Relations

8.1. Introduction to Relations

8.2 Properties of Relations

8.3 Equivalence Relations

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

Relations

8.2 Properties of Relations

In this lecture:

- ❑ Part 1: **Properties: Reflexivity, Symmetry, Transitivity**
- ❑ Part 2: Proving Properties of Relations
- ❑ Part 3: Transitive Closure

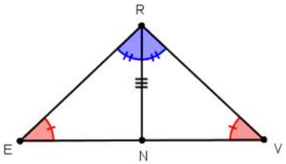
Reflexivity, Symmetry, and Transitivity

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x \, R \, x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, ***if*** $x \, R \, y$ then $y \, R \, x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, ***if*** $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

Because of the equivalence of the expressions $x \, R \, y$ and $(x, y) \in R$ for all x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive	$\Leftrightarrow$	for all x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.



Reflexivity تناظر

R is reflexive $\Leftrightarrow$ for all x in A, $(x, x) \in R$.

R is Reflexive: Each element is related to itself.

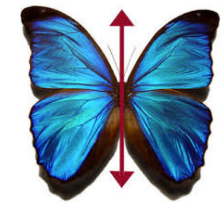
علاقة ثنائية على مجموعة ما ، وكل عنصر في المجموعة مرتبط بنفسه في إطار هذه العلاقة.

R is not reflexive: there is an element x in A such that $x \not R x$ [*that is, such that* $(x, x) \notin R$] .

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

Symmetry تماثل



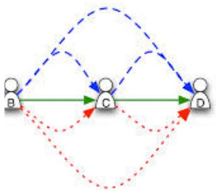
R is symmetric $\Leftrightarrow$ for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.

R is Symmetric: If any one element is related to any other element, then the second is related to the first.

***R* is not Symmetric:** there are elements x and y in A such that x R y but y ~~R~~ x *[that is, such that* $(x, y) \in R$ *but* $(y, x) \notin R$*]*.

Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?



Transitivity تعدي

R is transitive ⇔ for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

R is Transitive: If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

***R* is not transitive:** there are elements x,y and z in A such that xRy and yRz but x ~~R~~ z *[that is, such that $(x,y) \in R$ and $(y,z) \in R$ but $(x , z) \notin R$]* .

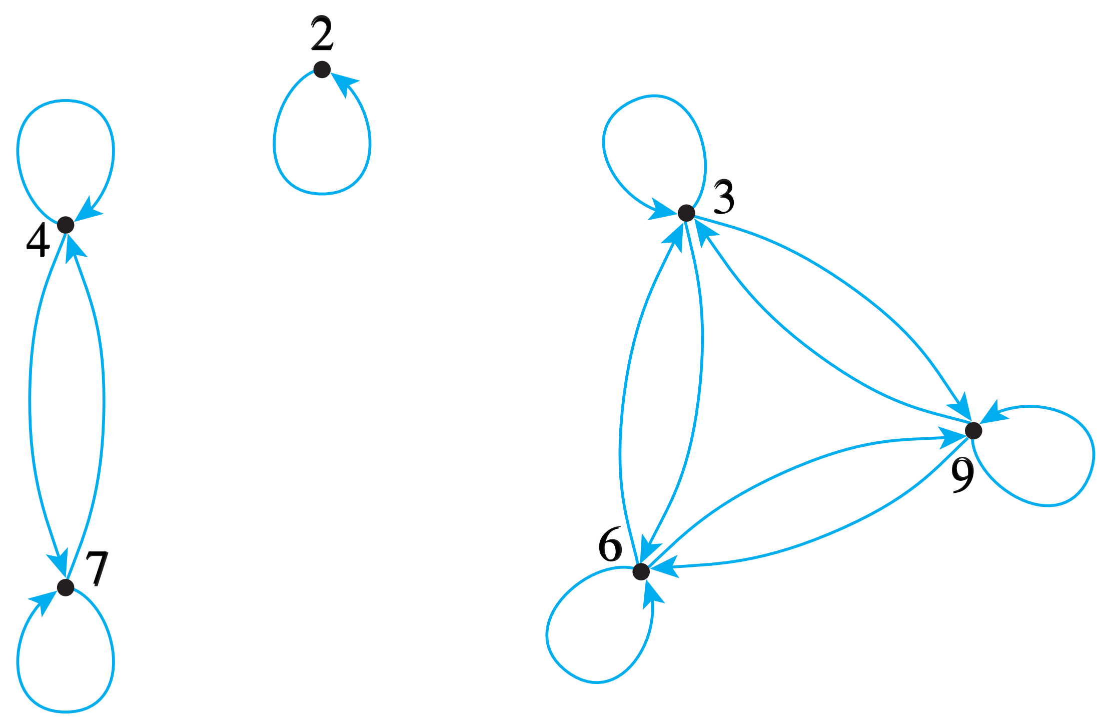
Examples:

