Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Functions

7.1. Introduction to Functions

7.2 One-to-One, Onto, Inverse functions

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: **One-to-one Functions**
- ❑ Part 2: Onto Functions
- ❑ Part 3: one-to-one Correspondence Functions
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

One-to-One Functions

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

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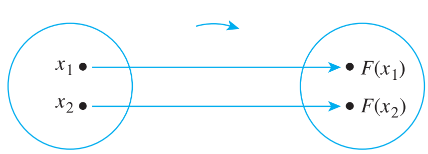
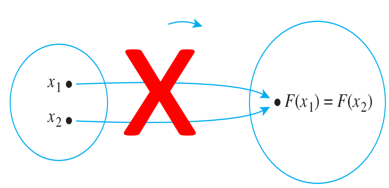
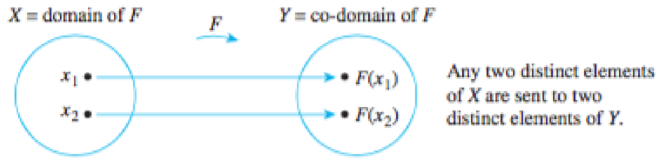
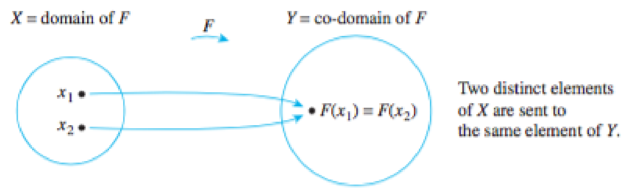




One-to-One Functions





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One-to-One Functions

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

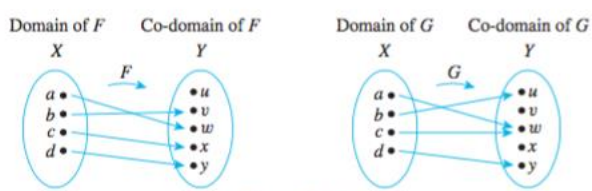

Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

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One-to-One Functions

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

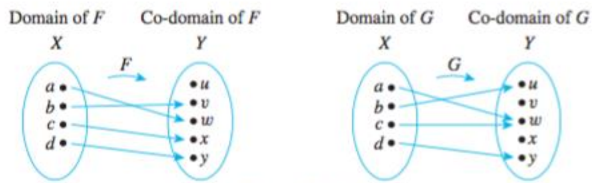

Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

(a) F is one-to-one but G is not. F is one-to-one because no two different elements of X are sent by F to the same element of Y. G is not one-to-one because the elements a and c are both sent by G to the same element of Y: $G(a) = G(c) = w$ but $a \neq c$.

(b) H is one-to-one but K is not. H is one-to-one because each of the three elements of the domain of H is sent by H to a different element of the co-domain: $H(1) \neq H(2)$, $H(1) \neq H(3)$, and $H(2) \neq H(3)$. K, however, is not one-to-one because $K(1) = K(3) = d$ but $1 \neq 3$.

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Proving/Disproving Functions are One-to-One

To prove f is one-to-one (Direct Method):

suppose x_1 and x_2 are elements of $X \mid f(x_1) = f(x_2)$, and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one:

Find elements x_1 and x_2 in X so $f(x_1)=f(x_2)$ but $x_1 \neq x_2$.

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Proving/Disproving Functions are One-to-One
Example 1

Define f : $\mathbf{R}\to\mathbf{R}$ by the rule
$f(x) = 4x-1$ for all $x\in\mathbf{R}$

Is f one-to-one? Prove or give a counterexample.

Proving/Disproving Functions are One-to-One
Example 1

Define f : $\mathbf{R}\to\mathbf{R}$ by the rule
$f(x) = 4x-1$ for all $x\in\mathbf{R}$

Is f one-to-one? Prove or give a counterexample.

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$.
[We must show that $x_1 = x_2$] By definition of f,

$4x_1 - 1 = 4x_2 - 1$. Adding 1 to both sides gives

$4x_1 = 4x_2$, and dividing both sides by 4 gives

$x_1 = x_2$, which is what was to be shown.

Proving/Disproving Functions are One-to-One
Example 2

Define $g : \mathbf{Z} \to \mathbf{Z}$ by the rule

$$g(n) = n^2 \qquad \text{for all } n \in \mathbf{Z}.$$

Is g one-to-one? Prove or give a counterexample.

