# Relations

## 8.1. Introduction to Relations

## 8.2 Properties of Relations

## 8.3 Equivalence Relations

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http://jarrar-courses.blogspot.com/2014/03/discrete-mathematics-course.html

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**Acknowledgement:**
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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# Relations

## 8.1 Introduction to Relations

**In this lecture:**

- Part 1: **What is a Relation**
- Part 2: Inverse of a Relation;
- Part 3: Directed Graphs;
- Part 4: n-ary Relations,
- Part 5: Relational Databases

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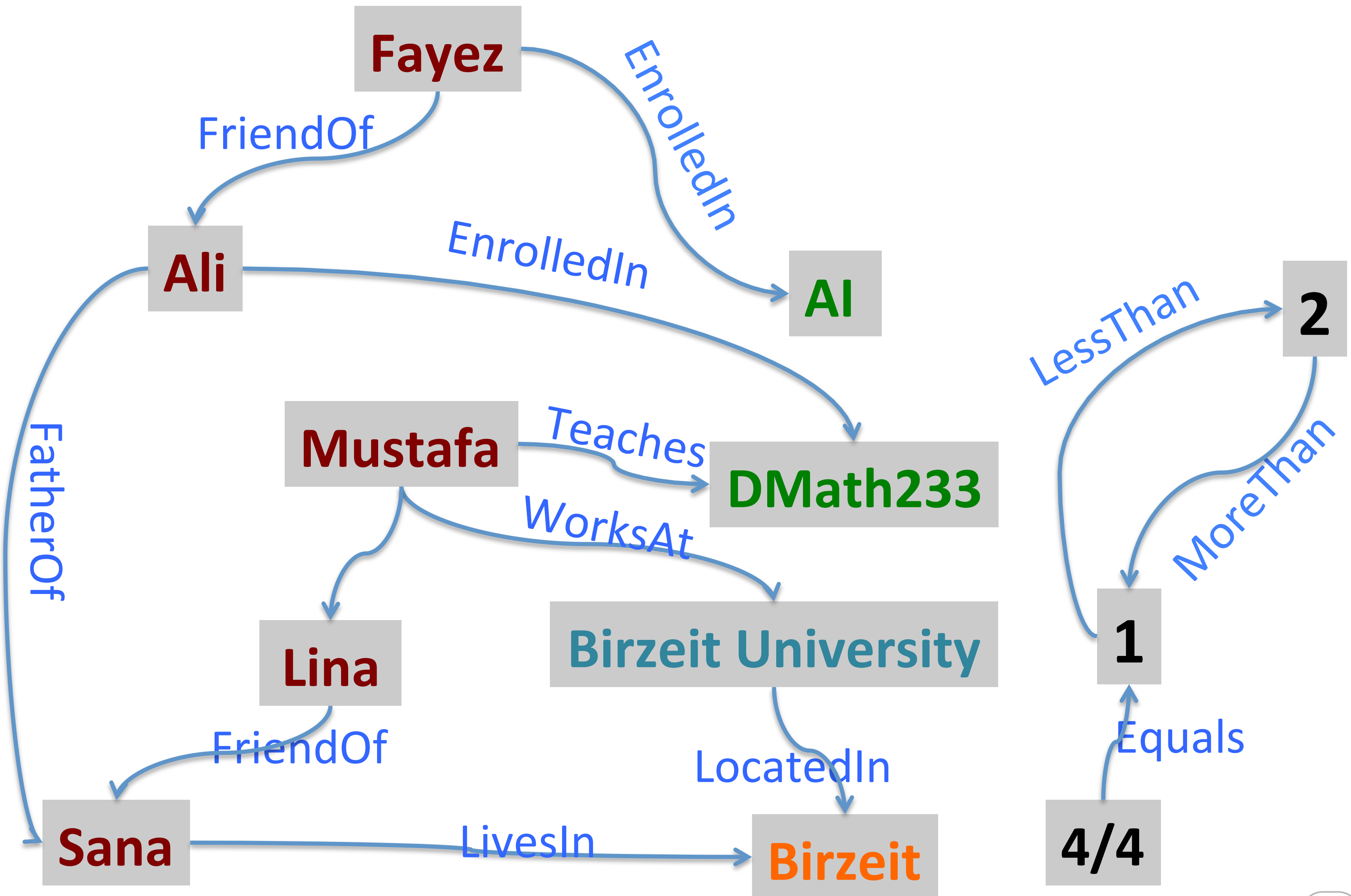
# What is a Relation?



# What is a Relation?

**• Definition**

Let $A$ and $B$ be sets. A **(binary) relation *R* from *A* to *B*** is a subset of $A \times B$. Given an ordered pair $(x, y)$ in $A \times B$, ***x* is related to *y* by *R***, written $x \; R \; y$, if, and only if, $(x, y)$ is in $R$.

$$x \; R \; y \quad \Leftrightarrow \quad (x, y) \in R$$

$$x \not R \; y \quad \Leftrightarrow \quad (x, y) \notin R$$

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# Example

## The Less-than Relation for Real Numbers

Define a relation $L$ from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers $x$ and $y$,

$$x\ L\ y \Leftrightarrow x < y.$$

a. Is $57\ L\ 53$? b. Is $(-17)\ L\ (-14)$? c. Is $143\ L\ 143$? d. Is $(-35)\ L\ 1$?