Likes?	MemberOf?	BrotherOf?
LocatedIn?	PartOf?	SonOf?
Kills?	SubSetOf?	FatherOf?
FreindOf?	SameAS?	RelativeOf?

Example

Let $A = \{2,3,4,6,7,9\}$ and define a relation R on A as:

For all $x, y \in A$, $x\ R\ y \Leftrightarrow 3 \mid (x-y)$.



Is R Reflexive? Symmetric? Transitive?

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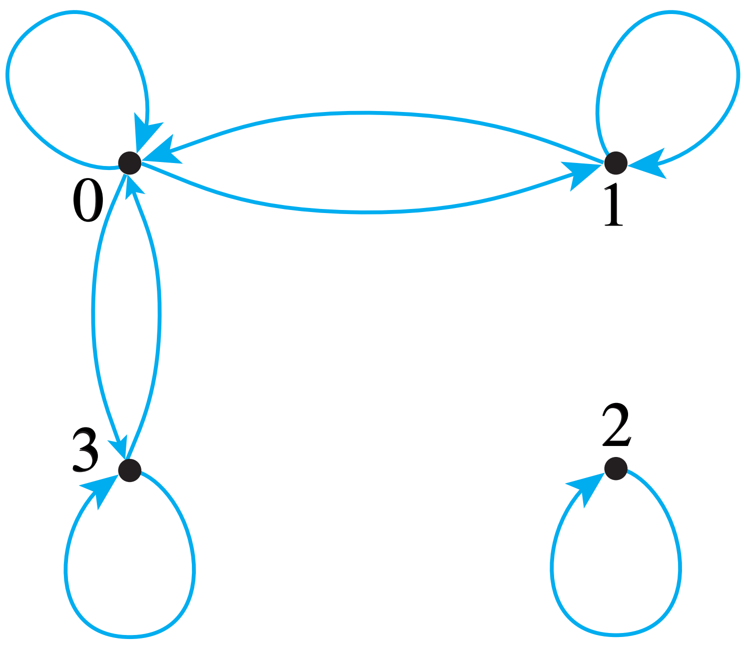
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$



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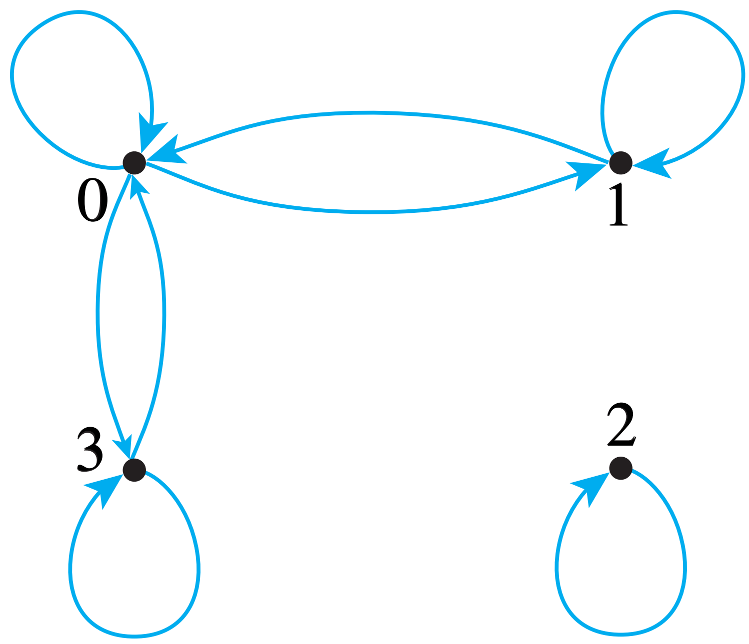
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$