Proving/Disproving Functions are One-to-One
Example 2

Define $g : \mathbf{Z} \to \mathbf{Z}$ by the rule

$$g(n) = n^2 \qquad \text{for all } n \in \mathbf{Z}.$$

Is g one-to-one? Prove or give a counterexample.

Counterexample:
Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \text{ and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $g(n_1) = g(n_2)$ but $n_1 \neq n_2$,
and so g is not one-to-one.

Proving/Disproving Functions are One-to-One
Example 3

Define g : **MobileNumber** → **People** by the rule
$g(x) = Person$ for all x∈**MobileNumber**

Is g one-to-one? Prove or give a counterexample.

Counter example:
0599123456 and 0569123456 are both for Sami

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Proving/Disproving Functions are One-to-One
Example 4

Define g : **Fingerprints** → **People** by the rule
$g(x) = Person$ for all x∈**R** **Fingerprint**

Is g one-to-one? Prove or give a counterexample.

Prove:
In biology and forensic science: "The flexibility of friction ridge skin means that no two finger or palm prints are ever exactly alike in every detail" [w].

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Functions

7.2 Properties of Functions

In this lecture:

Onto Functions

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

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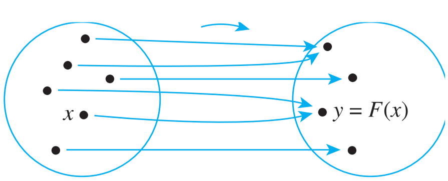

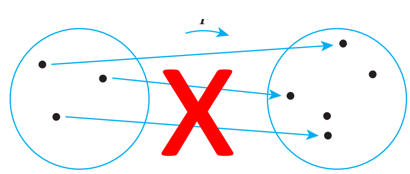

Onto Functions

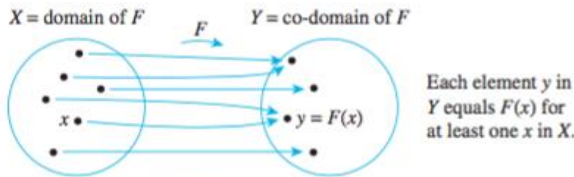

Figure 7.2.3(a) A Function That Is Onto

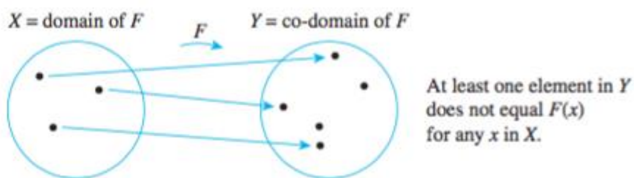

Figure 7.2.3(b) A Function That Is Not Onto

Onto Functions

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

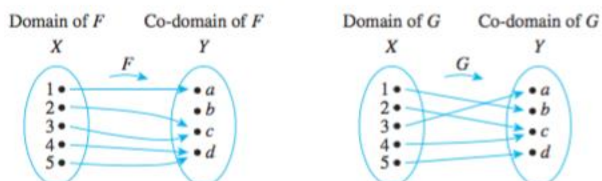

Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

Onto Functions

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

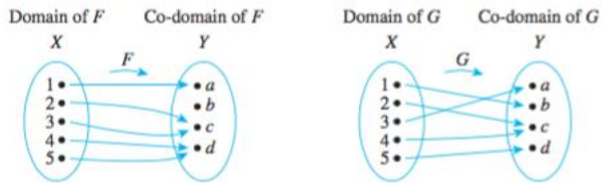

Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

(a) F is not onto because $b \neq F(x)$ for any x in X. G is onto because each element of Y equals $G(x)$ for some x in X: $a = G(3)$, $b = G(1)$, $c = G(2) = G(4)$, and $d = G(5)$.

(b) H is onto but K is not. H is onto because each of the three elements of the co-domain of H is the image of some element of the domain of H: $a = H(2)$, $b = H(4)$, and $c = H(1) = H(3)$. K, however, is not onto because $a \neq K(x)$ for any x in $\{1,2,3,4\}$.

Proving/Disproving Functions are Onto

To prove F is onto, (**method of generalizing from the generic particular**)

suppose that y is any element of Y

show that there is an element x of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y | $y \neq F(x)$ for *any* x in X.

Proving/Disproving Functions are Onto
Example 1

Define f : $\mathbf{R} \to \mathbf{R}$

$f(x) = 4x - 1$ for all $x \in \mathbf{R}$

Is f onto? Prove or give a counterexample.