,

# Example

## The Less-than Relation for Real Numbers

Define a relation $L$ from **R** to **R** as follows: For all real numbers $x$ and $y$,

$$x \; L \; y \Leftrightarrow x < y.$$

a. Is $57 \; L \; 53$? b. Is $(-17) \; L \; (-14)$? c. Is $143 \; L \; 143$? d. Is $(-35) \; L \; 1$?

a. No, $57 > 53$ b. Yes, $-17 < -14$ c. No, $143 = 143$ d. Yes, $-35 < 1$

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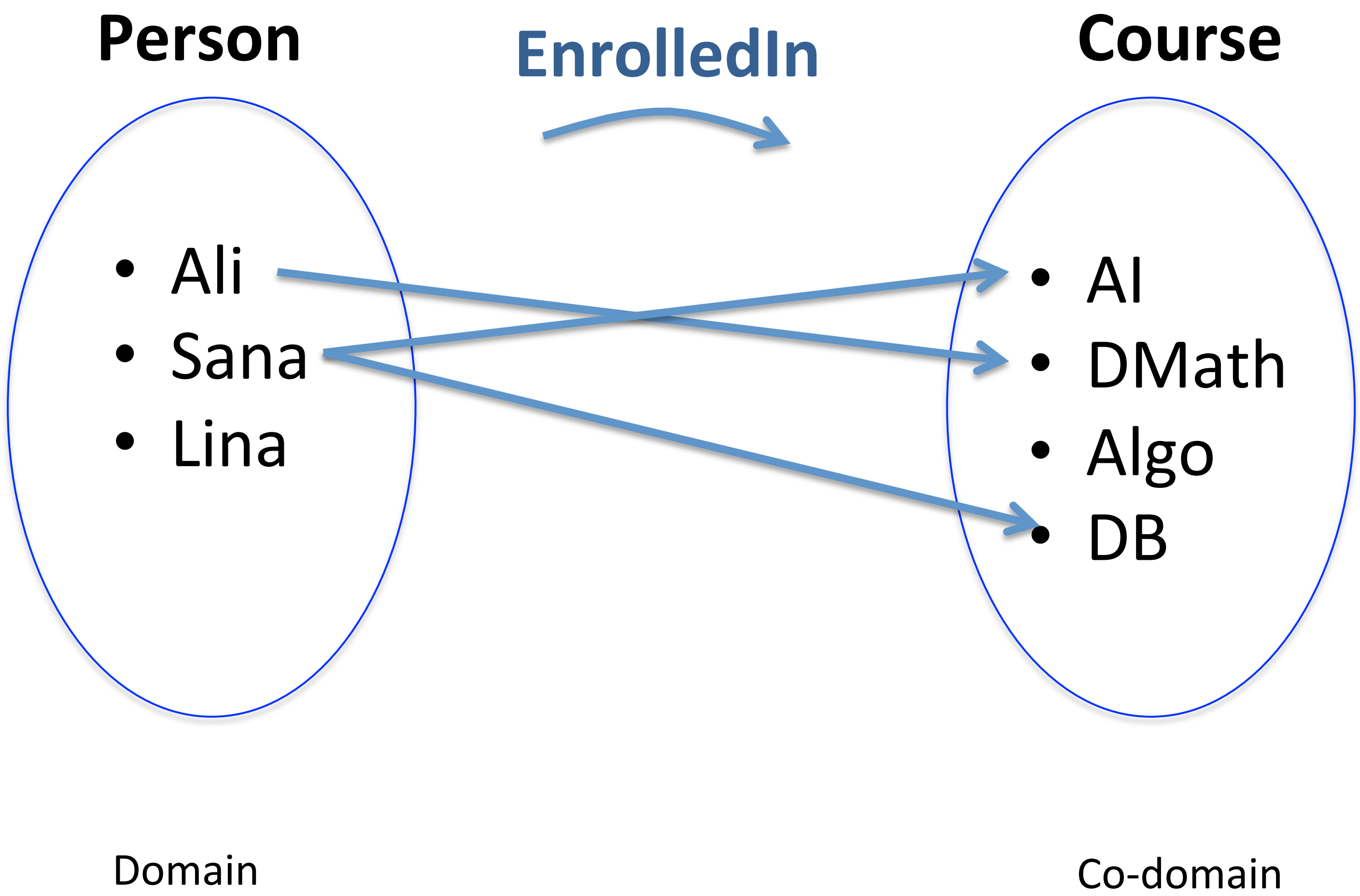
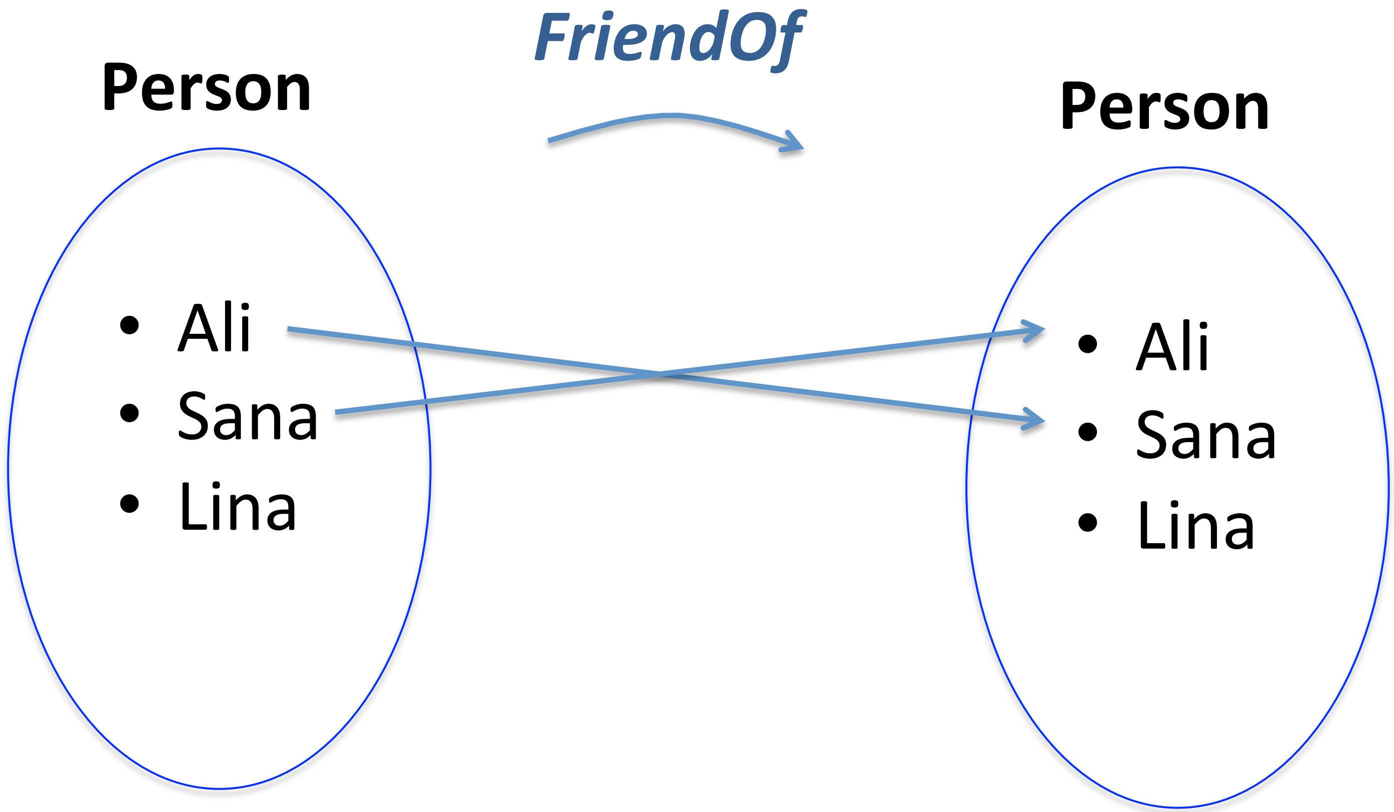
# Example