Is R Reflexive? ☑ *Symmetric?* ☑ *Transitive?* ☒

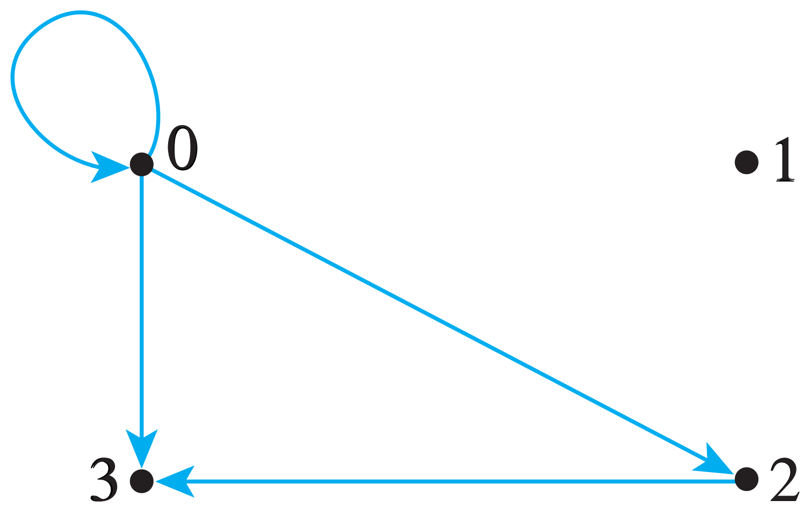


Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$

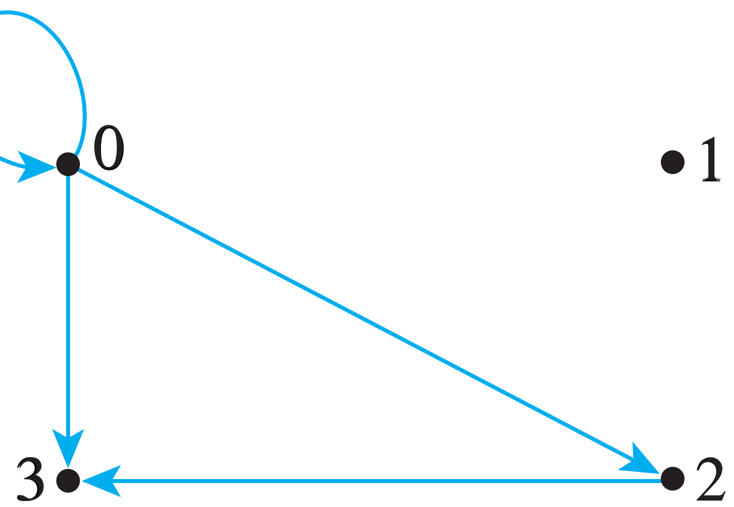
Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$



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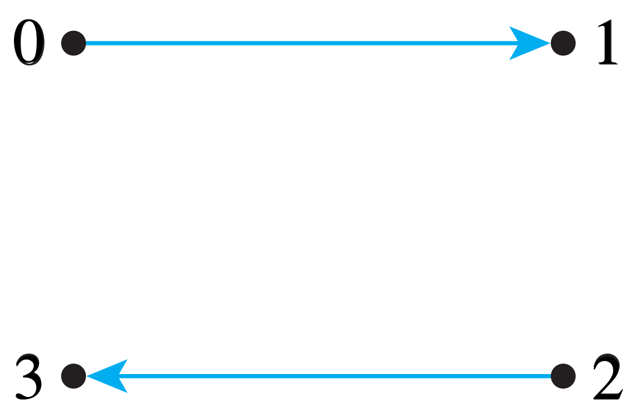
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0,1),(2,3)\}$



