in A and x is not in B.

By De Morgan's laws of logic this implies that

it is false that (x is in A or x is in B),

which can be written $x \notin A \cup B$

by definition of union. Hence, by definition of complement, $x \in (A \cup B)^c$ *[as was to be shown]*. It follows that $A^c \cap B^c \subseteq (A \cup B)^c$ by definition of subset.

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

Prove at home

Proof: If every person is a student, then the set of persons and students are students

Part (a): Suppose A and B are sets with $A \subseteq B$. To show part (a) we must show both that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. We already know that $A \cap B \subseteq A$ by the inclusion of intersection property. To show that $A \subseteq A \cap B$, let $x \in A$. *[We must show that $x \in A \cap B$.]* Since $A \subseteq B$, then $x \in B$ also. Hence

$$x \in A \quad \text{and} \quad x \in B,$$

and thus

$$x \in A \cap B$$

by definition of intersection *[as was to be shown]*.

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Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by Contradiction:

Suppose not. [We take the negation of the theorem and suppose it to be true.]
That is, Suppose: E with no elements, and E $\nsubseteq$ A.
assuming (E $\nsubseteq$ A) means there $x \in$E and this x $\notin$ A [by definition of subset].

But there can be no such element since E has no elements. **This is a contradiction.**

Hence the supposition that there are sets E and A, where E has no elements and E $\nsubseteq$ A, is false, and so the theorem is true.

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Proving: **Uniqueness of the Empty Set**

Corollary 6.2.5 Uniqueness of the Empty Set

There is only one set with no elements.

Proof:

Suppose E_1 and E_2 are both sets with no elements.

By Theorem 6.2.4, $E_1 \subseteq E_2$ since E_1 has no elements.

Also $E_2 \subseteq E_1$ since E_2 has no elements.

Thus $E_1 = E_2$ by definition of set equality.

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Proving: **a Conditional Statement**

Example: If every student is smart and every smart is not-foolish, then there are no foolish students

Proposition 6.2.6

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof:

Suppose not, Suppose there is an element x in $A \cap C$.

Then $x \in A$ and $x \in C$ (By definition of intersection).

As $A \subseteq B$ then $x \in B$ (by definition of subset).

Also, as $B \subseteq C^c$, then $x \in C^c$ (by definition of subset).

So, $x \notin C$ (by definition of complement)

Thus, $x \in C$ and $x \notin C$, which is a contradiction.

So the supposition that there is an element x in $A \cap C$ is false,
and thus $A \cap C = \emptyset$ [as was to be shown].

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