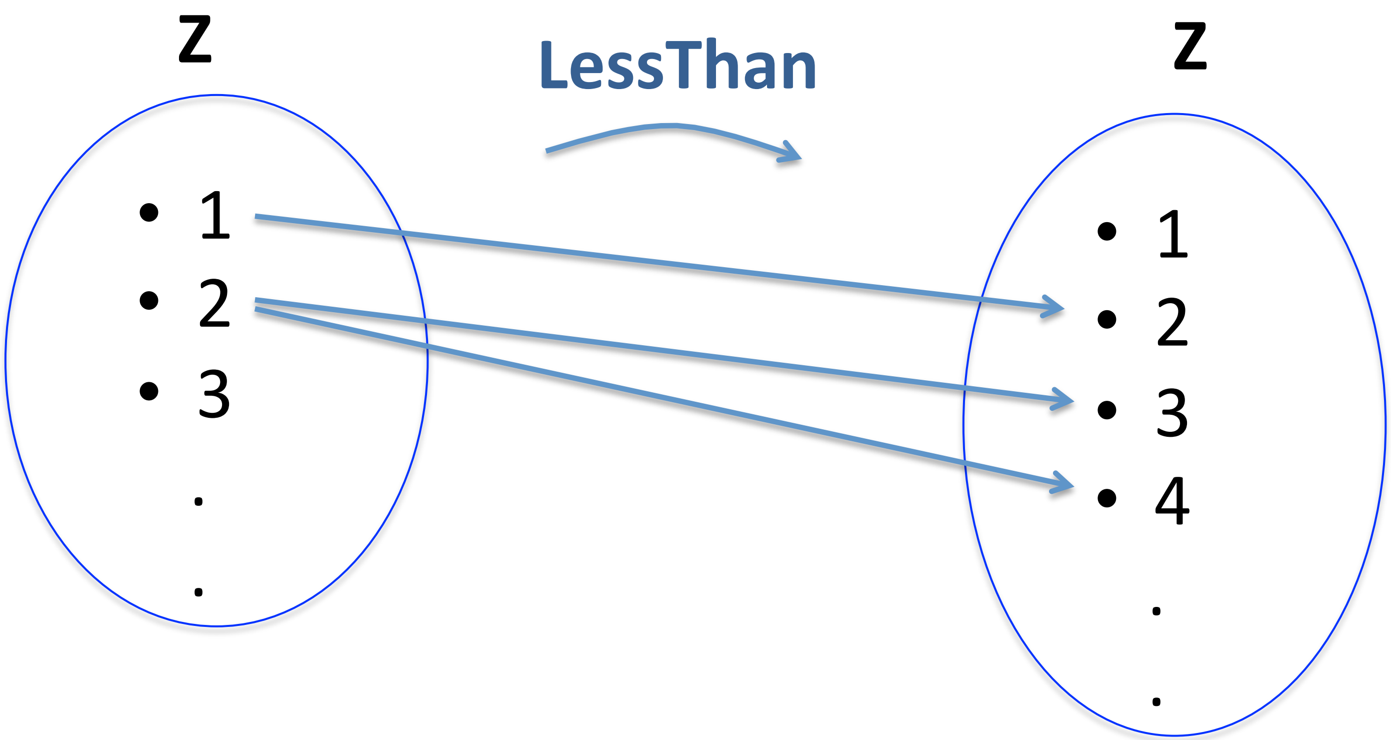

Person
EnrolledIn
Course
Ali
Sana
Lina
AI
DMath
Algo
DB
Domain
Co-domain

# Example




$$FriendOf^{I} = \{(\text{Ali}, \text{Sana}), (\text{Sana},\text{Ali})\}$$

# Example

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# Example

Define a relation $E$ from **Z** to **Z** as follows: For all $(m,n) \in \mathbf{Z} \times \mathbf{Z}$, $mEn \Leftrightarrow m-n$ is even.

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by $E$ to 1.

c. Prove that if $n$ is any odd integer, then $n\ E\ 1$.

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# Example

Define a relation $E$ from **Z** to **Z** as follows: For all $(m,n) \in \mathbf{Z} \times \mathbf{Z}$, $mEn \Leftrightarrow m-n$ is even.

a. Is 4 $E$ 0? Is 2 $E$ 6? Is 3 $E$ (−3)? Is 5 $E$ 2?

b. List five integers that are related by $E$ to 1.