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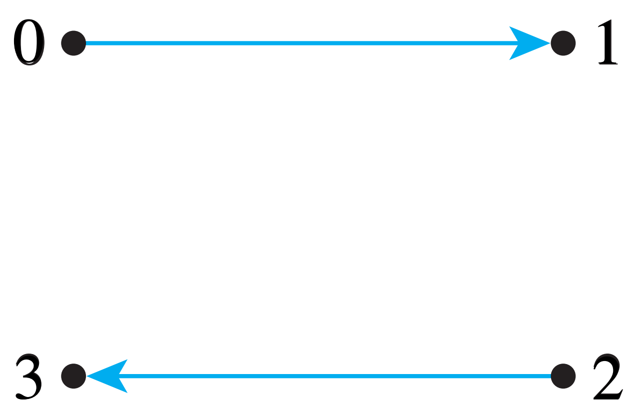
Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0,1),(2,3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



T **is transitive by default because it is *not* not transitive!**

Relations

8.2 Properties of Relations

In this lecture:

- ❑ Part 1: Properties: Reflexivity, Symmetry, Transitivity
- ❑ Part 2: **Proving Properties of Relations**
- ❑ Part 3: Transitive Closure

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Proving Properties on Relations on Infinite Sets

Until Now we discussed relation on Finite Sets

Next, we discussed relation on infinite Sets

To prove a relation is reflexive, symmetric, or transitive, first write down what is to be proved, in **First Order Logic**.

For instance, for symmetry

$\forall x, y \in A$, if $x\, R\, y$ then $y\, R\, x$.

Then use **<u>direct methods</u>** of proving

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? Symmetric? Transitive?

***R is reflexive*: R is reflexive if, and only if, the following statement is true: For all $x \in \mathbf{R}$, $x \, R \, x$. And since $x \, R \, x$ just means that $x = x$, this is the same as saying For all $x \in \mathbf{R}$, $x = x$. Which is true; every real number is equal to it

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Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

***R is symmetric*:** R is symmetric if, and only if, the following statement is true:
For all $x,y \in \mathbf{R}$, **if** $x \, R \, y$ then $y \, R \, x$.

By definition of R, $x \, R \, y$ means that $x = y$ and $y \, R \, x$ means that $y = x$.
Hence R is symmetric if, and only if,
For all $x,y \in \mathbf{R}$, **if** $x=y$ then $y=x$.

This statement is true; if one number is equal to a second, then the second is equal to the first.

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows: For all real numbers x and y. $x\ R\ y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

***R is transitive*:** R is transitive if, and only if, the following statement is true: For all $x,y,z \in \mathbf{R}$, **if** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

By definition of R, $x\ R\ y$ means that $x = y$, $y\ R\ z$ means that $y = z$, and $x\ R\ z$ means that $x = z$. Hence R is transitive iff the following statement is true: For all $x,y,z \in \mathbf{R}$, **if** $x=y$ and $y=z$ then $x=z$.

This statement is true: If one real number equals a second and the second equals a third, then the first equals the third.

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Properties of Less Than

Define a relation R on $\mathbf{R}$ (the set of all real numbers) as follows:
For all $x, y \in R$, $x\, R\, y \Leftrightarrow x < y$.

Is R Reflexive? *Symmetric?* *Transitive?*

***R* is not reflexive:** R is reflexive if, and only if, $\forall x \in \mathbf{R}$, $x \; R \; x$. By definition of R, this means that $\forall x \in \mathbf{R}$, $x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

***R* is not symmetric:** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x \; R \; y$ then $y \; R \; x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

***R* is transitive:** R is transitive if, and only if, for all $x, y, z \in \mathbf{R}$, if $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$. By definition of R, this means that for all $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive. ■

Properties of Congruence Modulo 3

Define a relation T on $\mathbf{Z}$ (the set of all integers) as follows: For all integers m and n,

$$m\ T\ n \Leftrightarrow 3 | (m-n).$$

Is R ***Reflexive?*** *Symmetric? Transitive?*

For all $m \in \mathbf{Z}$, $3 | (m-m)$.