Proving/Disproving Functions are Onto
Example 1

Define f : $\mathbf{R} \to \mathbf{R}$

$f(x) = 4x - 1$ for all $x \in \mathbf{R}$

Is f onto? Prove or give a counterexample.

Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in $\mathbf{R}$ such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$f(x) = f\left(\frac{y+1}{4}\right) \quad \text{by substitution}$$

$$= 4 \cdot \left(\frac{y+1}{4}\right) - 1 \quad \text{by definition of } f$$

$$= (y+1) - 1 = y \quad \text{by basic algebra.}$$

[This is what was to be shown.]

Proving/Disproving Functions are Onto
Example 2

Define $h : \mathbf{Z} \to \mathbf{Z}$ by the rules

$$h(n) = 4n - 1 \qquad \text{for all } n \in \mathbf{Z}.$$

Is h onto? Prove or give a counterexample.

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Proving/Disproving Functions are Onto
Example 2

Define $h : \mathbf{Z} \to \mathbf{Z}$ by the rules

$$h(n) = 4n - 1 \qquad \text{for all } n \in \mathbf{Z}.$$

Is h onto? Prove or give a counterexample.

Counterexample:

The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer n. For if $h(n) = 0$, then

$$4n - 1 = 0 \qquad \text{by definition of } h$$

which implies that

$$4n = 1 \qquad \text{by adding 1 to both sides}$$

and so

$$n = \frac{1}{4} \qquad \text{by dividing both sides by 4.}$$

But $1/4$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

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Proving/Disproving Functions are Onto
Example 3

Define g : **MobileNumber** → **People** by the rule
$g(x) = Person$ for all $x \in$ **MobileNumber**

Is g onto? Prove or give a counterexample.

Counter example:
Sami does not have a mobile number

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Proving/Disproving Functions are Onto
Example 4

Define g : **Fingerprints** → **People** by the rule
$g(x) = Person$ for all $x \in$ **Fingerprint**

Is g onto? Prove or give a counterexample.

Prove:
In biology and forensic science: there is no person without fingerprint

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Functions

7.2 Properties of Functions

In this lecture:

- ❑ Part 1: One-to-one Functions
- ❑ Part 2: Onto Functions
- ❑ Part 3: **one-to-one Correspondence Functions**
- ❑ Part 4: Inverse Functions
- ❑ Part 5: Applications: Hash and Logarithmic Functions

One-to-One Correspondences

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

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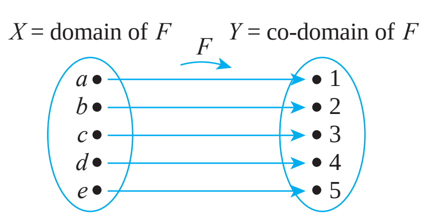

String-Reversing Function

Let T be the set of all finite strings of x's and y's. Define

$g : T \rightarrow T$ by the rule: For all strings $s \in T$,

$g(s)$ = the string obtained by writing the characters of s in reverse order. E.g., g("Ali") = "ilA"

Is g a one-to-one correspondence from T to itself?

1. We have to show that if g is **one-to-one** & **onto**

(a) one-to-one: suppose that for some strings $s1$ and $s2$ in T, $g(s1) = g(s2)$. *[We must show that* $s1 = s2$*.]* Now to say that $g(s1) = g(s2)$ is the same as saying that the string obtained by writing the characters of $s1$ in reverse order equals the string obtained by writing the characters of $s2$ in reverse order. But if $s1$ and $s2$ are equal when written in reverse order, then they must be equal to start with. In other words, $s1 = s2$ *[as was to be shown]*.

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String-Reversing Function

Let T be the set of all finite strings of x's and y's. Define

$g : T \rightarrow T$ by the rule: For all strings $s \in T$,

$g(s)$ = the string obtained by writing the characters of s in reverse order. E.g., g("Ali") = "ilA"

(b) onto: suppose t is a string in T. *[We must find a string s in T such that* $g(s) = t$*.]* Let $s = g(t)$. By definition of g, $s = g(t)$ is the string in T obtained by writing the characters of t in reverse order. But when the order of the characters of a string is reversed once and then reversed again, the original string is recovered. Thus

$g(s) = g(g(t))$ = the string obtained by writing the characters of t in reverse order and then writing those characters in reverse order again

$= t$.

This is what was to be shown.