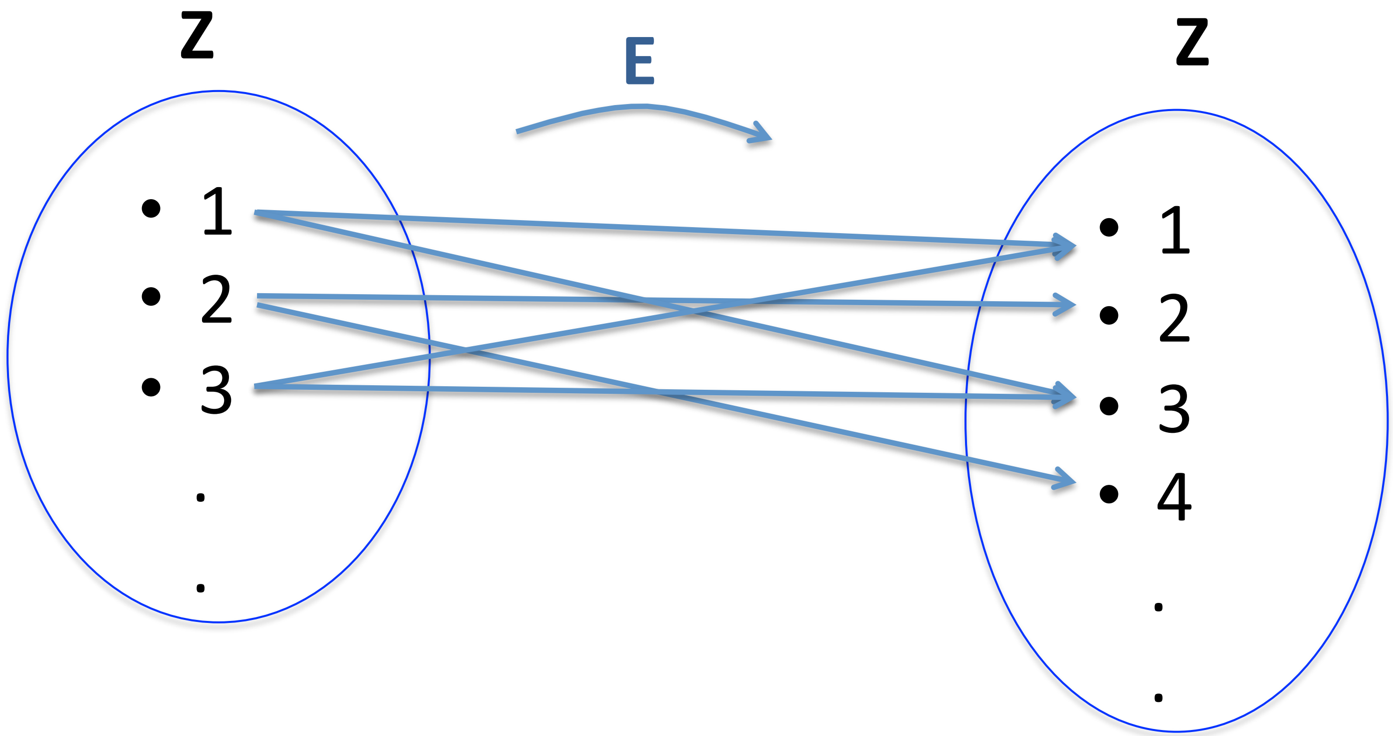
c. Prove that if $n$ is any odd integer, then $n$ $E$ 1.

a. Yes, 4 $E$ 0 because $4-0=4$ and 4 is even. Yes, 2 $E$ 6 because $2-6=-4$ and −4 is even. Yes, 3 $E$ (−3) because $3-(-3)=6$ and 6 is even. No, 5 $E$ 2 because $5-2=3$ and 3 is not even.

b. 1 because $1-1=0$ is even, 3 because $3-1=2$ is even, 5 because $5-1=4$ is even, −1 because $-1-1=-2$ is even, −3 because $-3-1=-4$ is even.

c. Suppose $n$ is any odd integer. Then $n = 2k + 1$ for some integer $k$. By definition of $E$, $n$ $E$ 1 if, and only if, $n - 1$ is even. By substitution, $n - 1 = (2k + 1) - 1 = 2k$, and since $k$ is an integer, $2k$ is even. Hence $n$ $E$ 1 *[as was to be shown]*.

# Example




Define a relation $E$ from $\mathbf{Z}$ to $\mathbf{Z}$ as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m \; E \; n \Leftrightarrow m-n$ is even.

# Example: a relation on a Power Set

Let $X = \{a, b, c\}$. Then $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation **S** from $P(X)$ to **Z** as follows: For all sets $A$ and $B$ in $P(X)$ (i.e., for all subsets $A$ and $B$ of $X$),

$$A \text{ S } B \Leftrightarrow A \text{ has at least as many elements as } B.$$

a. Is $\{a,b\}$ S $\{b,c\}$?

b. Is $\{a\}$ S $\emptyset$?

c. Is $\{b,c\}$ S $\{a,b,c\}$?

d. Is $\{c\}$ S $\{a\}$?

# Example: a relation on a Power Set

Let $X = \{a, b, c\}$. Then $P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define a relation **S** from $P(X)$ to $\mathbf{Z}$ as follows: For all sets $A$ and $B$ in $P(X)$ (i.e., for all subsets $A$ and $B$ of $X$),

$$A \text{ S } B \Leftrightarrow A \text{ has at least as many elements as } B.$$

✓ a. Is $\{a,b\}$ S $\{b,c\}$? Yes, both sets have two elements.

✓ b. Is $\{a\}$ S $\varnothing$? Yes, $\{a\}$ has one element and $\varnothing$ has zero elements, and $1 \geq 0$.

✗ c. Is $\{b,c\}$ S $\{a,b,c\}$? No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$.

✓ d. Is $\{c\}$ S $\{a\}$? Yes, both sets have one element.

# Example

Let A = {1, 2, 3} and B = {1, 3, 5}, define relations **S** and **T** from A to *B* as follows: For all $(x, y) \in A \times B$,

$(x, y) \in S \Leftrightarrow x < y$ S is a “LessThan” relation.

$T = \{(2, 1), (2, 5)\}$.

