Set Theory

6.1. Basics of Set Theory

6.2 Properties of Sets and Element Argument

6.3 Algebraic Proofs

6.4 Boolean Algebra

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in “Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)”.

Set Theory

6.4 Boolean Algebra

In this lecture:

- ❑ Part 1: **History of Algebra**
- ❑ Part 2: What is Boolean Algebra
- ❑ Part 3: Proving Boolean Algebra Properties

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What is Algebra?

Al-Khwarizmi 850 – 780 (Baghdad)

الكتاب المختصر في حساب الجبر والمقابلة

The Compendious Book on Calculation by Completion and Balancing

Developed an advanced arithmetical system with which they were able to do calculations in an algorithmic fashion.

Statements to describe relationships between things

Symbols and the rules for manipulating these symbols

Do you know any algebra (جبر)?

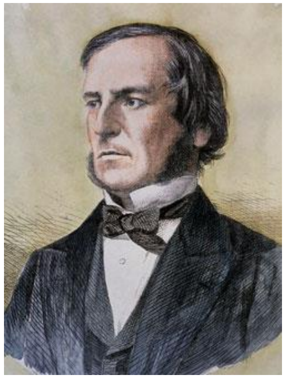
Set Theory

6.4 Boolean Algebra

In this lecture:

- ❑ Part 1: History of Algebra
- ❑ Part 2: **What is Boolean Algebra**
- ❑ Part 3: Proving Boolean Algebra Properties

Boolean Algebra



George Boole
1815-1864,
England

Introduced by George Boole in his first book The Mathematical Analysis of Logic (1847),

A structure abstracting the computation with the truth values false and true.

Instead of elementary algebra where the values of the variables are numbers, and the main operations are addition and multiplication, the main operations of Boolean algebra are the conjunction (∧) the disjunction (∨) and the negation not (¬).

Used extensively in the simplification of logic Circuits

Compare

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t}, \equiv \mathbf{t}$	a. $A \cup U = U$

Compare

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge$	a. $A \cup A^c = U$
$\sim(\sim$	
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t}, \equiv \mathbf{t}$	a. $A \cup U = U$

Both are special cases of the same general structure, known as a *Boolean Algebra*.

Boolean Algebra

• Definition: Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a + b = b + a \quad \text{and} \quad \text{(b) } a \cdot b = b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a + b) + c = a + (b + c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a + (b \cdot c) = (a + b) \cdot (a + c) \quad \text{and} \quad \text{(b) } a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for all a in B,

$$\text{(a) } a + 0 = a \quad \text{and} \quad \text{(b) } a \cdot 1 = a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a + \overline{a} = 1 \quad \text{and} \quad \text{(b) } a \cdot \overline{a} = 0.$$

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Properties of a Boolean Algebra

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.
3. *Double Complement Law:* For all $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Law:* For all $a \in B$,

$$\text{(a) } a + a = a \quad \text{and} \quad \text{(b) } a \cdot a = a.$$

5. *Universal Bound Law:* For all $a \in B$,

$$\text{(a) } a + 1 = 1 \quad \text{and} \quad \text{(b) } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\text{(a) } \overline{a + b} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \text{(b) } \overline{a \cdot b} = \overline{a} + \overline{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$\text{(a) } (a + b) \cdot a = a \quad \text{and} \quad \text{(b) } (a \cdot b) + a = a.$$

8. *Complements of* 0 *and* 1:

$$\text{(a) } \overline{0} = 1 \quad \text{and} \quad \text{(b) } \overline{1} = 0.$$

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Proving of Boolean Algebra Properties

Uniqueness of the Complement Law: For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.

Proof:

Suppose a and x are particular, but arbitrarily chosen, elements of B that satisfy the following hypothesis: $a + x = 1$ and $a \cdot x = 0$. Then

$x = x \cdot 1$	because 1 is an identity for $\cdot$
$= x \cdot (a + \overline{a})$	by the complement law for $+$
$= x \cdot a + x \cdot \overline{a}$	by the distributive law for $\cdot$ over $+$
$= a \cdot x + x \cdot \overline{a}$	by the commutative law for $\cdot$
$= 0 + x \cdot \overline{a}$	by hypothesis
$= a \cdot \overline{a} + x \cdot \overline{a}$	by the complement law for $\cdot$
$= (\overline{a} \cdot a) + (\overline{a} \cdot x)$	by the commutative law for $\cdot$
$= \overline{a} \cdot (a + x)$	by the distributive law for $\cdot$ over $+$
$= \overline{a} \cdot 1$	by hypothesis
$= \overline{a}$	because 1 is an identity for $\cdot$.

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Proving of Boolean Algebra Properties

Theorem 6.4.1(3) Double Complement Law

For all elements a in a Boolean algebra B, $\overline{(\overline{a})} = a$.

Proof:

Suppose B is a Boolean algebra and a is any element of B. Then

$$\begin{aligned} \overline{a} + a &= a + \overline{a} && \text{by the commutative law} \\ &= 1 && \text{by the complement law for 1} \end{aligned}$$

and

$$\begin{aligned} \overline{a} \cdot a &= a \cdot \overline{a} && \text{by the commutative law} \\ &= 0 && \text{by the complement law for 0.} \end{aligned}$$

Thus a satisfies the two equations with respect to $\overline{a}$ that are satisfied by the complement of $\overline{a}$. From the fact that the complement of a is unique, we conclude that $\overline{(\overline{a})} = a$.

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