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Functions

7.2 Properties of Functions

In this lecture:

- Part 1: One-to-one Functions
- Part 2: Onto Functions
- Part 3: one-to-one Correspondence Functions
- Part 4: **Inverse Functions**
- Part 5: Applications: Hash and Logarithmic Functions

Inverse Functions

Theorem 7.2.2

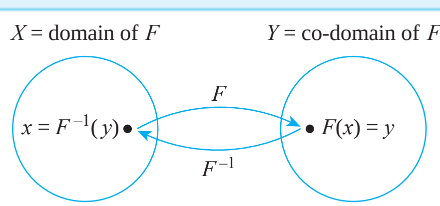
Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows:

Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$



➔ Is it always that the inverse of a function is a function?

Inverse Functions

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows:

Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \iff y = F(x).$$

Finding Inverse Functions

The function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1 \text{ for all real numbers } x$$

(was shown one-to-one and onto)

Find its inverse function?

Finding Inverse Functions

The function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by the formula
$f(x) = 4x-1$ for all real numbers x

(was shown one-to-one and onto)
Find its inverse function?

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$
\begin{aligned}
& f(x) = y \\
\Leftrightarrow\quad & 4x - 1 = y && \text{by definition of } f \\
\Leftrightarrow\quad & x = \frac{y+1}{4} && \text{by algebra.}
\end{aligned}
$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

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Functions

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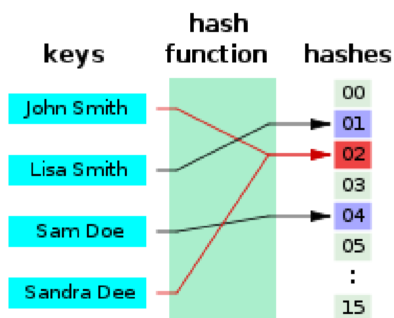
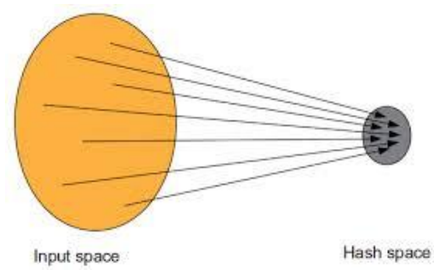
- ❑ Part 4: Inverse Functions
- ❑ Part 5: **Applications: Hash and Logarithmic Functions**

Hash Functions

- Maps data of arbitrary length to data of a fixed length.
- Very much used in databases and security





Hash Functions

How to store long (ID numbers) for a small set of people

For example: $\boldsymbol{n}$ is an ID number, and $\boldsymbol{m}$ is number of people we have

$$\text{Hash}(n) = n \bmod m$$

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

collision?

Exponential and Logarithmic Functions

$$\text{Log}_b\, x = y \iff b^y = x$$

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Relations between Exponential and Logarithmic Functions

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v} \quad 7.2.1$$

$$(b^u)^v = b^{uv} \quad 7.2.2$$

$$\frac{b^u}{b^v} = b^{u-v} \quad 7.2.3$$

$$(bc)^u = b^u c^u \quad 7.2.4$$

The exponential and logarithmic functions are one-to-one and onto. Thus the following properties hold:

For any positive real number b with $b \neq 1$,

if $b^u = b^v$ then $u = v$ for all real numbers u and v, 7.2.5

and

if $\log_b u = \log_b v$ then $u = v$ for all positive real numbers u and v. 7.2.6

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Relations between Exponential and Logarithmic Functions

We can derive additional facts about exponents and logarithms, e.g.:

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a\log_b x$

d. $\log_c x = \frac{\log_b x}{\log_b c}$

How to prove this?

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Using the One-to-Oneness of the Exponential Function

Prove that:
$$\log_c x = \frac{\log_b x}{\log_b c}.$$

Solution Suppose positive real numbers b, c, and x are given. Let

(1) $u = \log_b c$ (2) $v = \log_c x$ (3) $w = \log_b x$.

Then, by definition of logarithm,

(1′) $c = b^u$ (2′) $x = c^v$ (3′) $x = b^w$.

Substituting (1′) into (2′) and using one of the laws of exponents gives

$$x = c^v = (b^u)^v = b^{uv}$$ by 7.2.2

But by (3), $x = b^w$ also. Hence

$$b^{uv} = b^w,$$

and so by the one-to-oneness of the exponential function (property 7.2.5),

$$uv = w.$$

Substituting from (1), (2), and (3) gives that

$$(\log_b c)(\log_c x) = \log_b x.$$

And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

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