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# Example

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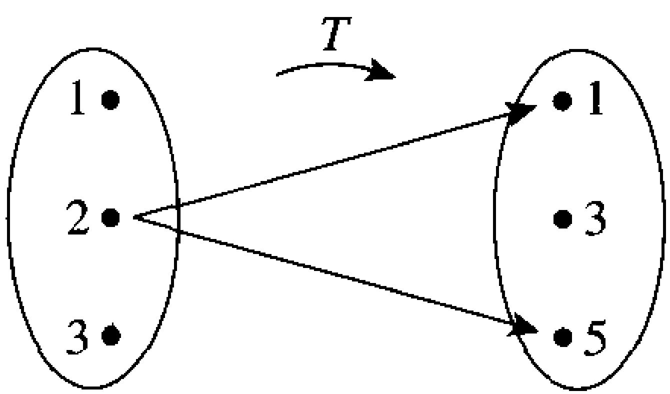


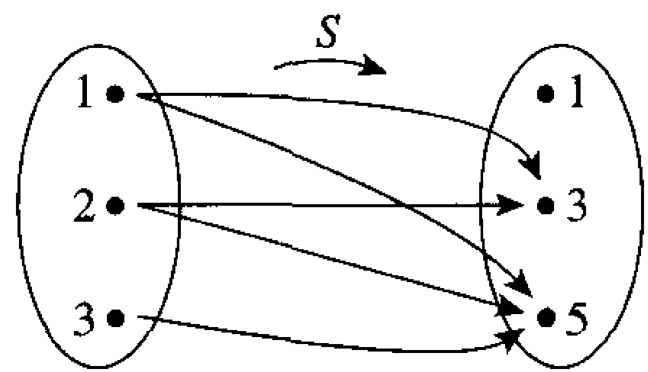


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# Relations and Functions

- **Definition**

A **function $F$ from a set $A$ to a set $B$** is a relation from $A$ to $B$ that satisfies the following two properties:

1. For every element $x$ in $A$, there is an element $y$ in $B$ such that $(x, y) \in F$.
2. For all elements $x$ in $A$ and $y$ and $z$ in $B$,

$$\text{if } (x, y) \in F \text{ and } (x, z) \in F, \text{ then } y = z.$$

If $F$ is a function from $A$ to $B$, we write

$$y = F(x) \quad \Leftrightarrow \quad (x, y) \in F.$$

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# Example

**Let A = {2 , 4, 6} and B = {1, 3 , 5}.**
**Is relation R a function from A to *B*?**

$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}.$$

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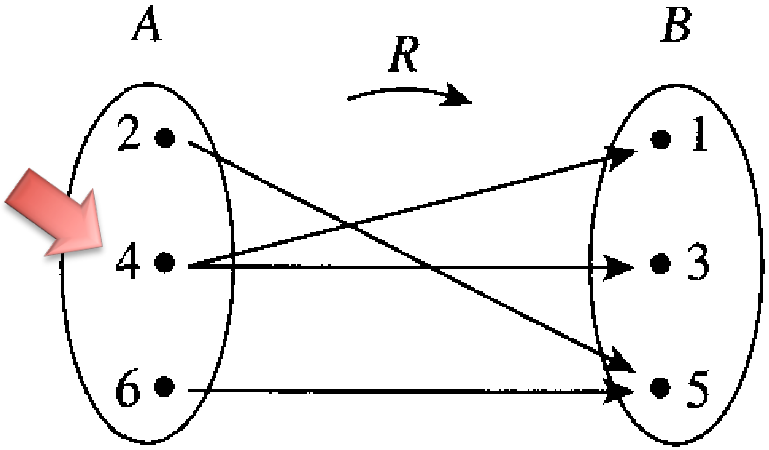
# Example

**Let A = {2 , 4, 6} and B = {1, 3 , 5}.**
**Is relation R a function from A to *B?***

$$R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}.$$ ✗

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# Example

**Let A = {2 , 4, 6} and B = {1, 3 , 5}.**
**Is relation R a function from A to *B*?**

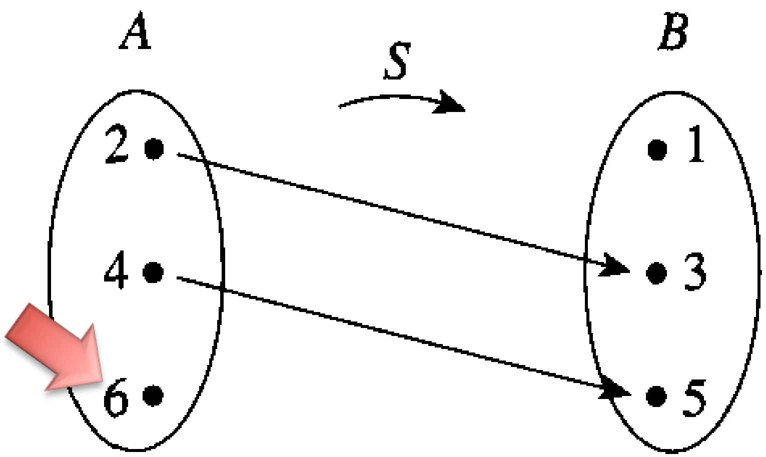
For all $(x,y) \in A \times B$, $(x,y) \in S$ $y=x+1$.

,

# Example

**Let A = {2 , 4, 6} and B = {1, 3 , 5}.**
**Is relation R a function from A to *B*?**

For all $(x,y) \in A \times B$, $(x,y) \in S$ $y=x+1$. ✗

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# Relations

## 8.1 Introduction to Relations

**In this lecture:**

- ❑ Part 1: What is a Relation
- ❑ Part 2: **Inverse of a Relation**
- ❑ Part 3: Directed Graphs
- ❑ Part 4: *n*-ary Relations
- ❑ Part 5: Relational Databases

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# Inverse Relation

**Definition**

Let $R$ be a relation from $A$ to $B$. Define the inverse relation $R^{-1}$ from $B$ to $A$ as follows:

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

For all $x \in A$ and $y \in B$, $(y,x) \in R^{-1} \Leftrightarrow (x,y) \in R$.



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# Example

Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let $R$ be the "divides" relation from $A$ to $B$: For all $(x, y) \in A \times B$,

$$x\ R\ y \Leftrightarrow x \mid y \qquad x \text{ divides } y.$$

a. State explicitly which ordered pairs are in $R$ and $R{-}1$, and draw arrow diagrams for $R$
and $R{-}1$
b. Describe $R{-}1$ in words.



