Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Functions

7.1. Introduction to Functions

7.2 One-to-One, Onto, Inverse functions

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in “Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)”.

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Functions

7.1 Introduction to Functions

In this lecture:

- ❑Part 1: **What is a function**
- ❑Part 2: Equality of Functions
- ❑Part 3: Examples of Functions
- ❑Part 3: Checking Well Defined Functions

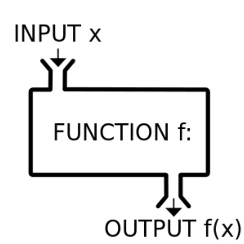
3

Motivation

Many issues in life can be mathematized and used as functions:

- Div(x), mod(x), ….
- FatherOf(x), TruthTable (x)

- In this lecture we focus on **discrete functions**

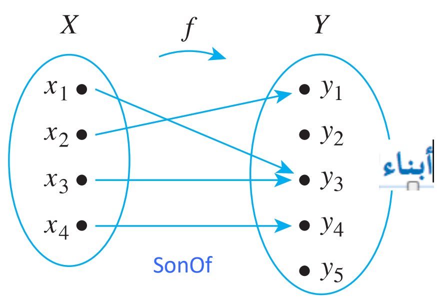


4

What is a Function

أبناء | أباء

Domain | **Co-domain**



علاقة بين عنصرين
كل عنصر في المجال يجب ان
يكون له صورة واحدة في المدى.
كل عنصر في المدى هو صورة
لعنصر (او اكثر) في المجال

A function is a relation from X, the domain, to Y, the co-domain, that satisfies 2 properties: 1) Every element is related to some element in Y; 2) No element in X is related to more than one element in Y

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Function Definition

• Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

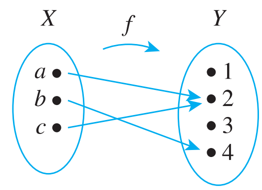
$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Example

Let $X = \{a, b, c\}$ and $Y = \{1,2,3,4\}$. Define a function f from X to Y



a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

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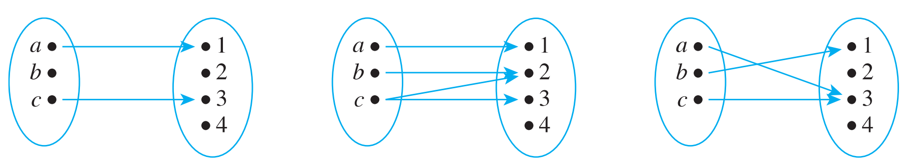
Example

Which are functions?



(a) (b) (c)

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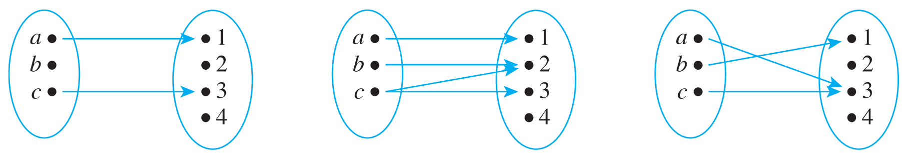
Example

Which are functions?



(a) (b) (c)

(a) There is an element x, namely b, that is not sent to any element in of Y (i.e., there is no arrow coming out of Y

(b) The element c isn’t sent to a unique element of Y: that is, there are two arrows coming out of c; one pointing to 2 and the other is pointing to 3

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9

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Functions

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- ❑Part 1: What is a function
- ❑Part 2: **Equality of Functions**
- ❑Part 3: Examples of Functions
- ❑Part 3: Checking Well Defined Functions

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10

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example:

Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x *in* J_3

$$f(x) = (x^2 + x + 1) \ mod \ 3 \quad \text{and } g(x) = (x + 2)^2 \ mod \ 3.$$

Does $f = g$?

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x	x^2+x+1	$f(x) = (x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x) = (x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

Equal functions in reality?

Equality of Functions

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Example:

Let F: $\mathbf{R} \to \mathbf{R}$ and G: $\mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G$: $\mathbf{R} \to \mathbf{R}$ and $G + F$: $\mathbf{R} \to \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

13

Equality of Functions

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$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

$$\begin{aligned} (F + G)(x) &= F(x) + G(x) && \text{by definition of } F + G \\ &= G(x) + F(x) && \text{by the commutative law for addition of real numbers} \\ &= (G + F)(x) && \text{by definition of } G + F \end{aligned}$$

Hence $F + G = G + F$.

14

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15

Examples of Functions

Identity Function

$$I_X(x) = x \text{ for all } x \text{ in } X.$$

Identity function send each element of X to the element that is identical to it

E.g., $I_x(y) = y$

E.g., Let X be any set and suppose that a_{ij}^k and $\varphi(z)$ are elements of X. Find $I_x(a_{ij}^k)$ and $I_x(\varphi(z))$

Sol. Whatever is input to the identity function comes out unchanged. So, $I_x(a_{ij}^k) = a_{ij}^k$ and $I_x(\varphi(z)) = \varphi(z)$

16

Examples of Functions
Sequences

An infinite sequence is a function defined on set of integers that are greater than or equal to a particular integer.