Suppose m is a particular but arbitrarily chosen integer. *[We must show that m T m.]*

Now, $m - m = 0$.

But $3 \mid 0$ since $0 = 3 \cdot 0$.

Hence $3 | (m-m)$.

Thus, by definition of T, $mT\ m$

[as was to be shown].

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Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n,

$$m \, T \, n \;\Leftrightarrow\; 3 \mid (m-n).$$

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m, n \in \mathbf{Z}$, **if** $3 \mid (m-n)$ then $3 \mid (n-m)$.

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m \, T \, n$.

[We must show that n T m.]

By definition of T, since $m \, T \, n$ then $3 \mid (m - n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k.

Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, $n \, T \, m$

[as was to be shown].

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n,

$$m\,T\,n \Leftrightarrow 3|(m-n).$$

Is R Reflexive? *Symmetric?* <u>*Transitive?*</u>

For all $m, n \in \mathbf{Z}$, **if** $3|(m-n)$ and $3|(n-p)$ then $3|(m-p)$.

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\,T\,n$ and $n\,T\,p$. *[We must show that $m\,T\,p$.]* By definition of T, since m T n *and* $n\,T\,p$, *then* $3|(m-n)$ *and* $3|(n-p)$. By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m-n)+(n-p)=3r+3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3|(m - p)$. Hence, by definition of T, $m\,T\,p$ *[as was to be shown]*.

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Relations

8.2 Properties of Relations

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- ❑ Part 2: Proving Properties of Relations
- ❑ Part 3: **Transitive Closure**

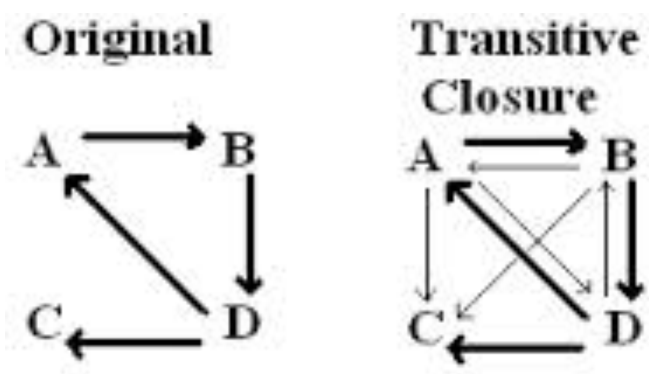
The Transitive Closure of a Relation

The **smallest** transitive relation that contains the relation.

• Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.



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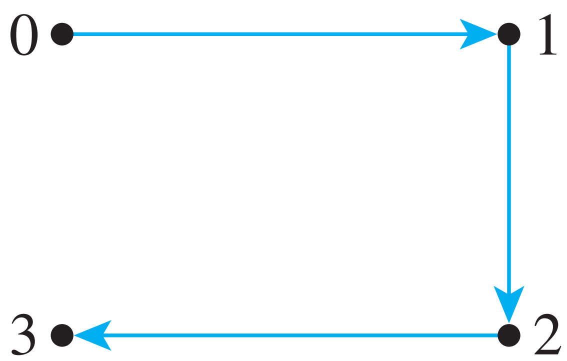
Exercise

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.



R

R^t

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Exercise

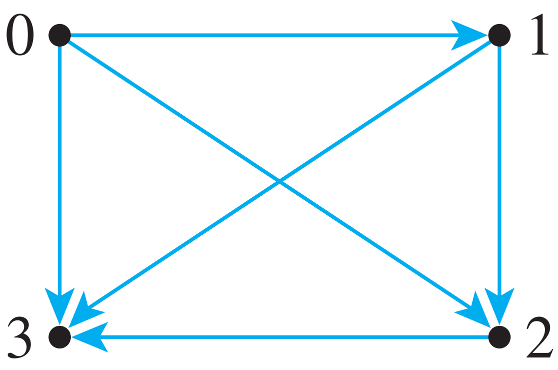
Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

$R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$

R



$\mathbf{R^t}$