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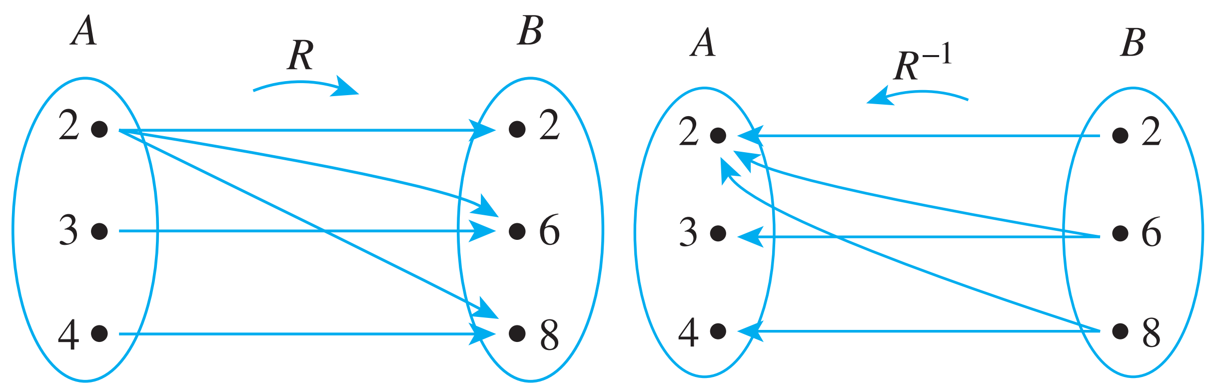
# Example

Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let $R$ be the "divides" relation from $A$ to $B$: For all $(x, y) \in A \times B$,

$$x\ R\ y \Leftrightarrow x \mid y \qquad x \text{ divides } y.$$

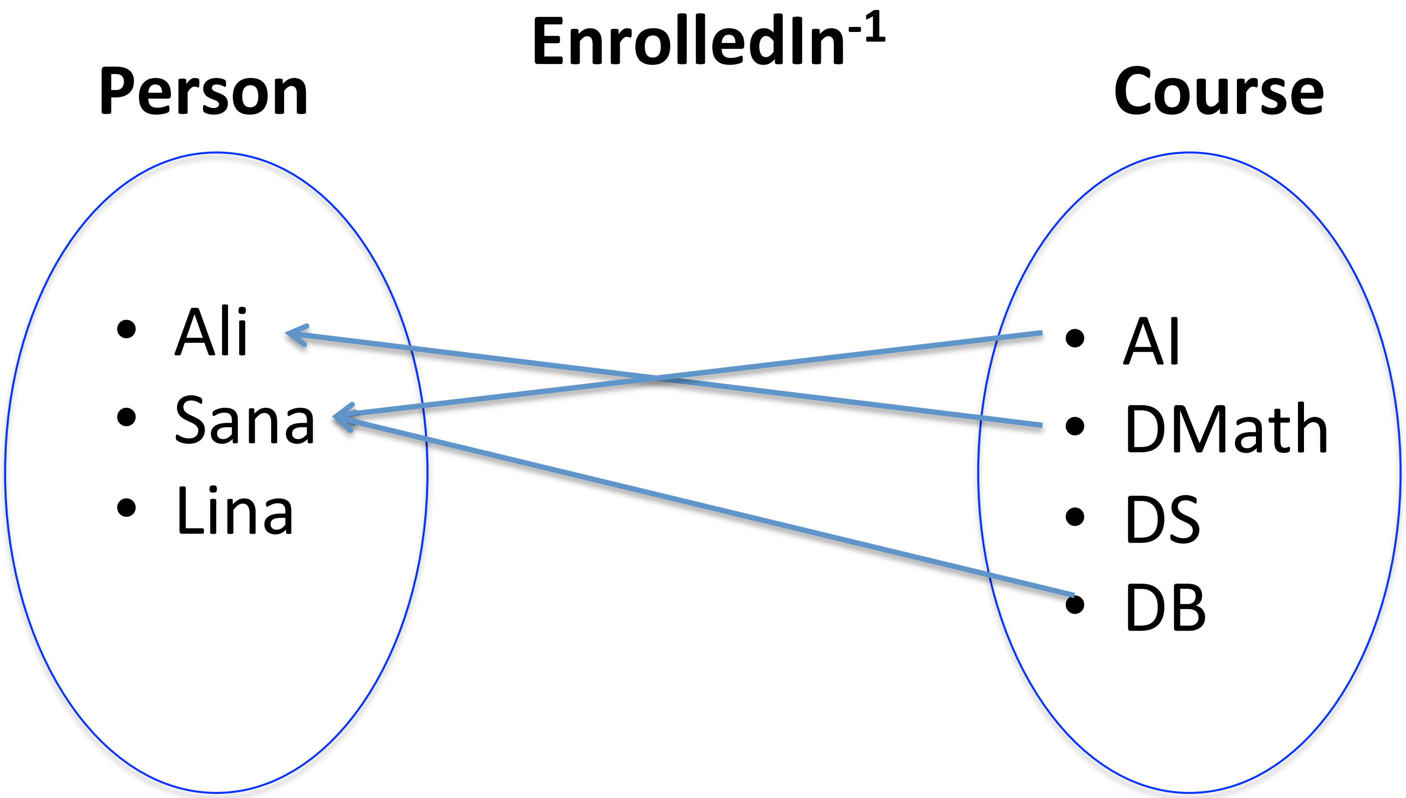
a. State explicitly which ordered pairs are in $R$ and $R-1$, and draw arrow diagrams for $R$
and $R^{-1}$

$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}$
$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}$




b. Describe $R-1$ in words.
For all $(y, x) \in B \times A$, y $R^{-1}$ $x \Leftrightarrow y$ is a multiple of $x$.