E.g., Define the following sequence as a function from the set of positive integers to the set of real numbers

s

can be thought as a function *f* from the nonnegative integers to the real numbers that associate 0 → 1, 1 → -1/2, 2 → 1/3, …

$f\colon \mathbf{Z}^{nonneg} \to \mathbf{R} \qquad n \geq 0$

send each integer n>= 0 to $f(n) = \frac{(-1)^n}{n+1}$

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17

Examples of Functions
Function Defined on a Power Set

Draw an arrow diagram for *F as follows:*

Define a function f: ℘({a, b, c}) → $\mathbf{Z}^{\text{nonneg}}$

as follows: for each x∈℘({a, b, c}) → $\mathbf{Z}^{\text{nonneg}}$

F(x) = the number of elements in X.

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18

Examples of Functions

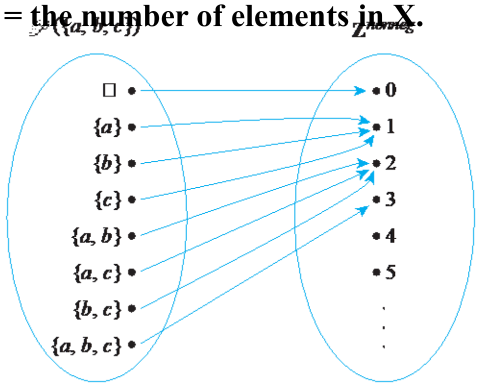
Function Defined on a Power Set

Draw an arrow diagram for *F as follows:*

Define a function f: $\wp(\{a, b, c\}) \rightarrow Z^{nonneg}$

as follows: for each $x \in \wp(\{a, b, c\}) \rightarrow Z^{nonneg}$

F(x) = the number of elements in X.



Examples of Functions

Cartesian product

Define functions $M: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and $R: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$ as follows: For all ordered pairs (a, b) of integers,

$$M(a, b) = ab \quad \text{and} \quad R(a, b) = (-a, b).$$

M is the multiplication function that sends each pair of real numbers to the product of the two. R is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis.

Find the following:

a. $M(-1, -1)$ b. $M\left(\frac{1}{2}, \frac{1}{2}\right)$ c. $M(\sqrt{2}, \sqrt{2})$

d. $R(2, 5)$ e. $R(-2, 5)$ f. $R(3, -4)$

Examples of Functions

Cartesian product

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c. $M(\sqrt{2}, \sqrt{2})$
d. $R(2, 5)$
e. $R(-2, 5)$
f. $R(3, -4)$

a. $(-1)(-1) = 1$
b. $(1/2)(1/2) = 1/4$
c. $\sqrt{2} \cdot \sqrt{2} = 2$
d. $(-2, 5)$
e. $(-(-2), 5) = (2, 5)$
f. $(-3, -4)$

21

Examples of Functions

String Functions

g: S → **Z**

g(s) = the number of a's in s.

Find the following.

a. g(e)
b. g(bb)
c. g(ababb)
d. g(bbbaa)

22

Examples of Functions

Logarithmic functions

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base *b* of *x***, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base *b*** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

- $\log_3 9 = 2$ because $3^2 = 9$.
- $\log_2 (1/2) = -1$ because $2^{-1} = ½$.
- $\log_{10}(1) = 0$ because $10^0 = 1$.
- $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m.
- $2^{\log_2 m} = m$ because $\log_2 m$ is the exponent to which 2 must be raised to obtain m.

23

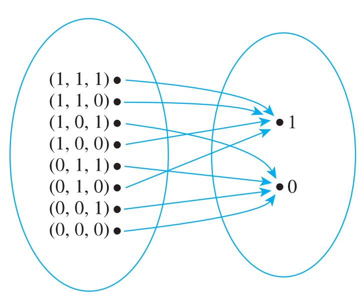
Examples of Functions

Boolean Functions

• Definition

An **(*n*-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

Input			Output
P	*Q*	*R*	*S*
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0



24

Examples of Functions

Boolean Functions

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \text{ } mod \text{ } 2.$$

Describe f using an input/output table.

Examples of Functions

Boolean Functions

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \text{ } mod \text{ } 2.$$

Describe f using an input/output table.

$f(1, 1, 1) = (1 + 1 + 1)$ *mod* $2 = 3$ *mod* $2 = 1$
$f(1, 1, 0) = (1 + 1 + 0)$ *mod* $2 = 2$ *mod* $2 = 0$
and so on to calculate the other values

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)$ *mod* 2
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

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- Part 3: **Checking Well Defined Functions**

Well-defined Functions

Checking Whether a Function Is Well Defined

A function is not well defined if it fails to satisfy at least one of the requirements of being a function

E.g., Define a function $f: \mathbf{R} \rightarrow \mathbf{R}$ by specifying that for all real numbers x, $f(x)$is the real number y such that $x^2+y^2=1$.

There are two reasons why this function is not well defined:
For almost all values of x either (1) there is no y that satisfies the given equation or (2) there are two different values of y that satisfy the equation

Consider when x=2
Consider when x=0

Well-defined Functions

Checking Whether a Function Is Well Defined

$f: \mathbf{Q} \to \mathbf{Z}$ defines this formula:

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Is f a well defined function?

It is not a well defined function since fractions have more than one representation as quotients of integers.

$$f\left(\tfrac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\tfrac{3}{6}\right) = 3,$$

$$f\left(\tfrac{1}{2}\right) \neq f\left(\tfrac{3}{6}\right).$$

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Well-defined Functions

Checking Whether a Function or not

Y= BortherOf(x)
Y= Parent Of(x)
Y= SonOf(x)
Y= FatherOf(x)
Y= Wife Of(x)
.
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