# Example

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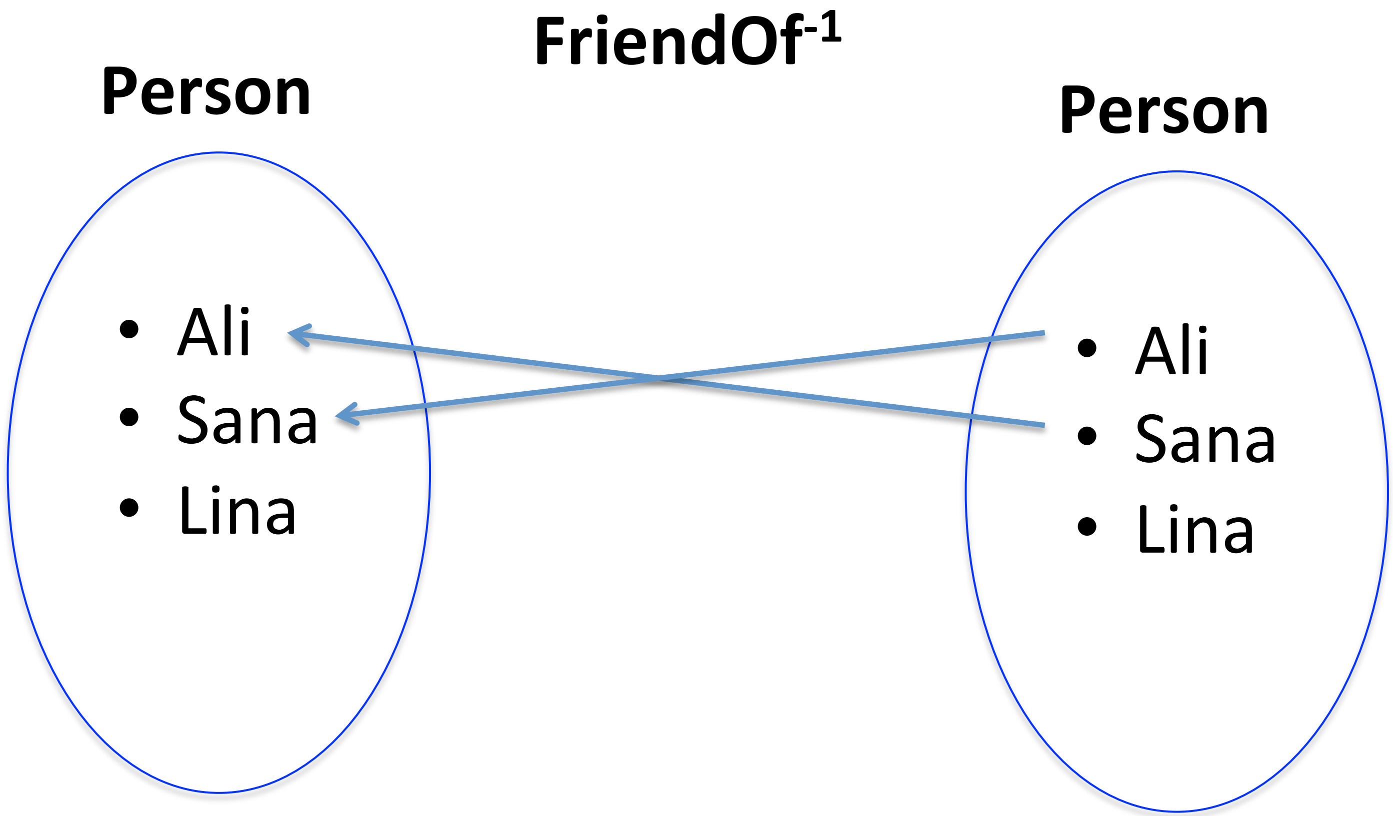
# Example

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# Inverse of Relations in Language

What would be the inverse of the following relations in English

SonOf $^{-1}$ = ?

WifeOf $^{-1}$ = ?

WorksAt $^{-1}$ = ?

EnrolledOf $^{-1}$ = ?

PresidentOf $^{-1}$ = ?

BrotherOf $^{-1}$ = ?

SisterOf $^{-1}$ = ?

….

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- ❑ Part 3: **Directed Graphs**
- ❑ Part 4: *n*-ary Relations
- ❑ Part 5: Relational Databases

# Directed Graph of a Relation

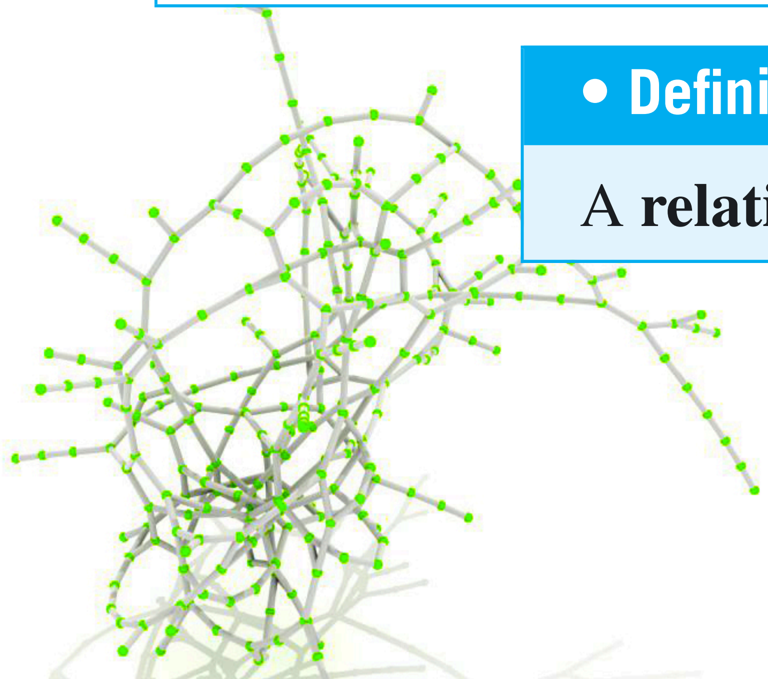
When a relation $R$ is defined *on* a set $A$, the arrow diagram of the relation can be modified so that it becomes a **directed graph.**

For all points $x$ and $y$ in $A$,

$$\text{there is an arrow from } x \text{ to } y \quad \Leftrightarrow \quad x\ R\ y \quad \Leftrightarrow \quad (x, y) \in R.$$

- **Definition**

A **relation on a set** $A$ is a relation from $A$ to $A$.



It is important to distinguish clearly between a relation and the set on which it is defined.

# Example

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation $R$ on $A$ as follows: For all $x, y \in A$, $x \, R \, y \Leftrightarrow 2 \mid (x-y)$.

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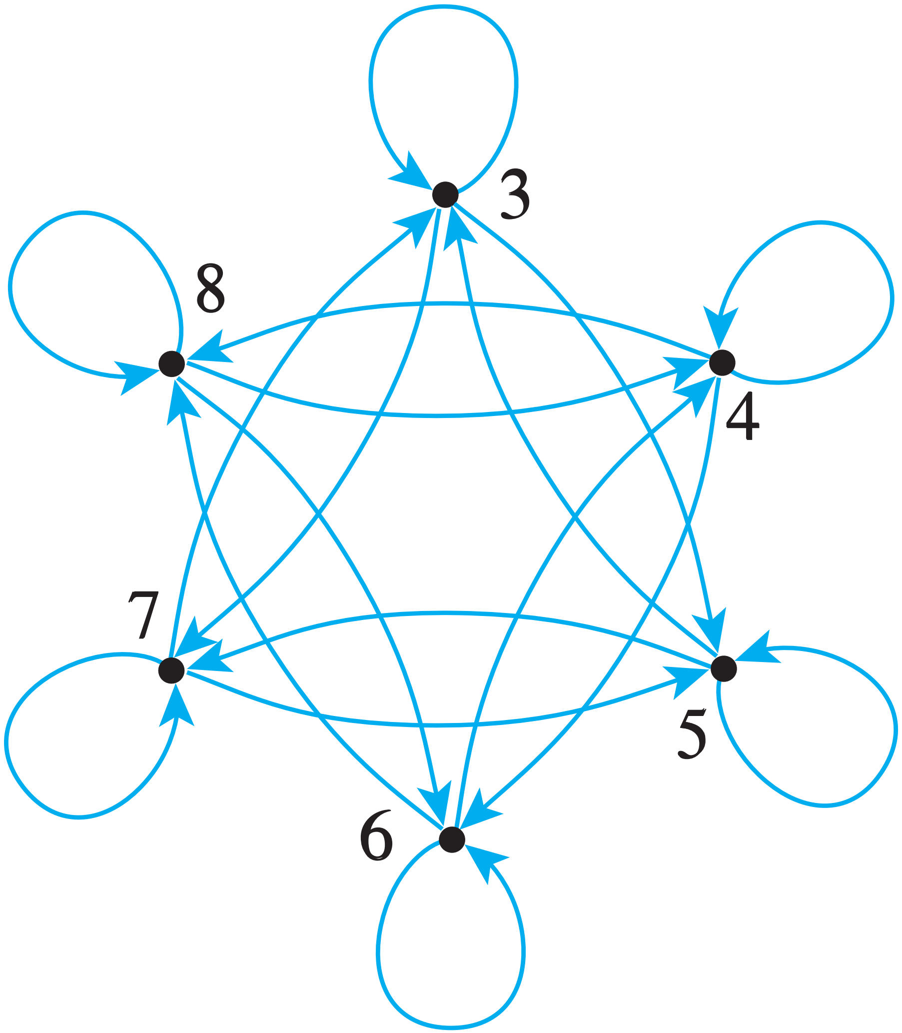
# Example

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation $R$ on $A$ as follows: For all $x$, $y \in A$, $x\ R\ y \Leftrightarrow 2 \mid (x-y)$.

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# Relations

## 8.1 Introduction to Relations

**In this lecture:**

- Part 1: What is a Relation
- Part 2: Inverse of a Relation
- Part 3: Directed Graphs
- Part 4: ***n*-ary Relations**
- Part 5: Relational Databases

# N-ary Relations

| | |
|---|---|
| EnrolledIn(Ali, Dmath)<br>EnrolledIn(Sami, DB) | Binary (2-ary) |
| Enrollment(Sami, DB, 99) | Ternary (3-ary) |
| Enrollment(Sami, DB, 99, 2014) | Quaternary (4-ary) |
| Enrollment(Sami, DB, 99, 2014,F) | 5-ary |
| R($a_1$, $a_2$, $a_3$, ….., $a_n$) | *n*-ary |

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# N-ary Relations

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** $R$ on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

# Relations

## 8.1 Introduction to Relations

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- ❑ Part 5: **Relational Databases**

# Relational Databases

Let $A1$ be a set of positive integers, $A2$ a set of alphabetic character strings, $A3$ a set of numeric character strings, and $A4$ a set of alphabetic character strings. Define a quaternary relation $R$ on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Patient(ID, Name, Date, **Diagnosis**)

(011985, John Schmidt, 020710, asthma)

(574329, Tak Kurosawa, 114910, pneumonia)

(466581, Mary Lazars, 103910, appendicitis)

(008352, Joan Kaplan, 112409, gastritis)

(011985, John Schmidt, 021710, pneumonia)

(244388, Sarah Wu, 010310, broken leg)

(778400, Jamal Baskers, 122709, appendicitis)



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# Relational Databases

$R$ on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Relation

Each row is called **tuple**

| Patient | | | |
|---|---|---|---|
| **ID** | **Name** | **Date** | **Diagnosis** |
| (011985, | John Schmidt, | 020710, | asthma) |
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| (466581, | Mary Lazars, | 103910, | appendicitis) |
| (008352, | Joan Kaplan, | 112409, | gastritis) |
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# Relational Databases

$R$ on $A1 \times A2 \times A3 \times A4$ as follows:

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Relation

| Patient | | | |
|---|---|---|---|
| ID | Name | Date | Diagnosis |
| (0 | | | |
| (5 | | | |
| (4 | | | |
| (0 | | | |
| (0 | | | |
| (2 | | | |
| (778400, | Jamal Baskers, | 122709, | appendicitis) |

Each row is called **tuple**

- Notice that **Tables** in this way are called **Relations**.
- Information stored in this way is called a "**Relational Database**"

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